Mathematisches Forschungsinstitut Oberwolfach

Report No. 24/2016

DOI: 10.4171/OWR/2016/24

Rough Paths, Regularity Structures and Related Topics

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1 May – 7 May 2016

Abstract. The purpose of the Oberwolfach workshop "Rough Paths and Regularity Structures" was to bring together these researchers, both young and senior, with the aim to consolidate progress in rough path theory and stochastic partial differential equations.

Mathematics Subject Classification (2010): 34A99,35R60, 60H10, 60H15, 65C99.

Introduction by the Organisers

Since its original development in the mid-nineties by T. Lyons the theory of rough paths, based on the profound insight that stochastic differential equations can be solved pathwise and that the solution map is continuous in suitable rough path metrics, has grown into a mature and widely applicable mathematical theory. Spectacular recent progress was made by Hairer (workshop participant, Fields Medal 2014) and then Gubinelli-Imkeller-Perkowski with their respective extensions of rough paths to "rough fields" (first presented in a similarly-spirited MFO meeting in 2012), capable of giving meaning and robust solutions theories to a number of singular non-linear stochastic partial differential equations (SPDEs).

The workshop was held between 1st-7th May 2016. Its aims were twofold: to develop insights and applications on classical rough path theory on the one side, and to investigate non-linear SPDEs and regularity structures on the other.

T. Lyons opened the meeting by giving a survey of recent applications of rough paths to the analysis of data-streams. Other participants presented recent work in rough differential equations and rough paths. For example, S. Aida, presented a theory of reflected rough differential equations, A. Deya reported ongoing work on the ergodic properties of rough systems. M. Hofmanova presented work on operator-valued rough paths, in particular applicable to (non-singular) SPDEs. Y. Inahama showed how rough path continuity can be used in short-time asymptotic expansions of hypoelliptic heat kernel at the cut locus. H. Kawabi discussed non-symmetric random walks on nilpotent covering graphs, naturally related to Brownian rough paths with perturbed Levy area.

Progress on the side of singular SPDEs was introduced with two survey lectures, given by H. Weber and A. Chandra on regularity structures and paracontrolled distributions. These were followed by talks of M. Hairer on a singular SPDE motivated by the geometrical evolution of random loops on manifolds and A. Kupiainen which explained an approach to singular SPDEs using Wilsonian renormalization group. L. Zambotti reported on the algebraic structures involved in the renormalization of a general class of SPDEs. While numerical approximation schemes for singular SPDEs were discussed by K. Matetski. With financial applications in mind, J. Teichmann discussed stochastic integration in the context of processes with values in spaces of modelled distributions.

Other talks were dedicated to problems in nearby areas of stochastic analysis. A. Abdesselam addressed the problem of defining pointwise products of singular random Schwartz distributions using local integrability of moment kernels and discussed connections with the description of non-linear operation on distribution given by the Operator Product Expansion. A. Nahmod gave a comprehensive introduction to the random data approach to the analysis of the low regularity theory for nonlinear dispersive equations.

Over 50 invited participants attended the workshop. These scientists came from a diverse set of countries and young mathematicians were especially well-represented among them. The Mathematisches Forschungsinstitut Oberwolfach provided the ideal environment for fruitful discussion between participants to develop new collaborations and to enhance the synergy among those working in these related areas.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows".

Workshop: Rough Paths, Regularity Structures and Related Topics

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Abstracts

Products of Random Distributions and Wilson's Operator Product Expansion

Abdelmalek Abdesselam

In this extended abstract we will report on the recent result obtained in [2] which is a generalization to the non-Gaussian setting of the Wick product construction for random distributional fields. Assume that we have a collection $(\mathcal{O}_A)_{A\in\mathcal{B}}$ of $\mathcal{S}'(\mathbb{R}^d)$-valued random variables on some probability space $(\Omega,\mathcal{F},\mathbb{P})$. Suppose they have moments of all orders, i.e., for all test function $f\in\mathcal{S}(\mathbb{R}^d)$, all $A\in\mathcal{B}$ and all $p\geq 1$, we have that the real-valued random variable $\mathcal{O}_A(f)\in L^p(\Omega,\mathcal{F},\mathbb{P})$. Let ρ be a mollifier, namely, a smooth compactly supported function $\mathbb{R}^d\to\mathbb{R}$ with $\int\rho=1$. For some fixed number $L>1$, and for each $r\in\mathbb{Z}$ define the rescaled function $\rho_r(x)=L^{rd}\rho(L^r x)$. We also define translates of the latter $\rho_{r,x}(y)=\rho_r(y-x)$ for each point $x\in\mathbb{R}^d$. The moments of the given random variables such as

$$\mathbb{E}\left[\mathcal{O}_{A_1}(f_1)\cdots\mathcal{O}_{A_n}(f_n)\right]$$

can be seen as continuous n-linear forms on Schwartz space and also, via the nuclear theorem, as elements of $\mathcal{S}'(\mathbb{R}^{dn})$. The pointwise correlations or moments are defined by the limit

$$\langle\mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_n}(x_n)\rangle=\lim_{r\to-\infty}\mathbb{E}\left[\mathcal{O}_{A_1}(\rho_{r,x_1})\cdots\mathcal{O}_{A_n}(\rho_{r,x_n})\right]$$

if it exists.

The first main assumption we make is that these pointwise correlations exist and are smooth functions on the configuration space Conf_n, i.e., the set of n-tuples $(x_1\ldots,x_n)$ made of distinct points in $\mathbb{R}^d$. Furthermore, we require that moments are given by integration against such pointwise correlations. More precisely, this includes the local integrability condition

$$\int_{K^n\cap\mathrm{Conf}_n}dx_1\ldots dx_n\ |\langle\mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_n}(x_n)\rangle|\ <\infty$$

for every compact $K\subset\mathbb{R}^d$, as well as the condition

$$\mathbb{E}\left[\mathcal{O}_{A_1}(f_1)\cdots\mathcal{O}_{A_n}(f_n)\right]=\int_{\mathrm{Conf}_n}dx_1\ldots dx_n\ \ \langle\mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_n}(x_n)\rangle\ \ f_1(x_1)\cdots f_n(x_n)$$

for all $f_1,\ldots,f_n\in\mathcal{S}(\mathbb{R}^d)$.

Given these hypotheses it is trivial to define more complicated pointwise correlations using formal multilinear expansion. For instance if $f(x,y)$ is a function on Conf_2 Then

$$\langle\ (\mathcal{O}_A(x)\mathcal{O}_B(y)-f(x,y)\mathcal{O}_C(y))\ \ (\mathcal{O}_D(z)-\mathcal{O}_E(u))\ \ \mathcal{O}_F(v)\ \rangle$$

is to be understood as

$$\begin{array}{rl} & \langle \mathcal{O}_A(x)\mathcal{O}_B(y)\mathcal{O}_D(z)\mathcal{O}_F(v)\rangle \\ - & \langle \mathcal{O}_A(x)\mathcal{O}_B(y)\mathcal{O}_E(u)\mathcal{O}_F(v)\rangle \\ - & f(x,y)\,\langle \mathcal{O}_C(y)\mathcal{O}_D(z)\mathcal{O}_F(v)\rangle \\ + & f(x,y)\,\langle \mathcal{O}_C(y)\mathcal{O}_E(u)\mathcal{O}_F(v)\rangle \end{array}$$

which is a well defined function of $(x,y,z,u,v) \in \mathrm{Conf}_5$. We also assume that for each field $\mathcal{O}_A$ we are given a number called the scaling dimension $[A]$ which governs the short distance singularities on the big diagonal. For instance, we are requiring that covariance kernels $\langle \mathcal{O}_A(x)\mathcal{O}_A(y)\rangle$ are bounded by $|x-y|^{-2[A]}$ (modulo eventual logarithmic corrections) for $|x-y|$ small.

We say that an abstract system of pointwise correlations (eventually with fields indexed by a set $\mathcal{A}$ containing $\mathcal{B}$) satisfies Wilson's operator product expansion (OPE) if there exists smooth functions $\mathcal{C}_{A,B}^{C}(x,y)$ on Conf_2 such that one has "inside correlations" an expansion of the form

$$\mathcal{O}_A(x)\mathcal{O}_B(y) = \sum_{[C]\le\Delta} \mathcal{C}_{A,B}^{C}(x,y)\mathcal{O}_C(y) + o(|x-y|^{\Delta-[A]-[B]})$$

for given cutoff Δ on scaling dimensions. This allows us to state the second main hypothesis of our theorem which is the bound $\exists\eta>0$, $\exists\gamma>0$, $\forall\epsilon>0$, $\exists k\in\mathbb{N}$, $\exists K>0$,

$$\prod_{i=1}^{m+n} \mathbb{1}\left\{|y_i-x_i| \le \eta \min_{j\neq i}|x_i-x_j|\right\} \times$$

$$\left|\left\langle \prod_{i=1}^{m}\mathrm{OPE}_i(y_i,x_i) \prod_{i=m+1}^{m+n}\mathrm{CZ}_i(y_i,x_i) \prod_{i=m+n+1}^{m+n+p}\mathcal{O}_{B_i}(x_i)\right\rangle\right| \le$$

$$K\prod_{i=1}^{m+n+p}\langle x_i\rangle^k \times \prod_{i=1}^{m+n}\langle y_i\rangle^k \times \prod_{i=1}^{m}\left\{|y_i-x_i|^{\Delta_i+\gamma-[A_i]-[B_i]}\times\left(\min_{j\neq i}|x_i-x_j|\right)^{-\Delta_i-\gamma-\epsilon}\right\}$$

$$\times\prod_{i=m+1}^{m+n}\left\{|y_i-x_i|^{\gamma}\times\left(\min_{j\neq i}|x_i-x_j|\right)^{-[B_i]-\gamma-\epsilon}\right\}\times\prod_{i=m+n+1}^{m+n+p}\left(\min_{j\neq i}|x_i-x_j|\right)^{-[B_i]-\epsilon}$$

where we used the notation $\langle x\rangle = \sqrt{1+|x|^2}$, as well as "OPE" for objects of the form

$$\mathrm{OPE}_i(y_i,x_i) = \mathcal{O}_{A_i}(y_i)\mathcal{O}_{B_i}(x_i) - \sum_{[C_i]\le\Delta_i}\mathcal{C}_{A_i,B_i}^{C_i}(y_i,x_i)\mathcal{O}_{C_i}(x_i)\ ,$$

and "CZ" for objects of the form

$$\mathrm{CZ}_i(y_i,x_i) = \mathcal{O}_{B_i}(y_i) - \mathcal{O}_{B_i}(x_i)\ .$$

The third needed hypothesis is a mild condition on the kernels $\mathcal{C}_{A,B}^{C}(x,y)$ which means that the corresponding distribution in $\mathcal{S}'(\mathbb{R}^{2d}) = \mathcal{S}'(\mathbb{R}^d)\,\widehat{\otimes}\,\mathcal{S}'(\mathbb{R}^d)$ in fact belongs to the smaller space $\mathcal{S}'(\mathbb{R}^d)\,\widehat{\otimes}\,\mathcal{O}_M(\mathbb{R}^d)$ (where $\mathcal{O}_M$ is the space of smooth temperate functions) together with a bound of the form $|x-y|^{[C]-[A]-[B]}$ near the diagonal (modulo eventual logarithmic corrections which explain the ϵ precaution in the bound above).

Suppose we have a system of abstract pointwise correlations indexed by $\mathcal{A} = \mathcal{B} \cup \{C_*\}$ satisfying the previous hypotheses and a pair $A, B \in \mathcal{B}$ such that $\mathcal{C}_{A,B}^{C_*}(x,y)$ is nonzero and obeys a lower bound of the form $|x-y|^{[C_*]-[A]-[B]}$. Our main theorem is a construction of the a priori "virtual" field $\mathcal{O}_{C_*}$ as a Borel measurable functional of the already existing fields $(\mathcal{O}_C)_{C\in\mathcal{B}}$ on our probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Essentially, we define it as a renormalized product of the fields $\mathcal{O}_A$ and $\mathcal{O}_B$, intuitively given by the formula

$$\mathcal{O}_{C_*}(x) = \lim_{y\to x} \frac{1}{\mathcal{C}_{A,B}^{C_*}(y,x)} \left(\mathcal{O}_A(y)\mathcal{O}_B(x) - \sum_{[C]\leq[C_*], C\neq C_*} \mathcal{C}_{A,B}^{C}(y,x)\mathcal{O}_C(x) \right) .$$

Of course, this is to be understood in the sense of distributions and needs proper smearing with a rescaled mollifier ρ_r, see [2, Theorem 1] for a more precise statement. We also require the scaling dimensions of the fields to belong to the interval $\left[0, \frac{d}{2}\right)$. Indeed, higher scaling dimensions would invalidate our local integrability hypothesis on pointwise correlations. The construction of renormalized Wick products for Gaussian fields as in [3] is the simplest application of our theorem. However, the latter should also apply in the case of non-Gaussian measures arising in Euclidean quantum field theory and scaling limits of lattice spin systems such as the critical long-range Ising model in three dimensions. See [1] for a broader perspective. Indeed, our new result [2, Theorem 1] shows how to deduce [1, Conjecture 9] from [1, Conjecture 8].

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Reflected rough differential equations via controlled paths

Shigeki Aida

Let $\mathbf{X}_{s,t} = (X_{s,t}, \mathbb{X}_{s,t})$ be a β-Hölder rough path $(1/3 < \beta \leq 1/2)$. Let us consider rough differential equations (=RDEs), $dY_t = \sigma(Y_t)d\mathbf{X}_t, Y_0 = \xi$. The existence of the solutions were proved by A.M. Davie (2008) under the assumption that $\sigma \in \mathrm{Lip}^{\gamma-1}$. Note that when $\gamma = n+\theta$ $(n \in \mathbb{Z}^+, 0 < \theta \leq 1)$, Lip^γ denotes the set of C_b^n functions such that the n-times deivative is θ-Hölder continuous. Also he proved that there exist infinitely many solutions to the RDE for a certain $\sigma \in \mathrm{Lip}^{2-\varepsilon}$ and for almost all Brownian rough paths. On the other hand, if the state space of stochastic processes is a domain of a Euclidean space, we need to study stochastic differential equations with (normal) reflections at the boundary. The equation contains the bounded variation term $\Phi(t)$ corresponding to the local time term. By using the Skorohod map, the SDE is transformed to a path-dependent SDE without reflection term. However we cannot expect the Lipschitz continuity of the Skorohod map generally. Therefore previous studies on RDEs cannot be applied directly to RDEs with normal reflection.

Reflected rough differential equations(=RRDEs) are defined on connected domains D. The boundary need not be smooth. As in the work by Lions-Sznitman and Saisho, we consider the following conditions (A), (B) on the boundary.

Definition 1. *We write* $B(z,r) = \{y \in \mathbb{R}^d \mid |y-z| < r\}$*, where* $z \in \mathbb{R}^d$*,* $r > 0$*. The set* $\mathcal{N}_x$ *of inward unit normal vectors at the boundary point* $x \in \partial D$ *is defined by*

$$\mathcal{N}_x = \cup_{r>0}\mathcal{N}_{x,r},$$
$$\mathcal{N}_{x,r} = \left\{\boldsymbol{n} \in \mathbb{R}^d \mid |\boldsymbol{n}| = 1, B(x - r\boldsymbol{n}, r) \cap D = \emptyset\right\}.$$

(A) *There exists a constant* $r_0 > 0$ *such that*

$$\mathcal{N}_x = \mathcal{N}_{x,r_0} \neq \emptyset \quad \text{for any } x \in \partial D.$$

(B) *There exist constants* $\delta > 0$ *and* $\delta' \leq 1$ *satisfying:*
for any $x \in \partial D$ *there exists a unit vector* l_x *such that*

$$(l_x, \boldsymbol{n}) \geq \delta' \qquad \text{for any } \boldsymbol{n} \in \cup_{y \in B(x,\delta)\cap\partial D}\mathcal{N}_y.$$

When the domain D satisfies the conditions (A) and (B), the Skorohod problem associated with a continuous path $w \in C([0,T], \mathbb{R}^d)$ with $w_0 \in \bar{D}$

$$y_t = w_t + \phi_t, \quad y_t \in \bar{D} \quad 0 \leq t \leq T,$$
$$\phi_t = \int_0^t 1_{\partial D}(y_s)\boldsymbol{n}(s)d|||\phi|||_{[0,s]}, \qquad \boldsymbol{n}(s) \in \mathcal{N}_{y_s} \text{ if } y_s \in \partial D$$

can be uniquely solved (Saisho, 1987). Here $|||\phi|||_{[s,t]}$ denotes the total variation of ϕ_u $(s \leq u \leq t)$. We write $L(w)(t) = \phi_t$. The following is our main theorem of this talk.

Theorem 1. *Assume* $\sigma \in \mathrm{Lip}^{\gamma-1}(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^d))$*. Then there exists a controlled path* $(Z, Z') \in \mathscr{D}_X^{2\beta}(\mathbb{R}^d)$ *and a bounded variation path* $\Phi \in \mathcal{V}_{1,\beta}(\mathbb{R}^d)$ *with* $\Phi_0 = 0$ *such that*

$$Z_t = \xi + \int_0^t \sigma(Z_s + \Phi_s)d\mathbf{X}_s, \; Z'_t = \sigma(Z_t + \Phi_t),$$
$$\Phi_t = L\left(\xi + \int_0^{\cdot} \sigma(Z_s + \Phi_s)d\mathbf{X}_s\right)_t.$$

Further there exist positive constants C_1, C_2 *such that*

$$\|Z\|_\beta + \|R^Z\|_{2\beta} + |||\Phi|||_{1,\beta} \leq C_1 e^{C_2\tilde{\rho}_\beta(\mathbf{X})}\tilde{\rho}_\beta(\mathbf{X}),$$

where C_1, C_2 *are constants which depend only on* σ, β, γ*.*

About the notation above:

$$\tilde{\rho}_\beta(\mathbf{X}) = \sum_{i=1}^{3} \rho_\beta(\mathbf{X})^i, \quad \rho_\beta(\mathbf{X}) = \|X\|_\beta + \sqrt{\|\mathbb{X}\|_{2\beta}},$$

where $\|\mathbb{X}\|_{2\beta} = \sup_{0<s<t<T} \frac{|\mathbb{X}_{s,t}|}{(t-s)^{2\beta}}$. $\mathcal{V}_{1,\beta}(\mathbb{R}^d)$ is a Banach space consisting of continuous bounded variation paths Φ satisfying

$$|||\Phi|||_{1,\beta} := \sup_{0<s<t<T} \frac{|||\Phi|||_{[s,t]}}{|t-s|^\beta} < \infty.$$

In [1], we prove the existence of solutions under that $\sigma \in C_b^3$ and the condition,

(H1) Assume (A) and the mapping L appeared in the Skorohod problem satisfies

$$|||L(w)|||_{[s,t]} \le C_D \max_{s\le u,v\le t} |w(v) - w(u)| \quad 0 \le s \le t \le T.$$

Tanaka (1979) proved that (H1) holds if D is convex and there exists a unit vector $l \in \mathbb{R}^d$ such that $\inf\{(l, \boldsymbol{n}(x)) \mid \boldsymbol{n}(x) \in \mathcal{N}_x, x \in \partial D\} > 0$. Clearly, these conditions imply (A) and (B).

Actually, the assumption $\sigma \in C_b^3$ can be relaxed to $\sigma \in \mathrm{Lip}^{\gamma-1}$ by a suitable modification of the proof in [1]. On the other hand, concerning the boundary condition, to prove Theorem 1, we use some properties of L which can be proved by using (A) and (B). The properties of L is much weaker than (H1). Hence Theorem 1 improves the existence theorem in [1] in this sense too.

Even if σ is sufficiently smooth, at the moment, it is not clear whether the uniqueness of solutions holds. However, under the assumption in Theorem 1, we can prove the existence of universally measurable solution mapping. About this version, we show the support theorem holds.

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SDE based regression for random PDEs

Christian Bayer

(joint work with Felix Anker, Martin Eigel, Marcel Ladkau, Johannes Neumann and John Schoenmakers)

In this presentation, we consider the problem of a PDE with random coefficients. In contrast to many other presentations at this workshop, we are working in a *linear* setting, and the noise is smooth. The purpose of this presentation is, hence, not to give a meaning to the (solution of the) equation, but rather to *solve* the equation numerically. To fix ideas, we shall consider one particular case, namely *Darcy's equation*,

$$-\nabla \cdot (\kappa(x)\nabla u(x)) = f(x), \quad x \in D, \tag{1a}$$

$$u(x) = g(x), \quad x \in \partial D. \tag{1b}$$

This equation is used to model groundwater flow, but also oil reservoir and many other similar objects, and has been established as the de-facto benchmark problem in uncertainty quantification, see for instance [1]. In (1), all the coefficients f, g, κ can be random fields, but we will mostly concentrate on the field κ. As indicated above, we assume that κ is smooth (e.g., C^2) in space (for instance as obtained from a Karhunen-Loeve expansion of a rougher fields, or simply by mollification), and positive (so that we can solve (1) for each individual ω).

Standard techniques for solving (1) include Monte Carlo simulation coupled with FEM, spectral methods or stochastic collocation methods. We propose an alternative based on

- point-wise stochastic representation of the solution by a Feynman-Kac formula (with random vector fields);
- spacial resolution of the solution by Monte Carlo regression.

Suppose that the actual quantity of interest is the function $v(x) = E[u(x)]$, $x \in D$—or a linear functional thereof. It is worth noting that the proposed method allows us to directly approximate v, i.e., we do not need a (nested) Monte Carlo simulation on u (on top of the Monte Carlo regression for u). Indeed, let

$$dX_t^x = \nabla\kappa(X_t)dt + \sqrt{2\kappa(X_t)}dW_t, \quad Z_t^x = \int_0^t f(X_s)ds, \tag{2}$$

started at $X_0 = x$, for a Brownian motion W independent of the random coefficients κ, f, g, then the Feynman-Kac representation implies

$$v(x) = E\left[\Phi^x\right], \quad \Phi^x \equiv g(X_\tau^x) + Z_\tau^x, \quad x \in D, \tag{3}$$

τ denoting the first hitting time of ∂D.

Regarding the second step, we assume we are given basis functions $\phi_1, \dots, \phi_K : D \to \mathrm{R}$. Using Monte Carlo regression, we compute the (L^2) projection of v to the span of these basis functions. To this end, we give ourselves a probability measure μ on D, sample points $x_1, \dots, x_M \in D$ and, starting from these points, sample Φ^{x_i} (with the Wiener process and the random coefficients chosen independent from the initial point). The principal idea is now to minimize the sum of squares

$$\sum_{i=1}^{M}\left(\Phi^{x_i} - \sum_{j=1}^{K}\alpha_j\phi(x_j)\right)^2 \to \min$$

in the coefficients $\alpha_1, \dots, \alpha_K$. However, since we have full control of both basis functions and sampling measure μ, it makes sense to choose them in such a way that the basis functions are orthonormal w.r.t. μ. Define the *semi-stochastic* regression coefficients $\overline{\gamma}$ by

$$\overline{\gamma} \equiv \frac{1}{M}\mathcal{M}^\top \mathcal{Y} \tag{4}$$

with

$$\mathcal{M} \equiv (\phi_k(x_i))_{1\le i\le M, 1\le k\le K}\,, \quad \mathcal{Y} \equiv (\Phi^{x_i})_{1\le i\le M}\,.$$

The corresponding approximate solution $\overline{v} \equiv \sum_{k=1}^{K} \overline{\gamma}_k \phi_k$ can then be seen to converge to the true *projection* with error proportional to $\frac{1}{M}$ in the MSE sense. The procedure is visualized In Figure 1.

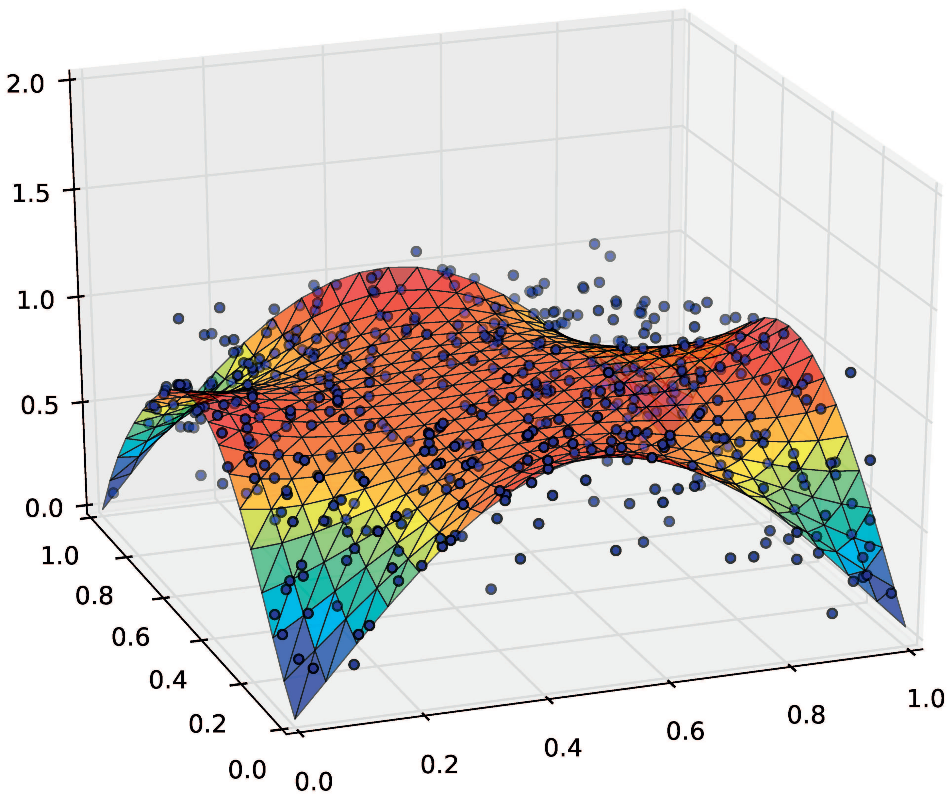

FIGURE 1. Regression procedure

Finally, the SDE (2) is solved numerically by an adaptive (to the distance to ∂D) Euler scheme.

The method proposed here is very general and only relies on the existence of a Feynman-Kac type stochastic representation. Therefore, we think that the method will also work for those types of rough PDEs, where such a representation exists, thereby reducing the problem of approximating the rough PDE to the problem of solving the corresponding rough DE.

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Factorial decay estimates for rough paths

Horatio Boedihardjo

In this talk we will discuss a class of factorial decay estimates in rough path theory. The motivation for these estimates arise from the study of linear differential equation. More precisely, let X be a α-Hölder geometric rough path in $\mathbb{R}^d$ and $f : \mathbb{R}^d \to L(\mathbb{R}^d, \mathbb{R}^{d'})$ be a linear vector field. One way to construct a solution for

$$\mathrm{d}Y_t = f(Y_t)\mathrm{d}X_t$$

is through Picard's iteration. Picard's iteration gives formally the following expansion for the solution Y_t

$$Y_t = Y_0 + \int_0^t f(Y_0)\mathrm{d}X_{s_1} + \int_0^t \int_0^{s_2} f[f(Y_0)\mathrm{d}X_{s_1}]\mathrm{d}X_{s_2} + \ldots.$$

This series for Y_t actually converges because of Lyons' estimate [7] that

$$\left\| \int_0^t \cdots \int_0^{s_3} \int_0^{s_2} \mathrm{d}X_{s_1} \otimes \ldots \otimes \mathrm{d}X_{s_n} \right\| \leq \frac{c_\alpha^n |X|_{\alpha-H\ddot{o}l}^n t^{\alpha n}}{(n!)^\alpha}$$

where $|X|_{\alpha-H\ddot{o}l}$ denote the Hölder norm of the branched rough path X. The first main result of this talk is an analogue of this factorial decay result for branched rough paths. Branched rough paths are introduced by M. Gubinelli in [4] as a theory of rough path calculus where the chain rule

$$\mathrm{d}(XY) = X\mathrm{d}Y + Y\mathrm{d}X$$

does not necessarily hold. The failure of the chain rule means that, for Branched rough paths, the Picard's iteration will involve a lot more terms. These extended set of terms will be indexed by rooted trees. Applying the Picard's iteration for the following differential equation driven by a branched rough path X

$$\mathrm{d}Y_t = f(Y_t)\mathrm{d}X_t \tag{1}$$

will give a series expansion looking like

$$Y_t = Y_0 + f^{\circ\tau_1}(Y_0)\mathbb{X}_{0,t}^{\tau_1} + f^{\circ\tau_2}(Y_0)\mathbb{X}_{0,t}^{\tau_2} + \ldots$$

where $\mathbb{X}_{0,t}^{\tau}$ denote the tree-indexed "iterated integral" of Branched rough path X and $f^{\circ\tau}$ denote some "tree derivative" of f. We will not go into the precise definition of $\mathbb{X}^\tau$ and $f^{\circ\tau}$—for our purpose, it's enough to know that they arise naturally from the Picard's iteration. There is a natural notion of factorial of trees (see [4]) defined in the following way. If $\bullet$ denote the tree with a single vertex, then

$$\bullet! \;=\; 1.$$

If $[\tau_1, \ldots, \tau_n]_\bullet$ is the tree obtained by joining the roots of n rooted trees $\tau_1, \ldots, \tau_n$ to a new root $\bullet$, and let $|\tau|$ denote the number of vertices of a rooted tree τ, then

$$[\tau_1, \ldots, \tau_n]_\bullet! = (|\tau_1| + \ldots + |\tau_n| + 1)\tau_1! \ldots \tau_n!.$$

The first main result of this talk is the following:

Theorem 1. *(Conjectured by Gubinelli [4], proof proposed in [1]) There exists $c_\gamma > 0$ such that for all γ-Hölder Branched rough path X and all rooted trees τ,*

$$|\mathbb{X}^\tau_{s,t}| \leq \frac{c_\gamma^{|\tau|} |X|^{|\tau|}_{\gamma-H\ddot{o}l} (t-s)^{\gamma|\tau|}}{(\tau!)^\gamma}$$

where $\mathbb{X}^\tau$ denote the iterated integrals of X associated with the tree τ and $|\tau|$ be the number of vertices in τ.

The interesting thing about this inequality is that the conventional approach for proving this inequality—via the "neoclassical" inequality—fails. Our proof is based on Lyons' proof [6] in 1994 that does not use the "neoclassical" inequality. Unfortunately, that proof was constructed for α-Hölder geometric rough paths where $\alpha > \frac{1}{2}$. The proof for main result Theorem 1 requires an extension of Lyons' 94 approach to the Hölder exponent $\alpha \leq \frac{1}{2}$ and (the difficiult bit) to the branched rough paths.

As mentioned, iterated integrals are interesting because they arise from Picard's iteration. There is a slightly different type of approximation, known as the Taylor expansion, for the rough differential equation (1) with nonlinear vector field f. To describe the Taylor expansion, we need some more notations. We define $\frac{\mathrm{d}^k Y}{\mathrm{d}X^k} : \mathbb{R}^d \to L((\mathbb{R}^d)^{\otimes m}, \mathbb{R}^{d'})$ inductively by

$$\begin{aligned} \frac{\mathrm{d}Y}{\mathrm{d}X} &= f \\ \frac{\mathrm{d}^{k+1}Y}{\mathrm{d}X^{k+1}} &= D(\frac{\mathrm{d}^k Y}{\mathrm{d}X^k})\frac{\mathrm{d}Y}{\mathrm{d}X} = D(\frac{\mathrm{d}^k Y}{\mathrm{d}X^k})f. \end{aligned} \tag{2}$$

We now define the order-n Taylor expansion by

$$\Gamma^{(n)}_{s,t} Y_s = \sum_{k=0}^{n} \frac{\mathrm{d}^k Y}{\mathrm{d}X^k}(Y_s) \int_s^t \ldots \int_s^{s_2} dX_{s_1} \otimes \ldots \otimes dX_{s_k}.$$

The inductive relation (2) is well defined as $\frac{\mathrm{d}^k Y}{\mathrm{d}X^k}$ is a function on $\mathbb{R}^d$. The second main result of this talk is:

Theorem 2. *(B., Lyons, Yang [2])Let X be a α-Hölder geometric rough path. Let f be a Lip$(n+1)$ vector field with $n > \alpha^{-1}$. Let Y be solution to*

$$\mathrm{d}Y_t = f(Y_t)\mathrm{d}X_t.$$

There exists $C_{\alpha,f,X} > 0$, depending only on α, f and X such that

$$|Y_t - \Gamma^{(n)}_{s,t} Y_s| \leq \frac{C^{n+1}_{\alpha,f,X}}{(n+1)!^\alpha} \|f\|_{\circ n+1} \|X\|^{n+1}_{\alpha-H\ddot{o}l} (t-s)^{\alpha(n+1)}$$

where

$$\|f\|_{\circ n+1} = \max_{n-\lfloor\alpha^{-1}\rfloor+2\le m\le n+1} |\frac{\mathrm{d}^m Y}{\mathrm{d}X^m}|_{Lip(\min(n+1-m,1))}.$$

This inequality builds on earlier work by A.M. Davie ($\alpha > \frac{1}{3}$)[2] and Friz-Victoir (all α)[3] who proved, under the same assumptions, that

$$\|Y_t - \Gamma^n_{s,t} Y_s\| \le C_{f,n,X,\alpha} \|X\|^{n+1}_{\alpha-H\ddot{o}l} (t-s)^{\alpha n}.$$

Our main contribution is in making explicit the dependence of the "constant" $C_{f,n,X}$ on n.

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Hopf algebras of coloured forests in Regularity Structures

Yvain Bruned

(joint work with Martin Hairer and Lorenzo Zambotti)

The regularity structures introduced in [5] allow us to describe the solution of a singular SPDE by a Taylor expansion with new monomials. These monomials are of the form $\Pi_x \tau$ where τ belongs to an abstract space T and where Π_x interprets τ as a distribution center at the point x. For solving a singular SPDE, we smooth the noise with a mollifier. This procedure depends on a small parameter ε and we want to pass to the limit for the $\Pi_x^{(\varepsilon)}\tau$. It happens that some of these monomials are singular and we need to perform a renormalisation procedure. We use Hopf algebras in order to build two groups:

- The structure group (Positive renormalisation) which defines Π_x and the map Γ_{xy} used for changing the point of our monomials. This construction has been performed in [5] and the coproduct is close to the Connes-Kreimer coproduct [4].
- The renormalisation group (Negative renormalisation) which acts on the model (Π_x, Γ_{xy}) for proving the convergence. This one is defined through the extraction-contraction coproduct in [1]. This coproduct has been used for B-series in [3].

In the case of the B-series, the interaction of these two Hopf algebras has been outlined in [2]. But for the framework of regularity structure, these coproducts act on decorated trees. In order to specify the interaction and to obtain a simple formula for the renormalised model, we introduce in [1] the notion of coloured trees. The colours allow us to order the two renormalisations and to remember which part of the tree has been renormalised before. This information is one of the key point for preparing the ground for the convergence of the renormalised model.

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Malliavin Calculus for Regularity Structures: the case of gPAM

Giuseppe Cannizzaro

(joint work with Peter K. Friz and Paul Gassiat)

As it is well-known, the theory of Regularity Structures and the paracontrolled distributions approach allowed to solve a number of ill-posed stochastic partial differential equations (SPDEs). Now that local existence and uniqueness have been established, the common aim is to move forward and investigate finer properties of the solutions to such equations. Our work tries, in a sense, to go in this direction; we introduce Malliavin calculus tools in the context of Regularity structures and prove that the solution u to the generalized parabolic Anderson equation (gPAM), one standard example to which the theory applies, admits a density with respect to the Lebesgue measure when evaluated at a space-time point.
Recall that gPAM is formally given by the following non-linear SPDE

$$(\partial_t - \Delta)u = g(u)\xi, \qquad u(0,\cdot) = u_0(\cdot). \tag{1}$$

for $t \geq 0$, g sufficiently smooth, spatial white noise $\xi = \xi(x,\omega)$ and fixed initial data u_0. Assuming periodic boundary conditions, write $x \in \mathbb{T}^d$, the d-dimensional torus. Now a.s. the noise is a Gaussian random distribution, of Hölder regularity $\alpha < -d/2$. Standard reasoning suggests that u (and hence $g(u)$) has regularity $\alpha + 2$, due to the regularization of the heat kernel. But the product of two such Hölder distributions is only well-defined, if the sum of the regularities is strictly positive - which is the case in dimension $d = 1$ but not when $d = 2$. Hence we focus on gPAM in dimension $d = 2$, along [2] and also Gubinelli et al. [1] in the already mentioned paracontrolled framework.

A necessary first step in employing Malliavin calculus in this context is an understanding of the perturbed equation, formally given by

$$(\partial_t - \Delta)u^h = g(u^h)(\xi + h), \qquad u(0,\cdot) = u_0^h(\cdot) \tag{2}$$

where $h \in \mathcal{H}$, the Cameron–Martin space, nothing but L^2 in the Gaussian (white) noise case. Proceeding on this formal level and setting $v^h = \frac{\partial}{\partial \varepsilon}\{u^{\varepsilon h}\}|_{\varepsilon=0}$, one is naturally lead to the following *tangent equation*

$$(\partial_t - \Delta)v^h = g(u)h + v^h g'(u)\xi, \qquad v_0^h(\cdot) = 0. \tag{3}$$

Readers familiar with Malliavin calculus will suspect (correctly) that $v^h = \langle Du, h\rangle_{\mathcal{H}}$, where Du is the Malliavin derivative (better: $\mathcal{H}$-derivative) of u, solution to gPAM as given in (1). Once in possession of a Malliavin differentiable random variable, such as $u = u(t,x;\omega)$ for a fixed (t,x), non-degeneracy of $\langle Du, Du\rangle_{\mathcal{H}}$ will guarantee existence of a density. We have, loosely stated,

Theorem 1. *In spatial dimension $d = 2$, equations (1),(2),(3) can be solved in a consistent, renormalized sense (as reconstruction of modelled distributions, on a suitably extended regularity structure). If the solution u to (1) exists on $[0,T)$, for some explosion time $T = T(u_0;\omega)$, then so does then v^h, for any $h \in L^2$, and v^h is indeed the $\mathcal{H}$-derivative of u in direction h. At last, conditional on $0 < t < T$, and for fixed $x \in \mathbb{T}^2$, the solution $u = u(t,x;\omega)$ to gPAM admits a density with respect to the Lebesgue measure.*

In order to prove a Theorem as the one stated above, there is a number of steps and technical aspects that one has to take into account.

- To solve (1), (2) and (3) it is necessary at first to construct a suitable regularity structure, which loosely speaking, consists of a list of symbols representing the abstract counterpart of the processes on which the solution to the equation we aim at solving, continuously depends. If for (gPAM) this was done in [2] (resulting in $\mathcal{T}_g$), for (2), the very presence of a perturbation $h \in L^2$ forces us to introduce a new symbol H, which in turn induces several more and the notion of structure group has to be revisited for the enlarged structure.
- Once the previous point has been settled, we have to give a meaning to these symbols. To this purpose, Hairer defines the set of *admissible models* for a generic regularity structure $\mathcal{T}$, $\mathscr{M}(\mathcal{T})$, containing those maps that associate to each symbol a suitable distribution, or better the local expansion of a distribution around every point and that encode the main features of our equation. Now, we do not want to consider *any* admissible model on our enlarged regularity structure, but only those for which H is associated to a Cameron-Martin path, i.e. a function $h \in L^2$. Therefore, we introduce two maps, the *extension map* E, which assigns to every function in $\mathcal{H}$ and admissible model on $\mathcal{T}_g$, a *unique* admissible model on the enlarged structure satisfying certain properties (among which the

one stated above), and the *translation map* T, which assigns to every function in $\mathcal{H}$ and admissible model on $\mathcal{T}_g$, a *unique* admissible model on $\mathcal{T}_g$. Moreover, we show that these maps are both jointly locally Lipschitz continuous in each of their arguments.

- The fact that it is possible to "lift" the white noise to a suitable model was shown in [2]. Such a model was obtained as the limit of *renormalized* smooth sequences. It is not hard to show that indeed the extension and translation map commute with the renormalization procedure and, thanks to their local Lipschitz continuity, we are able to extend and translate this limiting gaussian model.
- We can now also "lift" our equations to the space of modelled distributions (the abstract counterpart of the space of Hölder functions), solve them through a fixed point procedure and show that the solution to the "abstract" tangent equation is indeed the derivative of the solution to the abstract (gPAM) via a simple application of the implicit function theorem. At last one reconstructs the modelled distributions so obtained, via the reconstruction operator (see [2]).
- Non-degeneracy of $\langle Du, Du\rangle_{\mathcal{H}}$ is established by a novel *strong maximum principle* for solutions to linear equations – on the level of modelled distributions – which may be of independent interest. Indeed, the argument we give, despite written in the context of gPAM, adapts immediately to other situations, such as the linear multiplicative stochastic heat equation in dimension $d = 1$ where we recover Mueller's work, [3], and to the linear PAM equation in dimensions $d = 2, 3$ for which the result appears to be new.

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Rough interacting particle systems

Rémi Catellier
(joint work with Ismaël Bailleul)

A classical problem when dealing with interacting particle systems consists at looking at the dynamic of n particles subject to mean-field interactions in the drift and driven by some independent Brownian motions $\big(X^{(i)}\big)_{i\geq 1}$, *i.e.*

$$\mathrm{d}Y_t^{(i),n} = b\big(Y_t^{(i),n}, \bar{\mu}_t^n\big)\mathrm{d}t + \sigma\big(Y_t^{(i),n}\big)\mathrm{d}X_t^{(i)}, \quad Y_0^{(i),n} \sim \mu_0. \tag{1}$$

Here

$$\bar{\mu}_t^n = \frac{1}{n}\sum_{k=1}^n \delta_{Y_t^{(k),n}} \in \mathcal{M}^1(\mathbb{R}^d)$$

is the empirical measure of the points $\left(Y^{(1),n}, \cdots, Y^{(n),n}\right)$,$\mathcal{M}^1(\mathbb{R}^d)$ is the set of all regular probability measures on $\mathbb{R}^d$, $\mu_0 \in \mathcal{M}^1(\mathbb{R}^d)$ and $b : \mathbb{R}^d \times \mathcal{M}^1(\mathbb{R}^d) \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \left(\mathbb{R}^d\right)^{\otimes 2}$ are two enough regular functions. One can think of b as

$$b(x,\mu) = \int_{\mathbb{R}^d} b(x,y)\mathrm{d}\mu(y), \tag{2}$$

but more general type of functions b will be considered.

A natural question for such a system is the typical behavior of a particle when n is large. Asymptotically, one expect *propagation of chaos*, which is equivalent to the convergence of the measure valued process $\bar{\mu}^n$ to a limit μ. One can also wonder more specific behavior of the empirical measure, such as large and moderate deviations, and central limit theorems.

In the Brownian case, those questions are generally answered thanks to martingale problems, by looking at non-linear partial differential equations such as McKean-Vlasov equations, see for example Sznitman [4] for more details. Nevertheless, when the processes $\left(X^{(i)}\right)$ are not Brownian motions, neither semi-martingales (one can think as fractional Brownian motion), those techniques are no longer available. A possible answer to that difficulty is to use rough paths theory, to make sense of the system and to study the empirical measure. Usually when dealing with rough paths, one regain some continuity properties of the solution map with respect to the data and one expect that this will also be the case in this setting.

Formally, if $\bar{\mu}^n$ converges to μ, one expect that μ satisfies the following couple equations :

$$\begin{cases} \mathrm{d}Y_t = b(Y_t,\mu_t)\mathrm{d}t + \sigma(Y_t)\mathrm{d}\mathbf{X}_t \\ \mathcal{L}(Y_t) = \mu_t, \end{cases} \tag{3}$$

where $\mathcal{L}(Y_t)$ denotes the law of Y_t. We present here a way to make sense of the previous non-linear rough differential equation, and apply this theory to the analysis of the interacting particle system (1).

Note that in the rough path setting, these questions have been studied previously by Cass and Lyons [2] when the drift b is of the form (2) and for general diffusivity σ. For the same kind of drifts, with $\sigma = id$ and for Brownian motions, Deuschel, Friz, Maurelli and Slowik [3] have studied some special kind of large deviation principles for the empirical measure.

The non-linear system: strategy. We want to solve the system (3) when $\mathbf{X}$ is a random $\frac{1}{p}$-Hölder geometric rough path. Furthermore we would like to have a continuity of the solution (as a measure valued process) with respect to the law of $\mathbf{X}$.

The strategy is described in the following steps. We fix a measure valued process $\mu : [0,T] \to \mathcal{M}^1(\mathbb{R}^d)$. Then we solve, in a pathwise sense, the rough differential equation

$$\mathrm{d}Y_t^\mu = b\big(Y_t^\mu, \mu_t\big)\mathrm{d}t + \sigma\big(Y_t^\mu\big)\mathrm{d}\mathbf{X}_t.$$

Since $\mathbf{X}$ is a random rough path, Y^μ is a random process, and it defines a measure valued process $\mathcal{L}(Y^\mu)$. Then we prove that the fixed point problem

$$\mathcal{L}(Y^\mu) = \mu$$

has a unique solution. Finally we show that this solution is a continuous function of the law of $\mathbf{X}$.

Topologies. To fulfill such a program, one need to specify the topologies one the different spaces which are involved. All of these topologies relies on the order of the rough path $p \geq 1$.

On the space $\mathcal{M}^1(\mathbb{R}^d)$ of regular probability measures on $\mathbb{R}^d$, we consider the trace of the topology of the dual space of $C_b^{[p]+1}$, *i.e.* for all $\mathcal{P}, \mathcal{Q}\ in \mathcal{M}^1(\mathbb{R}^d)$,

$$d\big(\mathcal{P}, \mathcal{Q}\big) = \sup_{\substack{f \in C_b^{[p]+1} \\ \|f\|_{C_b^{[p]+1}} \leq 1}} \big|\langle f, \mathcal{P} - \mathcal{Q}\rangle\big|.$$

Let $\rho \geq 1$. Let ν be a probability measure on the space of $\frac{1}{p}$-Hölder geometric rough paths. We say that ν has a finite ρ-moment, and we note $\nu \in E_\rho$ if

$$\int \big(1 + d(\mathbf{X}, 0)^\rho\big)\mathrm{d}\nu(\mathbf{X}) < +\infty,$$

where d is the homogeneous distance on the space of rough paths. On E_ρ we consider the weak topology for the convergence of probability measures, *i.e.* a sequence (ν^n) of elements of E_ρ converges to $\nu \in E_\rho$ if for all bounded and continuous function f from the space of rough paths to $\mathbb{R}$, we have

$$\langle f, \nu^n\rangle \underset{n\to\infty}{\longrightarrow} \langle f, \nu\rangle.$$

We are now able to state the main theorem of this analysis.

Theorem. Let $p \geq 1$. Let us suppose that $x \to b\big(x, \mathcal{P}\big)$ is in $C_b^{3+[p]}\big(\mathbb{R}^d, \mathbb{R}^d\big)$ uniformly in $\mathcal{P}$, that $\mathcal{P} \to b\big(x, \mathcal{P}\big)$ is Lipschitz continuous from $\mathcal{M}^1(\mathbb{R}^d)$ to $\mathbb{R}^d$ uniformly in x and that σ is in $C_b^{2([p]+1)}\big(\mathbb{R}^d, (\mathbb{R}^d)^{\otimes 2}\big)$.

Then there exists $\rho > p$ such that for all $\nu \in E_\rho$ and all random rough path $\mathbf{X}$ of law ν, there exists a unique solution $\mu \in \mathcal{C}^{\frac{1}{p}}\big([0,T]; \mathcal{M}^1(\mathbb{R}^d)\big)$ to the non-linear problem (3). Furthermore, the following function is continuous :

$$F : \begin{cases} E_\rho & \to \quad C\big([0,T]; \mathcal{M}^1(\mathbb{R}^d)\big) \\ \nu & \to \quad \mu = F(\nu). \end{cases}$$

Corollary: the interacting particle system. Thanks to the previous theorem, we are now able to study the empirical measure of the system (1). We use here a trick due to Cass and Lyons [2].

Take a sequence of rough paths $\big(\mathbf{X}^{(i)}\big)_{i\geq 1}$, fix $n \geq 1$ and consider the empirical measure

$$\bar{\nu}^n = \frac{1}{n}\sum_{i=1}^{n} \delta_{\mathbf{X}^{(i)}}.$$

Then $\bar{\nu}^n \in E_\rho$ for all $\rho > 1$. If the coefficients satisfy the previous hypothesis, one can show that the solution of the non-linear system (3) driven by a random rough path $\mathbf{X}$ of law $\bar{\nu}^n$ coincides with the empirical measure of the interacting particle system (1), hence we have

$$\bar{\mu}^n = F(\bar{\nu}^n).$$

Since F is a continuous function the asymptotic properties of $\bar{\nu}^n$ can be easily transfered to $\bar{\mu}^n$. This can be applied when $\big(\mathbf{X}^{(i)}\big)_{i\geq 1}$ is a sequence of i.i.d. $\frac{1}{p}$-Hölder geometric rough paths of common law $\nu \in E_\rho$, where ρ is as in the theorem. In that setting, $\bar{\nu}^n \to \nu$ almost surely and verifies a large deviation principle, hence $\bar{\mu}^n \to F(\nu)$ almost surely and thanks to the contraction principle, $\bar{\mu}^n$ also verifies a large deviation principle. Note also that this does not relies and the independence of the previous sequence, but really on its asymptotic behavior.

Extension. When $1 \leq p < 2$, one can also consider interactions in the diffusivity, and σ has to fulfills the same hypothesis than b. By slightly changing the space E_ρ, and by asking for one degree more of regularity for the coefficients, both in the measure and the space variables, one can show that F is a *continuously* differentiable function. Hence one can also prove moderate deviation principles and central limit theorems for the empirical measure $\bar{\mu}^n$.

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Regularity Structure II

Ajay Chandra

The bulk of my talk was a continuation of the introduction to SPDEs and Martin Hairer's theory of regularity structures given by Hendrik Weber. I began by recalling why the Da Prato - Debussche method for the Φ^4_2 stochastic quantization equation completely breaks down when treating Φ^4_3 - regardless of where one truncates the Wild expansion for the solution ϕ the fixed point problem for the remainder is always ill-posed because the remainder has insufficient regularity to define products that appear in this fixed point problem. Here we are operating in Holder-Besov spaces where there is a canonical product $\mathcal{C}^\alpha \times \mathcal{C}^\beta \to \mathcal{C}^{\alpha\wedge\beta}$ if and only if $\alpha + \beta > 0$

We adopted the point of view that regularity was the wrong property to demand of our remainder since the regularity of a product of a pair of functions/distributions, when it can be defined, is determined by the how singular the worst of the pair is. On the other hand, if we are willing to adopt a more local point of view, the quality of satisfying homogeneity bound at a specific point behaves better under the products - given a pair of objects satisfying a homogeneity bound at a space-time point x with exponents α and β, we are guaranteed their product will satisfy a homogeneity bound at x with exponent $\alpha + \beta$.

Using this as motivation, we move away from using the global Wild expansion and instead seek to describe ϕ as a jet of local expansions - here one generalizes the classical notion of Taylor series by including indeterminants which represent certain Gaussian polynomials (Wild trees) built out of the linear solution as monomials in addition to classical polynomials. A key idea is that when the right deterministic and quantitative notion of ϕ being locally well-approximated by explicit Gaussian processes is combined with a probablistic algorithm for defining Gaussian polynomials one gets a method of defining products of ϕ. One can view this entire procedure as defining a new notion of regularity in which ϕ has positive regularity.

After this I described the concept of a "model" which is what allows one to associate concrete objects to abstract jets. The key content of a model is information is (i) a map which associates to each Wild tree indeterminant a concrete space-time distribution which is the "homogenous incarnation" of that tree and (b) a family of parallel transport maps which allow one to move this these local expansions from point to point. In order to define the map of item (a) one must mollify the underlying driving white noise at some scale ϵ and then subtract renormalization constants (which diverge as $\epsilon \downarrow 0$) in order to guarantee a limit as the mollification is removed. However convergence is not enough, one must also perform "recenterings" of this process so that these approximations satisfy, uniformly in ϵ, a homogeneity bound. I then discussed results, obtained in collaboration with Martin Hairer, which use multiscale techniques from constructive field theory in order to show that one can define an automatic procedure to perform these renormalizations and check that after recentering one again gets convergence along with the necessary homogeneity bounds.

Characteristic functions of path signatures

Ilya Chevyrev
(joint work with Terry Lyons)

The signature of a geometric p-rough path $\mathbf{x} : [0, T] \mapsto G^{\lfloor p \rfloor}(\mathrm{R}^d)$ is the solution to a universal linear differential equation

$$dS(\mathbf{x})_t = S(\mathbf{x})_t \otimes d\mathbf{x}_t, \quad S(\mathbf{x})_0 = \mathbf{1},$$

and is given concretely by the sequence of iterated integrals of $\mathbf{x}$. Let $E(\mathrm{R}^d)$ denote the algebra of tensor series over R^d with an infinite radius of convergence, and $G(\mathrm{R}^d)$ the subset of group-like elements of $E(\mathrm{R}^d)$. A result of Chen [4] implies that $S(\mathbf{x})$ take values in the group $G(\mathrm{R}^d)$, and it has furthermore recently been shown that the solution to any rough differential equation driven by $\mathbf{x}$ is completely determined by $S(\mathbf{x})_T$ [1].

For a one-dimensional path $\mathbf{x} : [0, T] \mapsto \mathrm{R}$, the signature captures precisely the powers of the increment $\mathbf{x}_T - \mathbf{x}_0$. In particular, for a random one-dimensional path $\mathbf{X} : [0, T] \mapsto \mathrm{R}$, the expected signature, whenever it exists, is given by the moments of $\mathbf{X}_T - \mathbf{X}_0$. It is natural then to interpret the expected signature of a general random geometric rough path $\mathbf{X} : [0, T] \mapsto G^{\lfloor p \rfloor}(\mathrm{R}^d)$ (or more generally of any $G(\mathrm{R}^d)$-valued random variable) as the natural generalization of moments.

To study the expected signature of a $G(\mathrm{R}^d)$-valued random variable, we introduce a suitable notion of a characteristic function. For a Hilbert space H, let $\mathfrak{u}(H)$ denote the Lie algebra of anti-Hermitian operators on H. Let $\mathcal{A}$ be the collection of all linear maps $M : \mathrm{R}^d \mapsto \mathfrak{u}(H)$, where H varies over all finite dimensional Hilbert spaces H. Every $M \in \mathcal{A}$ canonically induces a finite-dimensional unitary representation of the group $G(\mathrm{R}^d)$. Our first result ensures that the map $M \mapsto \mathbb{E}[M(X)]$ is a meaningful characteristic function of a $G(\mathrm{R}^d)$-valued random variable X.

Theorem 1 ([6] Corollary 4.12)**.** *Let X and Y be $G(\mathrm{R}^d)$-valued random variables. Then $X \overset{\mathcal{D}}{=} Y$ if and only if $\mathbb{E}[M(X)] = \mathbb{E}[M(Y)]$ for all $M \in \mathcal{A}$.*

The proof rests crucially on the fact that $\mathcal{A}$ separates the points of $E(\mathrm{R}^d)$, which in turn follows from the study polynomial identities in unitary Lie algebras [9]. Denoting the expected value of a $G(\mathrm{R}^d)$-valued random variable X by $\mathrm{ESig}[X]$, the following partial solution to the moment problem is an immediate consequence of Theorem 1.

Corollary 2 ([6] Proposition 6.1)**.** *Let X and Y be $G(\mathrm{R}^d)$-valued random variables such that $\mathrm{ESig}[X] = \mathrm{ESig}[Y]$. If $\mathrm{ESig}[X]$ has an infinite radius of convergence, then $X \overset{\mathcal{D}}{=} Y$.*

The characteristic function $M \mapsto \mathbb{E}[MS(\mathbf{X}_T)]$ has been explicitly determined for all Lévy p-rough paths $\mathbf{X}$ in [5], while the associated expected signature has been determined under suitable integrability conditions in [8] (see also [6] Example 6.2).

To apply Corollary 2, it is important to control the radius of convergence of the expected signature. In this direction, we have the following result stated in terms of the local p-variation functional introduced in [2].

Theorem 3 ([6] Corollaries 6.6, 6.18)**.** *Let $p \geq 1$ and $N = N_{1,[0,T],p}(\mathbf{X})$ denote the local p-variation of a random geometric p-rough path $\mathbf{X} : [0, T] \mapsto G^{\lfloor p \rfloor}(\mathrm{R}^d)$.*

(1) *If $\mathbb{E}\left[e^{\lambda N}\right] < \infty$ for all $\lambda > 0$, then* $\mathrm{ESig}\left[S(\mathbf{X})_T\right]$ *has an infinite radius of convergence.*

(2) *If $\mathbb{E}\left[e^{\lambda N}\right] < \infty$ for some $\lambda > 0$, then* $\mathrm{ESig}\left[S(\mathbf{X})_T\right]$ *has a non-zero radius of convergence and the map $r \mapsto \mathbb{E}\left[(rM)S(\mathbf{X})\right]$ is analytic on* R.

Part (1) of Theorem 3 allows us to apply Corollary 2 to solve the moment problem, whilst part (2) of Theorem 3 allows us to solve the moment problem within the subclass of $G(\mathrm{R}^d)$-valued random variables with analytic characteristic functions.

The integrability condition $\mathbb{E}\left[e^{\lambda N}\right] < \infty$ for all $\lambda > 0$ has been shown to hold for Markovian rough paths [3] and a wide class of Gaussian rough paths [2, 7] (see also [6] Examples 6.7, 6.8), whilst the integrability condition $\mathbb{E}\left[e^{\lambda N}\right] < \infty$ for some $\lambda > 0$ holds for Markovian rough paths first stopped upon exiting a domain (see [6] Example 6.20).

We note that in the multi-dimensional case $d \geq 2$, unlike in the one-dimensional case $d = 1$, analyticity of the characteristic function at the origin does not guarantee analyticity on all of R for a general $G(\mathrm{R}^d)$-valued random variable (see [6] Example 6.10).

Finally, we can demonstrate the following analogue of the classical method of moments.

Theorem 4 ([6] Theorem 6.31)**.** *Let $(X_n)_{n\geq 1}$ be a sequence of $G(\mathrm{R}^d)$-valued random variables such that $\mathbb{E}\left[X_n\right] \in E(\mathrm{R}^d)$ for all $n \geq 1$. Suppose that $\mathbb{E}\left[X_n\right]$ converges to some $x \in E(\mathrm{R}^d)$ in the weak topology of $E(\mathrm{R}^d)$. Then there exists a unique $G(\mathrm{R}^d)$-valued random variable X such that $X_n \overset{\mathcal{D}}{\to} X$ and $x = \mathbb{E}\left[X\right]$.*

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Continuous Schrödinger operator with white noise potential

Khalil Chouk
(joint work with Romain Allez)

The aim of my talk is to explain the construction of the schrödinger operator with white noise potential on the two dimensional torus $\mathbb{T}_R^2$ of size R performed in [1]. This operator is formally given by :

$$\mathscr{H} = -\Delta + \eta$$

Let us first observe that this operator was already constructed in [2] the one dimensional setting with Dirichlet boundary on an interval of length R by using the theory of Dirichlet form. Moreover the author show that it have compact resolvent and compute explicitly the integrated density of state in this case.In dimension two t he main difficulty with the operator is that the white noise is a distribution and actually $\eta \in \mathscr{C}^{-1-\varepsilon}$ for every $\varepsilon > 0$ almost surely. a naive approach is to define the operator on the Sobolev space $H^{1+\varepsilon}$ so that the product $f\eta$ for $f \in H^{1+\varepsilon}$ is well-defined and actually lie in the space $H^{-1-\varepsilon}$. Unfortunately we can see immediately yield to a "non robust" definition of the operator actually the operaor construct in this way is not lower semi-bounded. Indeed let

$$q(f) = \int |\nabla f|^2 + \int f^2\eta$$

and X_N a solution of

$$-\Delta X_N = \Pi_N \eta$$

where Π_N is the projection on Fourier mode less than N. Now pick $f_N = \exp(\theta X_N)$ for some constant θ. A quick computation show that

$$q(f_N) = \theta^2 \int |\nabla X_N|^2 \exp(2\theta X_N) + \int \exp(2\theta X_N)\eta$$

the main observation now is that when N goes to the infinity

$$\int \exp(2\theta X_N)\eta = \theta \int X_N\eta + O(1)$$

now the point is that

$$\mathbb{E}|\nabla X_N|^2 = \mathbb{X}_N\eta = \frac{1}{2\pi}\log N + O(1)$$

and actually we can prove that $\int |\nabla X_N|^2 \exp(2\theta X_N) \sim^{N\to+\infty} \int X_N\eta \exp(2\theta X_N)$ $\sim \frac{1}{2\pi}\log N \int e^{2\theta X}$ where X is defined by $-\Delta X = \eta$. Thus for θ small enough $q(f_N) \to -\infty$.

so to avoid this kind problem the philosophical idea is to choose a space of function f for which the less regular part of $-\Delta f$ cancel the less regular part of $f\eta$. This heuristic can be made rigorous by using the Bony paraproduct decomposition and actually we will take as a domain of the operator the Hilbert space $\mathscr{D}_\eta$ formed by the function in $H^{1-\varepsilon}$ for which $f - f \prec X \in H^{2-\varepsilon}$ on this Hilbert space we can define the operator $\mathscr{H}$ in rigorous. And a flavor of this construction is that we can see our operator as the limit (in the resolvent sense) of classical Schrödinger operator $\mathscr{H}_\delta = -\Delta + \eta_\delta + c_\delta$ where η_δ is a smooth mollification of the white noise and c_δ a deterministic diverging constant. Actually we can choose c_δ such that the limit operator $\mathscr{H}$ does not depend on the way that we have mollified η. Moreover we can prove that the ground state energy Λ_1^R of the operator $\mathscr{H}$ satisfies that

$$\sup_R \frac{|\Lambda_1^R|}{\log R} < +\infty \text{ almost surely },$$

and

$$\exp(-c_2^R x(1+o(1))) \leq \mathbb{P}(\Lambda_1^R \leq -x) \leq \exp(-c_1^R x(1+o(1)))$$

when $x \to +\infty$, $c_1, c_2 > 0$ and where we recall that R is the size of the Torus.

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Sticking rough solutions driven by a fBm and related ergodic issues

Aurélien Deya

(joint work with Fabien Panloup and Samy Tindel)

The purpose of the talk was to give an idea of some of the technical difficulties related to the ergodic analysis of the rough stochastic system

$$dY_t = b(Y_t)\,dt + \sigma(Y_t)\,dB_t^H \quad , \quad Y_0 = a \in \mathbb{R}^d \ , \tag{1}$$

where $(B_t^H)_{t\geq 0}$ stands for an $\mathbb{R}^d$-valued fractional Brownian motion (fBm) with Hurst index $H > \frac{1}{3}$, and $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$ satisfy standard regularity, dissipativity and non-degeneracy conditions.

When $H \neq \frac{1}{2}$, it is a well-known fact that $(B_t^H)_{t\geq 0}$ is not a Markov process, and therefore classical ergodic theory and results cannot be applied to the system (1) in this situation. Some ten years ago, M. Hairer [4] introduced an alternative strategy based on a sophisticated coupling procedure. His results then included existence and uniqueness of an invariant measure, as well as a bound on the rate of convergence (in law) of any solution toward the stationary solution. However, the analysis in [4] is limited to the sole case of a constant diffusion coefficient σ (in particular, rough paths interpretation is not even required in this situation).

Extending the procedure of [4] to a more general vector field $\sigma : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$ (that is, to the general rough system), which was essentially the aim of our analysis in [1], is the source of several additional difficulties. The most delicate transition issue lies in the so-called sticking (or hitting) step of the machinery. The challenge here can be loosely summed up as follows: given an $\mathbb{R}^d$-valued fBm B^H on $[0,1]$ and two (deterministic) initial conditions $a, \tilde{a} \in \mathbb{R}^d$, construct a continuous (random) function $G = G(B^H, a, \tilde{a}) : [0,1] \to \mathbb{R}^d$ such that the respective solutions $Y, \widetilde{Y}$ of the two systems

$$\begin{aligned} dY_t &= b(Y_t)\, dt + \sigma(Y_t)\, dB_t^H \qquad &, \qquad Y_0 = a \ , \\ d\widetilde{Y}_t &= b(\widetilde{Y}_t)\, dt + \sigma(\widetilde{Y}_t)\, \big(dB_t^H + G_t\, dt\big) \qquad &, \qquad \widetilde{Y}_0 = \tilde{a} \ , \end{aligned} \tag{2}$$

meet at time 1, that is $Y_1 = \widetilde{Y}_1$, with strictly positive probability. Combining such a construction with a Girsanov-type property then provides us with the desired result, namely a control on the possibility to stick any general solution with the stationary solution, a first step toward the asymptotic properties of the system (1).

In [1], our strategy to construct the above drift function G is based on the consideration of the intermediate functional-valued system

$$\begin{cases} dY_t^\xi = \big[b(Y_t^\xi) - \int_0^\xi d\eta\, J_t^\eta\big]\, dt + \sigma(Y_t^\xi)\, dB_t^H \\ dJ_t^\xi = \nabla b(Y_t^\xi)\, J_t^\xi\, dt + \nabla\sigma(Y_t^\xi)\, J_t^\xi\, dB_t^H \end{cases} , \tag{3}$$

with initial condition

$$Y_0^\xi = a + \xi(\tilde{a} - a) \quad , \quad J_0^\xi = \tilde{a} - a \quad , \quad \text{for every } \xi \in [0,1] \ .$$

It can indeed be checked that if $(Y_t^\xi, J_t^\xi)_{t\in[0,1],\xi\in[0,1]}$ is a solution of (3) (understood in the rough sense) and if we define G as $G_t := -\sigma(Y_t^1)^{-1} \int_0^1 d\eta\, J_t^\eta$, then the processes $Y_t := Y_t^0$ and $\widetilde{Y}_t := Y_t^1$ do satisfy the two equations in (2), as well as the sticking condition $Y_1 = \widetilde{Y}_1$.

The advantage of the new formulation (3) of the problem lies in the disappearance of any terminal condition: in brief, the problem is reduced to solving a particular rough system. Unfortunately, the vector fields involved in (3), as well as their derivatives, are not uniformly bounded, so that existence of a global solution on the fixed interval $[0,1]$ cannot be derived from standard rough paths results, and small-time explosion phenomenon may actually occurs for some particular choices of a rough driver (when considering the general rough system).

One of the main results of our analysis consisted in the exhibition of a deterministic constant $M = M(a, \tilde{a}) > 0$ such that if $\|\mathbf{B}^H\|_{\gamma;[0,1]} \leq M$, then the system (3) indeed admits a unique solution defined on $[0,1]$, and that this solution also satisfies appropriate regularity conditions with respect to the parameter ξ. Here, γ is a fixed parameter in $(\frac{1}{3}, H)$ and the notation $\|\mathbf{B}^H\|_{\gamma;[0,1]}$ refers to the usual γ-Hölder norm on $[0,1]$ of the canonical rough path above the fBm B^H. Once endowed with this existence result, the conclusion immediately follows from the strict positivity of the probability $\mathbb{P}(\|\mathbf{B}^H\|_{\gamma;[0,1]} \leq M)$ (as shown in [3]).

This simplified formulation of the problem however hides a certain number of additional constraints induced by the non-Markovianity of the fBm and the necessary control on the past dependence throughout the procedure. A possible way to express this difficulty is to go back to the Mandelbrot-Van-Ness representation of the fBm, that is

$$B_t^H = \alpha_H \int_{-\infty}^0 (-r)^{H-\frac{1}{2}} (dW_{r+t} - dW_r) ,$$

where $(W_s)_{s\in\mathbb{R}}$ is a two-sided Wiener process. With this representation in mind, the actual quantity that we need to control is the conditional probability $\mathbb{P}(Y_1 = \widetilde{Y}_1 | (W_s)_{s\leq 0})$, where Y and $\widetilde{Y}$ still stand for the solutions of the two equations (2). As a consequence, our analysis first appeals to the decomposition of B^H as the sum of a smooth past component and a rough innovation component, respectively

$$D_t^H := \alpha_H \int_{-\infty}^0 \left\{(t-r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}}\right\} dW_r \quad \text{and} \quad Z_t^H := \alpha_H \int_0^t (t-r)^{H-\frac{1}{2}} \, dW_r \, .$$

The argument then involves an extension of the above-described sticking procedure that takes this past-innovation splitting into account.

By combining these technical ingredients with the general strategy displayed in [2], we have indeed been able to extend all the results of [4] (for the additive case) to a general vector fields $\sigma : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$, at least in the situation where $H > \frac{1}{3}$.

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The parabolic Anderson model on 2-dimensional Riemmanian manifolds

Joscha Diehl

(joint work with A. Dahlqvist, B. Driver)

The theory of regularity structures [1] is a recent development to solve singular stochastic partial differential equations. A model equation is the two dimensional parabolic Anderson model on the 2-dimensional torus. Formally

$$\text{(PAM)} \qquad \partial_t u = \Delta u + u\xi,$$

where $u : [0,1] \times \mathbb{T}^2 \to \mathrm{R}$ and ξ is (space) white noise on $\mathbb{T}^2$. This equation is singular in the sense that the solution u is not expected to have enough regularity to define canonically the product with the distribution ξ. Note that since we are dealing with white noise in space, Itō's theory of stochastic integration is also of no help here.

Using the theory of regularity structures one handles such equations in the following way

- assume that u “looks like“ the solution $\nu_t = \int_0^t P_s \xi ds$ to the additive-noise equation

$$\partial_t \nu = \Delta \nu + \xi$$

- under this assumption, if we somehow *define* $\nu\xi$, then $u\xi$ is automatically defined
- define $\nu\xi$ probabilistically
- close the fixpoint argument, i.e.
  (1) u ”looks like“ ν
  (2) $w := P_t u_0 + \int_0^t P_{t-s}[u_s \xi] ds$, then w ”looks like“ ν

We carry out this program for a 2-dimensional Riemannian manifold M. The first hurdle to overcome is the fact that polynomials are an essential part for the local description of u (u “looks like” ν above). There is no canonical (global) replacement for them on curved spaces. Nonetheless there exist appropriate local replacements. To be more specific, define

$$\mathcal{T} := \mathrm{R} \oplus T^* M,$$

with $\mathcal{T}_x$ its fiber at $x \in M$. Then for $a + b \in \mathcal{T}_x$,

$$\Pi_x(a + b) := a + b \exp_x^{-1}(\cdot)$$

works as a good replacement for linear polynomials. Classical polynomials have the property that they can be re-expanded around any basepoint. In our case this holds only true up to a certain order. In particular for $a + b \in \mathcal{T}_x$ and y close to x there is $\tilde{a} + \tilde{b} \in \mathcal{T}_y$ such that

$$\Pi_x(a + b) - \Pi_y(\tilde{a} + \tilde{b}) = O(|\cdot - y|^2).$$

This is enough to make reconstruction work, i.e. the construction of a unique function/distribution after only knowing its local description (for example in terms of polynomials and ν in the setting above).

In order to carry out a fixpoint argument for the equation, one needs to quantify how the convolution with the heat kernel improves regularity. This “Schauder estimate” has to be carried out on the level of the abstract spaces that locally describe the distributions under considerations. Here we use classical results on asymptotics of the heat kernel on Riemannian manifolds [2] and show the improvement “by hand” for every term in the regularity structure for PAM. The heat kernel asymptotics are finally also used for renormalization, i.e. the probablistic construction of $\nu\xi$ mentioned above.

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Regularization by noise for stochastic Hamilton-Jacobi equations

Paul Gassiat, Benjamin Gess

The questions of regularizing effects and well-posedness by noise for (stochastic) partial differential equations have attracted much interest in recent years. The principle idea is that the inclusion of stochastic perturbations may lead to more regular solutions and in some cases even to uniqueness of solutions. However, the type of noise leading to such an effect is a-priori unclear. Historically, possible regularizing effects of additive noise have been investigated, e.g. for (stochastic) reaction diffusion equations

$$dv = \Delta v\, dt + f(v)\, dt + dW_t$$

in [11] and for Navier-Stokes equations in [6, 7]. In [4, 3, 1], well-posedness and regularization by linear multiplicative noise for transport equations, that is for

$$dv = b(x)\nabla_x v\, dt + \nabla v \circ d\beta_t,$$

have been obtained. We refer to [5] for more details on the literature. Only very recently, regularizing effects of *non-linear* stochastic perturbations in the setting of (stochastic) scalar conservation laws have been discovered in [9]. In particular, in [9] it has been shown that quasi-solutions to

$$dv + \frac{1}{2}\partial_x v^2 \circ d\beta_t = 0 \quad \text{on } \mathbb{T} \tag{1}$$

where $\mathbb{T}$ is the one-dimensional torus, enjoy fractional Sobolev regularity of the order

$$v(t) \in W^{\alpha,1}(\mathbb{T}) \quad \text{for all } \alpha < \frac{4}{5},\ t > 0,\ \mathbb{P}\text{-a.s.}. \tag{2}$$

This is in contrast to the deterministic case, in which examples of quasi-solutions to

$$\partial_t v + \frac{1}{2}\partial_x v^2 = 0 \quad \text{on } \mathbb{T}$$

have been given in [2] such that, for all $\alpha > \frac{1}{3}$,

$$v(t) \notin W^{\alpha,1}(\mathbb{T}) \quad \text{for all } t > 0.$$

In this sense, the stochastic perturbation introduced in (1) has a regularizing effect. In [9], the question of optimality of the estimate (2) remained open.

Subsequently, the results and techniques developed in [9] have been (partially) extended in [10] to a class of parabolic-hyperbolic SPDE, as a particular example including the SPDE

$$dv + \frac{1}{2}\partial_x v^2 \circ d\beta_t = \frac{1}{6}\partial_{xx} v^3\, dt \quad \text{on } \mathbb{T}. \tag{3}$$

In [10], the regularity of solutions to (3) was analyzed. More precisely, it was shown that

$$v(t) \in W^{\alpha,1}(\mathbb{T}) \quad \text{for all } \alpha < \frac{2}{3},\ \mathbb{P}\text{-a.s.}. \tag{4}$$

However, neither optimality of these results nor regularization by noise could be observed in this case. That is, the regularity estimates for solutions to (3) proven in [10] did not exceed the known regularity for the solutions to the non-perturbed cases

$$\partial_t v + \frac{1}{2}\partial_x v^2 = \frac{1}{6}\partial_{xx} v^3 \quad \text{or} \quad \partial_t v = \frac{1}{6}\partial_{xx} v^3 \text{ on } \mathbb{T}. \tag{5}$$

The purpose of this project is to provide sharp regularity estimates to a class of SPDE, in particular including (3), and to prove regularization by noise in this case. More precisely, sharp estimates are obtained for the L^∞ norm of the second derivative of solutions to SPDE of the type

$$du + F(x,u,Du,D^2u)\,dt + \frac{1}{2}|Du|^2 \circ d\xi_t = 0 \quad \text{on } \mathbb{R}^N,$$

for F satisfying appropriate assumptions detailed below and ξ being a continuous function. Before stating the main theorem in detail let us first consider some concrete examples.

As a first example, as mentioned above, the results answer the question of optimal regularity and regularization by noise for (3). Indeed, let u be the unique viscosity solution to the SPDE

$$du + \frac{1}{2}(\partial_x u)^2 \circ d\beta_t = \frac{1}{6}\partial_x(\partial_x u)^3 dt, \quad \text{on } \mathbb{R}.$$

Then, informally, $v = \partial_x u$ is a solution to (3). Our results yield that if $\beta = \sigma B$ where B is a standard Brownian motion, then

$$\sigma^2 > 2 \Rightarrow v(t) \in W^{1,\infty} \quad \mathbb{P}\text{-a.s.},$$

whereas (at least for some choice of initial conditions)

$$\sigma^2 \leq 2 \Rightarrow \mathbb{P}\text{-a.s.} \quad \exists T > 0, \forall t \geq T, v(t) \notin W^{1,\infty}.$$

One actually has the sharp bound

$$\|\partial_x v(t)\|_{L^\infty} \leq \frac{1}{L^+(t) \wedge L^-(t)},$$

where L^+, L^- are the solutions to the reflected (at 0^+) SDE with dynamics on $(0,\infty)$ given by

$$dL^+ = -\frac{1}{L^+(t)}dt + d\beta_t, \;\; dL^- = -\frac{1}{L^-(t)}dt - d\beta_t$$

and initial conditions

$$L^+(0) = \frac{1}{\|(\partial_x v_0)_+\|_{L^\infty}}, \;\; L^-(0) = \frac{1}{\|(\partial_x v_0)_-\|_{L^\infty}}.$$

This demonstrates that, when the noise coefficient is large enough, the stochastic perturbation in (3) has a regularizing effect as compared to the non-perturbed situation

$$\partial_t w = \partial_{xx} w^3, \quad \text{on } \mathbb{R}$$

for which solutions are known to develop singularities in terms of a blow-up of $\|\partial_x w\|_{L^\infty}$.

As a second example, consider hyperbolic SPDE of the form

$$du + F(x, Du)\, dt + \frac{1}{2}|Du|^2 \circ d\beta_t^H = 0, \quad \text{on } \mathbb{R}^N, \tag{6}$$

where β^H is a fractional Brownian motion with Hurst parameter $H \in (0, 1)$. Typically, the solutions to the deterministic counterpart

$$\partial_t w + F(x, Dw) + \frac{1}{2}|Dw|^2 = 0 \quad \text{or} \quad \partial_t w + F(x, Dw) = 0 \quad \text{on } \mathbb{R}^N$$

develop singularities in terms of shocks of the derivative, that is, Dw will become discontinuous for large times, even if w_0 is smooth. In contrast, our results yield that

$$\mathbb{P}(\|D^2u(t,\cdot)\|_{L^\infty} < \infty) = 1 \quad \forall t > 0,$$

for u being a solution to (6).

We now proceed to the details of our main result. Roughly speaking, we assume that F satisfies the usual assumptions from the theory of stochastic viscosity solutions (cf. e.g. [14, 13, 8]) and allows for a control on the rate of loss of semiconvexity, in the sense that there is a $V_F \in Lip_{loc}(\mathbb{R}_+ \setminus \{0\})$ such that, for $\ell_0 > 0$,

$$D^2 g \le \frac{Id}{\ell_0} \quad \Rightarrow \quad D^2(S_F(t, g)) \le \frac{Id}{\ell(t)}, \tag{7}$$

where $t \mapsto S_F(t, g)$ denotes the solution to

$$\begin{aligned} \partial_t w + F(x, w, Dw, D^2w) &= 0 \\ w(0) &= g \end{aligned}$$

and ℓ the solution to

$$\begin{aligned} \dot{\ell}(t) &= V_F(\ell(t)) \\ \ell(0) &= \ell_0. \end{aligned}$$

The main theorem then reads

Theorem 1. *Assume* (7)*, let ξ be a continuous path and let u be the solution to*

$$\begin{aligned} &du + F(x, u, Du, D^2u)\, dt + \tfrac{1}{2}|Du|^2 \circ d\xi_t = 0, \\ &u(0,\cdot) = u_0, \end{aligned}$$

with $u_0 \in BUC(\mathbb{R}^N)$ such that $D^2u_0 \le \frac{Id}{\ell_0}$ ($\ell_0 \in [0, \infty)$). Then, for each $t \ge 0$, one has

$$D^2u(t,\cdot) \le \frac{Id}{L_t}, \tag{8}$$

where L is the maximal solution to the reflected differential equation on $[0, \infty)$

$$\begin{aligned} &dL_t = V_F(L_t)dt + d\xi_t \ \text{ on } \{L > 0\}, \quad \text{with } L_t \ge 0, \\ &L_0 = \ell_0. \end{aligned} \tag{9}$$

The proof of this result relies on a two-step approximation procedure. In a first step, the possibly singular (at zero) vector field V is approximated by Lipschitz continuous approximations. In the second approximation step, the solution u is approximated via a Trotter-Kato approach. That is, the approximating solutions u^n are constructed via a time-discretization and an operator splitting approach, thus iteratively solving, for each discrete time-step, the purely deterministic PDE

$$\partial_t w + F(t, x, w, Dw, D^2 w) = 0 \tag{10}$$

and the pure noise part

$$dz + \frac{1}{2}|Dz|^2 \circ d\xi_t = 0. \tag{11}$$

For the approximations u^n, the bound (8) can then be shown by applying Assumption (7) for the deterministic step and using semiconvexity estimates similar to those established in [12] for the stochastic step (11). This yields a time-discrete version of the bound (8) of the form

$$D^2 u^n(t, \cdot) \leq \frac{Id}{L^n_t}$$

where L^n can be regarded as a time-discrete approximation of L. The remaining technical part of the proof then is to prove convergence of the approximations $u^n \to u$ and $L^n \to L$.

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The Signature of a Rough Path: Uniqueness and Reconstruction

Xi Geng

The *signature* of a path is a non-commutative tensor series of iterated path integrals. Heuristically speaking, the description of a path involves its local behaviors and their interactions, while the signature is a global quantity as it consists of the total increment, geometric signed area and all higher order "areas" of the underlying path. It is widely believed (and surprisingly) that the signature contains essentially all information about the underlying path.

In as early as 1958, when restricted to the class of piecewise regular and irreducible paths, Chen [2] proved that a path is uniquely determined by its signature up to translation and reparametrization. However, the class of paths he studied is very special as it does not reveal a crucial invariance property of the signature map: a piece along which the path x goes out and traces back does not contribute to the signature of x. It was after more than five decades that Hambly and Lyons [4] first gave a complete characterization of this invariance property for the class of paths with finite length, in terms of a "tree-like" property defined by a height function.

Since the work of Hambly and Lyons, it has been an important problem to extend their result to the rough path setting. The fundamental difficulty in the extension lies in the fact that their technique relies on the nice regularity of the underlying path and the coarea formula in a crucial way. Moreover, their characterization does not hold any more beyond the bounded variation case since their tree-like notion actually forces the path to have finite length. To understand the rough path situation, we need to identify the right notion of the above invariance property and develop new ideas for the proof.

In a joint work with Boedihardjo, Lyons and Yang [1] in 2015, we proved that the signature of a rough path $\mathbf{X}$ over some Banach space is trivial if and only if $\mathbf{X}$ can be realized as a loop in some real tree. In particular, a rough path is uniquely determined by its signature up to tree-like equivalence in this sense. For the sufficiency part, the main idea is to properly construct a piecewise geodesic approximation of a loop in some real tree. This step involves the basic theory of real trees. For the necessity part, the main idea is to lift the underlying rough path to its full signature path in the tensor algebra, and show that every signature path has a unique reduced form in the sense of non-self-intersection. This gives rise to a real tree structure on the signature group in a canonical way in terms of tree-reduced paths.

The uniqueness result for signature establishes an isomorphism between the space of rough paths modulo tree-like equivalence and the corresponding signature group. However, our proof of the uniqueness result does not contain any information on how to "see" the trajectory of a tree-reduced rough path from its signature. This drawback is mainly due to the use of the homomorphism property of the signature map as a starting point. In a recent work of myself [3], we developed a method to reconstruct a tree-reduced rough path from its signature in an explicit and universal way. More precisely, let g be the signature of a tree-reduced rough path $\mathbf{X}$ with roughness p. Then we were able to construct a sequence $\mathbf{X}^n$ of piecewise linear paths in the truncated tensor algebra up to degree $\lfloor p \rfloor$ from the knowledge of the signature g only, such that

$$d([\mathbf{X}^n],[\mathbf{X}]) \leqslant \frac{68 D_{N(g)}^{\frac{3}{2}}}{n}.$$

Here d is the parametrization-free uniform distance, $N(g)$ is a positive integer determined by the signature g, and $D_{N(g)}$ is the dimensional of the truncated tensor algebra up to degree $N(g)$. Inspired by a series of probabilistic works, our method is based on the understanding on certain stability properties associated with the discrete route of a path in any given geometric configuration of disjoint compact domains. Hopefully this work will provide us with a constructive understanding on the uniqueness result for signature and in particular on the inverse of the signature map.

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Random Strings

Martin Hairer

(joint work with Yvain Bruned, Ajay Chandra and Lorenzo Zambotti)

Given a Riemannian manifold $(\mathcal{M}, g)$, take a collection of vector fields σ_i on $\mathcal{M}$ with the property that, interpreting the σ_i as first-order differential operators, one has $\sum_i \sigma_i^2 = \Delta$, where Δ is the Laplace-Beltrami operator on $\mathcal{M}$. Given these vector fields, it is natural to consider the stochastic perturbation to the usual length-shortening flow written in local coordinates as

$$(\star) \qquad \partial_t u^\alpha = \partial_x^2 u^\alpha + \Gamma^\alpha_{\beta\gamma}(u) \partial_x u^\beta \partial_x u^\gamma + \sigma_i^\alpha(u)\, \xi_i\ ,$$

where the ξ_i are independent space-time white noises and $\Gamma^\alpha_{\beta\gamma}$ denotes the Christoffel symbols for the Lévy-Civita connection of $\mathcal{M}$.

Unfortunately, it turns out that $(\star)$ is hopelessly ill-posed. For example, the best one could hope for, based on the behaviour of the one-dimensional stochastic heat equation, is that solutions to $(\star)$ are Hölder regular for exponents less than

1/2 in space and 1/4 in time. In particular, one expects that the derivatives $\partial_x u^\beta$ are very badly behaved distributions, so that their product is devoid of canonical meaning. One natural way of trying to nevertheless give a meaning to ($\star$) is in a more pragmatic way. Take a sequence $\xi_i^{(\varepsilon)}$ of smooth approximations to the ξ_i (say by space-time convolution with a fixed mollifier) and consider the corresponding sequence of solutions u_ε to ($\star$). One then has the following result [1, 2].

Theorem 1. *There exists a stochastic process u with values in $\mathcal{M}$-valued loops such that, as $\varepsilon \to 0$, u_ε converges weakly to u.*

One major difficulty arising when trying to prove such a result with the help of the theory of regularity structures is that a very high number of renormalisation constants are needed in order to build a convergent model. One would therefore like to have a systematic way of a) building a "sufficiently large" group $\mathcal{G}_-$ acting on the space of all admissible models for the type of regularity structure $(\mathcal{T}, \mathcal{G}_+)$ associated to a singular SPDE like ($\star$) and b) having a systematic way of showing that, given suitable regular approximations $\xi_i^{(\varepsilon)}$ to space-time white noise, one can find elements g_ε in $\mathcal{G}_-$ so that, when acting with g_ε on the canonical lift of $\xi_i^{(\varepsilon)}$, one obtain a convergent sequence in the space of models.

This is achieved in [1, 2] via a construction with the following ingredients. Write $\hat{\mathcal{T}}_-$ for the free algebra generated by all formal expressions appearing in the usual construction of the regularity structure $(\mathcal{T}, \mathcal{G}_+)$ and $\mathcal{I}_+ \subset \hat{\mathcal{T}}_-$ for the ideal generated by expressions of positive degree. One then constructs

- A map $\Delta^- : \hat{\mathcal{T}}_- \to \hat{\mathcal{T}}_- \otimes \hat{\mathcal{T}}_-$ endowing $\hat{\mathcal{T}}_-$ with a bialgebra structure and $\mathcal{T}_- = \hat{\mathcal{T}}_- / \mathcal{I}_+$ with a Hopf algebra structure.
- A map $\Delta^- : \mathcal{T} \to \mathcal{T}_- \otimes \mathcal{T}$ yielding an action of the group $\mathcal{G}_-$ of characters for $\mathcal{T}_-$ onto $\mathcal{T}$.
- A projection $\hat{\pi}_- : \hat{\mathcal{T}}_- \to \hat{\mathcal{T}}_-$ onto the subspace spanned by the empty product, as well as products of strictly negative total degree.

The two maps Δ^- are formally defined in virtually the same way and consist essentially in "extracting and contracting subforests", as is familiar from renormalisation theory in perturbative QFT. One then shows that the action of $\mathcal{G}_-$ on $\mathcal{T}$ yields indeed in a natural way an action of $\mathcal{G}_-$ on the space of admissible models for $(\mathcal{T}, \mathcal{G}_+)$. Furthermore, given a random linear map $\mathbf{\Pi} : \mathcal{T} \to \mathcal{C}^\infty$ satisfying suitable admissibility and stationarity properties, one can associate to it a character $g_{\mathbf{\Pi}}^-$ for $\hat{\mathcal{T}}_-$ by setting

$$g_{\mathbf{\Pi}}^-(\tau) = \mathcal{E}(\mathbf{\Pi}\tau)(0) ,$$

and then extending this multiplicatively to all of $\hat{\mathcal{T}}_-$. It turns out that there exists a "pseudo-antipode" $\hat{\mathcal{A}}_- : \mathcal{T}_- \to \hat{\mathcal{T}}_-$ such that, or a very large class of approximations $\xi_i^{(\varepsilon)}$ with canonical lifts $\mathbf{\Pi}^{(\varepsilon)}$, $g_{\mathbf{\Pi}^{(\varepsilon)}}^- \hat{\mathcal{A}}_-$ allows to renormalise the model in such a way that it converges to a non-trivial limit as $\varepsilon \to 0$.

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Rough Gronwall Lemma

Martina Hofmanová

(joint work with Aurélien Deya, Massimiliano Gubinelli and Samy Tindel)

Lyons [2] introduced rough paths to give a description of solutions to ordinary differential equation driven by external time varying signals which is robust enough to allow very irregular signals like the sample paths of a Brownian motion. However, attempts to use the rough path theory to study rough path driven PDEs have been so far limited by two factors: the first one is the need to look at RPDEs as evolutions in Banach spaces perturbed by one parameter rough signals (in order to keep rough paths as basic objects), the second one is the need to avoid unbounded operators by looking at mild formulations or Feynman–Kac formulas or transforming the equation in order to absorb the rough dependence into better understood objects (e.g. flow of characteristic curves). These requirements pose strong limitations on the kind of results one is able to obtain and the proof strategies are very different from the classical PDE proofs.

In [1] we developed several general tools to deal with RPDEs in the context of weak solutions, that is distributional relations satisfied by the unknown together with its weak derivatives. The key problem with rough paths and rough integrals is the absence of very basic and effective tools like Gronwall lemma. Skimming over any book on PDEs shows how fundamental this tool is for any nontrivial result on weak solutions. Our main technical contribution is a strategy to obtain a priori estimates of Gronwall type for RPDEs via a *Rough Gronwall lemma*. This result is not very difficult to prove but, as the standard one, it is the cornerstone of various arguments aiming at establishing properties of weak solutions to RPDEs leading to well–posedness.

In order to present the main ideas of our approach towards rough PDEs, let us consider a toy model of a linear rough heat equation of the form

$$
\begin{aligned}
\mathrm{d}u &= \Delta u \,\mathrm{d}t + V \cdot \nabla u \,\mathrm{d}z, \qquad x \in \mathbb{R}^N,\ t \in (0,T), \\
u(0) &= u_0,
\end{aligned} \tag{1}
$$

where $V = (V^1, \dots, V^K)$ is a family of sufficiently smooth and bounded vector fields on $\mathbb{R}^N$ and $z = (z^1, \dots, z^K)$ can be lifted to a geometric p-rough path for $p \in [2,3)$. We denote by $\mathbf{Z} = (Z^1, Z^2)$ its rough path lift. Let us insist on the fact that the linearity of the leading order operator does not play any role and the discussion below can be easily adapted to quasilinear elliptic or monotone operators.

Setting (using the Einstein summation convention)

$$A^1_{st}u := Z^{1,k}_{st}\, V^k \cdot \nabla u, \qquad A^2_{st}u := Z^{2,jk}_{st}\, V^k \cdot \nabla(V^j \cdot \nabla u) \tag{2}$$

we observe that $\mathbf{A} = (A^1, A^2)$ defines an unbounded p-rough driver in the scale $E_n = W^{n,2}(\mathbb{R}^N)$ and we are lead to the following formulation of (1) which should be satisfied for every $0 \leqslant s \leqslant t \leqslant T$ and every test function $\varphi \in W^{3,2}$:

$$\delta u(\varphi)_{st} = \int_s^t u_r(\Delta\varphi)\mathrm{d}r + u_s(A^{1,*}_{st}\varphi) + u_s(A^{2,*}_{st}\varphi) + u^\natural_{st}(\varphi) \tag{3}$$

where $u^\natural$ is an $W^{-3,2}$-valued 2-index map such that for every $\varphi \in E_3$ the map $u^\natural_{st}(\varphi)$ possesses sufficient time regularity, namely, it has finite local $\frac{q}{3}$-variation for some $q < 3$ and some regular control.

In order to understand (somehow heuristically) the rough path mechanism, let us assume that we are able to derive rigorously the equation for u^2. This should yield the following dynamics:

$$\begin{aligned} \delta u^2(\varphi)_{st} &= -2\int_s^t |\nabla u_r|^2(\varphi)\,\mathrm{d}r - 2\int_s^t (u\nabla u_r)(\nabla\varphi)\,\mathrm{d}r \\ &\quad + u^2_s(A^{1,*}_{st}\varphi) + u^2_s(A^{2,*}_{st}\varphi) + u^{2,\natural}_{st}(\varphi) \end{aligned} \tag{4}$$

for some new remainder $u^{2,\natural}$. Remark that since u^2 is expected to belong to $L^1(\mathbb{R}^N)$, the corresponding scale of test function spaces here is $E_n = W^{n,\infty}(\mathbb{R}^N)$, unlike in (3) where we considered $E_n = W^{n,2}(\mathbb{R}^N)$. Choosing $\varphi = 1$ and $s = 0$ leads to

$$\begin{aligned} \|u_t\|^2_{L^2} + 2\int_0^t \|\nabla u_r\|^2_{L^2}\,\mathrm{d}r &= \|u_0\|^2_{L^2} + u^2_0(A^{1,*}_{0t}1) + u^2_0(A^{2,*}_{0t}1) + u^{2,\natural}_{0t}(1) \\ &\lesssim \|u_0\|^2_{L^2}\Big(1 + |Z^1_{0t}|\|V\|_{W^{1,\infty}} + |Z^2_{0t}|\|V\|^2_{W^{2,\infty}}\Big) + |u^{2,\natural}_{0t}(1)|. \end{aligned} \tag{5}$$

In order to achieve a rough Gronwall lemma type argument, it is now easily seen that we need to estimate the remainder $u^{2,\natural}$ in terms of the left hand side in (5). Otherwise stated, our problem is reduced to show that:

$$|u^{2,\natural}_{0t}(1)| \lesssim_t \sup_{0\le r\le t} \|u_r\|^2_{L^2} + 2\int_0^t \|\nabla u_r\|^2_{L^2}\,\mathrm{d}r$$

in such a way that the proportional constant can be made sufficiently small provided t is small. This estimate is the key element of our theory and its applications are rather wide. The result is presented in full detail and generality in [1, Theorem 2.5].

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Short time full asymptotic expansion of hypoelliptic heat kernel at the cut locus

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(joint work with Setsuo Taniguchi)

We discuss a short time asymptotic expansion of a hypoelliptic heat kernel on an Euclidean space and on a compact manifold. We study the "cut locus" case, namely, the case where energy-minimizing paths which join the two points under consideration form not a finite set, but a compact manifold. Under mild assumptions we obtain an asymptotic expansion of the heat kernel up to any order. Our approach is probabilistic and the heat kernel is regarded as the density of the law of a hypoelliptic diffusion process, which is realized as a unique solution of the corresponding stochastic differential equation (SDE). Our main tools are S. Watanabe's distributional Malliavin calculus and T. Lyons' rough path theory.

Our work has the following three features. To our knowledge, there are no works which satisfy all of these conditions simultaneously:

(1) The manifold and the hypoelliptic diffusion process on it are rather general. In other words, this is not a study of special examples.
(2) The "cut locus" case is studied. More precisely, we mean by this that the set of energy-minimizing paths (or controls) which connect the two points under consideration becomes a compact manifold of finite dimension.
(3) The asymptotic expansion is full, that is, the polynomial part of the asymptotics is up to any order.

On an Euclidean space, however, there are two famous results which satisfy (2), (3) and the latter half of (1). Both of them are probabilistic and use generalized versions of Malliavin calculus. One is Takanobu and Watanabe [3]. They use Watanabe's distributional Malliavin calculus. The other is Kusuoka and Stroock [2]. They use their version of generalized Malliavin calculus. We use the former.

Though we basically follow Takanobu-Watanabe's argument in [3], the main difference is that we use T. Lyons' rough path theory together, which is something like a deterministic version of the SDE theory. The main advantage of using rough path theory is that while the usual Itô map i.e., the solution map of an SDE is discontinuous, the Lyons-Itô map i.e., the solution map of a rough differential equation (RDE) is continuous.

This fact enables us to do "local analysis" of the Lyons-Itô map (for instance, restricting the map on a neighborhood of its critical point and doing a Taylor-like expansion) in a somewhat similar way we do in the Fréchet calculus. Recall that in the standard SDE theory, this type of local operation is very hard and sometimes impossible, due to the discontinuity of the Itô map. For this reason, the localization procedure in [3] looks so complicated that it might be difficult to generalize their method if rough path theory did not exist. Of course, there is a possibility that our main result can be proved without rough path theory, but we believe that the theory is quite suitable for this problem and gives us a very clear view (in particular, in the manifold case).

A detailed proof can be found in our preprint [1]. We first reprove and generalize the main result in [3] in the Euclidean setting by using rough path theory. Then, we study the manifold case. Recall that Malliavin calculus for a manifold-valued SDE was studied by Taniguchi [4]. Even in this Euclidean setting, many parts of the proof are technically improved, thanks to rough path theory. We believe that the following are worth mentioning: (i) Large deviation upper bound. (ii) Asymptotic partition of unity. (iii) A Taylor-like expansion of the Lyons-Itô map and the uniform exponential integrability lemma for the ordinary and the remainder terms of the expansion. (iv) Quasi-sure analysis for the solution of the SDE.

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From non-symmetric random walks on nilpotent covering graphs to rough paths via discrete geometric analysis

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(joint work with Satoshi Ishiwata and Ryuya Namba)

As a fundamental problem in the theory of random walks, Donsker's invariance principle has been studied intensively and extensively. In rough path theory, Friz and coauthors [1, 2] captured the Brownian rough path as the standard CLT-scaling limit for random walks on a step-2 nilpotent Lie group satisfying some standard conditions (including zero mean condition). In this report, we discuss this problem from a viewpoint of *discrete geometric analysis* initiated by Sunada [6]. We consider a certain class of non-symmetric random walks on nilpotent covering graphs and obtain a distorted Brownian rough path as the CLT-scaling limit. In particular, we observe that an effect of the non-symmetry appears only on the second level path of this rough path. It is closely related to the *magnetic Brownian rough path* which was obtained by Friz–Gassiat–Lyons [3] as the small mass limit of the canonical rough path lift of physical Brownian motion in a magnetic field.

Let $X = (V, E)$ be a locally finite connected graph, where V and E denote the sets of vertices and oriented edges, respectively. For each edge $e \in E$, the origin, the terminus and the inverse edge are denoted by $o(e)$, $t(e)$ and $\overline{e}$, respectively. We call X a *nilpotent covering graph* if there exists a finitely generated torsion free nilpotent group Γ acting on X on the left freely and its quotient $\Gamma \setminus X$ is a finite graph. In other words, X is a covering graph of a finite graph $X_0 = (V_0, E_0)$

whose transformation group is a finitely generated torsion free nilpotent group Γ. Throughout this report, we assume that Γ is step-2 nilpotent.

We introduce a random walk on X. A transition probability is given by a Γ-invariant non-negative function $p : E \to \mathbb{R}$ satisfying $\sum_{e \in E_x} p(e) = 1$ $(x \in V)$ and $p(e) + p(\overline{e}) > 0$ $(e \in E)$, where $E_x = \{e \in E| \ o(e) = x\}$. Here $p(e)$ stands for the probability that a particle at $o(e)$ moves to $t(e)$ along the edge e in one unit time. It induces the probability measure $\mathbb{P}_x$ on the set $\Omega_x(X)$ of all infinite paths starting from $x \in V$. The random walk associated with p is the (time homogeneous) Markov chain $(\Omega_x(X), \mathbb{P}_x, \{w_n\}_{n=0}^{\infty})$ with values in X. Here w_n is defined by $w_n(c) := o(c(n+1))$, where $c(n)$ is the nth edge of the infinite path $c \in \Omega_x(X)$. Then the n-step transition probability $p(n, x, y)$ $(n \in \mathbb{N}, x, y \in V)$ is given by $p(n, x, y) = \mathbb{P}_x(w_n = y)$. Since p is Γ-invariant, we may project the random walk on X_0.

We assume that the random walk on X_0 is *irreducible*, that is, for every $x, y \in V_0$, there exists some $n = n(x, y) \in \mathbb{N}$ such that $p(n, x, y) > 0$. (If $p(e) > 0$ for each $e \in E_0$, this assumption holds.) Then by the Perron–Frobenius theorem, we find a unique positive function $m : V_0 \to \mathbb{R}$, called the *invariant measure*, satisfying

$$m(x) = \sum_{e \in (E_0)_x} p(\overline{e})m(t(e)) \quad (x \in V_0), \qquad \sum_{x \in V_0} m(x) = 1.$$

We also write $m : V \to \mathbb{R}$ for the (Γ-invariant) lift of the invariant measure, and set $\widetilde{m}(e) = p(e)m(o(e))$. Let $\mathrm{H}_1(X_0, \mathbb{R})$ and $\mathrm{H}^1(X_0, \mathbb{R})$ be the first homology group and the first cohomology group, respectively. We define the *homological direction* γ_p by

$$\gamma_p := \sum_{e \in E_0} \widetilde{m}(e)e \in \mathrm{H}_1(X_0, \mathbb{R}).$$

It should be noted that $\gamma_p = 0$ if and only if the random walk is m-symmetric, i.e., $\widetilde{m}(e) = \widetilde{m}(\overline{e})$ $(e \in E_0)$. By virtue of the discrete Hodge–Kodaira theorem, we identify $\mathrm{H}^1(X_0, \mathbb{R})$ with the space of modified harmonic 1-forms on X_0 Hence we may equip $\mathrm{H}^1(X_0, \mathbb{R})$ with the flat metric

$$\langle\!\langle \omega_1, \omega_2 \rangle\!\rangle := \sum_{e \in E_0} \omega_1(e)\omega_2(e)\widetilde{m}(e) - \langle \gamma_p, \omega_1 \rangle \langle \gamma_p, \omega_2 \rangle \qquad (\omega_1, \omega_2 \in \mathrm{H}^1(X_0, \mathbb{R})).$$

By Mal'cev's theorem, there exists a connected and simply connected nilpotent Lie group $G = G_\Gamma$ such that Γ is a cocompact lattice in G. Let $\mathfrak{g}$ be the Lie algebra of G. Since Γ is step-2 nilpotent, $\mathfrak{g}$ has a direct sum decomposition of the form

$$\mathfrak{g} = \mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)} = \mathfrak{g}^{(1)} \oplus [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}].$$

Let $\hat{\pi} : G \to G/[G, G]$ be the canoncal projection and we identify $G/[G, G]$ with $\mathfrak{g}^{(1)}$. Then $\hat{\pi}(\Gamma)$ is also lattice of $G/[G, G]$ and there is an isomorphism between $\hat{\pi}(\Gamma) \otimes \mathbb{R}$ and $\mathfrak{g}^{(1)}$. Since X is a covering graph of X_0 with the transformation group Γ, there exists a surjective homomorphism from the fundamental group $\pi_1(X_0)$ to Γ. This yields a surjective linear map $\rho_{\mathbb{R}} : \mathrm{H}_1(X_0, \mathbb{R}) \to \hat{\pi}(\Gamma) \otimes \mathbb{R} \cong \mathfrak{g}^{(1)}$. By using the map $\rho_{\mathbb{R}}$, we can induce the flat metric from $\mathrm{H}^1(X_0, \mathbb{R})$ to $\mathfrak{g}^{(1)}$. We call the induced metric the *Albanese metric* on $\mathfrak{g}^{(1)}$. We equip G with the Carnot–Carathéodory metric

$$d_{\mathrm{CC}}(x,y)=\inf\Big\{\int_0^1|\dot{c}(t)|_{\mathfrak{g}^{(1)}}\,dt\Big|c\in AC([0,1],G),\\ c(0)=x,c(1)=y,\dot{c}(t)\in\mathfrak{g}^{(1)}_{c(t)}\Big\}\qquad(x,y\in G),$$

where $AC([0,1],G)$ and $\mathfrak{g}^{(1)}_{c(t)}$ denote the set of absolutely continuous curves and the evaluation of $\mathfrak{g}^{(1)}$ at $c(t)$, respectively. Let $\{V_1,\ldots,V_d\}$ be an orthonormal basis of $\mathfrak{g}^{(1)}$ with respect to the Albanese metric. We extend $V_i\in\mathfrak{g}$ $(i=1,\ldots,d)$ to a left invariant vector field on G in the usual way. Then we have an identification of G with $\mathbb{R}^{d(d+1)/2}$ as differential manifold given by

$$(x_i;x_{ij})_{1\le i<j\le d}\mapsto\exp\Big(\sum_{1\le i\le d}x_iV_i+\sum_{1\le i<j\le d}x_{ij}[V_i,V_j]\Big),$$

which is called the canonical coordinates of the first kind.

Let $\Phi:X\to G$ be a Γ-equivalent map such that $\log\Phi|_{\mathfrak{g}^{(1)}}:X\to\mathfrak{g}^{(1)}$ is *modified harmonic*. Namely, $\Phi(\gamma x)=\gamma\Phi(x)$ $(x\in V,\gamma\in\Gamma)$ and

$$\sum_{e\in E_x}p(e)\log\big(\Phi(o(e))^{-1}\cdot\Phi(t(e))\big)\big|_{\mathfrak{g}^{(1)}}=\rho_{\mathbb{R}}(\gamma_p)\qquad(x\in V).$$

Here $\rho_{\mathbb{R}}(\gamma_p)$ is called the ($\mathfrak{g}^{(1)}$-)*asymptotic direction*. Although the symmetry of the random walk implies $\rho_{\mathbb{R}}(\gamma_p)=0$, the converse does not hold in general. There exists such a map Φ and it is uniquely determined on $\mathfrak{g}^{(1)}$ up to parallel transform (cf. [4, 5]). We set

$$\beta(\Phi):=\sum_{e\in E_0}\widetilde{m}(e)\log\big(\Phi(o(e))^{-1}\cdot\Phi(t(e))\big)\big|_{\mathfrak{g}^{(2)}}=\sum_{1\le i<j\le d}\beta(\Phi)^{ij}[V_i,V_j].$$

Note that $\beta(\Phi)=0$. holds provided the random walk is m-symmetric.

We now fix a reference point $x_*\in V$ such that $\Phi(x_*)=\mathbf{1}_G$ and put

$$\xi_n(c):=\Phi(w_n(c)),\quad \Xi_n(c):=\log(\xi_n(c))\qquad(c\in\Omega_{x_*}(X)).$$

We define $\mathcal{Y}^{(n)}_t:=\tau_{n^{-1/2}}(\exp(\mathcal{X}^{(n)}_t))$ $(n\in\mathbb{N},t\ge0)$, where τ_ε $(\varepsilon>0)$ is the dilation operator on G and $\mathcal{X}^{(n)}_t=\Xi_{[nt]}+(nt-[nt])(\Xi_{[nt]+1}-\Xi_{[nt]})$.

We consider a G-valued diffusion process starting from $\mathbf{1}_G$ which solves

$$dY_t=\sum_{i=1}^dV_i(Y_t)\circ dB^i_t+\sum_{1\le i<j\le d}\beta(\Phi)^{ij}[V_i,V_j](Y_t)dt,$$

where $(B_t)_{t\ge0}=(B^1_t,\ldots,B^d_t)_{t\ge0}$ is a standard Brownian motion on $\mathbb{R}^d$. If we use the canonical coordinates of the first kind introduced above, (Y_t) is given by

$$Y_t=\Big(B^i_t\,;\,\frac{1}{2}\int_0^t(B^i_s\circ dB^j_s-B^j_s\circ dB^i_s)+t\beta(\Phi)^{ij}\Big)_{1\le i<j\le d}\quad(t\ge0).$$

Our main result in this report is the following:

Theorem 1. *Under $\rho_{\mathbb{R}}(\gamma_p) = 0$, we have*

$$(\mathcal{Y}_t^{(n)})_{t\geq 0} \underset{n\to\infty}{\Longrightarrow} (Y_t)_{t\geq 0} \quad \text{in } \mathcal{C}^{\alpha}([0,1], G) \text{ for all } \alpha < 1/2.$$

In other words, $\mathcal{Y}^{(n)}$ converges to a distorted Brownian rough path

$$\overline{\mathbf{B}} = \Big(B_t - B_s, \int_s^t \int_s^u dB_v \otimes \circ dB_u + (t-s)\overline{\beta}(\Phi)\Big)_{0\leq s\leq t\leq 1}$$

in α-Hölder topology for all $\alpha < 1/2$, where $\overline{\beta}(\Phi) = (\beta(\Phi)^{ij})_{i,j=1}^d$ is an anti-symmetric matrix defined by $\beta^{ji}(\Phi) = -\beta^{ij}(\Phi)$ for $j \geq i$.

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Renormalization Group and SPDE's

Antti Kupiainen

Nonlinear stochastic PDE's driven by a space time white noise have been under intensive study in recent years [1, 2, 3, 4, 5]. These equations are of the form

$$\partial_t u = \Delta u + V(u) + \Xi \tag{1}$$

where $u(t,x) \in \mathbb{R}^n$ is defined on $\Lambda \subset \mathbb{R}^d$, $V(u)$ is a function of u and possibly its derivatives which can also be non-local and Ξ is white noise on $\mathbb{R} \times \Lambda$, formally

$$\mathbb{E}\ \Xi_\alpha(t',x')\Xi_\beta(t,x) = \delta_{\alpha\beta}\delta(t'-t)\delta(x'-x). \tag{2}$$

In order to be defined these equations in general require renormalization. One first regularizes the equation by e.g. replacing the noise by a mollified version $\Xi^{(\epsilon)}$ which is smooth on scales less than ϵ and then replaces V by $V^{(\epsilon)} = V + W^{(\epsilon)}$ where $W^{(\epsilon)}$ is an ϵ-dependent "counter term". One attempts to choose this so that solutions converge as $\epsilon \to 0$.

The rationale of such counterterms is that although they diverge as $\epsilon \to 0$ their effect on solutions on scales much bigger than ϵ is small. They are needed to make the equation well posed in small scales but they disturb it little in large scales.

Such a phenomenon is familiar in quantum field theory. For instance in quantum electrodynamics the "bare" mass and charge of the electron have to be made cutoff dependent so as to have cutoff independent measurements at fixed scales.

The modern way to do this is to use the Renormalization Group (RG) method of Wilson [8] which constructs a one parameter family of *effective theories* describing how the parameters of the theory vary with scale.

Such a RG method was applied to SPDE's in [5] for the case $n = 1$, $d = 3$ and

$$V(u) = u^3.$$

In that case

$$W^{(\epsilon)} = (a\epsilon^{-1} + b \log \epsilon)u$$

and path wise solutions were constructed recovering earlier results by [1, 2]. In [9] we considered the equations of Stochastic Hydrodynamics recently introduced by Spohn [6]. They give rise to the problem (1) with $n = 3$, $d = 1$ and

$$V(u) = (\partial_x u, M \partial_x u) \tag{3}$$

where $(\cdot, \cdot)$ denotes the standard inner product in R^3 and $M = (M^{(1)}, M^{(2)}, M^{(3)})$ with $M^{(i)}$ are symmetric matrices, so that (3) can be read component-wise as $V_i(u) = (\partial_x u, M^{(i)} \partial_x u)$ for $i = 1, 2, 3$. We construct path wise solutions in this case by taking

$$W^{(\epsilon)} = a\epsilon^{-1} + b \log \epsilon.$$

The case $n = 1$ is the KPZ equation and this was constructed before by Hairer [7]. In that case $b = 0$. For a generic $M_{\alpha\beta\gamma}$ in (3) $b \neq 0$. This counter term is third order in the nonlinearity. Thus in this case the simple Wick ordering of the nonlinearity does not suffice to make the equation well posed.

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Weakly asymmetric bridges and the KPZ equation

Cyril Labbé

We consider a discrete bridge from $(0,0)$ to $(2N,0)$, that is, a piecewise linear function $S : [0,2N] \to \mathrm{R}$ with slope ± 1 on each interval $[k, k+1)$ such that $S(0) = S(2N) = 0$. Then, we let this bridge evolve according to the corner growth dynamics: namely, each downwards corner flips into an upwards corner at rate p_N, and conversely, each upwards corner flips into a downwards corner at rate $1-p_N$. We refer to Figure for an illustration. This defines a continuous-time Markov chain on the finite state-space of discrete bridges. It is simple to check that this is an irreducible chain, whenever $p_N \neq 0, 1$. Let us point out that this process is intimately related to the exclusion process with N particles on $2N$ sites.

We choose to parametrise the asymmetry of the jump rates as follows

$$p_N = \frac{1}{2} + \frac{\sigma}{(2N)^\alpha}\ , \quad \sigma, \alpha \geq 0\ .$$

The important parameter is α. When it equals $+\infty$, we are in the symmetric regime, while $\alpha = 0$ corresponds to a strong asymmetry. Since these two extreme values lead to well-known behaviour, in the present work we focus on $\alpha \in (0, \infty)$.

As we are considering an irreducible continuous time Markov chain on a finite state-space, there exists a unique invariant measure μ_N whose expression is explicit. It happens that this measure is also reversible. Our first result concerns the scaling limit of the invariant measure. To state our result, let us define Σ_α^N as the mean of S under μ_N. For $\alpha \geq 1$, we rescale the lattice $[0, 2N]$ onto $[0, 1]$, and $x \mapsto \Sigma_\alpha^N(x)$ is a function from $[0, 1]$ into R, then we set

$$u^N(x) := \frac{S(x2N) - \Sigma_\alpha^N(x)}{\sqrt{2N}}\ , \quad x \in [0,1]\ .$$

For $\alpha < 1$, we need to zoom in a window of order N^α centred at site N in order to see non-trivial fluctuations and therefore $x \mapsto \Sigma_\alpha^N(x)$ is a function from

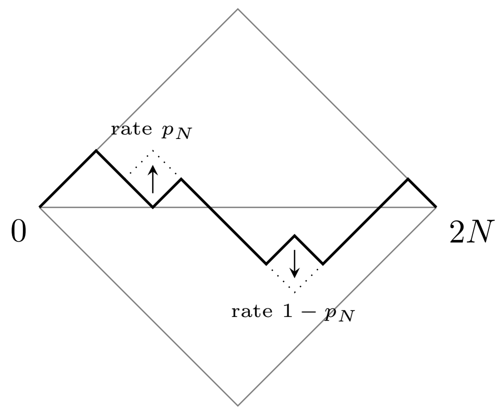

Figure 1. An example of interface.

$[-N/(2N)^\alpha, N/(2N)^\alpha]$ into R. Then, we set

$$u^N(x) := \frac{S(N + x(2N)^\alpha) - \Sigma_\alpha^N(x)}{(2N)^{\frac{\alpha}{2}}}\;, \quad x \in [-N/(2N)^\alpha, N/(2N)^\alpha]\;.$$

Theorem 1. *Under the invariant measure μ_N, we have $u^N \Rightarrow B_\alpha$ as $N \to \infty$. The process B_α is a Brownian bridge on $[0,1]$ when $\alpha > 1$. For $\alpha = 1$, resp. $\alpha < 1$, it is the image of a Brownian bridge on $[0,1]$ through a deterministic time change that maps $[0,1]$ onto itself, resp. onto* R.

We now turn our attention to the dynamics. First, let us mention that it is possible to characterise the dynamics at equilibrium: one gets fluctuations that evolve according to a linear stochastic heat equation in the scaling limit. We do not provide the details here, and refer the interested reader to [9].

Let us ask the following question: suppose we start from some initial profile S_0 at time 0, how does the dynamical interface reach its stationary state ? In other terms, we want to characterise the hydrodynamical limit of our process. To that end, we set

$$m^N(t,x) := \frac{S(t(2N)^2, x2N)}{(2N)^{2-\alpha}}\;, \quad t \geq 0\;, \quad x \in [0,1]\;,$$

when $\alpha \in [1, 3/2)$. In the case $\alpha < 1$, we set

$$m^N(t,x) := \frac{S\big(t(2N)^{\alpha+1}, x2N\big)}{2N}\;, \quad t \geq 0\;, \quad x \in [0,1]\;.$$

Theorem 2. *The process m^N converges in probability, in the Skorohod space $\mathbb{D}([0,\infty), \mathcal{C}([0,1]))$, to the deterministic process m where:*

(1) $\partial_t m = \frac{1}{2}\partial_x^2 m + \sigma$, *when* $\alpha \in (1, 3/2)$,
(2) $\partial_t m = \frac{1}{2}\partial_x^2 m + \sigma\big(1 - (\partial_x m)^2\big)$, *when* $\alpha = 1$,
(3) $\partial_t m = \sigma\big(1 - (\partial_x m)^2\big)$, *when* $\alpha < 1$.

Additionally, in all three cases, m satisfies Dirichlet boundary conditions.

Compare these three PDEs and observe that, as α decreases, the asymmetric term becomes predominant. Actually, the last equation is not well-defined and we need to work at the level of the derivative of the interface in order to establish a rigorous result. We refer to [9] for more details.

We now concentrate on the case $\alpha < 1$. It happens that the solution of the PDE in that case is explicit:

$$m(t,x) = \sigma t \wedge x \wedge (1-x)\;. \tag{1}$$

Essentially, the interface grows evenly, at speed σ, until it reaches the maximal shape $x \wedge (1-x)$. It is then natural to ask for the fluctuations around this hydrodynamical limit.

It appears that this question is related to a famous result of Bertini and Giacomin [1] on the Kardar Parisi Zhang (KPZ) equation, that we now briefly recall. Consider a corner growth process on the infinite lattice $\mathbb{Z}$ with jump rates $1/2+\sqrt{\epsilon}$

upwards and 1/2 downwards. If one starts from a flat initial profile, then results in [3, 2] ensure that the hydrodynamic limit grows evenly at speed $\sqrt{\epsilon}$. Then, Bertini and Giacomin look at the fluctuations around this hydrodynamic limit and show that the random process $\sqrt{\epsilon}\,(S(t\epsilon^{-2}, x\epsilon^{-1}) - \epsilon^{-3/2}t)$ converges to the solution of the KPZ equation, whose expression is given below (in Bertini and Giacomin's case, $\sigma = 1/2$).

Although our setting is similar to the one considered by Bertini and Giacomin, our Dirichlet boundary condition induces a major difference: our process admits a reversible probability measure, while this is not the case on the infinite lattice $\mathbb{Z}$. However, if one starts the interface "far" from equilibrium, then we are in an irreversible setting up to the time needed by the interface to reach the stationary regime, and one would expect the fluctuations to be described by the KPZ equation.

Bertini and Giacomin's result suggests to rescale the height function by $1/(2N)^\alpha$, the space variable by $(2N)^{2\alpha}$ and the time variable by $(2N)^{4\alpha}$. The space scaling immediately forces one to take $\alpha \le 1/2$ since, otherwise, the lattice $\{0, 1, \ldots, 2N\}$ would be mapped onto a singleton in the limit. It happens that the geometry of our model imposes a further constraint: Theorem 2 and Equation (1) show that the interface reaches the stationary state in finite time in the time scale $(2N)^{\alpha+1}$; therefore, as soon as $4\alpha > \alpha + 1$, Bertini and Giacomin's scaling yields an interface which is already at equilibrium in the limit $N \to \infty$. Consequently, we have to restrict α to $(0, 1/3]$ for this scaling to be meaningful.

We set

$$h^N(t,x) := \gamma_N S\big(t(2N)^{4\alpha}, N + x(2N)^{2\alpha}\big) - \lambda_N t\ ,$$

where $\lambda_N \sim 2\sigma^2(2N)^{2\alpha}$.

Theorem 3. *Take $\alpha \le 1/3$. As $N \to \infty$, the sequence h^N converges in distribution to the solution of the KPZ equation:*

$$\begin{cases} \partial_t h = \frac{1}{2}\partial_x^2 h - \sigma(\partial_x h)^2 + \dot{W}\ , \quad x \in \mathrm{R}\ , \quad t > 0\ , \\ h(0,x) = 0\ . \end{cases}$$

The convergence holds on $\mathbb{D}\big([0,T), \mathcal{C}(\mathrm{R})\big)$ where $T = 1/(2\sigma)$ when $\alpha = 1/3$, and $T = \infty$ when $\alpha < 1/3$.

The point which may be surprising in this theorem is that, at time T and in the case $\alpha = 1/3$, the fluctuations suddenly vanish. This is due to the fact that when $\alpha = 1/3$, the time scales of the hydrodynamical limit and of the KPZ fluctuations coincide: since the hydrodynamical limit reaches its stationary state in finite time, the irreversible (in nature) KPZ fluctuations suddenly vanish at this time.

Let us recall that the KPZ equation is a singular SPDE. While it was introduced in the physics literature [8] by Kardar, Parisi and Zhang, a first rigorous definition was given by Bertini and Giacomin [1] through the so-called Hopf-Cole transform. There exists a more direct definition of this SPDE (restricted to a bounded domain) due to Hairer [6, 7] via his theory of regularity structures. Let us also mention the notion of "energy solution" introduced by Gonçalves and Jara [4], for which uniqueness has been proved by Gubinelli and Perkowski [5]. It provides a new framework for characterising the solution to the KPZ equation but it requires the equation to be taken under its stationary measure: this is not the case in our setting. Hence, we adapt the proof of Bertini and Giacomin to establish our theorem.

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Rough paths, Signatures and the modelling of functions on streams

Terry Lyons

Rough path theory is focused on capturing and making precise the interactions between highly oscillatory and non-linear systems. The techniques draw particularly on the analysis of LC Young and the geometric algebra of KT Chen. The concepts and theorems, and the uniform estimates, have found widespread application; the first applications gave simplified proofs of basic questions from the large deviation theory and substantially extending Ito's theory of SDEs; the recent applications contribute to (Graham) automated recognition of Chinese handwriting and (Hairer) formulation of appropriate SPDEs to model randomly evolving interfaces. At the heart of the mathematics is the challenge of describing a smooth but potentially highly oscillatory and vector valued path xt parsimoniously so as to effectively predict the response of a nonlinear system such as $\partial y_t = f(y_t)\partial x_t,\ y_0 = a$. The Signature is a homomorphism from the monoid of paths into the grouplike elements of a closed tensor algebra. It provides a graduated summary of the path x. Hambly and Lyons have shown that this non-commutative transform is faithful for paths of bounded variation up to appropriate null modifications. Among paths of bounded variation with given Signature there is always a unique shortest representative. These graduated summaries or features of a path are at the heart of the definition of a rough path; locally they remove the need to look at the fine structure of the path. Taylor's theorem explains how any smooth function can, locally, be expressed as a linear combination of certain special functions (monomials based at that point). Coordinate iterated integrals form a more subtle algebra of features that can describe a stream or path in an analogous way; they allow a definition of rough path and a natural linear "basis" for functions on streams that can be used for machine learning.

Linking modelled and paracontrolled distributions

Jörg Martin
(joint work with Nicolas Perkowski)

The recent years have seen remarkable new approaches for the solution of singular stochastic partial differential equations. The theory of regularity structures, developed by Martin Hairer in [1], yields a general yet quite algebraic solution theory. Another approach, developed by Massimiliano Gubinelli, Peter Imkeller and Nicolas Perkowski in [2] relies on paracontrolled calculus to develop a more lightweighted method to solve singular stochastic PDEs.

While Hairer's regularity structures builds on a Taylor-like expansion of the solution to stochastic PDEs, leading to the concept of a *modelled distribution*, the paracontrolled approach uses an expansions in paraproducts which leads to *paracontrolled distributions*. Despite the different spirit behind these expansions, concrete examples such as 2d PAM [2] and the Φ^4_3 model [3] suggest a link between these notions. We show that this is no coincidence, in fact we show that there is a one-to-one correspondence between modelled and paracontrolled distributions.

To this end we consider as in [1] a solution f to an SPDE rather as a jet F in a regularity structure and rewrite the paraproduct expansion as a linear map acting on F. As a consequence we are able to reformulate the paracontrolled requirement in the language of regularity structures. We then show that a modelled distribution F automatically satisfies this condition. However, it turns out that an additional requirement on a paracontrolled distribution is needed to establish a one-to-one correspondence. As an example recall that in the theory of regularity structures polynomials in the jet F play a crucial rule. However, since the paraproduct with polynomials vanishes the polynomial terms of F will not appear in the paracontrolled description. We therefore have to introduce a second condition for paracontrolled distributions which, loosely speaking, fixes the polynomial terms of F. It turns out that this is enough to close the circle and we see that every paracontrolled distribution is modelled.

Beyond the intrinsic charm of this duality, it can be seen as a suggestion how paracontrolled distributions with several paraproducts could be defined and we currently try to apply it to extend the class of Schauder estimates for modelled/paracontrolled distributions.

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An invariant measure for the Φ^4_3 equation

Konstantin Matetski
(joint work with Martin Hairer)

A general framework for spatial discretisations of locally subcritical parabolic stochastic PDEs whose solutions are provided by the theory of regularity structures and which are functions in the time variable has been developed in [1]. A particular example of such equations is the dynamical Φ^4_3 model on the torus $\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3$ which can be formally written as

$$\partial_t \Phi = \Delta\Phi + \infty\Phi - \Phi^3 + \xi\,, \qquad \Phi(0,\cdot) = \Phi_0(\cdot)\,, \tag{1}$$

where Δ is the Laplace operator on $\mathbb{T}^3$, ξ is a space-time white noise and Φ_0 is an initial value. The "infinite constant" ∞ refers to the limit of a diverging renormalisation constant which should be added to the equation with a mollified noise in order to have a non-trivial limit. Due to the low regularity of the driving noise, the equation (1) is ill-posed, in the sense that a solution to the linearised equation is a distribution in space whose third power is undefined. A notion of solution to the Φ^4_3 equation was provided in [2] using the theory of regularity structures.

We consider spatial discretisations of the dynamical Φ^4_3 model on the dyadic grid $\mathbb{T}^3_\varepsilon \subset \mathbb{T}^3$ with the mesh size $\varepsilon > 0$ of the form

$$\frac{d}{dt}\Phi^\varepsilon = \Delta^\varepsilon \Phi^\varepsilon + C^\varepsilon \Phi^\varepsilon - \big(\Phi^\varepsilon\big)^3 + \xi^\varepsilon\,, \qquad \Phi^\varepsilon(0,\cdot) = \Phi^\varepsilon_0(\cdot)\,, \tag{2}$$

where Δ^ε is the nearest-neighbor approximation of the Laplacian Δ, Φ^ε_0 is some periodic initial value and the discretisation of the noise ξ is given by

$$\xi^\varepsilon(t,x) = \varepsilon^{-3}\langle \xi(t,\cdot), \mathbf{1}_{|\cdot - x|_\infty \le \varepsilon/2}\rangle\,, \qquad (t,x) \in \mathbb{R}\times\mathbb{T}^3_\varepsilon\,,$$

where $|\cdot|_\infty$ is the supremum norm in $\mathbb{R}^3$. We identify the discrete objects with their piece-wise linear extensions off the grid. A result concerning the discrete dynamical Φ^4_3 model can be formulated as follows:

Theorem 1. *In the described settings, let $\Phi_0 \in \mathcal{C}^\eta(\mathbb{R}^3)$ almost surely, for some $\eta > -\frac{2}{3}$, let Φ be the unique maximal solution of (1) on a random time interval $[0, T_\star)$, and let Φ^ε be the unique global solution of (2). If the initial data satisfies*

$$\lim_{\varepsilon\to 0} \|\Phi_0 - \Phi^\varepsilon_0\|_{\mathcal{C}^\eta} = 0$$

almost surely, then for every $\alpha < -\frac{1}{2}$ there is a sequence of renormalisation constants $C^\varepsilon \sim \varepsilon^{-1}$ in (2) and a sequence of stopping times T_ε satisfying $\lim_{\varepsilon\to 0} T_\varepsilon =$

$T_\star$ in probability such that, for every $\bar\eta < \eta \wedge \alpha$, and for any $\delta > 0$ small enough, one has the limit in probability

$$\lim_{\varepsilon\to 0} \|\Phi - \Phi^\varepsilon\|_{\mathcal{C}^\delta_{\bar\eta}([0,T_\varepsilon],\mathcal{C}^\alpha)} = 0 ,$$

where $\bar\eta$ is the blow-up at time $t = 0$.

Our main motivation to prove this convergence result goes back to the seminal article [3], where the authors prove that lattice approximations μ_ε to the Φ^4_3 measure are tight as the mesh size ε goes to 0. These measures are given by

$$\mu_\varepsilon(\Phi^\varepsilon) = e^{-S_\varepsilon(\Phi^\varepsilon)} \prod_{x\in\mathbb{T}^3_\varepsilon} d\Phi^\varepsilon(x)/Z_\varepsilon ,$$

where Φ^ε is any function on $\mathbb{T}^3_\varepsilon$, Z_ε is a normalisation factor, called "partition function", and the "action" S_ε is defined by

$$S_\varepsilon(\Phi^\varepsilon) = \frac{\varepsilon}{2}\sum_{x\sim y}\big(\Phi^\varepsilon(x) - \Phi^\varepsilon(y)\big)^2 - \frac{C^\varepsilon\varepsilon^3}{2}\sum_{x\in\mathbb{T}^3_\varepsilon}\Phi^\varepsilon(x)^2 + \frac{\varepsilon^3}{4}\sum_{x\in\mathbb{T}^3_\varepsilon}\Phi^\varepsilon(x)^4 , \tag{3}$$

with the first sum running over all the nearest neighbours on the grid, when each pair x, y is counted twice. Since these measures are invariant for the finite difference approximations (2), showing that these converge to (1) straightforwardly implies that any accumulation point of μ_ε is invariant for the solutions of (1). These accumulation points are known to coincide with the Φ^4_3 measure μ, see [4], thus showing that μ is indeed invariant for (1), as one might expect. Heuristically, the measure μ can be written as

$$\mu(\Phi) \sim e^{-S(\Phi)} \prod_{x\in\mathbb{T}^3} d\Phi(x) ,$$

for every $\Phi \in \mathcal{D}'(\mathbb{T}^3)$. In this case the "action" S is a limit of its finite difference approximations (3), i.e. it is formally given by

$$S(\Phi) = \int_{\mathbb{T}^3}\left(\frac{1}{2}\big(\nabla\Phi(x)\big)^2 - \frac{\infty}{2}\Phi(x)^2 + \frac{1}{4}\Phi(x)^4\right) dx .$$

With this notation at hand, an important corollary of Theorem 1 is the following result.

Corollary 2. *In the described context, for μ-almost every initial condition Φ_0, the solution of* (1) *constructed in* [2] *is almost surely global in time. In particular, this yields a reversible Markov process on $\mathcal{C}^\alpha(\mathbb{T}^3)$, with α as in Theorem 1, for which the Φ^4_3 measure is invariant.*

Since our framework is not designed specifically for the Φ^4_3 equation, it lays the foundations of a systematic approximation theory which can in principle be applied to many other singular stochastic PDEs, e.g. stochastic Burgers-type equations, the KPZ equation, or the continuous parabolic Anderson model.

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Enhanced Sanov theorem and large deviations for interacting particles

Mario Maurelli
(joint work with Jean-Dominique Deuschel, Peter K. Friz, Martin Slowik)

In this talk we analyze large deviations associated with an ensemble of weakly interacting particles and their joint iterated integrals, in the limit of large number of particles.

We consider the interacting particle system (of mean field type) on $\mathbb{R}^d$

$$dX^{i,N} = \frac{1}{N}\sum_{j=1}^{N} b(X^{i,N}, X^{j,N})dt + dW^i, \quad i = 1, \dots N,$$
$$X_0^{i,N} \text{ i.i.d. with law } \mu,$$

where W^i are independent Brownian motions and b is a bounded regular drift. For this system the asymptotic behaviour (for N large) of the random empirical measure

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{X^{i,N}}$$

is of interest and several classical results are available: law of large number ([10] for example), large deviations ([4], [5], [6], [11] among others). Recently asymptotic results were proven also in the context of rough interacting particles ([1], [3]).

Here we are interested in the behaviour not only of the particles $X^{i,N}$ but also of the joint iterated integrals $\int_0^t X_r^{i,N} \circ dX_r^{j,N}$, more precisely we consider the enhanced empirical measure

$$\frac{1}{N^2}\sum_{,ji=1}^{N} \delta_{S(X^{i,N}, X^{j,N})}$$

where $S(X^{i,N}, X^{j,N}) \sim (X^{i,N}, X^{j,N}, \int_0^\cdot X_r^{i,N} \circ dX_r^{j,N})$ is the (Stratonovich) rough path lifting of $(X^{i,N}, X^{j,N})$. Our main results (contained in [7]) are two large deviation principles (LDP) for these enhanced empirical measures, in the limit $N \to \infty$: the first result (enhanced Sanov theorem) in the case of no interaction, the second in the general case.

Compared to other large deviation results, the peculiarity of our result is precisely the study of the joint iterated integrals. As a corollary we get the convergence of the enhanced empirical measure (which seems not an easy consequence of any law of large numbers). Another application may be in large deviation estimates for SDEs driven by $(X^{i,N})_{i=1,2}$, taking advantage of the continuity of the rough integral. Moreover as a byproduct of our LDP we get a robust proof of classical large deviation principle for empirical measures (extending the proof in [4]), although other short proofs are also available.

The proof of the enhanced Sanov theorem is obtained starting from classical Sanov theorem (for the classical Brownian empirical measure) and applying the extended contraction principle: Sanov theorem can be transferred to the enhanced empirical measure, once we prove this is an "almost continuous" function of the empirical measure. For this we need two ingredients, one coming from large deviation one from rough paths: the Hoeffding decomposition ([9]) allows to split the ensemble $S(X^{i,N}, X^{j,N})$ into subgroups of independent variables, thus recovering the key independence structure of B^i; the exponential convergence of piecewise linear approximations to the Brownian rough path (in [8]) gives the needed approximation in the extended contraction principle.

The LDP in the general interacting case is obtained from the enhanced Sanov theorem: we use Girsanov theorem and the mean field type of interaction to relate the enhanced empirical measures in the interacting and non-interacting cases, then we apply Varadhan lemma, where we take advantage of the continuity of the rough integral. A technical difficulty arises in the more than linear growth of the rough integral and is treated via special tools coming from [2].

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Long time dynamics of random data nonlinear dispersive equations

Andrea R. Nahmod

In this talk we convey the main ideas behind Bourgain's approach in the nineties[1] to invariant Gibbs measures and almost sure global well-posedness for Hamiltonian dispersive PDE. We do not go over all details or precise definitions but rather give the main flavors. The focus is on the nonlinear Schrödinger equation (NLS).

We then present some recent probabilistic well posedness results for the NLS on $\mathbb{T}^d$ ($d = 3, 2, 1$) (joint work with G. Staffilani). Here we describe two type of results. One is the a.s local well posedness and long time well-posedness with positive probability for the quintic NLS on $\mathbb{T}^3$ with random data below H^1. The other one is for the defocusing cubic NLS on $\mathbb{T}^2$ and the focusing quintic NLS on $\mathbb{T}^1$. Here we explain a new probabilistic preservation of regularity argument which allow us to close an important gap between the deterministic global well-posedness (gwp) theory and the a.s gwp one proved by Bourgain via the invariance of their associated Gibbs measures. We indicate along the way why the theory regularity structures and of paracontrolled distributions is poised to play a fundamental role in two challenging open problems. Namely proving the a.s. gwp of the quintic NLS on $\mathbb{T}^2$ and of the cubic NLS on $\mathbb{T}^3$ with invariance of the associated Gibbs measures.

We conclude describing work in progress on a nondeterministic approach to the $2D$ wave constant mean curvature equation (CMC) of Chanillo et al.. We address only the first part of this program concerning the a.s local well-posedness below the energy space H^1 (joint with M. Czubak, D. Mendelson and G. Staffilani)[2]. The latter result has interest in its own right. The geometry of the problem induces a structure on the nonlinearity, known as a *null form*, which renormalizes the quadratic derivative nonlinearity naturally removing the resonant interactions needed for an improved a.s. lwp at a supercritical regime.

The nonlinear Schrodinger equation (NLS) is

$$\text{(NLS)} \qquad \begin{cases} i\, u_t + \Delta u = \pm |u|^{p-1} u, \\ u(0,x) = u_0(x) \in H^s \qquad x \in \mathcal{M}^d, \end{cases}$$

where $u : \mathbb{R} \times \mathcal{M}^d \to \mathbb{C}$ and $\mathcal{M}^d = \mathbb{R}^d$, $\mathbb{T}^d$ or any compact manifold with Δ the Laplace Beltrami. It has conservation of mass $\int |u(t,x)|^2 dx$ and of Hamiltonian

$$H(u(t)) := \frac{1}{2} \int |\nabla u(t,x)|^2 \, dx \pm \frac{1}{p+1} \int |u(t,x)|^{p+1} dx$$

where the positive sign yields a defocusing equation (gives a global in time bound for the H^1 norm of $u(t,x)$) and the minus a focusing one (energy could be negative and blow up may occur). The equation also enjoys *time reversibility* and on $\mathbb{R}^d$ we have the following *scaling*: $u_\lambda(x,t) = \lambda^{-\frac{2}{p-1}} u\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right)$, $u_{0,\lambda} = \lambda^{-\frac{2}{p-1}} u_0(\frac{x}{\lambda})$ whence the scale invariant problem corresponds to regularities $s_c = \frac{d}{2} - \frac{2}{p-1}$ called the

[1]After the works of Lebowitz-Rose-Speer and of Zhidkov for Hamiltonian PDE and of Glimm-Jaffe and others for the ϕ^4 model.

[2]The long term program is joint with S. Chanillo as well.

critical exponent. Regularities $s > s_c$ are called subcritical (scaling help us) while $s < s_c$ supercritical (scaling is against us). The characteristic feature of dispersive equations is the fact that different frequencies travel at different velocities. On $\mathbb{R}^d$ dispersion implies decay of the solution as time evolves (while conserving mass and Hamiltonian) and the Strichartz estimates yield local well-posedness in subcritical and small data global well posedness in certain critical regimes (mass, energy). Furthermore, in the last 15 years there has been lots of progress to prove large data gwp and scattering for the defocusing NLS on $\mathbb{R}^d$ for energy ($s_c = 1$) and mass critical ($s_c = 0$) problems.

On compact domains, wave packets have no escape from interacting indefinitely with each other. Dispersion does not necessarily translate to decay. Dispersion is weaker and in fact a limited number of Strichartz estimates survive. These were proved by Bourgain (93') for the square torus $\mathbb{T}^d$ and the whole range recently completed by Bourgain-Demeter (14') for irrational tori as well. These are sufficient for local well-posedness (lwp) at subcritical regimes $s > s_c$ and global in H^1 for $s_c < 1$ but in general much less is known. For example, lwp for $s_c = 0$ -say the cubic NLS on $\mathbb{T}^2$ or the quintic NLS on $\mathbb{T}$ are unknown. We thus turn to a nondeterministic approach to well-posedness. The starting point is the work of Bourgain (94'-96') who studied longtime dynamics of periodic NLS (d=1,2)[3] in the almost sure sense and capturing generic behavior of the flow. He proved for example that the cubic NLS on $\mathbb{T}^2$ is gwp for a set of data of full Gibbs measure and that the (Gibbs) measure is invariant under the flow. And a similar result for the *focusing* cubic or quintic NLS on $\mathbb{T}$ (with suitable restriction of the mass). The invariance of the Gibbs measure, just like the usual conserved quantities, controls the growth in time of those solutions in its support and thus allows one to extend the local in time solutions to global ones almost surely at a level of regularity where there are no conservation laws. The approach via invariant measures has challenges and limitations. In the particular case of the periodic NLS, we know that there are no Gibbs measure for $d \geq 4$ nor for -say- the defocusing quintic NLS on $\mathbb{T}^3$. There are no Gibbs measures either for the focusing cubic NLS on $\mathbb{T}^2$ (Brydges-Slade). A major difference between Bourgain's work in 1D and 2D was the fact that while in 1D there was a deterministic local well-posedness in place on $H^s(\mathbb{T}), s > 0$ and hence on $H^{1/2-}(\mathbb{T})$ where the Gibbs measure lives, in 2D there is no deterministic local well posedness theory at the level of $H^{-\varepsilon}(\mathbb{T}^2)$ which lies in the supercritical regime. Bourgain's point (96') was then to prove *probabilistic local well posedness* for typical elements in the support of the measure. That is, almost a.s. for random data $\phi^\omega(x) = \sum_{n\in\mathbb{Z}^2} \frac{g_n(\omega)}{\sqrt{1+|n|^2}} e^{in\cdot x}$ in $H^{-\varepsilon}(\mathbb{T}^2)$. Randomization does not improve regularity in terms of derivatives. The improvement is with respect to L^p spaces a.s. which in turn, imply better estimates than the deterministic ones[4] Bourgain's strategy consists then in looking for solutions of the form $u = S(t)\phi^\omega + w$, use that the linear evolution of random data, $S(t)\phi^\omega$ has

[3]As well as other dispersive Hamiltonian PDE.

[4]as is well known from classical results of Rademacher, Kolmogorov, Paley and Zygmund showing that random series enjoy better L^p bounds than deterministic ones.

better L^p estimates a.s to show that $w = u - S(t)\phi^\omega$ solves a difference equation $iw_t + \Delta w = \mathcal{N}(S(t)\phi^\omega + w)$ that lives in a smoother space. A deterministic lwp w in the smoother space is also needed. As a consequence Bourgain showed that a.s. in ω the nonlinear part of the solution $w = u - S(t)\phi^\omega$ is smoother than the linear part. Together with G. Staffilani we consider the quintic NLS on $\mathbb{T}^3$ with random data $\phi^\omega \in H^\gamma(\mathbb{T}^3), \gamma < 1$ and first establish a similar probabilistic local well posedness. In 3D Wick ordering does not remove bad resonant frequencies of the quintic nonlinearity which correspond to certain double pairs of frequencies collapsing simultaneously. Instead, we use a gauge transformation to renormalize the quintic nonlinearity to: $N(v) := v|v|^4 - 3v\left(\int_{\mathbb{T}^3} |v|^4\, dx\right)$ and this is the one we estimate following Bourgain strategy as above. In case we do not have a Gibbs measure, and the difference equation satisfies no conservation law. Thus extending the local solutions to global ones a.s is a challenging problem. Nonetheless we are able to prove large data long time existence with positive probability; i.e. that there are some large infinite energy data evolving to solutions for long times[5].

The second result we discuss also joint with Staffilani is a new probabilistic propagation of regularity method which we apply to prove that the cubic defocusing NLS is a.s gwp in $H^s(\mathbb{T}^2)$, $s > 0$ and that the quintic focusing NLS is a.s gwp in $H^s(\mathbb{T})$, $s > \frac{1}{2}$. These results close an important gap between the a.s. gwp of Bourgain and the known deterministic gwp. For example, in 2D, deterministic methods yield lwp for $s > 0$ (Bourgain) and gwp for $s > 2/3$ (De Silva-Pavlovic-Staffilani-Tzirakis) via the so called I-method of *almost conservation laws*. Data randomization and the invariance of the Gibbs measure yield a.s. gwp in $H^{-\epsilon}$ (Bourgain) as described above. Our result thus fill the gap a.s. for $0 < s \le 2/3$. One should note that our set of initial data $\Sigma \subset H^s$, $s > 0$ is not seen by the Gibbs measure so the result is not trivial. Large data gwp at a critical or supercritical regularity level for NLS is a challenging question which is not made any easier by assuming higher regularity of the initial data[6]. To prove large data gwp one has to start with data at the regularity level of some conserved quantity such as, for example, the mass (L^2) or the Hamiltonian (H^1). It is only after one has proved such global result, that smooth global solutions can be obtained by a standard preservation of regularity argument based on differentiation of the equation. This is a purely deterministic approach. The procedure we implement in our proof is not based on differentiation of the equation as in the deterministic preservation of regularity argument. Our key idea instead is to suitably decompose the data into a term that is close to the support of the invariant measure in the rougher topology, and a smoother remainder term to which deterministic arguments can be applied. Then, a nondeterministic perturbation argument is used to conclude. We believe that this argument could be applied to other problems for which an a.s. gwp is proved using an invariant or almost invariant measure.

[5]As $T \to \infty$ the size of the set of initial data giving rise to solutions on the whole interval $[0, T]$ shrinks to zero.

[6]The flow does not improve the regularity of the data so the solution is at each time t as regular as the data.

The first step in our nondeterministic study of the wave CMC concerns the nonlinear wave equation (NLW) in 2D: $\Box u = 2u_x \wedge u_y$ which unlike general quadratic derivative nonlinearity inherits a special structure from its geometric framework. For general quadratic derivative nonlinearities deterministic lwp holds for regularities $s > \frac{7}{4}$ only (by Strichartz) even though the scaling of the problem is $s_c = 1$. This can be improved if there is structure in quadratic derivative nonlinearity. For the one that concern us, to $s > \frac{5}{4}$ and this is sharp in the Sobolev class. We consider random data $u_0^\omega = \sum_{n\in\mathbb{Z}^2} \frac{g_n(\omega)}{|n|^2} e^{in\cdot x} \in H^{1-\epsilon}$ and establish a.s lwp. Let us denote by $W(t)$ the linear evolution. The key aspect of the null form structure $u_x \wedge u_y$ revels itself while estimating random-random interactions $(W(t)u_0^\omega)_x \wedge (W(t)u_0^\omega)_y$, for it provides the necessary independence in the large deviation estimates to prove that outside a small set of omega's $\left|\sum_{S(m,\tau)} \frac{g_n(\omega)}{|n|^2}\frac{g_k(\omega)}{|k|^2}(n_1k_2 - n_2k_1)\right|^2$ can be bounded by $\sum_{S(m,\tau)} \frac{1}{|n|^4}\frac{1}{|k|^4}(n_1k_2 - n_2k_1)^2$ where $S(m,\tau) = \{(n,k) \;/\; m = n+k \;\; \tau = |n| + |k|\}$. And this one we can control thanks to the strong decay and favorable integer lattice counting estimates on $\mathbb{Z}^2$.

Regression on the Path Space

Hao Ni

In this talk, we bring the theory of rough paths to the study of non-parametric statistics on streamed data. We discuss the problem of regression where the input variable is a stream of information, and the dependent response is also (potentially) a stream.

A certain graded feature set of a stream, known in the rough path literature as the signature, has a universality that allows formally, linear regression to be used to characterise the functional relationship between independent explanatory variables and the conditional distribution of the dependent response.

This approach, via linear regression on the signature of the stream, is almost totally general, and yet it still allows explicit computation. The grading allows truncation of the feature set and so leads to an efficient local description for streams (rough paths). In the statistical context this method offers potentially significant, even transformational dimension reduction, which is demonstrated in the example of learning the solution to unknown controlled differential equations.

By way of illustration, our approach is applied to stationary time series including the familiar AR model and ARCH model. In the numerical examples we examined, our predictions achieve similar accuracy to the Gaussian Process (GP) approach with much lower computational cost especially when the sample size is large.

Lastly although our approach provides a systematic treatment of a general regression problem on the paths space, sometimes it requires to use the signature of a path up to very high degree to achieve certain fitting accuracy. It potentially results in the overfitting issue due to high dimensionality of truncated signatures. To overcome this difficulty, we propose a variant of the signature feature set of data streams - so called dyadic path signature features, which is a collection of signatures of a path of lower degree over dyadic time partitions. It turns out to be a more efficient features for a path over a long time interval in terms of the dimensionality. We combine this feature set with deep learning to solve the online Chinese character recognition, which outperforms the existing method and significantly reduces test errors.

Hairer-Quastel universality with energy solutions

Nicolas Perkowski
(joint work with Massimiliano Gubinelli)

Consider the following stochastic PDE which models the fluctuations in random interface growth:

$$\partial_t v = \Delta v + \varepsilon^{1/2}\partial_x F(v) + \partial_x \rho * \xi \tag{1}$$

on $[0,\infty)\times\mathbb{T}_\varepsilon$ with $\mathbb{T}_\varepsilon = \mathbb{R}/(\varepsilon^{-1}\mathbb{Z})$, where ξ is a space-time white noise, ρ is a smooth kernel, and F is an even polynomial. The nonlinearity $\varepsilon^{1/2}\partial_x F$ is a weak asymmetry that breaks the time-reversibility of the linear system. The celebrated Hairer–Quastel universality result [HQ15] states that there exist constants $c_1, c_2 \in \mathbb{R}$ such that the rescaled process $\varepsilon^{-1/2}v_{t\varepsilon^{-2}}((x-c_1\varepsilon^{-1/2}t)\varepsilon^{-1})$ converges to the solution u of the stochastic Burgers equation

$$\partial_t u = \Delta u + c_2\partial_x u^2 + \xi.$$

We give an alternative proof of this result using the notion of *energy solutions* to the stochastic Burgers equation, which formulate it as a martingale problem [GJ13a, GJ13b, GP15]. So consider the equation

$$\partial_t u^\varepsilon = \Delta u^\varepsilon + \varepsilon^{-1}\partial_x\Pi_0^N(F(\varepsilon^{1/2}u^\varepsilon) - c_1(F)\varepsilon^{1/2}\partial_x u^\varepsilon) + \partial_x\Pi_0^N\xi, \qquad u_0^\varepsilon = \Pi_0^N\eta, \tag{2}$$

on $\mathbb{R}_+\times\mathbb{T}$ (for $\mathbb{T}=\mathbb{T}_1$), where Π_0^N denotes the projection on the Fourier modes $\{-N,\dots,N\}\setminus\{0\}$ for $N=\varepsilon^{-1}$ and η is a space white noise, independent of ξ. This is the equation we obtain for $u^\varepsilon(t,x) = \varepsilon^{-1/2}v_{t\varepsilon^{-2}}((x-c_1\varepsilon^{-1/2}t)\varepsilon^{-1})$ if v solves (1) and ρ is the Fourier truncation operator. The only difference is that now we also mollified the nonlinearity by applying the truncation operator to it as well. In this way we can guarantee that the law of $\Pi_0^N\eta$ is invariant under the dynamics of u^ε, which will be important in what follow. If $\varphi\in C^\infty(\mathbb{T})$ is a test function, then (2) translates to

$$u_t^\varepsilon(\varphi) = u_0^\varepsilon(\varphi) + \int_0^t u_s^\varepsilon(\Delta\varphi)\mathrm{d}s - \int_0^t \varepsilon^{-1}(F(\varepsilon^{1/2}u_s^\varepsilon) - c_1(F)\varepsilon^{1/2}u_s^\varepsilon)(\Pi_0^N\partial_x\varphi)\mathrm{d}s + M_t^\varphi,$$

where M^φ is a martingale in the filtration generated by u^ε, with quadratic variation $\mathrm{d}\langle M^\varphi\rangle_t = 2\|\partial_x\Pi_0^N\varphi\|^2\mathrm{d}t$. Our strategy for proving the convergence of (u^ε) is now as follows:

– Show the tightness of u^ε in $C([0,T],\mathcal{D}')$, where $\mathcal{D}'$ denotes the distributions on $\mathbb{T}$. For that purpose it suffices to show the tightness of $u^\varepsilon(\varphi)$ in

$C([0,T],\mathbb{R})$ for any $\varphi \in C^\infty(\mathbb{T})$, which in turn follows once we get the joint tightness of

(3)
$$\left(u_0^\varepsilon(\varphi), \int_0^\cdot u_s^\varepsilon(\Delta\varphi)\mathrm{d}s, \int_0^\cdot \varepsilon^{-1}\left(F\left(\varepsilon^{1/2}u_s^\varepsilon\right) - c_1(F)\varepsilon^{1/2}u_s^\varepsilon\right)(\Pi_0^N\partial_x\varphi)\mathrm{d}s, M^\varphi\right).$$

– Once we showed tightness of u^ε, verify that every limit point of u^ε is an energy solution to stochastic Burgers equation.
– Use the uniqueness of energy solutions to conclude that (u^ε) converges weakly to the solution of the stochastic Burgers equation.

The most difficult step is as usual to prove tightness. All of the terms in (3) are very easy to control, except the nonlinear one. To bound it, we need to introduce a new tool. One can show that under stationary initial conditions the time-reversed process $\hat{u}_t^\varepsilon = u_{T-t}^\varepsilon$ solves the equation

$$\partial_t\hat{u}^\varepsilon = \Delta\hat{u}^\varepsilon - \varepsilon^{-1}\Pi_0^N\partial_x(F(\varepsilon^{1/2}\hat{u}^\varepsilon) - c_1(F)\varepsilon^{1/2}\hat{u}^\varepsilon) + \partial_x\Pi_0^N\hat{\xi},$$

where $\hat{\xi}$ is now a space-time white noise in the backward filtration. We write $\mathscr{L}_0^\varepsilon$ for the generator of the solution X^ε to $\partial_t X^\varepsilon = \Delta X^\varepsilon + \partial_x\Pi_0^N\xi$, and define for $\Psi \in L^2(\mu^\varepsilon)$, with μ^ε denoting the law of $\Pi_0^N\eta$,

$$\mathcal{E}^\varepsilon(\Psi) := \sum_{0<|k|\leqslant N} k^2|\mathrm{D}_k\Psi|^2,$$

where D_k is the directional derivative of Φ in the k-th Fourier monomial e_k. If $\Psi \in \mathrm{dom}\,(\mathscr{L}_0^\varepsilon)$, then we apply Itô's formula to $\Psi(u^\varepsilon)$ and to $\Psi(\hat{u}^\varepsilon)$, and adding the resulting expressions up we cancel exactly the contribution of the nonlinear term and thus obtain the following lemma by applying the Burkholder-Davis-Gundy inequality to the remaining forward and backward martingale terms.

Lemma 1 (Itô trick)**.** *For $\Psi \in \mathrm{dom}\,(\mathscr{L}_0^\varepsilon)$ and $T > 0$, $p \geqslant 1$ we have*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t \mathscr{L}_0^\varepsilon\Psi(u_s^\varepsilon)\mathrm{d}s\right|^p\right] \lesssim T^{p/2}\mathbb{E}[\mathcal{E}^\varepsilon(\Psi)^{p/2}].$$

The Itô trick is a very powerful tool for controling additive functionals of u^ε, provided that we can solve the Poisson equation $\mathscr{L}_0^\varepsilon\Psi = \Phi$ for a given Φ. Here it turns out to be very helpful that we only have to solve the Poisson equation in $L^2(\mu^\varepsilon)$, where we have a lot of additional structure (μ^ε is a Gaussian measure) and in particular we can work with the chaos expansion. In our case we want to control

$$\mathbb{E}\left[\left|\int_0^T \varepsilon^{-1}(F(\varepsilon^{1/2}u_s^\varepsilon) - c_1(F)\varepsilon^{1/2}u_s^\varepsilon)(\Pi_0^N\partial_x\varphi)\mathrm{d}s\right|^p\right].$$

Using the fact that if η denotes the coordinate map on $C(\mathbb{T},\mathbb{R})$, then $\varepsilon^{1/2}\eta(x)$ is a standard Gaussian under μ^ε, we can explicitly derive the chaos expansion of the integrand G^ε:

$$G^{\varepsilon}(\eta) = \sum_{n \geqslant 2} c_n(F)\varepsilon^{n/2} \sum_{k_1,\dots,k_n} \hat{\varphi}(-k_1 - \dots - k_n) I_n(e_{-k_1} \otimes \dots \otimes e_{-k_n}), \tag{4}$$

where I_n is a multiple stochastic integral and $c_n(F) = \frac{1}{k!}\mathbb{E}[F(U)H_k(U)]$, with H_k being the k-th Hermite polynomial and $U \sim \mathcal{N}(0,1)$. Here we used that $\Pi_0^N \partial_x \varphi$ vanishes when tested against constants, and that the correction term $-c_1(F)\varepsilon^{-1/2}\eta(\Pi_0^N \partial_x \varphi)$ exactly cancels the first chaos contribution of $\varepsilon^{-1}F(\varepsilon^{1/2}\eta)(\Pi_0^N \partial_x \varphi)$. Solving the Poisson equation is now an easy exercise, because it turns out that

$$\mathscr{L}_0^{\varepsilon} I_n(f_n) = I_n(\Delta f_n),$$

where Δ denotes the Laplacian on $\mathbb{T}^n$. Using Fourier coordinates we can even write down the solution explicitly. Moreover, all terms in (4) with $n > 2$ come with a factor $\varepsilon^{1/2}$, and therefore they vanish as we send $\varepsilon \to 0$. Consequently, we only remain with the quadratic contribution and we see that the nonlinear term is for small ε well approximated by

$$\int_0^T c_2(F)(u_s^{\varepsilon})^2 (\Pi_0^N \partial_x \varphi) \mathrm{d}s.$$

This allows us to show the tightness of the nonlinear term, and finally we deduce that (u^{ε}) is tight and any limit point u is a *controlled process:*

- the law of u_t is the white noise μ for all $t \in [0,T]$;
- for any $\varphi \in C^{\infty}(\mathbb{T})$ the process $t \mapsto \langle \mathcal{A}_t, \varphi \rangle$ has zero quadratic variation, $\langle \mathcal{A}_0, \varphi \rangle = 0$ and

$$\langle u_t, \varphi \rangle = \langle u_0, \varphi \rangle + \int_0^t \langle u_s, \Delta \varphi \rangle \mathrm{d}s + \langle \mathcal{A}_t, \varphi \rangle - \langle M_t, \partial_x \varphi \rangle, \tag{5}$$

  where $(\langle M_t, \partial_x \varphi \rangle)_{0 \leqslant t \leqslant T}$ is a martingale with quadratic variation $[\langle M, \partial_x \varphi \rangle]_t = 2t \|\partial_x \varphi\|^2_{L^2(\mathbb{T})}$;
- the reversed processes $\hat{u}_t = u_{T-t}$, $\hat{\mathcal{A}}_t = -(\hat{\mathcal{A}}_T - \mathcal{A}_{T-t})$ satisfy the same equation in their own filtration.

Moreover, we get for any limit point u that

$$\mathcal{A}_t(\varphi) = \lim_{M \to \infty} -\int_0^t \langle (\Pi_0^M u_s)^2, \partial_x \varphi \rangle \mathrm{d}s.$$

As it turns out, there exists only one controlled process which satisfies the stochastic Burgers equation in this martingale sense [GP15], and therefore our proof of convergence is complete:

Theorem 2 ([GP16]). *Let F be almost everywhere differentiable and assume that F' has polynomial growth. Then the solution u^{ε} to (2) converges in distribution to the unique equilibrium energy solution to*

$$\partial_t u = \Delta u + c_2(F)\partial_x u^2 + \xi,$$

where ξ is a space-time white noise with variance 2 and for $U \sim \mathcal{N}(0,1)$ and H_2 the second Hermite polynomial we have $c_2(F) = \mathbb{E}[F(U)H_2(U)]/2$.

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Continuity of the Itô map on Nikolskii spaces

David J. Prömel

(joint work with Peter K. Friz)

Stochastic differential equations (SDEs) are central objects in stochastic analysis since, in particular, they allow for modeling many real world phenomena as for instance appearing in finance or physics. From an analytical point of view SDEs are controlled ordinary differential equations (ODEs) which have random driving signals. Assuming the driving signal is a continuous deterministic path $X\colon [0,T] \to \mathbb{R}^n$, the dynamics of a controlled ODE is given by

$$\mathrm{d}Y_t = V(Y_t)\,\mathrm{d}X_t, \quad Y_0 = y_0, \quad t \in [0,T], \tag{1}$$

where $y_0 \in \mathbb{R}^m$ and $V\colon \mathbb{R}^m \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is some smooth enough vector field. Here $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ denotes the space of linear operators from $\mathbb{R}^n$ to $\mathbb{R}^m$.

One of the original motivations of rough path theory is to provide a rigorous mathematical meaning and the existence of a unique solution $Y\colon [0,T] \to \mathbb{R}^m$ to the controlled ODE (1) with the additional aim to cover driving signals X which are as irregular as sample paths of semi-martingales or Gaussian processes.

For this purpose the free nilpotent group of step N over $\mathbb{R}^n$ is denoted by $G^N(\mathbb{R}^n)$ and equipped with the Carnot-Caratheodory metric d_{cc}. Roughly speaking, Lyons [2, 3] introduced the space of weakly geometric p-rough paths, for $p \in [0,\infty)$, which is the space $\mathcal{V}^p([0,T], G^{\lfloor p \rfloor}(\mathbb{R}^n))$ of continuous paths X with values in $G^{\lfloor p \rfloor}(\mathbb{R}^n)$ of finite p-variation, i.e.,

$$\|X\|_{p\text{-var};[0,T]} := \Bigg(\sup_{\mathcal{P}\subset[0,T]} \sum_{[u,v]\in\mathcal{P}} d_{cc}(X_u, X_v)^p \Bigg)^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all partitions $\mathcal{P}$ of the interval $[0,T]$. Assuming the driving signal $X \in \mathcal{V}^p([0,T], G^{\lfloor p \rfloor}(\mathbb{R}^n))$, Lyons first proved the existence of a unique solution Y to the equation (1) and furthermore established the local Lipschitz continuity of the Itô map Φ, which is given by

$$\Phi\colon \mathbb{R}^m \times \mathrm{Lip}^\gamma \times \mathcal{V}^p([0,T], G^{\lfloor p \rfloor}(\mathbb{R}^n)) \to \mathcal{V}^p([0,T];\mathbb{R}^m)$$
$$\text{via} \quad \Phi(y_0, V, X) := Y,$$

where Y denotes the solution to equation (1) given the input (y_0, V, X).

Based on Lyons' estimates, which more precisely used general control functions, Friz [1] deduced the local Lipschitz continuity of the Itô map defined on the space $\mathcal{C}^\delta([0,T], G^{\lfloor 1/\delta \rfloor}(\mathbb{R}^n))$ of Hölder weakly geometric rough paths, i.e., $X \in \mathcal{C}^\delta([0,T], G^{\lfloor 1/\delta \rfloor}(\mathbb{R}^n))$ if

$$\|X\|_{\mathcal{C}^\delta;[0,T]} := \sup_{u,v\in[0,T],\, u<v} \frac{d_{cc}(X_u, X_v)}{|v-u|^\delta} < \infty, \quad \delta \in (0,1].$$

The Hölder space and the p-variation space can be somehow seen as the two extreme spaces on which the continuity of the Itô map can be restored. Especially, one obviously has the following relation

$$\mathcal{C}^\delta([0,T], G^{\lfloor 1/\delta \rfloor}(\mathbb{R}^n)) \subset \mathcal{V}^{1/\delta}([0,T], G^{\lfloor 1/\delta \rfloor}(\mathbb{R}^n))$$

for $\delta \in (0,1]$ and it is well-known that there exist many classical functions spaces "interpolating" between these two spaces. To unify the picture about the continuity of the Itô map, we investigate the continuity of the Itô map with respect to (classical) Nikolskii distances. In the case of weakly geometric p-rough paths for $p \in (2,3)$ a basically complete picture was very recently provided by [4] using Besov spaces.

In order to obtain the continuity for general weakly geometric p-rough paths, we introduce the Nikolskii regularity of a path $X \in \mathcal{V}^p([0,T], G^{\lfloor p \rfloor}(\mathbb{R}^n))$ in terms of p-variation and define a "mixed Hölder-variation semi-norm" by

$$\|X\|_{\tilde{N}^{\delta,p};[0,T]} := \left(\sup_{\mathcal{P}\subset[0,T]} \sum_{[u,v]\in\mathcal{P}} \frac{\|X\|^p_{\frac{1}{\delta}\text{-var};[u,v]}}{|v-u|^{\delta p -1}} \right)^{\frac{1}{p}}, \quad \delta \in (0,1],\, p \in (1,\infty),$$

and the corresponding space is denoted by $\tilde{N}^{\delta,p}([0,T]; G^{\lfloor 1/\delta \rfloor}(\mathbb{R}^n))$. This new type of measuring regularity turns out to be an equivalent characterization of classical Nikolskii spaces $N^{\delta,p}([0,T]; G^{\lfloor 1/\delta \rfloor}(\mathbb{R}^n))$ if $\delta \in (0,1]$ and $p \in (1,\infty)$ with $\delta > 1/p$. Notice that the Nikolskii space $N^{\delta,p}([0,T]; G^{\lfloor 1/\delta \rfloor}(\mathbb{R}^n))$ corresponds to the Besov space $B^\delta_{p,\infty}([0,T]; G^{\lfloor 1/\delta \rfloor}(\mathbb{R}^n))$. Furthermore, in contract to the classical Nikolskii regularity, it provides an exact interpolation between the Hölder space $C^\delta([0,T]; G^{\lfloor 1/\delta \rfloor}(\mathbb{R}^n))$ and the p-variation space $\mathcal{V}^{1/\delta}([0,T]; G^{\lfloor 1/\delta \rfloor}(\mathbb{R}^n))$ as we see that

$$\begin{aligned} C^\delta([0,T]; G^{\lfloor 1/\delta \rfloor}(\mathbb{R}^n)) &= \tilde{N}^{\delta,\infty}([0,T]; G^{\lfloor 1/\delta \rfloor}(\mathbb{R}^n)) \\ &\subset \tilde{N}^{\delta,p}([0,T]; G^{\lfloor 1/\delta \rfloor}(\mathbb{R}^n)) \\ &\subset \tilde{N}^{\delta,1/\delta}([0,T]; G^{\lfloor 1/\delta \rfloor}(\mathbb{R}^n)) = \mathcal{V}^{1/\delta}([0,T]; G^{\lfloor 1/\delta \rfloor}(\mathbb{R}^n)) \end{aligned}$$

for $p \in (1/\delta, \infty)$. Based on this new characterization of Nikolskii spaces and Lyons' sophisticated estimates, we deduce the locally Lipschitz continuity of the Itô map in Nikolskii topology.

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Rough differential equations with unbounded drift

Sebastian Riedel

We consider a rough differential equation of the form

$$\begin{aligned} dy &= b(y)\,dt + \sigma(y)\,d\mathbf{x}_t; \quad t \in [0,T] \\ y_0 &= \xi \in \mathbb{R}^m \end{aligned} \tag{1}$$

where $\mathbf{x}$ is a generic p-rough path (in the sense of T. Lyons), σ is the diffusion coefficient and b is the drift term. In this talk, σ will be smooth and bounded with bounded derivatives. Using the flow decomposition, we can show that (1) induces a flow provided b is locally Lipschitz continuous and has at most linear growth. Assuming only that b satisfies the classical one-sided growth condition

$$\langle b(v), v \rangle \leq C_1(1 + |v|^2) \quad \text{for all } v \in \mathbb{R}^m \tag{2}$$

and the additional condition

$$\left| b(v) - \frac{\langle b(v), v\rangle v}{|v|^2} \right| \leq C_2(1 + |v|) \quad \text{for all } v \in \mathbb{R}^m \setminus \{0\} \tag{3}$$

which bounds the growth of b at v in the orthogonal directions of v, we can show that (1) still induces a semiflow. We further present an example by [Cox, Hutzenthaler, Jentzen; arXiv 2013] which shows that assuming only (2) is in general not enough to guarantee non-explosion for equations of the form (1).

SPDEs with Three Types of Multiplicative Noises

Hao Shen
(joint work with Ajay Chandra and Martin Hairer)

One of the important objects in the study of rough path theory is the stochastic ODE: $dX = G(X)dB$ where B is a driven signal such as a Brownian motion and G is a smooth enough function. In this talk we consider the stochastic PDE analogue of this problem

$$\partial_t u = \Delta u + G(u)\zeta \tag{1}$$

We consider three types of noises ζ

- Space-time white noise
- Mixing non-Gaussian noise
- Gaussian multiplicative chaos

The study of all these cases relies on the theory of regularity structures [1].

When $\zeta = \xi$ is space-time white noise, Eq.(1) has the notion of Itô solution. [2] proved the following interesting result. Let $\xi_\varepsilon = \xi * \varphi_\varepsilon$ be a molification of ξ. Then the solution u_ε to

$$\partial_t u_\varepsilon = \partial_x^2 u_\varepsilon - C_\varepsilon G'(u_\varepsilon)G(u_\varepsilon) - c_1 G'(u_\varepsilon)^3 G(u_\varepsilon) - c_2 G''(u_\varepsilon)G'(u_\varepsilon)G^2(u_\varepsilon) + G(u_\varepsilon)\,\xi_\varepsilon$$

converges to the Itô solution of Eq.(1), where $C_\varepsilon \sim \varepsilon^{-1}$ and c_1, c_2 are finite constants.

Assume now that ζ is a random field with suitable regularity, integrability and mixing condition, that is not necessarily Gaussian distributed. The rescaled field $\zeta_\varepsilon = \varepsilon^{-\frac{3}{2}}\zeta(\varepsilon^{-2}t, \varepsilon^{-1}x)$ converges to ξ by central limit theorem. With A.Chandra [3] we showed that in order to obtain Itô solution as $\varepsilon \to 0$, the above equation with ξ_ε replaced by ζ_ε need the following extra correction terms

$$-O(\varepsilon^{-\frac{1}{2}})G'(u_\varepsilon)^2 G(u_\varepsilon) - O(\varepsilon^{-\frac{1}{2}})G''(u_\varepsilon)G^2(u_\varepsilon) - c_3 G'''(u)G^3(u)$$

and modifications to values of c_1, c_2 are also necessary. The new constants arise from the higher cumulants of ζ. Our method follows (and generalizes) that used in [4] for the KPZ equation.

Finally, we consider the case that $\zeta = \Psi$ - a version of Gaussian multiplicative chaos (GMC). More precisely, let Φ_ε be the stationary solution to

$$\partial_t \Phi_\varepsilon = \frac{1}{2}\Delta\Phi_\varepsilon + \xi_\varepsilon$$

where ξ_ε is a mollified space-time white noise. Then Hairer and myself showed that as $\varepsilon \to 0$, the process $c_\varepsilon e^{i\beta\Phi_\varepsilon}$ converges to a limiting process called Ψ in the space $\mathcal{C}^{-\frac{\beta^2}{4\pi}}$ where $c_\varepsilon \sim \varepsilon^{-\frac{\beta^2}{4\pi}}$, which is a version of GMC. The motivation is to study the sine-Gordon equation [5]

$$\partial w_\varepsilon = \frac{1}{2}\Delta w_\varepsilon + c_\varepsilon \sin(\beta w_\varepsilon) + \xi_\varepsilon$$

Indeed, letting $w_\varepsilon = \Phi_\varepsilon + u_\varepsilon$, then u_ε satisfies an equation of the form (1) driven by noise $c_\varepsilon e^{i\beta\Phi_\varepsilon}$. The larger β is, the more singular the noise Ψ is; so β plays the role of Hurst number in the study of the SDEs driven by fractional Brownian motions. Hopefully this similarity would be of interest for both people working on regularity structures and rough paths.

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Stochastic Analysis with Modelled Distributions

Josef Teichmann

(joint work with David J. Prömel)

We introduce Sobolev-Slobodeckij type norms on spaces of modelled distributions in the framework of Martin Hairer's regularity structures. We show directly that on these spaces reconstruction is still possible and we prove furthermore that those spaces are of martingale type 2 or UMD, which guarantees a rich stochastic integration theory for stochastic processes with values therein.

There are two competing notions of integration along trajectories, which do not satisfy the finite variation paradigm: stochastic integration theory and rough paths theory (or its generalization, the theory of regularity structures). Stochastic integration theory relies on orthogonality of martingale increments and works for the large set of all bounded predictable integrands. Stochastic Integration is probabilistic in spirit, i.e. limits are in probability and to be understood almost surely. By its very nature the stochastic integral is tailormade to express properly gains and loss processes in Finance. Rough paths theory on the other hand is a pathwise way to construct integration along rough trajetories in terms of local expansions by iterated integrals up to appropriate orders of accuracy. The integral is not limited to predictable integrands but subtle analytic properties have to be satisfied (leading to the notion of controlled rough paths). From several geometric, analytic, or numerical viewpoints the rough paths approach with its continuity properties provides valuable and fruitful insights.

In this talk we try to combine both approaches, i.e. the goal is to consider SPDEs with time and spatial variables, where we allow for stochastic integration with respect to time and where we apply the theory of regularity structures with respect to space variables. In more technical terms: we aim to consider stochastic processes with values in modelled distributions.

Modelled distributions are the spine of the theory of regularity structures [Hai14]: they constitute a way to describe, by means of functions taking values in a graded vector space which satisfy certain graded estimates, generalized functions of certain degrees of (ir-)regularity (which is hard-coded in the given graded vector space structure). So far modelled distributions come with Hölder type norms, which is most natural from the point of view of the reconstruction theorem (see Theorem 3.10 in [Hai14]). However, with stochastic analysis in mind, Sobolev-Slobodeckij type norms are a more natural choice. It is the goal of this article to show that reconstruction still works by a direct proof, which, of course, mimics Martin Hairer's original proof on the existence of the reconstruction operator.

The reconstruction operator $\mathcal{R}$ maps modelled distributions to generalized functions in a linear and bounded way with additional continuous dependence on the underlying model. The reconstruction operator can be considered as an abstract integration operation, which, depending on the particular regularity structure. It generalizes Young integration [You36], or controlled rough paths [Lyo98, Gub04], etc. The main result of this article is a Fubini theorem, which asserts for bounded modelled distribution valued predictable processes H and Brownian motion W that the order of "integration" can be interchanged

$$\Big\langle \mathcal{R}\big((H \bullet W)\big), \psi \Big\rangle = \Big(\langle \mathcal{R}(H), \psi \rangle \bullet W \Big)$$

for every test function ψ. This Fubini theorem only makes sense if the space of modelled distributions $\mathcal{D}^\gamma_p$ has, e.g., some martingale type 2 structure, such that a rich stochastic integration theory is at hand. There are many approaches to stochastic integration for Banach space valued processes, some of them involve properties of the Banach space like martingale type 2 or UMD. It depends on the purpose in mind, which property is actually needed, but for integrals with respect to Brownian motion martingale type 2 or UMD is favorable, see, e.g. [Brz95]. We shall prove here that the space of modelled distributions $\mathcal{D}^\gamma_p$ (for $p \geq 2$) are of martingale type 2 or UMD, which suffices to define a rich stochastic integration theory as needed for the treatment of stochastic differential equations with Brownian drivers like in the books of DaPrato-Peszat-Zabczyk [PZ07, DZ14].

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Parabolic Anderson model with rough dependence in space

Samy Tindel
(joint work with X. Chen, Y. Hu and D. Nualart)

This research proposal is concerned with the following stochastic heat equations on $\mathrm{R}_+ \times \mathrm{R}$, formally written as:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + u\,\dot{W}\,, \quad t \geq 0, \quad x \in \mathbb{R}\,, \tag{1}$$

In equation (1), $\dot{W}$ is a noise which is white in time and colored in space, and we are interested in regimes where the spatial behavior of $\dot{W}$ is rougher than white noise. More specifically, our noise can be seen as the formal space-time derivative of a centered Gaussian process whose covariance is given by:

$$\mathbb{E}\left[W(s,x)W(t,y)\right] = \frac{1}{2}\left(|x|^{2H} + |y|^{2H} - |x-y|^{2H}\right)(s \wedge t), \tag{2}$$

where $\frac{1}{4} < H < \frac{1}{2}$. That is, W is a standard Brownian motion in time and a fractional Brownian motion with Hurst parameter H in the space variable. Notice that the spatial covariance of $\dot{W}$, which is formally equal to $H(2H-1)|x-y|^{2H-2}$, is not locally integrable when $H < \frac{1}{2}$. Therefore the stochastic integration with respect to W cannot be handled by classical theories (see e.g. [5, 7]). However, we have recently been able (cf. [6]) to give a proper definition of equation (1) and to solve it in a space of Hölder continuous processes (see also the recent work [1], covering the linear case (1)). We shall take those results for granted.

Let us now highlight the fact that space-time asymptotics for stochastic heat equations like (1) have attracted a lot of attention in the recent past. This line of research stems from different motivations, and among them let us quote the following: For a fixed $t > 0$, the large scale behavior of the function $x \mapsto u(t,x)$ is dramatically influenced by the presence of the noise $\dot{W}$ in (1) (as opposed to a deterministic equation with no noise). One way to quantify this assertion is to analyze the asymptotic behavior of $x \mapsto u(t,x)$ as $|x| \to \infty$. Results in this sense include intermittency results, upper and lower bounds for $M_R \equiv \sup_{|x|\leq R} u(t,x)$ contained in [3], and culminate in the sharp results obtained in [2]. Roughly speaking, in case of a white noise in time like in (2), those articles establish that $\ln(M_R)$ behaves like $[\ln(R)]^{\psi}$, for an exponent ψ which depends on the spatial covariance structure of $\dot{W}$. In particular if the spatial covariance of $\dot{W}$ is described by the Riesz kernel $|x|^{-\alpha}$ for $\alpha \in (0,1)$, one gets $\psi = \frac{2}{4-\alpha}$. This interpolates between a regular situation in space ($\alpha = 0$ and $\psi = 1/2$) and the KPZ or white noise setting ($\alpha = 1$ and $\psi = 2/3$). In any case those results are in sharp contrast with the deterministic case, for which $x \mapsto u(t,x)$ stays bounded.

With these preliminaries in mind, the current contribution completes the space-time asymptotics picture for the stochastic heat equation, covering very rough situations like the ones described by (2). Namely, we get the following spatial asymptotics:

Theorem 1. *Let $\dot{W}$ be the noise given by the covariance* (2). *Let u be the unique solution to equation* (1) *driven by $\dot{W}$ with initial condition $u_0 = 1$, and consider $t > 0$. Then:*

$$\lim_{R\to\infty} (\log R)^{-\frac{1}{1+H}} \log\left(\max_{|x|\le R} u(t,x)\right) = \mathcal{E}_H \qquad a.s., \tag{3}$$

where $\mathcal{E}_H$ is a variational constant which can be described precisely.

Notice that the exponent ψ ranges in the interval $(2/3, 4/5)$ when $H \in (1/4, 1/2)$, indicating a possible superdiffusive behavior of the corresponding directed polymer.

Let us say a few words about our strategy in order to prove Theorem 1. It can be roughly be divided in two main steps:

(i) Tail estimate for $u(t,x)$. Let us fix $t \in \mathrm{R}_+$ and $x \in \mathrm{R}$. Our first main objective is to prove that for large a, we have

$$\mathbb{P}\left(\log(u(t,x)) \ge a\right) \asymp \exp\left(-\frac{\hat{c}_{H,t}\, a^{1+H}}{t^H}\right), \tag{4}$$

where $\hat{c}_{H,t}$ is determined by a variational problem. This stems, via some large deviation arguments, from a sharp analysis of the high moments of $u(t,x)$. Namely, our main effort in order to get the tail behavior is to prove that for large $m \in \mathbb{N}$, we have:

$$\mathbb{E}\left[(u(t,x))^m\right] \simeq \exp\left(c_H t m^{1+\frac{1}{H}}\right), \tag{5}$$

with a variational expression for c_H. Towards this aim, we resort to a Feynman-Kac representation for the moments of $u(t,x)$, which involves a kind of intersection local time for a m-dimensional Brownian motion weighted by a singular potential. We are thus able to relate the quantity $\mathbb{E}[(u(t,x))^m]$ to a semi-group on $L^2(\mathrm{R}^m)$, and this semi-group admits a generator A_m which can be expressed as the Laplace operator on R^m perturbed by a singular distributional potential. Then we shall get our asymptotic result (5) thanks to a careful spectral analysis of A_m.

(ii) Spatial behavior. Once the tail of $\log(u(t,x))$ has been sharply estimated, we can complete the study of the asymptotic behavior in the following way: on the interval $[-M, M]$ for large M, we are able to produce some random variables $X_1, \ldots, X_{\mathcal{N}}$ such that:

- $\mathcal{N}$ is of order $2M$.
- $X_1, \ldots, X_{\mathcal{N}}$ are i.i.d, and satisfy approximately (4).
- $X_1, \ldots, X_{\mathcal{N}}$ are approximations of $u(t, x_1), \ldots u(t, x_{\mathcal{N}})$ with some elements $x_1, \ldots, x_{\mathcal{N}}$ of $[-M, M]$.
- Fluctuations of u in small boxes around $x_1, \ldots, x_{\mathcal{N}}$ are small.

With this information in hand, the behavior $\ln(R)^{\frac{1}{1+H}}$ in Theorem 1 can be heuristically understood as follows: for an additional parameter λ, we have

$$\mathbb{P}\left(\max_{j\leq\mathcal{N}}\log(X_j)\leq\lambda[\ln(R)]^{\frac{1}{1+H}}\right)=\left[1-\mathbb{P}\left(\log(X_j)\geq\lambda[\ln(R)]^{\frac{1}{1+H}}\right)\right]^{\mathcal{N}},$$

and thanks to the tail estimate (4), we obtain:

$$\mathbb{P}\left(\max_{j\leq\mathcal{N}}\log(X_j)\leq\lambda[\log(R)]^{\frac{1}{1+H}}\right)\simeq\left[1-\exp\left(-\hat{c}_{H,t}\lambda^{1+H}\log(R)\right)\right]^{\mathcal{N}}.$$

With some elementary calculus considerations, and playing with the extra parameter λ, one can now easily check that for large enough R:

$$\mathbb{P}\left(\max_{j\leq\mathcal{N}}\log(X_j)\leq\lambda[\log(R)]^{\frac{1}{1+H}}\right)\leq\exp(-R^{\nu}),$$

with a positive ν. Otherwise stated, we obtain an exponentially small probability of having $\log(X_j)$ of order less than $[\log(R)]^{\frac{1}{1+H}}$. Using a Borel-Cantelli type argument and the fact that fluctuations of u in small boxes around $x_1,\ldots,x_{\mathcal{N}}$ are small, we thus prove Theorem 1.

As already mentioned, the spatial covariance γ of the noise $\dot{W}$ driving equation (1) is a non positive distribution. With respect to smoother cases such as the ones treated in [2], this induces some serious additional difficulties which can be summarized as follows. First, the variational asymptotic results involving the generator A_m cannot be reduced to a one-dimensional situation due to the singularities of γ. We thus have to handle a family of optimization problems in $L^2(\mathrm{R}^m)$ for arbitrarily large m. Then, still in the part concerning the asymptotic behavior of $m\mapsto\mathbb{E}[u(t,x)]$, the upper bound obtained in [2] relied heavily on a compactfication by folding argument for which the positivity of γ was essential. This approach is no longer applicable here, and we have to replace it by a coarse graining procedure. Finally, the localization procedure and the study of fluctuations in the spatial behavior step of our proof, though similar in spirit to the one in Conus et al. [4], is more involved in its implementation. More specifically, in our case the moment estimates cannot be obtained by using sharp Burkholder inequalities, because of the roughness of the noise. For this reason we use Wiener chaos expansions and hypercontractivity, which are more suitable methods in our context. The fluctuation estimates alluded to above are also obtained through chaos expansions.

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On a non-zero sum stochastic differential game

Nizar Touzi

We consider a general formulation of the Principal-Agent problem from Contract Theory, on a finite horizon. We show how to reduce this non-zero sum Stackelberg stochastic differential game to a stochastic control problem which may be analyzed by the standard tools of control theory. In particular, Agent's value function appears naturally as a controlled state variable for the Principal's problem. Our argument relies on the Backward Stochastic Differential Equations approach to non-Markovian stochastic control, and more specifically, on the most recent extensions to the second order case.

On the Eyring–Kramers law for renormalised SPDEs

Hendrik Weber
(joint work with Nils Berglund and Giacomo Di Gesù)

The derivation of precise asymptotics for the expected transition times of a gradient diffusion driven by a small noise term

$$\mathrm{d}x(t) = -\nabla V(x(t))\mathrm{d}t + \sqrt{2\varepsilon}\mathrm{d}w(t) \tag{1}$$

is a classical problem. It is by now well-known [7, 5, 6] that if τ denotes the first hitting time of a neighbourhood of a local minimiser y of V for a diffusion x started in another local minimiser x, then (under suitable assumptions on V) the Kramers-Eyring law

$$\mathbb{E}[\tau] = \frac{2\pi}{|\lambda_0(z)|}\sqrt{\frac{|\det D^2V(z)|}{\det D^2V(x)}}e^{[V(z)-V(x)]/\varepsilon}[1+O(\varepsilon)]\ , \tag{2}$$

holds. Here z denotes the (by assumption) unique saddle connecting x and y and $\lambda_0(z)$ is the (by assumption) unique negative eigenvalue of $D^2V(z)$.

In this work we aimed to extend this result to an infinite dimensional system. The Allen-Cahn equation

$$\partial_t\phi(t,x) = \Delta\phi(t,x) - (\phi(t,x)^3 - \phi(t,x)) + \sqrt{2\varepsilon}\xi(t,x)\ , \tag{3}$$

driven by space time-white noise ξ constitutes a natural example. In the case where the spatial variable x runs over a one-dimensional interval this equation determines a reversible diffusion process and a Kramers-Eyring law similar to (2) has been previously established to govern transitions between the stable configurations $\phi_\pm =$

± 1 [2, 1, 4]. In these results the Hessian D^2V has to be replaced by the linearisation of the potential

$$V(\phi) = \int \Big(\frac{1}{2}|\partial_x \phi|^2 - \frac{1}{2}\phi^2 + \frac{1}{4}\phi^4\Big) dx \tag{4}$$

around the minimiser ϕ_- and the saddle $\phi_0 = 0$, i.e. the operators $-\partial_x^2 + 2$ and $-\partial_x^2 - 1$. In the case where x runs over a two-dimensional domain two problems appear: On the one hand, for spatial dimension ≥ 2 equation (3) is not well-posed as it stands. In order to construct a well-defined and non-trivial solution to (3), the equation has to be renormalised by adding an "infinite constant" into the potential. On the level of an approximating Galerkin scheme this reads

$$\partial_t \phi_N = \Delta \phi_N - (P_N \phi_N^3 - 3\varepsilon C_N \phi_N - \phi_N) + \sqrt{2\varepsilon}\xi_N, \tag{5}$$

where C_N diverges as $N \to \infty$. On the other hand, the ratio of determinants of operators for $-\Delta + 2$ and $-\Delta - 1$ does not converge for $d \geq 2$. Our main result shows that the renormalisation procedure (i.e. the introduction of the infinite constant C_N) also corrects for this divergence. We consider the approximations (5) over a two-dimensional torus $\mathrm{R}^2/L\mathbb{Z}^2$ of size $L < 2\pi$ and show that the Kramers-Eyring formula

$$\mathbb{E}[\tau_B] \leq \frac{2\pi}{|\lambda_0|}\sqrt{\prod_k \frac{|\lambda_k|}{\nu_k} \exp\Big(\frac{\nu_k - \lambda_k}{|\lambda_k|}\Big)} e^{[V(\phi_0) - V(\phi_-)]/\varepsilon}\big(1 + o(\varepsilon)\big) \tag{6}$$

for transitions between suitable neighbourhoods A, B of $\varphi_\pm$ holds uniformly in the discretisation parameter N. Here ν_k and λ_k denote the eigenvalues of $(-\Delta + 2)$ and $(-\Delta - 1)$. The extra factor $\exp\Big(\frac{\nu_k - \lambda_k}{|\lambda_k|}\Big)$ is caused by the renormalisation and precisely ensures that the product converges as $N \to \infty$.

This talk was based on joint work [3] with N. Berglund and G. Di Gesù.

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Regularity structures and the ϕ^4 equation

Hendrik Weber

This was an introductory talk summarising some aspects of the recent developments in the theory of singular stochastic PDE. This was set in the context of the dynamic φ^4_3 model which is given by the stochastic PDE

$$\partial_t \varphi = \Delta \varphi - \varphi^3 + \xi, \tag{1}$$

where ξ denotes a space-time white noise over $\mathrm{R} \times \mathrm{R}^3$. At least formally (1) describes a Markov process which is reversible with respect to the Euclidean ϕ^4_3 quantum field theory. The definition of solutions to (1) and a short time existence and uniqueness theory were one of the first applications of Hairer's celebrated theory of regularity strucures [3]. Similar results were then obtained using the notion of paracontrolled distributions [2, 1] as well as ideas from renormalisation group theory [4]. Global-in-time solutions over the three-dimensional torus were constructed in [5]

In this talk some of the aspects involved in the derivation of these results were reviewed. It was shown in particular how manipulations on graphs can be used to efficiently perform stochastic moment estimates involved in the construction of solutions. Also the paracontrolled ansatz was explained an compared to Hairer's regularity structures.

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Large scale behaviour of phase coexistence models

Weijun Xu

(joint work with Martin Hairer and Hao Shen)

Consider microscopic phase coexistence models of the type

$$\partial_t u = \Delta u - \epsilon V'_\theta(u) + \eta \tag{1}$$

in three spatial dimensions. Here, η is a space-time Gaussian random field with smooth covariance and correlation length 1, and the potential $(\theta, u) \mapsto V_\theta(u)$ is a polynomial whose coefficients depend smoothly on the parameter θ. Let Ψ be the stationary solution to the linearised equation

$$\partial_t \Psi = \Delta \Psi + \eta,$$

and μ denote the stationary measure of Ψ. We further define the averaged potential $\langle V_\theta \rangle$ by

$$\langle V_\theta \rangle(x) := \int_{\mathrm{R}} V(x+y)\mu(dy).$$

In the work [3], we obtained the following theorem.

Theorem 1. *Suppose the average potential $\langle V_\theta \rangle$ has a pitchfork bifurcation at the original, and let P denote the heat kernel. Let A be the quantity given by*

$$A = \int P(z)\mathcal{E}\big(V_0'(\Psi_0)V_0''(\Psi_z)\big)dz,$$

where the integration is taken over the space-time domain $\mathrm{R} \times \mathrm{R}^3$. Let u be the microscopic process given by (1). If $A = 0$, then $\exists c > 0$ such that for $\theta = -c\epsilon|\log \epsilon| + \mathcal{O}(\epsilon)$, the rescaled process

$$u_\epsilon(t,x) := \epsilon^{-\frac{1}{2}} u(t/\epsilon^2, x/\epsilon)$$

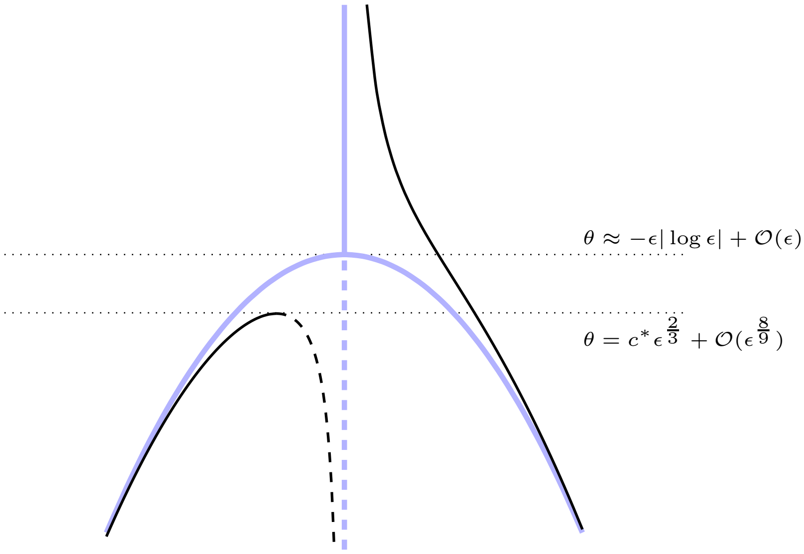
converges to the solution of Φ^4_3 equation. On the other hand, if $A \neq 0$, then $\exists h_\epsilon$ and $\theta = c'\epsilon^{\frac{2}{3}} + \mathcal{O}(\epsilon^{\frac{8}{9}})$ such that the process

$$u_\epsilon(t,x) := \epsilon^{-\frac{5}{12}} \big(u(t/\epsilon^{\frac{5}{3}}, x/\epsilon^{\frac{5}{6}}) - h_\epsilon\big)$$

converges to the solution of Φ^3_3 equation.

The proof is based on the theory of regularity structures [1] as well as the recent work on universality of KPZ equation [2]. In the subsequent work [4], we extend the symmetric case ($A = 0$) to non-Gaussian noise η with suitable integrability and mixing conditions (also symmetric noise). We expect to observe Φ^3_3 in the generic non-symmetric case.

The above result could be illustrated by the following figure. The light shaded curve corresponds to the symmetric case ($A = 0$), and the black curve represents the generic case ($A \neq 0$) when $\langle V \rangle$ undergoes a pitchfork bifurcation. Here, the field Φ is represented on the horizontal axis and the bifurcation parameter θ on the vertical axis (with positive direction pointing downwards). We can see that the asymmetry in the potential separates one local minimum from two other critical points in the pitchfork bifurcation, and forms a saddle point.



The intuitive explanation of why this is so is that $\langle V\rangle$ is really only a 0-th order approximation to the "real" effective potential felt by the system at large scales. Since pitchfork bifurcations are structurally unstable, small generic perturbations tend to turn them into a saddle-node bifurcation taking place very close to a local minimum.

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The theory of rough paths via one-forms

Danyu Yang
(joint work with Terry Lyons)

We propose a one-form approach [10] to rough paths theory, and reduce the rough integration to an inhomogeneous analogue of the classical Young integral [14]. The approach is simple and can be used as an efficient tool to tackle problems in rough paths theory. In [11], we provide an overview of the approach, and provide a simple proof of the existence, uniqueness and stability of the solution to rough differential equations.

The key idea is about exact one-forms. The integration of an exact one-form along any continuous path is well-defined, and simply gives back the difference of the values of the function at the end points of the path. For example, a constant one-form c on a vector space is an exact one-form, and $\int_0^1 c dy = c(y_1 - y_0)$ for any continuous path y on $[0, 1]$. Then for a given continuous path y on $[0, 1]$, it is possible to vary the constant one-form slowly with time to incorporate a large family of integrable one-forms. The integral makes sense when the one-form and the path satisfy a compensated regularity condition. For example, when the time-varying constant one-form and the continuous path satisfy Young's condition [14], the integral is well-defined as Young integral. Rough paths can be viewed as paths taking values in a group. By lifting polynomial one-forms to exact one-forms on the group, we interpret rough integration as an analogue of Young integration: the integration of a slowly-varying exact one-form against a continuous path.

In Young integration, when x is α-Hölder and y is β-Hölder for $\alpha+\beta>1$, the integral $\int xdy$ is well-defined as the limit of Riemann sums. The idea of Young integral is that

$$x_s\left(y_t-y_s\right)=x_s\left(y_u-y_s\right)+x_u\left(y_t-y_u\right)+O\left(\left|t-s\right|^{\alpha+\beta}\right)\text{ for }s<u<t.$$

Since $\alpha+\beta>1$, we can keep on inserting partition points, and get a consistent integral in the limit. The following example shares the same spirit with our group-valued integration. Suppose x is a continuous path taking values in a differentiable manifold, and α is a continuous path taking values in exact one-forms on the differentiable manifold i.e. for any time t, α_t is an exact one-form. If

$$\int_{r\in[s,t]}\alpha_s dx_r\approx\int_{r\in[s,u]}\alpha_s dx_r+\int_{r\in[u,t]}\alpha_u dx_r\text{ for }s<u<t,$$

then as in Young integration, we can keep on inserting partition points, and the integral exists:

$$\int_{r\in[0,1]}\alpha_r dx_r:=\lim_{|D|\to 0}\sum_{k,t_k\in D}\int_{r\in[t_k,t_{k+1}]}\alpha_{t_k}dx_r.$$

Suppose x is a continuous path with finite length. The lifting g of x is a path on $[0,T]$ given by

$$g_t=1+\sum_{l=1}^{n}x_t^l\text{ with }x_t^l=\int_{0<u_1<\cdots<u_l<t}dx_{u_1}\cdots dx_{u_l}.$$

Based on Chen [1], g takes values in the free nilpotent Lie group. Suppose p is a polynomial one-form i.e. a polynomial taking values in continuous linear mappings. Then it can be computed that

$$\int_{r\in[s,t]}p\left(x_r\right)dx_r=\sum_{k=1}^{n}\left(D^{k-1}p\right)\left(x_s\right)x_{s,t}^{k+1}:=P\left(g_s,g_{s,t}\right).$$

where $g_{s,t}=g_s^{-1}g_t$ based on Chen's identity. P is well-defined for any continuous path g, and the value of $P\left(g_s,g_{s,t}\right)$ only depends on the end points of $g|_{[s,t]}$. In fact P is the exact one-form induced by the polynomial function on the group

$$a\mapsto P\left(1,a\right),$$

where 1 denotes the identity in the group. As a consequence, we can lift a polynomial one-form, which is by no-means exact in the classical setting, to an exact one-form on the free nilpotent Lie group. The integration of an exact one-form along any continuous path is well-defined. For a given continuous path, it is possible to vary an exact one-form slowly with time to incorporate a large family of integrable one-forms.

Based on Stein [13] and Hairer [6], a Lipschitz one-form is a function taking values in polynomial one-forms, with the Lipschitz degree describing the varying speed of polynomials. The higher the Lipschitz degree, the slower the varying speed of polynomials. Since we can lift polynomial one-forms to exact one-forms, we can lift a Lipschitz one-form to a function taking values in exact one-forms. When the Lipschitz degree is high, we can lift a Lipschitz one-form to a slowly-varying exact one-form. Suppose α is a Lipschitz one-form with lifting β and x is a continuous path of finite length with lifting g. Then we have the equality holds

$$\int_{r\in[0,1]} \alpha\left(x_r\right) dx_r = \int_{r\in[0,1]} \beta\left(g_r\right) dg_r .$$

The equality holds because we just rewrite the integral of p against x as the integral of P against g, and the equality holds based on the comparison of local expansions. The point of this rewriting is that, when x is not regular enough, the left hand does not have a proper meaning but the right hand side has a meaning. The right hand side is actually the rough integration.

Consider the exact one-forms derived from functions taking values in another group. We call an exact one-form a cocyclic one-form when it is derived from a polynomial function that takes values in another group. More specifically, we say β is a cocyclic one-form on group G if

$$\beta\left(a, bc\right) = \beta\left(a, b\right) \beta\left(ab, c\right), \forall a, b, c \in G.$$

Intuitively, if we start from point a and go in the direction of bc, it is equivalent that we start from a go in the direction of b, and then we start from ab and go in the direction of c. The polynomial function that induces β is given by

$$a \mapsto \beta\left(1, a\right).$$

For a given group-valued path, we can vary an exact one-form slowly with time, and the integral still makes sense. Suppose g is a continuous path taking values in the group G, and β is a slowly-varying cocyclic one-form. Suppose the generalized Young condition holds:

$$\beta_s\left(g_s, g_{s,t}\right) \approx \beta_s\left(g_s, g_{s,u}\right) \beta_u\left(g_u, g_{u,t}\right) \text{ for } s < u < t.$$

Then the integral exists as the limit

$$\int_0^1 \beta_u\left(g_u\right) dg_u = \lim_{|D|\to 0} \beta_{t_0}\left(g_{t_0}, g_{t_0,t_1}\right) \cdots \beta\left(g_{t_{n-1}}, g_{t_{n-1},t_n}\right).$$

In particular, this integration generalizes the rough integration.

By viewing Lipschitz functions as slowly-varying polynomial functions and by lifting polynomial one-forms to exact one-forms, we encapsulate the nonlinearity of the integral to the structure of the group and to the exact one-forms on the group so that the idea behind the generalized integral is clearer and bears a similar form to the classical Young integral.

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On the renormalisation group in regularity structures

Lorenzo Zambotti

(joint work with Yvain Bruned and Martin Hairer)

I want to present a general construction of the renormalisation group in regularity structures based on Hopf algebras of labelled rooted forests. This construction allows to unify the renormalisation group and the structure group giving further insight in the algebraic properties of regularity structures. This is based on joint work with Yvain Bruned and Martin Hairer.

In four celebrated papers (1954, 1957, 1958, 1971) Kuo-Tsai Chen discovered that the family of iterated integrals of a smooth path in $\mathbb{R}^d$ has a number of algebraic properties.

Let $s \le t$ and $X : [s,t] \to \mathbb{R}^d$ a smooth path. Set

$$\mathbf{X}_{st}(\emptyset) := 1,$$

$$\mathbf{X}_{st}(i_1,\ldots,i_n) := \int_s^t \dot{X}^{i_1}_{r_1}\partial r_1 \int_s^{r_1} \dot{X}^{i_2}_{r_2}\partial r_2 \cdots \int_s^{r_{n-1}} \dot{X}^{i_n}_{r_n}\partial r_n,$$

with $n \in \mathbb{N}$, $i_k \in \{1,\ldots,d\}$.

Let $V := \mathrm{Span}\{(i_1, \dots, i_n), n \geq 0\}$ and V^* its dual. Then $\mathbf{X}$ is a function $(s,t) \mapsto \mathbf{X}_{st} \in V^*$: $\langle \mathbf{X}_{st}, \tau \rangle := \mathbf{X}_{st}(\tau)$.

If $u \in [s,t]$ and $X_{[s,t]} := (X_r, r \in [s,t])$ we write

$$X_{[s,t]} = X_{[s,u]} * X_{[u,t]}, \tag{1}$$

the concatenation of $X_{[s,u]}$ and $X_{[u,t]}$. We have

$$\begin{aligned}
&\mathbf{X}_{st}(i_1, \dots, i_n) = \\
&= \sum_{k=0}^{n} \int_s^t \dot{X}_{r_1}^{i_1} \partial r_1 \int_s^{r_1} \dot{X}_{r_2}^{i_2} \partial r_2 \cdots \int_s^{r_{n-1}} \dot{X}_{r_n}^{i_n} \partial r_n \; \mathbb{1}_{(r_{k+1} \leq u < r_k)} \\
&= \sum_{k=0}^{n} \mathbf{X}_{ut}(i_1, \dots, i_k)\, \mathbf{X}_{su}(i_{k+1}, \dots, i_n),
\end{aligned}$$

$$\text{i.e.} \qquad \langle \mathbf{X}_{st}, \tau \rangle = \langle \mathbf{X}_{su} \otimes \mathbf{X}_{ut}, \Delta\tau \rangle, \qquad \forall \tau \in V$$

where $\Delta : V \to V \otimes V$ is the deconcatenation coproduct

$$\Delta(i_1, \dots, i_n) := \sum_{k=0}^{n} (i_{k+1}, \dots, i_n) \otimes (i_1, \dots, i_k).$$

By duality, Δ defines a product on V^*:

$$\langle A \star B, \tau \rangle := \langle A \otimes B, \Delta\tau \rangle.$$

With this notation, $\mathbf{X}_{st} = \mathbf{X}_{su} \star \mathbf{X}_{ut}$, which is the analog of (1).

On V we have a product $\sqcup\!\sqcup$ (shuffle) and a coproduct Δ, which satisfy suitable properties: V is endowed with the structure of a bialgebra.

$\mathbf{X}$ is a V^*-valued function with the following properties:

- $\langle \mathbf{X}_{st}, \tau \rangle = \langle \mathbf{X}_{su} \otimes \mathbf{X}_{ut}, \Delta\tau \rangle, \quad \forall \tau \in V,\ s \leq u \leq t.$
- $\langle \mathbf{X}_{st}, \tau_1 \sqcup\!\sqcup \tau_2 \rangle = \langle \mathbf{X}_{st}, \tau_1 \rangle \langle \mathbf{X}_{st}, \tau_2 \rangle.$

Terry Lyons defines a (weak) geometric rough path of regularity $\gamma > 0$ as a V^*-valued function $\mathbf{X}$ satisfying the above properties plus

- $\sup_{s \neq t} [|\langle \mathbf{X}_{st}, (i_1, \dots, i_n) \rangle| / |t-s|^{n\gamma}] < +\infty$, for all $(i_1, \dots, i_n) \in V$.

(Notations from [Hairer-Kelly 2013]). Smooth paths are dense.

By the property $\langle \mathbf{X}_{st}, \tau_1 \sqcup\!\sqcup \tau_2 \rangle = \langle \mathbf{X}_{st}, \tau_1 \rangle \langle \mathbf{X}_{st}, \tau_2 \rangle$, $\mathbf{X}$ takes values in the characters on the algebra $(V, \sqcup\!\sqcup)$.

It turns out that the set of characters is a group for the multiplication

$$\langle A \star B, \tau \rangle = \langle A \otimes B, \Delta\tau \rangle$$

with identity element $\mathbf{1}^*(\tau) := \mathbb{1}_{(\tau = \emptyset)}$ and inverse

$$\langle A^{-1}, \tau \rangle = \langle A, S\tau \rangle$$

where $S : V \to V$ is the antipode map given in this case by

$$S(i_1, \dots, i_n) = (-1)^n (i_n, \dots, i_1).$$

Then V is a Hopf algebra. In particular, setting $\mathbf{X}_t := \mathbf{X}_{0,t}$

$$\mathbf{X}_{st} = \mathbf{X}_s^{-1} \star \mathbf{X}_t.$$

Massimiliano Gubinelli had the idea of considering a general rough path as an element of K^*, where K is a larger bialgebra than V.

K is the vector space generated by decorated forests. We define a product $\cdot$ given by the disjoint union and a suitable deconcatenation coproduct $\Delta : K \to K \otimes K$.

It turns out that K has an antipode and therefore

- K is a Hopf algebra
- the set of characters on K is a group

Hopf algebras and groups of characters based on trees and forests were already known to play an important role in ODEs and numerical analysis since the work of Cayley (1889), Butcher (1972) and Hairer-Wanner (1974).

Kreimer and Connes-Kreimer (1998) used the same space in QFT.

Therefore, Massimiliano defines a branched rough path of regularity $\gamma > 0$ as a function $\mathbf{X} : [0,T]^2 \to K^*$ s.t.

- $\langle \mathbf{X}_{st}, \tau \rangle = \langle \mathbf{X}_{su} \otimes \mathbf{X}_{ut}, \Delta\tau \rangle, \quad \forall \tau \in K.$
- $\langle \mathbf{X}_{st}, \tau_1 \cdot \tau_2 \rangle = \langle \mathbf{X}_{st}, \tau_1 \rangle \langle \mathbf{X}_{st}, \tau_2 \rangle.$
- $\sup_{s \neq t} [|\langle \mathbf{X}_{st}, \tau \rangle|/|t-s|^{\gamma|\tau|}] < +\infty$, for all $\tau \in K$, where $|\tau|$ is the number of nodes of the forest τ.

In particular

$$\mathbf{X}_{st} = \mathbf{X}_s^{-1} \star \mathbf{X}_t.$$

Notations and presentation follow here [Hairer-Kelly 2013].

Around 2010, Martin and Massimiliano, among others, try to generalise the previous setting to (singular) stochastic PDEs like KPZ, PAM and Φ^4.

$$\begin{aligned} &(\text{KPZ}) \qquad \partial_t u = \Delta u + (\nabla u)^2 + \xi, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \\ &(\text{PAM}) \qquad \partial_t u = \Delta u + u\,\xi, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^2, \\ &(\Phi^4_3) \qquad \partial_t u = \Delta u - u^3 + \xi, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^3. \end{aligned}$$

This needs two generalisations:

- The rough path must be parametrized by $\mathbb{R}^d$ with $d \geq 2$
- $\langle \mathbf{X}_{st}, \tau \rangle$ can become a distribution, say, in t for fixed s, i.e. we want to allow that $\sup_{s \neq t} [|\langle \mathbf{X}_{st}, \tau \rangle|/|t-s|^{\alpha_\tau}] < +\infty$ with $\alpha_\tau \in \mathbb{R}$.

Two new theories are born: regularity structures and paraproducts.

In regularity structures, we have a linear space $\mathcal{H}$ of trees, which represent distributions on $\mathbb{R}^d$. This space should play the role of K, so that the rough path $\mathbf{X}$ should be a $\mathcal{H}^*$-valued distribution on $\mathbb{R}^d$ with good properties. More precisely, we can think of $\mathbf{X}$ as a map $\mathcal{H} \to \mathcal{S}'(\mathbb{R}^d)$.

However K^* is an algebra with the $\star$ product, while we do not expect to multiply all $\mathcal{H}^*$-valued distributions. Therefore we do not expect $\mathcal{H}$ to have a coproduct.

Instead, we consider a Hopf algebra $\mathcal{H}_+$ and a right coaction $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}_+$. Then the character group $G_+ \subset \mathcal{H}_+^*$ of $\mathcal{H}_+$ acts on the right on $\mathcal{H}^*$: $G_+ \to \mathrm{End}(\mathcal{H}^*, \mathcal{H}^*)$, $\mathcal{H}^* \ni h \mapsto hg \in \mathcal{H}^*$

$$hg(\tau) := (h \otimes g)\Delta\tau, \quad \tau \in \mathcal{H}, \quad (hg_1)g_2 = h(g_1 g_2).$$

By duality, G_+ acts on the left on $\mathcal{H}$. G_+ is the structure group.

Then our rough path $x \mapsto \mathbf{X}_x$ becomes here

$$\mathbb{R}^d \ni x \mapsto (h_x, g_x) \in \mathcal{H}^* \times G_+, \qquad G_+ \subset \mathcal{H}_+^*$$

and we write (recalling that $\mathbf{X}_{xy} = \mathbf{X}_x^{-1} \star \mathbf{X}_y$)

$$\Pi_x \tau(y) := (h_y \otimes g_x)\Delta\tau, \qquad \Gamma_{xz}\tau = (\mathrm{id} \otimes g_x^{-1} g_z)\Delta\tau$$

where $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}_+$ is a right coaction. Then the formulae

$$\Pi_z \Gamma_{zx}\tau(y) = \Pi_x\tau(y), \qquad \Gamma_{xz}\Gamma_{zy} = \Gamma_{xy}$$

are the natural generalisation to this setting of the Chen formula

$$\mathbf{X}_{xz} \star \mathbf{X}_{zy} = \mathbf{X}_{xy}.$$

Note: the right coaction is equivalent to the left action of G_+ on $\mathcal{H}$ and replaces the coproduct on $\mathcal{H}$.

The fact that now $\mathbf{X} = (h, g)$ can contain distributions has important consequences: if we approximate with smooth $\mathbf{X}^\varepsilon = (h^\varepsilon, g^\varepsilon)$, some quantity might diverge in order for generalised functions to appear. We must modify $\mathbf{X}^\varepsilon$ in order to make it convergent.

But how? Well, the algebraic structure must be preserved. We need to understand the morphisms of our structure. This is the Renormalisation step.

We need now a left action of another group G_- on $\mathcal{H}^*$ and on G_+

$$G_- \to \mathrm{End}(\mathcal{H}^*, \mathcal{H}^*), \qquad G_- \to \mathrm{Hom}(G_+, G_+),$$

$$\ell_1(\ell_2 h) = (\ell_1\ell_2)h, \quad \ell_1(\ell_2 g) = (\ell_1\ell_2)g \quad \ell(g_1 g_2) = (\ell g_1)(\ell g_2).$$

where $\ell \in G_-$, $g \in G_+$, $h \in \mathcal{H}^*$.

Then the renormalised rough path is

$$\Pi_x^\ell \tau(y) := (\ell h_y \otimes \ell g_x)\Delta\tau, \qquad \Gamma_{xz}^\ell \tau = (\mathrm{id} \otimes (\ell g_x)^{-1}\ell g_z)\Delta\tau.$$

Again we have

$$\Pi_z^\ell \Gamma_{zx}^\ell \tau(y) = \Pi_x^\ell \tau(y), \qquad \Gamma_{xz}^\ell \Gamma_{zy}^\ell = \Gamma_{xy}^\ell.$$

G_- is the renormalisation group.

It would be nice if we had

$$\Pi_x^\ell \tau(y) = \Pi_x(\tau\ell),$$

where $\mathcal{H} \ni \tau \mapsto \tau\ell$ is the dual action of $\mathcal{H}^* \ni h \mapsto \ell h$, namely

$$\ell h(\tau) := h(\tau\ell).$$

For this we need the compatibility condition

$$\ell(hg) = (\ell h)(\ell g).$$

This is reminiscent of another formula

$$\ell(g_1 g_2) = (\ell g_1)(\ell g_2).$$

In fact, we can realise all the above actions and interactions with two operators Δ_1, Δ_2 on suitable space of decorated forests, satisfying

- co-associativity
$$(\mathrm{id} \otimes \Delta_i)\Delta_i = (\Delta_i \otimes \mathrm{id})\Delta_i, \qquad i = 1, 2,$$
which is responsible for
$$g_1(g_2\tau) = (g_1g_2)\tau, \quad \ell_1(\ell_2 h) = (\ell_1\ell_2)h, \quad \ell_1(\ell_2 g) = (\ell_1\ell_2)g,$$
- compatibility (for lack of a more precise term)
$$(\mathrm{id} \otimes \Delta_2)\Delta_1 = \mathcal{M}^{(13)(2)(4)}(\Delta_1 \otimes \Delta_1)\Delta_2,$$
which is responsible for
$$\ell(g_1g_2) = (\ell g_1)(\ell g_2), \qquad \ell(hg) = (\ell h)(\ell g).$$

The theory of paraproducts has no G_+ but $\mathcal{H}$ is there and the action of G_- is arguably the same.

Also in regularity structures, we can treat only globally defined distributions
$$\Pi\tau(y) = h_y(\tau), \qquad \Pi^\ell\tau(y) = \ell h_y(\tau) = h_y(\tau\ell).$$

The above structure can be summarised in a left action of the semidirect product $G_- \ltimes G_+$ on $\mathcal{H}$, where
$$(\ell_1, g_1)(\ell_2, g_2) = (\ell_1\ell_2, g_1(\ell_1 g_2)).$$

Fully nonlinear SPDEs and RPDEs: Classical and viscosity solutions

Jianfeng Zhang

(joint work with Rainer Buckdahn, Christian Keller and Jin Ma)

This talk concerns the following fully nonlinear stochastic partial differential equations with initial condition $u(0, x, \o) = u_0(x)$:

$$du(t, x, \o) = f(t, x, \o, u, \partial_x u, \partial^2_{xx} u)dt + g(t, x, \o, u, \partial_x u) \circ dB_t, \tag{1}$$

where B is a standard Brownian motion, $\circ\, dB_t$ is the Stratonovich integration, and f is increasing in $\partial^2_{xx}u$. Such equation typically does not have classical solution. The goal of this work is to establish the viscosity theory, and for that it is more convenient to study the corresponding rough partial differential equations:

$$du(t, x, \o) = f(t, x, \o, u, \partial_x u, \partial^2_{xx} u)dt + g(t, x, \o, u, \partial_x u) \circ d\o_t, \tag{2}$$

where $\o$ is viewed as a rough path and $d\o$ is the rough path integration corresponding to Stratonovich integration.

Such problems were introduced by [8, 9], and two approaches were proposed. The first one is to mollify the (rough) path $\o$ and prove the solutions of the approximating equations converge. The work [3] follows this approach. The other approach is to transform the SPDE (or rough PDE) to a standard PDE with random coefficients by using characteristics. We shall follow this approach, in particular, we will define viscosity solutions via test functions. The recent work [6] is also in this direction.

In this talk, we first introduce the pathwise Ito calculus of [7], based on the pathwise Taylor expansion of [1, 2]. This unifies the path derivatives of Dupire [4] and the controlled rough paths of Gubinelli [5]. Next, assuming f and g are smooth enough in the sense of [7], by using characteristic we show that (2) admits a global classical solution when g is semilinear:

$$g = \sigma(, x, \emptyset)\partial_x u + g_0(t, x, \emptyset, u), \tag{3}$$

and admits a local classical solution for general case.

To introduce viscosity solution, we assume g is still smooth but f may not be. The test functions are smooth functions (again in the sense of [7]) whose path derivative coincides with g. We show that the definition is equivalent to the alternative definition defined through semi-jets, and is consistent with classical solution. We establish the partial comparison principle (comparison between a viscosity subsolution and a classical supersolution) and stability under mild conditions. Finally, we show that full comparison principle: globally in the semilinear case (3) and locally in general case.

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