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Algebraic K-theory and Motivic Cohomology

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Abstract. Algebraic K-theory and motivic cohomology have developed together over the last thirty years. Both of these theories rely on a mix of algebraic geometry and homotopy theory for their construction and development, and both have had particularly fruitful applications to problems of algebraic geometry, number theory and quadratic forms. The homotopy-theory aspect has been expanded significantly in recent years with the development of motivic homotopy theory and triangulated categories of motives, and K-theory has provided a guiding light for the development of non-homotopy invariant theories. 19 one-hour talks presented a wide range of latest results on many aspects of the theory and its applications.

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Introduction by the Organisers

Algebraic K-theory and motivic cohomology have developed together over the last thirty years. Both of these theories rely on a mix of algebraic geometry and homotopy theory for their construction and development, and both have had particularly fruitful applications to problems of algebraic geometry, number theory and quadratic forms. The homotopy-theory aspect has been expanded significantly in recent years with the development of motivic homotopy theory and triangulated categories of motives, and K-theory has provided a guiding light for the development of non-homotopy invariant theories.

The workshop program presented a varied series of lectures on the latest developments in the field. The 53 participants came mostly from Europe, but there were large contingents from the USA and Japan, as well as additional participants from Argentina, Russia and India. The participants ranged from leading experts in the field to younger researchers and also some graduate students. 19 one-hour talks presented a wide range of the latest results on the theory and its applications, reflecting a good mix of nationalities and age groups.

Here is a more detailed description of the talks.

Computations in K-theory and in the homology of linear groups. Hesselholt (with M. Larsen and A. Lindestrauss) constructed a reduced norm map for p-adic K-theory of a division ring over a local field with residue field of characteristic $p > 0$, using the cyclotomic trace map. Morrow extended known results on mod p^n-K-theory to the p-adic case and used this to prove continuity results, infinitesimal versions of weak Lefschetz for the Chow groups and a p-adic version of a conjecture of Kato-Saito. Wendt described his work giving a presentation of the homology of GL_3 of a elliptic curve and its relation to the construction of an elliptic dilogarithm complex. Schlichting described a refinement of Suslin's results on the homology of GL_n to the case of SL_n and E_n.

Categorical constructions. Zakharevich described her approach to the study of $K_0(Var)$ via a categorical scissors congruence construction. Yamazaki presented a framework (developed jointly with Kahn and Saito) for a triangulated category of "motives with modulus", which hopefully will give a good framework for studying non-homotopy invariant phenomena, such as wild ramification. Déglise showed how to construct a category of effective motives over a general base (joint with Bondarko), which admits a reasonable t-structure. Ivorra gave a description of the nearby cycles functor in terms of tubes in non-archimedean geometry. Panin presented aspects of his work with Garkusha on a new description of the motivic stable homotopy category in terms of "framed correspondences" and drew an analogy with Segal's machine for constructing infinite loop spaces.

Theories for topological rings. Tamme presented his work (with Kerz and Saito) describing the "generic fiber" of the comparison map from the K-theory of a smooth proper scheme over a complete DVR its continuous K-theory in terms of a K-theory of the associated rigid analytic space. Using the theory of bornological algebras, Cortiñas showed how to define a theory of rigid cohomology in the non-commutative setting, and gave a description of this in terms of cyclic homology (a joint work with Cuntz, Meyer and Tamme).

Motives and algebraic cycles. Kohrita extended the classical theory of cycles algebraically equivalent to zero and the notion of a universal regular homomorphism, for smooth projective varieties, to the case of smooth varieties and motivic cohomology. He showed the existence of the universal regular homomorphism to a semi-abelian variety for motivic cohomology in a certain range. Tabuada discussed non-commutative versions of Grothendieck's conjecture on the equality of numerical and homological equivalence, and Voevodsky's smash nilpotence conjecture. He showed that the non-commutative and commutative conjectures are equivalent, gave applications to quadric fibrations and stacks, and showed that under certain conditions, these conjectures are invariant under homological projective duality.

Ancona presented his work proving Grothendieck's conjecture on the positive definiteness of the intersection pairing in a number of new cases. Kahn showed how to interpret the Griffiths group of a smooth projective threefold X (assuming a Chow-Künneth decomposition) as the group of morphisms in the category of motives modulo algebraic equivalence from the Lefschetz motive to the transcendental part of $h_3(X)$. Vial discussed the question: is the field of definition of the intermediate Jacobian (assuming this to be algebraic, or its algebraic part if it is not) of a smooth projective complex variety X the same as the field of definition of X, and showed that this is the case for J^3_{alg}. In addition, he showed (with Achter and Casalaina-Martin) that the associated Abel-Jacobi map is Galois equivariant.

Arithmetic. Schmidt (with Stix) considered a more general formulation of Grothendieck's section conjecture, involving the entire étale homotopy type rather than just the fundamental group, and obtained a number of "anabelian" statements for varieties that can be embedded as locally closed subschemes of a product of hyperbolic curves. Zhao gave an extension of duality in local class field theory to higher dimensional regular schemes in positive characteristic. Morin described a unified framework and series of conjectures (developed with Flach) for the zeta-values of arithmetic schemes.

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Workshop: Algebraic K-theory and Motivic Cohomology

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Abstracts

K-theory of division algebras over local fields

Lars Hesselholt
(joint work with Michael Larsen and Ayelet Lindenstrauss)

Suppose that K is a complete discrete valuation field with finite residue field of characteristic p and that D is a central division algebra over K of finite index d. Thirty years ago, Suslin and Yufryakov [3] proved that for all prime numbers $\ell \neq p$ and all positive integers j, there is an isomorphism of ℓ-adic K-groups

$$K_j(D, \mathbb{Z}_\ell) \xrightarrow{\mathrm{Nrd}_{D/K}} K_j(K, \mathbb{Z}_\ell)$$

such that $d\,\mathrm{Nrd}_{D/K}$ is equal to the norm $N_{D/K}$. We prove the following analogous result for the p-adic K-groups [1].

Theorem. *For every positive integer j, there is an isomorphism*

$$K_j(D, \mathbb{Z}_p) \xrightarrow{\mathrm{Nrd}_{D/K}} K_j(K, \mathbb{Z}_p)$$

with the property that $d\,\mathrm{Nrd}_{D/K} = N_{D/K}$.

By contrast with the norm $N_{D/K}$ homomorphism, we do not know that the reduced norm isomorphism $\mathrm{Nrd}_{D/K}$ is induced by a map of K-theory spectra. Therefore, we are not able to conclude that a reduced norm isomorphism between the integral K-groups exists; for it is unknown if the p-primary torsion subgroups of the integral K-groups are of bounded exponent.

To prove the ℓ-adic statement, Suslin-Yufryakov use Gabber-Suslin rigidity to reduce the statement to one concerning the ℓ-adic K-groups of the respective residue fields. The additional tool that makes it possible to now prove the p-adic statement is the cyclotomic trace map of Bökstedt-Hsiang-Madsen from K-theory to topological cyclic homology. We let $S \subset K$ be the valuation ring, let $A \subset D$ be the maximal S-order, and consider the cyclotomic trace map

$$K_j(D, \mathbb{Z}_p) \longrightarrow \mathrm{TC}_j(A \,|\, D; p, \mathbb{Z}_p)$$

in the case of the exact category of perfect complexes of left A-modules with the chain maps that become quasi-isomorphisms after extension of scalars to D as the weak equivalences; compare [2, Section 1]. It is an isomorphism, for all positive integers j, as is the cyclotomic trace map

$$K_j(K, \mathbb{Z}_p) \longrightarrow \mathrm{TC}_j(S \,|\, K; p, \mathbb{Z}_p)$$

The theorem is now proved by showing that, as a graded $\mathrm{TC}_*(S \,|\, K; p, \mathbb{Z}_p)$-module, $\mathrm{TC}_*(A \,|\, D; p, \mathbb{Z}_p)$ is free on a single generator in degree 0. We remark that said generator is not in the image of the cyclotomic trace map and that, by comparison, the graded $K_*(K, \mathbb{Z}_p)$-module $K_*(D, \mathbb{Z}_p)$ is neither free nor finitely generated, except in the trivial case $d = 1$.

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The annihilator of the Lefschetz motive

Inna Zakharevich

The Grothendieck ring of varieties is defined to be the free abelian group generated by k-varieties, modulo the relation that for any closed immersion $Y \hookrightarrow X$, we impose the relation that $[X] = [Y] + [X \backslash Y]$; the ring structure is defined by $[X][Y] = [X \times Y]$. In 2014 two longstanding questions about the Grothendieck ring of varieties were answered:

(1) If two varieties X and Y are piecewise isomorphic then they are equal in the Grothendieck ring; does the converse hold?
(2) Is the class of the affine line a zero divisor?

Both questions were answered by Borisov, who constructed an element in the kernel of multiplication by the affine line; coincidentally, the proof also constructed two varieties whose classes in the Grothendieck ring are the same but which are not piecewise isomorphic. In this talk we will investigate these questions further by constructing a topological analog of the Grothendieck ring and analyzing its higher homotopy groups. Using this extra structure we will sketch a proof that Borisov's coincidence is not a coincidence at all: that any element in the annihilator of the Lefschetz motive can be represented by a difference of varieties which are equal in the Grothendieck ring but not piecewise isomorphic.

We define an *assembler* in the following manner:

Let $\mathcal{C}$ be a Grothendieck site with an initial object. We denote the full subcategory of noninitial objects by $\mathcal{C}^\circ$. We say that a family of maps $\{A_i \to A\}_{i \in I}$ is a *covering family* if it generates a covering sieve in the topology; it is *finite* if I is finite. A covering family is *disjoint* if for any two morphisms $A_i \to A$ and $A_j \to A$, the pullback $A_i \times_A A_j$ exists an is equal to the initial object.

An *assembler* is a small Grothendieck site $\mathcal{C}$ satisfying the following extra conditions:

(I) $\mathcal{C}$ has an initial object $\emptyset$, and the empty family is a covering family of $\emptyset$.
(R) For any A, any two finite disjoint covering families of A have a common refinement which is itself a finite disjoint covering family.
(M) All morphisms in $\mathcal{C}$ are monomorphisms.

An assembler encodes all of the formal information necessary to keep track of how varieties decompose, while forgetting the particular geometric nature of varieties. The category of assemblers comes equipped with a functor K to spectra such that for an assembler $\mathcal{C}$,

$$\pi_0 K(\mathcal{C}) \cong \text{free abelian group on objects of } \mathcal{C}/\sim,$$

where the equivalence relation says that given a finite disjoint covering family $\{A_i \to A\}_{i=1}^n$, $[A] = \sum_{i=1}^n [A_i]$ [ZakB]. We write $K_i(\mathcal{C}) = \pi_i K(\mathcal{C})$. Let $\mathcal{V}_k$ be the assembler whose objects are varieties and whose morphisms are locally closed embeddings, $K_0(\mathcal{V}_k)$ is the Grothendieck ring of varieties.

In addition, $K(\mathcal{C})$ preserves further information about the structure of the problem; for example, any element of $K_1(\mathcal{C})$ is given by an object X and two decompositions of X into the same set of pieces $X_1, \ldots, X_n$ [ZakC]; in other words, an element of $K_1(\mathcal{C})$ is represented by a piecewise automorphism of X. Thus $K_1(\mathcal{C})$ keeps track of the different ways an object can be decomposed into the same finite set of pieces in two different ways. These higher homotopy groups of $K(\mathcal{C})$ thus preserve a significant amount of data about scissors congruence which can be used to analyze the problems in more detail.

Assemblers also retain sufficient combinatorial data to allow analysis of the homotopy type of $K(\mathcal{C})$; in particular, we have the following localization and dévissage theorems:

Theorem 1 ([ZakB, Theorem C]). *The functor K extends to a functor on simplicial assemblers. For any morphism of simplicial assemblers $g : \mathcal{C}. \to \mathcal{D}.$ there exists a simplicial assembler $(\mathcal{D}/g).$ and a morphism of assemblers $\mathcal{D}. \to (\mathcal{D}/g).$ such that the sequence*

$$K(\mathcal{C}.) \to^g K(\mathcal{D}.) \to K((\mathcal{D}/g).)$$

is a cofiber sequence of spectra.

The set of varieties comes with a filtration by the dimension of the variety. This induces a filtration on the Grothendieck ring $K_0(\mathcal{V}_k)$; unfortunately, it is difficult to learn anything directly about $K_0(\mathcal{V}_k)$ from this filtration, since it is prohibitively difficult to compute the associated graded of the filtration. However, this filtration also gives a filtration on the assembler $\mathcal{V}_k$ and therefore a filtration on the K-theory $K(\mathcal{V}_k)$. It is possible to prove the following localization theorem on the assembler of varieties:

Theorem 2 ([ZakA, Theorem A]). *Let $\mathcal{V}_k^{(n)}$ be the assembler of varieties of dimension at most n, and let $\iota_n : \mathcal{V}_k^{(n-1)} \to \mathcal{V}_k^{(n)}$ be the inclusion of assemblers. Write B_n for the set of birational isomorphism classes of varieties of dimension n. Then the cofiber of the map*

$$K(\iota_n) : K(\mathcal{V}_k^{(n-1)}) \to K(\mathcal{V}_k^{(n)})$$

is weakly equivalent to $\bigvee_{[X]\in B_n} \Sigma_+^\infty B\mathrm{Aut}\, k(X)$.

This theorem computes the associated graded of the dimension filtration on $K(\mathcal{V}_k)$. It produces a homologically-graded spectral sequence where the 0-th diagonal converges to the associated graded of the Grothendieck ring of varieties, starting from groups of the form $\mathbf{Z}[B_n]$. The -1-st diagonal is 0; the 1-st diagonal contains groups of the form

$$\bigoplus_{[X]\in B_n} (\operatorname{Aut} k(X))^{ab} \oplus \mathbf{Z}/2.$$

This spectral sequence is the connection between the naïve analysis from the beginning of this section and the actual associated graded of the Grothendieck ring of varieties.

The spectral sequence can also be used to analyze the annihilator of the Lefschetz motive $\mathbb{L} = [\mathbb{A}^1]$. This kernel is important because many of the computational techniques that use the Grothendieck ring require it to be localized at $\mathbb{L}$. Thus any geometric information present in the annihilator of $\mathbb{L}$ is destroyed and can't be detected by these computational techniques. In [Bor], Borisov constructed an element $[X]-[Y]$ in the annihilator of $\mathbb{L}$, thus showing that this kernel is nontrivial. Surprisingly enough, his proof also showed that $X \times \mathbb{A}^1$ and $Y \times \mathbb{A}^1$ are not birational, thus constructing an element in the kernel of ι_n for some n. Using assemblers we can show that this is not a coincidence:

Theorem 3 ([ZakA, Theorem D]). *Suppose that $\alpha \in K_0(\mathcal{V}_k)$ is in the kernel of multiplication by $\mathbb{L}$, and write $\alpha = [X]-[Y]$ with X and Y of minimal dimension. Then $X \times \mathbb{A}^1$ and $Y \times \mathbb{A}^1$ are not scissors congruent.*

Thus any minimal representation of an element in the kernel of multiplication by $\mathbb{L}$ gives an element in the kernel of ι_n. The proof of this theorem relies heavily on the higher homotopy groups of $K(\mathcal{V}_k)$, and thus illustrates the principle that the spectrum $K(\mathcal{V}_k)$ contains useful information which is lost in the Grothendieck ring $K_0(\mathcal{V}_k)$.

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Algebraic part of motivic cohomology

Tohru Kohrita

For a smooth proper connected scheme X over an algebraically closed field k, the subgroup $A^r(X)$ of the Chow group $CH^r(X)$ consisting of cycles algebraically equivalent to zero is called the *algebraic part* of $CH^r(X)$, and a group homomorphism ϕ from the algebraic part $A^r(X)$ to the group of rational points of an abelian variety A is called a *regular homomorphism* if for any connected smooth proper scheme T over k pointed at a rational point t_0, and for any cycle $Y \in CH^r(T \times X)$, the composition

$$T(k) \xrightarrow{w_Y} A^r(X) \xrightarrow{\phi} A(k)$$

is induced by a scheme morphism $T \longrightarrow A$ [Sam, Section 2.5]. Here, the map w_Y sends $t \in T(k)$ to $Y(t) - Y(t_0)$, where $Y(t)$ stands for the image of the intersection of cycles $(t \times X) \cdot Y \in CH^{\dim T + r}(T \times X)$ under the proper pushforward along the projection $T \times X \longrightarrow X$. A regular homomorphism $\phi : A^r(X) \longrightarrow A(k)$ is said *universal* if, given any regular homomorphism $\phi' : A^r(X) \longrightarrow A'(k)$, there is a unique homomorphism of abelian varieties $h : A \longrightarrow A'$ such that $h \circ \phi = \phi'$ holds. It is classical that the universal regular homomorphism exists for $r = 1$ and $\dim X$. The case $r = 1$ is the theory of Picard varieties and the case $r = \dim X$ coincides with that of Albanese varieties. The existence of universal regular homomorphism for $r = 2$ is also known by Murre [Mur, Theorem A]. It would be worth noting that one of the important ingredients of Murre's proof is the theorem of Merkurjev and Suslin [MS] on norm residue homomorphisms. We also know that universal regular homomorphisms are isomorphisms if $r = 1$, and isomorphisms on torsion if $r = d$ ([Ro, Bl, Mi]) or $r = 2$ ([Mur]).

In this talk, we consider an analogue for arbitrary smooth (not necessarily proper) schemes by replacing Chow groups with motivic cohomology with compact supports. The classical definitions of algebraic part of Chow groups and regular homomorphisms can readily be adapted by using Voevodsky's tensor triangulated category $DM^-_{Nis}(k)$ of motives over k (see [V] and [MVW]). The tensor structure of $DM^-_{Nis}(k)$ is denoted by $\otimes$, and $M(X)$ (resp., $M^c(X)$) stands for the motive (resp., motive with compact supports) of a scheme X over k.

Given a zero-cycle z on a smooth scheme T over k (or its image in $H_0(T, \mathbb{Z})$) and $Y \in Hom_{DM^-_{Nis}(k)}(M(T) \otimes M^c(X), \mathbb{Z}(n)[m])$, we write the composition

$$M^c(X) \cong \mathbb{Z} \otimes M^c(X) \xrightarrow{z \otimes id_{M^c(X)}} M(T) \otimes M^c(X) \xrightarrow{Y} \mathbb{Z}(n)[m]$$

of morphisms in $DM^-_{Nis}(k)$ as Y_t. Note that Y_t is by definition an element of $H^m_c(X, \mathbb{Z}(n))$.

Definition 1. *Let X be a smooth scheme over k. The* **algebraic part** *of the motivic cohomology group with compact supports $H^m_c(X, \mathbb{Z}(n))$ is defined as*

$$H^m_{c,alg}(X, \mathbb{Z}(n)) := \bigcup_{\substack{T,\ smooth \\ connected}} \mathrm{im}\{H_0(T, \mathbb{Z})^0 \times Hom_{DM^-_{Nis}(k)}(M(T) \otimes M^c(X), \mathbb{Z}(n)[m]) \longrightarrow H^m_c(X, \mathbb{Z}(n))\},$$

where the map sends a pair (z, Y) with

$$z \in H_0(T, \mathbb{Z})^0 := \ker\{H_0(T, \mathbb{Z}) \xrightarrow{str_*} H_0(\mathrm{Spec}\ k, \mathbb{Z})\}$$

and

$$Y \in Hom_{DM^-_{Nis}(k)}(M(T) \otimes M^c(X), \mathbb{Z}(n)[m])$$

to

$$Y_z \in Hom_{DM^-_{Nis}(k)}(M^c(X), \mathbb{Z}(n)[m]) \stackrel{def}{=} H^m_c(X, \mathbb{Z}(n)).$$

Under the assumption of resolution of singularities, it can be shown that for any smooth connected scheme of dimension d over k, the Suslin-Friedlander duality isomorphism ([V, Theorem 4.3.7]) induces an isomorphism between $H^{2d}_{c,alg}(X, \mathbb{Z}(d))$ and $H_0(X, \mathbb{Z})^0$. If X is, in addition, proper and $(m, n) = (2r, r)$ for some integer r, then there is a canonical isomorphism $H^{2r}_{c,alg}(X, \mathbb{Z}(r)) \cong A^r(X)$. The notion of regular homomorphisms also naturally generalizes to this setting.

Definition 2. *Let X be a smooth scheme over k and let S be a semi-abelian variety over k. A group homomorphism $\phi : H^m_{c,alg}(X, \mathbb{Z}(n)) \longrightarrow S(k)$ is called* **regular** *if for any smooth connected scheme T over k pointed at $t_0 \in T(k)$ and any $Y \in Hom_{DM^-_{Nis}(k)}(M(T) \otimes M^c(X), \mathbb{Z}(n)[m])$, the composition*

$$T(k) \xrightarrow{w_Y} H^m_{c,alg}(X, \mathbb{Z}(n)) \xrightarrow{\phi} S(k)$$

is induced by some scheme morphism $T \longrightarrow S$. Here, w_Y sends $t \in T(k)$ to $Y_t - Y_{t_0} \in H^m_{c,alg}(X, \mathbb{Z}(n))$.

A regular homomorphism $\Phi^{m,n}_{c,X} : H^m_{c,alg}(X, \mathbb{Z}(n)) \longrightarrow Alg^{m,n}_{c,X}(k)$ is said *universal* if for any regular homomorphism ϕ, there exists a unique homomorphism h of semi-abelian varieties as indicated in the diagram

$$\begin{array}{ccc} H^m_{c,alg}(X, \mathbb{Z}(n)) & \xrightarrow{\Phi^{m,n}_{c,X}} & Alg^{m,n}_{c,X}(k) \\ & \searrow^{\forall \phi} & \vdots\ \exists! h \\ & & S(k) \end{array}$$

With this setup, we can prove the following.

Theorem 3 (Existence)**.** *Let X be a smooth connected scheme of dimension d over an algebraically closed field k. Then, there exists a universal regular homomorphism*

$$\Phi^{m,n}_{c,X} : H^m_{c,alg}(X, \mathbb{Z}(n)) \longrightarrow Alg^{m,n}_{c,X}(k)$$

if $m \leq n + 2$ or $(m, n) = (2d, d)$.

Under the assumption of resolution of singularities, one can show that the (Serre's) Albanese map $alb_X : H_0(X, \mathbb{Z})^0 \longrightarrow Alb_X(k)$ constructed in [SS, Ra] agrees with our universal regular homomorphism for $(m, n) = (2d, d)$ via the isomorphism $H^{2d}_{c,alg}(X, \mathbb{Z}(d)) \cong H_0(X, \mathbb{Z})^0$ mentioned right after Definition 1.

For the case $(m, n) = (2, 1)$, we may interpret the universal regular homomorphism for $H^2_{c,alg}(X, \mathbb{Z}(1))$ in terms of the relative Picard group of a smooth compactification $\bar{X}$ of X with a simple normal crossing boundary divisor Z. Below, let us assume that Z is non-empty. Otherwise, it would be the classical proper case, and one would also needs to take fppf sheafification in cosidering a Picard functor below. Let $Pic^0_{\bar{X},Z,red}$ be the reduction of the identity component of the group scheme representing the functor that sends $T \in Sch/k$ to the relative Picard group $Pic(T \times \bar{X}, T \times Z)$ as in [B-VS]. We can show:

Theorem 4. *Assume resolution of singularities. If X is a smooth connected scheme over k with a smooth compactification $\bar{X}$ with a non-empty simple normal crossing boundary divisor Z, then there is a canonical regular homomorphism*

$$\phi_0 : H^2_{c,alg}(X, \mathbb{Z}(1)) \longrightarrow Pic^0_{\bar{X},Z,red}(k)$$

and it is universal. In particular, $Pic^0_{\bar{X},Z,red}$ depends only on X. Also, ϕ_0 is an isomorphism.

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Algebraic K-theory and motivic cohomology of formal schemes in characteristic p

Matthew Morrow

The talk was a survey of the main results of the preprint [6].

1. Pro Geisser–Levine and Bloch–Kato–Gabber theorem

If A is any $\mathbb{F}_p$-algebra, then we may consider the natural homomorphisms

$$K_n(A)/p^r \longleftarrow K_n^M(A)/p^r \stackrel{\text{dlog}[\cdot]}{\longrightarrow} W_r\Omega^n_{A,\log},$$

where $W_r\Omega^n_{A,\log}$ (also denoted by $\nu_r^n(A)$ in the literature) is the subgroup of the Hodge–Witt group $W_r\Omega^n_A$ consisting of elements which can be written étale locally as sums of dlog forms, and the map dlog[$\cdot$] is given by $\{a_1, \ldots, a_n\} \mapsto$

$\text{dlog}[a_1]\cdots\text{dlog}[a_n]$ as usual. If A is regular and local then both of these homomorphisms are known to be isomorphisms: this reduces, via Gersten sequences, to the case that A is a field, in which case the leftwards isomorphism is due to Geisser and Levine [3], who also proved that $K_n(A)$ is p-torsion-free, and the rightwards isomorphism is the Bloch–Kato–Gabber theorem (see [6, Thm. 5.1] for more details and references; also, to avoid issues caused by finite residue fields, we use Kerz–Gabber's improved Milnor K-theory throughout). Still assuming that A is regular and local, it follows that $K_n(A;\mathbb{Z}/p^r)\cong W_r\Omega^n_{A,\log}$, which satisfactorily calculates the p-adic part of the algebraic K-theory of A.

The focus of the talk was the following analogous calculation for formal schemes:

Theorem 1. *Let A be a regular local $\mathbb{F}_p$-algebra, and $I\subseteq A$ an ideal; assume that A/I is both F-finite (i.e., finitely generated over its subring of p^{th}-powers) and a "generalised normal crossing divisor" (i.e., whenever $Y_1,\dots,Y_N$ are some of the irreducible components of $\operatorname{Spec} A/I$, then $(\bigcap_i Y_i)_{\text{red}}$ is regular).*

Then the homomorphisms of pro abelian groups

$$\{K_n(A/I^s)/p^r\}_s \longleftarrow \{K_n^M(A/I^s)/p^r\}_s \xrightarrow{\text{dlog}[\cdot]} \{W_r\Omega^n_{A/I^s,\log}\}_s$$

are surjective and have the same kernel, thereby inducing an isomorphism

$$\{K_n(A/I^s)/p^r\}_s \xrightarrow{\simeq} \{W_r\Omega^n_{A/I^s,\log}\}_s.$$

Moreover, the pro abelian group $\{K_n(A/I^s)\}_s$ is p-torsion-free.

In other words, although the homomorphisms will not be isomorphisms for A/I^s for any fixed $s>1$, the obstruction to having an isomorphism $K_n(A/I^s;\mathbb{Z}/p^r)\cong W_r\Omega^n_{A/I^s,\log}$ is Mittag-Leffler zero as $s\to\infty$.

In the special case that I is principal and A/I is regular, Theorem 1 can be improved: the stated homomorphisms of pro abelian groups are not merely surjective but even isomorphisms. This is likely to be true without these additional assumptions on I and A/I, but certainly necessary results on the p-torsion in the Milnor K-theory of multivariable truncated polynomial algebras remain to be established.

2. Applications

Theorem 1 leads to a number of applications concerning K-theory in the presence of infinitesimal thickenings.

2.1. **The continuity problem.** The continuity problem in algebraic K-theory asks whether the canonical map $K(A)\to\operatorname{holim}_s K(A/I^s)$ is a weak equivalence after pro-finite completion whenever A is an I-adically complete ring. If A is an $\mathbb{F}_p$-algebra then it is sufficient to work with the p-adic part of the K-theories, by Gabber rigidity. The first application of Theorem 1 is an affirmative answer to this question in certain cases:

Theorem 2. *With A and I as in Theorem 1, assume moreover that A is I-adically complete. Then the canonical maps*

$$K_n(A;\mathbb{Z}/p^r) \longrightarrow \pi_n \operatorname{holim}_s K_n(A/I^s;\mathbb{Z}/p^r) \longrightarrow \varprojlim_s K_n(A/I^s;\mathbb{Z}/p^r)$$

are isomorphisms for all $n \geq 0$, $r \geq 1$.

In the case in which A/I is regular, this was proved earlier by Geisser and Hesselholt [2].

2.2. **The infinitesimal part of the weak Lefschetz conjecture for Chow groups.** Let X be a smooth, projective, d-dimensional variety over a perfect field of characteristic p, and $Y \hookrightarrow X$ a smooth ample divisor. Then the Lefschetz conjecture for Chow groups predicts that the restriction map $CH^n(X) \to CH^n(Y)$ is an isomorphism rationally if $2n < d-1$. According to the philosophy of Bloch, Esnault, and Kerz proposed in [1], such conjectures can be split into an algebrisation and an infinitesimal part by noting that the map may be rewritten (by the Bloch–Quillen formula) as $H^i_{\mathrm{Zar}}(X,\mathcal{K}_{n,X}) \to H^i_{\mathrm{Zar}}(Y,\mathcal{K}_{n,Y})$, which factors through $\varprojlim_s H^i_{\mathrm{Zar}}(Y_s,\mathcal{K}_{n,Y_s})$, i.e., the "Chow group of the formal completion of X along Y". The second application of Theorem 1 resolves the infinitesimal part of the conjecture:

Theorem 3. *The restriction map $\varprojlim_s H^i_{\mathrm{Zar}}(Y_s,\mathcal{K}_{n,Y_s}) \to H^i_{\mathrm{Zar}}(Y,\mathcal{K}_{n,Y})$ is an isomorphism rationally if $2n < d-1$. In fact, whenever $i+n < d-1$ the restriction map of pro abelian groups*

$$\{H^i_{\mathrm{Zar}}(Y_s,\mathcal{K}_{n,Y_s})\}_s \longrightarrow \{H^i_{\mathrm{Zar}}(Y,\mathcal{K}_{n,Y})\}_s$$

has kernel and cokernel killed by a power of p.

2.3. **Kato–Saito's conjecture.** Let X be a smooth, d-dimensional variety over a perfect field of characteristic p. A standard consequence of Gersten's conjecture (or of the structure of $\mathcal{K}_n$ as a homotopy invariant presheaf with transfer) is that the canonical maps $H^i_{\mathrm{Zar}}(X,\mathcal{K}_n) \to H^i_{\mathrm{Nis}}(X,\mathcal{K}_{n,\mathrm{Nis}})$ are isomorphisms for all $i,n \geq 0$, and similarly for Milnor K-theory.

Now let $Y \hookrightarrow X$ be a normal crossing divisor. Then it was conjectured by Kato and Saito [5, pg. 256], as part of their higher dimensional class field theory, that the analogous maps

$$\varprojlim_s H^i_{\mathrm{Zar}}(X,\mathcal{K}^M_{n,(X,Y_s)}) \longrightarrow \varprojlim_s H^i_{\mathrm{Nis}}(X,\mathcal{K}^M_{n,(X,Y_s),\mathrm{Nis}})$$

would also be isomorphisms if the base field were finite and $i = n = d$. The new theory of reciprocity sheaves [4] even predicts that $H^i_{\mathrm{Zar}}(X,\mathcal{K}^M_{n,(X,Y_s)}) \xrightarrow{\simeq} H^i_{\mathrm{Nis}}(X,\mathcal{K}^M_{n,(X,Y_s),\mathrm{Nis}})$ for each fixed $s \geq 1$.

The final application of Theorem 1 which we present here is a p-adic form of Kato–Saito's conjecture which is valid in all degrees but only established at this time for Quillen K-theory:

Theorem 4. *With X, Y as immediately above, the canonical map of pro abelian groups*

$$\{H^i_{\mathrm{Zar}}(X, \mathcal{K}_{n,(X,Y_s)}/p^r)\}_s \xrightarrow{\simeq} \{H^i_{\mathrm{Nis}}(X, \mathcal{K}_{n,(X,Y_s),\mathrm{Nis}}/p^r)\}_s$$

is an isomorphism for all $i, n \geq 0$, $r \geq 1$.

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Remarks on Grothendieck's standard conjecture of type D and on Voevodsky's nilpotence conjecture

Gonçalo Tabuada

Let k be a base field of characteristic zero. Given a smooth projective k-scheme X, let us denote by $Z^*(X)_\mathbb{Q}$ the $\mathbb{Q}$-vector space of algebraic cycles on X. Following Grothendieck [2] (see also Kleiman [5, 6]), the standard conjecture of type D, denoted by $D(X)$, asserts that $Z^*(X)_\mathbb{Q}/_{\sim \mathrm{hom}} = Z^*(X)_\mathbb{Q}/_{\sim \mathrm{num}}$, where the homological equivalence relation $_{\sim \mathrm{hom}}$ is taken with respect to a classical Weil cohomology theory. Following Voevodsky [17], the nilpotence conjecture, denoted by $V(X)$, asserts that $Z^*(X)_\mathbb{Q}/_{\sim \mathrm{nil}} = Z^*(X)_\mathbb{Q}/_{\sim \mathrm{num}}$, where $_{\sim \mathrm{nil}}$ stands for the smash-nilpotence equivalence relation introduced in *loc. cit.* By construction, we have $V(X) \Rightarrow D(X)$. Thanks to the work of Lieberman [13], the conjecture $D(X)$ holds when $\dim(X) \leq 4$ or when X is an abelian variety. In the same vein, thanks to the work of Kahn-Sebastian [3], Voevodsky [17], and Voisin [18], the conjecture $V(X)$ holds when $\dim(X) \leq 2$ or when X is an abelian 3-fold. Beyond the aforementioned cases, the preceding conjectures remain wide open.

A *differential graded (=dg) category* $\mathcal{A}$ is a category enriched over complexes of k-vector spaces; consult Keller's ICM survey [4]. Every (dg) k-algebra A gives naturally rise to a dg category with a single object. Another source of examples is provided by schemes since the category of perfect complexes $\mathrm{perf}(X)$ of every quasi-compact quasi-separated k-scheme X admits a canonical dg enhancement[1] $\mathrm{perf}_{\mathrm{dg}}(X)$. Following Kontsevich [7, 8, 9], a dg category $\mathcal{A}$ is called *smooth* if it is compact as a bimodule over itself and proper if $\sum_n \dim H^n\mathcal{A}(x,y) < \infty$ for every

[1] When X is quasi-projective this dg enhancement is unique; see Lunts-Orlov [14, Thm. 2.12].

pair of objects (x, y). Examples include the dg categories $\mathrm{per}_{\mathrm{dg}}(X)$ associated to smooth proper (quasi-compact quasi-separated) k-schemes X.

Given a smooth proper dg category $\mathcal{A}$, let $K_0(\mathcal{A})_{\mathbb{Q}}$ be the $\mathbb{Q}$-linearization of the Grothendieck group of $\mathcal{A}$. Following [1, 16], this $\mathbb{Q}$-vector space comes equipped with a smash-nilpotence equivalence relation $\sim_{\mathrm{nil}}$, with an homological equivalence relation $\sim_{\mathrm{hom}}$, and also with a numerical equivalence relation $\sim_{\mathrm{num}}$. Motivated by the original conjectures of Grothendieck and Voevodsky, we hence introduce the following (noncommutative) conjectures:

Conjecture $D_{\mathrm{nc}}(\mathcal{A})$*:* We have $K_0(\mathcal{A})_{\mathbb{Q}}/_{\sim \mathrm{hom}} = K_0(\mathcal{A})_{\mathbb{Q}}/_{\sim \mathrm{num}}$.

Conjecture $V_{\mathrm{nc}}(\mathcal{A})$*:* We have $K_0(\mathcal{A})_{\mathbb{Q}}/_{\sim \mathrm{nil}} = K_0(\mathcal{A})_{\mathbb{Q}}/_{\sim \mathrm{num}}$.

Similarly to the commutative case, we have $V_{\mathrm{nc}}(\mathcal{A}) \Rightarrow D_{\mathrm{nc}}(\mathcal{A})$. Our first main result, whose proof makes use of the recent theory of noncommutative motives [15], is the following (consult [1, Thm. 1.1] and [16, Thm. 1.1]):

Theorem 1. *Given a smooth projective k-scheme X, we have the equivalences of conjectures $D(X) \Leftrightarrow D_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(X))$ and $V(X) \Leftrightarrow V_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(X))$.*

Roughly speaking, Theorem 1 extends Grothendieck's standard conjecture of type D and Voevodsky's nilpotence conjecture from the realm of schemes to the broad setting of dg categories. This noncommutative viewpoint enables the proof of the original conjectures of Grothendieck and Voevodsky in several (new) cases. Here are some families of examples:

Quadric fibrations. Let S be a smooth projective k-scheme and $q\colon Q \to S$ a flat quadric fibration of relative dimension d. Recall from Kuznetsov [11, §3] the construction of the sheaf $\mathcal{C}l_0(q)$ of even Clifford algebras associated to q. Our second main result is the following (consult [1, Thm. 1.2] and [16, Thm. 1.2]):

Theorem 2. *We have the following equivalences of conjectures:*

$$D(Q) \Leftrightarrow D_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(S, \mathcal{C}l_0(q))) + D(S) \quad V(Q) \Leftrightarrow V_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(S, \mathcal{C}l_0(q))) + V(S)\,.$$

Moreover, when d is even, the discriminant divisor of q is smooth, and $\dim(S) \leq 4$, resp. $\dim(S) \leq 2$, the conjecture $D(Q)$, resp. $V(Q)$, holds.

Homological projective duality. Let V be a finite dimensional k-vector space and X a smooth projective k-scheme equipped with a map $f\colon X \to \mathbb{P}(V)$; we write $\mathcal{O}_X(1)$ for the line bundle $f^*\mathcal{O}_{\mathbb{P}(V)}(1)$. Assume that the triangulated category $\mathrm{perf}(X)$ admits a Lefschetz decomposition $\langle \mathbb{A}_0, \mathbb{A}_1(1), \ldots, \mathbb{A}_{i-1}(i-1)\rangle$ with respect to $\mathcal{O}_X(1)$ in the sense of Kuznetsov [12, Def. 4.1]. Following [12, Def. 6.1], let $Y \to \mathbb{P}(V^*)$ be the HPD-dual[2] of X. Given a generic linear subspace $L \subset V^*$, consider the linear sections $X_L := X \times_{\mathbb{P}(V)} \mathbb{P}(L^\perp)$ and $Y_L := Y \times_{\mathbb{P}(V^*)} \mathbb{P}(L)$. Our third main result is the following (consult [1, Thm. 1.12] and [16, Thm. 1.4]):

Theorem 3. *Let X and Y be as above. Assume that the linear sections X_L and Y_L are smooth, that $\mathrm{codim}(X_L) = \dim(L)$ and $\mathrm{codim}(Y_L) = \dim(L^\perp)$, and that*

[2] The HPD-dual Y of X is in general a noncommutative variety in the sense of [10, §2.4].

the conjecture $D_{\mathrm{nc}}(\mathbb{A}_{0,\mathrm{dg}})$*, resp.* $V_{\mathrm{nc}}(\mathbb{A}_{0,\mathrm{dg}})$*, holds*[3]*. Under these assumptions, we have the equivalence of conjectures* $D(X_L) \Leftrightarrow D(Y_L)$*, resp.* $V(X_L) \Leftrightarrow V(Y_L)$*.*

Intuitively speaking, Theorem 3 shows that Grothendieck's standard conjecture of type D and Voevodsky's nilpotence conjecture are invariant under homological projective duality. All the assumptions of Theorem 3 are known to hold in the case of linear duality, Veronese-Clifford duality, Grassmannian-Pfaffian duality, spinor duality, Segre-determinantal duality, etc; consult Kuznetsov's ICM survey [10] and the references therein. In the particular case of the Veronese-Clifford duality, we hence conclude, for example, that the intersection of up to five quadric hypersurfaces in an odd-dimensional ambient projective space satisfies Grothendieck's standard conjecture of type D; consult [1, Thm. 1.4]. In the same vein, the intersection of up to three quadric hypersurfaces in an odd-dimensional ambient projective space satisfies Voevodsky's nilpotence conjecture; consult [16, Thm. 1.7].

Stacks. Theorem 1 allows us to easily extend the conjectures of Grothendieck and Voevodsky from schemes to stacks. Concretely, given a smooth proper Deligne-Mumford stack $\mathcal{X}$, let $D(\mathcal{X}) := D_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(\mathcal{X}))$ and $V(\mathcal{X}) := V_{\mathrm{nc}}(\mathrm{perf}_{\mathrm{dg}}(\mathcal{X}))$. These extended conjectures can be proved, for example, in the case of intersections of bilinear divisors. Let W be a finite dimensional k-vector space, and $\mathcal{X}$ the associated smooth proper Deligne-Mumford stack $(\mathbb{P}(W) \times \mathbb{P}(W))/\mu_2$ equipped with the map $\mathcal{X} \to \mathbb{P}(S^2W), ([w_1],[w_2]) \mapsto [w_1 \otimes w_2 + w_2 \otimes w_1]$. Given a generic linear subspace $L \subset S^2W^*$, the linear section $\mathcal{X}_L$ corresponds to the intersection of the $\dim(L)$ bilinear divisors in $\mathcal{X}$ parametrized by L. Our fourth main result is the following (consult [16, Thm. 1.10]):

Theorem 4. *Assume that* $\dim(W)$ *is odd and that* $\dim(L) \leq 5$*, resp.* $\dim(L) \leq 3$*. Under these assumptions, the conjecture* $D(\mathcal{X}_L)$*, resp.* $V(\mathcal{X}_L)$*, holds.*

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On the standard conjecture of Hodge type for abelian varieties

Giuseppe Ancona

The aim of this talk is to give some new (and very partial) results on a classical conjecture on algebraic cycles, due to Grothendieck, called standard conjecture of Hodge type. The text is divided in four sections. In the first two we recall the conjecture and give the list of known results. In the last two we state our contribution and give some ideas of its proof.

In what follows k is a base field and (X, L) is a polarized smooth, projective and geometrically connected variety over k of dimension g. By algebraic cycle we will mean an algebraic cycle with rational coefficients.

1. The Conjecture

Consider the algebraic cycles of codimension i whose cohomology class belongs to the primitive cohomology $H^{2i,prim}_{\ell}(X)$. Consider on these cycles the numerical equivalence and let $V^i(X)$ be the quotient vector space[1]. For $2i \leq g$, we can define a pairing

$$P : V^i(X) \times V^i(X) \to \mathbb{Q}$$

$$Z_1, Z_2 \mapsto (-1)^i \cdot \deg(Z_1 \cdot Z_2 \cdot L^{g-2i}).$$

The standard conjecture of Hodge type predicts that this pairing is positive definite.

[1] A priori this space (as well as the conjecture) depends on ℓ. When $X = A$ is an abelian variety the independency of ℓ is known.

1.1. **Known cases.** When Grothendieck formulated the conjecture he knew the following three facts.

(1) The conjecture holds in characteristic zero, as a consequence of the Hodge Index Theorem.
(2) The conjecture holds for surfaces, by Segre, Hodge et al.
(3) If we know the conjecture for a fiber of a family, then we know it for the generic fiber, in particular the conjecture is reduced to the case when k is finite.

To the author's knowledge the only result in the literature which has been added to this list since then is a theorem of Milne [1].

2. Milne's work

Definition 1. *Define $L^i(X) \subset V^i(X)$ to be the subspace of cycles that can be written as linear combinations of intersections of divisors. These cycles are called Lefschetz classes. Define $E^i(X) \subset V^i(X)$ to be the subspace orthogonal to $L^i(X)$ with respect to the pairing P. These cycles are called exotic classes.*

Theorem 1. *(Milne [1]) Let $X = A$ be an abelian variety, then the pairing P restricted to $L^i(A)$ is positive definite.*

Thanks to Milne's result, in order to fully prove the conjecture for abelian varieties one is reduced to study the pairing P restricted to the space $E^i(A)$.

3. Main result

Theorem 2. *Suppose that k is finite and $X = A$ is a simple, ordinary, polarized abelian variety. Let K be the biggest totally real field inside $End(A)_{\mathbb{Q}}$ and $\widetilde{K}$ its Galois closure over $\mathbb{Q}$. If $[\widetilde{K} : \mathbb{Q}]$ is odd, then (for all integers i and n) the pairing P restricted to the space $E^i(A^n)$ is not negative definite.*

Some comments on this statement.

- Because of 1.1(3), it is very natural to assume k finite.
- Ordinary abelian varieties form an open dense subset of the moduli space of all abelian varieties (of given positive characteristic), so this is also a mild hypothesis.
- Asking that $[\widetilde{K} : \mathbb{Q}]$ is odd is a very restrictive hypothesis. It implies in particular that A is of odd dimension.
- It can happen that the space of exotic classes is reduced to zero (in which case the result of Milne suffices for the conjecture). As soon as this space is not reduced to zero our theorem produces a new class on which the pairing P takes positive value.
- There are examples of abelian varieties satisfying our hypothesis and for which there are (or there should be) some exotic classes. In a 1975 letter to Masser, Serre gives an example of a nine dimensional ordinary abelian variety with exotic classes and such that $End(A)_{\mathbb{Q}} = \mathbb{Q}(\zeta_{19})$, which implies $\widetilde{K} = K = \mathbb{Q}(\zeta_{19}) \cap \mathbb{R}$ and $[\widetilde{K} : \mathbb{Q}] = 9$.

4. On the strategy

4.1. **Milne's strategy.** Suppose that our abelian variety A lifts to a complex abelian variety $\widetilde{A}$ and all divisors also lift. Then Milne's theorem clearly holds because the intersection products one has to compute can be computed on $\widetilde{A}$, where one can use Hodge Index Theorem.

In general one cannot find such a $\widetilde{A}$ but (roughly speaking) for each divisor D_i on A one can find, by Honda-Tate, $(\widetilde{A}_i, \widetilde{D}_i)$ over $\mathbb{C}$ lifting (A, D_i). In order to put the informations of each $\widetilde{D}_i$ together Milne uses a Tannakian argument.

4.2. **Some remarks.** If one was able to lift also the exotic classes to $\mathbb{C}$, Milne's argument would go through also in this case. For example, he shows that Hodge conjecture for complex abelian varieties implies the standard conjecture of Hodge type for abelian varieties. Unfortunately lifting exotic classes seems a problem out of reach with the present technology.

Our starting point is a simple remark. Hodge Index Theorem is a cohomological theorem which gives information on all classes of singular cohomology (and not just algebraic ones). Then understanding how far an algebraic class in $H^{2i}_{\ell}(A)$ is rational (with respect to the rational structure given by the singular cohomology of $\widetilde{A}$) will give informations on the signature of P.

4.3. **Our strategy.** The idea is to compare two $\mathbb{Q}$-quadratic forms, q_1 given by P on $V^i(A)$ and the q_2 given by the analogous pairing on the singular cohomology of $\widetilde{A}$. Then one does the following steps.

(1) Compare $q_1 \otimes \mathbb{Q}_p$ with $q_2 \otimes \mathbb{Q}_p$ (for each p). When the prime is different from the characteristic this will be rather formal. For p equal to the characteristic we will use that A is ordinary.
(2) Compute $q_2 \otimes \mathbb{R}$. This is done via the Hodge Index Theorem.
(3) Write the classical product formula on Hilbert symbols for q_1 and q_2. Thanks to the previous steps all the terms in this product are known except $q_1 \otimes \mathbb{R}$.
(4) To deduce from the product formula that there is at least a vector on which q_1 is positive one will need the odd hypothesis of the theorem.

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On the Chow motive of a 3-fold

Bruno Kahn

Let X be a smooth projective 3-fold over a field k. Assuming the existence of a Chow-Künneth decomposition for its Chow motive $h(X)$

$$h(X) = \bigoplus_{i=0}^{6} h_i(X)$$

we decompose $h_3(X)$ further into two parts

$$h_3(X) = t_3(X) \oplus h_1(J^2)(1)$$

where J^2 is a certain (isogeny class of) abelian variety and $t_3(X)$ is the *transcendental part* of $h_3(X)$. This is a higher-dimensional analogue of the transcendental part of the motive of a surface studied in [2].

The hypothesis on X is satisfied when X is an abelian variety, a complete intersection in projective space or the product of a curve and a surface.

Let Ab^2 be the abelian variety constructed by Murre in [3]: it is the universal abelian variety receiving a regular morphism from algebraically trivial cycles of codimension 2 on X. There is a surjection $\mathrm{Ab}^2 \twoheadrightarrow J^2$, which is an isomorphism under the generalised Hodge or Tate conjecture for $H^3(X)$. (The latter are verified when X is a product of 3 elliptic curves.)

The main result of the talk is an isomorphism

$$\operatorname{Griff}(X) \simeq \mathcal{M}_{\mathrm{alg}}(\mathbb{L}, t_3(X)). \qquad (1)$$

Here $\operatorname{Griff}(X)$ is the group of numerically trivial cycles of codimension 2 on X, modulo algebraic equivalence, $\mathcal{M}_{\mathrm{alg}}$ is the category of pure motives modulo algebraic equivalence and $\mathbb{L}$ is the Lefschetz motive. (Since we work with $\mathbf{Q}$ coefficients, the above isomorphism is to be taken up to isogeny.)

The Bloch-Beilinson–Murre conjectures on filtrations on Chow groups imply that in fact

$$\mathcal{M}(\mathbb{L}, t_3(X)) \xrightarrow{\sim} \mathcal{M}_{\mathrm{alg}}(\mathbb{L}, t_3(X))$$

where $\mathcal{M}$ is the category of Chow motives. This would imply the following striking fact: up to isogeny, the subgroup of $CH^2(X)$ consisting of cycles algebraically equivalent to 0 is the image of an idempotent algebraic correspondence.

In the special case $X = C \times S$, where C is a curve and S is a surface, (1) yields a surjection (up to isogeny)

$$T(S_{k(C)})/T(S) = \mathcal{M}(h_1(C), t_2(S)) \twoheadrightarrow \mathcal{M}_{\mathrm{alg}}(h_1(C), t_2(S)) = \operatorname{Griff}(C \times S)$$

where $T(S)$ is the Albanese kernel of S. This provides a relationship between the Albanese kernel and the Griffiths group.

Let

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{\ \pi\ } & S \\ {\scriptstyle f}\downarrow & & \\ D & & \end{array}$$

be a Lefschetz pencil, with generic fibre Γ. It seems that the methods of [1], combined with the absolute irreducibility of the monodromy on the vanishing cohomology of $H^1(\Gamma)$ imply the following: if k is infinite but finitely generated over its prime field, there is a thin subset $T \subset D(k)$ such that $0 \neq \widetilde{\iota}_u \in \mathrm{Griff}(\Gamma_u \times S)$ for $u \notin T$, where $\Gamma_u = f^{-1}(u)$ and $\widetilde{\iota}_u$ is the class of the graph of the closed immersion $\iota_u : \Gamma_u \hookrightarrow S$, modified by the Chow-Künneth projectors of Γ_u and S defining $h_1(\Gamma_u)$ and $t_2(S)$.

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Anabelian geometry with étale homotopy types

Alexander Schmidt

(joint work with Jakob Stix)

Grothendieck's anabelian philosophy [Gr83] predicts the existence of a class of *anabelian* varieties X that are reconstructible from their étale fundamental group $\pi_1^{\mathrm{et}}(X, \bar{x})$. All examples of anabelian varieties known so far are of type $K(\pi, 1)$, i.e., their higher étale homotopy groups vanish. For general varieties X, the homotopy theoretic viewpoint suggests to ask the modified question, whether they are reconstructible from their *étale homotopy type* X_{et} instead of only $\pi_1^{\mathrm{et}}(X, \bar{x})$. For varieties X of type $K(\pi, 1)$ this makes no difference since then X_{et} is weakly equivalent to the classifying space $B\pi_1^{\mathrm{et}}(X, \bar{x})$.

Recall that the *étale topological type* X_{et} of a scheme X is an object in $\mathsf{pro\text{-}ss}$, the pro-category of simplicial sets. Any geometric point $\bar{x}$ of X defines a point $\bar{x}_{\mathrm{et}}$ on X_{et}. If X is locally noetherian, the fundamental group $\pi_1(X_{\mathrm{et}}, \bar{x}_{\mathrm{et}})$ is the usual (in the sense of [SGA3] X §6) étale fundamental group $\pi_1^{\mathrm{et}}(X, \bar{x})$ and the higher homotopy groups of X_{et} are the higher étale homotopy groups of X by definition, cf. [AM69], [Fr82]. Isaksen [Is01] defined a model structure on $\mathsf{pro\text{-}ss}$ and we denote the associated homotopy category by $\mathrm{Ho}(\mathsf{pro\text{-}ss})$. When considered as an object of $\mathrm{Ho}(\mathsf{pro\text{-}ss})$, we refer to X_{et} as the *étale homotopy type* of X. For a pro-simplicial set B, we denote the category of morphisms to B in $\mathrm{Ho}(\mathsf{pro\text{-}ss})$ by $\mathrm{Ho}(\mathsf{pro\text{-}ss}) \downarrow B$.

For the rest of this talk the letter k always denotes a *finitely generated field extension of* $\mathbb{Q}$.

In the language of étale homotopy theory, the theorem of Mochizuki and Tamagawa on the anabelian geometry of hyperbolic curves [Mo99, Ta97] can be reformulated as follows.

Theorem 1 ([SS15], Thm. 1.1)**.** *Let X and Y be smooth hyperbolic curves over k. Then the natural map*

$$\mathrm{Isom}_k(X,Y) \longrightarrow \mathrm{Isom}_{\mathrm{Ho}(\textsf{pro-ss})\downarrow k_{\mathrm{et}}}(X_{\mathrm{et}}, Y_{\mathrm{et}})$$

is bijective.

Theorem 2 below constitutes a first step towards a generalisation of Theorem 1 to higher dimensional varieties.

Theorem 2 ([SS15], Thm. 1.2)**.** *Let X and Y be smooth, geometrically connected varieties over k which can be embedded as locally closed subschemes into a product of hyperbolic curves over k. Then the natural map*

$$(*) \qquad \mathrm{Isom}_k(X,Y) \longrightarrow \mathrm{Isom}_{\mathrm{Ho}(\textsf{pro-ss})\downarrow k_{\mathrm{et}}}(X_{\mathrm{et}}, Y_{\mathrm{et}})$$

is a split injection with a functorial retraction

$$r : \mathrm{Isom}_{\mathrm{Ho}(\textsf{pro-ss})\downarrow k_{\mathrm{et}}}(X_{\mathrm{et}}, Y_{\mathrm{et}}) \longrightarrow \mathrm{Isom}_k(X,Y).$$

We obtain the following weakly anabelian statement as a trivial corollary.

Corollary 3. *Let X and Y be smooth, geometrically connected varieties over k which can be embedded as locally closed subschemes into a product of hyperbolic curves over k.*

If $X_{\mathrm{et}} \cong Y_{\mathrm{et}}$ in $\mathrm{Ho}(\textsf{pro-ss}) \downarrow k_{\mathrm{et}}$, then X and Y are isomorphic as k-varieties.

Remark 4. *By* [Is04]*, the functor $X \mapsto X_{\mathrm{et}}$ from smooth k-schemes to* $\mathrm{Ho}(\textsf{pro-ss}) \downarrow k_{\mathrm{et}}$ *factors through the $\mathbb{A}^1$-homotopy category of Morel and Voevodsky* [MV99]*. In particular, it is not faithful. However, this does not affect Theorem 2 since the schemes occurring there are $\mathbb{A}^1$-local.*

In order to investigate whether the map $(*)$ of Theorem 2 is bijective, we may (by Corollary 3) assume that $X = Y$ and consider the kernel of the retraction r. A first piece of information is provided by

Theorem 5 ([SS15], Thm. 1.9)**.** *Let X be a smooth, geometrically connected variety over k which can be embedded as a locally closed subscheme into a product of hyperbolic curves over k. Let γ be in the kernel of the retraction map of Theorem 2:*

$$r : Aut_{\mathrm{Ho}(\textsf{pro-ss})\downarrow k_{\mathrm{et}}}(X_{\mathrm{et}}) \longrightarrow Aut_k(X).$$

Then the induced automorphism $\pi_1(\gamma) \in \mathrm{Aut}^{\mathrm{out}}_{Gal_k}(\pi_1^{\mathrm{et}}(X))$ is class-preserving.

We get more information if X is a *strongly hyperbolic Artin neighbourhood*:

Definition 6. *A* strongly hyperbolic Artin neighbourhood *is a smooth variety X over k such that there exists a sequence of morphisms*

$$X = X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = Spec(k)$$

such that for all i

(i) *the morphism* $X_i \to X_{i-1}$ *is an elementary fibration into hyperbolic curves, and*

(ii) X_i *admits an embedding into a product of hyperbolic curves.*

Theorem 7 ([SS15], Thm. 6.2). *Let* X *be a strongly hyperbolic Artin neighbourhood over* k *and let* $\gamma \in Aut_{\mathrm{Ho(pro\text{-}ss)}\downarrow k_{\mathrm{et}}}(X_{\mathrm{et}})$ *be an automorphism with* $r(\gamma) = \mathrm{id}_X$*. Then* $\pi_1(\gamma) \in \mathrm{Aut}^{\mathrm{out}}_{Gal_k}(\pi_1^{\mathrm{et}}(X))$ *is the identity.*

Since strongly hyperbolic Artin-neighbourhoods are of type $K(\pi, 1)$, we can deduce:

Corollary 8. *Let* X *and* Y *be strongly hyperbolic Artin neighbourhoods over* k*. Then the natural map*

$$\mathrm{Isom}_k(X, Y) \longrightarrow \mathrm{Isom}_{\mathrm{Ho(pro\text{-}ss)}\downarrow k_{\mathrm{et}}}(X_{\mathrm{et}}, Y_{\mathrm{et}}) = \mathrm{Isom}^{\mathrm{out}}_{Gal_k}\left(\pi_1^{\mathrm{et}}(X), \pi_1^{\mathrm{et}}(Y)\right)$$

is bijective.

We remark that by different techniques Hoshi proves in [Ho14] §3 a statement similar to Corollary 8 but restricted to dimension ≤ 4. Corollary 8 implies the following statement predicted by Grothendieck in his letter to Faltings [Gr83]:

Corollary 9. *Let* X *be a smooth, geometrically connected variety over* k*. Then every point of* X *has a basis of Zariski-neighbourhoods consisting of anabelian varieties, in the sense that* k*-isomorphisms between any two of these are in bijection with outer* G_k*-isomorphisms of their respective étale fundamental groups.*

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Motives with modulus

Takao Yamazaki
(joint work with Bruno Kahn, Shuji Saito)

We report our attempt [2] to generalize Voevodsky's triangulated category $\mathbf{DM}_{\mathrm{gm}}^{\mathrm{eff}}$ of mixed motivs over a field [6] in such a way as to emcompass non-homotopy invariant phenomena. Here a traditional notion of "modulus" plays a central role. In the first section, we review how the concept of modulus was used in the theories of generalized Jacobian and Albanese. In the second section, after a brief review of Voevodsky's $\mathbf{DM}_{\mathrm{gm}}^{\mathrm{eff}}$, we survey our generalization of $\mathbf{DM}_{\mathrm{gm}}^{\mathrm{eff}}$ "with modulus". Throughout, we work over a perfect field k.

1. Modulus in the theory of generalized Jacobian and Albanese

1.1. **Generalized Jacobian.** Let C be a smooth proper curve over k, D an effective divisor on it, and x_0 a (fixed) k-rational point on $C \setminus |D|$. The generalized Jacobian $J(C, D)$ of C with modulus D is a commutative algebraic group over k that can be characterized by the universality explained in the next subsection [5]. Its group of k-rational points admits a description

$$J(C, D)(k) \cong \mathrm{Div}^0(C \setminus |D|)/\{\mathrm{div}(f) \mid f \in k(C)_D^\times\},$$

where $k(C)_D^\times := \{f \in k(C)^\times \mid f \equiv 1 \mod D\}$. The following statemens hold.

- If $D = 0$, $J(C, D)$ is the classical Jacobian of C (hence an abelian variety).
- If D is reduced, $J(C, D)$ is semi-abelian. The converse holds too.

1.2. **Universality.** Let G be a commutative algebraic group over k and let $f : C \setminus |D| \to G$ be a k-morphism. Given $E = \sum_i n_i x_i \in \mathrm{Div}(C \setminus |D|)$, we write $f(E) := \sum_i \mathrm{Tr}_{k(x_i)/k} n_i f(x_i) \in G(k)$. A divisor D' such that $|D'| \subset |D|$ is called a modulus for f if $f(\mathrm{div}(g)) = 0$ holds for any $g \in k(C)_{D'}^\times$. There exists the minimum effective divisor $\mathrm{Mod}(f)$ that is a modulus for f. For example, one has $\mathrm{Mod}(f) = 0$ if f can be extended to a morphism $C \to G$. Thus, if G is an abelian variety, one always has $\mathrm{Mod}(f) = 0$. Similarly, if G is semi-abelian, $\mathrm{Mod}(f)$ is always reduced. If G is not semi-abelian, $\mathrm{Mod}(f)$ can be non-reduced in general.

Now we can state the universality. The generalized Jacobian $J(C, D)$ comes equipped with a k-morphism $\iota : C \setminus |D| \to J(C, D)$ such that $\iota(x_0) = 0$ and $\mathrm{Mod}(f) \leq D$. (On rational points, ι simply sends x to the class of $x - x_0$.) Any k-morphism $f : C \setminus |D| \to G$ with $f(x_0) = 0$ and $\mathrm{Mod}(f) \leq D$ uniquely factors as $f = g \circ \iota$ for some morphism $g : J(C, D) \to G$ of commutative algebraic groups.

1.3. **Generalized Albanese.** In a series of papers published in 2008–2013 (see [4] and references therein), Kato and Russell generalized this theory to higher dimensional varieties. Indeed, everything stated in the previous subsection remains true when C is replaced by any smooth projective variety, except that the definition of $\mathrm{Mod}(f)$ should be replaced by a new, much more involved one. Note that for semi-abelian varieties (corresponding to the case of reduced D) this is already established in 1958 by Serre, and of course for abelian varieties (correspoinding to the case $D = 0$) it goes back further to the classical theory of Albanese varieties. Nevertheless a generalization to non-reduced D has emerged only in 21st century.

2. Motives with modulus

2.1. **Voevodsky's motive and generalized Jacobian.** Voevodsky's definition of $\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ is, against expectations, rather simple. We first introduce a category **Cor** of finite correspondences. It is an additive category having the same objects as **Sm** (the category of smooth varieties) and finite correspondences as morphisms. Then $\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ is defined to be the pseudo-abelian envelope of the localization of the homotopy category of bounded complexes $K^b(\mathbf{Cor})$ by two relations arising from Nisnevich Mayer-Vietoris and homotopy invariance. For each $X \in \mathbf{Sm}$, there is an object $M(X) \in \mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ called the motive of X. Let $X, Y \in \mathbf{Sm}$ and assume that X is proper and equidimensional of dimension d. Then one has

$$\mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}}(M(Y), M(X)[-j]) \cong CH^d(X \times Y, j).$$

This is one of the most important properties of Voevodsky's category $\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$.

A contravariant functor from **Cor** to **Ab** that is a sheaf for Nisnevich topology is called a Nisnevich sheaf with transfers. They form an abelian category **NST**. The Yoneda functor induces a full faithful functor $i : \mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}} \to D(\mathbf{NST})$. A Nisnevich sheaf with transfers F is said to be homotopy invariant if $F(X) \cong F(X \times \mathbb{A}^1)$ for any smooth variety X. For any $K \in \mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$, all homology sheaves of $i(K)$ are homotopy invariant. Now let us consider a smooth proper connected curve C having a k-rational point, and a non-empty effective reduced divisor D on it. Then one has $i(M(C \setminus |D|)) \cong (J(C, D) \oplus \mathbb{Z})[0]$, where $J(C, D)$ is identified with the Nisnevich sheaf with transfers represented by it. Since a commutative algebraic group (regarded as an object of **NST**) is homotopy invariant if and only if it is semi-abelian, $J(C, D)$ belongs to $\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ if and only if D is reduced.

2.2. **Motives with modulus.** We construct our "modulus versions" of categories **Sm**, **Cor**, $\mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ and **NST**, which are denoted by **MSm**, **MCor**, $\mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}}$ and **MNST**. An object (X, D) of **MSm** is called a modulus pair and consists of a locally integral k-variety X and an effective Cartier divisor D such that $X \setminus |D| \in \mathbf{Sm}$. Starting from this category, we develop construction of categories **MCor**, $\mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}}$, **MNST** and of functors $M : \mathbf{MSm} \to \mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}}$, $i : \mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}} \to D(\mathbf{MNST})$. Roughly speaking it is done by following Voevodsky's strategy, but largely different in details, for which we refer the readers to [2].

The functor $\omega : \mathbf{MSm} \to \mathbf{Sm}, \omega(X, D) = X \setminus |D|$ induces $\omega_{\mathrm{gm}} : \mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}} \to \mathbf{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ and $\omega_! : D(\mathbf{MNST}) \to D(\mathbf{NST})$. They provide us with a very useful relation of our categories with Voevodsky's. They are also important ingredients in the proof of the following result: Let X be a smooth proper and equidimensional variety of dimension d, and let (Y, E) be any modulus pair. Then one has

$$\mathrm{Hom}_{\mathbf{MDM}^{\mathrm{eff}}_{\mathrm{gm}}}(M(Y, E), M(X, 0)[-j]) \cong CH^d(X \times (Y \setminus |E|), j).$$

It is an important open question to generalize this formula to the case of non-trivial divisor on X. We expect it should be described by Binda-Saito's higher Chow group with modulus [1].

Now let us consider a smooth proper connected curve C having a k-rational point, and a non-reduced effective divisor D on it. We expect that $\omega_! i(M(C,D)) \cong (J(C,D) \oplus \mathbb{Z})[0]$. Unfortunately we are unable to prove this statement, but an evidence is given in [3]. A more challenging open problem is to describe $i(M(C,D)) \in D(\mathbf{MNST})$ as a complex of $\mathbf{MNST}$.

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Zeta-values of arithmetic schemes

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(joint work with Matthias Flach)

1. Philosophy

This talk is based on the preprint [2]. We conjecture that on the category of arithmetic schemes (separated schemes of finite type over the integers), there exist two cohomologies with compact support:

Weil-Arakelov cohomology with compact support: $R\Gamma_{ar,c}(-, A(n))$ with values in the bounded derived category $\mathrm{D}^b(\mathrm{LCA})$ of the quasi-abelian category LCA of locally compact abelian groups [5], for any $A \in \mathrm{LCA}$ of finite ranks and any $n \in \mathbb{Z}$;

Weil-étale cohomology with compact support: $R\Gamma_{W,c}(-, \mathbb{Z}(n))$ with values in the bounded derived category $\mathrm{D}^b(\mathrm{Ab})$ of abelian groups, for any $n \in \mathbb{Z}$;

such that the following properties hold.

(1) $R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}(n))$ is a perfect complex of abelian groups for any $\mathcal{X}, n$.

(2) $R\Gamma_{ar,c}(\mathcal{X}, \mathbb{R}(n))$ is a perfect complex of $\mathbb{R}$-vector spaces for any $\mathcal{X}, n$.

(3) There is a fundamental class $\theta \in H^1(\overline{\mathrm{Spec}(\mathbb{Z})}, \mathbb{R}(0))$ such that

$$\cdots \xrightarrow{\cup\theta} H^i_{ar,c}(\mathcal{X}, \mathbb{R}(n)) \xrightarrow{\cup\theta} H^{i+1}_{ar,c}(\mathcal{X}, \mathbb{R}(n)) \xrightarrow{\cup\theta} \cdots$$

is an acyclic complex.

(4) Weil-Arakelov cohomology is exact with respect to coefficients: for example we have an exact triangle

$$R\Gamma_{ar,c}(\mathcal{X},\mathbb{Z}(n)) \to R\Gamma_{ar,c}(\mathcal{X},\mathbb{R}(n)) \to R\Gamma_{ar,c}(\mathcal{X},\mathbb{R}/\mathbb{Z}(n)) \to$$

(5) There is a natural transformation $R\Gamma_{ar,c}(-,\mathbb{Z}(n)) \to R\Gamma_{W,c}(-,\mathbb{Z}(n))$ such that, for $\mathcal{X}/\mathbb{Z}$ proper and regular, we have an exact triangle

$$R\Gamma_{dR}(\mathcal{X}_{\mathbb{R}}/\mathbb{R})/F^n[-2] \to R\Gamma_{ar,c}(\mathcal{X},\mathbb{Z}(n)) \to R\Gamma_{W,c}(\mathcal{X},\mathbb{Z}(n)) \to$$

(6) For $\mathcal{X}/\mathbb{Z}$ proper regular of pure dimension d, there is a perfect pairing of locally compact abelian groups

$$H^i_{ar}(\overline{\mathcal{X}},A(n)) \times H^{2d+1-i}_{ar}(\overline{\mathcal{X}},A^D(d-n)) \to H^{2d+1}_{ar}(\overline{\mathcal{X}},\mathbb{R}/\mathbb{Z}(d)) \to \mathbb{R}/\mathbb{Z}$$

where $A^D := \underline{\mathrm{Hom}}(A,\mathbb{R}/\mathbb{Z})$ is the Pontryagin dual, and a quasi-isomorphism of perfect complexes of abelian groups

$$R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z}(n)) \xrightarrow{\sim} R\mathrm{Hom}(R\Gamma_W(\overline{\mathcal{X}},\mathbb{Z}(d-n)),\mathbb{Z}[-2d-1]).$$

(7) For $\mathcal{X}/\mathbb{Z}$ proper regular, there is an isomorphism (induced by generalized versions of (5) and (6))

$$R\Gamma_{ar,c}(\mathcal{X},\mathbb{R}/\mathbb{Z}(n)) \simeq R\Gamma_{W,c}(\mathcal{X},\mathbb{Z}(n)) \otimes^L \mathbb{R}/\mathbb{Z}.$$

(8) The vanishing order of the zeta function at $s=n$ is given by

$$\mathrm{ord}_{s=n}\zeta(\mathcal{X},s) = \sum_{i\in\mathbb{Z}}(-1)^i \cdot i \cdot \dim_{\mathbb{R}} H^i_{ar,c}(\mathcal{X},\mathbb{R}(n)).$$

(9) The special value $\zeta^*(\mathcal{X},n)$ at $s=n$ is given up to sign by

$$\lambda_{X,n}(\zeta^*(\mathcal{X},n)^{-1}\cdot c(\mathcal{X},n)\cdot\mathbb{Z}) = \Delta(\mathcal{X}/\mathbb{Z},n)$$

where $\Delta(\mathcal{X}/\mathbb{Z},n) := (\det_{\mathbb{Z}} R\Gamma_{W,c}(\mathcal{X},\mathbb{Z}(n))) \otimes_{\mathbb{Z}} (\det_{\mathbb{Z}} R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^n)$, the isomorphism $\lambda_{\mathcal{X},n} : \mathbb{R} \simeq \Delta(\mathcal{X}/\mathbb{Z},n)\otimes\mathbb{R}$ is defined below and $c(\mathcal{X},n) \in \mathbb{Q}^*$ is an explicit non-zero rational number, such that $c(\mathcal{X},n)=1$ if $n\leq 0$ and $c(\mathcal{X},n)=1$ if $\mathcal{X}$ has characteristic p.

The trivialization $\lambda_{\mathcal{X},n}$ is defined as follows. For $C \in \mathrm{D}^b(\mathrm{LCA})$ of finite ranks we consider its tangent complex $T_\infty C := R\underline{\mathrm{Hom}}(R\underline{\mathrm{Hom}}(C,\mathbb{R}/\mathbb{Z}),\mathbb{R})$. Property (4) above and the fact that T_∞ is triangulated gives an exact triangle

$$T_\infty R\Gamma_{ar,c}(\mathcal{X},\mathbb{Z}(n)) \to T_\infty R\Gamma_{ar,c}(\mathcal{X},\mathbb{R}(n)) \to T_\infty R\Gamma_{ar,c}(\mathcal{X},\mathbb{R}/\mathbb{Z}(n)) \to$$

By properties (5), (2) and (7), this triangle can be identified with

$$R\Gamma_{dR}(\mathcal{X}_{\mathbb{R}}/\mathbb{R})/F^n[-2] \to R\Gamma_{ar,c}(\mathcal{X},\mathbb{R}(n)) \to R\Gamma_{W,c}(\mathcal{X},\mathbb{Z}(n))_{\mathbb{R}} \to$$

The map $\lambda_{\mathcal{X},n}$ is the following composition of isomorphisms:

$$\lambda_{\mathcal{X},n} : \mathbb{R} \xrightarrow{\sim} \det_{\mathbb{R}} R\Gamma_{ar,c}(\mathcal{X},\mathbb{R}(n))$$

$$\xrightarrow{\sim} (\det_{\mathbb{R}} R\Gamma_{W,c}(\mathcal{X},\mathbb{Z}(n))_{\mathbb{R}}) \otimes_{\mathbb{R}} (\det_{\mathbb{R}} R\Gamma_{dR}(\mathcal{X}_{\mathbb{R}}/\mathbb{R})/F^n) \xrightarrow{\sim} \Delta(\mathcal{X}/\mathbb{Z},n)_{\mathbb{R}}$$

where the first isomorphism is induced by the acyclic complex of Property (3) and the second isomorphism is given by the last exact triangle above. Properties (1)-(9) extend to arbitrary $n\in\mathbb{Z}$ the conjectural picture suggested by Lichtenbaum in [6] for $n=0$.

We now briefly explain a connexion with Deninger's program [1]. Recall that C. Deninger conjectured the existence of a cohomology theory $\mathcal{X} \mapsto H^i_c(\mathcal{X}, \mathcal{C})$ on the category of arithmetic schemes, which takes values in ∞-dimensional $\mathbb{C}$-vector spaces with an endomorphism Θ, such that for $\mathcal{X}$ of dimension d, we have

$$\zeta(\mathcal{X}, s) = \prod_{i=0}^{i=2d} \det_\infty \left(\frac{s-\Theta}{2\pi} \mid H^i_c(\mathcal{X}, \mathcal{C}) \right)^{(-1)^{i+1}}$$

where $\det_\infty$ is a zeta-regularized determinant. We expect a long exact sequence

$$\cdots \to H^i_{ar,c}(\mathcal{X}, \mathbb{C}(n)) \to H^i_c(\mathcal{X}, \mathcal{C}) \xrightarrow{\frac{n-\Theta}{2\pi}} H^i_c(\mathcal{X}, \mathcal{C}) \to H^{i+1}_{ar,c}(\mathcal{X}, \mathbb{C}(n)) \to \cdots$$

such that $\cup\theta : H^i_{ar,c}(\mathcal{X}, \mathbb{C}(n)) \to H^{i+1}_{ar,c}(\mathcal{X}, \mathbb{C}(n))$ coincides with the composite map

$$H^i_{ar,c}(\mathcal{X}, \mathbb{C}(n)) \to \mathrm{Ker}(n-\Theta) \hookrightarrow H^i_c(\mathcal{X}, \mathcal{C}) \to \mathrm{Coker}(n-\Theta) \to H^{i+1}_{ar,c}(\mathcal{X}, \mathbb{C}(n)).$$

Moreover we expect semi-simplicity in the sense that $\mathrm{Ker}(n-\Theta) \hookrightarrow H^i_c(\mathcal{X}, \mathcal{C}) \to \mathrm{Coker}(n-\Theta)$ is an isomorphism. This would explain the vanishing order conjecture (8) and would identify $\lambda_{\mathcal{X},n} \otimes \mathbb{C}$ with an isomorphism

$$\mathbb{C} \xrightarrow{\sim} \bigotimes_i \det_{\mathbb{C}}^{(-1)^i} [H^i_c(\mathcal{X}, \mathcal{C}) \xrightarrow{\frac{n-\Theta}{2\pi}} H^i_c(\mathcal{X}, \mathcal{C})] \xrightarrow{\sim} \Delta(\mathcal{X}/\mathbb{Z}, n)_{\mathbb{C}}$$

where the first isomorphism is induced by semi-simplicity.

2. A conditional result

Let $\mathcal{X}/\mathbb{Z}$ be a proper regular arithmetic scheme of pure dimension d. For $n \geq 0$, we denote by $\mathbb{Z}(n) := z^n(-, 2n - *)$ Bloch's cycle complex of sheaves on the étale site $\mathcal{X}_{et}$. For $n < 0$, we set $\mathbb{Z}(n) := \bigoplus_p j_{p,!} \mathrm{colim}\, \mu_{p^r}^{\otimes n}[-1]$, where p runs over the set of prime numbers and $j_p : \mathcal{X}[1/p] \to \mathcal{X}$ is the open immersion. We also denote by $\mathbb{Z}(n)$ the extension of the motivic complex to the Artin-Verdier étale topos $\overline{\mathcal{X}}_{et}$ defined in ([2] App. A). We set $R\Gamma_{dR}(\mathcal{X}/\mathbb{Z})/F^n := R\Gamma(\mathcal{X}_{Zar}, L\Omega^*_{\mathcal{X}/\mathbb{Z}}/F^n)$ where $L\Omega^*_{\mathcal{X}/\mathbb{Z}}$ denotes Illusie's derived de Rham complex. We need the following

Conjecture AV$(\mathcal{X}, n)$**.** *There is a perfect pairing of finite groups*

$$H^i(\overline{\mathcal{X}}_{et}, \mathbb{Z}/m(n)) \times H^{2d+1-i}(\overline{\mathcal{X}}_{et}, \mathbb{Z}/m(d-n)) \to H^{2d+1}(\overline{\mathcal{X}}_{et}, \mathbb{Z}/m(d)) \to \mathbb{Q}/\mathbb{Z}.$$

Conjecture L$(\mathcal{X}, n)$**.** $H^i(\overline{\mathcal{X}}_{et}, \mathbb{Z}(n))$ *is finitely generated for any* $i \leq 2n+1$.

Conjecture B$(\mathcal{X}, n)$**.** *The pairing*

$$H^i_c(\mathcal{X}, \mathbb{R}(n)) \times H^{2d-i}(\mathcal{X}, \mathbb{R}(d-n)) \to H^{2d}_c(\mathcal{X}, \mathbb{R}(d)) \to \mathbb{R}$$

is a perfect pairing of finite dimensional $\mathbb{R}$*-vector spaces.* Here $H^i_c(\mathcal{X}, \mathbb{R}(n))$ denotes the cohomology of the mapping fiber of the regulator map.

Theorem. *Let $\mathcal{X}/\mathbb{Z}$ be a proper regular scheme of pure dimension d and let $n \in \mathbb{Z}$. If $\mathcal{X}$ satisfies* $\mathrm{AV}(\mathcal{X}, n)$, $\mathrm{L}(\mathcal{X}, n)$, $\mathrm{L}(\mathcal{X}, d-n)$ *and* $\mathrm{B}(\mathcal{X}, n)$ *then there exist Weil-Arakelov complexes* $R\Gamma_{ar,c}(\mathcal{X}, A(n))$ *and* $R\Gamma_{ar}(\overline{\mathcal{X}}, A(n))$ *for* $A = \mathbb{Z}, \mathbb{R}, \mathbb{R}/\mathbb{Z}$ *and Weil-étale complexes* $R\Gamma_{W,c}(\mathcal{X}, \mathbb{Z}(n))$ *and* $R\Gamma_W(\overline{\mathcal{X}}, \mathbb{Z}(n))$*, such that:*

- *Properties (1)–(7) hold.*
- *(8) is equivalent to Soulé's conjecture* [7].
- *If $\mathcal{X}$ has characteristic p, then (9) is equivalent to the Milne-Lichtenbaum-Geisser conjecture* [4].
- *If $\mathcal{X}$ is smooth proper over a number ring $\mathcal{O}_F$, then (9) is compatible with the Bloch-Kato conjecture* [3] *for the L-function* $L(\oplus_i h^i(\mathcal{X}_F)(n)[-i], s)$.

In particular, (1)–(8) hold for $\mathcal{X}/\mathbb{Z}$ proper regular of dimension ≤ 1. If $F/\mathbb{Q}$ is an abelian number field, then (9) holds for $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_F)$ and any $n \in \mathbb{Z}$, and one has $c(\mathcal{X}, n) = (n-1)!^{-[F:\mathbb{Q}]}$ for $n \geq 1$.

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Cyclic homology and rigid cohomology

Guillermo Cortiñas

(joint work with Joachim Cuntz, Ralf Meyer, Georg Tamme)

The problem of defining a cohomology theory with good properties for an algebraic variety over a field k of non-zero characteristic has a long history. In the breakthrough paper [8] by Monsky and Washnitzer, such a theory for smooth affine varieties was constructed as follows. Take a complete discrete valuation ring V with uniformizer π and residue field $k = V/\pi V$ (for example, V the Witt ring $W(k)$ if k is perfect). Let K be the fraction field of V. Choose a V -algebra R which is a lift mod π of the coordinate ring of the variety and which is smooth over V (such a lift exists by [4]). Monsky–Washnitzer then introduce the 'weak' or dagger-completion $R^\dagger$ of R and define their cohomology as the de Rham cohomology of $R^\dagger \otimes_V K$. The construction of a weak completion has become a basis for the definition of cohomology theories in this context ever since. The

Monsky–Washnitzer theory has been generalized by Berthelot [1] to "rigid cohomology," which represents a satisfactory cohomology theory for general varieties and schemes over k.

With the advent of cyclic homology around 1980 it was immediately realized that this represents a new approach to de Rham theory. Periodic cyclic homology is defined for arbitrary non-commutative algebras over a field K of characteristic 0, but when specialized to the coordinate ring of a smooth variety (say over $\mathbb{C}$), it reproduces de Rham theory, see [2],[5], [6]. More generally, it naturally gives the infinitesimal cohomology, in the sense of Grothendieck, for an arbitrary affine variety [5].

Our aim in this project is to develop a version of periodic cyclic theory that reproduces Berthelot's rigid cohomology for commutative algebras, but which is defined also for non-commutative V -algebras. One starting point is the fact that already in the classical case in characteristic 0 it was found that a natural framework for periodic cyclic homology consists of bornological algebras, see [7]. One of our first results shows that $R^\dagger \otimes_V K$ is a special case of a bornological I -adic completion of $R \otimes_V K$ for an ideal I in R. We use Große-Klönne's description of Berthelot's theory to show that in general such bornological I -adic completions can be used very naturally to associate a complete bornological K -algebra B to a given k -algebra A, where K is the quotient field of V, such that the de Rham theory of B gives the rigid cohomology of A. However, to get the analogue of infinitesimal cohomology, we must consider inverse systems representing an 'infinitesimal neighbourhood' of an embedding of B into a smooth algebra. This leads to the use of pro-algebras. Accordingly, we have to use the periodic cyclic homology theory developed in [3] for pro-algebras over K, in order to compare periodic cyclic homology for an arbitrary affine variety over k to its rigid cohomology.

Our main result describes rigid cohomology in terms of cyclic homology, as follows. Given a commutative k -algebra A of finite type, write it as a quotient R/J, where R is a smooth (for example, free commutative) V -algebra. Then the rigid cohomology of A is the periodic cyclic homology of the pro-algebra $\widetilde{R_\infty}$ (this notation here is preliminary) defined by the bornological J^m -adic completions of R. We do this by comparing the cyclic homology of $\widetilde{R_\infty}$ to its de Rham cohomology. The de Rham cohomology of $\widetilde{R_\infty}$ recovers the rigid cohomology of A. This description of $H^*_{\mathrm{rig}}(A)$ is in some sense analogous to Grothendieck's definition of the infinitesimal cohomology of a variety in characteristic 0. However, our definition remains less natural than the description of infinitesimal cohomology via periodic cyclic theory in characteristic 0. This suggests that we may not have the ultimate result.

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On descending cohomology geometrically

Charles Vial
(joint work with Jeffrey Achter and Sebastian Casalaina-Martin)

Recall that, for a smooth complex projective variety X, the intermediate Jacobian is a complex torus defined by

$$J^{2n+1}(X) := F^{n+1}\mathrm{H}^{2n+1}(X,\mathbb{C})\backslash \mathrm{H}^{2n+1}(X,\mathbb{C})/\mathrm{H}^{2n+1}(X,\mathbb{Z}),$$

where $F^\bullet$ denotes the Hodge filtration, and that there is an *Abel–Jacobi map*

$$\operatorname{Ker}\big(\mathrm{CH}^n(X) \to \mathrm{H}^{2n}(X,\mathbb{Z})\big) \longrightarrow J^{2n+1}(X).$$

A choice of polarization on X induces a polarization on the Hodge structure $H := \mathrm{H}^{2n+1}(X,\mathbb{Z})$. In the case where H has Hodge level 1, *i.e.*, $H\otimes_{\mathbb{Z}}\mathbb{C} = \mathrm{H}^{n+1,n}\oplus \mathrm{H}^{n,n+1}$, the well-known equivalence of categories between the category of polarized $\mathbb{Z}$-Hodge structures such that $H\otimes_{\mathbb{Z}}\mathbb{C} = \mathrm{H}^{1,0}\oplus\mathrm{H}^{0,1}$ and the category of polarized complex abelian varieties shows that $J^{2n+1}(X)$ is in fact a complex abelian variety. At Joe Harris' 60-th birthday conference, Barry Mazur [5] posed the following:

Question 1. Let X be a smooth projective variety defined over a subfield $K \subseteq \mathbb{C}$, and let $X_{\mathbb{C}}$ denote the base-change of X to the field of complex numbers. Assume that $\mathrm{H}^{2n+1}(X(\mathbb{C}),\mathbb{Z})$ has Hodge level 1. Does the abelian variety $J^{2n+1}(X_{\mathbb{C}})$ admit a model over K ?

As pointed out by Mazur, this question can be traced back at least to [4], where Deligne gave a positive answer to that question for complete intersections X of odd dimension $2n+1$ and of Hodge level 1, thereby establishing the Weil conjectures for such X. (Of course, Deligne established the Weil conjectures in full generality not long after.) Deligne proceeded with a proof using the universal family, which consisted in using the irreducibility of the monodromy action on $\mathrm{H}^{2n+1}(X(\mathbb{C}),\mathbb{Q})$ and on $\mathrm{H}^{2n+1}(X(\mathbb{C}),\mathbb{Z}/\ell)$ for all primes ℓ. It turns out that for such X, one can in fact prove that $\mathrm{H}^{2n+1}(X(\mathbb{C}),\mathbb{Q})$ is supported in codimension n, or in other words that $\mathrm{H}^{2n+1}(X(\mathbb{C}),\mathbb{Q})$ has geometric coniveau n. The *geometric coniveau filtration* is defined by:

$$\mathrm{N}^{\nu}\,\mathrm{H}^{i}(X(\mathbb{C}),\mathbb{Q}) := \sum_{\substack{Z\subseteq X \\ \text{closed, codim } \geq \nu}} \mathrm{Ker}\left(\mathrm{H}^{i}(X(\mathbb{C}),\mathbb{Q}) \to \mathrm{H}^{i}((X\backslash Z)(\mathbb{C}),\mathbb{Q})\right).$$

Resolution of singularities and the formalism of Deligne's mixed Hodge structures show, in particular, that $\mathrm{N}^{n}\,\mathrm{H}^{2n+1}(X(\mathbb{C}),\mathbb{Q})$ is a Hodge structure of Hodge level 1. Note that the generalized Hodge conjecture predicts that $\mathrm{N}^{n}\,\mathrm{H}^{2n+1}(X(\mathbb{C}),\mathbb{Q})$ should be the largest sub-Hodge structure of $\mathrm{H}^{2n+1}(X(\mathbb{C}),\mathbb{Q})$ of Hodge level 1. The Hodge structure $\mathrm{N}^{n}\,\mathrm{H}^{2n+1}(X(\mathbb{C}),\mathbb{Q})$ can also be characterized in terms of the Abel–Jacobi map. Define the *algebraic intermediate Jacobian* $J_a^{2n+1}(X_{\mathbb{C}})$ to be the complex torus that is the image of the Abel–Jacobi map restricted to algebraically trivial cycles

$$\mathrm{AJ} : \mathrm{A}^{n+1}(X_{\mathbb{C}}) \to J^{2n+1}(X_{\mathbb{C}}).$$

Then $\mathrm{H}^1(J_a^{2n+1}(X_{\mathbb{C}}),\mathbb{Q})$ is naturally isomorphic to $\mathrm{N}^{n}\,\mathrm{H}^{2n+1}(X(\mathbb{C}),\mathbb{Q}(n))$ as Hodge structures, and AJ is surjective if and only if $\mathrm{H}^{2n+1}(X,\mathbb{Q})$ has geometric coniveau n. One is thus led to ask the following alternative question:

Question 2. Given a smooth projective variety X defined over a subfield $K \subseteq \mathbb{C}$, does the algebraic intermediate Jacobian $J_a^{2n+1}(X_{\mathbb{C}})$ admit a model J over K? In terms of Galois representations, which was also a motivation for Mazur's question, do there exist an abelian variety J over K and an isomorphism of Galois-representations $\mathrm{N}^{n}\,\mathrm{H}^{2n+1}_{\acute{e}t}(X_{\bar{K}},\mathbb{Q}_\ell(n)) \cong \mathrm{H}^1_{\acute{e}t}(J_{\bar{K}},\mathbb{Q}_\ell)$?

In forthcoming work [2], we provide an affirmative answer to Question 2, and as a corollary, give an alternate proof to Deligne's theorem on complete intersections of Hodge level 1.

However, two non-isomorphic abelian varieties over K may become isomorphic after base-change of field. We therefore ask the more precise :

Question 3. Does $J_a^{2n+1}(X_{\mathbb{C}})$ admit a *distinguished* model J over K, in the sense that the Abel–Jacobi map

$$\mathrm{AJ} : \mathrm{A}^{n+1}(X_{\mathbb{C}}) \to J(\mathbb{C})$$

is $\mathrm{Aut}(\mathbb{C}/K)$-equivariant?

The Picard scheme and the Albanese scheme classically give a positive answer for codimension-1 cycles and for dimension-0 cycles, respectively. In [1], we use Murre's work [6] on algebraic representatives for codimension-2 cycles on smooth projective varieties defined over algebraically closed fields to give a positive answer for codimension-2 cycles:

Theorem. Let X be a smooth projective variety over a field $K \subseteq \mathbb{C}$. The algebraic intermediate Jacobian $J_a^3(X_{\mathbb{C}})$ has a distinguished model J over K. Moreover, there is a correspondence γ on $J \times_K X$ inducing for all primes ℓ a split injective morphism of Galois-representations $\mathrm{H}^1_{\acute{e}t}(J_{\bar{K}},\mathbb{Q}_\ell) \hookrightarrow \mathrm{H}^3_{\acute{e}t}(X_{\bar{K}},\mathbb{Q}_\ell(1))$ with image $\mathrm{N}^1\,\mathrm{H}^3_{\acute{e}t}(X_{\bar{K}},\mathbb{Q}_\ell(1))$, and with splitting induced by a correspondence γ' on $X \times_K J$.

In categorical terms, we also establish that there exist an abelian variety J over K such that $J_{\mathbb{C}} \cong J_a^{2n+1}(X_{\mathbb{C}})$ and a natural transformation of contravariant functors, extending the Abel–Jacobi map $\mathrm{AJ} : \mathrm{A}^{n+1}(X_{\mathbb{C}}) \to J(\mathbb{C})$, from the functor, defined on smooth integral schemes over K, $T \mapsto \mathrm{A}^{n+1}(X \times_K T)$ to the functor $T \mapsto J(T)$. An important aspect of our proof, which we will come back to in [3], consists in exhibiting suitable parameter spaces for algebraically trivial cycles. Precisely, we prove and use the following:

Proposition. Let X be a smooth projective variety defined over a perfect field K and consider an algebraically trivial cycle class $\alpha \in \mathrm{A}^{n+1}(X_{\bar{K}})$. Then there exist an abelian variety A over K, a cycle class $Z \in \mathrm{CH}^{n+1}(A \times_K X)$, and a pair of $\bar{K}$-points $t_0, t_1 \in A(\bar{K})$, such that $\alpha = Z_{t_1} - Z_{t_0}$. (Here, Z_{t_i} denotes the Gysin fiber of Z over t_i pushed forward to $X_{\bar{K}}$.)

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Étale duality for p-torsion sheaves on semistable schemes over $\mathbb{F}_q[[t]]$

Yigeng Zhao

1. Motivation

Let K be a local field, i.e., a complete discrete valuation field of characteristic $p > 0$, let $\mathcal{O}_K$ be its ring of integers, let k be its residue field, and let ν_K be its valuation. We fix a uniformizer $\pi \in \mathcal{O}_K$. Recall that in local class field theory, we have the following theorem:

Theorem.(Artin-Schreier-Witt) *There is a perfect pairing of topological groups, that we call the Artin-Schreier-Witt symbol*

$$(1) \qquad W_n(K)/(1-F)W_n(K) \times K^\times/(K^\times)^{p^n} \longrightarrow \mathbb{Z}/p^n\mathbb{Z}$$
$$(a, b) \mapsto [a, b) := (b, L/K)(\alpha) - \alpha$$

where $(1-F)(\alpha) = a$, for some $\alpha \in W_n(K^{sep})$, $L = K(\alpha)$, $(b, L/K)$ is the norm residue of b in L/K, and the topological structure on the first term is discrete, on the second term is induced from $K^\times$.

Moreover, on both $W_n(K)$ and $K^\times$, we have filtrations:

$$\mathrm{fil}_m W_n(K) := \{(a_{n-1},\dots,a_0) \in W_n(K) | p^i \nu_K(a_i) \geq m\};$$

$$K^\times \supset \mathcal{O}_K^\times =: U_K^0 \supset U_K^m := \{x \in \mathcal{O}_K^\times | x \cong 1 \mathrm{mod} \pi^m\}.$$

Theorem. ([1]) *With respect to the Artin-Schreier-Witt symbol (1), the orthogonal complement of $fil_{m-1} H^1(K, \mathbb{Z}/p^n\mathbb{Z})$ is $U_K^m \cdot (K^\times)^{p^n}/(K^\times)^{p^n}$, for any $m \geq 1$.*

We want to generalize those results to higher dimensional regular schemes.

2. Main results

Let $X \to \mathrm{Spec}(\mathbb{F}_q[[t]])$ be a projective strictly semistable scheme of relative dimension d, and let $i : X_s \hookrightarrow X$ be its special fiber. Let D be an effective Cartier divisor on X such that $\mathrm{Supp}(D)$ has simple normal crossing, and let $j : U \hookrightarrow X$ be its open complement.

Definition. *We define $W_n\Omega^r_{X|D,\log} \subset j_* W_n \Omega^r_{U,\log}$ be the subsheaf étale locally generated by the symbols $\frac{d[x_1]_n}{[x_1]_n} \cdots \frac{d[x_r]_n}{[x_r]_n}$, where $x_1 \in 1 + \mathcal{O}_X(-D)$ and $x_i \in j_*\mathcal{O}_U$ for all i.*

If $D_1 \geq D_2$, then $W_n\Omega^r_{X|D_1,\log} \subset W_n\Omega^r_{X|D_1,\log}$. Under this order, we obtain a pro-system "$\varprojlim_D$" $W_m\Omega^r_{X|D,\log}$, where D runs over the subset of effective divisor of X such that $\mathrm{Supp}(D)$ has simple normal crossing and is contained in $X - U$.

Our main theorem is:

Main Theorem.([4]) *Let $X \to Spec(\mathbb{F}_q[[t]])$ be as above. Then there is a perfect pairing of topological $\mathbb{Z}/p^n\mathbb{Z}$-modules*

$$H^i(U, W_n\Omega^r_{U,\log}) \times \varprojlim_D H^{d+2-i}_{X_s}(X, W_n\Omega^{d+1-r}_{X|D,\log}) \to H^{d+2}_{X_s}(X, W_n\Omega^{d+1}_{X,\log}) \xrightarrow{Tr} \mathbb{Z}/p^n\mathbb{Z},$$

where the first term is endowed with the discrete topology, and the second term is endowed with the profinite topology.

The trace morphism in the above theorem is given by the following purity result and Moser-Sato trace map[2, 3] for $\nu^d_{n,X_s} := \mathrm{Ker}(\oplus_{x \in X_s^0} i_x^* W_n\Omega^d_{x,\log} \xrightarrow{\partial} \oplus_{x \in X_s^1} i_x^* W_n \Omega^{d-1}_{x,\log})$, where ∂ is the Kato residue map on logarithmic de Rham-Witt sheaves.

Theorem(Purity). *There is a canonical isomorphism*

$$Gys^{log}_{i,n} : \nu^d_{n,X_s}[-1] \xrightarrow{\cong} Ri^! W_n\Omega^{d+1}_{X,\log}$$

in $D^+(X_s, \mathbb{Z}/p^n\mathbb{Z})$.

We denote this pairing in our Main Theorem as $< \cdot, \cdot >$.

For any $\chi \in H^i(U, W_n\Omega^r_{U,\log})$, we define the higher Artin conductor

$$\mathrm{ar}(\chi) := \min\{D \mid \ < \chi, \cdot > \text{ factors through } H^{d+2-i}_{X_s}(X, W_n\Omega^{d+1-r}_{X|D,\log})\},$$

Then we define

$$\mathrm{Fil}_D H^i(U, W_n\Omega^r_{U,\log}) := \{\chi \in H^i(U, W_n\Omega^r_{U,\log}) | \ \mathrm{ar}(\chi) \leq D\},$$

and

$$\pi_1^{\mathrm{ab}}(X, D)/p^n := \mathrm{Hom}(\mathrm{Fil}_D H^1(U, \mathbb{Z}/p^n\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$$

endowed with the usual profinite topology of the dual.

The quotient $\pi_1^{\mathrm{ab}}(X, D)/p^n$ classifies abelian étale coverings of U of degree p^n with ramification bounded by the divisor D.

3. Application

Let $X = B = \mathrm{Spec}\mathbb{F}_q[[t]]$, let $D = X_s = (t)$ be the unique closed point. Then $U = \mathrm{Spec}(\mathbb{F}_q((t)))$. We denote $mD := m(t)$ for any $m \in \mathbb{N}$. Our Main Theorem in this setting is:

$$H^i(K, W_n\Omega^r_{K,\log}) \times \varprojlim_m H^{2-i}_{X_s}(B, W_n\Omega^{1-j}_{B|mD,\log}) \to \mathbb{Z}/p^n\mathbb{Z}$$

is a perfect pairing of topological abelian groups.

In particular, if $i = 1, j = 0$. We are back to the Artin-Schreier-Witt symbol.

Proposition. *For $m \geq 1$, we have*

$$Fil_{mD} H^1(K, \mathbb{Z}/p^n\mathbb{Z}) = fil_{m-1} H^1(K, \mathbb{Z}/p^n\mathbb{Z}),$$

i.e., Our new filtration is same as the Brylinski-Kato filtration.

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K-theory for non-archimedean algebras and spaces

Georg Tamme
(joint work with Moritz Kerz, Shuji Saito)

Let K be a complete discretely valued field with ring of integers $\mathcal{O}_K$, uniformizer π, and residue field k.

1. Motivation

Let $\mathcal{X}/\mathcal{O}_K$ be a smooth proper scheme. Denote by $\mathcal{X}_n := \mathcal{X} \otimes_{\mathcal{O}_K} \mathcal{O}_K/(\pi^n)$ the infinitesimal thickenings of the special fibre $\mathcal{X}_k$, by $X := \mathcal{X} \otimes_{\mathcal{O}_K} K$ the generic fibre of $\mathcal{X}$, and by X^{rig} its associated rigid analytic space. By definition, the continuous K-theory of $\mathcal{X}$ is the pro-spectrum $K^{\mathrm{cont}}(\mathcal{X}) := \{K(\mathcal{X}_n)\}_{n\in\mathbb{N}} \in \mathrm{Pro(Sp)}$. The study of the comparison map $K(\mathcal{X}) \to K^{\mathrm{cont}}(\mathcal{X})$ plays an important role in the strategy to attack Grothendieck's variational Hodge conjecture and its variants proposed by Bloch, Esnault, and Kerz [1, 2]. Our goal here is to understand 'the generic fibre' of this comparison map. More precisely, we want to construct a version of K-theory for the rigid space X^{rig} which fits in a homotopy cartesian square

$$\begin{array}{ccc} K(\mathcal{X}) & \longrightarrow & K^{\mathrm{cont}}(\mathcal{X}) \\ \downarrow & & \downarrow \\ K(X) & \longrightarrow & K\text{-theory of } X^{\mathrm{rig}}. \end{array}$$

2. Constructions

We first define a K-theory for affinoid algebras, and then globalize it to all rigid spaces. Let A be an affinoid K-algebra. For an integer j define

$$A\langle x\rangle_{\pi^j} := \{\sum_i a_i x^i \in A[[x]] \,|\, a_i\pi^{ij} \to 0 \text{ for } i \to \infty\},$$

the algebra of power series with coefficients in A converging on a closed disk of radius $|\pi^{-j}|$. Set $A\langle\Delta^n\rangle_{\pi^j} := A\langle x_0, \dots, x_n\rangle_{\pi^j}/(x_0 + \cdots + x_n - 1)$. For varying n these assemble into a simplicial ring $A\langle\Delta^\bullet\rangle_{\pi^j}$. For each j, the classifying space of the simplicial group $\mathrm{GL}(A\langle\Delta^\bullet\rangle_{\pi^j})$ is the infinite loop space of a connective spectrum which we denote by $KV^{(j)}(A)$.

Remark. The homotopy groups of the spectrum $KV^{(0)}(A)$ are the K-groups of the Banach algebra A defined by Karoubi-Villamayor [3].

For varying j we get a pro-spectrum $KV^{\mathrm{an}}(A) := \{KV^{(j)}(A)\}_{j\in\mathbb{N}}$ called *analytic KV-theory.* In order to globalize to all rigid spaces, we need a descent result for affinoid coverings. It turns out that one needs a non-connective variant of analytic KV-theory to achieve this. We construct this using an analog of the classical Bass construction, replacing (Laurent) polynomial rings by rings of convergent power series $A\langle x\rangle_1$ and $A\langle x, x^{-1}\rangle_1$. The resulting pro-spectrum $KH^{\mathrm{an}}(A)$ is called *analytic KH-theory.*

3. Results

In this section we assume that the characteristic of k is 0.

Theorem 1. *Let A_0 be an algebra of topologically finite type over $\mathcal{O}_K$ and assume that $A := A_0 \otimes_{\mathcal{O}_K} K$ is regular. Then there is a homotopy cartesian square*

$$\begin{array}{ccc} K^B(A_0) & \longrightarrow & K^{B,\mathrm{cont}}(A_0) \\ \downarrow & & \downarrow \\ K^B(A) & \longrightarrow & KH^{\mathrm{an}}(A) \end{array}$$

in $\mathrm{Pro}(\mathrm{Sp}^+)$.

Here K^B is the non-connective Bass K-theory introduced by Thomason-Trobaugh [5] and $K^{B,\mathrm{cont}}$ is defined similarly as before. $\mathrm{Pro}(\mathrm{Sp}^+)$ is the category of pro-objects of bounded above spectra. It is a localization of $\mathrm{Pro}(\mathrm{Sp})$.

Theorem 2. *The functor $KH^{\mathrm{an}}\colon$ (affinoid algebras) $\to \mathrm{Pro}(\mathrm{Sp}^+)$ satisfies descent for affinoid coverings.*

The second theorem allows us to extend analytic KH-theory to all rigid spaces. Theorem 1 shows that the resulting theory fits in the desired homotopy cartesian square.

The proofs of these results use pro-cdh descent [4], Weibel's K-dimension conjecture, and resolution of singularities. This is where we need the characteristic 0 assumption.

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Homotopy t-structure and effectivity

Frédéric Déglise
(joint work with Mikhail Bondarko)

1. Introduction

In this talk, I have presented a work in collaboration with Mikhail Bondarko in the context of relative motives.

Voevodsky's theory of motivic complexes, over a perfect field, first comes as a category of effective objects in which the Tate twist is non invertible for the tensor product. On the other hand, it is equipped with a canonical t-structure called the *homotopy t-structure*. This t-structure allows us to have a deeper understanding of motivic complexes, in particular in the case of curves. The t-structure can be extended to the non effective case, but then its properties are less strong; in particular, geometric motives are not bounded for sure.

The work done in collaboration with Bondarko aims at extending the notion of effectivity to the case of motives relative to an arbitrary base. Though candidates for this notion already exist, they do not satisfy good functorial properties. We define a notion of effective motives over an arbitrary base scheme S as a full subcategory of the category motives over S. The definition allows us to control the behaviour of these motives under the six operations constructed in the stable case along the lines of Ayoub.

Then we are able to extend the definition of Voevodsky's t-structure to these effective motives, and obtain an extension of certain good properties proved by Voevodsky over a field to the case of an arbitrary base. Besides, this new homotopy t-structure shares good similar properties to that of perverse sheaves, extending the work of Ayoub on the *perverse homotopy t-structure* (cf. [Ayo07]).

2. Effective motives and dimension functions

For a base scheme S (excellent noetherian finite dimensional scheme), we let $DM(S)$ be the triangulated category of mixed motives with rational coefficients (cf. [CD12]) or, if S is an $\mathbb{F}$-scheme for $\mathbb{F}$ a prime field of exponential characteristic p, with $\mathbb{Z}[1/p]$-coefficients (cf. [CD15]). We will use a special motive associated with a separated S-scheme of finite type type $f : X \to S$, called the *Borel-Moore motive* of X/S:

$$M^{BM}(X/S) := f_!(\mathbb{1}_X),$$

where $\mathbb{1}_X$ is the constant motive over X.

Our main tool to define effective motives is that of a *dimension function*[1] $\delta : S \to \mathbb{Z}$ over the scheme S. Once such a dimension function is chosen, the definition goes on as follows.

[1] see [ILO14, chap. 2], or [BD15, 1.1] for recall;

Definition 1. *(cf.* [BD15, Def. 2.2.1]*) We define the category of* δ-effective motives over S*, denoted by* $DM^{\delta-eff}(S)$*, as the localizing triangulated subcategory of* $DM(S)$ *generated by motives of the form* $M(X/S)(n)$ *for any separated* S*-scheme* X *of finite type and any integer* $n \geq \delta(X)$*.*[2]

Example 2. *When* $S = \mathrm{Spec}(k)$ *is the spectrum of a perfect field, and* $\delta = 0$ *is the obvious dimension funtion, we get that* $DM^{\delta-eff}(k)$ *is equivalent to Voevodsky's category of (unbounded) motivic complexes (cf.* [BD15, Ex. 2.3.12]*).*

The category of δ-effective motives satisfies good functorial properties:

(1) Given a closed immersion i with complementary open immersion j, it is stable under the functors i^*, $i_* = i_!$, $j_!$, $j^* = j^!$. In particular, the 2-functor $DM^{\delta-eff}$ satisfies the so-called *localization property* (cf. [BD15, Cor. 2.2.10]).

(2) More generally, it is stable under the functor $f_!$ (resp. $f^*(d)$) where f is a separated morphism of finite type (resp. any morphism of finite type whose dimension of fibers is bounded by d) (cf. [BD15, Cor. 2.2.6]).

3. The δ-homotopy t-structure in the effective case

The triangulated category $DM(S)$, as well as $DM^{\delta-eff}(S)$ defined above, is compactly generated. It is well known that one can define t-structure on these type of triangulated categories $\mathscr{T}$ by choosing homologically positive objects[3], as a full sub-category of $\mathscr{T}$ which is stable under direct sums, positive shift and extensions (see [BD15, §1.1] for recall). After Keller and Vossieck, we will call such a sub-category an *aisle* of $\mathscr{T}$.

Definition 3. *Let* S *be a scheme equipped with a dimension function* δ*.*

One defines the δ*-homotopy* t*-structure on* $DM^{\delta-eff}(S)$ *as the unique one such that the (homologically) positive objects are the smallest aisle containing motives of the form:* $M^{BM}(X/S)(\delta(X)+n)[2\delta(X)+n]$ *for any separated* S*-scheme* X *and any integer* $n \geq 0$*.*

Example 4. *In the condition of the example in the previous section, the* δ-homotopy t-structure *on* $DM^{\delta-eff}(k)$ *coincides with Voevodsky's homotopy* t*-structure.*

The δ-homotopy t-structure has good functorial properties; we refer the reader to the introduction of [BD15] for details. We are able to prove these properties as we can control the homology with respect to this t-structure by the following tool.

Definition 5. *Let* M *be a* δ*-effective motive over* S *and* $i \in \mathbb{Z}$ *an integer. Given a couple* (x, n) *where* $x \in S(E)$ *is a point with values in a field* E *(finitely generated*

[2] The notation $\delta(X)$, for any pro-étale S-scheme stands for the maximum of the integers tr.deg.$\big(\kappa(x)/\kappa(x)\big)$ where x runs over the points of X and s is the projection of X to S.

[3] Our conventions for t-structure will be homological;

over S) and $n \leq 0$ an integer, we define the δ-homology at (x, n) with coefficients in M as follows:

$$\widehat{H}_i(M)(x,n) := \varinjlim_{X} [\mathrm{Hom}\big(M^{BM}(X/S)(\delta(X)-n)[2\delta(X)-n+i], M\big)]$$

where $X = \mathrm{Spec}(A)$ runs over S-models of x.[4] *We look at $\widehat{H}_i(M)$ as a functor on the discrete category whose objects are the couples (x, n) as above.*

Our main theorem is the following one.

Theorem 1. *(*[BD15, Th. 3.3.1]*) Under the notations of the previous definition, the following conditions on a δ-effective motive M over S are equivalent:*

(i) *M is positive (resp. negative) for the δ-homotopy t-structure;*
(ii) *for any integer $i \leq 0$ (resp. $i \geq 0$), $\widehat{H}_i^\delta(M) = 0$.*

In other words, $\widehat{H}_i^\delta$ behaves like the homology with respect to the δ-homotopy t-structure. During the talk, we have drawn several consequences of this theorem.

(1) *Functorial properties.–* Let $f : T \to S$ is smooth of finite type (resp. proper and with dimension of fibers bounded by d) then $f^! : DM^{\delta-eff}(T) \to DM^{\delta-eff}(S)$ is t-exact (resp. $f_* : DM^{\delta-eff}(S) \to DM^{\delta-eff}(T)$ has homological amplitude $[0, d]$) for the δ-homotopy t-structure.
(2) *Boundedness.–* If S is regular and $\delta = -\mathrm{codim}_S$, the constant motive $\mathbb{1}_S$ in $DM^{\delta-eff}(S)$ belongs to the heart of the δ-homotopy t-structure.[5]
(3) *Link with perversity.–* A δ-effective motive M over S is homologically positive (resp. negative) for the δ-homotopy t-structure if and only if the following condition holds:

$$\forall x \in S, H_p^{\delta_x}(i_x^! M) = 0 \text{ when } p \leq \delta(x) \text{ (resp. } p \geq \delta(x)).$$

Here δ_x is the obvious dimension function induced by δ on the spectrum of the residue field $\kappa(x)$ of x in S, and we have put $i_x^! = j^* i^!$ for the factorization $\mathrm{Spec}(\kappa(x)) \xrightarrow{j} \overline{\{x\}} \xrightarrow{i} S$.

It is striking that the last property is the exact analog of the characterization of the perverse t-structure as defined by Beilinson, Bernstein and Deligne.

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[4] *i.e.* affine S-schemes of finite type such that $A \subset E$.

[5] One will be careful that this result is false if we drop the assumption that S is regular or if we consider the analog of the δ-homotopy t-structure on the stable category $DM(S)$.

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Homology of GL_3 of elliptic curves

Matthias Wendt

In the talk, I reported some progress in the ongoing project of constructing the elliptic dilogarithm complex using group homology. Here, *elliptic dilogarithm complex* refers to a complex computing the weight-two parts of the K-theory of an elliptic curve. Such a complex has been described by Goncharov and Levin in their work on Zagier's conjecture for elliptic curves [1]. The overall goal of the project is to identify this (essentially two-term) complex as a differential in a spectral sequence computing group homology of $\mathrm{GL}_4(k[E])$, similar to the group homology approach to the motivic weight-two complex over fields due to Suslin and Dupont–Sah. Hopefully, different realizations of the elliptic dilogarithm complex and the resulting alternative presentations for the first and second K-groups of elliptic curves can help to improve our understanding of K-theory of elliptic curves.

For the formulation of the results of the group homology computation, let k be an arbitrary field, let $\overline{E}/k$ be an elliptic curve with fixed k-rational point P, and set $E = \overline{E} \setminus \{P\}$. The computations so far describe the homology of the group $\mathrm{GL}_3(k[E])$, using its action on the Bruhat–Tits building $\mathfrak{B}$ associated to GL_3 and the field $k(E)$ equipped with the valuation v_P associated to the k-rational point P. The key result is the following computation of the action of $\mathrm{GL}_3(k[E])$ on the building $\mathfrak{B}$.

Theorem 1. (1) *The quotient* $\mathrm{GL}_3(k[E])\backslash\mathfrak{B}$ *has the homotopy type* $\Sigma\mathcal{F}l_3(k)$, *the suspension of the flag complex for the vector space* k^3. *This is essentially the union of the links of the two stable rank* 3 *bundles on* $\overline{E}$ *appearing in the building.*

(2) *The subcomplex of cells with non-trivial stabilizer deformation-retracts equivariantly to a graph of groups* Γ_E. *The underlying graph of* Γ_E *is given by the following diagram of moduli spaces of vector bundles on* $\overline{E}$*:*

$$\mathcal{M}_{2,1}(\overline{E}) \leftarrow \mathcal{M}_{2,0}(\overline{E}) \rightarrow \mathcal{M}_{3,0}(\overline{E}).$$

Here $\mathcal{M}_{r,d}(\overline{E})$ *denotes the moduli space of vector bundles of rank* r *and degree* d *on* $\overline{E}$. *However, the (semi-)stable bundles* $\mathcal{V}$ *in* $\mathcal{M}_{2,i}$ *actually correspond to the rank* 3 *bundles* $\mathcal{V} \oplus \det^{-1}\mathcal{V}$ *on* $\overline{E}$. *This becomes a graph of groups by mapping vector bundles to their automorphism groups.*

The basic proof steps are the following: first, the identification of orbits of vertices with suitable vector bundles on the curve $\overline{E}$ is classical; this identification moreover identifies the stabilizers of vertices as automorphism groups of the corresponding bundles. Using Atiyah's classification of vector bundles on elliptic curves, explicit computations of the action of automorphism groups on the links and explicit identification of the corresponding elementary transformations of vector bundles, one obtains a complete description of the quotient. The simplified description in the theorem then arises from an explicit deformation retraction which moves unstable bundles towards semistable bundles.

From the isotropy spectral sequence for the action of $\mathrm{GL}_3(k[E])$ on $\mathfrak{B}$ together with classical stabilization results in group homology, we get an exact sequence, where the morphism in brackets is the group homology version of the elliptic dilogarithm complex:

$$\mathrm{H}_2(\Gamma_E) \to \mathrm{K}_2(E) \to \left[\mathrm{St}_3(k)/\mathcal{R} \xrightarrow{\partial} \mathrm{H}_1(\Gamma_E)\right] \to \mathrm{K}_1(E) \to 0.$$

Here $\mathrm{H}_1(\Gamma_E)$ is the first homology group of the graph of groups (which is defined via the cone of the inclusion map from edge groups into vertex groups). The explicit identification of the graph Γ_E allows to identify $\mathrm{H}_1(\Gamma_E)$ with the subspace of $\mathrm{Jac}_{\overline{E}}(\overline{k})\otimes\overline{k}^\times$ generated by $Q\otimes u$ where Q is a point of degree ≤ 3 and $u \in k(Q)^\times$. This presentation is a lot smaller (in terms of number of generators) than the usual presentation of $\mathrm{K}_1(\overline{E})$ as Somekawa K-group.

The Steinberg representation $\mathrm{St}_3(k)$ (which arises from the homology of the flag complex $\mathcal{F}l_3(k)$) gives rise to a geometric interpretation of K_2-classes as triangles in $\mathbb{P}^2$. The relations $\mathcal{R}$ are not explicitly described for now; for this, computations of the action of $\mathrm{GL}_4(k[E])$ on the associated building are necessary. Part of the differential ∂ can be seen in this geometric description as mapping a side L of the triangle to the vector bundle associated to the degree 3 divisor obtained by intersecting the line L with the curve $E \subset \mathbb{P}^2$.

In future work, it remains to explicitly compute ∂ and $\mathcal{R}$. Naturally, the expectation would be that there is a morphism of complexes from the elliptic dilogarithm complex described by Goncharov and Levin [1] to the complex above arising from group homology. This morphism of complexes should induce the restriction map $\mathrm{K}_i(\overline{E}) \to \mathrm{K}_i(E)$ on homology. Note, however, that the above complex does in fact provide integral computations, where Goncharov and Levin compute rational K-theory.

Finally, as an interesting aside, the computation of the quotient $\mathrm{GL}_3(k[E])\backslash\mathfrak{B}$ actually allows to completely determine the structure of $\mathrm{H}^\bullet(\mathrm{GL}_3(k[E]);\mathbb{F}_\ell)$ (as module over the ring of Chern classes $\mathbb{F}_\ell[c_1,c_2,c_3]$) when $\overline{E}$ is an elliptic curve defined over the finite field $k = \mathbb{F}_q$. When $\ell \mid q-1$, this provides new counterexamples to a function field analogue of Quillen's conjecture on cohomology of arithmetic groups; the reason behind the counterexamples is essentially that the non-contractibility of both the quotient $\mathrm{GL}_3(k[E])\backslash\mathfrak{B}$ as well as the graph of vector bundles with non-trivial automorphisms implies the existence of many cohomology classes which are torsion for the Chern class ring $\mathbb{F}_\ell[c_1,c_2,c_3]$.

A preliminary version of Theorem 1 (without proofs) and further discussion may be found in a note on the arXiv [2].

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Motives, nearby cycles and Milnor fibers

Florian Ivorra

(joint work with Joseph Ayoub and Julien Sebag)

In this talk, we give a survey of the main results of [4, 8] relating tubes in non-archimedean geometry, nearby motivic sheaves and motivic nearby cycles in the sense of Denef-Loeser.

Let k be a field of characteristic zero, $R = k[[t]]$ be the ring of formal power series and $K = k((t))$ be its fraction field. In this exposition, we restrict ourselves to the stable homotopy category of schemes $\mathbf{SH}(k)$ introduced by Morel and Voevodsky and its rigid analytic version $\mathbf{RigSH}(K)$ introduced by Ayoub.

1. Recollections on rigid motives

Let X be a smooth rigid variety over K. We denote by $\mathrm{M}_{\mathrm{rig}}(X)$ the homological motive associated with X and by

$$\mathrm{M}^{\vee}_{\mathrm{rig}}(X) := \underline{\mathrm{Hom}}(\mathrm{M}_{\mathrm{rig}}(X), \mathbb{1}_{\mathsf{Spm}(K)})$$

its dual. Recall that, given a separated K-scheme of finite type X, there is an associated rigid analytic K-variety X^{an}. The functor $X \mapsto X^{\mathrm{an}}$ extends into a triangulated functor

$$\mathrm{Rig}^* : \mathbf{SH}(K) \to \mathbf{RigSH}(K)$$

such that $\mathrm{Rig}^*(\mathrm{M}(X)) = \mathrm{M}_{\mathrm{rig}}(X^{\mathrm{an}})$ where $\mathrm{M}(X)$ is the usual (algebraic) motive of the K-scheme X.

Let $\mathbf{QUSH}(k)$ be the category of quasi-unipotent (algebraic) motives, i.e., the full triangulated subcategory of $\mathbf{SH}(\mathbf{G}_{m,k})$ closed under infinite direct sums and generated by the objects of the form $\mathrm{Sus}^p_T(Q^{gm}_r(X,g) \otimes \mathbb{1})$ where X is a smooth k-scheme, $g \in \mathscr{O}(X)^{\times}$, $r \in \mathbf{N}^{\times}$ and $Q^{gm}_r(X,g)$ is the smooth $\mathbf{G}_{m,k}$-scheme

$$Q^{gm}_r(X,g) := \mathrm{Spec}(\mathscr{O}_X[T,T^{-1},V]/(V^r - gT)) \to \mathrm{Spec}(k[T,T^{-1}]) = \mathbf{G}_{m,k}.$$

In [3], Ayoub shows that the composition of the three functors

$$\mathbf{QUSH}(k) \hookrightarrow \mathbf{SH}(\mathbf{G}_{m,k}) \xrightarrow{t^*} \mathbf{SH}(K) \xrightarrow{\mathrm{Rig}^*} \mathbf{RigSH}(K)$$

is an equivalence of categories. We fix a quasi-inverse to the above equivalence

$$\mathfrak{R} : \mathbf{RigSH}(K) \xrightarrow{\sim} \mathbf{QUSH}(k)$$

and are interested in the composition

$$1^* \circ \mathfrak{R} : \mathbf{RigSH}(K) \to \mathbf{SH}(k)$$

where $1 : \mathrm{Spec}(k) \to \mathbf{G}_{m,k}$ is the unit section.

2. Tubes and nearby motivic sheaves

Let $f\colon X \to \mathrm{Spec}(R)$ be a separated finite type R-scheme and denote by X_η and X_σ the generic and special fibers of X.

By t-adic completion, one obtains a separated formal R-scheme topologically of finite type $\mathscr{X} \to \mathsf{Spf}(R)$ and we denote by $\mathscr{X}_\eta$ the generic fiber of $\mathscr{X}$. The rigid analytic variety $\mathscr{X}_\eta$ is an open analytic subvariety of the analytification X_η^{an} of the algebraic generic fiber X_η. With a locally closed subset $Z \subset X_\sigma$, we may associate its tube $]Z[$ which is an open rigid analytic subvariety of $\mathscr{X}_\eta$.

Assume that the rigid analytic variety $\mathscr{X}_\eta$ is smooth over K. One of the main result of [4] is the following theorem that relates rigid motives of tubes with the nearby motivic sheaf $\Psi_f(\mathbb{1}_{X_\eta})$ introduced by Ayoub in [2].

Theorem 1 (see [4])**.** *Denote by $z : Z \hookrightarrow X_\sigma$ the inclusion. Then, there is a canonical isomorphism*

$$1^* \circ \mathfrak{R}(\mathrm{M}^\vee_{\mathrm{rig}}(]Z[)) \simeq (f_\sigma)_* z_* z^* \Psi_f(\mathbb{1}_{X_\eta})$$

in the category of motives $\mathbf{SH}(k)$*.*

Taking $Z = X_\sigma$, one gets that the cohomological motive $\mathrm{M}^\vee_{\mathrm{rig}}(\mathscr{X}_\eta)$ is related to the nearby motivic sheaf by a canonical isomorphism

$$1^* \circ \mathfrak{R}(\mathrm{M}^\vee_{\mathrm{rig}}(\mathscr{X}_\eta)) \simeq (f_\sigma)_* \Psi_f(\mathbb{1}_{X_\eta})$$

in $\mathbf{SH}(k)$.

Our main theorem is a motivic analog of a theorem of Berkovich that we recall now. Let $\overline{K}$ be the completion of an algebraic closure of the valued field K and let $\overline{k}$ be its residue field. Set $\overline{Z} = Z \times_k \overline{k}$ and $\overline{]Z[} =]Z[\widehat{\times}_K \overline{K}$. In [5, 6], Berkovich constructed a canonical isomorphism of étale cohomology groups

$$\mathrm{H}^i_{\mathrm{\acute{e}t}}(\overline{]Z[}, \mathbf{Q}_\ell) \simeq \mathbb{H}^i_{\mathrm{\acute{e}t}}(\overline{Z}, R\Psi_f(\mathbf{Q}_{\ell,X_\eta})|_{\overline{Z}}).$$

It is worth noting that Berkovich's theorem holds over general non-archimedean fields whereas, to state Theorem 1, we need to assume equal characteristic zero. Indeed, this is required in [3] to ensure the existence of the equivalence $\mathfrak{R}$.

3. Relation with the work of Denef and Loeser

Let X be a semi-stable R-scheme. Denote by $(D_i)_{i\in I}$ be the irreductible components of its special fiber and set

$$D_J := \cap_{j\in} D_j \quad D_J^\circ := D_J \setminus \cup_{i\notin J} D_i$$

for a non-empty subset $J \subseteq I$.

Theorem 2 (see [4, 8])**.** *Let X be a semi-stable R-scheme. For $\emptyset \neq J \subset I$, let $\rho_J : \widetilde{D}_J^\circ \to D_J^\circ$ be the étale finite cover defined as in* [8, §3.1.3]*. Then, one has the formula*

$$[\Psi_f(\mathbb{1}_{X_\eta})] = \sum_{\emptyset\neq J\subset I} (-1)^{|J|-1} \left[\mathrm{M}_{X_\sigma,c}(\widetilde{D}_J^\circ \times_k \mathbf{G}_{m,k}^{|J|-1})\right]$$

in $\mathrm{K}_0(\mathbf{SH}_{\mathsf{ct}}(X_\sigma))$*.*

Note that $\mathbf{SH}_{\mathsf{ct}}(X_\sigma)$ is the full triangulated subcategory of $\mathbf{SH}(X_\sigma)$ formed by the constructible motives. In [7], Denef and Loeser have introduced a notion *motivic nearby cycles* which is an element ψ_f in the Grothendieck ring of the special fiber $\mathscr{M}_{X_\sigma}$ and is defined in terms of the arc scheme as the limit of the motivic zeta function associated with f.

Corollary 1 (see [4, 8])**.** *Let X be a finite type R-scheme with smooth generic fiber and denote by $f : X \to \mathrm{Spec}(R)$ its structural morphism. We have the equality*

$$[\Psi_f(\mathbb{1}_{X_\eta})] = \chi_{X_\sigma,c}(\psi_f)$$

in $\mathrm{K}_0(\mathbf{SH}_{\mathsf{ct}}(X_\sigma))$.

Here

$$\chi_{X_\sigma,c} : \mathscr{M}_{X_\sigma} \to \mathrm{K}_0(\mathbf{SH}_{\mathsf{ct}}(X_\sigma))$$

is the motivic Euler characteristic over the special fiber X_σ. A particular case of Theorem 1 gives an isomorphism of motives

$$1^* \circ \mathfrak{R}(\mathrm{M}^\vee_{\mathrm{rig}}(\mathscr{F}_x)) \simeq x^*\Psi_f(\mathbb{1}_{X_\eta})$$

where $\mathscr{F}_x$ is the analytic Milnor fiber at x introduced by Nicaise-Sebag in [9]. From this and Corollary 1, we deduce the following formula

$$[1^* \circ \mathfrak{R}(\mathrm{M}^\vee_{\mathrm{rig}}(\mathscr{F}_x))] = \chi_{k,\mathrm{c}}(\psi_{f,x})$$

where $\psi_{f,x} := x^*\psi_f$ in $\mathscr{M}_k$ and $\chi_{k,\mathrm{c}} : \mathscr{M}_k \to \mathrm{K}_0(\mathbf{SH}_{\mathsf{ct}}(k))$ is the motivic Euler characteristic over k. This formula expresses the fact that the motivic Milnor fiber $\psi_{f,x}$ of Denef-Loeser, at least as a class in the Grothendieck ring of constructible motives, is determined by the rigid motive of the analytic Milnor fiber.

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A motivic version of Segal's theorem

Ivan Panin

(joint work with G.Garkusha)

Using the machinery of framed sheaves developed by Voevodsky [Voe2] we state and prove a motivic Segal's theorem. It states particularly that for any infinite perfect field k and any integer $n > 0$ a natural morphism

$$\underline{Hom}(\Delta^\bullet \times -, \Omega^\infty_{\mathbb{P}^1}\Sigma^\infty_T(T^n)) \to \Omega^\infty_{\mathbb{P}^1,mot}\Sigma^\infty_T(T^n)$$

of motivic spaces is a stalk-wise weak equivalence for the Nisnevich topology. Here $T^n = \mathbb{A}^n/(\mathbb{A}^n - \{0\})$ is the motivic $(2n, n)$-sphere, $\Omega_{\mathbb{P}^1}$ is the naive $\mathbb{P}^1$-loop-functor and $\Omega_{\mathbb{P}^1,mot}$ is the motivic $\mathbb{P}^1$-loop-functor. To prove this result a triangulated category of framed motives $SH^{fr}_{S^1}(k)$ is introduced and studied.

We also construct a compactly generated triangulated category of framed bispectra $SH^{fr}(k)$. Using an extension of the above theorem it is proved that $SH^{fr}(k)$ reconstructs the Voevodsky category $SH(k)$ in the case of an infinite and perfect field.

As a topological application, it is proved that in the complex number case for any integer $n > 0$ there is an equality in the ordinary homotopy category of simplicial sets

$$Fr(\Delta^\bullet_{\mathbb{C}}, T^n) = \Omega^\infty_{S^1}\Sigma^\infty_{S^1}(S^{2n}),$$

where S^{2n} is the ordinary $2n$-sphere.

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Euler class groups and the homology of elementary groups

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A ring R *has many units* if for every integer $n \geq 1$, there are central elements $a_1, ..., a_n \in R$ such that for all $\emptyset \neq I \subset \{1, ..., n\}$, the partial sum $a_I = \sum_{i \in I} a_i$ is a unit in R. For instance, any infinite field, any commutative local ring with infinite residue field, any (possibly non-commutative) algebra over a ring with many units has many units. For a ring R, denote by $E_n(R)$ the subgroup of $GL_n(R)$ generated by the elementary matrices. For a commutative ring R and integer $n \geq 1$, we let $SL_n(R)$ denote the subgroup of $GL_n(R)$ consisting of matrices of determinant one (for the definition of SL_0; see below). For a discrete group G, we denote by $H_n(G)$ its integral homology groups, that is, $\mathrm{Tor}_n^{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z})$. Finally, denote by $sr(R) \in \mathbb{N}$ the stable range of R. For instance, $sr(R) = 1$ for any commutative local ring, $sr(R) = 1$ for any (possibly non-commutative) artinian ring, and $sr(R) \leq 1 + \dim R$ when R is commutative noetherian.

Theorem 1 ([Sch]). *Let R be ring with many units. Then*

(1) $H_i(E_n R, E_{n-1} R) = 0, \qquad i \leq n - sr(R)$

(2) *If R is commutative, then*

$$H_i(SL_n R, SL_{n-1} R) = 0, \qquad i \leq n - sr(R).$$

When R is a field of characteristic zero, Theorem 1 was previously proved by Hutchinson and Tao [HT10]. The theorem implies a conjecture of Bass [Bas73].

Theorem 2 ([Sch]). *Let R be a ring with many units. Then*

$$\pi_i(BGL_n^+ R, BGL_{n-1}^+) = 0 \qquad i \leq n - sr(R).$$

Bass actually conjectured Theorem 2 for commutative noetherian rings without the hypothesis of "many units". However, in this general case, $R = \mathbb{Z}$ already provides a counter example.

Let A be a commutative ring, let $\mathbb{Z}[A^*]$ be the group ring of the group of units A^* in A with standard $\mathbb{Z}$-basis $\langle a \rangle$, $a \in A^*$, multiplication $\langle a \rangle \cdot \langle b \rangle = \langle ab \rangle$, and $\langle 1 \rangle = 1$. Let $I_{A^*} = \mathrm{Ker}(\mathbb{Z}[A^*] \to \mathbb{Z} : \langle a \rangle \mapsto 1)$ be the augmentation ideal, and let $[a] = \langle a \rangle - 1 \in I_{A^*}$.

Definition 3 ([Sch]). *Let A be a commutative ring (with infinite residue fields). We define the graded $\mathbb{Z}[A^*]$-algebra*

$$\widehat{K}_*^{MW}(A) = \mathrm{Tens}_{\mathbb{Z}[A^*]}\, I_{A^*} / \,\mathrm{Steinberg}$$

as the quotient of the tensor algebra of I_{A^*} over the group ring $\mathbb{Z}[A^*]$ modulo the ideal generated by the Steinberg relations $[a] \otimes [1 - a]$ for $a, 1 - a \in A^*$.

For instance,

$$\widehat{K}_0^{MW}(A) = \mathbb{Z}[A^*], \quad \widehat{K}_1^{MW}(A) = I_{A^*}, \quad \widehat{K}_2^{MW}(A) = I_{A^*} \otimes_{A^*} I_{A^*} / \,\mathrm{Steinberg}\,.$$

We prove in [Sch] that the group $\widehat{K}_n^{MW}(A)$ is isomorphic to the Morel-Hopkins Milnor-Witt K-group $K_n^{MW}(A)$ when $n \geq 2$ and A is a field or local ring with residue field $\neq \mathbb{F}_2, \mathbb{F}_3$. We have the following strengthening of Theorem 1. Here, SL_0A is defined to be the discrete set of units A^* in A, so as to make the formula $H_i(SL_nA) = \mathrm{Tor}_i^{\mathbb{Z}[GL_nA]}(\mathbb{Z}[A^*], \mathbb{Z})$ hold for all i, n.

Theorem 4 ([Sch])**.** *Let A be a commutative local ring with infinite residue field. Then there are isomorphisms of A^*-modules for all $n \geq 0$*

$$H_i(SL_nA, SL_{n-1}A; \mathbb{Z}) \cong \begin{cases} 0 & i < n \\ \widehat{K}_n^{MW}(A) & i = n. \end{cases}$$

This theorem is the SL_n-analogue of a theorem of Nesterenko-Suslin [NS89]. A version of the theorem was proved for fields of characteristic zero by Hutchinson-Tao [HT10].

Let R be a noetherian ring with infinite residue fields, and let P be an oriented projective R-module of rank n. We define the "Euler class" $e(P)$ of P as a certain Zariski cohomology class

$$e(P) \in H^n_{Zar}(R, \mathcal{K}_n^{MW})$$

where $\mathcal{K}_n^{MW}$ denotes the Zariski sheaf associated with the presheaf $A \mapsto \widehat{K}_n^{MW}(A)$. Using the previous theorem, we can show the following.

Theorem 5 ([Sch])**.** *Let R be a commutative noetherian ring of dimension $n \geq 2$. Assume that all residue fields of R are infinite. Let P be an oriented rank n projective R-module. Then*

$$P \cong Q \oplus R \Leftrightarrow e(P) = 0 \in H^n_{Zar}(R, \mathcal{K}_n^{MW}).$$

Using $\mathbb{A}^1$-homotopy theory, a version of this theorem was proved by Morel [Mor12] for smooth algebras over infinite perfect fields.

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