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## Measured Group Theory

Organised by  
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28 August – 3 September 2016

ABSTRACT. The workshop aimed to study discrete and Lie groups and their actions using measure theoretic methods and their asymptotic invariants, such as  $\ell^2$ -invariants, the rank gradient, cost, torsion growth, entropy-type invariants and invariants coming from random walks and percolation theory. The participants came from a wide range of mathematics: asymptotic group theory, geometric group theory, ergodic theory,  $\ell^2$ -theory, graph convergence, representation theory, probability theory, descriptive set theory and algebraic topology.

*Mathematics Subject Classification (2010):* 05C25, 20E05, 20E26, 20F69, 22D40, 22E15, 22E40, 37A20, 37A50.

### Introduction by the Organisers

The workshop ‘Measured group theory’ organized by Miklos Abert (Budapest), Damien Gaboriau (Lyon) and Andreas Thom (Dresden) was held 28 August - 2 September 2016. The event was an important next stage for the recently emerging field of measured group theory.

As measured group theory is progressing, the participants of the meetings understand each other’s language better and better. Still, for a uniform random participant and talk of the event, it was a first date with probability at least  $1/2$ .

The organizers made an effort to keep the number of talks down (half-successfully) and ask young people for talks (quite successfully).

It is natural to arrange the workshop material along the following subtopics.

### **Space of subgroups, invariant random subgroups and Benjamini–Schramm convergence.**

Mikolaj Fraczyk talked about his result that for a rank 1 simple Lie group, all sequences of distinct congruence locally symmetric spaces Benjamini-Schramm converge to the full symmetric space of the Lie group.

Vadim Alekseev talked about his joint with Miklos Abert, Andreas Thom and Rahel Brugger on invariant random positive definite functions and disintegration rigidity. An invariant random positive definite function is a generalization of a character the same way an invariant random subgroup is a generalization of a normal subgroup.

Gabor Kun talked about his result that any sofic approximation (a graph sequence that Benjamini–Schramm converges to a Cayley graph of the group) of a property (T) group is close to a disjoint union of expander graphs.

Nicolas Matte Bon talked about his joint work with Adrien Le Boudec on subgroup dynamics and  $C^*$ -simplicity of groups of homeomorphisms. As a tool, they use uniformly recurrent subgroups, a topological version of invariant random subgroups recently introduced by Eli Glasner and Benjy Weiss.

Emmanuel Breuillard talked about his joint work with Mehrdad Kalantar, Matthew Kennedy and Narutaka Ozawa on  $C^*$ -simplicity and the unique trace property for discrete groups.

### **Growth of homology and torsion, $L^2$ theory.**

Roman Sauer talked about his joint with Uri Bader and Tsachik Gelander on torsion homology growth in negative curvature. They show that for the family of closed hyperbolic 3-manifolds, the volume-normalized log torsion can be arbitrarily large, even when one assumes Benjamini-Schramm convergence to  $H^3$ . However, starting from dimension 4, the picture changes and they show that this invariant stays bounded, in the spirit of Betti numbers in the Ballmann-Gromov-Schroeder theorem.

Nikolay Nikolov talked about his joint with Miklos Abert and Tsachik Gelander. They found a new vanishing condition on the first homology torsion growth for finitely generated groups that applies to a large class of higher rank irreducible lattices.

Wolfgang Lück talked about his joint work with Stefan Friedl on universal  $L^2$ -torsion, twisted  $L^2$ -Euler characteristic, Thurston norm and polytopes.

Andrei Jaikin-Zapirain talked about the base change in the strong Atiyah conjecture and the Lück approximation conjecture, in which he extends the known results on the strong Atiyah conjecture to complex coefficients.

### **Borel and p.m.p. actions of countable groups and their graphings.**

Oleg Pikhurko talked about his joint recent breakthrough work with Lukasz Grabowski and Andras Mathe on squaring the circle (disc) with measurable pieces.

Robin Tucker-Drob talked about ergodic hyperfinite subgraphs and primitive subrelations. Among other things, he showed that every ergodic graphing contains an ergodic hyperfinite subgraphing.

Lukasz Grabowski talked about his joint work with Endre Csoka, Andras Mathe, Oleg Pikhurko and Konstantinos Tyros on the Borel Local Lemma. The Lovasz Local Lemma is a fundamental tool in discrete mathematics, and various Borel (measurable) generalizations have been used successfully in measured group theory.

Anush Tserunyan talked about finite generating partitions for actions of countable groups. In particular, she proved that if a Borel action of a countable group with a  $\sigma$ -compact realization admits no invariant measure, then it has a size 32 generating partition.

Clinton Conley talked about the measurable coloring theory of graphings, and analyzed measurable chromatic numbers.

Andrew Marks talked about his joint work with Clinton Conley, Steve Jackson, David Kerr, Brandon Seward, and Robin Tucker-Drob on measurable tilings of free pmp actions of amenable groups.

### **Entropy theory of group actions.**

Brandon Seward talked about his recent result that positive Rokhlin entropy actions of countable groups factor onto Bernoulli shifts. Note that his generalization of Sinai's theorem (that proves the same for  $\mathbb{Z}$ ) does not assume the soficity of the group, as most of the known entropy results do.

Felix Pogorzelski talked about his joint work joint with Amos Nevo on subadditive convergence and cocycle entropy, in which they define a new entropy notion for invariant coloring processes on countable groups.

Ben Hayes talked about his work on Fuglede-Kadison determinants and sofic entropy. His work completes the computation of both the topological and measure entropy of algebraic actions in terms of spectral measure.

### **Probability on groups.**

Lison Jacoboni talked about metabelian groups where the return probability is asymptotically as large as possible, assuming the group has exponential growth.

Adam Timar talked about indistinguishability (or ergodicity) of infinite clusters in random spanning forests. In particular, he showed that if  $p_c < p_u$  for a Cayley graph, then the infinite clusters of the free minimal spanning forest are indistinguishable.

Gady Kozma talked about his joint work with Gideon Amir, Itai Benjamini, Hugo Duminil-Copin, Ariel Yadin and Tianyi Zheng on harmonic functions and the log log law. They analyze the possible growth functions of harmonic functions on Cayley graph in terms of the word metric, for various classes of groups.

### **The group of talks that can not be put in a group.**

Uri Bader talked about his recent joint work with Pierre-Emmanuel Caprace and Jean Lecureux on the non-linearity of  $\tilde{A}_2$ -lattices. While the results are not inherently measure theoretic, the proof uses ergodic theory in a surprising way.

Friedrich Martin Schneider talked about his joint with Andreas Thom, in which they find a new Følner-type condition that is equivalent to amenability for Hausdorff topological groups.

Adrien Le Boudec talked about his joint work with Yves Cornuier on infinite presentability and the relation range of groups.

Damian Osajda talked about his construction of finitely generated groups with isometrically embedded expanders. The construction, that uses a version of small cancellation, has various applications, e.g. they do not coarsely embed into a Hilbert space.

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## Abstracts

### Positive entropy actions of countable groups factor onto Bernoulli shifts

BRANDON SEWARD

For a countably infinite group  $G$  and a standard probability space  $(L, \lambda)$ , the *Bernoulli shift* over  $G$  with base space  $(L, \lambda)$  is the product measure space  $(L^G, \lambda^G)$  together with the left shift-action of  $G$ : for  $g \in G$  and  $x \in L^G$ ,  $g \cdot x$  is defined by  $(g \cdot x)(t) = x(g^{-1}t)$  for  $t \in G$ . The *Shannon entropy* of the base space  $(L, \lambda)$  is defined as

$$H(L, \lambda) = \sum_{\ell \in L} -\lambda(\ell) \log \lambda(\ell),$$

if  $\lambda$  has countable support, and  $H(L, \lambda) = \infty$  otherwise.

The *Kolmogorov–Sinai entropy*  $h_{\mathbb{Z}}^{\text{KS}}(X, \mu)$  of a probability-measure-preserving action  $\mathbb{Z} \curvearrowright (X, \mu)$  was introduced by Kolmogorov in 1958 (and the definition was improved by Sinai in 1959). Entropy provided a useful tool for studying dynamics of  $\mathbb{Z}$ -actions, and it eventually provided powerful insight into the nature of Bernoulli shifts over  $\mathbb{Z}$ . One of the most well known results in entropy theory is the following theorem of Sinai, which reveals a significant and unexpected structural consequence of having positive entropy.

**Theorem 1** (Sinai’s factor theorem, 1962 [7]). *If  $\mathbb{Z} \curvearrowright (X, \mu)$  is an ergodic probability-measure-preserving action and  $(L, \lambda)$  is a probability space with  $H(L, \lambda) \leq h_{\mathbb{Z}}^{\text{KS}}(X, \mu)$ , then  $\mathbb{Z} \curvearrowright (X, \mu)$  factors onto the Bernoulli shift  $(L^{\mathbb{Z}}, \lambda^{\mathbb{Z}})$ .*

This theorem was extended to actions of countable amenable groups by Ornstein and Weiss in 1987 [5].

In essence, this is a structure theorem which indicates that Bernoulli shifts are the source of all positive entropy (this can be made precise using the language of relative entropy). Sinai’s theorem also admits many applications, as it is relatively easy to compute the entropy of a  $\mathbb{Z}$ -action but extremely difficult to build a factor which is Bernoulli. In fact, prior to this theorem it was not even known that Bernoulli shifts over  $\mathbb{Z}$  of large entropy factor onto all Bernoulli shifts of smaller entropy.

Sinai’s theorem is also of historical importance, as it serves as a critical foundation to the development of Ornstein theory. Ornstein theory is generally considered to be the deepest and greatest achievement of entropy theory. It began with Ornstein’s famous isomorphism theorem [3, 4] (which states that two Bernoulli shifts over  $\mathbb{Z}$  are isomorphic if and only if they have equal entropy) but over time evolved into a collection of necessary and sufficient conditions for a probability-measure-preserving  $\mathbb{Z}$ -action to be isomorphic to a Bernoulli shift. This led to the surprising discovery that factors of Bernoulli shifts over  $\mathbb{Z}$  are Bernoulli, inverse limits of Bernoulli shifts over  $\mathbb{Z}$  are Bernoulli, and that many interesting and natural actions of  $\mathbb{Z}$  are isomorphic to Bernoulli shifts (such as ergodic automorphisms

of compact metrizable groups, mixing Markov shifts, geodesic flows on surfaces of negative curvature, Anosov flows with smooth measures, and two dimensional billiards with a convex scatterer).

For a few decades the notion of entropy was restricted to the realm of actions of amenable groups. However, recent ground-breaking work of Bowen [1], together with improvements by Kerr and Li [2], created the notion of *sofic entropy* for probability-measure-preserving actions of sofic groups. The class of sofic groups contains the countable amenable groups, and sofic entropy coincides with Kolmogorov–Sinai entropy for actions of amenable groups. This has led to a surge of new research into entropy theory where, for the first time, the acting groups are non-amenable. We draw our motivation from this, but we work with an alternate notion of entropy previously introduced by the speaker [6].

For any countable group  $G$  (not necessarily sofic) and any ergodic probability-measure-preserving action  $G \curvearrowright (X, \mu)$ , we define the *Rokhlin entropy* of  $G \curvearrowright (X, \mu)$  as

$$h_G^{\text{Rok}}(X, \mu) = \inf \left\{ H(\alpha) : \alpha \text{ is a countable partition with } \sigma\text{-alg}_G(\alpha) = \mathcal{B}(X) \right\}.$$

Here  $H(\alpha) = \sum_{A \in \alpha} -\mu(A) \log \mu(A)$  is the Shannon entropy of  $\alpha$ ,  $\sigma\text{-alg}_G(\alpha)$  denotes the smallest  $G$ -invariant  $\sigma$ -algebra containing  $\alpha$ , and  $\mathcal{B}(X)$  denotes the Borel  $\sigma$ -algebra of  $X$ . For free actions of countable amenable groups, Rokhlin entropy coincides with Kolmogorov–Sinai entropy, and for free actions of sofic groups it is greater than or equal to sofic entropy. For free actions of sofic groups, it is unknown if Rokhlin entropy coincides with sofic entropy (when sofic entropy is not minus infinity).

The following is the main theorem.

**Theorem 2.** *Let  $G$  be a countably infinite group, let  $G \curvearrowright (X, \mu)$  be a free ergodic probability-measure-preserving action, and let  $(L, \lambda)$  be a probability space. If  $H(L, \lambda) \leq h_G^{\text{Rok}}(X, \mu)$ , then  $G \curvearrowright (X, \mu)$  factors onto the Bernoulli shift  $G \curvearrowright (L^G, \lambda^G)$ .*

Since sofic entropy is bounded above by Rokhlin entropy, this theorem remains true if Rokhlin entropy is replaced by sofic entropy.

As with the original Sinai theorem, before obtaining this theorem it was not known if each Bernoulli shift factors onto all smaller entropy Bernoulli shifts.

All prior proofs of Sinai's theorem have relied critically upon special properties of actions of amenable groups (such as the Rokhlin lemma, the Shannon–McMillan–Breiman theorem, and the monotonicity of Kolmogorov–Sinai entropy under factor maps). The proof of the above theorem relies upon an entirely new technique.

The development of an Ornstein theory for actions of non-amenable groups, possibly based off of the above theorem and its proof, is a possible direction for future research.



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## Homology torsion growth in right angled lattices

NIKOLAY NIKOLOV

(joint work with Miklós Abért and Tsachik Gelander)

A group  $G$  is right angled if  $G$  can be generated by elements  $g_1, \dots, g_k$  of infinite order such that  $[g_i, g_{i+1}] = 1$  for  $i = 1, \dots, k - 1$ . Right angled groups were first studied by D. Gaboriau who proved that they have fixed price 1. Building on his argument and developing the theory of combinatorial cost initiated by G. Elek we prove the following

**Theorem 1** ([2]). *Let  $(H_i)$  be a Farber sequence in a right angled group  $G$ . Then  $\lim_i \frac{d(H_i)-1}{[G:H_i]} = 0$ .*

Here  $d(G)$  denotes the minimal size of a generating set of a group  $G$ . A sequence  $(H_i)$  of finite index subgroups of  $G$  is Farber if the Schreier graphs  $Sch(G/H_i, S)$  (with respect to some fixed generating set  $S$  of  $G$ ) form a sofic approximation to the Cayley graph  $Cay(G, S)$ .

We can apply our tools to the study of growth of torsion in homology. It is easy to see that for a finitely presented group  $G$  with a finite index subgroup  $M$  the torsion  $torH_1(M, \mathbb{Z})$  of  $H^{ab} \simeq H_1(M, \mathbb{Z})$  is bounded above by an exponential function of the index  $[G : M]$ .

**Theorem 2** ([2]). *Let  $G$  be a finitely presented right angled group and let  $(M_i)$  be a Farber sequence in  $G$ . Then  $\lim_i \frac{\log torH_1(M_i, \mathbb{Z})}{[G:M_i]} = 0$ .*

Note that if  $G$  is not finitely presented then there is no general bound on  $torH_1(M_i, \mathbb{Z})$  in terms of  $[G : M_i]$  even for solvable groups  $G$ , see [3].

We apply these results to lattices in simple Lie groups. While some lattices are right angled (for example  $SL(n, \mathbb{Z})$  with  $n \geq 3$ ), all known ones until now were non-cocompact. We construct the first examples of cocompact right angled lattices:

**Theorem 3** ([2]). *Let  $\mathcal{G}$  be  $SL(n, \mathbb{R})$ , ( $n > 2$ ) or  $SO(n, 2)$ , ( $n \geq 7$ ). Then  $\mathcal{G}$  has cocompact right angled lattices.*

By combining our methods with the results on invariant random subgroups of lattices in higher rank Lie groups [1] we deduce the following:

**Corollary 4** ([2]). *Let  $G$  be a right angled lattice in a higher rank simple Lie group with trivial center. Let  $(M_i)$  be any sequence of finite index subgroups of  $G$  with  $[G : M_i] \rightarrow \infty$ . Then*

$$\lim_{i \rightarrow \infty} \frac{d(M_i) - 1}{[G : M_i]} = \lim_{i \rightarrow \infty} \frac{\log \text{tor} H_1(M_i, \mathbb{Z})}{[G : M_i]} = 0.$$

We finish with the following ambitious conjecture:

**Conjecture 5.** *Let  $\mathcal{G}$  be a higher rank simple Lie group and let  $(M_i)$  be any sequence of lattices of  $\mathcal{G}$  with unbounded covolume. Then  $\lim_i \frac{d(M_i) - 1}{\text{vol}(\mathcal{G}/M_i)} = 0$ .*

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### Torsion homology growth in negative curvature

ROMAN SAUER

(joint work with Uri Bader and Tsachik Gelander)

The topological complexity of Hadamard manifolds is controlled, to some extent, by the volume. This phenomenon is most nicely illustrated in the case of surfaces of constant negative curvature. Indeed, the Gauss–Bonnet theorem implies that the volume coincides (up to a normalization) with the Euler characteristic, which in turn determines the homeomorphism type of the manifold. In much greater generality, a celebrated theorem of Ballmann, Gromov and Schroeder [1] says that the Betti numbers of a negatively<sup>1</sup> curved manifold are bounded by the volume.

By *normalized bounded negative curvature* we mean that the sectional curvature is contained in a closed sub-interval of  $[-1, 0)$ .

**Theorem 1** (Ballmann–Gromov–Schroeder). *For every  $d \in \mathbb{N}$  there exists  $C = C_d > 0$  such that for every complete  $d$ -dimensional Riemannian manifold of normalized bounded negative curvature and for every degree  $k$ ,*

$$\text{rank } H_k(M; \mathbb{Q}) \leq C \text{ vol}(M).$$

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<sup>1</sup>A more general version of this theorem holds for non-positively curved manifolds.

That is, the abelian group  $H_k(M; \mathbb{Z})$  is isomorphic to  $\mathbb{Z}^{b_k} \oplus \text{tors}_k$  where  $b_k \leq C \text{vol}(M)$  and  $\text{tors}_k$  denotes the torsion part. In recent years there has been a growing interest in the size of the torsion part  $\text{tors}_k(M)$  motivated by number theory and topology. However,  $\text{tors}_k$  is much harder to control than  $b_k$ . In dimension  $\neq 3$  we can prove a universal upper bound in all degrees:

**Theorem 2.** *For every  $d \neq 3$ , there exists  $C = C_d > 0$  such that for every complete  $d$ -dimensional Riemannian manifold of normalized bounded negative curvature and for every degree  $k$ ,*

$$\log |\text{tors } H_k(M; \mathbb{Z})| \leq C \text{vol}(M).$$

The most complicated case in the above theorem is  $k = d - 2$ . This is also manifested in dimension  $d = 3$  where the above theorem drastically fails:

**Theorem 3.** *There exists a sequence of closed hyperbolic 3-manifolds  $M_n$  that converges in the Benjamini–Schramm topology to  $\mathbb{H}^3$  such that*

$$\lim_{n \rightarrow \infty} \frac{\log |\text{tors } H_1(M_n, \mathbb{Z})|}{\text{vol}(M_n)} = \infty.$$

*Furthermore, for any function  $f: (0, \infty) \rightarrow (0, \infty)$  there is such a sequence  $M_n$  with*

$$\log |\text{tors } H_1(M_n, \mathbb{Z})| > f(\text{vol}(M_n)).$$

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### Ergodic Hyperfinite Subgraphs and Primitive Subrelations

ROBIN TUCKER-DROB

We show that every ergodic p.m.p. graph contains an ergodic hyperfinite subgraph. This implies a conjecture of Bowen: every ergodic p.m.p. treeable equivalence relation contains an ergodic hyperfinite primitive subrelation. We also obtain the following strengthening of Hjorth’s Lemma on cost attained: every ergodic p.m.p. treeable equivalence relation of cost  $n$  is generated by a free action of the free group of rank  $n$  in which one of the generators acts ergodically.

## Finite generating partitions for actions of countable groups

ANUSH TSERUNYAN

In search of a concrete model for a given Borel action of a countable group  $G$  on a standard Borel space  $X$ , one may wonder if it is Borel-embeddable into a finite shift action of  $G$ . This is equivalent to the existence of a finite Borel generating partition, i.e. a finite partition of  $X$  into Borel sets such that every point in  $X$  is determined by its trajectory through the partition when acted upon by  $G$ .

For  $G := \mathbb{Z}$ , or more generally, for any amenable group, the existence of such a partition is obstructed by the existence of an invariant probability measure of infinite entropy. First, I showed that the question of whether this is the only obstruction boils down to proving the existence of a finite generating partition for continuous actions on Polish spaces that do not admit any invariant probability measure whatsoever. It was asked by B. Weiss [3] in the 80s whether this is true for  $G := \mathbb{Z}$  and it was asked again in [1] for arbitrary countable groups. I proved in [2] that the answer is positive (in fact, there is a 32-generator) for continuous actions of arbitrary countable groups on  $\sigma$ -compact Polish spaces.

I also showed in [2] that finite generating partitions always exist in the context of Baire category, thus answering a question of A. Kechris asked in the 90s. The precise statement is that any continuous aperiodic action of a countable group on a Polish space admits a 4-generator on an invariant comeager set.

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## Non-linearity of $\tilde{A}_2$ -lattices

URI BADER

(joint work with Pierre-Emmanuel Caprace, Jean Lecureux)

Let  $X$  be a locally finite irreducible affine building of dimension  $\geq 2$  and  $\Gamma \leq \text{Aut}(X)$  be a discrete group acting cocompactly. We address the following question: When is  $\Gamma$  linear? More generally, when does  $\Gamma$  admit a finite-dimensional representation with infinite image over a commutative unital ring? If  $X$  is the Bruhat–Tits building of a simple algebraic group over a local field and if  $\Gamma$  is an arithmetic lattice, then  $\Gamma$  is clearly linear. We prove that if  $X$  is of type  $\tilde{A}_2$ , then the converse holds. In particular, cocompact lattices in exotic  $\tilde{A}_2$ -buildings are non-linear. As an application, we obtain the first infinite family of lattices in exotic  $\tilde{A}_2$ -buildings of arbitrarily large thickness, providing also a partial answer to a question of W. Kantor from 1986. We also show that if  $X$  is Bruhat–Tits of arbitrary type, then the linearity of  $\Gamma$  implies that  $\Gamma$  is virtually contained in the

linear part of the automorphism group of  $X$ ; in particular  $\Gamma$  is an arithmetic lattice. The proofs are based on the machinery of algebraic representations of ergodic systems recently developed by U. Bader and A. Furman. The implementation of that tool in the present context requires the geometric construction of a suitable ergodic  $\Gamma$ -space attached to the building  $X$ , which we call the *singular Cartan flow*.

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## Fuglede-Kadison determinants and sofic entropy

BEN HAYES

I discussed my results in [6] computing the entropy of certain algebraic actions. Given a countable, discrete, group  $G$ , an *algebraic action* is an action of  $G$  by automorphisms on a compact, metrizable, abelian group  $X$ . By Pontryagin duality, every such action arises in the following manner: consider a countable  $\mathbb{Z}(G)$ -module  $A$  and let  $X$  be the group of homomorphisms from  $A$  into  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$  (this is typically denoted by  $\widehat{A}$ ). One particular class of modules that one can take are those of the form  $A_f = \mathbb{Z}(G)^{\oplus n}/\mathbb{Z}(G)^{\oplus k} f$  for some  $f \in M_{k,n}(\mathbb{Z}(G))$ . In a certain sense this is the “generic”  $\mathbb{Z}(G)$ -module, since every finitely presented  $\mathbb{Z}(G)$  module is of such a form, and so a general  $\mathbb{Z}(G)$ -module is an increasing union of inverse limits of such modules. The dual of  $A_f$  is usually denoted by  $X_f$  and many previous works have tackled the problem of computing the measure-theoretic (or topological) entropy of  $G$  acting on  $X_f$ , equipping  $X_f$  with the Haar measure  $m_{X_f}$  (see e.g. [9],[4],[5],[7],[3]) with the amenable case only being recently settled in 2014 by the work of Li-Thom [8].

In this talk, I presented my work which completes the computation of entropy for these actions.

**Theorem 1.** *Let  $f \in M_{k,n}(\mathbb{Z}(G))$  and let  $G$  be a sofic group. Then*

- (1) *The topological entropy of the action of  $G$  on  $X_f$  is finite if and only if  $f$  is injective as a convolution operator  $\ell^2(G)^{\oplus n} \rightarrow \ell^2(G)^{\oplus k}$ .*
- (2) *If  $f$  is injective as a convolution operator on  $\ell^2(G)^{\oplus n} \rightarrow \ell^2(G)^{\oplus k}$ , then the topological entropy of the action of  $G$  on  $X_f$  is at most the logarithm of the Fuglede-Kadison determinant of  $f$ .*
- (3) *If  $k = n$  and  $f$  is injective as a convolution operator on  $\ell^2(G)^{\oplus n} \rightarrow \ell^2(G)^{\oplus k}$ , then the topological entropy of the action of  $G$  on  $X_f$  equals the measure-theoretic entropy of the action of  $G$  on  $X_f$  and both are the logarithm of the Fuglede-Kadison determinant of  $f$ .*

Many of the previous works on this problem focused on the case  $k = n = 1$  (such algebraic actions are called principal algebraic actions).

In the talk I discussed the definition of a sofic group, as well as the definition of sofic entropy (properly speaking what I defined is not the sofic entropy as defined by Bowen in [2], but the model-measure entropy defined in [1]). Sofic groups are a vastly larger class of groups than amenable groups and they include all free groups, all residually finite groups, all linear groups, all amenable groups and are closed under free products with amalgamation over amenable subgroups. Thus entropy for actions of sofic groups may be considered to be a significant generalization of entropy for actions of amenable groups. Afterwards, I defined the Fuglede-Kadison determinant for elements in  $M_n(\mathbb{C}(G))$ , using the intuition of finite-dimensional linear algebra. I then stated the main theorem (described above). When  $G$  is amenable, one can relate the  $L^2$ -torsion of  $A$  to the entropy of the action of  $G$  on  $\widehat{A}$ . I discussed why, in the sofic case, such a connection is *impossible* e.g. it necessarily fails when  $G$  is a cocompact lattice in  $SO(n, 1)$  and  $n \equiv 1 \pmod{4}$ .

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### Invariant random positive definite functions and disintegration rigidity

VADIM ALEKSEEV

(joint work with Miklós Abért, Andreas Thom, Rahel Brugger)

The study of group representations had motivated a great amount of interest in positive definite class functions (characters) on groups since the work of Godement and Thoma. The main object of interest in this research are characters of discrete groups:

**Definition 1.** Let  $\Gamma$  be a discrete group. A function  $\varphi: \Gamma \rightarrow \mathbb{C}$  with  $\varphi(e) = 1$  is called

- (1) positive definite, if the matrix  $[\varphi(g_j^{-1}g_i)]_{i,j}$  is positive semidefinite for all  $g_1, \dots, g_n \in \Gamma$ ;
- (2) conjugation invariant if  $\varphi(ghg^{-1}) = \varphi(h)$  for all  $g, h \in \Gamma$ ;
- (3) a character, if it's positive definite and conjugation invariant.

Characters form a closed convex set (in fact, a Choquet simplex) with respect to the weak\* topology on  $\ell^\infty(\Gamma)$ ; in particular, the study of characters reduces to the study of extreme points in the space of characters. Two basic examples of characters are the *trivial* character  $\tau_\varepsilon$  mapping every group element to 1 and the *left regular* character  $\tau_\lambda$  mapping the identity to 1 and all other group elements to zero. The terminology is justified by the following correspondence between characters and representations into finite von Neumann algebras:

**Theorem 2** (Thoma). *Let  $\Gamma$  be a discrete group. There is a one-to-one correspondence between*

- (1) characters  $\varphi: \Gamma \rightarrow \mathbb{C}$  and
- (2) equivalence classes of representations  $\pi_\varphi: \Gamma \rightarrow \mathcal{U}(M)$ , where  $(M, \tau)$  is finite von Neumann algebra such that  $\pi_\varphi(\Gamma)'' = M$ .

*This correspondence is given by  $\varphi = \tau \circ \pi_\varphi$ . Moreover, extreme characters correspond to representations where  $M$  is a factor.*

In the recent years, the phenomenon of *character rigidity* has received much attention:

**Definition 3.** A discrete group  $\Gamma$  is character rigid if the only extreme characters are the trivial one  $\tau_\varepsilon$  and the left regular one  $\tau_\lambda$ .

It was shown by Bachir Bekka that  $PSL(n, \mathbb{Z})$  is character rigid for  $n \geq 3$ ; later on, these results were extended by Jesse Peterson and Andreas Thom to  $SL(2, k)$  or  $SL(n, \mathcal{O}_K)$ ,  $n \geq 2$ , where  $k$  is an infinite field and  $\mathcal{O}_K$  is the ring of integers of an algebraic number field  $K$  containing infinitely many units. Recently, Jesse Peterson has used operator algebraic techniques to prove character rigidity for arbitrary lattices in higher rank simple Lie groups.

The ongoing joint project with Miklós Abért, Andreas Thom and Rahel Brugger is concerned with a refinement of character rigidity called decomposition rigidity which stems from the question how a given character of the group (for instance, the regular one) can be decomposed into a conjugation-invariant combination of states (this realises the idea of “invariant random positive definite functions”).

**Definition 4.** Let  $\Gamma$  be a discrete group, let  $\Gamma \curvearrowright (\Omega, \nu)$  be a p.m.p. action. An invariant random positive definite function is a  $\Gamma$ -equivariant map  $\Phi: \Omega \rightarrow \text{PD}(\Gamma)$ , where  $\text{PD}(\Gamma)$  is the space of positive definite functions on  $\Gamma$ . It is called ergodic if the action  $\Gamma \curvearrowright (\Omega, \nu)$  is ergodic.

Notice that if  $\Phi$  is a random invariant positive definite function, then its conditional expectation

$$\varphi = \mathbb{E}(\Phi) := \int_{\Omega} \Phi_{\omega} d\nu(\omega)$$

is a character on  $\Gamma$ . Therefore we can think of a random invariant positive definite function as of a disintegration of a given character. This motivates the following definition.

**Definition 5.** A character  $\varphi: \Gamma \rightarrow \mathbb{C}$  is called *disintegration rigid* if for every ergodic invariant random positive definite function  $\Phi$  with  $\mathbb{E}(\Phi) = \varphi$  we have  $\Phi_\omega = \varphi$  almost everywhere.

A discrete group  $\Gamma$  is called *disintegration rigid* if its left regular character is disintegration rigid.

To study disintegration rigidity, we establish the correspondence between invariant random characters and representations  $\pi: \Gamma \rightarrow \mathcal{N}_M(A)$ , where  $A \subset M$  is an inclusion of a (diffuse) abelian von Neumann algebra into a finite one:

**Theorem 6.** Let  $\alpha: \Gamma \curvearrowright (\Omega, \nu)$  be an ergodic p.m.p. action and let  $A = L^\infty(\Omega, \nu)$ . There is a one-to-one correspondence between:

- (1) invariant random positive definite functions  $\varphi: \Omega \rightarrow \text{PD}(\Gamma)$ ,
- (2)  $\Gamma$ -equivariant positive definite functions  $\varphi: \Gamma \rightarrow A$ :

$$\varphi(g^{-1}hg) = \alpha_g(\varphi(h)), \quad g, h \in \Gamma,$$

- (3) equivalence classes of equivariant representations  $\pi: \Gamma \rightarrow \mathbb{B}({}_A H)$  into a (left) Hilbert  $A$ -module  $H = {}_A H$  with a distinguished cyclic vector  $\xi_0 \in {}_A H$ ,
- (4) equivalence classes of representations  $\pi: \Gamma \rightarrow \mathcal{N}_M(A)$  for a trace-preserving inclusion  $A \subset M$  into a tracial von Neumann algebra  $(M, \tau)$  such that  $(A \cup \pi(\Gamma))'' = M$ . Here  $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) \mid uAu^* = A\}$  is the normaliser of  $A$  inside  $M$ .

This allows us to deduce another characterisation of disintegration rigidity: a group  $\Gamma$  is disintegration rigid if and only if for any representation  $\pi: \Gamma \rightarrow \mathcal{N}_M(A)$  for a trace-preserving inclusion  $A \subset M$  into a tracial von Neumann algebra  $(M, \tau)$  such that  $(A \cup \pi(\Gamma))'' = M$  and  $\pi(\Gamma)'' = L\Gamma$ , we have that  $M = A \rtimes \Gamma$ .

The results above open the way to generalise the techniques of Bachir Bekka and Jesse Peterson to allow for disintegration rigidity results. In the joint ongoing work with Miklós Abért and Andreas Thom we apply the techniques developed by Bekka to prove disintegration rigidity for  $PSL(n, \mathbb{Z})$ ,  $n \geq 3$  as well as the techniques by Peterson and Thom for  $SL(2, k)$  or  $SL(n, \mathcal{O}_K)$ ,  $n \geq 2$ , where  $k$  is an infinite field and  $\mathcal{O}_K$  is the ring of integers of an algebraic number field  $K$  containing infinitely many units. In the ongoing joint work with Rahel Brugger we apply Peterson's techniques to prove disintegration rigidity for lattices in higher rank Lie groups.

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## Measurable chromatic numbers

CLINTON T. CONLEY

Since Kechris-Solecki-Todorćević’s seminal paper [KST99], there has been considerable investigation into Borel colorings of Borel graphs on standard Borel spaces. In this talk we discuss analogs on standard probability spaces: the  $\mu$ -measurable chromatic number in which one is allowed to discard a  $\mu$ -null set before coloring in a Borel fashion, and the  $\mu$ -approximate chromatic number in which one is allowed to discard sets of arbitrarily small measure. We discuss joint work with Kechris and Tucker-Drob characterizing amenability, property (T), and the HAP in terms of these chromatic numbers for Cayley graphings of bipartite groups.

We also pay special attention to the hyperfinite setting. Recall that a graph is called *hyperfinite* when its connectedness relation may be realized by a Borel action of the integers; if null sets may be discarded this aligns with amenability. Joint work with B. Miller shows that such graphs have  $\mu$ -measurable chromatic number at most  $2\chi - 1$ , where  $\chi$  is the real chromatic number of the graph (without definability constraints). On the other hand, recent work with Jackson, Marks, Seward, and Tucker-Drob shows that no such uniform bound is possible for Borel colorings.

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## Equidecompositions with Lebesgue measurable pieces

OLEG PIKHURKO

(joint work with Łukasz Grabowski and András Máthé)

Two subsets  $A$  and  $B$  of  $\mathbb{R}^k$  are (*set-theoretically*) *equidecomposable* if it is possible to find a partition of  $A$  into finitely many pieces and rearrange these pieces using isometries to form a partition of  $B$ . The most famous result about equidecomposable sets is known as the *Banach-Tarski paradox*: in  $\mathbb{R}^3$ , the unit ball and two disjoint copies of the unit ball are equidecomposable. It is a special case of the following theorem.

**Theorem 1** (Banach and Tarski [1]). *When  $k \geq 3$ , any two bounded sets with non-empty interiors in  $\mathbb{R}^k$  are equidecomposable. When  $k \leq 2$ , equidecomposable sets which are (Lebesgue) measurable have the same measure.*

In view of this result, Tarski [9] formulated the following problem, known as *Tarski's circle squaring*: *is the unit disk in  $\mathbb{R}^2$  equidecomposable to a square of the same area?* Some 65 years later, Laczkovich [6] showed that Tarski's circle squaring is possible. In fact, his main result (coming from the papers [6, 7, 8]) is much more general and stronger. In order to state it, we need some definitions.

We call two sets  $A, B \subseteq \mathbb{R}^k$  *equivalent* (and denote this by  $A \overset{\text{Tr}}{\sim} B$ ) if they are equidecomposable using translations, that is, there are partitions  $A = A_1 \cup \dots \cup A_m$  and  $B = B_1 \cup \dots \cup B_m$ , and vectors  $\mathbf{v}_1, \dots, \mathbf{v}_m \in \mathbb{R}^k$  such that  $B_i = A_i + \mathbf{v}_i$  for each  $i \in \{1, \dots, m\}$ . Let  $\lambda = \lambda_k$  denote the Lebesgue measure on  $\mathbb{R}^k$ . The *box* (or *grid*, or *upper Minkowski*) *dimension* of  $X \subseteq \mathbb{R}^k$  is

$$\Delta(X) := k - \liminf_{\epsilon \rightarrow 0^+} \frac{\log \lambda(\{\mathbf{x} \in \mathbb{R}^k \mid \text{dist}(\mathbf{x}, X) \leq \epsilon\})}{\log \epsilon},$$

where  $\text{dist}(\mathbf{x}, X)$  means e.g. the  $L^\infty$ -distance from the point  $\mathbf{x}$  to the set  $X$ . Let  $\partial X$  denote the topological boundary of  $X$ . It is easy to show that if  $A \subseteq \mathbb{R}^k$  satisfies  $\Delta(\partial A) < k$ , then  $A$  is measurable and, furthermore,  $\lambda(A) > 0$  if and only if  $A$  has non-empty interior. With these observations, the result of Laczkovich can be formulated as follows.

**Theorem 2** (Laczkovich [6, 7, 8]). *Let  $k \geq 1$  and let  $A, B \subseteq \mathbb{R}^k$  be bounded sets with non-empty interior such that  $\lambda(A) = \lambda(B)$ ,  $\Delta(\partial A) < k$ , and  $\Delta(\partial B) < k$ . Then  $A$  and  $B$  are equivalent.*

In the same work Laczkovich asked whether Tarski's circle squaring is possible with measurable pieces. Similar questions have been asked about other classical equidecomposition results. For example, the following "measurable version" of Hilbert's third problem has been asked by Wagon [10, Question 3.14]: *is a regular tetrahedron in  $\mathbb{R}^3$  measurably equidecomposable to a cube of the same volume?*

There are various results which imply the impossibility of measurable equidecompositions when additional regularity of the pieces is requested. Examples include Dehn's theorem [2] solving Hilbert's third problem and the result of Dubins, Hirsch and Karush [3] which shows that Tarski's circle squaring is not possible with Jordan domains.

On the other hand, until recently there have been very few general positive results on the existence of measurable equidecompositions, although a related problem of measurable equidecompositions with countably many parts was studied already by Banach and Tarski [1, Théorème 42]. For more historical information we recommend Wagon's monograph [10].

In [4, 5] we give new criteria for the existence of a measurable equidecomposition. The first one states that under the same assumption as in Theorem 2 one can additionally require that all parts are measurable.

**Theorem 3** (Grabowski, Máthé and Pikhurko [4]). *Let  $k \geq 1$  and let  $A, B \subseteq \mathbb{R}^k$  be bounded sets with non-empty interior such that  $\lambda(A) = \lambda(B)$ ,  $\Delta(\partial A) < k$ , and  $\Delta(\partial B) < k$ . Then  $A \overset{\text{Tr}}{\sim} B$  with parts that are measurable.*

The above theorem implies that measurable Tarski's circle squaring is possible, and settles Wagon's measurable version of Hilbert's third problem.

In [5] we give an equidecomposability criterion for  $k \geq 3$ . The most important feature of it when compared to Theorem 3 is that for many sets  $A \subseteq \mathbb{R}^k$ ,  $k \geq 3$ , we are able to completely characterize sets  $B$  which are measurably equidecomposable to  $A$ . Furthermore, we do not need to assume anything about the boundaries of the sets.

We say that a set  $A \subseteq \mathbb{R}^k$  covers another set  $B$  if  $B$  is contained in the union of finitely many sets congruent to  $A$ .

**Theorem 4** (Grabowski, Máthé and Pikhurko [5]). *Let  $k \geq 3$ , let  $A \subseteq \mathbb{R}^k$  be a bounded measurable set which covers a non-empty open set. Then a set  $B \subseteq \mathbb{R}^k$  is measurably equidecomposable to  $A$  if and only if  $A$  and  $B$  cover each other and  $B$  is a measurable set of the same measure as  $A$ .*

Note that both covering each other and having equal measures are obvious necessary conditions for the existence of a measurable equidecomposition between  $A$  and  $B$ .

A result analogous to Theorem 4 is proved in [5] for equidecompositions of sets on the unit sphere  $\mathbb{S}^{k-1}$ ,  $k \geq 3$ , and in  $\mathbb{R}^2$ , but in the latter case only when we allow moving the pieces by arbitrary measure-preserving affine transformations. These results follow from a more general theorem, in which the space  $\mathbb{R}^k$  is replaced by a general measure space  $\Omega$  on which a group  $\Gamma$  acts in a measure-preserving way.

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## Benjamini-Schramm convergence of sequences of arithmetic 3-manifolds

MIKOLAJ FRACZYK

Let  $G$  be a semisimple real Lie group and  $K$  its maximal compact subgroup. Write  $X$  for the symmetric space  $G/K$ . A (finite volume) locally symmetric space is an orbifold of the form  $\Gamma \backslash X$  where  $\Gamma$  is a lattice in  $G$ . For a real number  $R > 0$  the  $R$ -thin part of a locally symmetric space  $\Gamma \backslash X$  is denoted by  $(\Gamma \backslash X)_{<R}$  and defined as the set of points  $x \in \Gamma \backslash X$  such that the ball of radius  $R$  around  $x$  is not isometric to the ball of radius  $R$  in  $X$ . We say that a sequence of finite volume locally symmetric spaces  $(\Gamma_i \backslash X)$  converges Benjamini-Schramm (B-S) to  $X$  if for every  $R > 0$

$$\lim_{i \rightarrow \infty} \frac{\text{Vol}((\Gamma_i \backslash X)_{<R})}{\text{Vol}(\Gamma_i \backslash X)} = 0.$$

The notion of Benjamini-Schramm convergence for locally symmetric spaces was studied in [1] where it was shown ([1, Theorem 1.5]) that if  $G$  is a higher rank group with property (T) then for every sequence of pairwise non-conjugate lattices  $(\Gamma_i)$  the orbifolds  $\Gamma_i \backslash X$  converge B-S to  $X$ .

In my talk we take a closer look at sequences of arithmetic lattices, with particular focus on the case  $G = \text{SL}(2, \mathbb{R})$  or  $\text{SL}(2, \mathbb{C})$ . One can ask what are the reasonable conditions that we should impose on a sequence of arithmetic lattices  $(\Gamma_i)$  in order to guarantee that  $\Gamma_i \backslash X$  converge B-S to  $X$ . The example of cyclic covers of a compact arithmetic hyperbolic surface shows that, in general, arithmeticity is not enough. On the other hand [1, Theorem 1.12] tells us that if  $(\Gamma_i)$  is a sequence of pairwise nonconjugate congruence lattices contained in a single arithmetic lattice  $\Gamma_0$  then  $(\Gamma_i \backslash X)$  converges to  $X$ . This suggests the following conjecture

**Conjecture 1.** *Let  $G$  be a semisimple real Lie group and let  $(\Gamma_i)$  be a sequence of pairwise nonconjugate congruence arithmetic lattices in  $G$ . Then the sequence  $(\Gamma_i \backslash X)$  converges Benjamini-Schramm to  $X$ .*

Similar statement was conjectured by Jean Raimbault in [2] but only for lattices defined over number fields of uniformly bounded degrees. In this direction he proved ([2, Theorem A]) that the Conjecture 1 holds for every sequence of congruence arithmetic lattices in  $\text{SL}(2, \mathbb{C})$  defined over a quadratic or cubic extension of  $\mathbb{Q}$ . It follows in particular that for  $G = \text{SL}(2, \mathbb{C})$  Conjecture 1 holds for non uniform lattices, because every such lattice is defined over a quadratic extension of  $\mathbb{Q}$ . For  $G = \text{SL}(2, \mathbb{R})$  or  $\text{SL}(2, \mathbb{C})$  I have shown the following:

**Theorem 2.** *Fix a positive real  $R$ . Let  $G = \text{SL}(2, \mathbb{R})$  or  $G = \text{SL}(2, \mathbb{C})$ . Write  $X$  for  $\mathbb{H}^2$  in the first and for  $\mathbb{H}^3$  in the second case. Then, for any uniform, torsion free congruence arithmetic lattice  $\Gamma$  in  $G$*

$$\text{Vol}((\Gamma \backslash X)_{<R}) \leq C_R \text{Vol}(\Gamma \backslash X)^{1-\delta},$$

where  $\delta$  is an absolute positive constant and  $C_R$  depends only on  $R$ .

It is known that for any  $N > 0$  there are only finitely many arithmetic lattices of covolume bounded by  $N$  so we deduce from Theorem 2 that the Conjecture 1 holds for  $G = \mathrm{SL}(2, \mathbb{R}), \mathrm{SL}(2, \mathbb{C})$  and sequences of torsion free, congruence arithmetic lattices.

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**Measurable tilings of free pmp actions of amenable groups**

ANDREW MARKS

(joint work with Clinton Conley, Steve Jackson, David Kerr, Brandon Seward,  
and Robin Tucker-Drob)

As part of their development of the ergodic theory of amenable groups, Ornstein and Weiss proved that every free pmp action of an amenable group can be quasitiled [3]. In their paper, they pose the problem of whether these quasitilings can be improved to tilings. Indeed, even the purely algebraic question of whether every amenable group admits a Følner sequence all of whose elements can tile the group remains open.

It was recently shown by Downarowicz, Huczek, and Zhang that if  $\Gamma$  is a finitely generated amenable group, and  $\epsilon > 0$ , then  $\Gamma$  admits a partition into  $\epsilon$ -Følner sets such that up to translation, only finitely many distinct parts appear [1]. We establish a strengthening of this result for measurable tilings of free measure preserving actions of amenable groups. Our proof uses a measurable matching lemma of Lyons and Nazarov [2] that we adapt to an asymmetric context.

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## Universal $L^2$ -torsion, Twisted $L^2$ -Euler characteristic, Thurston norm and polytopes

WOLFGANG LÜCK

(joint work with Stefan Friedl)

We want to investigate and compare the following four invariants of 3-manifolds which are of rather different nature: the Thurston norm and polytope, see [10] the degree of higher order Alexander polynomials in the sense of Cochran and Harvey, see [1, 6], the degree of the  $L^2$ -torsion function, see [2, 3, 9], and a version of the  $L^2$ -Euler characteristic, see [4]. We explain that the  $L^2$ -Euler characteristic encompasses the degree of higher order Alexander polynomials. We relate all these invariants by inequalities and equalities. In particular we show that they agree for the universal coverings and (for many other coverings) of a compact connected irreducible orientable 3-manifold with infinite fundamental group and empty or toroidal boundary. We will explain universal  $L^2$ -torsion which encompasses all the invariants above and is based on localizations techniques applied to group rings and  $K_1$ . Some of these results have been conjectured in [2]. For basic introduction to  $L^2$ -invariants we refer to [8].

Behind all these invariants is the universal  $L^2$ -torsion  $\rho_u^2(\overline{M}; \mathcal{N}(G)) \in \text{Wh}^w(G)$  of a  $G$ -covering  $\overline{X} \rightarrow X$  of a finite connected CW-complex  $X$  such that all its  $L^2$ -Betti numbers  $b_n^{(2)}(\overline{X}; \mathcal{N}(G))$  vanish, see [5]. Here  $\text{Wh}^w(G)$  is a variation of the classical Whitehead group, where one considers instead of matrices  $A \in M_{n,n}(\mathbb{Z}G)$ , which are invertible, those ones, for which the induced  $G$ -operator  $r_A: L^2(G)^n \rightarrow L^2(G)^n$  is a weak isomorphism.

In the sequel we assume that the torsionfree group  $G$  satisfies the Atiyah Conjecture about the integrality of  $L^2$ -Betti numbers. This is for instance the case if  $G$  is residually torsionfree elementary amenable or the fundamental group of an irreducible 3-manifold which is not a closed graph manifold.

If  $\mathcal{D}(G)$  is the division closure of  $\mathbb{Z}G$  in the algebra  $\mathcal{U}(G)$  of operators affiliated to the group von Neumann algebra  $\mathcal{N}(G)$ , then  $\mathcal{D}(G)$  is a skewfield and there is an isomorphism, see [7],

$$\text{Wh}^w(G) \cong \text{Wh}(\mathcal{D}(G)) = K_1(\mathcal{D}(G))/\{\pm g \mid g \in G\}.$$

The Dieudonné determinant yields an isomorphism

$$\text{Wh}^w(\mathcal{D}(G)) \cong \mathcal{D}(G)^\times / [\mathcal{D}(G)^\times, \mathcal{D}(G)^\times] \cdot \{\pm g \mid g \in G\}.$$

Let  $\mathcal{P}(H_1(G; \mathbb{R}))$  be the Grothendieck group of the abelian monoid of polytopes in  $H_1(G; \mathbb{R})$  under the Minkowski sum. We define a group homomorphism

$$P': \mathcal{D}(G)^\times \rightarrow \mathcal{P}(H_1(G; \mathbb{R})).$$

From these data we obtain a homomorphism

$$P: \text{Wh}^w(G) \rightarrow \mathcal{P}(H_1(G; \mathbb{R})).$$

Hence we can consider  $P(\rho_u^{(2)}(\overline{X})) \in \mathcal{P}(H_1(G; \mathbb{R}))$ .

One of our main theorems says

**Theorem.** Let  $M$  be a compact connected orientable irreducible 3-manifold with infinite fundamental group  $\pi$  and empty or incompressible torus boundary which is not a closed graph manifold.

Then there is a virtually finitely generated free abelian group  $\Gamma$ , and a factorization  $\pi_1(M) \xrightarrow{\alpha} \Gamma \xrightarrow{\beta} H_1(M)_f := H_1(M)/\text{tors}(H_1(M))$  of the canonical projection into epimorphisms such that the following holds:

For any factorization of  $\alpha: \pi \rightarrow \Gamma$  into group homomorphisms  $\pi \xrightarrow{\mu} G \xrightarrow{\nu} \Gamma$  for a torsionfree group  $G$  satisfying the Atiyah Conjecture the composite

$$\text{Wh}^w(G) \xrightarrow{P} \mathcal{P}(H_1(G; \mathbb{R})) \xrightarrow{\mathcal{P}(H_1(\beta \circ \nu; \mathbb{R}))} \mathcal{P}(H_1(M; \mathbb{R}))$$

sends  $\rho_u^{(2)}(\overline{M}; \mathcal{N}(G))$  to the class of the Thurston polytope of  $M$ .

Notice that it applies in particular to the universal covering, i.e.,  $G = \pi_1(M)$ ,  $\mu = \text{id}$  and  $\overline{M} = \widetilde{M}$ .

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### Metabelian groups with large return probability

LISON JACOBONI

Let  $G$  be a finitely generated group and  $\mu$  be a symmetric probability measure on  $G$  with generating support. We study the return probability to the origin of the random walk on  $G$ , driven by  $\mu$ . The simple random walk on a Cayley graph of  $G$  provides a fundamental example.

A theorem of Pittet and Saloff-Coste ([5]) asserts that any two symmetric and finitely supported probability measures with generating support give rise to equivalent return probabilities. Let  $p_{2n}$  denote this invariant of the group. Here, we

say that two monotone functions  $\varphi, \psi$  are equivalent, denoted by  $\varphi \sim \psi$ , if there exists positive constants  $c$  and  $C$  such that  $C\varphi(Ct) \leq \psi(t) \leq c\varphi(ct)$ .

Understanding how the random walk behaves allows to have insight into the large-scale geometry of the group. Indeed, Kesten ([4]) proved that non-amenable groups are characterized by the fact that they behave the worst, for their return probability decays exponentially fast. From another side, a nilpotent group, which has polynomial growth, behaves like  $\mathbb{Z}^d$ , where  $d$  is the degree of the growth (Varopoulos [6]). Hebisch and Saloff-Coste ([2]) proved that if  $G$  has exponential growth, then

$$p_{2n} \lesssim \exp(-n^{\frac{1}{3}}).$$

Lamplighter groups  $F \wr \mathbb{Z}$ , with  $F$  finite, polycyclic groups, discrete solvable subgroups of Lie groups, solvable Baumslag-Solitar and solvable groups of finite Prüfer rank (a group has finite Prüfer rank if there exists an integer  $r$  such that any finitely generated subgroup can be generated by at most  $r$  elements) achieve this bound whenever they have exponential growth.

From another side, there are examples of amenable groups with a smaller return probability. For example, if  $F$  is finite, the wreath product  $F \wr \mathbb{Z}^d$  satisfies  $p_{2n}^{F \wr \mathbb{Z}^d} \sim \exp(-n^{\frac{d}{d+2}})$ . This example illustrates the fact that among the simplest solvable groups, namely the metabelian ones, there exist groups whose return probability exponent is as close as one may want to 1, the return probability exponent of non amenable groups.

**Question 1.** *Which finitely generated amenable groups of exponential growth satisfies  $p_{2n} \sim \exp(-n^{\frac{1}{3}})$ ?*

In [3], we answer this question in the case of metabelian groups. Let  $B_d$  be the free metabelian group of rank  $d$  and  $B_2^{(p)} = B_d/[B_d, B_d]^p$  denote the free  $p$ -metabelian group of rank  $d$ . We prove :

**Theorem 2.** *Let  $G$  be a finitely generated metabelian group. Then, either*

$$p_{2n}^G \lesssim \exp(-n^{\frac{1}{3}}),$$

*or  $G$  contains one of the three following groups*

$$\mathbb{Z} \wr \mathbb{Z}, (\mathbb{Z}/p\mathbb{Z}) \wr \mathbb{Z}^2, \text{ or } B_2^{(p)}, \text{ for some prime } p.$$

A metabelian group  $G$  is an extension of an abelian group  $Q$  by another abelian group  $A$  :

$$A \hookrightarrow G \twoheadrightarrow Q.$$

The subgroup  $A$  carries a natural structure of  $\mathbb{Z}Q$ -module, coming from the action by conjugation. We show that the Krull dimension of this module, when non-zero, does not depend on the exact sequence representing  $G$ . We call it the *Krull dimension of  $G$* , denoted  $\text{Krull}(G)$ .

It is a special case of the definition of Krull dimension for groups, introduced in [3], by generalizing the usual Krull dimension for modules. We define  $\text{Krull}(G)$  to be the deviation of the poset of normal subgroups of  $G$ . Note that it can be



an ordinal number and even that this poset may not have a deviation : in this last case, we say that the group does not admit a Krull dimension. A necessary condition for a group to admit a Krull dimension is to satisfy the maximal property on normal subgroups.

For metabelian groups,  $\text{Krull}(G)$ , satisfies the following characterization :

i) If the Krull dimension of the  $\mathbb{Z}G_{ab}$ -module  $[G, G]$  is positive, then

$$\text{Krull}(G) = \text{Krull}_{\mathbb{Z}G_{ab}}([G, G]).$$

ii) Otherwise, when the module  $[G, G]$  has Krull dimension zero,  $G$  has dimension 0 as well if it is finite, and has dimension 1 if it is infinite.

Studying the impact of dimension on the structure of the group allows one to find interesting subgroups.

**Proposition 3.** *Let  $G$  be a metabelian group of Krull dimension  $k$ .*

*If  $k \geq 2$ , then  $G$  contains either  $\mathbb{Z} \wr \mathbb{Z}$  or  $\mathbb{Z}/p\mathbb{Z} \wr \mathbb{Z}^2$  or  $B_2^{(p)}$ , for some prime  $p$ .*

Theorem 2 now rephrases as :

**Theorem 4.** *Let  $G$  be a finitely generated metabelian group. Then,*

$$\text{Krull}(G) \leq 1 \Leftrightarrow p_{2n}^G \asymp \exp(-n^{\frac{1}{3}}).$$

One implication is given by the previous proposition. To produce lower bounds for the return probability, we construct sequences of Følner pairs so as to use [1]. This is possible once we can reduce to the split case. To this end, we prove a variation of a theorem by Kaloujnine and Krasner.

**Theorem 5.** *Every finitely generated metabelian group, which is the extension of a finitely generated group  $Q$  by another abelian group, can be embedded inside a finitely generated split metabelian group  $B \rtimes Q$ , of the same Krull dimension, with  $B$  abelian.*

More precisely, the construction of Følner pairs runs in every dimension and we prove :

**Theorem 6.** *Let  $G$  be a metabelian group of Krull dimension  $k$ . Assume that  $[G, G]$  is torsion. Then,*

$$p_{2n}^G \asymp \exp(-n^{\frac{k}{k+2}}).$$

*If the group splits,  $p_{2n}^G \sim \exp(-n^{\frac{k}{k+2}})$ .*

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## On Følner sets in topological groups

FRIEDRICH MARTIN SCHNEIDER

(joint work with Andreas Thom)

The study of amenable (discrete) groups benefits from a vast variety of possible viewpoints – ranging from analytic to combinatorial – that allow for numerous approaches to problems and give rise to many surprising applications. Among the most important and fundamental amenability criteria is Følner’s theorem [1], which characterizes amenability by the existence of almost invariant finite subsets. In [4] we extend Følner’s insight to the realm of topological groups and in turn develop a new, more combinatorial perspective on topological amenability.

Our topological version of Følner’s criterion is in terms of topological matchings. To be more precise, let  $\mathcal{B} = (X, Y, R)$  be a *bipartite graph*, i.e., a triple consisting of two finite sets  $X$  and  $Y$  and a relation  $R \subseteq X \times Y$ . A *matching* in  $\mathcal{B}$  is an injective map  $\varphi: D \rightarrow Y$  with  $D \subseteq X$  and  $(x, \varphi(x)) \in R$  for all  $x \in D$ . We denote by  $\mu(\mathcal{B})$  the maximum cardinality of (the domain of) a matching in  $\mathcal{B}$ , i.e.,

$$\mu(\mathcal{B}) := \sup\{|\text{dom}(\varphi)| \mid \varphi \text{ matching in } \mathcal{B}\}.$$

Given an identity neighborhood  $U$  in a topological group  $G$  along with finite subsets  $E, F \subseteq G$ , let us consider the bipartite graph

$$\mathcal{B}(E, F, U) := (E, F, R(E, F, U))$$

with the relation defined by

$$R(E, F, U) := \{(x, y) \in E \times F \mid yx^{-1} \in U\},$$

and let  $\mu(E, F, U) := \mu(\mathcal{B}(E, F, U))$ .

The following theorem from [4] may be regarded as a topological version of Følner’s amenability criterion for discrete groups [1]. Recall that a topological group  $G$  is *amenable* if every continuous action of  $G$  by affine homeomorphisms on a non-void compact convex subset of a locally convex topological vector space has a fixed point, or equivalently, if the space of bounded uniformly continuous real-valued functions on  $G$  admits a left-invariant mean.

**Theorem 1** (Theorem 4.5 in [4]). *Let  $G$  be a Hausdorff topological group. The following are equivalent.*

- (1)  $G$  is amenable.
- (2) For every  $\theta \in (0, 1)$ , every finite subset  $E \subseteq G$ , and every identity neighborhood  $U$  in  $G$ , there exists a finite non-empty subset  $F \subseteq G$  such that

$$\forall g \in E: \quad \mu(F, gF, U) \geq \theta|F|.$$

This result has a number of non-trivial consequences: as applications, we obtain a topological version [4, Corollary 5.12] of Whyte’s geometric solution to von Neumann’s problem [5] and provide an affirmative answer [4, Theorem 6.1] to a question posed by Rosendal [2, Problem 40] concerning the coarse geometry of amenable topological groups. The theorem above also improves on some of our earlier work [3, Theorem 6.1].

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### Infinite presentability and relation range of groups

ADRIEN LE BOUDEC

(joint work with Yves Cornulier)

If  $G$  is a group and  $S$  a generating subset, a relation in  $G$  is an element of the kernel of the natural map  $F_S \rightarrow G$ , where  $F_S$  is the free group over  $S$ . The relation range of a finitely generated group, introduced by Bowditch in order to distinguish a continuum of quasi-isometry classes of small cancellation groups [1], is the set of lengths of new relations in the group. Here new relations means relations that are not consequences of relations of smaller length. Up to a natural equivalence relation, the relation range does not depend on the choice of a finite generating subset, and is actually a quasi-isometry invariant.

In joint work with Yves Cornulier, we investigate the class of groups whose relation range is as large as possible, called densely related groups. These are groups satisfying a strong negation of finite presentability, in the sense that new relations appear at all scales. Any non-trivial standard wreath product is densely related. The Grigorchuk group of intermediate growth also is an example of densely related group. A group that is not densely related is called lacunary presented.

**Theorem 1.** *If  $G$  has (at least) one simply connected asymptotic cone, then  $G$  is lacunary presented.*

For example any lacunary hyperbolic group (group with one asymptotic cone a real tree) is lacunary presented. The class of lacunary presented groups is therefore much larger than the class of finitely presented groups, and enjoys various stability properties. For instance it is stable under taking direct products with finitely presented groups.

For several classes of finitely generated groups, we show that a group that is not finitely presented must be densely related.

**Theorem 2.** *Let  $G$  be a finitely generated group. Assume that  $G$  is metabelian; or that  $G$  is nilpotent-by-cyclic. Then  $G$  is either finitely presented or densely related.*

Recall that it follows from Bieri-Strebel theorem that a (non virtually cyclic) finitely generated group  $G$  that is (locally finite)-by-cyclic is infinitely presented [3]. There is no hope to obtain any restriction on the relation range of a (locally finite)-by-cyclic group in full generality. Nevertheless, we show that under the additional assumption that the group satisfies a law, the relation range is forced to be as large as possible.

**Theorem 3.** *Let  $G$  be a finitely generated group that is (infinite locally finite)-by-cyclic. If  $G$  satisfies a law, then  $G$  is densely related. In particular  $G$  has no simply connected asymptotic cone.*

This contrasts with a construction due to Olshanskii-Osin-Sapir, who gave examples of lacunary hyperbolic groups which are (locally finite)-by-cyclic [2].

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### C\*-simplicity for discrete groups

EMMANUEL BREUILLARD

(joint work with Mehrdad Kalantar, Matthew Kennedy and Narutaka Ozawa)

A discrete group is said to be *C\*-simple* if the reduced C\*-algebra of the group is simple, and is said to have the *unique trace property* if the reduced C\*-algebra has a unique trace. The problem of which groups have these properties captured the interest of mathematicians in 1975 with Powers' proof [8] that the free group on two generators is both C\*-simple and has the unique trace property. Since then the problem has received a great deal of attention, and many more examples of groups with these properties have been found.

The following theorem gives a necessary and sufficient condition for the C\*-simplicity of a group.

**Theorem 1** ([6]). *A discrete group is C\*-simple if and only if it has a topologically free boundary action.*

This theorem was established using Hamana's theory of injective envelopes of  $G$ -operator systems. In the talk we presented a new, more direct, proof of this result.

It turns out that it is often possible to prove the existence of a topologically free boundary action for a given group without actually having to construct one. This makes Theorem 1 useful in practice for establishing  $C^*$ -simplicity.

Day ([2], Lemma 4.1) showed that every discrete group  $G$  has a largest amenable normal subgroup, called the *amenable radical* of  $G$ , that contains every amenable normal subgroup of  $G$ .

Our results show the  $C^*$ -simplicity of a large class of groups.

**Theorem 2** ([1]). *A discrete group with trivial amenable radical having either non-trivial bounded cohomology or non-vanishing  $\ell^2$ -Betti numbers is  $C^*$ -simple.*

The next result implies the  $C^*$ -simplicity of (torsion-free) Tarski monster groups and free Burnside groups  $B(m, n)$  for  $m \geq 2$  and  $n$  odd and sufficiently large ([7]).

**Theorem 3** ([1]). *A discrete group with only countably many amenable subgroups is  $C^*$ -simple if and only if its amenable radical is trivial.*

The next result provides a negative answer to [5], Question (Q).

**Theorem 4** ([1]). *Let  $G$  be a discrete group and let  $N < G$  be a normal subgroup. Then  $G$  is  $C^*$ -simple if and only if both  $N$  and  $C_G(N)$  are  $C^*$ -simple, where  $C_G(N)$  denotes the centralizer of  $N$  in  $G$ . In particular,  $C^*$ -simplicity is closed under extension.*

The methods typically used to establish the  $C^*$ -simplicity of a group often also imply that the group has the unique trace property. However, it has been an open problem for some time to determine if this is true in general. We prove that this question has an affirmative answer and, more generally, completely settle the problem of which groups have the unique trace property.

**Theorem 5** ([1]). *A discrete group has the unique trace property if and only if its amenable radical is trivial. In particular, every  $C^*$ -simple group has the unique trace property.*

It is well known that if  $G$  is  $C^*$ -simple, then the amenable radical of  $G$  is necessarily trivial. Le Boudec [3] has recently constructed the first examples of groups showing that the converse is not true, thus settling a longstanding problem. His proof utilizes some of the results presented above.

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## Subgroup dynamics and C\*-simplicity of groups of homeomorphisms

NICOLÁS MATTE BON

(joint work with Adrien Le Boudec)

The talk was based on the preprint [4] joint with Adrien Le Boudec, in which we study the dynamics of the conjugation action on the Chabauty space of a class of groups of homeomorphisms, and give applications to C\*-simplicity.

A countable group  $G$  is said to be C\*-simple if its reduced C\*-algebra is simple. There is a considerable literature on the problem of deciding which groups have this property. Recently Kalantar and Kennedy [2] showed the following topological dynamical characterisation of C\*-simplicity.

**Theorem 1** (Kalantar–Kennedy). *A group  $G$  is C\*-simple if and only if there exists a topologically free  $G$ -boundary. Equivalently if  $G$  acts freely on its universal Furstenberg boundary.*

In some cases, a topologically free boundary action may be difficult to identify concretely. For this reason it is useful to have indirect criteria to establish its existence. To this end, consider the space  $\text{Sub}(G)$  of all subgroups of  $G$ , endowed with the Chabauty topology. This topology makes  $\text{Sub}(G)$  a compact space, on which  $G$  acts continuously by conjugation. A *uniformly recurrent subgroup* (URS for short) is a closed minimal  $G$ -invariant subset of  $\text{Sub}(G)$ . Kennedy showed the following criterion in [3]

**Theorem 2** (Kennedy). *A countable group is C\*-simple if and only if it admits no non-trivial amenable URS.*

After outlining a short argument to deduce this theorem directly from Theorem 1, I explained the following theorem, which gives a way to study URS's in a class of groups of homeomorphisms.

**Theorem 3** (LB–MB). *Let  $G$  be a countable group acting faithfully by homeomorphisms on a Hausdorff space  $X$ . For every open set  $U \subset X$  denote by  $G_U$  its rigid stabiliser in  $G$ , i.e. the point-wise fixator of  $X \setminus U$ . For every subgroup  $H \leq G$ , one of the following possibilities holds.*

- (1) *The closure of the conjugacy class of  $H$  in  $\text{Sub}(G)$  contains the trivial subgroup  $\{1\}$ .*
- (2) *There exists a non-empty open set  $U \subset X$  such that  $H$  admits a finite index subgroup of  $G_U$  as a subquotient.*

In particular this provides a new sufficient condition for C\*-simplicity.

**Corollary 4** (LB–MB). *Under the same assumptions, assume that for every non-empty open set  $U \subset X$  the rigid stabilizer  $G_U$  is non-amenable. Then  $G$  is  $C^*$ -simple.*

This criterion implies the  $C^*$ -simplicity of many groups that are naturally defined by an action by homeomorphisms, for which previously known criteria failed. Among its applications I mentioned the  $C^*$ -simplicity of Thompson's group  $V$  is  $C^*$ , of a class of groups of piecewise-projective homeomorphisms, and of non-amenable branch groups.

Assume that one is given a  $G$ -boundary  $X$  which is *not* topologically free. What can be said on the  $C^*$ -simplicity of  $G$ ? Breuillard–Kalantar–Kennedy–Ozawa [1] showed that if point-stabilisers are amenable, then  $G$  is *not*  $C^*$ -simple. The converse does not hold. I explained how Corollary 4 implies that the converse does hold under the additional assumptions.

**Corollary 5.** *Let  $X$  be a faithful extreme  $G$ -boundary (i.e. the action is minimal and extremely proximal). Then  $G$  is  $C^*$ -simple if and only if either the action on  $X$  is topologically free, or its the point-stabilisers are non-amenable.*

As an example of an extreme boundary action, consider Thompson's group  $T$  acting on the circle. Then the above corollary shows that  $T$  is  $C^*$ -simple if and only if Thompson's group  $F$  is non-amenable, in which case it is also  $C^*$  simple by Corollary 4 (one implication was already obtained by Haagerup and Olesen).

To conclude the talk, I mentioned how using Theorem 3 as a key tool, we obtained a complete classification of URS's of Thompson's groups:

**Theorem 6** (LB–MB). *Consider Thompson's groups  $F < T < V$ . Then*

- (1) *the only URS's of the group  $F$  are its normal subgroups;*
- (2) *the only URS's of the group  $T$  are  $\{1\}, \{T\}$  and the stabiliser URS arising from its standard action on the circle;*
- (3) *the only URS's of the group  $V$  are  $\{1\}, \{V\}$  and the stabiliser URS arising from its standard action on the Cantor set.*

This yields a rigidity property for their minimal actions on compact spaces:

**Corollary 7** (LB–MB). (1) *Every faithful, minimal action of the group  $F$  on a compact space is topologically free.*

- (2) *A minimal action of the group  $T$  on a compact space is either topologically free or factors onto its standard action on the circle.*
- (3) *A minimal action of the group  $V$  on a compact space is either topologically free or factors onto its standard action on the Cantor set.*

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## On sofic approximations of Property (T) groups

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A sequence of graphs is locally convergent if for every  $r$  the isomorphism class of a rooted  $r$ -ball centered at a vertex chosen uniformly at random converges in distribution. Our main result is the proof of Bowen's following conjecture [2].

**Theorem 1.** *Let  $\Gamma$  be a countably infinite Property (T) group and  $\{G_n\}_{n=1}^\infty$  a sequence of finite, bounded degree graphs that locally converges to a Cayley graph of  $\Gamma$ . Then there exists a  $\gamma > 0$  and a sequence of finite  $d$ -regular graphs  $\{G'_n\}_{n=1}^\infty$  such that*

- (1)  $V(G_n) = V(G'_n)$
- (2)  $\lim_{n \rightarrow \infty} \frac{|E(G_n) \Delta E(G'_n)|}{|V(G_n)|} = 0$
- (3) *For every  $n$  the graph  $G'_n$  is a vertex-disjoint union of  $\gamma$ -expander graphs.*

A group is called **sofic** if any of its labeled Cayley graphs admits a local approximation by finite labeled graphs. Sofic groups were introduced by Gromov [6], see also Weiss [8]. Many classical conjectures are proved for sofic groups: Gottschalk's conjecture (Gromov [6]), Kaplansky's direct finiteness conjecture (Elek, Szabó [3]) and Connes' embedding conjecture (Elek, Szabó [4]). It is not known if every group is sofic, but it is generally believed that non-sofic groups exist.

Our theorem is the first one for sparse graphs that implies quasirandom global structure under local conditions. The notion of quasirandomness is at the heart of Szemerédi's regularity lemma and the limit theory of dense graphs: These rely on the fact that quasirandomness of dense graphs can be implied by local conditions. See the book of Lovász for the details [7].

Graphs with large girth may or may not be close to a vertex-disjoint union of expanders. Hence graphs close to a vertex-disjoint union of expanders (or an expander) can not be characterized by local conditions. However, we can give a characterization in terms of the Markov operator.  $M$  denotes the Markov operator, and  $\|*\|$  denotes the  $L_2$  norm with respect to the uniform probability distribution.

**Theorem 2.** *Let  $\{G_n\}_{n=1}^\infty$  be a sequence of  $d$ -regular graphs. Then the following are equivalent:*

- (1) *There exists an  $\varepsilon > 0$  such that for every  $\delta > 0$  for all, but finitely many  $n$  and for every function  $f : V(G_n) \rightarrow [0; 1]$  the inequality  $\|M^2 f - Mf\| \leq (1 - \varepsilon)\|Mf - f\| + \delta$  holds.*
- (2) *There exists a  $\gamma > 0$  and a sequence of  $d$ -regular graphs  $\{G'_n\}_{n=1}^\infty$  such that  $V(G_n) = V(G'_n)$ ,  $\lim_{n \rightarrow \infty} \frac{|E(G_n) \Delta E(G'_n)|}{|V(G_n)|} = 0$  and every  $G'_n$  is a vertex-disjoint union of  $\gamma$ -expanders.*



Note that if we had no  $\delta$  in (1) then  $G_n$  would be a vertex-disjoint union of expander graphs with second eigenvalue less than  $(1 - \varepsilon)$ .

Decompositions of graphs into graphs with good expansion properties is one of the main directions to attack Khot's **Unique Games Conjecture**, see [1] for the best known algorithm. Sequences of graphs that are essentially disjoint union of expanders have a much better decomposition, but this holds for special graphs only.

The first theorem gives an **ergodic decomposition theorem** for certain non-separable probability measure spaces with an invariant group action: Given a sequence of finite labeled graphs that locally approximates the labeled Cayley graph of  $\Gamma$ , the ultraproduct of these graphs will admit an almost free, measure-preserving  $\Gamma$ -action. The support of almost every ergodic measure in the decomposition of the ultraproduct space will be almost an ultraproduct of expanders. Hence these supports can be disjoint in the decomposition. See [2, 5] on ultraproducts of graphs.

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### The base change in the strong Atiyah and the Lück approximation conjectures

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The strong Atiyah conjecture arised from a question of M. F. Atiyah [1, page 72] about whether  $L^2$ -Betti numbers of a manifold  $Y$  with a cocompact proper  $G$ -action can be irrational. In [2] J. Dodziuk reformulated the Atiyah question in a question about  $CW$ -complexes of finite type and this problem received the name of the Atiyah conjecture. The formulation of the strong Atiyah conjecture, that refined the previous one, is due to W. Lück and T. Schick.

Let  $F$  be a free finitely generated group, freely generated by a finite set  $S$ , and  $N$  is its normal subgroup. We denote the quotient group  $F/N$  by  $G$ . Let  $K$  be

a subfield of  $\mathbb{C}$ . For a countable set  $X$ , let  $l^2(X)$  denote the Hilbert space with Hilbert basis the elements of  $X$ ; thus  $l^2(X)$  consists of all square summable formal sums  $\sum_{x \in X} a_x x$  with  $a_x \in \mathbb{C}$  and inner product

$$\left\langle \sum_{x \in X} a_x x, \sum_{y \in X} b_y y \right\rangle = \sum_{x \in X} a_x \overline{b_x}.$$

The group  $G$  (and so  $F$ ) acts by the multiplications on the left and right sides on  $l^2(G)$ . The right action of  $G$  on  $l^2(G)$  extends to an action of  $\mathbb{C}[G]$  on  $l^2(G)$  and so we obtain that the group algebra  $\mathbb{C}[G]$  acts faithfully as bounded linear operators on  $l^2(G)$ .

A finitely generated **Hilbert**  $G$ -module is a closed subspace  $V \leq (l^2(G))^n$ , invariant by the left action of  $G$ . We put

$$\dim_G V := \sum_{i=1}^n \langle \text{proj}_V \mathbf{1}_i, \mathbf{1}_i \rangle_{(l^2(G))^n},$$

where  $\mathbf{1}_i$  is the element of  $(l^2(G))^n$  having 1 in the  $i$ th entry and 0 in the rest of the entries. The number  $\dim_G V$  is the **von Neumann dimension** of  $V$ .

Let  $A \in \text{Mat}_{n \times m}(\mathbb{C}[F])$  be a matrix over  $\mathbb{C}[F]$ . By right multiplication,  $A$  induces a bounded linear operator  $\phi_G^A : (l^2(G))^n \rightarrow (l^2(G))^m$ . Our aim is to understand properties of  $\ker_G \phi_G^A$ . Our first motivation is the strong Atiyah conjecture.

**Conjecture 1** (The strong Atiyah conjecture with coefficients in  $K$  for a group  $G$ ). *Assume that there exists an upper bound for the orders of finite subgroups of  $G$  and let  $\text{lcm}(G)$  be the least common multiple of these orders. Then for every  $A \in \text{Mat}_{n \times m}(K[F])$ , we have that*

$$\dim_G \ker \phi_G^A \in \frac{1}{\text{lcm}(G)} \mathbb{Z}.$$

In particular, the conjecture predicts that if  $G$  is torsion free, then  $\dim_{G_k} \ker \phi_{G_k}^A$  is an integer. This is known to imply the Kaplansky zero-divisor conjecture for  $K[G]$ .

J. Dodziuk et al. [3] proved Conjecture 1 for groups from the class  $\mathcal{D}$  with coefficients in  $\bar{\mathbb{Q}}$ , the field of algebraic numbers. The class  $\mathcal{D}$  is the smallest non-empty class of groups such that:

- (1) If  $G$  is torsion-free and  $A$  is elementary amenable, and we have a projection  $p : G \rightarrow A$  such that  $p^{-1}(E) \in \mathcal{D}$  for every finite subgroup  $E$  of  $A$ , then  $G \in \mathcal{D}$ .
- (2)  $\mathcal{D}$  is subgroup closed.
- (3) Let  $G_i \in \mathcal{D}$  be a directed system of groups and  $G$  its (direct or inverse) limit. Then  $G \in \mathcal{D}$ .

The class  $\mathcal{D}$  contains, for example, residually torsion-free solvable groups. The proof of [3] uses the Lück approximation that we will introduce below.

It is a standard fact that if  $G$  satisfies the strong Atiyah conjecture, then a subgroup  $H$  of  $G$ , satisfying  $\text{lcm}(H) = \text{lcm}(G)$ , does. The question whether the

strong Atiyah Conjectures holds for a group  $G$  if it holds for a subgroup of finite index is a very delicate question. Some partial results were obtained in [8, 10]. Using these results the strong Atiyah conjectures with coefficients in  $\bar{\mathbb{Q}}$  is proved for Artin’s braid groups [8] and for finite extensions of fundamental groups of compact special cube complexes [10].

In our first result we prove the strong Atiyah conjecture with arbitrary coefficients in all the cases that we have described above.

**Theorem 2.** *Let  $G$  be a group belonging to on the following families*

- (1) *the class  $\mathcal{D}$ ;*
- (2) *Artin’s braid groups;*
- (3) *finite extensions of fundamental groups of compact special cube complexes.*

*Then  $G$  satisfies the strong Atiyah conjecture with coefficients in  $\mathbb{C}$ .*

Now we introduce the Lück approximation conjecture.

**Conjecture 3** (The Lück approximation conjecture with coefficients in  $K$  for a group  $G$ ). *Let  $F > N_1 > N_2 > \dots$  be a chain of normal subgroups of  $F$  with the intersection  $N = \cap N_i$ . Put  $G_i = F/N_i$  and  $G = F/N$ . Then for every  $A \in \text{Mat}_{n \times m}(K[F])$ ,*

$$\lim_{k \rightarrow \infty} \dim_{G_k} \ker \phi_{G_k}^A = \dim_G \ker \phi_G^A.$$

W. Lück gave a proof of the conjecture in the case when  $\{G_k\}$  are finite groups and  $A \in \text{Mat}_{n \times m}(\mathbb{Q}[F])$  ([9]). G. Elek and E. Szabó [5] observed that Lück’s method can be applied in a more general situation when the groups  $\{G_k\}$  are sofic. In [3] Dodziuk at al. developed a method that allowed to prove the Lück approximation conjecture with coefficients in  $\bar{\mathbb{Q}}$  when  $\{G_k\}$  are in  $\mathcal{D}$ . Combining the ideas of [5] and [3] one can unify the results of these papers and show that the conjecture holds when  $\{G_k\}$  are sofic and the coefficients of  $A$  are in  $\bar{\mathbb{Q}}$ .

At this moment there are only two cases for which the Lück approximation conjecture is known to be true in full generality when the coefficients are arbitrary: when  $G$  is amenable by a result of G. Elek [4], and when  $G$  is free by a result of A. Jaikin-Zapirain [6].

Our second theorem is the following.

**Theorem 4.** *The Lück approximation conjecture with arbitrary coefficients holds if  $G$  belongs to the class  $\mathcal{D}$  and  $G_i$  are sofic.*

Our main tool is the theory of epic  $*$ -regular  $R$ -rings. This new tool allows us to reformulate Conjecture 3 in a pure algebraic way and use the algebraic methods in the proofs of Theorems 2 and 4. The details of the proofs of the presented results can be found in [7].

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## Subadditive convergence and cocycle entropy

FELIX POGORZELSKI

(joint work with Amos Nevo)

The classical entropy theory for point measure preserving (p.m.p.)  $\mathbb{Z}$ -actions on probability spaces goes back to groundbreaking work of Kolomogorov and Sinai from the late 50ies, cf. [Ko58, Ko59, Si59]. There, the notion of measure entropy was defined via subadditive convergence lemmas along suitable averaging sequences. This method can be applied in higher generality for p.m.p. actions of countable, amenable groups. However, there is no immediate extension to the non-amenable world. In fact, due to some phenomena which are absent for amenable groups it was a long-standing question in the community whether a meaningful entropy theory for actions of non-amenable groups can exist. The fundamental developments of sofic entropy [Bo10, KL11, KL13, Ke13] for sofic groups and Rokhlin entropy [Se15, Se16] for general countable groups show that in fact, the answer to this question is positive.

In a project with Amos Nevo [NP16], we put forward a new notion of entropy, called *cocycle entropy*, for p.m.p. actions of arbitrary countable groups. Using certain cocycles defined via hyperfinite, p.m.p. measurable equivalence relations, we obtain entropy values via subadditive convergence theorems. This and the validity of a corresponding Shannon-McMillan-Breiman theorem indicate that cocycle entropy is the natural extension of the Kolmogorov-Sinai approach.

The first part of the talk aims at explicating the situation for the free group  $F_2$  on two generators. The action of this group on its Gromov boundary  $(\partial F_2, \nu)$ , endowed with a suitable probability measure gives rise to the so-called *synchronous tail relation*  $\mathcal{R} \subset \partial F_2 \times \partial F_2$ . The latter equivalence relation is measurable with countable fibers. Further, it is p.m.p. and hyperfinite in the sense of [CFW81], i.e.

the fibers of equivalent points are exhausted by an increasing sequence  $(\mathcal{R}_n)$  of finite equivalence classes which are uniformly bounded in size for every  $n \in \mathbb{N}$ . We write  $\mathcal{R} = \bigsqcup_{n=1}^{\infty} \mathcal{R}_n$ . These observations for the free group motivate the following generalized set (A) of assumptions.

- (A) Let  $\Gamma$  be a countable group. Assume further that there is a measurable, countable fiber, hyperfinite, p.m.p. equivalence relation  $\mathcal{R} = \bigsqcup_{n=1}^{\infty} \mathcal{R}_n$  over some probability space  $(Y, \nu)$  such that  $\sup_y |\mathcal{R}_n(y)| < \infty$  for each  $n \in \mathbb{N}$ . Suppose that  $\Gamma$  and  $\mathcal{R}$  are linked via a measurable, class injective cocycle  $\alpha : \mathcal{R} \rightarrow \Gamma$ .

The assumptions (A) can be guaranteed to hold for vast classes of countable groups such as hyperbolic groups or even Markov groups. For a probability space  $(X, \mathcal{B}, \mu)$  and a partition  $\mathcal{P}$  over  $X$ , we define the *Shannon entropy* as  $H(\mathcal{P}) := -\sum_{P \in \mathcal{P}} \mu(P) \log \mu(P)$ . Now cocycle entropy values for partitions are obtained by the following theorem.

**Theorem 1** (Nevo, P.). *Assume (A). If  $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$  is p.m.p., then for all partitions  $\mathcal{P}$  with  $H(\mathcal{P}) < \infty$ , the limit*

$$h_{\mathcal{P}}^{coc} = \lim_{n \rightarrow \infty} \int_Y \frac{H\left(\bigvee_{z \in \mathcal{R}_n(y)} \alpha(z, y)^{-1} \mathcal{P}\right)}{|\mathcal{R}_n(y)|} d\nu(y)$$

*exists and is independent of the choice of the sequence  $(\mathcal{R}_n)$ .*

We state the theorem in the talk and give a rough outline of the proof. It is based on a new Ornstein-Weiss lemma for subadditive functions defined on finite subequivalence relations of  $\mathcal{R}$ . Corresponding results for functions defined on finite subsets in amenable groups can for example be found in [Gr99, LW00]. Given the above theorem, we can define cocycle entropy for p.m.p. actions.

**Definition 2** (Cocycle entropy). Given a class injective cocycle  $\alpha : \mathcal{R} \rightarrow \Gamma$  satisfying (A), the *cocycle entropy* for a p.m.p. action  $\Gamma \curvearrowright X$  is defined as

$$h_{\Gamma \curvearrowright X}^{coc} := \inf \{h_{\mathcal{P}}^{coc} \mid \mathcal{P} \text{ generating partition}\}.$$

It will be shown that in generic cases, cocycle entropy coincides with Rokhlin entropy. In those situations, the dependence on the cocycle vanishes. The proof of the following theorem is based on a nice theorem by Seward, cf. Theorem 1.5 in [Se16].

**Theorem 3** (Nevo, P.). *If  $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$  is p.m.p. and free, then for all class injective cocycles  $\alpha : \mathcal{R} \rightarrow \Gamma$  satisfying (A), we have*

$$h_{\Gamma \curvearrowright X}^{coc} = h_{\Gamma \curvearrowright X}^{Rok}.$$

Under additional ergodicity assumptions, one also obtains a Shannon-McMillan-Breiman theorem. The latter is concerned with the so-called *information function* which is defined as follows. Given a finite partition  $\mathcal{P}$  and  $x \in X$ , set  $\mathcal{J}(\mathcal{P}(x)) =$

$-\log \mu(P)$ , where  $P \in \mathcal{P}$  is the unique element such that  $P \ni x$ . It is easily verified that

$$\int_X \mathcal{J}(\mathcal{P}(x)) d\mu(x) = H(\mathcal{P}).$$

We state the following theorem.

**Theorem 4** (Nevo, P.). *Assume (A) and suppose that  $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$  is p.m.p. If the extended equivalence relation  $\mathcal{R}^* \subset (X \times Y) \times (X \times Y)$  induced by the cocycle  $\alpha$  is ergodic and  $\lim_n |\mathcal{R}_n| / \log n = \infty$ , then for all finite partitions  $\mathcal{P}$ , we have*

$$\lim_{n \rightarrow \infty} \frac{\mathcal{J}(\bigvee_{z \in \mathcal{R}_n(y)} \alpha(z, y)^{-1} \mathcal{P})(x)}{|\mathcal{R}_n(y)|} = h_{\mathcal{P}}^{\text{coc}}, \quad (\mu \otimes \nu)\text{-a.e. } (x, y).$$

The overall strategy of the proof is an extension of Lindenstrauss' line of argumentation in [Li01]. In the latter paper, the Shannon-McMillan-Breiman theorem is proven for tempered Følner sequences in countable amenable groups.

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## Indistinguishable clusters in random spanning forests

ADAM TIMAR

It was proved by Lyons and Schramm that the infinite components of Bernoulli percolation on a Cayley graph are indistinguishable. This means that any invariantly defined property either holds for every infinite component or for none of them. Indistinguishability of clusters is the same as the ergodicity of the cluster equivalence relation. The perhaps most important invariant random spanning forests of a Cayley graph are the Uniform Spanning Forest (USF) and the Minimal Spanning Forest (MSF). Benjamini, Lyons, Peres and Schramm asked whether these forests satisfy indistinguishability.

We prove indistinguishability and 1-infinity laws for the components (*clusters*) of random spanning forests of Cayley graphs, given that the forest has a property that we call weak insertion tolerance, and it has a tree with infinitely many ends.

We say that a random forest of a unimodular quasitransitive graph  $G$  is *weakly insertion tolerant* if for any  $\{x, y\} = e \in E(G)$ ,  $r$  nonnegative integer, and configuration  $\omega$  such that  $x$  and  $y$  are in different components, there exists an  $f = f(\omega, e, x, r) \in E(G)$  with  $\text{dist}(x, f) = r$  such that the following holds. Fixing  $e, x, r$  and looking at  $f$  as a function of  $\omega$ , it is measurable. If  $A$  is such that  $\mathbf{P}(A) > 0$  and for almost every configuration in  $A$ ,  $C_x \neq C_y$ , then  $\mathbf{P}(\omega \cup \{e\} \setminus \{f\} : \omega \in A) > 0$ . Furthermore, an endpoint of  $f$  is in the same component of  $\omega \cap B(x, r)$  as  $x$  for almost every  $\omega \in A$ .

We show that the Free and the Wired Uniform Spanning Forest (FUSF and WUSF) and the Free and the Wired Minimal Spanning Forest (FMSF and WMSF) satisfy weak insertion tolerance. See [3] for the definitions as well as the importance and some main properties of these forests.

**Theorem 1.** *Suppose that the FUSF and WUSF are different for some unimodular quasitransitive graph  $G$ . Then the following hold:*

- (1) *The FUSF has either 1 or infinitely many components.*
- (2) *Every component of the FUSF has infinitely many ends.*
- (3) *More generally, no two components of the FUSF can be distinguished by any invariantly defined property.*

The condition  $\text{FUSF} \neq \text{WUSF}$  is equivalent to that there exist nonconstant harmonic Dirichlet functions on  $G$ , or, in different terms, that the first  $L^2$  Betti number is nonzero. This was shown by Benjamini, Lyons, Peres and Schramm, see [1]. The previous theorem was proved, without any condition on the FUSF and WUSF, by Hutchcroft and Nachmias, [2].

**Theorem 2.** *Suppose that the FMSF and WMSF are different for some unimodular quasitransitive graph  $G$ . Then the following hold:*

- (1) *The FMSF has either 1 or infinitely many components.*
- (2) *Every component of the FMSF has infinitely many ends.*
- (3) *More generally, no two components of the FMSF can be distinguished by any invariantly defined property.*

The condition  $\text{FMSF} \neq \text{WMSF}$  is equivalent to  $p_c < p_u$ , as shown by Lyons, Peres and Schramm. Here  $p_c$  and  $p_u$  are respectively the critical probability and uniqueness critical probability for Bernoulli percolation on  $G$ . The condition  $p_c < p_u$  is conjecturally equivalent to  $G$  being nonamenable, and is known to hold for *some* Cayley graph of every nonamenable group. See [4] for more details.

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### Harmonic functions and the log log law

GADY KOZMA

(joint work with Gideon Amir, Itai Benjamini, Hugo Duminil-Copin, Ariel Yadin, Tianyi Zheng)

For a finitely-generated group  $G$ , a symmetric set of generators  $S$  and an increasing function  $\omega : \mathbb{N} \rightarrow \mathbb{R}$  we say that  $\text{hg}(G) = \omega$  (this might depend on  $S$ , but we suppress it in the notation) if the following two conditions hold:

- (1) Any function  $f : G \rightarrow \mathbb{R}$  which is harmonic and satisfies  $f(x) = o(\omega(|x|))$  is constant.
- (2) There exists an  $f : G \rightarrow \mathbb{R}$  harmonic, non-constant with  $f(x) = O(\omega(|x|))$ .

Here and below, a function is harmonic with respect to a symmetric set of generators  $S$  if  $f(x) = \frac{1}{|S|} \sum_{s \in S} f(xs)$ . The notation  $|x|$  is the word length with respect to the generators  $S$ .

We discussed the following results.  $\text{hg}((\mathbb{Z}/2) \wr \mathbb{Z}) = x$ ,  $\text{hg}((\mathbb{Z}/2) \wr \mathbb{Z}^2) = \log x$ ,  $\text{hg}(\mathbb{Z} \wr \mathbb{Z}) = x^{2/3}(\log \log x)^{1/3}$  and versions for iterated wreath products. Further, we discussed the structure of the space of harmonic function of minimal growth and analogous results for *positive* harmonic functions.

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## Borel Local Lemma

LUKASZ GRABOWSKI

(joint work with Endre Csóka, András Máthé, Oleg Pikhurko,  
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We prove a Borel version of the local lemma, i.e. we show that, under suitable assumptions, if the set of variables in the local lemma has a structure of a Borel space, then there exists a satisfying assignment which is a Borel function. The main tool which we develop for the proof, which is of independent interest, is a parallel version of the Moser-Tardos algorithm which uses the same random bits to resample clauses that are far enough in the dependency graph.

Let us start by recalling a version of the local lemma. The first version of the local lemma was proved by Erdős and Lovász [3]. The version we present follows from the subsequent improvement of Lovász (published by Spencer [9]). For more historical context we refer to the classical exposition in [1].

Let  $G$  be a graph and let  $b$  be a natural number greater than 1. The elements of the vertex set  $V(G)$  should be thought of as variables which can take values in the set  $b = \{0, 1, 2, \dots, b - 1\}$ . Let  $\mathbf{R}$  be a function whose domain is  $V(G)$  and such that for  $x \in V(G)$  we have that  $\mathbf{R}(x)$  is a set of  $b$ -valued functions defined on the neighbourhood of  $x$ . Such a function  $\mathbf{R}$  is an example of a *local rule* on  $G$ . We say that  $f: V(G) \rightarrow b$  *satisfies*  $\mathbf{R}$  if for every  $x \in V(G)$  the restriction of  $f$  to the neighbourhood of  $x$  belongs to  $\mathbf{R}(x)$ .

The local lemma gives a condition under which a satisfying assignment exists. For  $x \in V(G)$  let  $p(x)$  be the probability of failure at  $x$ , i.e.  $p(x) := 1 - \frac{|\mathbf{R}(x)|}{b^{\deg(x)}}$ , where  $\deg(x)$  is the degree of  $x$ . Let  $\text{Rel}(G)$  be the graph whose vertex set is  $V(G)$  and such that there is an edge between  $x$  and  $y$  if and only if the neighbourhoods of  $x$  and  $y$  are not disjoint (we allow  $x$  and  $y$  to be equal). Finally, let  $\Delta$  be the maximal vertex degree in  $\text{Rel}(G)$ .

**Theorem 1** (Lovász's Local Lemma [9]). *If for all  $x \in V(G)$  we have  $p(x) < \frac{1}{e\Delta}$  then there exists  $f: V(G) \rightarrow b$  which satisfies  $\mathbf{R}$ .*

In order to motivate our *Borel version* of Theorem 1, let us recall a classical application of the local lemma to colorings of Euclidean spaces from [3]. A *b-coloring* of  $\mathbb{R}^n$  is a function  $f: \mathbb{R}^n \rightarrow b$ . We say that a set  $U \subset \mathbb{R}^n$  is *multicolored* with respect to a  $b$ -coloring if  $U$  contains points of all  $b$  colors.

**Corollary 2** ([3]). *Let  $T \subset \mathbb{R}^n$  be a finite set of vectors and let  $b \in \mathbb{N}$  be such that  $b(1 - \frac{1}{b})^{|T|} < \frac{1}{e \cdot |T|^2}$ . Then there exists a  $b$ -coloring of  $\mathbb{R}^n$  such that for every  $x \in \mathbb{R}^n$  the set  $x + T$  is multicolored.*

*Proof.* See [1] □

Our Borel version of the local lemma allows to deduce that the  $b$ -coloring in Corollary 2 can be demanded to be a Borel function.

## 1. BOREL LOCAL LEMMA.

Let  $G$  be a graph as before, but let us additionally assume that  $V(G)$  is a standard Borel space and that  $\mathbf{R}$  is a *Borel local rule*. Since it is notationally involved to precisely define what it means for a local rule to be Borel, in this abstract we only assure the reader that in all naturally occurring applications of the local lemma the local rules are in fact Borel.

We say that a graph  $G$  is of *uniformly subexponential growth* if for every  $\varepsilon > 0$  there exists  $r$  such that for all  $R > r$  and all vertices  $v \in V(G)$  the set of vertices of  $G$  at distance at most  $R$  from  $v$  has cardinality less than  $(1 + \varepsilon)^R$ .

**Theorem 3** (Borel Local Lemma). *Let  $G$  be a graph such that  $V(G)$  is a standard Borel space and let  $\mathbf{R}$  be a Borel local rule on  $G$ . Furthermore let us assume that the graph  $\text{Rel}(G)$  is of uniformly subexponential growth, and let  $\Delta$  be the maximal degree in  $\text{Rel}(G)$ .*

*If for all  $x \in V(G)$  we have  $p(x) < \frac{1}{e\Delta}$  then there exists a Borel function  $f: V(G) \rightarrow b$  which satisfies  $\mathbf{R}$ .*

Repeating the proof of Corollary 2 from [1] we obtain the following corollary.

**Corollary 4.** *In Corollary 2 we can additionally demand the  $b$ -coloring of  $\mathbb{R}^n$  to be a Borel function.*

**Remarks 5.** (i) One can weaken the assumption on  $p(x)$  in the local lemma to  $p(x) < \frac{(\Delta-1)^{\Delta-1}}{\Delta^\Delta}$ , and this is best possible [7]. The same is true in the case of Theorem 3.

(ii) If the set  $V(G)$  of vertices is countable, it can be regarded as a standard Borel space when we equip it with the discrete Borel structure. As will turn out, in this situation all local rules on  $G$  are Borel. If  $V(G)$  is finite then  $\text{Rel}(G)$  is also finite, and hence of uniformly subexponential growth. Thus Theorem 3 includes Theorem 1 as a special case when  $V(G)$  is a finite set.

**1.1. The Moser-Tardos algorithm with limited randomness.** The technique we use to prove Theorem 3 is a modified Moser-Tardos algorithm, and it is of independent interest.

The Moser-Tardos algorithm (MTA) is a randomized algorithm for *finding* the satisfying assignment under the assumptions of the local lemma. A version of it has been first described by Moser [5], and a modified version has been described by Moser and Tardos [6]. We refer to the introduction of [6] for the history of attempts to find a constructive version of the local lemma.

To motivate our modified MTA let us recall the parallel version of the MTA. Let us assume that the set  $V(G)$  of vertices is finite. We start by “sampling” each point of  $V(G)$  at random, i.e. we choose uniformly at random a function  $f_0 \in b^{V(G)}$ . Now we choose a subset  $W_0 \subset V(G)$  which is maximal among the subsets of  $V(G)$  satisfying the following two properties:

- (1) The function  $f_0$  violates the local rule at all points of  $W_0$ .
- (2)  $W_0$  is an independent set in the graph  $\text{Rel}(G)$ .

We define  $f_1$  by “resampling  $f_0$  at variables in  $W_0$ ”. More precisely, we start by defining  $X_0$  to be the union of neighbourhoods of points in  $W_0$ , and we let  $Y_0 := V(G) \setminus X_0$ . Now, we define  $f_1$  to be equal to  $f_0$  on  $Y_0$ . Finally, we choose uniformly at random a function in  $b^{X_0}$ , and we define  $f_1$  to be equal to that function on  $X_0$ .

We repeat this procedure with  $f_1$  in place of  $f_0$ , and so on, until we end up with a satisfying assignment. With overwhelming probability a satisfying assignment will be found in a time which is linear in  $\log(|V(G)|)$ .

Let us informally describe how we modify the MTA. We partition  $V(G)$  into  $p$  disjoint parts  $V_0, \dots, V_{p-1}$  with the property that for every  $i \in p$  and all distinct  $x, y \in V_i$  we have that  $x$  and  $y$  are at least  $r$  far away from each other in the graph  $\text{Rel}(G)$ , where the choice of  $r$  depends only on the growth of the balls in  $\text{Rel}(G)$ , but not on  $|V(G)|$ . Now we assign to each part  $V_i$  a “source of randomness”, i.e. an element  $\text{rnd}$  of  $b^{\mathbb{N}}$ , and we use this single sequence for resamplings of *all* points  $x$  which are in  $V_i$ .

Thus in our modified version of the MTA, the points which lie in the same parts are no longer resampled independently from each other.

**1.2. Previous results and open questions.** To our best knowledge, there is no previous work which establishes any *Borel* variants of the local lemma.

However, measurable variants of the local lemma have been studied by Kun [4] and very recently by Bernshteyn [2]. Let us discuss those of the results of [4] and [2] which are related to the present work.

We warn the reader that the following definition is not equivalent to the similar notion in [2].

**Definition 6.** Let  $\Gamma$  be a countable group, let  $X$  be a standard Borel space, let  $\nu$  be a probability measure on  $X$  and let  $\rho: \Gamma \curvearrowright X$  be an action by measure-preserving Borel bijections. For a sequence  $\gamma_0, \dots, \gamma_{k-1}$  of elements of  $\Gamma$  we define  $G = G(\gamma_0, \dots, \gamma_{k-1})$  to be the graph with  $V(G) := X$  and  $(x, y) \in E(G)$  if for some  $i \in k$  we have  $\gamma_i.x = y$ .

We say that *the measurable local lemma holds for the action  $\rho$*  if for all sequences  $\gamma_0, \dots, \gamma_{k-1}$  and all Borel local rules on  $G(\gamma_0, \dots, \gamma_{k-1})$  such that  $p(x) < \frac{1}{e\Delta}$  for all  $x \in X$ , there exists a measurable function  $f: X \rightarrow b$  which satisfies **R**.

The methods which we use to prove Theorem 3 can be rather easily modified to prove the following theorem.

**Theorem 7** (Measurable Local Lemma). *Let  $\Gamma$  be a countable amenable group, let  $X$  be a standard Borel space, let  $\nu$  be a probability measure on  $X$  and let  $\rho: \Gamma \curvearrowright X$  be an action by measure-preserving Borel bijections which is essentially free, i.e. for  $\nu$ -almost all  $x \in X$  we have that the map  $\Gamma \rightarrow X$  given by  $\gamma \mapsto \gamma.x$  is a bijection. Then the measurable local lemma holds for  $\rho$ .*

In [4] it is shown that the standard Moser-Tardos algorithm can be applied in the setting of an infinite countable graph. As a corollary, the measurable local lemma holds for the Bernoulli action  $\Gamma \curvearrowright [0, 1]^\Gamma$ , where  $\Gamma$  is an arbitrary (not

necessarily amenable) group  $\Gamma$ . In [2] it is shown that the measurable local lemma holds for any action  $\Gamma \curvearrowright X$  for which there exists a  $\Gamma$ -equivariant Borel surjection onto  $[0, 1]^\Gamma$ .

In particular, if  $\Gamma$  is an amenable group then the results of [4] and [2] imply the measurable local lemma only for the actions  $\Gamma \curvearrowright X$  which have infinite entropy. This is a rather large constrain - many natural actions of amenable groups which are covered by Theorems 3 and Theorems 7 do not have infinite entropy. For example, as far as we know, it is impossible to deduce the measurable, let alone Borel, version of Corollary 2 from [4] or [2].

On the other hand, the results in [4] and [2] do not require the group  $\Gamma$  to be amenable. It is very interesting open problem to determine whether the measurable local lemma holds for all probability measure preserving actions of all groups.

Very recently Andrew Marks and Alexander S. Kechris (private communication) observed that the Borel version of the local lemma does not hold for arbitrary Borel actions of the free group on two generators. It would be interesting to determine whether the Borel version of the local lemma holds for actions of arbitrary amenable groups.

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## A construction of finitely generated groups with isometrically embedded expanders

DAMIAN OSAJDA

I describe here a construction of finitely generated groups containing an infinite family of finite connected graphs of bounded degree [10]. It provides first examples of groups containing isometric copies of expanding families of graphs, and other exotic finitely generated groups.

Let  $\Theta = (\Theta_n)_{n \in \mathbb{N}}$  be a family of disjoint finite connected graphs of bounded degree. We assume that there exists a constant  $A > 0$  such that  $\text{diam } \Theta_n \leq A \text{ girth } \Theta_n$ , where  $\text{diam}$  denotes the diameter, and  $\text{girth}$  is the length of the shortest simple cycle. We fix a small cancellation constant  $\lambda \in (0, 1/6]$ , and we assume that  $1 < \lfloor \lambda \text{ girth } \Theta_n \rfloor < \lfloor \lambda \text{ girth } \Theta_{n+1} \rfloor$ .

**Theorem 1.** *There exists a  $C'(\lambda)$ -small cancellation labeling of  $(\Theta_n)_{n \in \mathbb{N}}$  over a finite set  $S$  of labels.*

With such labelled graph family  $\Theta$  we associate a graphical small cancellation presentation:

$$(1) \quad \mathcal{P} = \langle S \mid \Theta \rangle.$$

**Theorem 2.** *For every  $n$ , the graph  $\Theta_n$  embeds isometrically into the Cayley graph  $\text{Cay}(G, S)$  of the group  $G$  defined by the presentation (1).*

For  $\Theta$  being an expander family, as an immediate corollary we obtain the following.

**Corollary 3.** *There exist finitely generated groups with expanders embedded isometrically into Cayley graphs.*

These are the first examples of such groups. In particular, they are not coarsely embeddable into a Hilbert space and do not satisfy the Baum-Connes conjecture with coefficients. The only other groups with such properties are the Gromov monsters [7] (see [1] for an explanation of the construction). The Gromov construction uses a graphical presentation with much weaker ‘small cancellation’ properties. Consequently, only a weak embedding of expanders is established for those examples. The isometric embedding of expanders for the groups from Corollary 3 is useful for analyses of the failure of the Baum-Connes conjecture – see e.g. [14],[5],[6],[8].

Using Sapir’s [13] version of Higman embedding we obtain the first examples of groups as follows.

**Corollary 4.** *There exist closed aspherical manifolds with expanders embedded quasi-isometrically into their fundamental groups.*

The group  $G$  defined by the presentation (1) is the limit of finitely presented groups  $G_i$  defined by presentations  $\langle S \mid (\Theta_n)_{n=1}^i \rangle$ . For  $\Theta$  being a family of  $d$ -regular graphs with  $d > 2$ , we obtain the first examples of groups as follows.

**Corollary 5.** *There exists a sequence  $G_1 \twoheadrightarrow G_2 \twoheadrightarrow G_3 \twoheadrightarrow \cdots$  of finitely presented groups with the following properties. For all  $i$ ,  $\text{asdim}(G_i) = 2$ , and the asymptotic dimension of the limit group  $G$  is infinite.*

Note that despite the group  $G$  above has infinite asymptotic dimension, it behaves in many ways as a two-dimensional group – see e.g. [11].

Using the construction of the small cancellation presentation (1) provided by Theorem 1, and the method of constructing walls for small cancellation groups developed in [15], [16], [3], and [4], we obtain the following.

**Theorem 6.** *There exist finitely generated groups acting properly on  $\text{CAT}(0)$  cubical complexes and not having property A.*

In particular, such groups have the Haagerup property, and thus admit an equivariant coarse embedding into a Hilbert space. This answers in the negative the well known question whether, for groups, Yu’s property A (equivalent e.g. to the exactness of the reduced  $C^*$ -algebra of the group) implies coarse embedding into a Hilbert space. For spaces, the answer to the corresponding question was already known by [9] and [2], [12].

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