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Mini-Workshop: Computations in the Cohomology of Arithmetic Groups

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Abstract. Explicit calculations play an important role in the theoretical development of the cohomology of groups and its applications. It is becoming more common for such calculations to be derived with the aid of a computer. This mini-workshop assembled together experts on a diverse range of computational techniques relevant to calculations in the cohomology of arithmetic groups and applications in algebraic K-theory and number theory with a view to extending the scope of computer aided calculations in this area.

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Introduction by the Organisers

The mini-workshop *Computations in the Cohomology of Arithmetic Groups* was attended by 16 participants from 7 different countries with various expertise on the topics of the workshop. The week was organized around 14 short talks, a series of 3 talks given by Professor Günter Harder, a software session presenting new tools which can be used for the explicit calculations in the cohomology of arithmetic groups and related topics, and a problem session (presented at the end of this report). The schedule also included large time periods for dicussions and collaborations. The speakers presented survey talks and new results including related problems. Some questions have been even addressed during the week (see for instance the abstract of J. Lannes related to a problem of G. Harder).

The cohomology of arithmetic groups is a rich subject with links to geometry, topology, ring theory and number theory. A classic example of such a group is the general linear group $GL_N(\mathbb{Z})$ over the integers. A theorem of Borel is that the rational homology of this group in degree d does not depend on N for sufficiently large N. Moreover, Borel explicitly computed the homology in these cases, called the stable range. If one replaces $GL_N(\mathbb{Z})$ by a congruence subgroup, then Borel's theorem still applies. Machine computations for congruence subgroups in the unstable range are being pursued by a number of independent research groups across Europe and the US. One motivation for such computations is a theorem of Franke that establishes a deep connection between the cohomology of congruence subgroups Γ_0 and the study of automorphic forms: the cohomology $H^*(\Gamma_0; M)$ with suitable coefficients can be contructed from certain automorphic forms, namely those of homological type. The cohomology can be used to test various number theoretic conjectures, in particular those concerning Hecke operators on $H^*(\Gamma_0; M)$. Work of Quillen, Charney, and van der Kallen implies that the integral cohomology of $GL_N(O_K)$ in degree d (for O_K the ring of integers of a number field) is also independent of N for large N. In this case, the cohomology groups are intimately related to the algebraic K-theory of O_K. Recent computational work has yielded the algebraic K-group $K_8(\mathbb{Z})$. While the algebraic K-theory of number rings can be deduced in large parts from the Bloch/Kato conjectures, several groups are still out of reach, such as the $K_{4n}(\mathbb{Z})$ related to the Kummer/Vandiver conjecture, and it is still difficult to compute explict K-theory classes or their associated regulators. Furthermore, knowledge on the algebraic K-theory cannot descend to the cohomology of the related arithmetic groups and even low dimensional homology groups of arithmetic groups are not fully understood.

On one hand, we would like to be able to test conjectures for higher rank arithmetic groups (including symplectic groups) or give explicit evidence of cohomological classes. On the other hand, we would like to be able to compute explicitly, even using advanced machine calculations on state-of-the-art computers, the full structure of the cohomology groups (possibly even their associated cohomology rings) or at least the "less trivial part". Those computations involve also numerous computations on the cohomology of finite groups, on which there has been much recent progress via their related topological and geometric models.

The recent (and on-going) works presented during this workshop aimed at initiating immediate progress on the above problems, adressing either theoretical or computational aspects, thereby setting the stage for new collaborations after the workshop.

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Mini-Workshop: Computations in the Cohomology of Arithmetic Groups

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Abstracts

Computing units in maximal orders using the Voronoï algorithm

Renaud Coulangeon
(joint work with Oliver Braun, Gabriele Nebe and Sebastian Schönnenbeck)

Our goal is to study the unit group of an order Λ in a semisimple finite dimensional algebra over $\mathbb{Q}$, using a combination of Voronoï theory of perfect forms and Bass-Serre theory of graphs of groups.

In a first part, we describe an algorithm to compute all conjugation classes of maximal finite subgroups of $\Lambda^\times$. We define a certain cone of "positive definite forms" adapted to this situation, for which a Voronoï theory is available (perfect forms, Voronoï domains, minimal classes). We show that all the maximal finite subgroups in $\Lambda^\times$ arise as stabilizers of well-rounded minimal classes. The conjugation classes of well-rounded minimal classes can be computed using a variant of the Voronoï algorithm, which provide a finite list of candidates. Then for each candidate G, we use an equivariant version of the Voronoï algorithm, due to Berg, Martinet and Sigrist ("G-perfect forms") to check maximality.

In a second part, elaborating on an original idea by Opgenorth, we present an algorithm which provides a presentation (generators and relations) of the group of units $\Lambda^\times$ in a (maximal) order Λ in a semisimple finite dimensional algebra over $\mathbb{Q}$. The idea is again to use the graph of perfect forms, viewed as a graph of groups, *i.e.* with stabilizers of vertices and edges attached. Then, the fundamental sequence

$$1 \longrightarrow \pi_1(X) \longrightarrow \pi_1(\Gamma\backslash\backslash X) \longrightarrow \Gamma \longrightarrow 1$$

of Bass-Serre theory provides the expected presentation. We illustrate our method with several examples, some of which were out of reach with the existing packages.

Tessellations, Bloch groups, homology groups

Rob de Jeu
(joint work with David Burns, Herbert Gangl, Alexander Rahm and Dan Yasaki)

Let k be an imaginary quadratic number field with ring of integers $\mathcal{O}$. We discuss how an ideal tessellation of hyperbolic 3-space on which $GL_2(\mathcal{O})$ acts (as can be algorithmically computed using the algorithm described in, e.g., [3]), gives rise to an explicit element β of infinite order in the (second) Bloch group for k, i.e., the kernel of the map

$$\begin{aligned}\overline{B}_2(k) &\to \frac{k^* \otimes_{\mathbb{Z}} k^*}{\langle (-c)\otimes c \text{ with } c \text{ in } k^*\rangle} \\ [a] &\mapsto \text{the class of } (1-a)\otimes a\,,\end{aligned}$$

where $\overline{B}_2(k)$ is the quotient of the free Abelian group $\mathbb{Z}[k^\flat]$ on $k^\flat = k \setminus \{0,1\}$ by the subgroup generated by the following three types of elements:

(1) $[x] - [y] + [y/x] - [(1-y)/(1-x)] + [(1-y^{-1})/(1-x^{-1})]$;
(2) $[x] + [x^{-1}]$;
(3) $[y] + [1-y]$.

This kernel is closely related to the Bloch group defined in [4], and for k as above turns out to be isomorphic to $\mathbb{Z}$.

By [2, Theorem 4.1], our explicit element β gives rise to an element γ in $K_3(k)/\text{torsion} \simeq \mathbb{Z}$, for which $\text{reg}(\gamma) = -12\zeta_k'(-1)$ holds, where reg denotes the normalized Beilinson regulator. The Lichtenbaum conjecture for k at -1 suggests that γ should be divided by the order of $K_2(\mathcal{O})$ in order to obtain a generator of $K_3(k)/\text{torsion}$. In [1], those orders were computed for a number of k, and using [5], we computed elements β for those and many more fields. The expected division could be carried out explicitly in several cases by dividing β in the kernel of the map given above. The most notable case is that of $\mathbb{Q}(\sqrt{-303})$, where $K_2(\mathcal{O})$ has order 22.

We also discuss how to obtain from the tesselation an explicit non-torsion element in the third homology group of GL_2 of a localisation of $\mathcal{O}$.

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Lattices and perfect form theory for cohomology computations

Mathieu Dutour-Sikirić

The computation of cohomology of a group Γ is important for many applications in number theory and topology. Regardless of the cohomology theory considered (Bredon, Farrell-Tate, etc.), the practical computation rely on having a polyhedral tessellation $\mathcal{T}$ satisfying following conditions:

- Each cell of $\mathcal{T}$ is mapped by elements of Γ to a cell of $\mathcal{T}$.
- Every cell of $\mathcal{T}$ has finite stabilizer.
- The stabilizer of a cell preserves the cell pointwise (this property is required for Bredon cohomology and equivariant cohomology computation).

For the group $\mathrm{GL}_n(\mathbb{Z})$ we have one decomposition coming from perfect form theory. Perfect form theory was originally introduced in order to deal with the problem of computing lattices of maximum packing density in a fixed dimension n. A perfect form A is a positive definite quadratic form whose set of shortest vectors define it up to a scalar.

The perfect forms induce

- a tesselation of the rational closure of the cone of positive definite forms by perfect domains,
- the vertices of the Ryshkov polyhedron defined as the positive definite forms satisfying $A[x] \geq 1$ for all $x \in \mathbb{Z}^n - \{0\}$.

The two above tesselations are dual to each other. By standard Voronoï theory the number of perfect forms is finite and this gives a tesselation which is equivariant. After an equivariant contraction to the well rounded retract we obtain a complex of dimension $n(n-1)/2$. By [8] this complex is minimal in dimension.

Thus the perfect form complex can be used for efficient computation of the cohomology of $\mathrm{GL}_n(\mathbb{Z})$ up to the primes occurring in the stabilizers of finite subgroups of $\mathrm{GL}_n(\mathbb{Z})$. It also can be used to compute the action of Hecke operators on the cohomology (see [6]).

On the other hand, there are many other possible decomposition of the cone of positive definite forms. Following are invariant under the action of $\mathrm{GL}_n(\mathbb{Z})$ and are linear:

- The perfect form theory (Voronoï I) for lattice packings (full face lattice known for $n \leq 7$, perfect domains known for $n \leq 8$). This is the one most commonly used.
- The central cone compactification (Igusa & Namikawa) (known for $n \leq 6$). It may be simpler than Voronoï I but it has never been used for cohomology computations.
- The L-type reduction theory (Voronoï II) for Delaunay tessellations (known for $n \leq 5$). It is relatively easy to compute, but the number of cells is too large to be of interest in cohomology computations.
- The C-type reduction theory (Ryshkov & Baranovski) for edges of Delaunay tessellations (known for $n \leq 5$). It is a little bit simpler than the L-type reduction theory but still completely unworkable for $n > 5$.
- The Minkowski reduction theory it uses the successive minima of a lattice to reduce it (known for $n \leq 7$) not face-to-face. Unfortunately, the fact that it is not face-to-face makes it unsuitable to cohomology computations. Also, it is quite impractical to compute.
- Venkov's reduction theory also known as Igusa's fundamental cone (finiteness proved by Crisalli and Venkov). No practical computation is known to this author.

It is important to remark that the above theory for $\mathrm{GL}_n(\mathbb{Z})$ can be generalized to:

- Finite index subgroups.

- Automorphism groups preserving self-dual cones (those are classified in [1] and [2])
- Groups of the form $\mathrm{GL}_n(R)$ with R a ring of algebraic integers.
- Normalizers of finite subgroups of $\mathrm{GL}_n(\mathbb{Z})$.

Essentially all of the above theory can be generalized but perfect form theory remains the simplest to deal with.

The central compactification that could be a contender is defined in the following way:

- We take the set S_n^I of n-dimensional quadratic forms having integer values on vectors $v \in \mathbb{Z}^n$.
- We define R_n^I to be the convex hull of the set of forms $A \in S_n^I$ with $A[x] \geq 1$ for all $x \in \mathbb{Z}^n$.
- The vertices of R_n^I are the equivalent of the perfect forms. The facets are more complicated than for perfect form theory.
- The root lattices define vertices of R_n^I

Computing with the central cone compactification is harder than for perfect but still feasible even though computationally intensive. This is done by solving integer programming problems.

For dimension 8 and 9 the computations are harder to do with a very large number of perfect forms. It is not reasonable to expect to be able to compute the full complex in those dimensions. However, following is known:

- For $n = 8$, the number of orbits of faces of rank $r = 8$, 9, 10, 11, 12 is 13 (Zaraheva & Martinet), 106, 783, 6167, 50645
- For $n = 9$, the number of orbits of faces of rank $r = 9$, 10, 11 is 44, 759, 13437.

These results are based on [3] and methods of [4].

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Computing cohomology of groups and 2-types

Graham Ellis
(joint work with Bui Anh Tuan, Mathieu Dutour-Sikirić and Le Van Luyen)

The talk describes a computer technique for calculating the integral cohomology of a group G equipped with a cellular action on a contractible cellular space for which the stabilizer groups are finite. The method is based on an old perturbation technique of C.T.C. Wall [8] and uses the theory of discrete vector fields due to Robin Forman [7]. An implementation of the method is used in [3] to compute $H_n(PSL_4(\mathbb{Z}),\mathbb{Z})$ for $n \leq 5$ and to compute the low-dimensional homology of several other arithmetic groups. An implementation of the method is used in [6] to compute $H_n(SL_2(\mathbb{Z}[1/m]),\mathbb{Z})$ for all $n \geq 0$ and $m \leq 50$ except $m = 30, 42$. In [2], the method is used to recover a small free $\mathbb{Z}G$-resolution of $\mathbb{Z}$ for an arbitrary Coxeter group G due to De Concini and Salvetti [1]. The final part of the talk explains how the method can be adapted to study the cohomology of homotopy 2-types, i.e. the cohomology of CW-spaces X with $\pi_i(X) = 0$ for $i \neq 1, 2$. Up to homotopy equivalence such a space can be represented by a category $\mathbb{G}$ internal to the category of groups (often called a *cat*^{1}*-group*). Using the classical Homological Perturbation Lemma and the Eilenberg-Zilber theorem, the homology groups $H_n(X,\mathbb{Z})$ can be computed from a filtered chain complex arising directly from $\mathbb{G}$. Details on a computer implementation of this are available in [4]. The low-dimensional homology of homotopy 2-types is used in [5] to obtain a partial classification of cat^1-groups $\mathbb{G}$, up to weak equivalence, for orders $|\mathbb{G}| \leq 127$, $|\mathbb{G}| \neq 32, 64, 81, 96$.

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Computing the cohomology of p-groups

David J. Green
(joint work with Simon King)

I gave a survey talk on the relevant aspects of the computer calculation of mod-p cohomology rings of finite groups.

Let $A^* = H^*(G, k)$ be the cohomology of the finite group G over a field k of characteristic p. Then A^* is a finitely-presented connective graded-commutative k-algebra. Because it is finitely presented, one can attempt to compute it degree by degree. For $N > 0$ let $\tau_N(A)^*$ be the approximation to A^* obtained by taking all those generators and relations which occur in degree $\leq N$. Then there is an induced map $\alpha_N \colon \tau_N(A^*) \to A^*$. This map is an isomorphism for sufficiently large N, and by construction it is always an isomorphism in degrees $\leq N$. I learnt the following broad strategy from Carlson:

(1) Choose a good value of N.
(2) **Low dimensional computation**: Compute $\tau_N(A)^*$.
(3) **Certification**: Decide whether the map α_N is an isomorphism or not. If yes, then done.
(4) Otherwise repeat with a larger value of N.

Certification. There are two broad methods:

(1) **Degree bound**: Some theorem tells you in advance what value of N to use.
(2) **Recognition**: If $\tau_N(A)^*$ passes a series of tests then some theorem tells you that it is A^*.

Clearly an effective degree bounds would be preferable: but the only known bound for group cohomology is the d_1 of Henn–Lannes–Schwartz [6], and in spite of Kuhn's improvements [9] this is still too high. So one has to use recognition instead. The first recognition principle was discovered by Carlson [3] and involved verifying two conjectures for the group concerned in the course of recognition. Benson then published a better test [1], which does not (now) depend on any conjectures. King made some improvements to Benson's test [7].

I described Benson's test, which involves choosing a system of parameters. The test and its proof depend on the following results from the cohomology of finite groups:

- Quillen's work on the spectrum of group cohomology, especially the Krull dimension, and the recognition of systems of parameters by restriction to elementary abelian subgroups [12].
- Benson–Carlson duality, especially the *last survivor* [2].
- Symonds' theorem that $H^*(G, k)$ has Castelnuovo–Mumford regularity zero [13].

Low-dimensional computation. In view of the Cartan–Eilenberg result on stable elements, one can concentrate on p-groups P here. Over a field k of characteristic p, any finite group has a minimal projective resolution. Free modules are easier to compute with than projective modules, but over a p-group all projectives are free. I described the construction of the minimal resolution using non-commutative Gröbner bases [4], giving an extended example for the quaternion group Q_8. I then presented our results on the groups of order 128 [5] and our work with Ellis on the third Conway group [8]. Finally I discussed some bottlenecks, including the mod-2 cohomology ring of the Mathieu group M_{24} and – inspired by a question of Mason [10] – the low-dimensional mod-2 cohomology of the Janko group J_4. Mason suggested that the integral cohomology of J_4 might be highly connected: currently the only known instance of this phenomenon is the Mathieu group M_{23} [11].

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Modular symbols and the Sharbly complex

Paul E. Gunnells

Let $\mathbf{G}/\mathbb{Q}$ be a reductive algebraic group defined over $\mathbb{Q}$, and let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup. Let M be a $\mathbb{Z}\Gamma$-module arising from a rational representation of $\mathbf{G}(\mathbb{Q})$. Then the cohomology groups $H^*(\Gamma; M)$ contain a direct connection to automorphic forms and are of interest to compute explicitly. For example, when Γ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and $M \subset \mathbb{C}[x, y]$ is the subspace of degree k symmetric polynomials, then by the Eichler–Shimura isomorphism (as extended by Halbritter) $H^1(\Gamma; M)$ can be computed in terms of holomorphic cuspforms and Eisenstein series of weight $k+2$ for Γ. In general, the cohomology spaces $H^*(\Gamma; M)$ are expected to be connected to Galois representations. That this is true for $\Gamma \subset \mathrm{SL}_n(\mathbb{Z})$ is a very recent theorem of Scholze [17].

In this talk we presented tools to compute cohomology explicitly in various cases; our emphasis was on computing the action of the Hecke operators on cohomology. We began by reviewing Ash–Rudolph's explicit theory of modular symbols [1] for $\mathbf{G} = \mathrm{SL}_n/\mathbb{Q}$. Let X be the global symmetric space $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$, let $\Gamma \subset \mathrm{SL}_n(\mathbb{Z})$ be a congruence subgroup, and let $Y = \Gamma\backslash X$ be the locally symmetric space associated to Γ. An n-tuple $\mathbf{v} = (v_1, \ldots, v_n)$ of primitive points from $\mathbb{Z}^n$ determines an apartment in the Tits building T_n for $\mathrm{SL}_n(\mathbb{Q})$. The fundamental class of this apartment $[\mathbf{v}] \in \tilde{H}_{n-2}(T_n)$ (i.e., in the *Steinberg module* St_n) determines a class in the relative homology $H_{n-1}(\overline{X}, \partial\overline{X})$, where $\overline{X}$ is the *Borel–Serre compactification* of X [9], via the homotopy equivalence $T_n \simeq \partial\overline{X}$ and the long exact sequence of the pair $(\overline{X}, \partial\overline{X})$. This then determines a class in $H_{n-1}(\overline{Y}, \partial\overline{Y}) \simeq H^\nu(Y) \simeq H^\nu(\Gamma)$, where $\nu = n(n-1)/2$ is the virtual cohomological dimension of Γ. This class is called the *modular symbol* associated to $\mathbf{v}$. Moreover, Ash–Rudolph give an explicit algorithm that allows one to write any modular symbol as a sum of *unimodular symbols*; by definition a symbol is unimodular if the vectors v_i form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. This algorithm, which generalizes the classical continued fraction algorithm to higher dimensions, gives an explicit way to compute the action of the Hecke operators on $H^\nu(\Gamma; M)$. This was used in [2] to investigate non-selfdual automorphic forms for $\mathrm{SL}_3/\mathbb{Q}$, which appear in H^3 of subgroups of $\mathrm{SL}_3(\mathbb{Z})$.

The modular symbols form the first group in a resolution of the Steinberg module called the *Sharbly complex* Sh_* [7]. As an abelian group, Sh_k, $k \geq 0$, is generated by symbols $[v_1, \ldots, v_{n+k}]$, where the v_i are nonzero vectors in $\mathbb{Q}^n$, modulo the submodule generated by the following relations: (i) $[v_{\sigma(1)}, \ldots, v_{\sigma(n+k)}] - (-1)^\sigma[v_1, \ldots, v_{n+k}]$ for all permutations σ; (ii) $[v_1, \ldots, v_{n+k}]$ if $v_1, \ldots, v_{n+k}$ do not span all of $\mathbb{Q}^n$; and (iii) $[v_1, \ldots, v_{n+k}] - [av_1, v_2, \ldots, v_{n+k}]$ for all $a \in \mathbb{Q}^\times$. Note that by (iii), we can assume each v_i is a primitive vector in $\mathbb{Z}^n$. The boundary map $\partial\colon Sh_k \to Sh_{k-1}$ is given in the usual way by $\partial([v_1, \ldots, v_{n+k}]) = \sum_{i=1}^{n+k}(-1)^i[v_1, \ldots, \widehat{v_i}, \ldots v_{n+k}]$. The modular symbols correspond to Sh_0. By Borel–Serre duality [9], one has $H_k(\Gamma; Sh_* \otimes M) \simeq H^{\nu-k}(\Gamma; M)$ if Γ is torsion-free (or if, more generally, all primes dividing orders of finite elements of Γ are invertible in M). The Hecke operators act on the Sharbly complex, and thus any suitable notion of "unimodular k-Sharbly" with an algorithm that allows one to write a general k-Sharbly cycle as a sum of unimodular k-Sharbly cycles allows one to explictly compute the Hecke action on $H^{\nu-k}(\Gamma; M)$. Such an algorithm for Sh_1 was presented in [10], and has successfully been used to compute the Hecke action on H^5 of subgroups of $\mathrm{SL}_4(\mathbb{Z})$ in a series of papers [3–6, 8]. Variations of this algorithm have also been used to compute the Hecke action on subgroups of $\mathrm{GL}_2(\mathcal{O}_F)$, where $\mathcal{O}_F$ is the ring of integers of certain number fields F [12–14].

In the second part of the talk we discussed analogous results for the symplectic group $\mathbf{G} = \mathrm{Sp}_{2n}/\mathbb{Q}$. In this case the relevant symmetric space is the Siegel upper halfspace $\mathfrak{H} = \mathrm{Sp}_{2n}(\mathbb{R})/\mathrm{U}(n)$; the quotients $Y = \Gamma\backslash\mathfrak{H}$ for $\Gamma \subset \mathrm{Sp}_{2n}(\mathbb{Z})$ are the Siegel modular varieties. Already the case of $H^3(\Gamma; M)$ for $\Gamma \subset \mathrm{Sp}_4(\mathbb{Z})$ is of particular interest, because of Harder's examples/conjectures about congruences between Siegel and elliptic modular forms and their connections with special values of automorphic L-functions [15].

We first described a version of Ash–Rudolph's algorithm for the symplectic group [11]. This allows one to compute the Hecke action on $H^\nu(\Gamma; M)$, where $\Gamma \subset \mathrm{Sp}_{2n}(\mathbb{Z})$ and now $\nu = n^2$ is the virtual cohomological dimension of Γ. We presented the relations among the modular symbols that play a key role in the algorithm; these come from the simplicial geometry of the Tits building for Sp_{2n}, and in the case $n = 2$ are related to the combinatorial data parameterizing cells in MacPherson–McConnell's explicit reduction theory for $\mathrm{Sp}_{2n}(\mathbb{Z})$ [16].

For Sp_4, for instance, the algorithm in [11] allows one to compute the Hecke action on H^4. Harder's conjectures, however, address cuspidal Siegel modular forms of weight 3, which show up in H^3. Thus one needs an analogue of the "codimension one" algorithm in [10]. In particular one needs an analogue of the Sharbly complex for the symplectic group. Such a complex will form a resolution of the Steinberg module, but not just any old resolution will do: one needs a resolution that reflects the cellular structure in [16], just as the relations among modular symbols do. We gave some indications of what such a complex might look like, but in general a good construction of this resolution, as well as an analogue of the algorithm in [10], remains open.

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Explicit computation of denominators of Eisenstein cohomology classes

Günter Harder

I think that that there is some demand for a computer program which computes the cohomology of arithmetic groups and the action of the Hecke operators on these cohomology groups. The benefit of such a program would be that we can compute denominators of Eisenstein classes and this in turn would allow us to verify experimentally the conjectural congruences between eigenvalues of Hecke operators on eigenclasses on different groups.

We start from an arithmetic group $\Gamma \subset G(\mathbb{R})$ and a Γ-module $\mathcal{M}$ which should be finitely generated as $\mathbb{Z}$-module. The group Γ acts on the symmetric space $X = G(\mathbb{R})/K_\infty$. The module provides a sheaf $\tilde{\mathcal{M}}$ on the locally symmetric space $\Gamma\backslash X$. If $\pi : X \to \Gamma\backslash X$ is the natural projection, then

$$\tilde{\mathcal{M}}(V) = \{f : \pi^{-1}(V) \to \mathcal{M} \mid f \text{ locally constant and } f(\gamma u) = \gamma f(u)\} \tag{1}$$

In this case we have a general theorem by Raghunathan which asserts that the cohomology groups $H^q(\Gamma\backslash X, \tilde{\mathcal{M}})$ are finitely generated $\mathbb{Z}$ -modules.

Task A): Develop some software which allows us to compute these cohomology groups explicitly for a large class of groups Γ and coefficient systems $\tilde{\mathcal{M}}$.

In general the quotient $\Gamma\backslash X$ is not compact, hence we can define the cohomology with compact supports. We also can compactify and add the Borel-Serre boundary $\partial(\Gamma\backslash X)$ at infinity and get the fundamental long exact sequence

(2)
$$\to H^{q-1}(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}) \to H^q_c(\Gamma\backslash X), \tilde{\mathcal{M}}) \to H^q(\Gamma\backslash \bar{X}, \tilde{\mathcal{M}}) \xrightarrow{r} H^q(\partial(\Gamma\backslash X), \tilde{\mathcal{M}}) \to \dots$$

Task A1) Compute all the modules in this exact sequence explicitly and compute the arrows between these modules.

Let us assume that $\mathcal{M} = \mathcal{M}_\lambda$ is a highest weight module over $\mathbb{Z}$. In [2], Chap. III, sec. 2, we define the action of the Hecke algebra on the above cohomology groups, more specifically we define operators $T^{\mathrm{coh},\lambda}_{p,\chi}$ which act on all the cohomology groups.

Task B) Give explicit expressions for the $T^{\mathrm{coh},\lambda}_{p,\chi}$ under the assumptions that A) and A1) are done.

In [2] Chap. II , Sec. 3 and 4, I discuss a general strategy which allows us to tackle these tasks in principle. The cohomology is computed from the Čech complex of an orbiconvex covering [2] Chap. II , loc.cit. and then we can also write a procedure which computes Hecke operators. I have no idea whether this strategy is effective or optimal.

In a joint effort Herbert Gangl and I investigated the "baby" case (For more details [2] 3.2): The group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, the symmetric space is the upper half plane $\mathbb{H}$ and $\mathcal{M} = \mathcal{M}_n = \{\sum_{\nu=0}^{n} a_\nu X^\nu Y^{n-\nu} \mid a_\nu \in \mathbb{Z}\}$ where n is even.

In this case the cohomology in degree one is of interest. Task A and Task A1 are relatively easy in this case. If we divide by the torsion and observe that $H^1(\partial(\Gamma\backslash X), \tilde{\mathcal{M}})/tors = \mathbb{Z}\omega_n$ and break the exact sequence (2), then we get

$$0 \to H^1_!(\Gamma\backslash X, \tilde{\mathcal{M}}) \to H^1(\Gamma\backslash \bar{X}, \tilde{\mathcal{M}})/tors \xrightarrow{r} \mathbb{Z}\omega_n \to 0 \tag{3}$$

For any prime p, we have defined the Hecke operator $T_p = T^{\mathrm{coh}}_{p,\chi}$, where

$$\chi(p) = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}.$$

We know (Manin-Drinfeld principle, estimates of the eigenvalues on cusp forms)

$$T_p\omega_n = (2^{n+1}+1)\omega_n \text{ and } \det((p^{n+1}+1)\mathrm{Id} - T_p)|_{H^1_!(\Gamma\backslash X, \tilde{\mathcal{M}})} \neq 0. \tag{4}$$

If we tensor by $\mathbb{Q}$, we get a splitting

$$H^1(\Gamma\backslash \bar{X}, \tilde{\mathcal{M}} \otimes \mathbb{Q}) = H^1_!(\Gamma\backslash X, \tilde{\mathcal{M}} \otimes \mathbb{Q}) \oplus \mathbb{Q}\tilde{\omega}_n \tag{5}$$

where $r(\tilde{\omega}_n) = \omega_n$ and $T_p\tilde{\omega}_n = (p^{n+1}+1)\tilde{\omega}_n$. The class $\tilde{\omega}_n$ is called the Eisenstein class, the smallest positive integer $\Delta(n)$ such that $\Delta(n)\tilde{\omega}_n \in H^1(\Gamma\backslash \bar{X}, \tilde{\mathcal{M}})/tors$ is called the *Denominator of the Eisenstein class.*

This denominator can be computed from any of the Hecke operators. With Gangl we wrote a program which accomplishes task A1 and for T_2 also task B. Hence we could compute the matrix of T_2 for a large number of n (I think $n \leq 150$) and we found experimentally

$$\Delta(n) = \text{Numerator of}(\zeta(-1-n)) \tag{6}$$

I hope that this can be proved using the theory of modular symbols.

Next we may consider the case $\Gamma = \mathrm{Sp}_2(\mathbb{Z})$ and $\mathcal{M} = \mathcal{M}_\lambda$ where $\lambda = n_\beta\gamma_\beta + n_\alpha\gamma_\alpha, \beta =$ short root $, n_\beta \equiv 0(2)$. In this case the cohomology of the boundary becomes much more complicated. In the description of the boundary cohomology some genus one modular cusp forms f of a certain weight depending on λ enter the stage. Then we expect some primes ℓ which divide certain critical values

$L(f,\nu)/\Omega(f)_{\epsilon(\nu)}$ should also divide the denominator $\Delta(f)$ of the Eisenstein class $\tilde{\omega}(f)$.

In [1], I discuss the special case $\lambda = 4\gamma_\beta + 7\gamma_\alpha$. In this case the genus one modular form is the modular cusp form f_{22} of weight 22 and we have $41|L(f_{22}, 14)/\Omega_+$. I give some speculative reasons why 41 should divide the denominator $\Delta(f_{22})$.

If we only could carry out task A) , A1) and B) in this case for one Hecke operator we would get a verification of the $41|\Delta(f)$.

It is not clear to me whether in this case we are already beyond the limit of capability of existing computers.

The resulting congruence for the Hecke eigenvalues of T_p have been verified by Faber and van der Geer for $p \leq 37$ and by Chenevier and Lannes for all primes p. (See talk of Lannes at this mini-conference).

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Calculating the cohomology of discrete groups via centralizer approximations

Hans-Werner Henn

Let p be a prime. In this talk we explained how one can study the mod-p cohomology of a discrete group G by using approximations via centralizers of elementary abelian p-subgoups of G. The results presented are not new but they may be combined with recently developed computer assisted methods to perform new calculations.

If X is a G-CW complex, EG is the universal covering space of G and M a trivial G-module then we denote by $H^*_G(X, M)$ the cohomology of the Borel construction $EG \times_G X$ with coefficients in M.

The Quillen category $\mathcal{A}(G)$ of elementary abelian p-subgroups of G has as its objects the non-trivial elementary abelian p-subgroups of G and as morphisms all group homomorphisms between non-trivial elementary abelian p-subgroups which are induced by conjugation by elements in G. If E is an elementary abelian p-subgroup; and if we denote by X^E the subcomplex of X which is pointwise fixed by E; and by $C_G(E)$ the centralizer of E in G, then we get a functor with values in abelian groups

$$\mathcal{A}(G) \to \mathcal{A}b, \quad E \mapsto H^*_{C_G(E)}(X^E, M)\ .$$

The inverse limit of such a functor has right derived functors which we denote by $\lim^s_{\mathcal{A}(G)} H^*_{C_G(E)}(X^E, M)$.

The subcomplex of all points in X whose stabilizer contains an element of order p is denoted $X_{p,s}$.

Theorem 1. *Let G be a discrete group, let X be any G-CW complex for which all point stabilizers are finite and let M be any $\mathbb{Z}_{(p)}$-module.*
(a) Then there is a first quadrant cohomological spectral sequence

$$E_2^{s,t} = \lim^s_{\mathcal{A}(G)} H^t_{C_G(E)}(X^E; M) \Longrightarrow H^{t+s}_G(X_{p,s}; M)$$

with $E_2^{s,t} = 0$ for all $s \geq r_p(G)$, the p-rank $r_p(G)$, and all $t \in \mathbb{Z}$.
(b) Furthermore, if X has finitely many G-cells and $M = \mathbb{F}_p$ then $E_2^{s,t} = 0$ whenever $s > 0$ and t is sufficiently large, and the canonical map

$$H^*_G(X_s; \mathbb{F}_p) \longrightarrow \lim_{\mathcal{A}(G)} H^*_{C_G(E)}(X^E; \mathbb{F}_p)$$

induced by the obvious inclusions on the space and group level is an isomorphism in suffiiciently large cohomological degrees.

Remarks: a) Note that if X is mod-p acyclic and finite dimensional then X^E is also mod-p acyclic. In the case of arithmetic groups and their symmetric spaces X, both X and X^E are typically contractible, so that we get
$E_2^{s,t} = \lim^s H^t(C_G(E); \mathbb{F}_p)$ and
$\lim_{\mathcal{A}(G)} H^*_{C_G(E)}(X^E; \mathbb{F}_p) \cong \lim_{\mathcal{A}(G)} H^*(C_G(E); \mathbb{F}_p)$.
b) In the situation of Remark a) the spectral sequence only converges towards $H^*_G(X_{p,s}; \mathbb{F}_p)$ and not to $H^*(G; \mathbb{F}_p) = H^*_G(X; \mathbb{F}_p)$. However, if one succeeds to analyze the relative cohomology $H^*_G(X, X_{p,s}; \mathbb{F}_p)$ and the long exact cohomology sequence of the pair $(X, X_{p,s})$ then the spectral sequence of the theorem is the first step towards calculating $H^*_G(X; \mathbb{F}_p)$.

Example: The theorem and the strategy outlined in Remark b) were used almost 20 years ago in [1] and [2] in order to calculate the mod-2 cohomology of the S-arithmetic group $G = SL_3(\mathbb{Z}[\frac{1}{2}])$. Most of the work was the analysis of the relative cohomology $H^*_G(X, X_{p,s}; \mathbb{F}_p)$ for X the product of the symmetric space for $SL_3(\mathbb{R})$ with the Bruhat Tits building for $SL_3(\mathbb{Q}_2)$.

The talk ended by suggesting other groups for which the method could be successfully applied. Particularly promising are the groups $G = SL_3(\mathbb{Z}[\frac{1}{2}, i])$ and $G = SL_4(\mathbb{Z}[\frac{1}{2}])/\{\pm 1\}$. Again the main work would be to analyze $H^*_G(X, X_{p,s}; \mathbb{F}_p)$. For these groups, the size of X makes it unreasonable to do this by hand as in [2]. The hope is rather that computer assisted methods developed in recent years could come to help.

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On the low-dimensional homology of SL_2 of S-integers

Kevin Hutchinson

Let F be a global field. Let S be a set of primes of F containing all archimedean primes. We suppose in any case that S contains at least two primes, including at least one nonarchimedean prime. Let

$$\mathcal{O}_{F,S} = \mathcal{O}_S = \{x \in F \mid v(x) \geq 0 \text{ for all } v \notin S\},$$

the ring of S-integers in F. We discuss what can be said about the structure of the integral homology groups $H_n(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z})$ when S is not too small and $k \leq 3$.

When $n = 1$, the work of Vaserstein and Liehl in the 1970s shows that $H_1(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z}) = 0$ — i.e. the group $\mathrm{SL}_2(\mathcal{O}_S)$ is perfect — if there exists a unit λ in $\mathcal{O}_S$ for which $\lambda^2 - 1$ is also a unit.

When $n = 2$, it is natural to guess that $H_k(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z}) \cong K_2(2, \mathcal{O}_S)$ the rank one, or symplectic, K_2 of the ring $\mathcal{O}_S$, at least when S is sufficiently large, but there are very few instances where this is proven. The symplectic case of the theorem of Matsumoto and Moore establishes this isomorphism in the limit; i.e. when S is the set of all places of F. In this case one has an isomorphism $H_2(\mathrm{SL}_2(F), \mathbb{Z}) \cong K_2(2, F)$. Furthermore, it is natural to guess that there is a short exact sequence

$$0 \to K_2(2, \mathcal{O}_S) \to K_2(2, F) \to \oplus_{v \notin S} k(v)^\times \to 0$$

when S is sufficiently large. Here the right-hand homomorphisms are the tame symbols $K_2(F) \to k(v)^\times$ combined with the natural map $K_2(2, F) \to K_2(F)$. Again, this is only proved in a few cases. For example, this latter statement has been proved by J. Morita [4] in the case where $F = \mathbb{Q}$ and $S = \{\infty, p_1, \ldots, p_n\}$ where $n \geq 2$ and $p_1 < \cdots < p_n$ are the first n prime numbers. In [2], we proved that for any global field F there exists a finite set of primes S_0 with the property that whenever $S_0 \subset S$ there is a natural short exact sequence

$$0 \to H_2(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z}) \to K_2(2, F) \to \oplus_{v \notin S} k(v)^\times \to 0.$$

When $F = \mathbb{Q}$, we show that one can take $S_0 = \{\infty, 2, 3\}$. It follows that when $6|m$, we have

$$H_2(\mathrm{SL}_2(\mathbb{Z}\left[\tfrac{1}{m}\right]), \mathbb{Z}) \cong \mathbb{Z} \oplus \left(\oplus_{p|m} \mathbb{Z}/(p-1)\right).$$

Combining this with the result of Morita, it follows that $H_2(\mathrm{SL}_2(\mathbb{Z}\left[\frac{1}{m}\right]), \mathbb{Z}) \cong K_2(2, \mathbb{Z}\left[\frac{1}{m}\right])$ when $n \geq 2$ and m is the product of the first n primes. We note that the recent calculations of Bui and Ellis [1] suggest that, when $F = \mathbb{Q}$, $S_0 = \{\infty, 2\}$ may suffice in the result above.

When $n = 3$, one cannot expect to describe the structure of the groups $H_3(\mathrm{SL}_2(\mathcal{O}_S), \mathbb{Z})$ in terms of K-theory: There is a natural surjective homomorphism from $H_3(\mathrm{SL}_2(F), \mathbb{Z})$ to the indecomposable quotient, ${K_3}^{\mathrm{ind}}(F)$, of $K_3(F)$. It is known that ${K_3}^{\mathrm{ind}}(F)$ is a finitely generated group of the form $\mathbb{Z}^{r_2} \oplus \mathbb{Z}/w_2(F)$ where r_2 is the number of complex embeddings and $w_2(F)$ is the number of roots of unity in the compositum of all quadratic extensions of F. We denote the kernel of the map $H_3(\mathrm{SL}_2(F),\mathbb{Z}) \to {K_3}^{\mathrm{ind}}(F)$ by $H_3(\mathrm{SL}_2(F),\mathbb{Z})_0$. It is shown in [3] that there is a natural surjective homomorphism

$$H_3(\mathrm{SL}_2(F),\mathbb{Z}\left[\tfrac{1}{2}\right])_0 \to \oplus_{v\notin S_\infty} P(k(v)) \otimes \mathbb{Z}\left[\tfrac{1}{2}\right]$$

where $P(k(v))$, the scissors congruence group of the field $k(v)$, is a cyclic group of order $|k(v)|+1$. In particular, the group $H_3(\mathrm{SL}_2(F),\mathbb{Z})_0$ cannot even be finitely-generated. If S is sufficiently large, then it can be shown that the induced map $H_3(\mathrm{SL}_2(\mathcal{O}_S),\mathbb{Z}) \to {K_3}^{\mathrm{ind}}(F)$ is still surjective, and if we again denote the kernel by $H_3(\mathrm{SL}_2(\mathcal{O}_S),\mathbb{Z})_0$ the natural map

$$\pi_S : H_3(\mathrm{SL}_2(\mathcal{O}_S),\mathbb{Z}\left[\tfrac{1}{2}\right])_0 \to \oplus_{v\in S\setminus S_\infty} P(k(v)) \otimes \mathbb{Z}\left[\tfrac{1}{2}\right]$$

is surjective. In particular, the groups $H_3(\mathrm{SL}_2(\mathcal{O}_S),\mathbb{Z})$ must grow with S. It is a natural question whether the map π_S is an isomorphism for all sufficiently large S.

We note, by way of contrast, that the situation for SL_3 is quite different. It is known that $H_3(\mathrm{SL}_3(F),\mathbb{Z}) \cong {K_3}^{\mathrm{ind}}(F) = {K_3}^{\mathrm{ind}}(\mathcal{O}_S)$ for any S. Thus we would expect that $H_3(\mathrm{SL}_3(\mathcal{O}_S),\mathbb{Z}) \cong {K_3}^{\mathrm{ind}}(F)$ for any sufficiently large S.

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Harder's congruences and related computations

Jean Lannes

(joint work with Gaëtan Chenevier)

This talk is mainly based on Chapter X of the memoir "Formes automorphes et réseaux unimodulaires pairs" which Gaëtan Chenevier and I wrote recently (arXiv:1409.7616).

Let p be a fixed prime; one says that two even unimodular lattices L and L' in the Euclidean space $\mathbb{R}^n$ are p-neighbors if $L \cap L'$ is of index p in L and L'. Let X_n denote the set of isometry classes of even unimodular lattices in $\mathbb{R}^n$. It is well known that X_n is empty if n is not divisible by 8; X_8, X_{16} and X_{24} have been determined. The determination of X_{24} was done by Niemeier in 1968; X_{24} contains 24 elements, the most famous of which is the (class of the) Leech lattice.

Let us denote by $[-]$ the class of a lattice; in this context the Hecke operator T_p is the endomorphism of $\mathbb{Z}[\mathrm{X}_n]$ defined by the formula:

$$\mathrm{T}_p[L] := \sum_{L' \ p\text{-neighbor of } L \text{ in } \mathbb{R}\otimes L} [L'] \quad .$$

The coefficient of $[L']$ in $\mathrm{T}_p[L]$ is denoted by $\mathrm{N}_p([L],[L'])$.

From now on we take $n = 24$.

The determination of T_2 was done in 2001 by Nebe and Venkov who showed that T_2 is diagonalisable over $\mathbb{Q}$ with eigenspaces of dimension 1.

(1) The fact that T_p commutes with T_2 implies readily that the eigenvalues of T_p, say $\lambda_1(p), \lambda_1(2), \ldots, \lambda_{24}(p)$ satisfy many congruences.

(2) The determination of the $\lambda_i(p)$'s involves heavily the theory of automorphic forms: it is the deep core of our memoir. We have 24 explicit formulae

$$\lambda_i(p) \;=\; \mathrm{C}_{i,0}(p) + \sum_{r=1}^{10} \mathrm{C}_{i,r}(p)\,\theta_r(p)$$

such that

– $\mathrm{C}_{i,r}(X)$ is a polynomial in $\mathbb{Q}[X]$

– for $r = 1,2,3,4,5$, $\theta_r(p)$ is the p-th coefficient of the q-expansion of the normalized cusp form, for the group $\mathrm{SL}_2(\mathbb{Z})$, respectively of weight $12, 16, 18, 20, 22$ (in particular $\theta_1(p) = \tau(p)$)

– $\theta_6(p) = \tau(p^2)$

– for $r = 7,8,9,10$, $\theta_r(p)$ is given by the action of an Hecke operator $\mathrm{T}(p)$on some vector-valued Siegel cusp form, let us say f_r, for the group $\mathrm{Sp}_4(\mathbb{Z})$: $\mathrm{T}(p)f_r = \theta_r(p)\,f_r$.

(3) From (1) and (2) we obtain a proof of the congruence

$$\tau_{4,10}(p) \equiv \tau_{22}(p) + p^8 + p^{13} \mod 41$$

which was foreseen by Günter Harder. Above $\tau_{4,10}(p)$ denotes one of the "mysterious" $\theta_r(p)$'s, $r \geq 7$, and $\tau_{22}(p)$ is a more transparent notation for $\theta_5(p)$.

Let us give some details. First we observe that there exist two indices i and j such that one has the following equality:

$$\lambda_j(p) - \lambda_i(p) \;=\; (p+1)\,(\tau_{4,10}(p) - \tau_{22}(p) - p^8 - p^{13})$$

which using (3) gives the congruence

$$(p+1)\,(\theta_{10}(p) - \theta_5(p) - p^8 - p^{13}) \equiv 0 \mod 41 \quad .$$

Next we get rid of the factor $p+1$ by a delicate argument involving modular Galois representations.

Remark. Recently Thomas Megarban (a student of Chenevier) got around this argument by solving the p-neighbor problem for even lattices of dimension 23 and determinant 2. Indeed in this case there exist two eigenvalues $\lambda_i(p)$ and $\lambda_j(p)$ with

$$\lambda_j(p) - \lambda_i(p) = \theta_{10}(p) - \theta_5(p) - p^8 - p^{13} \quad .$$

As precedently he gets the congruence

$$\theta_{10}(p) - \theta_5(p) - p^8 - p^{13} \equiv 0 \mod 9840$$

which is the best possible because the value of the first member for $p = 2$ is -9840.

(4) In fact the $\theta_r(p)$'s, $r = 7, 8, 9, 10$, are difficult to compute...

So we changed strategy and deduced these $\theta_r(p)$'s from the computation of the $\mathrm{N}_p(L, \mathrm{Leech})$ for the four Niemeier lattices L with the highest Coxeter numbers. We found a method to determine these N_p's for $p \leq 113$; our methods fails for $p \geq 127$ essentially because the Ramanujan inequalities for the mysterious $\theta_r(p)$'s are no longer powerful enough.

Again let us give some details. Let L be a Niemeier lattice with roots and step$\,(L; p)$ the integer defined by

$$\mathrm{step}\,(L;p) \ := \ \frac{|\mathrm{W}(L)|}{\text{g.c.d.}\,(p-1, 24\,\mathrm{h}(L), |\mathrm{W}(L)|)} \qquad .$$

First we show (under a minor assumption) that $\mathrm{N}_p(L, \mathrm{Leech})$ is divisible by $\mathrm{step}(L; p)$; so we introduce the integer $\mathrm{n}_p(L)$ defined by $\mathrm{N}_p(L, \mathrm{Leech}) = \mathrm{n}_p(L)\,\mathrm{step}(L; p)$. Next, using Ramanujan inequalities for the mysterious $\theta_r(p)$'s we get an explicit bounding

$$\nu_p^{\inf}(L) \ \leq \ \mathrm{n}_p(L) \ \leq \ \nu_p^{\sup}(L)$$

which can be very efficient.

Example. We have $\nu_{47}^{\inf}(\mathrm{E}_{24}) \approx 0.99992$ and $\nu_{47}^{\sup}(\mathrm{E}_{24}) \approx 1.00006$; hence $\mathrm{n}_{47}(\mathrm{E}_{24}) = 1$ and $\mathrm{N}_{47}(\mathrm{E}_{24}, \mathrm{Leech}) = 1131456179644927440637132800000$.

Let L_1, L_2, L_3, L_4 be respectively the lattices D_{24}^+, $\mathrm{D}_{16}^+ \oplus \mathrm{E}_8$, $\mathrm{E}_8 \oplus \mathrm{E}_8 \oplus \mathrm{E}_8$, A_{24}^+. Using the above bounding (and some other minor tricks) we determine the 4-uple $(\mathrm{n}_p(L_i))_{1 \leq i \leq 4}$ for $p \leq 113$ and compute the mysterious $\theta_r(p)$'s by solving a linear system of rank 4.

Construction of discrete chamber transitive actions on affine buildings

Gabriele Nebe

(joint work with Markus Kirschmer)

Kantor, Liebler and Tits [1] classified discrete groups acting chamber transitively on the affine building of a simple adjoint algebraic group of relative rank ≥ 2 over a locally compact local field K. Such groups are very rare and hence this situation is an interesting phenomenom. One major disadvantage of the existing literature is that the proof is very sketchy, essentially the authors limit the possibilities that need to be checked to a finite number and leave the details to the reader. For the case $\mathrm{char}(K) = 0$ we give a number theoretic argument for this classification. The main work has been done in the habilitation thesis of Markus Kirschmer who classified one class genera of Hermitian lattices over number fields. Chambers in the building correspond to certain lattice chains; and the S-arithmetic group defined by this lattice chain acts chamber transitively on the building, if and only if this lattice chain represents a genus of lattice chains with class number 1.

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A new algorithm for computing Farrell–Tate and Bredon homology for arithmetic groups

Alexander D. Rahm

(joint work with Anh Tuan Bui and Matthias Wendt)

Understanding the structure of the cohomology of arithmetic groups is a very important problem with relations to number theory and various K-theoretic areas. Explicit cohomology computations usually proceed via the study of the actions of the arithmetic groups on their associated symmetric spaces, and recent years have seen several advances in algorithmic computation of equivariant cell structures for these actions. To approach computations of Farrell–Tate and Bredon (co)homology of arithmetic groups, one needs cell complexes having a *rigidity property*: cell stabilizers must fix their cells pointwise. The known algorithms (using Voronoi decompositions and such techniques, cf. e.g. [6], [5]) do not provide complexes with this rigidity property, and this leads to a significant bottleneck, both for the computation of Farrell–Tate cohomology (resp. the torsion at small prime numbers in group cohomology) of arithmetic groups as well as for the computation of Bredon homology.

In theory, it is always possible to obtain this rigidity property via the barycentric subdivision. However, the barycentric subdivision of an n-dimensional cell complex can multiply the number of cells by $(n+1)!$ and thus easily let the memory stack overflow. We provide an algorithm, called *rigid facets subdivision*, which subdivides cell complexes for arithmetic groups such that stabilisers fix their cells pointwise, but only leads to a controlled increase (in terms of sizes of stabilizer groups) in the number of cells, avoiding an explosion of the data volume. An implementation of the algorithm, cf. [2], shows that cases like $\mathrm{PSL}_4(\mathbb{Z})$ or $\mathrm{PGL}_3(\mathbb{Z}[i])$ can effectively be treated with it, using commonly available machine resources.

Computations of Farrell–Tate cohomology. Farrell–Tate cohomology is a modification of cohomology of arithmetic groups which is particularly suitable to investigate torsion related to finite subgroups (in particular, the torsion in cohomological degrees above the virtual cohomological dimension). While the known cell complexes for arithmetic groups can deal very well with the rational cohomology and torsion at primes which do not divide orders of finite subgroups, computations with these complexes run into serious trouble for small prime numbers because the differentials in the relevant spectral sequence are too complicated to evaluate. There is a suitable new technique called *torsion subcomplex reduction*, cf. [8], which produces significantly smaller cell complexes and therefore simplifies the equivariant spectral sequence calculations. To apply this simplification, however, one needs cell complexes with the abovementioned rigidity property. Using the rigid facets subdivision, applied to cell complexes for $\mathrm{PSL}_4(\mathbb{Z})$ and $\mathrm{PGL}_3(\mathbb{Z}[i])$, we have computed the Farrell–Tate cohomology of these groups, at the primes 3 and 5 for $\mathrm{PSL}_4(\mathbb{Z})$ and at the prime 3 for $\mathrm{PGL}_3(\mathbb{Z}[i])$. These results can be found in the full paper [3]. Since the computation proceeds through a complete description of the reduced torsion subcomplex, we can compute the torsion above the virtual cohomological dimension in all degrees.

In the cases which are effectively of rank one (5-torsion in $\mathrm{PSL}_4(\mathbb{Z})$, 3-torsion in PGL_3 over imaginary quadratic integers), we can check the results of the cohomology computation using torsion subcomplex reduction by comparing to a computation using Brown's formula. For this, we outline a generalization of a theorem of Reiner [11], giving a description of conjugacy classes of cyclic subgroups and the group structure of their normalizers. These results are proved in the full paper [3] and provide generalizations of the computations in [10].

Computations of Bredon homology. For any group G, Baum and Connes introduced a map from the equivariant K-homology of G to the K-theory of the reduced C^*-algebra of G, called the assembly map. For many classes of groups, it has been proven that the assembly map is an isomorphism; and the Baum–Connes conjecture claims that it is an isomorphism for all finitely presented groups G (counter-examples have been found only for stronger versions of the Baum–Connes conjecture). The assembly map is known to be injective for arithmetic groups. For an overview on the conjecture, see the monograph [7].

The geometric-topological side of Baum and Connes' assembly map, namely the equivariant K-homology, can be determined using an Atiyah–Hirzebruch spectral sequence with E_2-page given by the Bredon homology $\mathrm{H}_n^{\mathfrak{Fin}}(\underline{\mathrm{E}}G;\, R_{\mathbb{C}})$ of the classifying space $\underline{\mathrm{E}}G$ for proper actions with coefficients in the complex representation ring $R_{\mathbb{C}}$ and with respect to the system $\mathfrak{Fin}$ of finite subgroups of G. This Bredon homology can be computed explicitly, as described by Sanchez-Garcia ([12], [13]).

While for Coxeter groups with a small system of generators [13] and arithmetic groups of rank 2 [9], general formulae for the equivariant K-homology have been established, the only known higher-rank case to date is the example $\mathrm{SL}_3(\mathbb{Z})$ in [12]. Although there are by now considerably more arithmetic groups for which cell complexes have been worked out ([6], [4], [5]), no further computations of Bredon homology $\mathrm{H}_n^{\mathfrak{Fin}}(\underline{\mathrm{E}}G;\, R_{\mathbb{C}})$ have been done since 2008 because the relevant cell complexes fail to have the rigidity property required for Sanchez-Garcia's method. We discuss an explicit example in the full paper [3], demonstrating that the rigidity property is essential for the computation of Bredon homology and cannot be circumvented by a different method.

The application of the rigid facets subdivision to cell complexes for $\mathrm{PSL}_4(\mathbb{Z})$ and $\mathrm{PGL}_3(\mathbb{Z}[i])$ leads to the following computations. Applying rigid facets subdivision to the cell complex for $\underline{\mathrm{E}}\mathrm{PSL}_4(\mathbb{Z})$ from [4], we obtain

$$\mathrm{H}_n^{\mathfrak{Fin}}(\underline{\mathrm{E}}\mathrm{PSL}_4(\mathbb{Z});\, R_{\mathbb{C}}) \cong \begin{cases} 0, & n \geq 5, \\ \mathbb{Z}^{10}, & n = 4, \\ \mathbb{Z}, & n = 3, \\ 0, & n = 2, \\ \mathbb{Z}^4, & n = 1, \\ \mathbb{Z}^{25} \oplus \mathbb{Z}/2, & n = 0. \end{cases}$$

Applying rigid facets subdivision to the cell complex for $\underline{\mathrm{E}}\mathrm{GL}_3(\mathbb{Z}[i])$ from ([14], [1]), we obtain

$$\mathrm{H}_n^{\mathfrak{Fin}}(\underline{\mathrm{E}}\mathrm{GL}_3(\mathbb{Z}[i]);\, R_{\mathbb{C}}) \cong \begin{cases} 0, & n \geq 5, \\ \mathbb{Z}^{20}, & n = 4, \\ \mathbb{Z}^4 \oplus (\mathbb{Z}/8)^4 \oplus (\mathbb{Z}/3)^4, & n = 3, \\ \mathbb{Z}^{20}, & n = 2, \\ \mathbb{Z}^{36}, & n = 1, \\ (\mathbb{Z}^{36})^3 \oplus (\mathbb{Z}/4)^8, & n = 0. \end{cases}$$

The correctness of our results depends of course heavily on the cell complexes for $\underline{\mathrm{E}}G$ that we take as input for the rigid facets subdivision algorithm and the subsequent calculations. Therefore, for $\underline{\mathrm{E}}\mathrm{GL}_3(\mathbb{Z}[i])$, we have compared two independent implementations, [14] and [4].

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S-arithmetic groups in quantum computing

Sebastian Schönnenbeck
(joint work with Alex Bocharov and Vadym Kliuchnikov)

1. Preliminaries

While the state of a classical computer is usually thought of as an element of $\mathbb{F}_2^n$ (where n of the number of available bits), the possible states of a quantum computer are complex linear combinations of those of a classical computer. Thus we can model them as elements of $\left(\mathbb{C}^2\right)^{\otimes n} \cong \mathbb{C}^{2^n}$ (this is called an n-qubit state). In the classical situation a computation is then modeled by applying $n\times n$-matrices over $\mathbb{F}_2$. The model in the quantum situation is a little more restrictive and we think of a computation here as applying a unitary $2^n \times 2^n$-matrix.

The architecture of our quantum computer now provides us with a finite set $\mathcal{G}$ (the so-called gate set) of elementary operations we are allowed to use when programming the device. We will denote the group of unitary operators we can implement exactly on our device by $G = \langle \mathcal{G} \rangle$. Since G is countable it obviously never coincides with the full unitary group $\mathrm{U}_{2^n}(\mathbb{C})$. However, if G is dense in $\mathrm{U}_{2^n}(\mathbb{C})$ (with respect to the usual Euclidean metric) we can at least approximate any operator arbitrarily well in which case $\mathcal{G}$ is called a universal gate set.

2. Exact synthesis and the Clifford+T gate set

The problem of *exact synthesis* which we are primarily concerned with is now to express a given operator $g \in G$ as a word in $\mathcal{G}$, i.e. to find $g_1, ..., g_r \in \mathcal{G}$ such that $g = g_1 g_2 ... g_r$ (this is sometimes called a *constructive membership* problem). Moreover, each gate in $\mathcal{G}$ comes with an associated cost (usually coming from experimental results) and thus we would like to solve this problem under the additional constraint of finding such a word with low (or ideally minimal) cost.

The most commonly known gate set currently considered is the Clifford+T gate set: Let

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $\mathcal{P}_1 := \langle X, Y, Z \rangle$ is the Pauli group on one qubit and $\mathcal{P}_n := \mathcal{P}_1^{\otimes n}$ is the Pauli group on n qubits. The Clifford group $\mathcal{C}_n$ on n qubits is defined to be the normalizer of $\mathcal{P}_n$ in the unitary group of degree 2^n over the field $K := \mathbb{Q}(\zeta_8)$. The Clifford+T gate set now consists of $\mathcal{C}_n$ and the so-called T-gates given by $\exp(\frac{i\pi}{8}(P+I)) \in \mathrm{U}_{2^n}(K),\ P \in \mathcal{P}_n$. The usual cost function for this gate set is called the T-count and assigns cost 1 to all T-gates while Clifford gates are free.

The group G_n of exactly expressible unitaries for this gate set has a convenient description as an S-arithmetic subgroup.

Theorem 1 ([1]).

$$G_n = \mathrm{U}_{2^n}(\mathbb{Z}[\sqrt{2}^{-1}, i]), \tag{1}$$

up to a constraint on the determinant.

This means that G_n is the group of all unitary matrices over K whose entries are integral away from $\mathfrak{p}$, where $\mathfrak{p} = \langle 1+\zeta_8 \rangle$ is the unique prime ideal above 2.

3. The algorithm

Since we have seen that G_n is an S-arithmetic subgroup we can exploit the action of G_n on the affine Bruhat-Tits building of U_{2^n} at $\mathfrak{p}$ to perform exact synthesis. To that end first note that we can also describe $\mathcal{C}_n$ as the automorphism group of the generalized Barnes-Wall lattice BW, an even, unimodular Hermitian lattice in K^n (cf. [2]). Since any element $g \in G_n$ is integral away from $\mathfrak{p}$ we have

$$BW/(BW \cap gBW) \cong \mathcal{O}_K/\mathfrak{p}^{d_1} \oplus \ldots \oplus \mathcal{O}_K/\mathfrak{p}^{d_{2^n-1}}, d_1 \geq \ldots \geq d_{2^n-1} \geq 0, \tag{2}$$

and in this situation we set $ED(BW, gBW) := x_1^{d_1} \ldots x_{2^n-1}^{d_{2^n-1}}$ and endow the space of these monomials with the graded lexicographical order.

In this notation we propose the following algorithm to perform exact synthesis in any gate set consisting of $\mathcal{C}_n$ and additional elements $\mathcal{T}' \subset G_n$.

Algorithm 1 Exact synthesis in Clifford+$\mathcal{T}'$

function ExactSythesis
  Input: $g \in \mathrm{U}_{2^n}\left(\mathbb{Z}\left[\frac{1}{\sqrt{2}}, i\right]\right)$
  Output: A sequence $(g_1, \ldots, g_r) \in \mathcal{T}'^r$ such that $g^{-1} g_1 \cdot \ldots \cdot g_r \in \mathcal{C}_n$
  $M \leftarrow gBW$
  $w \leftarrow [\,]$
  while $M \neq BW$ **do**
    $h \leftarrow \operatorname{argmin}_{x \in \mathcal{T}'} ED(xBW, M)$
    $M \leftarrow h^{-1} M$
    Append h to w
  return w

If we use just the set of T-gates for the set $\mathcal{T}'$ in the above algorithm it will not necessarily terminate on any given input. However if we slightly enlarge the set of T-gate to include some closely related gates we appear to obtain a working algorithm.

Theorem 2. *For $n = 2$ and an appropriate set $\mathcal{T}'$ containing the set of T-gates the above algorithm always terminates and solves the exact synthesis problem.*

Moreover our experiments suggest that we obtain solutions with significantly lower T-count compared to other known algorithms.

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Computation of certain modular forms using Voronoi polytopes

Dan Yasaki

Let F be a number field with ring of integers $\mathcal{O}$. Borel conjectured, and Franke proved [2], that the complex cohomology of arithmetic subgroups of $\mathrm{GL}_n(\mathcal{O})$ can be computed in terms of certain automorphic forms. This allows for the possibility of computational investigations of Hecke eigensystems using topological techniques. A familiar incarnation of these ideas can be found in the study of classical holomorphic modular forms. The Eichler-Shimura isomorphism identifies the cohomology of congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ with holomorphic modular forms. There are many ways to compute these cohomology groups. One approach uses an explicit fundamental domain that can be computed using reduction theory of binary quadratic forms. The associated tiling of of the complex upper half-plane $\mathbb{H}$ is shown in Figure 1 on the left. Another approach uses the familiar trivalent tree shown in bold in Figure 1 on the right. Under the identification of points in $\mathbb{H}$ with binary quadratic forms, the tree can be viewed as the well-rounded forms. These are the forms whose minimal vectors span $\mathbb{R}^2$. This tree is dual to the tessellation of $\mathbb{H}$ given by $\mathrm{SL}_2(\mathbb{Z})$-translates of the ideal triangle with vertices $\{0, 1, \infty\}$. The cohomology can be computed using these explicit decompositions of $\mathbb{H}$, and modular symbols can be used to compute the Hecke eigensystems [18].

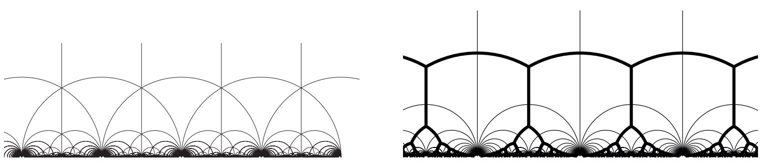

Figure 1. The figure on the left shows the usual fundamental domain in $\mathbb{H}$ for the action of $\mathrm{SL}_2(\mathbb{Z})$. On the right is the trivalent tree of well-rounded binary quadratic forms (shown in **bold**) with dual tessellation by ideal hyperbolic triangles.

Returning to the general situation where Γ is an arithmetic subgroup of $\mathrm{GL}_n(\mathcal{O})$, let X be the symmetric space associated to $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_n)(\mathbb{R})$, the real points of the Weil restriction. Our main tool is the polyhedral reduction theory for Γ developed by Ash [8, Ch. II] and Koecher [16], generalizing the work of Voronoi [19]. The reduction theory gives rise to a tessellation of X by ideal polytopes, analogous to the tessellation of $\mathbb{H}$ by triangles. Modular symbols are replaced by elements of the *sharbly complex*, a resolution of the Steinberg module [1] that can be used to compute the cohomology of Γ. In this setting, computing Hecke operators requires a technique for "reducing" sharblies [12]. This is a subtle issue that has been successfully addressed in practice for small values of n and small degree number fields. An incomplete list of references is given below.

For $F = \mathbb{Q}$, explicit computations for the cohomology as a Hecke module has been carried out for $n = 2, 3, 4$ [2–7] and without Hecke action for $n = 5, 6, 7$ [11]. For F an imaginary quadratic field and $n = 2$, analogous techniques were developed for the Euclidean cases in [9]. These techniques have been extended to several other imaginary quadratic fields, and partial results are available for $n = 3, 4$ [10, 17]. For higher degree fields, the computations are more arduous, and fewer examples have been computed. For $n = 2$ and F a cubic field of signature $(1, 1)$, the associated symmetric space is 6-dimensional. In [14], we develop 1-sharbly reduction for the complex cubic field of discriminant -23. For $n = 2$ and F a CM quartic field, The associated symmetric space is 7-dimensional. In [13], we develop 1-sharbly reduction techniques for the cyclotomic field $\mathbb{Q}(\zeta_5)$. This technique is modified in [15] for the cyclotomic field $\mathbb{Q}(\zeta_{12})$.

More extensive investigations are needed for more number fields of small degree. This will allow for further development of sharbly reduction techniques in order to compute the cohomology of these arithmetic groups as Hecke modules. An initial step in these investigations is to compute an explicit description of the cell structure of the tessellations of the corresponding symmetric space. We close with recent computations enumerating the number of equivalence classes of k cells for the first few CM quartic fields, ordered by discriminant. This table gives a rough idea of the growth in complexity one can expect going from the cyclotomic fields $\mathbb{Q}(\zeta_5)$ and $\mathbb{Q}(\zeta_{12})$ [13, 15] show in the first two rows of Table 1 to CM fields of larger discriminant.

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TABLE 1. Data for CM quartic fields F with real subfield K, including the absolute discriminant $|D_*|$, the class number h_F, the size of torsion unit group $\#\mu$, and the number of $\mathrm{GL}_2(\mathcal{O}_F)$-orbits of k-cells n_k.

$\|D_F\|$	$\|D_K\|$	h_F	$\#\mu$	n_1	n_2	n_3	n_4	n_5	n_6	n_7
125	5	1	10	1	4	9	11	10	5	1
144	12	1	12	1	4	13	17	14	6	2
225	5	1	6	1	5	11	14	10	6	2
256	8	1	8	2	8	24	37	34	14	3
400	5	1	4	2	9	23	28	19	8	2
441	21	1	6	2	8	25	35	23	10	3
576a	8	1	6	6	33	96	134	90	33	7
576b	24	1	6	3	19	61	99	72	25	5
784	28	1	4	10	62	196	288	199	57	8
1025	5	1	2	5	33	89	109	65	21	3
1088	8	1	2	7	45	134	180	112	33	4
1089	33	1	6	12	113	539	1116	1082	469	71
1225	5	1	2	3	28	85	125	85	32	5
1521	13	2	6	27	342	1726	3674	3677	1641	250
1525	5	1	2	9	81	253	352	234	74	10
1600	5	1	2	11	107	346	484	311	88	9
1936	44	1	4	7	89	391	719	603	230	33
2048	8	1	2	10	121	474	778	582	198	28
2197	13	1	2	22	238	970	1700	1371	481	49
2304a	12	2	2	13	166	669	1108	818	252	26
2304b	24	2	4	21	249	1008	1718	1334	433	40
2312	17	1	2	66	798	3341	5856	4696	1609	174
2601	17	1	6	53	754	3717	7865	8058	3833	673
2704	13	1	4	40	606	2890	5620	5271	2225	360
2725	5	1	2	23	278	1022	1621	1225	424	53
2873	13	1	2	27	335	1311	2197	1728	628	84

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Open problems in the cohomology of arithmetic groups

Problem session

Problem 1 (communicated by Günter Harder). Let $\Gamma := \mathrm{SL}_2(\mathbb{Z}[i])$, and let p be a prime number splitting as $p = y \cdot \overline{y}$ in $\mathbb{Z}[i]$. We impose $p \equiv 5 \mod 8$. Let the congruence subgroups $\Gamma_0(p)$, $\Gamma_0(y)$ of Γ act on hyperbolic 3-space $\mathcal{H}^3$. Then

$$\mathrm{H}^1({}_{\Gamma}\backslash\mathcal{H}^3; \mathrm{Ind}_{\Gamma_0(p)}^{\Gamma}\chi_4) \cong \mathrm{H}^1({}_{\Gamma_0(y)}\backslash\mathcal{H}^3; \mathbb{Z}),$$

for a non-trivial character χ_4. We compute the right hand side of this equation; Yasaki did find no cuspidal cohomology here for any $p < 16000$ — all the spaces were exhausted by Eisenstein cohomology. *Is there any prime number $p \equiv 5 \mod 8$ such that $H^1({}_{\Gamma_0(y)}\backslash\mathcal{H}^3; \mathbb{Z})$ contains cuspidal classes ? Or can it be proven that there is none ?*

Problem 2 (communicated by Mathieu Dutour Sikirić). Let h be an integer quadratic form of signature (p, q). *Find a generating set*

$$\Gamma := \{u \in \mathrm{GL}_n(\mathbb{Z}) \mid h(u(x)) = h(x)\}$$

and compute the cohomology as well as the Hecke operators.
Known:

- For $p = 0$, the group is finite.
- For $p = 1$, the group is hyperbolic and acts on hyperbolic space. Perfect form theory is available.
- For $(p, q) \in \{(2, 2), (2, 3), (3, 3)\}$, we can use the exceptional Lie isomorphisms.

Problem 3 (communicated by Günter Harder). *Can Hecke operators and their eigenvalues be computed for all arithmetic groups for which one can calculate the cohomology ?* A constructive answer to this question would be implementations of add-ons to current software for the computation of the cohomology of arithmetic groups, with which to compute Hecke operators.

Problem 4 (communicated by Philippe Elbaz-Vincent). Let Vor_Γ be a Voronoï type cell complex related to a modular group Γ. Let C_d be the set of representatives of d-cells of Vor_Γ (mod Γ). *Is there an algorithm for checking in subexponential runtime that the set C_d is complete ?*

Problem 5 (communicated by Paul Gunnells). Apply the "Discrete vector field" technique to cohomological computations with the Voronoï cell complex.

Problem 6 (communicated by Paul Gunnells). Consider Koecher reduction theory in the Voronoï setting for general number fields F. It is known that perfect forms do not come from F, but are to be defined over the Galois closure of F. *Can this help the computations with the Voronoï cell complex ?*

Problem 7 (communicated by Renaud Coulangeon). *Can you predict anything about the field of definition of a eutactic form ? Over $\mathbb{Q}$, or over any other number field ?*

Problem 8 (communicated by Paul Gunnells). Suppose we are given a form of SL_3 that has $\mathbb{Q}$-rank 1, in the setting of [1] (see reference below). Consider the field $L = \mathbb{Q}(\sqrt{d})$ for a square-free integer $d > 0$. Consider $J = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$ and the Galois conjugate of the transpose $g^* = (g^t)^\sigma$. Then we set $H(x, y) = x^* J y$,

$$G(\mathbb{Q}) = \{g \in \mathrm{SL}_3(L) \mid g^* J g = J\}.$$

Let Γ be a subgroup of $G(\mathbb{Q})$.

Can one carry out explicit reduction theory for such Γ ?

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