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Asymptotic Phenomena in Local Algebra and Singularity Theory

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Abstract. The goal of this workshop was to highlight, and further, the interactions between local algebra and singularity theory. The timing was serendipitous for both subjects have witnessed tremendous progress recently, much of which was reported at the workshop.

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Introduction by the Organisers

In the recent past there have been a number of spectacular developments in local algebra and singularity theory, the subject of this Oberwolfach workshop. Three outstanding conjectures have been settled (in the past six months) and there has been significant progress on two others; what is more, at Oberwolfach we got news that there has been a major breakthrough on another long-standing open problem. Fortunately, all the researchers responsible for these developments had been invited to the workshop, and most were able to participate and present their work. All this made for a lively and memorable gathering. There were more than fifty participants, from across the world; twenty three lectures, each an hour long, were scheduled. Here are some of the highlights.

Hochster's direct summand conjecture. The direct summand conjecture asserts that regular local rings are direct summands of their module-finite extensions. It is one in a network of homological conjectures in local algebra, formulated by Hochster, that have generated a tremendous amount of activity in the last 50 years. They had largely been resolved for commutative rings that contain a field or in low dimension. The direct summand conjecture (which has been known to be equivalent to various other homological conjectures, such as the canonical element conjecture and the improved new intersection theorem, among others) remained open in mixed characteristic until this past summer when André announced a proof of it based on Scholze's theory of perfectoid spaces. Bhatt delivered the opening lecture of the workshop, outlining a simplified proof of this conjecture, and an extension to the geometric setting; this again makes critical use of perfectoid spaces. By popular demand, Bhatt gave a second, more informal, lecture on Tuesday evening, giving an overview of perfectoid theory. It is clear that perfectoid theory is going to have an enormous impact on commutative algebra.

Eisenbud-Goto regularity conjecture. Over thirty years ago, Eisenbud and Goto conjectured a bound for the Castelnuovo-Mumford regularity of a prime ideal in terms of its multiplicity. This bound was proved by Gruson, Lazarsfeld, and Peskine for curves, by Pinkham and Lazarsfeld for smooth complex surfaces, and for some smooth 3-folds by Ran. Furthermore in dimensions three and four, Kwak proved regularity bounds that are only slightly worse than the one suggested by the conjecture. McCullough and Peeva (who were both at the workshop, as was Eisenbud) found a counterexample this summer, using an ingenious construction that yields, moreover, examples where the regularity is not bounded by any polynomial function of the multiplicity. McCullough gave a wonderful talk on these, and more recent, developments on this topic.

Resolution of singularities. In the early 1960s, Hironaka proved the existence of a resolution of singularities of a reduced algebraic scheme over a field of characteristic zero. This result has had spectacular application in many areas of mathematics. Because of the lack of hypersurfaces of maximal contact in positive characteristic, this problem has remained open over fields of positive characteristic, not to mention mixed characteristic. Abhyankar proved resolution of singularities for 3-folds of characteristic larger than five in the mid 1960s and Lipman proved resolution for reduced excellent surfaces in the mid 1970s. Recently, resolution for reduced excellent schemes of dimension three has been proven by Cossart and Piltant. This is the most general theorem which can be true in dimension three. Schober, who works with Cossart and Piltant, gave a talk in the workshop on this result, explaining the result and giving an outline of the proof.

Lech's multiplicity conjecture. Fifty years ago Lech conjectured that if $R \to S$ is a flat local ring extension, the multiplicities satisfy $e(R) \leq e(S)$; he proved it when $\dim R \leq 2$. Ma spoke about his recent work establishing an amazing inequality when R contains a field; this settles the case $\dim R = 3$. His proof uses Hilbert-Kunz multiplicities and local Chern characters to tackle rings of positive characteristic; he deduces the result for rings containing a field of characteristic zero using Artin approximation and reduction to positive characteristic.

Stillman's projective dimension conjecture. About ten years ago, Stillman conjectured that there is an upper bound on the projective dimension of an ideal in a polynomial ring, in terms of the number of generators of the ideal and their degrees; the remarkable point is that the bound is independent of the dimension of the ring. A few months ago Ananyan and Hochster posted a preprint on the arXiv that settles this conjecture. Regrettably, neither of them were present at this workshop. We were fortunate that Caviglia agreed to give an overview of the proof, which is a spectacular tour de force, involving a subtle study of subalgebras generated by regular sequences, and a complicated induction scheme.

Buchsbaum-Eisenbud-Horrocks conjecture on Betti numbers. This conjecture, from the 1970s, states that the ith Betti numbers of a module of finite length over a regular local ring of dimension d is at least $\binom{d}{i}$; in particular, the total Betti number is at least 2^d. It is related to the problem of finding minimal ranks of syzygies of vector bundles on the punctured spectrum of the ring. It can also be viewed as the local algebra analogue of Halperin's conjecture that if a real torus of dimension d acts almost freely on a finite CW complex X, then the total rank of the rational homology of torus is at least 2^d. At the workshop Walker announced that he had settled the conjecture concerning total Betti numbers just a few days ago! He gave a beautiful talk, with a complete proof of his result that covers also the case of modules of finite projective dimension over general local rings. The key new idea comes from K-theory, mainly the use of Adams operations on perfect complexes. It is clear that this opens up a whole new arsenal of techniques for use in local algebra, with a promise of further progress in this direction.

In addition to these topics, the talks in the workshop covered a range of topics of current interest in local algebra and singularity theory.

Presentations were given on recent progress towards resolution of singularities in positive characteristic and higher dimension. Teissier discussed his proof of local uniformisation along a maximal rank valuation in all dimensions and in positive characteristic by a close analysis of the associated graded ring of a local ring with respect to a valuation. Hauser gave an example showing that one of the main invariants used in resolution of singularities can have unexpected pathological behavior in higher dimension and positive characteristic. Villamayor discussed a systematic way to improve invariants of resolution under permissible blow ups in a ramified map to a nonsingular scheme of positive characteristic.

Singularities in positive characteristic were also the focus of the talks of Dao and Tucker. The latter discussed his recent result that the étale local fundamental group of an F-regular scheme (which can be viewed as the positive characteristic analog of a Kawamata log terminal singularity) at a geometric point is finite. Dao talked about the asymptotic behavior of the local cohomology modules of thickenings (that is to say, powers, or Frobenius powers) of an ideal in a local ring.

The structure of free resolutions of modules over local rings continues to be an active area of research, and many of the talks in the workshop were related to this topic. Şega's talk focused on the question of the rationality of Poincaré series of modules over compressed local rings. Berkesch explained some special features of homological algebra in multigraded settings. Avramov, Eisenbud, and Peeva spoke about (invariants of) minimal free resolutions of modules over complete intersection rings. On the more representation theoretic side of things, Buchweitz presented a complete description of the graded maximal Cohen-Macaulay modules over graded one dimensional Gorenstein rings.

The interaction between commutative algebra, algebraic geometry and combinatorics has a long tradition dating back, al least, to the 1970's with the pioneering work of Stanley and Hochster. Four talks at the meeting can be broadly considered as part of this area: Murai's talk on the properties of double h-polynomial of Buchsbaum complexes, Varbaro's presentation on a surprising connection between the Castelnuovo-Mumford regularity of monomial ideals and the virtual cohomological dimension of hyperbolic Coxeter groups, Huh's proof of the "top-heavy" conjecture of Dowling and Wilson for certain matroids employing geometric methods, and Srinivasan's analysis of homological properties of certain monomial curves.

Finally, two talks were devoted to introduce the audience to new research directions. In Erman's talk the problem of detecting Noether normalizations over finite fields was discussed and a solution based on a variant of Poonen's closed point sieve was presented. Römer presented results concerning Hilbert series of algebraic objects that have a finite description up to the action of an infinite group as part of the recent theory of "finite up-to symmetry" ideals which is connected with stable asymptotic behaviour of group representations.

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Workshop: Asymptotic Phenomena in Local Algebra and Singularity Theory

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Abstracts

The Direct Summand Conjecture and its Derived Variant

Bhargav Bhatt

The goal of this talk was to sketch a proof of Hochster's direct summand conjecture:

Theorem 1 (André). *Let R be a regular ring, and let $f : R \to S$ be a finite injective map. Then f admits an R-module splitting.*

Theorem 1 is straightforward in characteristic 0 or if $\dim(R) \leq 2$. The case of equicharacteristic $p > 0$ was settled in [Ho], while Heitmann settled dimension 3. The above result was established recently by Yves André in [An2] using [An1]. In this talk, we explained a proof of Theorem 1 avoiding [An1], following the method of [Bh2]. A similar method, when combined with a vanishing theorem due to Scholze, also leads to the following derived version, conjectured by de Jong:

Theorem 2. *Let R be a regular ring, and let $f : X \to \mathrm{Spec}(R)$ be a proper surjective map. Then $R \to R\Gamma(X, \mathcal{O}_X)$ splits in the derived category $D(R)$.*

The main new idea is systematically use Scholze's theory of perfectoid spaces, especially their analytic geometry. We first explain the characteristic p case in §1 using perfect rings. Replacing perfect rings with perfectoid rings, whose relevant properties are discussed in §2, leads to the general statement in §3.

1. Characteristic p

We prove Theorem 1 under the following assumptions: R has equicharacteristic p, and $f[\frac{1}{g}]$ is étale for some nonzero $g \in R$. Let $\mathrm{ob}_f \in \mathrm{Ext}^1_R(S/R, R)$ be the extension class defined by f. We must show $\mathrm{ob}_f = 0$.

The proof has two parts. First, one obtains an "almost splitting" for general reasons after passing to the perfection $R_{perf} = \varinjlim_{\mathrm{Frob}} R = R^{\frac{1}{p^\infty}}$. Using the regularity of R, we descend this to a genuine splitting over R.

Almost splitting after perfection. Let $f_{perf} : R_{perf} \to S_{perf}$ be the map on perfections induced by f. Our generic separability assumption that $g^k \in \mathrm{Tr}_{R_{perf}}(S_{perf}) \subset R_{perf}$ for some $k > 0$. As both R_{perf} and S_{perf} are perfect, it is immediate that the trace ideal $\mathrm{Tr}_{R_{perf}}(S_{perf}) \subset R_{perf}$ is closed under taking p-th roots. Thus, $g^{\frac{k}{p^n}} \in \mathrm{Tr}_{R_{perf}}(S_{perf})$ for all $n \geq 0$. On the other hand, it is also a standard fact that any element of $\mathrm{Tr}_{R_{perf}}(S_{perf})$ annihilates $\mathrm{ob}_{f_{perf}}$. It follows that all small powers of g annihilate $\mathrm{ob}_{f_{perf}}$. By a diagram case, the same holds true for $\mathrm{ob}_f \otimes_R R_{perf}$. In other words, the map $f \otimes_R R_{perf}$ is *almost split* with respect to the ideal $g^{\frac{1}{p^\infty}}$ in the sense of Faltings' almost mathematics.

Descent to the real world. As R is regular, the map $R \to R_{perf}$ is faithfully flat. Thus, the annihilator $I := \mathrm{Ann}_{R_{perf}}(\mathrm{ob}_f \otimes_R R_{perf}) \subset R_{perf}$ of $\mathrm{ob}_f \otimes_R R_{perf}$ comes from $\mathrm{Ann}_R(\mathrm{ob}_f) \subset R$, and is thus finitely generated. On the other hand, we showed above that $g^{\frac{k}{p^n}} \in I$ for all I. One may then readily show, using the faithful flatness of $R \to R_{perf}$ and Krull's intersection theorem, then I must be the unit ideal. Thus, $\mathrm{ob}_f \otimes_R R_{perf} = 0$, and thus $\mathrm{ob}_f = 0$ by faithful flatness.

2. Perfectoid spaces

Let $C = \widehat{\overline{\mathbf{Q}_p}}$ be a completed algebraic closure of $\mathbf{Q}_p$, and let $\mathcal{O}_C \subset C$ be the valuation ring. In the entire discussion below, this choice of C is not essential, and one may use other sufficiently ramified fields instead. The key objects are:

Definition 3 (Scholze)**.** *An $\mathcal{O}_C$-algebra R is* perfectoid *if it satisfies:*

(1) *R is p-adically complete and p-torsionfree.*
(2) *The Frobenius induces an isomorphism $R/p^{\frac{1}{p}} \to R/p$.*

A standard example is $R = \widehat{\mathcal{O}_C[x^{\frac{1}{p^\infty}}]}$, where the completion is p-adic. The most important tool in working with these algebras is the almost purity theorem:

Theorem 4 (Faltings, Scholze, Kedlaya-Liu)**.** *Let R be a perfectoid $\mathcal{O}_C$-algebra, and let $f : R \to S$ be a finite injective map with $f[\frac{1}{p}]$ étale. Then the p-adic completion of the integral closure of R in $S[\frac{1}{p}]$ is almost finite étale over R. In particular, the map $f : R \to S$ is almost split, i.e., the $p^{\frac{1}{p^n}} \cdot \mathrm{ob}_f = 0$ for all $n \geq 0$.*

For our purposes, this helps establish Theorem 1 in an important special case; this is a mild variant of the case handled in [Bh1]:

Corollary 5. *Theorem 1 holds true if $f[\frac{1}{p}]$ is étale.*

Proof. We may assume that R is local and complete. One may then construct a faithfully flat map $R \to R_\infty$ with R_∞ being perfectoid; this construction has been explored in recent work of Shimomoto. The map $f \otimes_R R_\infty$ is almost split by Theorem 4. The argument in §1 then gives $\mathrm{ob}_f = 0$. □

For the general case, we need two results. The first is:

Theorem 6 (André)**.** *Let R be a perfectoid $\mathcal{O}_C$-algebra. Fix some $g \in R$. Then there exists an almost faithfully flat (modulo p) map $\alpha : R \to R_\infty$ of perfectoid $\mathcal{O}_C$-algebras such that α is almost faithfully flat modulo p.*

The second result is a quantitative form of Scholze's Riemann extension theorem. For this, fix a perfectoid algebra R and a perfect element $g \in R$. Consider

$$R\langle \frac{p^n}{g} \rangle := \widehat{R[(\frac{p^n}{g})^{\frac{1}{p^\infty}}]},$$

where the completion is p-adic. These are the rings of bounded functions of the rational open subset $U_n := \{x \in X \mid |p^n| \leq |g(x)|\}$ of the perfectoid space X

attached to R. Note that $\cup_n U_n = \{x \in X \mid |g(x)| \neq 0\}$ is the Zariski open set $X[\frac{1}{g}]$ defined by g. In this setup, we have:

Theorem 7. *The natural map*

$$\{R\} \to \{R\langle \frac{p^n}{g} \rangle\}$$

of projective systems is an almost-pro-isomorphism modulo any nonzero power of p (where almost mathematics is measured with respect to $(pg)^{\frac{1}{p^\infty}}$). In particular, for any R-module M and $i \geq 0$, we have

$$\mathrm{Ext}^i_R(M, R) \overset{a}{\simeq} \lim \mathrm{Ext}^i_R(M, R\langle \frac{p^n}{g} \rangle).$$

The case $M = R$ and $i = 0$ is Scholze's Riemann extension theorem: it asserts that $R \overset{a}{\simeq} \lim R\langle \frac{p^n}{g} \rangle$, i.e., any bounded function on the Zariski open set $X[\frac{1}{g}] \subset X$ (almost) extends uniquely to X. For Theorem 1, the case $i = 1$ is crucial.

3. Mixed characteristic

We now sketch a proof of Theorem 1. For notational ease, we assume $R = \mathbf{Z}_p[\widehat{x_1, ..., x_d}]$ is the p-adic completion of a polynomial ring; Hochster had previously explained a reduction to mild variants of such rings. Fix a nonzero element $g \in R$ such that $f[\frac{1}{g}]$ is finite étale; such a g always exists as the generic characteristic of R is 0. We will show that $\mathrm{ob}_f = 0$ by mimicing the arguments in §1.

The replacement of R_{perf} arises by the following construction. Set $R_{\infty,0}$ to be the p-adic completion of $\mathcal{O}_C[x_1^{\frac{1}{p^\infty}}, ..., x_d^{\frac{1}{p^\infty}}]$, so $R_{\infty,0}$ is a perfectoid $\mathcal{O}_C$-algebra that is faithfully flat over R. Using Theorem 6, we can find an almost faithfully flat (modulo p) extension $R_{\infty,0} \to R_\infty$ such that the image of g in R_∞ is perfect.

Consider the base changes $f_n := f \otimes_R R_\infty\langle \frac{p^n}{g} \rangle$ and $f_\infty = f \otimes_R R_\infty$. By flatness consideration and a slight variant of the argument used in §1, it suffices to show that f_∞ is almost split with respect to $(pg)^{\frac{1}{p^\infty}}$. Now each f_n is almost split with respect to powers of p by almost purity, i.e., by Theorem 4: the map $f_n[\frac{1}{p}]$ is finite étale as $g \mid p^n$ in $R_\infty\langle \frac{p^n}{g} \rangle$. By two applications of Theorem 7, it follows that $f_\infty \overset{a}{\simeq} \lim f_n$ is almost split with respect to $(pg)^{\frac{1}{p^\infty}}$, as wanted.

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Rees-like Algebras and the Eisenbud-Goto Conjecture

Jason McCullough
(joint work with Irena Peeva)

Let $S = K[x_1, \dots, x_n]$ denote a polynomial ring over a field K, viewed as a standard graded ring, and let $I = (f_1, \dots, f_m)$ denote a homogeneous ideal of S. There are many invariants of I (or its associated projective variety) that one can define using a minimal graded free resolution or the graded Betti numbers. Of particular interest are the projective dimension and regularity as both are measures of the complexity of I. Let $\mathbf{F}_\bullet$ denote the minimal graded free resolution of S/I and write $F_i = \oplus_j S(-i)^{\beta_{ij}}$. Then we can define $\mathrm{pd}(S/I) = \max\{i \,|\, \beta_{ij} \neq 0 \text{ for some } j\}$ and $\mathrm{reg}(S/I) = \max\{j \,|\, \beta_{i,i+j} \neq 0 \text{ for some } i\}$. The projective dimension of an ideal is well controlled by the Hilbert Syzygy Theorem but the most general regularity bound on ideals is doubly exponential:

$$\mathrm{reg}(L) \leq (2\,\mathrm{maxdeg}(I))^{2^{n-2}}.$$

It is due to Bayer-Mumford [2], based on work by Giusti [6] and Galligo [5], if $\mathrm{char}(k) = 0$, and by Caviglia-Sbarra [3] in any characteristic. Here $\mathrm{maxdeg}(I)$ denotes the maximal degree of a minimal generator of I. This bound is nearly the best possible, due to examples based on the Mayr-Meyer construction [10]; for example, for each positive integer r there exists an ideal I_r in $22r - 1$ variables, generated by $22r - 3$ quadrics and one linear form, for which $\mathrm{maxdeg}(I_r) = 2$ and

$$\mathrm{reg}(I_r) \geq 2^{2^{r-1}}$$

by Koh [8]. However, better bounds were expected for geometrically nice ideals. In particular, Eisenbud and Goto made the following conjecture:

Conjecture 1. (Eisenbud-Goto [4], 1984) *Suppose that the field k is algebraically closed. If $P \subset (x_1, \dots, x_p)^2$ is a homogeneous prime ideal in S, then*

$$\mathrm{reg}(P) \;\leq\; \deg(S/P) - \mathrm{codim}(P) + 1,$$

where $\deg(S/P)$ *is the multiplicity of S/P (also called the degree of S/P) and* $\mathrm{codim}(L)$ *is the codimension (also called height) of L.*

The Regularity Conjecture holds if S/P is Cohen-Macaulay by [4]. It is proved for curves by Gruson-Lazarsfeld-Peskine [7], completing classical work of Castelnuovo. It also holds for smooth surfaces by Lazarsfeld [9] and Pinkham [11], and for most smooth 3-folds by Ran [12] along with many other special cases and related results.

We construct counterexamples to the Eisenbud-Goto conjecture as follows. First we define the *Rees-like algebra* of I to be $S[It, t^2] \subset S[t]$, where t is a new variable of degree 1. We introduce a new polynomial ring

$$T = S[y_1, \ldots, y_m, z]$$

graded by $\deg(z) = 2$ and $\deg(y_i) = \deg(f_i) + 1$ for every i. Now consider the graded homomorphism (of degree 0)

$$\begin{aligned} \varphi :\ T \ &\to S[It, t^2] \\ y_i &\longmapsto f_i t \\ z &\longmapsto t^2 . \end{aligned}$$

The homogeneous ideal $Q = \ker(\varphi)$ is prime. Unlike with the defining ideals of the Rees algebra $S[It]$ of I, the minimal generators of Q are easy to describe.

Proposition 2. *Let $F^r \xrightarrow{(c_{ij})} F^m \to I \to 0$ be a minimal presentation of I. Then Q is minimally generated by the following elements:*

$$\left\{ y_i y_j - z f_i f_j \,\middle|\, 1 \le i, j \le m \right\} \text{ and } \left\{ \sum_{i=1}^{m} c_{ij} y_i \,\middle|\, 1 \le j \le r \right\} .$$

We also construct the minimal free resolution of T/Q over T by showing that the minimal free resolution $\bar{T}/Q\bar{T}$ over $\bar{T} = T/(z)$ is the mapping cone of a map of complexes, the first being the complex obtained by truncating $\mathbf{F}_\bullet$ and tensoring by a koszul complex in $y_1, \ldots, y_m$, and the second being the resolution of $\bar{T}/(y_1, \ldots, y_m)^2\bar{T}$. Therefore we know the graded Betti numbers of T/Q, but Q is a prime ideal in a positively graded ring. To create a prime ideal P in a standard graded polynomial ring with the same graded Betti numbers, we show that a *step-by-step homogenization* technique preserves primeness and graded Betti numbers.

Finally, starting with Koh's ideal I_r, this process produces a prime ideal P_r in a standard graded ring R_r with $\deg(R_r/P_r) < 3^{50r}$ and $\operatorname{reg}(I_r) \ge 2^{2^{r-1}}$. In particular:

Theorem 3. *Over any field k (in particular, over $k = \mathbb{C}$), the regularity of non-degenerate homogeneous prime ideals is not bounded by any polynomial function of the multiplicity.*

However, one can ask whether there is *any* bound on the regularity of such non-degenerate homogeneous primes purely in terms of the multiplicity. We show that a positive answer to this question implies a positive answer to Stillman's Question; that is, can one bound the regularity (or projective dimension) of any ideal purely in terms of the degrees of the generators? The question was recently resolved in the affirmative by Ananyan-Hochster [1] but perhaps this approach will lead to smaller bounds.

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Ananyan-Hochster Proof of Stillman's Conjecture

Giulio Caviglia

Recently Tigran Ananyan and Melvin Hochster gave a proof of Stillman's conjecture [1]. This conjecture can be found in [6, 14] but has been informally circulating at least since 2000; it asserts that there exists an upper bound on the projective dimension of an ideal I of forms in a polynomial ring $K[X_1, \ldots, X_N]$ over a field depending only on the number and the degrees of the generators of I but not on the number N of variables. Until now the work on this problem could be broadly divided into two groups: establishing the existence of bounds for large classes of ideals, in general (most likely) very far away from being sharp, see [2] where the problem is solved for quadrics, and establishing the existence of bounds, sharp or close to be so, for ideals generated by a small number of quadrics and cubics see [3, 6, 7, 8, 9, 10, 12, 13]. The fact that the restriction on the degree of the forms is needed follows from work predating the conjecture [4, 5, 11].

Since both Ananyan and Hochster could not attend the workshop I was asked by organizers to present their proof. The notation and the content of my talk, including what follows, is taken from their paper [1].

Let n, d, η be positive integers. Ananyan and Hochster show that in a polynomial ring R in N variables over an algebraically closed field K of arbitrary characteristic, any K-subalgebra of R generated over K by at most n forms of degree at most d, is contained in a K-subalgebra of R generated by $B \leq {}^{\eta}\mathcal{B}(n,d)$ forms $G_1, \dots, G_B$ of degree $\leq d$, where ${}^{\eta}\mathcal{B}(n,d)$ does not depend on N or K, such that these forms are a regular sequence and such that for any ideal J generated by forms that are in the K-span of $G_1, \dots, G_B$, the ring R/J satisfies the Serre condition R_η. The proofs depend on giving a very special criterion for R/I, where I is generated by n forms of degree at most d, to satisfy R_η: there is a function ${}^{\eta}\mathcal{A}(n,d)$, independent of K and N, such that if no homogeneous generator of I is in an ideal generated by ${}^{\eta}\mathcal{A}(n,d)$ forms of strictly lower degree, then R/I satisfies R_η. These results imply Stillman's conjecture. They also show, and this is crucial for a certain inductive step of the proof, that there is a primary decomposition of the ideal such that all numerical invariants of the decomposition (e.g., the number of primary components and the degrees and numbers of generators of all of the prime and primary ideals occurring) are bounded independent of N.

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Resolution of Singularities for Arithmetic Threefolds

Bernd Schober

Resolution of singularities is an important method for studying the geometry of singular schemes. In his celebrated paper [12], Hironaka proves the existence of resolution of singularities over fields of characteristic zero. While the original proof is complicated and very technical, Hironaka's theorem is nowadays quite well understood (see, for example, the book by Cutkosky [9] or by Kollár [16]). In contrast to this, very little is known in positive and mixed characteristic. Recently, the following theorem has been established:

Theorem 1 (Cossart-Piltant, [6] Theorem 1.1)**.** *Let $\mathcal{X}$ be a reduced, separated, quasi-excellent, Noetherian scheme of dimension at most three. There exists a proper, birational morphism $\pi : \mathcal{X}' \to \mathcal{X}$ such that*

(i) *$\mathcal{X}'$ is everywhere regular,*
(ii) *π is an isomorphism outside of the singular locus, $\pi^{-1}(\mathrm{Reg}(\mathcal{X})) \cong \mathrm{Reg}(\mathcal{X})$,*
(iii) *$\pi^{-1}(\mathrm{Sing}(\mathcal{X}))$ is a simple normal crossing divisor on $\mathcal{X}'$.*

This extends previous work by the same authors [3] and [4], which deals with the situation over a field that is differentially finite over a perfect field.

If the dimension of $\mathcal{X}$ is four or larger, then the statement of the theorem is an important open problem. In [11], de Jong proves a weaker version using alterations which is valid over fields and in any dimension. Strengthened versions of the result are given by Gabber [14] and by Temkin [18].

Theorem 1 is only on the existence of π and its proof is neither functorial nor constructive. In particular, if $\mathcal{X}$ is embedded in some regular scheme $\mathcal{Z}$, $\mathcal{X} \subset \mathcal{Z}$, then π is not obtained as a sequence of blowing ups in regular centers and the embedding is not necessarily preserved. In fact, embedded resolution of singularities is only known up to dimension two (see [1], [2], [8], or [15]) and remains an important open problem in dimension three or larger.

Following *Zariski's program* Cossart and Piltant split the proof into two part: first, they prove *local uniformization*, a local variant of resolution of singularities, and then they show a *patching theorem* ([6] Proposition 4.4) in order to obtain a global morphism $\pi : \mathcal{X}' \to \mathcal{X}$ as desired.

Theorem 2 (Local Uniformization)**.** *Let (A, m, k) be quasi-excellent local domain of dimension three and $K := QF(A)$ its field of fractions. For every valuation v of K, with valuation ring $(\mathcal{O}_v, m_v, k_v)$ such that*

$$A \subset \mathcal{O}_v \subset K, \quad m_v \cap A = m, \quad \textit{and} \quad k_v \mid k \ \textit{algebraic},$$

there exists a finitely generated A-algebra T, $A \subseteq T \subseteq \mathcal{O}_v$, such that T_P is regular, where $P := m_v \cap T$.

The proof of this theorem can be reduced to the following situation (see [6] Theorem 1.4 and Propostion 4.8): Let (S, m_S, k) be an excellent regular local ring of dimension three with quotient field $K := QF(S)$ and residue characteristic $\mathrm{char}(k) = p > 0$. Let

$$h := X^p + f_1 X^{p-1} + \ldots + f_p \in S[X], \quad f_1, \ldots, f_p \in S,$$

be a reduced polynomial, $\mathcal{X} := \mathrm{Spec}(S[X]/(h))$, and $L := \mathrm{Tot}(S[X]/(h))$ be its total quotient ring. Assume one of the following conditions holds:

(i) $\mathrm{char}(K) = p$ and $f_1 = \ldots = f_{p-1} = 0$, or
(ii) $\mathcal{X}$ is G-invariant, where $G := \mathrm{Aut}_K(L) = \mathbb{Z}/p$.

Then Cossart and Piltant explicitly construct a sequence of blowing ups in regular centers solving the local uniformization problem in this particular case. Here, the centers are independent of the valuation and depend only on certain local invariants (the mulitplicity, τ, ω, κ).

As the case of multiplicity smaller than p is relatively easy to handle (see [5]), one may assume the multiplicity to be p. Applying embedded resolution in dimension two to the discriminant of h resp. the locus of multiplicity p points of $\mathcal{X}$, they deduce a certain condition (E) ([6] Corollary 4.13) stating that the latter are contained in the exceptional divisors and which is one of the inputs to reduce to the equi-characteristic situation.

By considering initial forms of h coming from Hironaka's characteristic polyhedron, Cossart and Piltant reduce to the case that S contains a field of positive characteristic ([6] Theorem 2.14). Hironaka's characteristic polyhedron is an important tool in the study of of singularities which is obtained by a particular projection of the Newton polyhedron of h, see [7], [13], [17]. It is then sufficient to investigate the behavior of the initial forms under blowing ups to construct a local uniformization. This is done in a very long and technical proof.

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Lech's Conjecture

Linquan Ma

Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d. The *Hilbert-Samuel multiplicity* of R is defined as:

$$e(R) = \lim_{t \to \infty} d! \cdot \frac{l(R/\mathfrak{m}^t)}{t^d}.$$

This is a classical invariant that measures the singularity of R. Morally speaking, the larger the multiplicity, the worse the singularity. In 1960, Lech made the following remarkable conjecture on Hilbert-Samuel multiplicities [6]:

Conjecture 1 (Lech's conjecture). *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local extension of local rings. Then $e(R) \leq e(S)$.*

It is very natural to expect that if $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a flat local extension, then R cannot have a worse singularity than S. Hence, Lech's conjecture seems quite natural and interesting. However, this conjecture has now stood for over fifty years and remains open in most cases! The best partial results are still those proved in Lech's original two papers [6] and [7]. There the conjecture was proved in the following cases:

(1) $\dim R \leq 2$;
(2) $S/\mathfrak{m}S$ is a complete intersection.

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In general, Lech proved that for $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ flat local extension with $d = \dim R$, we always have $e(R) \leq d! \cdot e(S)$ [6]. The conjecture has been of great interests to commutative algebraists, and throughout the years partial positive answers have been obtained. For example, it follows from results of [5] that Lech's conjecture holds when the base ring R is a strict complete intersection (i.e., $gr_{\mathfrak{m}}R$ is a complete intersection). The conjecture was also proved when R is a three-dimensional $\mathbb{N}$-graded k-algebra generated over k by one forms in characteristic $p > 0$ [2]. The central idea in the proof of these results is the construction of *Ulrich module*: a finitely generated maximal Cohen-Macaulay module M with multiplicity equal to its minimal number of generators. An observation of Hochster-Huneke shows that the existence of Ulrich module over $(R, \mathfrak{m})$ implies Lech's conjecture for any flat local extension $(R, \mathfrak{m}) \to (S, \mathfrak{n})$.

Other related results on Lech's conjecture can be found in [3], [4], [8]. However, the conjecture remains open as long as $\dim R \geq 3$. Our recent result settles the three-dimensional equal characteristic case, and provides substantial partial estimates in higher dimensions:

Theorem 2 ([9])**.** *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local extension between local rings of equal characteristic. If $\dim R = d$, then we have $e(R) \leq \max\{1, d!/2^d\} \cdot e(S)$. In particular, if $\dim R = 3$, then $e(R) \leq e(S)$.*

The strategy of the proof of Theorem 2 in [9] is to first tackle the case R has equal characteristic $p > 0$ using Frobenius methods, and then use a careful reduction to characteristic $p > 0$ procedure to get the characteristic 0 case.

Theorem 3 ([9])**.** *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local map between local rings of equal characteristic with $\dim R = d$. In order to prove Lech's conjecture $e(R) \leq e(S)$, or more generally, to prove $e(R) \leq C \cdot e(S)$ for certain constant C depending only on d, it suffices to prove the case when R has equal characteristic $p > 0$.*

The advantage of working in characteristic $p > 0$ is because we have a closely related invariant called the *Hilbert-Kunz multiplicity*, which is defined using the colength of Frobenius powers of ideals instead of ordinary powers:

$$e_{HK}(R) = \lim_{e \to \infty} \frac{l_R(R/\mathfrak{m}^{[p^e]})}{p^{ed}}.$$

It is a result of Monsky that this limit always exists [10]. In general, Hilbert-Kunz and Hilbert-Samuel multiplicities are related by the inequality:

$$\frac{e(R)}{d!} \leq e_{HK}(R) \leq e(R).$$

Because of this, Hilbert-Kunz theory allows us to prove estimates on Hilbert-Samuel multiplicities for flat local extensions. We should point out that, although Hilbert-Kunz multiplicity is in general hard to study, the analog of Lech's conjecture for Hilbert-Kunz multiplicity turns out to be true: if $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a flat local extension of local rings of characteristic $p > 0$, then $e_{HK}(R) \leq e_{HK}(S)$ [2].

Another important tool we used in the proof is the *Cohen-factorization* developed in [1]: for any local homomorphism of local rings $(R,\mathfrak{m}) \to (S,\mathfrak{n})$ with S complete, the map can be factored as:

$$(R,\mathfrak{m}) \to (T,\mathfrak{n}) \to (S,\mathfrak{n})$$

where $(R,\mathfrak{m}) \to (T,\mathfrak{n})$ is flat local with $T/\mathfrak{m}T$ regular and $S = T/J$. It is not hard to show that $e(R) = e(T)$, and when S is flat over R, J is a *perfect ideal* in T, i.e., the projective dimension of T/J is equal to the $depth_J T$ [1], [9]. Thus we arrive at the following natural conjecture:

Conjecture 4. *Let $(T,\mathfrak{n})$ be a local ring and J a perfect ideal of T. Then $e(T) \leq e(T/J)$.*

The above discussion shows Conjecture 4 implies Conjecture 1. We were not aware of any counter-example to Conjecture 4. It holds in many cases, for example when J is generated by a regular sequence or when T is standard graded over a field and J is homogeneous. On the other hand, we do not know a good general result, even when J is $\mathfrak{n}$-primary. In this case, the multiplicity $e(T/J)$ is simply the length $l(T/J)$, and we expect this case is more approachable.

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Approximating Rational Valuations by Abhyankar Valuations

Bernard Teissier

Let k be an algebraically closed field and let $X \subset \mathbf{A}^n(k)$ be an affine algebraic variety. According to the "viewpoint on resolution of singularities" of [2], one hopes to prove embedded resolution of singularities by proving the existence of re-embedding $X \subset \mathbf{A}^N(k)$ such that there exist coordinate systems on $\mathbf{A}^N(k)$ such that the intersection of X and the torus (complement of the coordinate hyperplanes) is dense in X and there exist proper birational toric maps $Z \to \mathbf{A}^N(k)$

of non singular toric varieties such that the strict transform of $X \subset \mathbf{A}^N(k)$ is non singular and transversal to the toric boundary of Z.
The same problem makes sense for projective varieties and Tevelev (see [5]) has proved that given an embedded resolution of an irreducible projective variety $X \subset \mathbf{P}^n(k)$, one can find re-embeddings $\mathbf{P}^n(k) \subset \mathbf{P}^N(k)$ (built from the given embedded resolution) and projective coordinates on $\mathbf{P}^N(k)$ such that the given embedded resolution is obtained by strict transforms from a proper birational toric map $Z \to \mathbf{P}^N(k)$ of non singular toric varieties. This means that toric embedded resolutions are in a sense "universal" among embedded resolutions.

In the absence of a given embedded resolution, how can one try to build suitable re-embeddings, say for $X \subset \mathbf{A}^n(k)$?

The idea is to first find "embedded local uniformizations" for valuations centered in X. This means to find re-embeddings $X \subset \mathbf{A}^N(k)$ and toric maps $Z \to \mathbf{A}^N(k)$ which will at least make the center of the valuation non singular on the strict transform $X' \subset Z$ of X, and X' transversal to the toric boundary of Z at that point. This strategy is suggested by the case of branches, corresponding to analytically irreducible one dimensional excellent local domains. In this case local embedded local uniformization of the unique valuation ν given by the normalization coincides with embedded resolution and the appropriate re-embedding is given by elements of the local ring R of the branch whose valuations (in the normalization) generate the semigroup of values $\Gamma = \nu(R \setminus \{0\}) \subset \mathbf{N}$.
Therefore we study valuations of the local ring R of X at a given closed singular point and we may assume that R is a domain. In [1, Proposition 3.20] it is shown that it suffices to uniformize *rational* valuations, which are those valuations centered in R for which the residual extension $R/m \subset R_\nu/m_\nu$ is trivial. These valuations correspond to rational points of the Zariski-Riemann manifold of the fraction field of R.

A valuation ν determines a filtration on each subring R' of R_ν by the ideals $\mathcal{P}_\phi(R') = \{x \in R'/\nu(x) \geq \phi\}$, and $\mathcal{P}^+_\phi(R') = \{x \in R'/\nu(x) > \phi\}$.

The graded ring $\mathrm{gr}_\nu R = \bigoplus_{\phi \in \Phi_{\geq 0}} \mathcal{P}_\phi(R)/\mathcal{P}^+_\phi(R)$ associated to the ν-filtration on R is the graded k-subalgebra of $\mathrm{gr}_\nu R_\nu$ whose homogeneous elements have degree in the semigroup $\Gamma = \nu(R \setminus \{0\})$. Since the valuation is rational, each homogeneous component of $\mathrm{gr}_\nu R$ is a one dimensional k-vector space and in fact $\mathrm{gr}_\nu R$ is isomorphic to the semigroup algebra $k[t^\Gamma]$. Since R is noetherian the semigroup Γ, which is not finitely generated in general, is well ordered and so has a minimal system of generators $\Gamma = \langle \gamma_1, \ldots, \gamma_i, \ldots \rangle$. We emphasize here that the γ_i are indexed by a countable ordinal $I \leq \omega^h$, where h is the rank (or height) of the valuation, which is less than its rational rank. In [1, Proposition 4.2], it is shown that the graded k-algebra $\mathrm{gr}_\nu R$ is then generated by homogeneous elements $(\overline{\xi}_i)_{i \in I}$ with $\deg \overline{\xi}_i = \gamma_i$ and we have a surjective map of graded k-algebras

$$k[(U_i)_{i \in I}] \longrightarrow \mathrm{gr}_\nu R, \quad U_i \mapsto \overline{\xi}_i,$$

where $k[(U_i)_{i\in I}]$ is graded by giving U_i the degree γ_i. Its kernel is generated by binomials $(U^{m^\ell} - \lambda_\ell U^{n^\ell})_{\ell\in L}$, $\lambda_\ell \in k^*$, where U^m represents a monomial in the U_i's. These binomials correspond to a generating system of relations between the generators γ_i of the semigroup.

By a result of Piltant (see [1, Proposition 3.1]), for rational valuations, the Krull dimension of the k-algebra $\mathrm{gr}_\nu R$ is the rational rank of the group Φ of the valuation ν, so that Abhyankar's inequality reduces to $\dim \mathrm{gr}_\nu R \le \dim R$. The valuations for which equality holds are called *Abhyankar valuations* and it was shown in [3] that embedded local uniformization holds for them.

The main purpose of the lecture was to explain how to approximate a rational valuation ν of rational rank r on a complete equicharacteristic noetherian local domain R of dimension d by Abhyankar semivaluations ν_B, that is, Abhyankar valuations ν_B on r-dimensional quotients R/K_B of R, indexed by certain finite subsets of the minimal set of generators of the semigroup Γ of ν on R, which are the generators of the value semigroups of the valuations ν_B and fill up the set of generators of Γ as B grows. The idea is that for large enough B an embedded local uniformization for ν_B will also uniformize ν. The reduction to the case of complete local domains is a separate issue which will not be discussed here.

Theorem 1. *Let R be a complete equicharacteristic noetherian local domain and let ν be a rational valuation centered in R, of rational rank $r = 1$. Let $\Gamma = \langle(\gamma_i)_{i\in L}\rangle$ be the minimal set of generators of Γ. There exist a collection $\mathcal{B}$ of finite subsets $B \subset I$ such that $I = \bigcup_{B\in\mathcal{B}} B$ and prime ideals K_B of R such that each quotient R/K_B is one dimensional and carries a rational Abhyankar valuation ν_B whose value semigroup is the semigroup $\langle(\gamma_i)_{i\in B}\rangle$, and for each $x \in R\setminus\{0\}$ there are $B \in \mathcal{B}$ such that $x \notin K_B$ and $\nu(x) = \nu_B(x \text{ mod}.K_B)$. One may choose the sets B to be nested, but there are no inclusions between the ideals K_B in general.*

Example 2. *If R is a power series ring in two variables over k, we recover the description of "infinitely singular" valuations as limits of "curve valuations", limits which is this case are understood in terms of infinite sequences of point blowing-ups. See* [1, Example 4.20].

We believe that the result is true for arbitrary rational rank.

The idea of the proof is to first present R as a quotient of a power series ring $S = k[[x_1,\dots,x_n]]$ by a prime ideal $P = (p_1,\dots,p_s)$ and apply the valuative Cohen theorem of [3, §4] to the valuation μ on S which is composed of the PS_P-adic valuation μ_1 on S and the valuation ν on S/P. The value group of μ is $\tilde{\Phi} = \mathbf{Z}\oplus\Phi$ with the *lex* order. One shows that one can choose a minimal system of generators $(p_a)_{a=1,\dots,s}$ of the ideal P such that their initial forms $\mathrm{in}_\mu p_a$ are part of a minimal system of generators of the graded k-algebra $\mathrm{gr}_\mu S$, as well as the $\overline{\xi}_i$ which generate the subalgebra $\mathrm{gr}_\nu R \subset \mathrm{gr}_\mu S$.

The valuative Cohen theorem gives us the existence of representatives in S

$$(\tilde{\xi}_i)_{i\in I}, p_1,\dots,p_s, (h_j)_{j\in J}$$

of the elements of a minimal system of generators of the graded k-algebra $\mathrm{gr}_\mu S$ such that there exists a surjective continuous map of k-algebras

$$\Pi\colon k[(u_i)_{i\in I}, \widehat{v_1,\ldots}, v_s, (z_j)_{j\in J}] \to S, \qquad u_i \mapsto \tilde{\xi}_i, v_a \mapsto p_a, z_j \mapsto h_j,$$

where the first algebra, whose definition is part of the theorem, is a generalized power series ring topologized by giving each variable the weight (e.g., $w(u_i) = \gamma_i$) of the corresponding element of the semigroup $\tilde{\Gamma}$ of μ and S is topologized by the filtration determined by μ.

The kernel of Π is generated up to closure by deformations of a set of binomials generating the kernel of the surjective map of graded k-algebras

$$k[(U_i)_{i\in I}, V_1, \ldots, V_s, (Z_j)_{j\in J}] \to \mathrm{gr}_\mu S$$

determined by the minimal set of generators of $\mathrm{gr}_\mu S$. The topological generators of the kernel of Π can be chosen in such a way that each involves only finitely many variables. For each v_a, among these generators must be series of the form

$$u^{m^\ell} - \lambda_\ell u^{n^\ell} + \sum_{w(p)>w(u^{m^\ell})} c_p u^p + \sum_q v^q = v_a.$$

In characteristic zero, the sum $\sum_q v^q$ does not appear. These equations and the variables they contain are used to define the ideals K_B.

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Resolution of Singularities in Positive Characteristic

Herwig Hauser

After reviewing the logical structure of Hironaka's proof of resolution in zero characteristic we explain how the invariant he uses for the induction can be defined in a characteristic free manner. The invariant is a string of positive integers (orders of ideals) associated to each point of the variety which, in zero characteristic, is upper semicontinuous and whose maximum decreases under blowup with center the stratum where the string attains lexicographically its maximal value. In positive characteristic, the invariant has a much more complicated behavior, is no longer upper semicontinuous and may increase under blowup.

We explain how this can be rearranged for surfaces so as to provide again a reliable resolution invariant. The idea is to consider local multiplicities not only at closed points but also along smooth curves. As for higher dimensions, we exhibit an example of Stefan Perlega from Vienna in five variables where the second component of the invariant, the so-called residual order, increases indefinitely. This is not yet a counter-example to resolution in positive characteristic. But it shows that the classical approaches need new ideas and methods so as to advance in this still open problem.

On the Multiplicity and Invariants of Singularities over Fields of Positive Characteristic

Orlando Villamayor U.

There are at least two ways to study the singularities of a variety X. One is by fixing locally at a point $x \in X$ an inclusion of X in a regular variety, say $X \subset W$, in which case the singularity at x is studied by looking at the equations in W defining X locally at such point. Another approach is to fix locally at a point $x \in X$ a dominant finite morphism $X \to W$; which enables us to consider X as finite (ramified) cover of the regular variety W.

The first approach was considered by Hironaka in his Theorem of Resolution of Singularities over fields of characteristic zero. There he considers the Hilbert Samuel function at a point $x \in X$ and uses a local immersion $X \subset W$ to describe in W the set of points of X with the same Hilbert-Samuel function (the Hilbert-Samuel stratum of X containing x).

If $Y \subset X (\subset W)$ is a regular subscheme in X, the blow-up of W at Y is a morphism

$$W \leftarrow W_1$$

and W_1 is also regular; Y also defines

$$X \leftarrow X_1$$

(the blow up of X at Y), together with a closed inmersion $X_1 \subset W_1$. Therefore X_1 is defined by equations in W_1.

Hironaka's strategy for resolution of singularities of schemes over fields of characteristic zero is to blow up at regular centers Y, included in X, chosen so that the Hilbert Samuel function at $x \in X$ is the same at each closed point $x \in Y$ (X is a normally flat along Y).

He shows that this Hilbert-Samuel function will ultimately improve by blowing up at suitable normally flat centers.

The second approach, that in which a variety is viewed as a ramified cover of a regular one, has been used historically in the formulation of the *multiplicity* at a point $x \in X$. In fact the notion of multiplicity of X at a point $x \in X$ was defined in terms of the finite morphism $X \to W$ in a neighborhood of the point.

One can read the multiplicity of X at a point x (at $x \in X$) from the Hilbert-Samuel function. But the multiplicity is a very elementary invariant which can be presented in many other ways, involving only basic concepts of commutative algebra as we indicate below.

There is an interesting similarity of the behavior of finite morphisms $X \to W$ with closed immersions $X \subset W$ when it comes to blow ups. To make this concept precise let us assume that the morphism is affine, so $X = Spec(B) \to W = Spec(S)$ where $S \subset B$ is a finite extension of a regular ring S. If the generic rank of B over S is n (i.e., if the dimension of $B \otimes_S K$ over K is n, where K is the quotient field of S), then the highest multiplicity at points of X is at most n, and the set of points of multiplicity n, say $F_n(X)$, is closed. Let us mention in passing that that the multiplicity at $x \in X$ is n if this is the smallest integer such that one can define locally a finite morphisms $X \to W$ of generic rank n.

If Y is a regular subscheme of $F_n(X)$, then Y defines two blow ups

$$W \leftarrow W_1 \quad X \leftarrow X_1,$$

together with finite morphisms $X \to W$ and $X_1 \to W_1$. Moreover, this morphisms produce a square and commutative diagram.

This says that $X_1 \to W_1$ is, in a natural way, the blow up of $X \to W$ at Y whenever we chose $Y \subset F_n$ regular. And this notion of blow ups of finite covers makes this second approach very useful in the study of the multiplicity; particularly to study its behavior of the multiplicity when blowing up X at regular equimultiple centers.

Over fields of characteristic zero one can also prove resolution of singularities by using the multiplicity as the main invariant (see [5]). This solves a question formulated by Hironaka in [3].

This latter approach has some advantages over that of Hironaka which uses the Hilbert-Samuel invariant, and the aim of this talk is to focus on one of them: We present invariants, related to the multiplicity, introduced when X is a variety over a perfect field of positive characteristic.

The open problem of resolution of singularities in positive characteristic would have a positive answer if one could prove reduction of the multiplicity by blowing up at regular equimultiple centers. The highest multiplicity of a variety X can be expressed, as indicated above, in terms of a finite morphism $X \to W$ over a regular variety W. In this talk we discuss about the open problem of reduction of the multiplicity of X for the case in which there is an affine morphism, say $X = Spec(B) \to W = Spec(S)$ where $S \subset B$ is a purely inseparable extensions of the regular ring S. We show that this opens the way to the new invariants, and we mention some tools and ideas developed in [1] and in [4].

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Layered Resolutions of Cohen-Macaulay Modules

Irena Peeva

(joint work with David Eisenbud)

Let S be a regular local ring and suppose that M is a finitely generated Cohen-Macaulay S-module of codimension c. Given a regular sequence $f_1, \dots, f_c$ in the annihilator of M we construct an S-free resolution

$$\mathbf{L}\!\uparrow^S(M, f_1, \dots, f_c),$$

and an $R := S/(f_1, \dots, f_c)$-free resolution

$$\mathbf{L}\!\downarrow_R(M, f_1, \dots, f_c)$$

of M. These resolutions are constructed through an induction on the codimension, and each of them comes with a natural filtration by subcomplexes; we call them *layered resolutions.*

The inductive construction of the resolutions follows a pattern often seen in results about complete intersections in singularity theory and algebraic geometry. It allows us to exploit the fact that we can choose the regular sequence to be in general position with respect to M. In this way we achieve minimality for high R-syzygies, and we give necessary and sufficient conditions for minimality in general.

We now explain the inductive constructions. We may harmlessly assume that M has no free summand as an R-module. For brevity, we will always abbreviate the phrase "maximal Cohen-Macaulay" to "MCM".

In the base case of the induction, $c = 0$, M is 0 and the layered resolutions are trivial. For the inductive step we think of R as a quotient, $R = R'/(f_c)$, where $R' = S/(f_1, \dots, f_{c-1})$ and consider the MCM approximation

$$\alpha : \; M' \oplus B_0 \twoheadrightarrow M$$

of M as an R'-module, in the sense of Auslander-Buchweitz: here B_0 is a free R'-module, M' is an MCM R'-module without free summand and the kernel B_1 of the surjection α has finite projective dimension. In our case B_1 is a free R'-module and we write $\mathbf{B^S}$ for the complex of free S-modules

$$\mathbf{B^S} : \ \mathbf{B_1^S} \to \mathbf{B_0^S}.$$

obtained by lifting the map $B_1 \xrightarrow{b} B_0$ back to S.

Layered resolution over S. For the layered resolution of M over S we let $\mathbf{K}$ be the Koszul complex resolving R' as an S-module and let $\mathbf{L}' = \mathbf{L}{\uparrow}^S(M', f_1, ..., f_{c-1})$, the layered resolution constructed earlier in the induction. There is an induced map $B_1^S \xrightarrow{\psi} L_0'$ which, in turn, induces a map of complexes $\mathbf{K} \otimes \mathbf{B^S} \to \mathbf{L}'$ whose mapping cone we define to be the layered S-free resolution of M with respect to $f_1, \dots, f_c$.

Layered resolution over R. For the layered resolution of M over R we let $\mathbf{T}' = \mathbf{L}{\downarrow}^{R'}(M', f_1, \dots, f_{c-1})$, the layered resolution constructed earlier in the induction. The layered R-free resolution of M with respect to $f_1, \dots, f_c$ is obtained from $\mathbf{T}'$ by the Shamash construction applied to the box complex

$$\begin{tikzcd} T' : & \cdots \to T_2' \arrow[r] & T_1' \arrow[rr] & & T_0' \\ & & \oplus & & \oplus \\ & & B_1 \arrow[rr, "b"'] \arrow[rru, "R' \otimes \psi"] & & B_0 , \end{tikzcd}$$

where b and ψ are the maps listed above.

Minimality. We give criteria for the minimality of the layered resolutions. They imply that, when the residue field of S is infinite, the layered resolutions can be taken to be minimal for any sufficiently high R-syzygy of a given R-module N. For such modules the layered resolutions coincide with the resolutions we have produced in terms of "higher matrix factorizations".

Tor as a Module over an Exterior Algebra

David Eisenbud

(joint work with Irena Peeva, Frank-Olaf Schreyer)

Write S for a regular local ring with maximal ideal $\mathfrak{m}$ and residue field k, and let $f_1, \dots, f_c \in S$ be a regular sequence. Set $I := (f_1, \dots, f_c) \subset S$ and consider the complete intersection $R := S/I$. Let M be a finitely generated S-module annihilated by I. We denote by E the exterior algebra

$$E := \wedge_k(I/\mathfrak{m}I) =: k\langle e_1, \dots e_c\rangle .$$

The finite-dimensional graded vector space $\mathrm{Tor}^S(M, k)$ has a natural E-module structure induced by the action of homotopies for the f_i on the minimal S-free resolution of M. When M is a high R-syzygy in the sense of [EP1] we show:

(i) The E-module $\operatorname{Tor}^S(M,k)$ is generated by $\operatorname{Tor}_0^S(M,k)$ and $\operatorname{Tor}_1^S(M,k)$, and its (Castelnuovo-Mumford) regularity is 1.

(ii) Let
$$T' := E \cdot \operatorname{Tor}_0^S(M,k) \subset \operatorname{Tor}^S(M,k)$$
and let
$$T'' := \operatorname{Tor}^S(M,k)/T'$$
be the quotient. Assuming that the field k is infinite and the generators of $(f_1,\dots,f_c)$ are chosen generally, we compute vector space bases of T' and T'', and show that, as E-modules, T' and T'' have Gröbner deformations to direct sums of copies of $E/(e_p,\dots,e_c)$ for $p=1,\dots,c$. It follows that, even when k is finite, T' and T'' have linear E-free resolutions, given explicitly in (iv) below.

(iii) We prove that the Betti numbers of the 0-linear strand of the minimal E-free graded resolution of $\operatorname{Tor}^S(M,k)$ are given by the even Betti numbers of M over R, and the Betti numbers of the 1-linear strand are given by the odd Betti numbers of M over R. That is:
$$\beta^E_{i,i}\Big(\operatorname{Tor}^S(M,k)\Big) = \beta^R_{2i}(M)$$
$$\beta^E_{i,i+1}\Big(\operatorname{Tor}^S(M,k)\Big) = \beta^R_{2i+1}(M)\,.$$

(iv) We show that the numerical statement in (iii) is a consequence of the structure of the E-free resolution of $\operatorname{Tor}^S(M,k)$ by proving that the minimal E-free resolution of $\operatorname{Tor}^S(M,k)$ is the mapping cone:

$$\begin{tikzcd}
\cdots \arrow[r, "t_2"] & \operatorname{Tor}_4^R(M,k)\otimes_R E \arrow[r, "t_2"] & \operatorname{Tor}_2^R(M,k)\otimes_R E \arrow[r, "t_2"] & \operatorname{Tor}_0^R(M,k)\otimes_R E \\
\cdots & \oplus & \oplus & \oplus \\
\cdots \arrow[r, "t_2"] & \operatorname{Tor}_5^R(M,k)\otimes_R E \arrow[r, "t_2"] \arrow[uur, "t_3"] & \operatorname{Tor}_3^R(M,k)\otimes_R E \arrow[r, "t_2"] \arrow[uur, "t_3"] & \operatorname{Tor}_1^R(M,k)\otimes_R E\,,
\end{tikzcd}$$

where the two rows are themselves (minimal) linear free resolutions of the E-submodule T' and the quotient T''. The maps labeled t_2 are the CI (=Complete Intersection) operators (also called Eisenbud operators), while the maps labeled t_3 between the two strands are higher CI-operators, introduced in [EPS].

Next we focus on $\operatorname{Ext}_R(M,k)$. The action of the CI operators makes the graded vector space $\operatorname{Ext}_R(M,k)$ into a finitely generated module over the ring
$$\mathcal{R} := \operatorname{Sym}_k\left((I/\mathfrak{m}I)^\vee\right) =: k[\chi_1,\dots,\chi_c]\,.$$
We prove that when M is a high R-syzygy, the minimal $\mathcal{R}$-free resolution of $\operatorname{Ext}_R^{\text{even}}(M,k)$ is obtained by the Bernstein-Gel'fand-Gel'fand (BGG) correspondence from the E-module structure of $T'^\vee$, and similarly for $\operatorname{Ext}_R^{\text{odd}}(M,k)$ and $T''^\vee$.

One Corollary doesn't even require the definition of T'. Write

$$\mu : E_1 \otimes_k \operatorname{Tor}_0^R(M,k) \to \operatorname{Tor}_1^R(M,k)$$

for the multiplication map and

$$\mu^\vee : \operatorname{Ext}_S^1(M,k) \to \operatorname{Ext}_S^1(M,k) \otimes \mathcal{R}_1$$

for its vector space dual. The $\mathcal{R}$-module $\operatorname{Ext}_R^{\mathrm{even}}(M,k)$ then has the (non-minimal) linear free presentation

$$\operatorname{Ext}_S^1(M,k) \otimes \mathcal{R}(-1) \xrightarrow{\tau} \hom(M,k) \otimes \mathcal{R} \to \operatorname{Ext}_R^{\mathrm{even}}(M,k) \to 0$$

where τ is the map of free modules whose linear part is $\mu^\vee$. This follows because $\mu^\vee$ is 0 on the submodule $T''^\vee$.

An essential ingredient in the proofs is a new theory of *higher CI operators*. Just as the Eisenbud-Shamash construction allows one to describe an R-free resolution of any R-module from the higher homotopies on an S-free resolution, one can describe an S-free resolution from the higher CI-operators on an R-free resolution. This construction was discovered independently by Jessie Burke [Bu]. The differentials in the E-free resolution of $\operatorname{Tor}_S(M,k)$ are related, as above, to the higher CI-operators.

We also use the "layered" structures of the minimal S-free and R-free resolutions of M [EP2], which come from the higher matrix factorizations of [EP1]

Related Work. Avramov and Buchweitz made use of the simple classification of modules over an exterior algebra on 2 generators to study free resolutions of modules over complete intersections of codimension 2 in [AB], and this study is carried further in [AZ]. For other points of view on the module structure of Tor see [Da, HW]. For further results on resolutions over exterior algebras, see for example [AI, Ei2, Fl].

Acknowledgements. Computations with Macaulay2 [M2] led us to guess the statements of our main theorems. Many of the constructions in this paper are coded in the packages BGG and CompleteIntersectionResolutions distributed with the Macaulay2 system. We want to express our gratitude to the authors Dan Grayson and Mike Stillman of Macaulay2 for their unfailing patience in answering our questions about the program.

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On Betti Numbers of Modules over Regular Rings

Mark E. Walker

Conjecture 1 (Weak Horrocks Conjecture)**.** *Assume $(R, \mathfrak{m}, k)$ is a local ring of dimension d. If M is a non-zero R-module of finite length and finite projective dimension, then*

$$\sum_i \beta_i(M) \geq 2^d,$$

where $\beta_i(M)$ is the i-th Betti number of M.

The stronger version of conjecture, known as "The Horrocks Conjecture" or sometimes as "The Buchsbaum-Eisenbud-Horrocks Conjecture" , asserts that

$$\beta_i(M) \geq \binom{d}{i}$$

holds for all i. The strong form first appears in print in the 1977 paper [3] by Buchsbaum and Eisenbud, and it also occurs (in the form of a question) in Hartshorne's 1979 problem list [6] where it is attributed to Horrocks. The first appearance in print of the weak form seems to be the 1985 book [4] of Evans and Griffith, where it is attributed to Avramov. It also appears in the 1993 paper [1] by Avramov and Buchweitz, in which they prove it holds for equi-characteristic rings of dimension at most 5. (They also prove a graded analogue of the conjecture in many cases.)

In this talk I will prove:

Theorem 2. *The Weak Horrocks Conjecture holds if $(R, \mathfrak{m}, k)$ is a regular local ring with* $\mathrm{char}(k) \neq 2$.

For any local ring $(R, \mathfrak{m}, k)$, let $\mathrm{Perf}^{\mathfrak{m}}(R)$ be the category of bounded complexes of finite rank free F-modules having finite length homology. For $F \in \mathrm{Perf}^{\mathfrak{m}}(R)$ define its Euler characteristic to be

$$\chi(F) := \sum_i (-1)^i \ell_R H_i(F)$$

where ℓ_R denotes the length of an R-module. The central technique in the proof is to exploit properties of a secondary invariant χ_2, which we now define.

For $F \in \mathrm{Perf}^{\mathfrak{m}}(R)$, we form the complex $F \otimes_R F$ and equip it with the action of the cyclic group C_2 or order two generated by τ by having τ act by

$$\tau \cdot (x \otimes y) = (-1)^{|x||y|} y \otimes x$$

where $| - |$ denote the degree of a homogeneous element in a complex. With this action $F \otimes_R F$ is a complex of $R[C_2]$-modules. If 2 is invertible in R then the elements $\frac{\tau+1}{2}$ and $\frac{\tau-1}{2}$ of $R[C_2]$ are mutually orthogonal idempotents that sum to one, and thus for any complex of $R[C_2]$-modules N, they induced a natural decomposition

$$N = N^+ \oplus N^-$$

of complexes of R-modules. In other words,

$$N^+ = \{m \in M \mid \tau \cdot m = m\} \text{ and } N^- = \{m \in M \mid \tau \cdot m = -m\}.$$

Given an object $F \in \mathrm{Perf}^{\mathfrak{m}}(R)$, we define $S^2(F) = (F \otimes_R F)^+$ and $\Lambda^2(F) = (F \otimes_R F)^-$, so that we have a natural decomposition

$$F \otimes_R F = S^2(F) \oplus \Lambda^2(F)$$

of complexes of R-modules. Since $F \in \mathrm{Perf}^{\mathfrak{m}}(R)$, the complex $F \otimes_R F$ also has finite length homology, and hence so do each of the summands $S^2(F)$ and $\Lambda^2(F)$.

Definition 3. *Assume $(R, \mathfrak{m}, k)$ is a local ring with* $\mathrm{char}(k) \neq 2$. *Given $F \in \mathrm{Perf}^{\mathfrak{m}}(R)$ define an integer*

$$\chi_2(F) := \chi(S^2(F)) - \chi(\Lambda^2(F)).$$

Remark 4. *The assignment $F \mapsto [S^2(F)] - [\Lambda^2(F)]$ induces an endomorphism on $K_0(\mathrm{Perf}^m(R))$, the Grothendieck group of* $\mathrm{Perf}^{\mathfrak{m}}(R)$*; see* [2]. *This endomorphism is the* second Adams operation, *written ψ^2.*

Proposition 5. *Assume $(R, \mathfrak{m}, k)$ is a local ring of dimension d with* $\mathrm{char}(k) \neq 2$.

(1) *If $F \xrightarrow{\sim} F'$ is a quasi-isomorphism of objects of* $\mathrm{Perf}^{\mathfrak{m}}(R)$*, then*

$$\chi_2(F) = \chi_2(F').$$

(2) *If $0 \to F' \to F \to F'' \to 0$ is a short exact sequence of objects of* $\mathrm{Perf}^{\mathfrak{m}}(R)$*, then*

$$\chi_2(F) = \chi_2(F') + \chi_2(F'').$$

(3) *If R is regular and K is the Koszul complex on a regular sequence of generators of the maximal ideal, then $\chi_2(K) = 2^d$.*

Versions of Proposition 5 have been known for some time. For example, using a different (but equivalent if $\mathrm{char}(k) \neq 2$) definition of the second Adams operation, this result may be found in [5]. The version stated here can be found in [2].

Corollary 6 (Corollary of Proposition 5). *Assume $(R, \mathfrak{m}, k)$ is a regular local ring of dimension d and M is R-module of finite length. If F is a free resolution of M, then*

$$\chi_2(F) = 2^d \cdot \ell_R(M).$$

Proof. We proceed by induction on $\ell_R(M)$. The case $\ell_R(M) = 1$ is given by parts (1) and (3) of the Proposition. If $\ell_R(M) > 1$, there is a short exact sequence $0 \to M' \to M \to M'' \to 0$ with $\ell_R(M'), \ell_R(M'') < \ell_R(M)$. The result follows by the Horseshoe Lemma and part (2) of the Proposition. $\square$

Proof of Theorem 2. Assume $(R, \mathfrak{m}, k)$ is a regular local ring, $\operatorname{char}(k) \neq 2$ and M is a non-zero R-module of finite length. Let F be the minimal free resolution of M. We have

$$\begin{aligned}
\chi_2(F) &= \chi(S^2(F)) - \chi(\Lambda^2(F)) \\
&= \sum_i (-1)^i \ell_R H_i(S^2(F)) - \sum_j (-1)^j \ell_R H_j(\Lambda^2(F)) \\
&\leq \sum_{i \text{ even}} \ell_R H_i(S^2(F)) + \sum_{j \text{ odd}} \ell_R H_j(S^2(F)) \\
&\leq \sum_{i \text{ even}} \ell_R H_i(F \otimes_R F) + \sum_{j \text{ odd}} \ell_R H_j(F \otimes_R F) \\
&= \sum_i \ell_R H_i(F \otimes_R F).
\end{aligned}$$

The second inequality holds since each of $S^2(F)$ and $\Lambda^2(F)$ is a summand of $F \otimes_R F$. We next use that

$$\ell_R H_i(F \otimes_R F) \leq \beta_j(M) \cdot \ell_R(M)$$

for each i. This follows from the fact that $H_i(F \otimes_R F) \cong H_i(F \otimes_R M)$ is a subquotient of $F_i \otimes_R M$ for each i, and the latter has length equal to $\operatorname{rank}(F_i) \cdot \ell_R(M) = \beta_i(M) \cdot \ell_R(M)$.

Combining these inequalities and the Corollary gives

$$\ell_R(M) \cdot 2^d \leq \ell_R(M) \cdot \sum_j \beta_j(M).$$

Since $\ell_R(M) > 0$ we obtain $\sum_j \beta_j(M) \geq 2^d$. $\square$

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Gorenstein Monomial Curves

Hema Srinivasan
(joint work with Philippe Gimenez)

1. Introduction and Notations

Let $\mathbf{a} = (a_0, \ldots a_n)$ be a sequence of positive integers and let k be an arbitrary field. If $\phi : k[x_0, \ldots, x_n] \to k[t]$ is the ring homomorphism defined by $\phi(x_i) = t^{a_i}$, then $I(\mathbf{a}) := \ker \phi$ is a prime ideal of height n in $R := k[x_0, \ldots, x_n]$. $I(\mathbf{a})$ is a weighted homogeneous binomial ideal with the weighting $\deg x_i := a_i$. It is the defining ideal of the affine monomial curve $\mathcal{C}(\mathbf{a}) \subset \mathbb{A}^n_k$ parametrically defined by $\mathbf{a}$ whose coordinate ring is $S(\mathbf{a}) := \mathrm{Im}\phi = k[t^{a_0}, \ldots, t^{a_n}] \simeq R/I(\mathbf{a})$. As $S(\mathbf{a})$ is isomorphic to $S(d\mathbf{a})$ for all integer $d \geq 1$, we will assume without loss of generality that $a_0, \ldots, a_n$ are relatively prime. Observe that $S(\mathbf{a})$ is also the semigroup ring of the numerical semigroup $\langle a_0, \ldots a_n \rangle \subset \mathbb{N}$ generated by $a_0, \ldots, a_n$.

In a series of papers beginning with [1] with Gimenez and Sengupta, we consider the resolutions of monomial curves. We explicitly construct the graded resolution for these curves when $\mathbf{a}$ is in arithmetic sequence and show that the entire minimal resolution is determined by the Cohen-Macaulay type and it is determined by a_0 modulo n. One of the many consequences of this is that the graded Betti numbers of arithmetic sequences with the same common difference are periodic in a_0 with period with period $a_n - a_0$. Thanh Vu generalized this to prove that graded Betti numbers of all monomial curves are eventually periodic in a_0 with period $a_n - a_0$, as conjectured by Herzog and Srinivasan.

In particular, we prove in [1] when $a_0, a_1, \ldots, a_n$ are in arithmetic sequence, with $a_i = a_0 + id$ $(a_0, d) = 1$ and $a_o = t$ modulo n, then the Cohen-Macaualy type of $S(\mathbf{a}) = t - 1$. In this case, the ideal $I(\mathbf{a}) = I_2(A) + I_2(B)$, the sum of two determinantal ideals, where $A = \begin{bmatrix} x_0 & x_1 & \ldots & x_{n-1} \\ x_1 & x_2 & \ldots & x_n \end{bmatrix}$. When $a_0 = 2$ modulo n, then we get a Gorenstein Monomial Curve. We also show that the resolution of $S(\mathbf{a})$ in this case, can also be obtained as a mapping cone of the Eagon Northcott Complex associated to the matrix A and its dual.

1.1. **Principal matrix.** Now consider the general $\mathbf{a} = \{a_0, a_1, \ldots, a_n\}$. Let us assume that $a_0, \ldots a_n$ are relatively prime and minimally generate the semigroup $\langle \mathbf{a} \rangle$. Then for each $i, 0 \leq i \leq n$, there exists a unique smallest positive integer r_i such that $r_i a_i = \sum_{j \neq i} r_{ij} a_j$ or equivalently $f_i = x_i^{r_i} - \prod_{j \neq i} x_j^{r_{ij}} \in I(\mathbf{a})$. These f_i are called principal relations and are part of a minimal generating set of $I(\mathbf{a})$.

Definition: The $n+1 \times n+1$ matrix $D(\mathbf{a}) := (\mathbf{r_{ij}})$ where $r_{ii} := -r_i$ is called a *principal matrix* associated to $\mathbf{a}$.

Observe that $D(\mathbf{a})$ is not unique. Although the diagonal entries $-r_i$ are uniquely determined, there is not a unique choice for r_{ij}. Since $D(\mathbf{a})[a_0, \dots, a_n]^T = 0$, it follows that the rank of $D(\mathbf{a})$ is $\leq n$.

We have the "map" $D : \mathbb{N}^{[n+1]} \to T_n$ from the set $\mathbb{N}^{[n+1]}$ of sequences of $n+1$ relatively prime positive integers to the subset T_{n+1} of $n+1 \times n+1$ matrices of rank n with negative integers on the diagonal and non negative integers outside the diagonal. One can recover $\mathbf{a}$ from $D(\mathbf{a})$ by factoring out the greatest common divisor of the $n+1$ maximal minors of the $n \times n$ submatrix of $D(\mathbf{a})$ obtained by removing the first row. By an abuse of notation, let $D^{-1} : T_{n+1} \to N^{[n+1]}$ the operation that, for $M \in T_{n+1}$, takes the first column of $adj(M)$ and then factors out the g.c.d. to get an element in $N^{[n+1]}$. However, not all matrices of this form, are principal matrices. For instance, $M = \begin{bmatrix} -4 & 0 & 1 & 1 \\ 1 & -5 & 4 & 0 \\ 0 & 4 & -5 & 1 \\ 3 & 1 & 0 & -2 \end{bmatrix}$ is not principal, for $D^{-1}(M) = (7, 11, 12, 16)$ and it is easy to check that $3(11) = 3(7) + 12$, so that $r_2 = 3 < 5$.

2. Gorenstein, non complete intersections

Now let $\mathbf{a}$ define a Gorenstein monomial curve. The numerical semigroup $S = \langle \mathbf{a} \rangle$ is called Symmetric if there is a positive integer s such that for all integers x either $x \in S$ or $s - x \in S$. It is a theorem of Kunz that $\langle \mathbf{a} \rangle$ is symmetric if and only if the semigroup ring $S(\mathbf{a})$ or the monomial curve $\mathcal{C}(\mathbf{a})$ is Gorenstein.

In [2], we strengthen a criterion of Brezinsky to prove the following:

Theorem 1. *Let A be a 4×4 matrix of the form*

$$A = \begin{bmatrix} -c_1 & 0 & d_{13} & d_{14} \\ d_{21} & -c_2 & 0 & d_{24} \\ d_{31} & d_{32} & -c_3 & 0 \\ 0 & d_{42} & d_{43} & -c_4 \end{bmatrix}$$

with $c_i \geq 2$ and $d_{ij} > 0$ for all $1 \leq i, j \leq 4$, and all the columns summing to zero. Then the first column of the adjoint of A (after removing the signs) defines a Gorenstein, non complete intersection, monomial curve if and only if these entries are relatively prime.

Brezinsky [3] proved that if $\mathbf{a}$ is symmetric and not Gorenstein, then its principal matrix has this form. We prove the converse, that is, any matrix of this form is a principal matrix of a monomial curve (hence a Gorenstein one) provided the entries of the first column of the adjoint are relatively prime.

Further, we prove that any matrix of this form defines a Gorenstein, non complete intersection ideal of height three which is not prime and it will be prime and the principal matrix of a Gorenstein monomial curve if and only if the entries of first (and hence every) column of the adjoint are relatively prime.

As a consequence, we also show that we can generate two families of Gorenstein non complete intersection curves in A^4 from a Gorenstein monomial curve by translation.

Theorem 2. *Given any Gorenstein non complete intersection monomial curve $\mathcal{C}(\mathbf{a})$ in $\mathbb{A}^4_k$, there exist two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{N}^4$ such that for all $t \geq 0$, $\mathcal{C}(\mathbf{a}+t\mathbf{u})$ and $\mathcal{C}(\mathbf{a}+t\mathbf{v})$ are also Gorenstein non complete intersection monomial curves whenever the entries of the corresponding sequence ($\mathbf{a}+t\mathbf{u}$ for the first family, $\mathbf{a}+t\mathbf{v}$ for the second) are relatively prime.*

We also write down precisely the resolution and the maps in the free resolution in terms of the principal matrix $D(\mathbf{a})$ for these Gorenstein ideals.

2.1. **Decomposable Gorenstein Sequences of length 5.** Let a_1, a_2, a_3, a_4, a_5 be a sequence of relatively prime positive integers which define a Gorenstein and non complete intersection monomial curve in A^5.

Question: Can one give a criterion for Gorenstein monomial curves defined by a_1, a_2, a_3, a_4, a_5 in terms of its principal matrix?

By a criterion of Delorme for Gorenstein monomial curves, if $a_1, a_2, a_3, a_4.a_5$ form a Gorenstein monomial curve which is not a complete intersection, then there are precisely two possibilities: Either no four of them have a common factor or after reordering there exists two relatively prime integers r and s, such that $\{a_1, a_2, a_3, a_4, a_5\} = \{rb_1, rb_2, rb_3, rb_4, s\}$ where b_1, b_2, b_3, b_4 are relatively prime and define a Gorenstein non complete intersection and s is in the semigroup $\langle b_1, b_2, b_3, b_4\rangle$. In the later case, we say the Goresntein sequence is decomposable.

Theorem 3. *Suppose $\mathbf{a}$ is a sequence of length 5 which is a Gorenstein non complete intersection that is decomposable. Then there is a principal matrix for $\mathbf{a}$ of the form*

$$\begin{bmatrix} A & 0 \\ B & s \end{bmatrix}$$

where A is a 4×4 pseudo Gorenstein matrix whose 3×3 minors of the any three rows are relatively prime. Conversely, if we have any such matrix, then we have a Gorenstein ideal I of height four which is the monomial prime defined by the 4×4 minors of the last four rows. Furthere, this ideal has a resolution

$$0 \to R \to R^6 \to R^{10} \to R^6 \to R \to R/I$$

Thus, one has that any Gorenstein, non complete intersection monomial curve in A^4 is defined by 5 equations and if it is decomposable one in A^5 it is given by 6 equations.

It is an open problem to determine if the number of generators for the ideals of an affine Gorenstein monomial curves in A^n are bounded.

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Points, Lines, Planes, etc.

June Huh
(joint work with Botong Wang)

I will give a precise statement of a conjecture proposed during the talk, referring [Oxl11] for undefined terms. Let M be a rank r simple matroid on E, $\mathcal{L}$ be the set of all flats of M, and $\mathcal{L}^p$ be the set of rank p flats of M.

I will define a graded analogue of the Möbius algebra for $\mathcal{L}$. Introduce symbols y_F, one for each flat F of M, and construct vector spaces

$$B^p(\mathrm{M}) = \bigoplus_{F\in\mathcal{L}^p} \mathbb{Q}\, y_F, \quad B^*(\mathrm{M}) = \bigoplus_{F\in\mathcal{L}} \mathbb{Q}\, y_F.$$

Equip $B^*(\mathrm{M})$ with the structure of a commutative graded algebra over $\mathbb{Q}$ by setting

$$y_{F_1}y_{F_2} = \begin{cases} y_{F_1\vee F_2} & \text{if } \mathrm{rank}(F_1)+\mathrm{rank}(F_2)=\mathrm{rank}(F_1\vee F_2),\\ 0 & \text{if } \mathrm{rank}(F_1)+\mathrm{rank}(F_2)>\mathrm{rank}(F_1\vee F_2).\end{cases}$$

Maeno and Numata introduced this algebra in a slightly different form in [MN12], who used it to show that modular geometric lattices have the Sperner property. Unlike its ungraded counterpart, which is isomorphic to the product of $\mathbb{Q}$'s as a $\mathbb{Q}$-algebra [Sol67], the graded Möbius algebra $B^*(\mathrm{M})$ has a nontrivial algebra structure. Define

$$L = \sum_{i\in E} y_i.$$

Conjecture 1. *For nonnegative integer p less than $\frac{r}{2}$, the multiplication map*

$$B^p(\mathrm{M}) \longrightarrow B^{r-p}(\mathrm{M}), \qquad \xi \longmapsto L^{r-2p}\,\xi$$

is injective.

Conjecture 1 for M implies the "top-heavy" conjecture of Dowling and Wilson for M [DW74, DW75]. When M is realizable over some field, Conjecture 1 can be deduced from the decomposition theorem package for ℓ-adic intersection complexes [HW].

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A Probabilistic Approach to Noether Normalization

Daniel Erman

(joint work with David J. Bruce)

Given a projective variety $X \subseteq \mathbb{P}^r$ over an infinite field, any generic collection of k polynomials of degree d will be a (partial) system of parameters, in the sense that the vanishing locus will have codimension k on X. We compute the corresponding probabilities over finite fields, relating this to the numerics of subvarieties in X. Throughout $\mathbb{F}_q$ will denote a finite field with q elements.

Theorem A. Let $X \subseteq \mathbb{P}^r_{\mathbb{F}_q}$ be an n-dimensional closed subscheme. Then the asymptotic probability that random polynomials $(f_0, \ldots, f_k)$ of degree d are parameters on X is

$$\lim_{d\to\infty} \operatorname{Prob}\begin{pmatrix} (f_0,\ldots,f_k) \text{ of degree } d \\ \text{are parameters on } X \end{pmatrix} = \begin{cases} 1 & \text{if } k < n \\ \zeta_X(n+1)^{-1} & \text{if } k = n \end{cases}$$

where $\zeta_X(s)$ is the arithmetic zeta function of X.

The maximal case $k = n$ is due to Bucur and Kedlaya [1, Theorem 1.2], and is proven via an application of Poonen's closed point sieve. For submaximal cases where $k < n$, we adapt Poonen's argument by sieving over closed subvarieties of dimension $n - k$.

We then apply these probabilistic results to provide an effective bound for Noether normalizations over a finite field, and to give a new proof of the existence of uniform Noether normalizations for projective families over the integers.

Corollary 1. Let $\mathbb{F}_q$ be a finite field and let $X \subseteq \mathbb{P}^r_{\mathbb{F}_q}$ where $\dim X = n$. If $\max\{d, q\} \geq \widehat{\deg}(X)$ and

$$d > \log_q \widehat{\deg}(X) + \log_q n + n \log_q d$$

then there exist $f_0, \ldots, f_n$ of degree d^n inducing a finite morphism $\pi : X \to \mathbb{P}^n_{\mathbb{F}_q}$.

Corollary 2. Let $X \subseteq \mathbb{P}^r_{\mathbb{Z}}$ be a closed subscheme. If each fiber of X over $\mathbb{Z}$ has dimension n, then for some d, there exist polynomials $f_0, f_1, \dots, f_n \in \mathbb{Z}[x_0, x_1, \dots, x_r]$ of degree d inducing a finite morphism $\pi : X \to \mathbb{P}^n_{\mathbb{Z}}$.

Remarks:

(1) Corollary 1 appears to be the first effective such bound in the literature.
(2) Corollary 2 is a special case of a recent result of Chinburg-Moret-Bailly-Pappas-Taylor [2, Theorem 1.2] and of Gabber-Liu-Lorenzini [3, Theorem 8.1].
(3) Corollary 2 holds with $\mathbb{Z}$ replaced by $\mathbb{F}_q[t]$ but it can fail if we replace $\mathbb{Z}$ by $\mathbb{Q}[t]$ or $\mathbb{Z}[t]$. See [2, 3].

Previous analogues include: [4] and [5] which prove something similar in the case where X is a normal projective curve over $\mathbb{Z}$ or more general Dedekind domains; [7], which shows that Noether normalizations of semigroup rings always exist over $\mathbb{Z}$; and [6, Theorem 14.4], which implies that given a family over any base, one can find a Noether normalization over an open subset of the base. A different analogue is Poonen's Bertini theorem over $\mathbb{Z}$ [8, Theorem 5.1], which is based on similar techniques, but is contingent on the abc conjecture.

Example 1. On $\mathbb{P}^1_{\mathbb{Z}}$ the forms $f_0 = ax^2 + bxy + cy^2$ and $f_1 = dx^2 + exy + fy^2$ will determine a finite map $\pi : \mathbb{P}^1_{\mathbb{Z}} \to \mathbb{P}^1_{\mathbb{Z}}$ if and only if a the determinant

$$\det \begin{pmatrix} a & b & c & 0 \\ 0 & a & b & c \\ d & e & f & 0 \\ 0 & d & e & f \end{pmatrix} = \pm 1.$$

The above determinant is the resultant of these two forms, and if they are divisible by some prime p, then the map $(x, y) \mapsto (f_0(x, y), f_1(x, y))$ will have a base point over $\mathbb{F}_p$.

Example 2. Let

$$X = [1 : 4] \cup [3 : 5] \cup [4 : 5] = V((4x - y)(5x - 3y)(5x - 4y)) \subseteq \mathbb{P}^1_{\mathbb{Z}}.$$

The fibers are 0-dimensional, so a finite map $X \to \mathbb{P}^0_{\mathbb{Z}}$ will be determined by a single polynomial f_0 that restrict to a unit on all of the points simultaneously. No linear form will work. In fact, there exists an $f_0(x, y)$ restricting to unit on X if and only if $\deg f_0$ is divisible by 60.

The proof of our main result is based on a computation of the probability that randomly chosen elements of degree d form a (partial) system of parameters, over a finite field. A key new idea in this computation is an adaptation Poonen's closed point sieve [8], instead sieving over higher dimensional varieties; this computes the desired probability via a zeta function type enumeration of subvarieties of a specified dimension and degree. In each fiber over $\mathbb{Z}$, the error in the sieve is bounded using geometric results about the locus of partial systems of parameters, while the global error bound over $\mathbb{Z}$ relies on a uniform convergence over $\mathbb{Z}$ obtained via uniform bounds on Hilbert functions.

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Free Complexes for Smooth Toric Varieties

Christine Berkesch Zamaere
(joint work with Daniel Erman, Gregory G. Smith)

The standard graded polynomial ring is the Cox ring for projective space. Minimal free resolutions over this ring enjoy a number of useful properties; for instance, they are

(i) acyclic complexes,
(ii) unique up to isomorphism,
(iii) short (in the sense of Hilbert's Syzygy Theorem),
(iv) measure vanishing (via Castelnuovo–Mumford regularity), and
(v) reflect geometry in a number of other ways, as well.

The coincidence of these properties is due in large part to the fact that in this setting, the maximal homogeneous ideal and irrelevant ideal for projective space coincide.

Fix a smooth projective toric variety X with Cox ring S and irrelevant ideal B. The local version of Hilbert's Syzygy Theorem implies that any coherent sheaf on X admits a locally free resolution of length at most $\dim X$. In contrast, the global version of Hilbert's Syzygy Theorem implies that every B-saturated S-module M has a minimal free resolution of length at most $\dim S - 1$. When $X = \mathbb{P}^m$, the minimal free resolution is optimal in both the geometric and algebraic settings. However, when the rank r of the Picard group of X is greater than 1, minimal free resolutions are longer, and typically much longer, than their geometric counterparts.

Definition 1. *A free complex* $F := [F_0 \leftarrow F_1 \leftarrow \cdots]$ *of* $\mathbb{Z}^r$*-graded* S*-modules is called a* free Cox complex *for a* $\mathbb{Z}^r$*-graded* S*-module* M *when the corresponding complex of vector bundles* $\widetilde{F}$ *is a resolution of the sheaf* $\widetilde{M}$*.*

For smooth toric varieties besides projective space, allowing irrelevant homology can yield shorter and simpler complexes. Our broad goal is to demonstrate that the right analogues over smooth toric varieties of these homological theorems use free Cox complexes rather than minimal free resolutions. In particular, while free Cox complexes of a $\mathbb{Z}^r$-graded S-module will typically not be (i) acyclic or (ii) unique up to isomorphism, they are the right replacement for minimal free resolutions to recover properties (iii), (iv), and (v) above.

We focus on the product of projective spaces $\mathbb{P}^{\mathbf{b}} := \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_r}$ with dimension vector $\mathbf{n} := (n_1, n_2, \ldots, n_r) \in \mathbf{N}^r$. Let $S := \Bbbk[x_{i,j} \mid 1 \leq i \leq s, 0 \leq j \leq n_i]$ be the Cox ring of $\mathbb{P}^{\mathbf{n}}$, and let $B := \bigcap_{i=1}^r \langle x_{i,0}, x_{i,1}, \ldots, x_{i,n_i} \rangle$ be its irrelevant ideal. We identify $\mathbb{Z}^r = \operatorname{Pic} \mathbb{P}^{\mathbf{n}}$ and partially order the elements using the componentwise order. If $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_r$ is the standard basis of $\mathbb{Z}^r$, then the polynomial ring S has the $\mathbb{Z}^r$-grading induced by $\deg(x_{i,j}) := \mathbf{e}_i$. We consider only finitely-generated $\mathbb{Z}^r$-graded S-modules.

Example 2. *Let C be a genus 4 hyperelliptic curve. This can be embedded as a curve of bi-degree $(2,8)$ in $\mathbb{P}^1 \times \mathbb{P}^2$, and we let $I \subseteq S$ be the B-saturated ideal defining C. A computation in Macaulay2 [2] yields such a curve, where the minimal free resolution of S/I is*

$$S^1 \leftarrow \begin{matrix} S(-3,-1)^1 \\ \oplus \\ S(-2,-2)^1 \\ \oplus \\ S(-2,-3)^2 \\ \oplus \\ S(-1,-5)^3 \\ \oplus \\ S(0,-8)^1 \end{matrix} \leftarrow \begin{matrix} S(-3,-3)^3 \\ \oplus \\ S(-2,-5)^6 \\ \oplus \\ S(-1,-7)^1 \\ \oplus \\ S(-1,-8)^2 \end{matrix} \leftarrow \begin{matrix} S(-3,-5)^3 \\ \oplus \\ S(-2,-7)^2 \\ \oplus \\ S(-2,-8)^1 \end{matrix} \leftarrow S(-3,-7)^1 \leftarrow 0\,.$$

The following free Cox complex of S/I has a much simpler form.

$$S^1 \xleftarrow{\psi} \begin{matrix} S(-3,-1)^1 \\ \oplus \\ S(-2,-2)^1 \\ \oplus \\ S(-2,-3)^2 \end{matrix} \xleftarrow{\varphi} S(-3,-3)^3 \leftarrow 0\,. \tag{1}$$

If $J \subseteq S$ is the image of ψ, then $J = I \cap Q$ for some ideal Q whose radical contains B. In fact, here S/J is Cohen–Macaulay, and thus J is the ideal of maximal minors of, for instance, the 4×3 matrix

$$\varphi = \begin{bmatrix} x_3^2 & x_4^2 & -x_2^2 \\ -x_1x_2 - x_1x_3 & 0 & x_0x_4 \\ x_0 & -x_1 & 0 \\ 0 & x_0 & x_1 \end{bmatrix}.$$

We now address property (iii) above, showing the existence of free Cox complexes of length at most $|\mathbf{n}| = \sum_{i=1}^{r} n_i = \dim \mathbb{P}^{\mathbf{n}}$.

Theorem 3. *Every B-saturated S-module M has a free Cox complex of length at most $|\boldsymbol{n}| = \dim \mathbb{P}^{\boldsymbol{n}}$.*

To prove this result, we adapt Beilinson's resolution of the diagonal $\mathbb{P}^{\mathbf{n}} \hookrightarrow \mathbb{P}^{\mathbf{n}} \times \mathbb{P}^{\mathbf{n}}$ to construct free Cox complexes that, while short, involve a large number of free summands that are generated in relatively high degrees. Our next two results produce free Cox complexes that not only avoid this problem but also address property (iv) by linking certain free Cox complexes to multigraded regularity, which is defined in terms of the vanishing of sheaf cohomology [1]. Let $\Delta_i \subseteq \mathbb{Z}^r$ denote the set of degrees of the generators of the i-th step of the minimal free resolution of the irrelevant ideal B.

Theorem 4. *Let M be a B-saturated S-module. The regularity of M contains $\boldsymbol{d} \in \mathbb{Z}^r$ if and only if $M(\boldsymbol{d})$ has a free Cox complex $F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_{|\boldsymbol{n}|} \leftarrow 0$, such that, for all $0 \leq i \leq |\boldsymbol{n}|$, the degree of each generator of F_i belongs to $\Delta_i + \mathbb{N}^r$.*

Our next result provides an analogue of the existence of linear resolutions on projective space, as well as a characterization of regularity via free Cox complexes.

Theorem 5. *Let M be a B-saturated S-module that is $\boldsymbol{d}$-regular. If G is the free subcomplex of a minimal free resolution of M consisting of summands generated in degree at most $\boldsymbol{d} + \boldsymbol{n}$, then G is a free Cox complex for M.*

The complex G, which we call the *free Cox complex of* $(M, \boldsymbol{d})$, can be very small in comparison to the minimal free resolution. In addition, Theorem 5 highlights the non-uniqueness of free Cox complexes. An S-module M will typically have several incomparable minimal elements in the regularity of M, each of which will yield an incomparable free Cox complex. We conclude with a result that extracts geometric information from certain free Cox complexes, as per property (v).

Theorem 6. *Let $Y \subseteq \mathbb{P}^{\boldsymbol{n}}$ and let I be the B-saturated ideal defining Y. Assume that I is defined in degrees $\boldsymbol{d}_1, \boldsymbol{d}_2, \ldots, \boldsymbol{d}_s$ and that the natural map $(S/I)_{\boldsymbol{d}_i} \to H^0(Y, \mathcal{O}_Y(\boldsymbol{d}_i))$ is an isomorphism for $i = 1, 2, \ldots, s$. If any of the following hold*

(1) $\operatorname{codim} Y = 2$ *and there is $\boldsymbol{d} \in \operatorname{reg}(S/I)$ such that the free Cox complex of $(S/I, \boldsymbol{d})$ has length* 2*, or*
(2) $\operatorname{codim} Y = 3$, $\min\{n_i\} \geq 2$*, and there is $\boldsymbol{d} \in \operatorname{reg}(S/I)$ such that the free Cox complex of $(S/I, \boldsymbol{d})$ is a self-dual complex (up to twist) of length* 3*, or*
(3) *in any codimension, there is $\boldsymbol{d} \in \operatorname{reg}(S/I)$ such that free Cox complex of $(S/I, \boldsymbol{d})$ is a Koszul complex of length* $\operatorname{codim} Y$*,*

then the embedded deformations of Y in $\mathbb{P}^{\boldsymbol{n}}$ are unobstructed and the component of the Hilbert scheme of $\mathbb{P}^{\boldsymbol{n}}$ containing the point Y is unirational.

Example 7. *Let C be the curve in Example 2 with defining ideal I. Since the complex* (1) *is the free Cox complex of $\big(S/I, (2,1)\big)$, Theorem 6 implies that the* *embedded deformations of the curve C are unobstructed. Moreover, the corresponding component of the Hilbert scheme of $\mathbb{P}^1 \times \mathbb{P}^2$ can be given an explicit unirational parametrization by simply varying the entries in the 4×3 matrix φ.*

Also among our results along the lines of property (v), every punctual scheme on projective space $\mathbb{P}^m$ is arithmetically Cohen–Macaulay, which means that the minimal free resolutions of their structure sheaves have projective dimension m. While this fails for points on products of projective spaces, we show that if we work with free Cox complexes instead of the minimal free resolution, there is an analogue of the arithmetic Cohen–Macaulay property for any punctual scheme. We then use this to show that there is a free Cox complex analogue of the Hilbert–Burch Theorem for S/I when X is a smooth toric surface and I is the B-saturated ideal of n points in general position.

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Fundamental Groups of *F*-regular Schemes

Kevin Tucker

(joint work with Bhargav Bhatt, Jose Carvajal-Rojas, Patrick Graf, and Karl Schwede)

In this talk, I would like to highlight the following restrictions on the étale fundamental groups of strongly F-regular schemes.

Theorem 1. [CST, main theorem] *Suppose X is a strongly F-regular variety over a perfect field of characteristic $p > 0$ with dimension at least two. Then for all $x \in X$, $\pi_1^{\text{ét}}(\operatorname{Spec}^\circ \mathcal{O}_{X,x}^{\text{sh}})$ is finite.*

Theorem 2. [BCGST, main theorem] *Suppose X is a strongly F-regular variety over a perfect field of characteristic $p > 0$.*

(1) *If $X \leftarrow X_1 \leftarrow X_2 \leftarrow \cdots$ is a sequence of finite Galois quasi-étale morphisms (i.e. étale in codimension one) of normal varieties, then all but finitely many of the morphisms in the sequence are étale.*
(2) *There exists a normal variety Y and a finite Galois quasi-étale morphism $Y \to X$ such that $\pi_1^{\text{ét}}(X_{\text{reg}}) = \pi_1^{\text{ét}}(Y_{\text{reg}}) = \pi_1^{\text{ét}}(Y)$.*

The first result is a local restriction, and is used in the proof of the the global statements of the second result.

One should view both of these results as positive characteristic versions of previously shown statements for complex algebraic varieties. Recall that, under the well-studied dictionary between singularities in characteristic zero and positive characteristic, one may characterize Kawamata log terminal (klt) singularities as those which are strongly F-regular after reduction to characteristic $p \gg 1$. The main result of [Xu] is but Theorem 1 for complex algebraic varieties, after replacing strongly F-regular with klt; similarly, the main result of [GKP] is essentially Theorem 2 with the same modification.

The proof of Theorem 1 follows from new transformation rules for the F-signature under finite local extensions. Recall that the F-signature of a finite local domain $(R, \mathfrak{m}, k)$ of characteristic $p > 0$ is the limit

$$s(R) = \lim_{\substack{e \to \infty \\ q = p^e}} \frac{\operatorname{frk}_R(R^{1/q})}{\operatorname{rk}_R(R^{1/q})}$$

where $\operatorname{rk}_R(M)$ denotes the torsion-free rank of an R-module M, and $\operatorname{frk}_R(M)$ is the so-called "maximal free rank" of M (the maximal rank of a free R-module G admitting a surjection $M \to G$). The F-signature was first introduced by Huneke-Leuschke, after being studied implicitly by Smith-Van den Bergh, and was shown to exist by the speaker in full generality.

Suppose $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ is a finite local inclusion of F-finite normal domains in characteristic $p > 0$. Previously, the best known result (due in stages to Huneke-Leuschke, Yao, Tucker) states that

$$\operatorname{frk}_R(S) \cdot s(S) \leq \operatorname{rk}_R(S) \cdot s(R)$$

with equality if S is regular. Roughly speaking, would like to achieve equality without this additional assumption, and in analogy to the situation for Hilbert-Kunz multiplicity. Recall that the Hilbert-Kunz multiplicity of $(R, \mathfrak{m}, k)$ is

$$e_{HK}(R) = \lim_{\substack{e \to \infty \\ q = p^e}} \frac{\mu_R(R^{1/q})}{\operatorname{rk}_R(R^{1/q})} = \lim_{\substack{e \to \infty \\ q = p^e}} \frac{\ell_R(R/\mathfrak{m}^{[q]})}{q^{\dim(R)}}$$

where $\mu_R(M)$ and $\ell_R(M)$ denote the minimal number of generators and length, respectively, of an R-module M, and $I^{[q]} = \langle i^q \mid i \in I \rangle$ denotes the expansion of I over the e-th iterate of Frobenius.

In the case of Hilbert-Kunz multiplicity, one readily generalizes the definition to finite colength ideals $I \subseteq R$, setting $e_{HK}(I, R) = \lim_{q \to \infty} \ell_R(R/I^{[q]})/q^{\dim(R)}$. For any finite local inclusion $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$, one can relate the Hilbert-Kunz multiplicity of an ideal

$$[l : k] \cdot e_{HK}(IS, S) = \operatorname{rk}_R(S) \cdot e_{HK}(I, R)$$

and its expansion. In the case that $\operatorname{Hom}_R(S, R) \simeq S$, we can do something similar for F-signature. For a finite colength ideal $I \subseteq R$, define

$$s(I, R) = \lim_{\substack{e \to \infty \\ q = p^e}} \frac{\ell_R(R/I_e)}{q^{\dim(R)}} \quad I_e = \langle r \in R \mid \phi(r^{1/q}) \in I \text{ for all } \phi \in \operatorname{Hom}_R(R^{1/q}, R) \rangle$$

and observe $s(\mathfrak{m}, R) = s(R)$ recovers the original notion of F-signature. When $\operatorname{Hom}_R(S, R) \simeq S$, let also $J = \langle s \in S \mid \phi(s) \in I \text{ for all } \phi \in \operatorname{Hom}_R(S, R) \rangle$; then

$$[l : k] \cdot s(J, S) = \operatorname{rk}_R(S) \cdot s(I, R)$$

in analogy to the result for Hilbert-Kunz multiplicity. In particular, specializing to the case where the extension is quasi-étale (so that $\mathrm{Hom}_R(S,R)$ is generated by the trace map), this imlies that the F-signature (of the maximal ideal) of a geometric point gets multiplied by the degree of a finite quasi-étale extension. Since the F-signature is always bounded above by one, this gives an upper bound on the degree of all quasi-étale extensions, and immediately implies Theorem 1.

In general, one can expect to deduce global results as in Theorem 2 from the local restrictions in Theorem 1 via stratification arguments. For example, [GKP] make use of Whitney stratifications to deduce the global results in characteristic zero. In [BCGST], we deduce Theorem 2 by first showing the existence of a stratification of a strongly F-regular scheme in terms of the local fundamental groups appearing in Theorem 1. The proof relies on a constructability result of Gabber, and also depends heavily upon knowing in advance that these local fundamental groups are bounded in size uniformly for all geometric points of a strongly F-regular scheme – which in turn follows from the local bound in Theorem 1 together with the lower semi-continuity of F-signature.

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Polynomial Growth of Betti Sequences over Local Rings

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(joint work with Alexandra Seceleanu and Zheng Yang)

Let $(R,\mathfrak{m})$ be a (noetherian) local ring. Let $\operatorname{mult} R$ denote the Hilbert-Samuel multiplicity of R, $\operatorname{edim} R$ the minimal number of generators of $\mathfrak{m}$, and set $\operatorname{codim} R = \operatorname{edim} R - \dim R$. When the completion $\widehat{R}$ is isomorphic to some regular local ring modulo a regular sequence, R is said to be c.i. (for *complete intersection*); an inequality $\operatorname{mult} R \geq 2^{\operatorname{codim} R}$ holds whenever R is c.i.

The asymptotic patterns of Betti sequences of finitely generated R-modules M reflect and affect the singularity of R. For instance, in 1974 Gulliksen showed that if R is c.i. of codimension c, then for each M there are polynomials $b^{\pm}_M(x)$ with $\deg b^{\pm}_M(x) \leq c-1$, such that $\beta^R_{2i}(M) = b^{+}_M(i)$ and $\beta^R_{2i+1}(M) = b^{-}_M(i)$ hold for all $i \gg 0$. In 1980 he proved a strong converse: If $\beta^R_i(R/\mathfrak{m})$ is bounded above by some polynomial of degree $c-1$, then R is c.i. with $\operatorname{codim} R \leq c$; furthermore, for

$i \geq d$ with $d = \dim R$ the function $i \mapsto \beta_i^R(R/\mathfrak{m})$ is equal to a polynomial in i of degree $c-1$ (Tate, 1957), and if $c = 1$ then $\beta_i^R(M) = \beta_d^R(M)$ (Eisenbud, 1980).

Our goal is to characterize those rings R over which the Betti numbers of every M are *eventually polynomial*, in the sense that for all $i \gg 0$ the function $i \mapsto \beta_i^R(M)$ is equal to $b_M(i)$ for some polynomial $b_M(x)$. To avoid boring special cases, we assume that $\mathfrak{m}$ is not principal and that k is infinite.

Our answers involve properties of $R^{\mathfrak{g}}$, the associated graded ring of R. To state them, let P be the symmetric algebra of $\mathfrak{m}/\mathfrak{m}^2$ over $R/\mathfrak{m}$, let I^* be the kernel of the canonical map $P \to R^{\mathfrak{g}}$, and let $I^*_{\langle 2\rangle}$ denote the ideal of P generated by I^*_2.

The proof of the first theorem below uses results of Avramov (1994, 1996).

Theorem 1. *For a positive integer c the following conditions are equivalent.*

(1) *The ring R is c.i. with* $\operatorname{codim} R = c \leq \dim_{R/\mathfrak{m}} I_2^* + 1$.

(2) *There is a real number b, such that for every real number $a > 0$ inequalities $ai^{c-2} < \beta_i^R(R/\mathfrak{m}^2) \leq bi^{c-1}$ hold all $i \gg 0$.*

(3) *The Betti numbers of $R/\mathfrak{m}^2$ are eventually polynomial of degree $c-1$.*

For the next theorem, we strengthen both conditions (1) and (3) of Theorem 1.

Theorem 2. *For a non-negative integer c the following conditions are equivalent.*

(1) *The ring R is c.i. with* $\operatorname{codim} R = c \leq \operatorname{height} I^*_{\langle 2\rangle} + 1$.

(2) *There is an isomorphism $\widehat{R} \cong Q/(g)$, where $(Q, \mathfrak{q})$ is a c.i. local ring with* $\operatorname{mult} Q = 2^{c-1}$*, and g is a non-zero-divisor in $\mathfrak{q}^2$.*

These conditions imply

(3) *The Betti numbers of each R-module M are eventually polynomial of degree at most $c-1$, and eventually polynomial of degree $c-1$ for some M.*

The fact that (2) implies (1) in Theorem 2 utilizes the structure of c.i. rings of minimal multiplicity, described by Rossi and Valla (1988). The converse implication may be viewed as a structure theorem for the rings described in (1).

The delicate part of the proof of Theorem 2 is to deduce the first assertion of (3) from (2). This means, to show that every series $P_M^R(z) = \sum_{i\geq 0} \beta_i^R(M) z^i$ satisfies $(1-z)^{c-1} P_M^R(z) \in \mathbb{Z}[z]$. After fixing M, we find a syzygy module $L = \Omega_R^s(M)$ such that $(1-z^2)\sum_{i\geq s} \beta_i^R(M) z^{i-s} = P_L^{Q'}(z)$ holds, where $(Q', \mathfrak{q}')$ is local with $\operatorname{mult} Q' = 2^{c-1}$ and $\widehat{R} \cong Q'/(g')$ for a non-zero-divisor $g' \in \mathfrak{q}'^2$. We then prove $(1-z)^{c-2} P_L^{Q'}(z) = (1+z) p_M(z)$ for some $p_M(z) \in \mathbb{Z}[z]$, by using results of Herzog and Iyengar (2005) and of Şega (2013). The desired property of $P_M^R(z)$ follows.

By combining the preceding theorems, we obtain

Corollary 3. *Assume that I^* is minimally generated by homogeneous forms $g_1, \ldots, g_c$ satisfying* $2 \leq \deg g_1 \leq \cdots \leq \deg g_c$.

All R-modules have eventually polynomial Betti numbers if and only if the sequence $g_1, \ldots, g_c$ is regular and $\deg g_1 = \cdots = \deg g_{c-1} = 2$.

The corollary settles a question in [1], and raises the question whether the conditions in Theorem 2 are always equivalent. We have an answer in low codimension.

Theorem 4. *If k is algebraically closed, R is c.i., and all cyclic modules have eventually polynomial Betti numbers, then the following inequality holds:*

$$\min\{\operatorname{codim} R, 4\} \leq \operatorname{height} I^{*}_{\langle 2\rangle} + 1\,.$$

In order to prove the last theorem, we set $c = \operatorname{codim} R$ and assume, by way of contradiction, that height $I^{*}_{\langle 2\rangle} \leq c-2$ holds for some $c \leq 4$. This implies that $I^{*}_{\langle 2\rangle}$ has a minimal prime ideals $\mathfrak{P}$ that is either degenerate (that is, contains non-zero linear forms) or satisfies height $\mathfrak{P} + 1 \leq \operatorname{mult}(P/\mathfrak{P}) \leq 4$. We use $\mathfrak{P}$ in order to produce ideals J in R, such that the ring $S = R/J$ is Golod; the classification of non-degenerate varieties of minimal or almost minimal multiplicity (by Del Pezzo, Bertini, Brodmann-Schenzel) is instrumental for these constructions.

The final step of the argument consists in showing that the Betti sequence of the R-module S is not eventually polynomial. This is equivalent to proving that the Poincaré series $P^R_S(z)$ has a pole at $z = -1$. We do it by explicitly computing the Poincaré series of Golod quotient rings of complete intersection rings, and verifying that in the cases of interest they do have poles at $z = -1$.

Proofs of the results reported in the talk will appear in [2].

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Asymptotic Behavior of Local Cohomology of Thickenings of Ideals

Hailong Dao

(joint work with Jonathan Montaño, Ilya Smirnov, Kei-ichi Watanabe)

Let $(R, \mathfrak{m})$ be a local ring of dimension d and $I \subset R$ an ideal. Let I_n be a decreasing sequence of ideals cofinal with I^n. For example, $I = I^n, I^{(n)}$ or $I^{[p^n]}$ when R has positive characteristic. In recent years the behavior of the length of $H^0_{\mathfrak{m}}(R/I_n)$ has attracted a lot of attention. For example, a striking recent result by Cutkosky states that when R is analytically unramified, the limit $\lim \frac{\ell(R/I_n)}{n^d}$ exists if $I_m I_n \subseteq I_{m+n}$ for all $m, n > 0$ and I_n are $\mathfrak{m}$-primary. Consequently, $\lim_{n\to\infty} \frac{\ell(H^0_{\mathfrak{m}}(R/I^n)}{n^d}$ exists for any ideal I. In this report we shall try to understand the behavior of higher local cohomology modules.

Question 1. *Assume that $\ell(H^i_{\mathfrak{m}}(R/I_n)) < \infty$ for $n \gg 0$. What can we say about the the asymptotic behavior of the sequence $l_n = \ell(H^i_{\mathfrak{m}}(R/I_n))$.*

The condition that $\ell(H^i_{\mathfrak{m}}(R/I_n) < \infty$ is not that restrictive. For example, it holds for I^n if I is generated by a regular sequence locally on the punctured spectrum. Or if char $R = p > 0$ and R/I satisfies Serre's condition (S_j), then $\ell(H^i_{\mathfrak{m}}(R/I^{[p^n]})) < \infty$ for all n and $i \leq j$.

Together with J. Montaño, we prove in [4]:

Theorem 2. *Let I a monomial ideal in a polynomial ring R. Assume that $\ell(H^i_{\mathfrak{m}}(R/I^n)) < \infty$ for $n \gg 0$. Then the sequence $\{\ell(H^i_{\mathfrak{m}}(R/I^n))\}_{n\gg 0}$ agrees with a quasi-polynomial with rational coefficients for $n \gg 0$. Moreover, this sequence has a rational generating function.*

Surprisingly, the proof uses the theory of Presburger language and Presburger counting function developed in [6].

The situation in characteristic $p > 0$ also holds great promise. In fact, length of local cohomology behaves well even for modules. In a recent work with Smirnov ([2]), generalizations of the classical Hilbert-Kunz functions are studied. Let M be a finitely generated R-module. Let $F^n_R(M) = M \otimes_R {}^n R$ denote the n-fold iteration of the Frobenius functor given by base change along the Frobenius endomorphism. Let $\dim R = d$ and $q = p^n$. The following introduced under different notations by Epstein and Yao ([5]):

and

$$e_{gHK}(M) := \lim_{n\to\infty} \frac{\ell H^i_{\mathfrak{m}}(F^n_R(M))}{p^{nd}},$$

which are called the generalized Hilbert-Kunz function and generalized Hilbert-Kunz multiplicity of M, respectively. We can prove ([2]):

Theorem 3. *Suppose that $M_{mathfrakp}$ has finite projective dimension for all non-maximal prime ideal $\mathfrak{p}$ (this always holds for example when R is an isolated singularity). Then*

(1) *If R is excellent, equidimensional and and locally Cohen-Macaulay on the punctured spectrum, $e_{gHK}(M)$ exists.*
(2) *If R is Cohen-Macaulay, $e_{gHK}(M)$ exists.*
(3) *If R is a complete intersection, then $e_{gHK}(M) = 0$ if and only if the projective dimension of M is less than* $\dim R$.

The limit exists for higher local cohomology modules and other homological functors. For example ([2]):

Theorem 4. *Let R be a local Cohen-Macaulay ring of dimension d and M a finitely generated R-module which is locally free on the punctured spectrum of R. Then for each $0 \le i \le d-1$.*

$$e^i_{gHK}(M) = \lim_{n\to\infty} \frac{\ell_R(H^i_{\mathfrak{m}}(F^n(M)))}{p^{nd}}$$

exists.

These limits were studied at the same time by Holger Brenner independently in [1]. What is interesting is that these notions seem to behave more geometrically than the old one. For example, Brenner made use of some of our results to construct examples of irrationality of the usual Hilbert-Kunz multiplicity, answering a long open problem in this area. The construction started by building a reflexive module M whose $e^2_{gHK}(M)$ is irrational.

One can give other strong applications. For example, we can give a quick proof that the Picard groups of the punctured spectrum of a complete intersection of dimension 3 is torsion-free. Or that the torsion elements in the class group of an isolated F-regular singularity is Cohen-Macaulay ([3]).

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Asymptotic Properties of Invariant Chains of Graded Ideals

Tim Römer

(joint work with Uwe Nagel)

Let K be a field and $c \in \mathbb{N}$. For $n \in \mathbb{N}$ let

$$K[X_n] = K[X_{i,j} : 1 \leq i \leq c,\ 1 \leq j \leq n]$$

be a polynomial ring in $c \times n$ many variables over K. We are interested in algebraic properties of ideals in a family $(I_n)_{n \geq 1}$, where $I_n \subseteq K[X_n]$ is an ideal for $n \geq 1$. Especially one would like to understand the "limit behavior" (if present) of the properties under consideration. In algebraic geometry and commutative algebra several examples of such situations exist. For a typical example let I_n be the ideal of 2-minors of the $c \times n$-matrix $(X_{i,j})_{1 \leq i \leq c, 1 \leq j \leq n}$.

Following an approach of Hillar-Sullivant in [4], an idea is to consider the polynomial ring

$$K[X] = K[X_{i,j} : 1 \leq i \leq c,\ j \geq 1]$$

and the ideal

$$I = \bigcup_{n \geq 1} I_n K[X].$$

In this generality not much can be said. One difficulty is, e.g., that $K[X]$ is not Noetherian. One has to find suitable assumptions under which the situation can be controlled in a better way.

For this let $\operatorname{Sym}(n)$ be the group of bijections $\pi \colon [n] \to [n]$ where $[n] = \{1, \ldots, n\}$. The group $\operatorname{Sym}(n)$ is naturally embedded into $\operatorname{Sym}(n+1)$ as the stabilizer of $\{n+1\}$. We set

$$\operatorname{Sym}(\infty) = \bigcup_{n \in \mathbb{N}} \operatorname{Sym}(n).$$

Then $\mathrm{Sym}(\infty)$ acts on $K[X]$ by $\sigma \cdot X_{i,j} = X_{i,\sigma(j)}$ for $\sigma \in \mathrm{Sym}(\infty)$. A first very interesting result is:

Theorem 1 (Hillar-Sullivant [4])**.** *Let $I \subseteq K[X]$ be a $\mathrm{Sym}(\infty)$-invariant ideal. Then I is finitely generated up to symmetry, i.e. there exist $f_1, \dots, f_m \in I$ such that $I = \langle \mathrm{Sym}(\infty) \cdot f_1, \dots, \mathrm{Sym}(\infty) \cdot f_m \rangle$.*

Note that in the theorem it is not assumed that I is directly defined by a family $(I_n)_{n \geq 1}$ as considered above. But given I one can naturally define a canonical/saturated family by setting

$$I_n = I \cap K[X_n] \text{ for } n \geq 1.$$

Observe that then

$$\mathrm{Sym}(n)(I_m) \subseteq I_n \text{ for every } m \leq n. \tag{1}$$

We say that this family is a $\mathrm{Sym}(\infty)$*-invariant filtration*. Conversely, given a family $(I_n)_{n \geq 1}$ of ideals which is a $\mathrm{Sym}(\infty)$-invariant filtration (i.e. (1) holds), then $I = \bigcup_{n \geq 1} I_n K[X]$ is again a $\mathrm{Sym}(\infty)$-invariant ideal of $K[X]$.

As a summary one should remember that for a given $\mathrm{Sym}(\infty)$-invariant filtration, there is the "global" object I which can be used to study algebraic properties of the ideals of the filtration. One should also mention that there exist also other relevant and very useful actions of suitable monoids, but to simplify things here, we consider only the defined $\mathrm{Sym}(\infty)$ action. See Draisma [3] for an overview article on relevant and related developments in the recent years. See also related approaches on twisted commutative algebras and GL_∞-algebras by Sam-Snowden [6, 7], and on FI-modules by Church-Ellenberg-Farb-Nagpal [1, 2].

Let fix us a $\mathrm{Sym}(\infty)$-invariant filtration $\mathcal{I} = (I_n)_{n \geq 1}$ of graded ideals, i.e. that additionally the ideals I_n are graded. Then we define its *bigraded Hilbert-series* to be

$$H_{\mathcal{I}}(s,t) = \sum_{n \geq 0,\, j \geq 0} \dim_K [K[X_n]/I_n]_j \cdot s^n t^j.$$

One of the main theorems in [5] is that this series is rational. More precisely, we know:

Theorem 2. *Let $\mathcal{I} = (I_n)_{n \in \mathbb{N}}$ be a $\mathrm{Sym}(\infty)$-invariant filtration of graded ideals. Then the bigraded Hilbert series $H_{\mathcal{I}}(s,t)$ of $\mathcal{I}$ is a rational function of the form*

$$H_{\mathcal{I}}(s,t) = \frac{g(s,t)}{(1-t)^a \cdot \prod_{j=1}^{b} [(1-t)^{c_j} - s \cdot f_j(t)]},$$

where a, b, c_j are non-negative integers with $c_j \leq c$, $g(s,t) \in \mathbb{Z}[s,t]$, and each $f_j(t)$ is a polynomial in $\mathbb{Z}[t]$ satisfying $f_j(1) > 0$.

A related result of [5] is that there exist $A, B \in \mathbb{Z}$ with $0 \leq A \leq c$ such that, for all $n \gg 0$,

$$\dim K[X_n]/I_n = An + B.$$

Note that also the multiplicities of the ideals in the filtration can be studied via this methods. For this and further results we refer to [5].

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A Duality of Buchsbaum Rings and Triangulated Manifold

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(joint work with Isabella Novik, Ken-ichi Yoshida)

The Dehn-Sommerville equation is one of the most famous symmetries in face enumeration theory. It was first proved by Dehn and Sommerville for simplicial polytopes, but later it was generalized to all triangulated manifolds. Here we introduce a simple algebraic way to express the Dehn-Sommerville equation for triangulated manifolds by using Matlis duality and Stanley–Reisner theory.

The Dehn-Sommerville equation for balls and spheres. We first recall the Dehn-Sommerville equation for balls and spheres. We fix a field $\mathbb{K}$. For a simplicial complex Δ of dimension $d-1$, let $f_{i-1}(\Delta)$ be the number of $(i-1)$-dimensional faces of Δ and let $h_i(\Delta) = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{i-j} f_{j-1}(\Delta)$ for $i = 0, 1, \dots, d$. For $F \in \Delta$, the simplicial complex $\mathrm{lk}_\Delta(F) = \{G \in \Delta : F \cup G \in \Delta, F \cap G = \emptyset\}$ is called the **link of** F in Δ. A simplicial complex Δ is said to be **pure** if all its facets have the same dimension. A pure simplicial complex Δ of dimension d is said to be a $\mathbb{K}$**-homology** d**-sphere** if $\widetilde{H}_*(\mathrm{lk}_\Delta(G)) \cong \widetilde{H}_*(\mathbb{S}^{d-|G|})$ for all faces $G \in \Delta$ (including $G = \emptyset$), where $\widetilde{H}_i(X)$ denotes the ith reduced homology group with coefficients in $\mathbb{K}$. A pure simplicial complex of dimension d is said to be a $\mathbb{K}$**-homology** d**-ball** if (1) the link of each face G of Δ has the homology of either $\mathbb{S}^{d-|G|}$ or $\mathbb{B}^{d-|G|}$, and (2) the set of all *boundary faces*, that is,

$$\partial\Delta := \left\{ G \in \Delta \ : \ \widetilde{H}_*(\mathrm{lk}_\Delta(G)) \cong \widetilde{H}_*(\mathbb{B}^{d-|G|}) \right\} \cup \{\emptyset\}$$

is a $\mathbb{K}$-homology $(d-1)$-sphere. For convention, for a homology sphere Δ, we set $\partial\Delta = \emptyset$. One version of the Dehn-Sommerville equation is the following.

Theorem 1 (Dehn-Sommerville equation for balls and spheres)**.** *If Δ is a $\mathbb{K}$-homology $(d-1)$-ball or a $\mathbb{K}$-homology $(d-1)$-sphere, then*

$$h_i(\Delta) = h_{d-i}(\Delta, \partial\Delta) \quad \textit{for } i = 0, 1, \dots, d. \tag{1}$$

This symmetry can be algebraically explained by using Matlis duality and canonical modules. For a $(d-1)$-dimensional simplicial complex Δ on $[n] = \{1, 2, \dots, n\}$, let $\mathbb{K}[\Delta] = S/I_\Delta$ be the Stanley–Reisner ring of Δ over $\mathbb{K}$, where $S = \mathbb{K}[x_1, \dots, x_n]$ and where $I_\Delta = (x_{i_1} \cdots x_{i_k} : \{i_1, \dots, i_k\} \notin \Delta)$ is the Stanley–Reisner ideal. For a subcomplex $\Gamma \subset \Delta$, the ideal $\mathbb{K}[\Delta, \Gamma] = I_\Gamma/I_\Delta$ of $\mathbb{K}[\Delta]$ is called the **Stanley–Reisner module of the pair** (Δ, Γ). Let $d = \dim \mathbb{K}[\Delta, \Gamma]$. Let $f_i(\Delta, \Gamma)$ be the number of i-dimensional faces in Δ but not in Γ. Define $h_i(\Delta, \Gamma)$ in the same way as the usual h-numbers. Then the h-numbers appears as the Hilbert series of $\mathbb{K}[\Delta, \Gamma]$. It is known that

$$\sum_{k=0}^{\infty} (\dim_{\mathbb{K}} \mathbb{K}[\Delta, \Gamma]_k) t^k = \frac{1}{(1-t)^d} \big\{ h_0(\Delta, \Gamma) + h_1(\Delta, \Gamma) t + \cdots + h_d(\Delta, \Gamma) t^d \big\}.$$

It was proved by Hochster [5, Ch. II §7] that if Δ is a $\mathbb{K}$-homology ball or a $\mathbb{K}$-homology sphere, then $\mathbb{K}[\Delta, \partial\Delta]$ is the canonical module of $\mathbb{K}[\Delta]$. This result implies that if we take a linear system of parameters Θ for $\mathbb{K}[\Delta]$, then we have an isomorphism

$$\mathbb{K}[\Delta]/\Theta\mathbb{K}[\Delta] \cong (\mathbb{K}[\Delta, \partial\Delta]/\Theta\mathbb{K}[\Delta, \partial\Delta])^\vee(-d), \tag{2}$$

where $N^\vee$ denotes the Matlis dual of a graded S-module N. This isomorphism (2) immediately implies (1) since h-numbers of Δ coincides with the Hilbert function of $\mathbb{K}[\Delta]/\Theta\mathbb{K}[\Delta]$ and h-numbers of $(\Delta, \partial\Delta)$ coincides with the Hilbert function of $\mathbb{K}[\Delta, \partial\Delta]/\Theta\mathbb{K}[\Delta, \partial\Delta]$.

The Dehn-Sommerville equation for manifolds. We say that a d-dimensional simplicial complex Δ is a **$\mathbb{K}$-homology d-manifold without boundary** if, for any non-empty face $G \in \Delta$, the link of G is an $\mathbb{K}$-homology $(d - |G|)$-sphere. An **$\mathbb{K}$-homology d-manifold with boundary** is a pure d-dimensional simplicial complex Δ satisfying (1) the link of each non-empty face G of Δ has the homology of either $\mathbb{S}^{d-|G|}$ or $\mathbb{B}^{d-|G|}$, and (2) the set of all **boundary faces**, that is,

$$\partial\Delta := \Big\{ G \in \Delta \ : \ \widetilde{H}_*(\mathrm{lk}_\Delta(G)) \cong \widetilde{H}_*(\mathbb{B}^{d-|G|}) \Big\} \cup \{\emptyset\}$$

is a $(d-1)$-dimensional $\mathbb{K}$-homology manifold without boundary. For homology manifold Δ without boundary, we set $\partial\Delta = \emptyset$. We now introduce a generalization of the Dehn-Sommerville equation for manifolds. Let $\Gamma \subset \Delta$ be a pair of simplicial complexes with $\dim \mathbb{K}[\Delta, \Gamma] = d$. We write $\tilde{\beta}_{j-1}(\Delta, \Gamma) = \dim_{\mathbb{K}} \widetilde{H}_{j-1}(\Delta, \Gamma)$. A connected $\mathbb{K}$-homology d-manifold is said to be **orientable** if $H_d(\Delta, \partial\Delta) \cong \mathbb{K}$. We define the **$h''$-numbers** of (Δ, Γ) as

$$h''_j(\Delta, \Gamma) = \begin{cases} h_j(\Delta, \Gamma) - \binom{d}{j} \sum_{l=1}^{j} (-1)^{j-l} \tilde{\beta}_{l-1}(\Delta, \Gamma), & \text{if } 0 \le j < d, \\ h_d(\Delta, \Gamma) - \sum_{l=1}^{d-1} (-1)^{j-l} \tilde{\beta}_{l-1}(\Delta, \Gamma), & \text{if } j = d. \end{cases}$$

Theorem 2 (Dehn-Sommerville equation for manifolds)**.** *If Δ is a connected orientable $\mathbb{K}$-homology $(d-1)$-manifold, then*

$$h''_i(\Delta) = h''_{d-i}(\Delta, \partial\Delta) \quad \text{for } i = 0, 1, \ldots, d. \tag{3}$$

The above equation was discovered by Klee [1] for manifolds without boundary and by Macdonald [2] for manifolds with boundary. While their formulations are different to (3) since they use f-numbers, the above simple formula in terms of h''-numbers were discovered in Novik [7] and Murai-Novik [6].

An algebraic Dehn-Sommerville equation for manifolds. Theorem 2 naturally leads us a question "can we express (3) using Matlis duality?" To answer this question, we need an idea considered by Goto [3] in 1983. For a finitely generated graded S-module M of Krull dimension d and its homogeneous system of parameters $\Theta = \theta_1, \ldots, \theta_d$, we define

$$\Sigma(\Theta; M) = \Theta M + \sum_{k=1}^{d} (\theta_1, \ldots, \hat{\theta}_i, \ldots, \theta_d)M :_M \theta_i.$$

This submodule was introduced by Goto to study Buchsbaum local rings. We say that a finitely generated graded S-module M of Krull dimension d is **Buchsbaum** if, for every homogeneous system of parameter $\Theta = \theta_1, \ldots, \theta_d$ for M,

$$(\theta_1, \ldots, \theta_{i-1})M :_M \theta_i = (\theta_1, \ldots, \theta_{i-1})M :_M \mathfrak{m}$$

for $i = 1, 2, \ldots, d$, where $\mathfrak{m} = (x_1, \ldots, x_n)$. We show the following result.

Theorem 3. *If $\mathbb{K}[\Delta, \Gamma]$ is Buchsbaum, then for any linear system of parameters Θ for $\mathbb{K}[\Delta, \Gamma]$, one has*

$$\dim_{\mathbb{K}}(\mathbb{K}[\Delta, \Gamma]/\Sigma(\Theta; \mathbb{K}[\Delta, \Gamma]))_i = h''_i(\Delta, \Gamma) \quad \text{for all } i.$$

By the work of Schenzel [8], if Δ is a $\mathbb{K}$-homology manifold, then both $\mathbb{K}[\Delta]$ and $\mathbb{K}[\Delta, \partial\Delta]$ are Buchsbaum. Hence the above theorem gives an algebraic way to study h''-vectors using Artinian graded algebra. Moreover, the following result holds.

Theorem 4. *Let $R = S/I$ be a Buchsbaum graded $\mathbb{K}$-algebra of Krull dimension $d \geq 2$, ω_R the canonical module of R, $\Theta = \theta_1, \ldots, \theta_d \in S$ a homogeneous system of parameters of R. If* $\operatorname{depth}(R) \geq 2$*, then*

$$\omega_R/\Sigma(\Theta; \omega_R) \cong \big(R/\Sigma(\Theta; R)\big)^{\vee}(-\delta),$$

where $\delta = \sum_{i=1}^{d} \deg \theta_i$.

It was proved by Gräbe [4] that the module $\mathbb{K}[\Delta, \partial\Delta]$ is the canonical module of $\mathbb{K}[\Delta]$ when Δ is a connected orientable homology manifold. Thus, by Theorem 4 we obtain the following strengthen of Theorem 2.

Theorem 5. *Let Δ be a connected orientable $\mathbb{K}$-homology $(d-1)$-manifold with non-empty boundary $\partial\Delta$ and Θ a linear system of parameters for $\mathbb{K}[\Delta]$. Then*

$$\mathbb{K}[\Delta]/\Sigma(\Theta; \mathbb{K}[\Delta]) \cong \big(\mathbb{K}[\Delta, \partial\Delta]/\Sigma(\Theta; \mathbb{K}[\Delta, \partial\Delta])\big)^{\vee}(-d).$$

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Rationality of Poincaré Series over Local Rings

Liana M. Şega

Let $(R, \mathfrak{m}, k)$ be a local Noetherian ring and M be a finitely generated R-module. We denote by $b_i(M)$ the ith *Betti number* of M, namely $b_i(M) = \dim_k \mathrm{Tor}_i^R(M, k)$. The *Poincaré series* of M is the formal power series

$$P_M^R(z) = \sum_{i=0}^{\infty} b_i(M) z^i \,.$$

In the 1950's Kaplansky and Serre asked if the Poincaré series $P_k^R(z)$ is always a rational function. This question was answered negatively by Anick in 1982. Subsequently, Bögvad constructed a Gorenstein ring R with $\mathfrak{m}^4 = 0$ and $P_k^R(z)$ transcendental. Many recent papers prove that certain families of rings have rational Poincaré series. Roos [6] calls a local ring R *good* if the Poincaré series of all finitely generated modules over R are rational, sharing a common denominator, and he gives an example of a ring R and a sequence of R-modules with rational Poincaré series which do not share a common denominator. In most known cases, the fact that a certain class of rings is good is established using a result a Levin, as a consequence of a remarkable structural property, namely that there exists a surjective *Golod homomorphism* from a complete intersection onto the ring. The existence of such a map is an important conclusion on its own, and has other important consequences.

We want to understand whether the property that a ring is good holds generically. We provide answers in the case of Artinian level k-algebras. Once the relevant parameters are fixed, these algebras are parameterized by a Grassmannian. We say that a property holds "generically" if it holds for all k-algebras that correspond to points on a non-empty open set of the Grassmanian. When k is infinite, it is known that one can choose a non-empty open subset of this Grassmannian so that its points correspond to *compressed level k-algebras*, see [4].

In [5] we extend the notion of compressed level k-algebra to the case of local rings that do not necessarily contain a field. To define this concept, we introduce first some terminology. The *embedding dimension* of R is the minimal number of generators of $\mathfrak{m}$. We say that R is a *level ring of socle degree* s if its socle is equal to $\mathfrak{m}^s$. In this case, we define the *socle dimension* of R to be the dimension of the k-vector space $\mathfrak{m}^s$. Also, we let $v(R)$ denote the largest integer i such that $I \subseteq \mathfrak{n}^i$, where $(Q, \mathfrak{n}, k)$ is a regular local ring and $I \subseteq \mathfrak{n}^2$.

Theorem 1. ([5, Theorem 3.5]) *Let* (e, s, c) *be integers and let* $(R, \mathfrak{m}, k)$ *be a level local Artinian ring with embedding dimension* e*, socle degree* s*, and socle dimension* c*. Then the following statements hold.*

(1) *The length of* R *satisfies*

$$(\#) \qquad \lambda_R(R) \leq \sum_{i=0}^{s} \min\left\{ \binom{(e-1)+i}{i}, c\binom{(e-1)+(s-i)}{s-i} \right\}.$$

(2) *Equality holds in* $(\#)$ *if and only if the Hilbert function of* R *is given by*

$$\dim_k(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \min\left\{ \binom{(e-1)+i}{i}, c\binom{(e-1)+(s-i)}{s-i} \right\}, \quad \textit{for } 0 \leq i \leq s\,.$$

(3) *If equality holds in* $(\#)$*, then*
   (a) *the parameter* $v(R)$ *satisfies* $s \leq 2v(R) - 1$*;*
   (b) $\operatorname{ann}(\mathfrak{m}^i) = \mathfrak{m}^{s-i+1}$ *for* $0 \leq i \leq s$*;*
   (c) *The associated graded ring* $\operatorname{gr}_{\mathfrak{m}}(R)$ *is a compressed level graded* k*-algebra.*

We say that R is a *compressed level local Artinian ring* if equality holds in $(\#)$.

The statements of the theorem are known when R is a k-algebra, in which case they can be elegantly proved using the Macaulay inverse system. Since we cannot use such methods in the case of rings that do not contain a field, our proof in [5] relies (in part) on writing R as a quotient of an Artinian Gorenstein ring of the same embedding dimension as R, and using Gorentsein duality.

We now come back to the main problem, namely understanding whether the property that there exists a Golod homomorphism from a complete intersection onto the ring (implying that the ring is good) holds generically, in the case of level Artinian k-algebras. In view of the discussion above, the results will be stated for compressed level local Artinian rings. Since the property that R is good is invariant with respect to faithfully flat extensions, there is no harm in assuming that k is algebraically closed. In the Theorem below, part (1) is contained in [3], part (2) is contained in [7] and part (2) is contained in [5].

Theorem 2. *Let* (e, s, c) *be integers and let* $(R, \mathfrak{m}, k)$ *be a compressed level local Artinian ring with embedding dimension* e*, socle degree* s*, and socle dimension* c*. Assume that* k *is algebraically closed. The following then hold.*

(1) *If* $c = 1$ *(equivalently,* R *is Gorenstein),* $s = 3$ *and there exists an element* $x \in \mathfrak{m}$ *such that* $\operatorname{ann}(x)$ *is principal, then there exists a Golod homomorphism onto* R *from a codimension* 2 *complete intersection.*

(2) *If* $c = 1$ *and* $s \neq 3$*, then there exists a Golod homomorphsim onto* R *from a codimension* 1 *complete intersection.*
(3) *For any* c*, if* s *is odd and* $s \neq 3$*, then the following hold:*
   (a) *If* $s = 2v(R) - 1$*, then there exists a Golod homomorphsim onto* R *from a codimension* 1 *complete intersection.*
   (b) *If* $s \neq 2v(R) - 1$*, then* R *is Golod (equivalently, there exists a Golod homomorphism onto* R *from a regular local ring).*

In view of the work in [1] and [2], the additional condition in (1) on the existence of the element x holds in the case of *generic* Gorenstein k-algebras of socle degree 3. This additional condition is necessary in (1), due to Bögvad's example.

The existence of the Golod homomorphisms implies that there exists a polynomial $d_R(z)$ such that $d_R(z)P^R_M(z) \in \mathbb{Z}[z]$ for all finitely generated R-modules M. In what follows, "case (1)" refers to the hypotheses of part (1) of Theorem 2 being satisfied, and so on. In each of four cases in the theorem, we can compute the common denominator $d_R(z)$ as follows:

$$d_R(z) = \begin{cases} 1 - et + et^2 - t^3 & \text{in case (1)} \\ 1 - z(P^Q_R(z) - 1) + cz^{e+1}(1+z) & \text{in cases (2) and (3a)} \\ 1 - z(P^Q_R(z) - 1) & \text{in case (3b)} \end{cases}$$

We provide some of the ingredients involved in the proof in case (3a). Assume s is odd and $s \neq 3$. Set $t = v(R)$, hence $s = 2t - 1$. Set $P = Q/(f)$ and $\mathfrak{p} = \mathfrak{n}/(f)$, where $f \in I \smallsetminus \mathfrak{n}^{t+1}$, and consider the induced map $\chi\colon P \to R$. To show that χ is Golod, we use the definition of the notion of Golod homomorphism, stated in terms of existence of certain trivial Massey operations. Once the following two claims are proved, the desired Massey operation can be constructed inductively.

Claim 1. The map $\operatorname{Tor}^P_i(R, k) \to \operatorname{Tor}^P_i(R/\mathfrak{m}^{t-1}, k)$ induced by the projection $R \to R/\mathfrak{m}^{t-1}$ is zero for all $i > 0$.

Claim 2. The map $\operatorname{Tor}^P_i(\mathfrak{m}^{2t-2}, k) \to \operatorname{Tor}^P_i(\mathfrak{m}^{t-1}, k)$ induced by the inclusion $\mathfrak{m}^{2t-2} \hookrightarrow \mathfrak{m}^{t-1}$ is zero for all $i \geq 0$.

The proof of Claim 1 follows by noting that the induced map $\operatorname{Tor}^P_i(\mathfrak{p}^t, k) \to \operatorname{Tor}^P_i(\mathfrak{p}^{t-1}, k)$ is zero for all $i > 0$. For Claim 2, we show that the induced map $\operatorname{Tor}^P_i(\mathfrak{m}^{t+1}, k) \to \operatorname{Tor}^P_i(\mathfrak{m}^t, k)$ is zero; the hypothesis $s \neq 3$ is necessary to conclude Claim 2 from here, as it guarantees that $2t-2 \geq t+1$. The main ingredient, proved using Gorenstein duality and Theorem 1, is the next lemma. Its statement can be further interpreted in terms of the Koszul homology of R with respect to a minimal generating set of $\mathfrak{m}$, towards further understanding of the relevant maps.

Lemma 3. *Let* $(R, \mathfrak{m}, k)$ *be a compressed level local Artinian ring with embedding dimension* e *and socle degree* s*. Set* $t = v(R)$*. Assume that* s *is odd and* $s = 2t-1$*. Decompose the maximal ideal* $\mathfrak{m}$ *as the sum of two subideals* $\mathfrak{m} = (x_1) + \mathfrak{m}'$ *with* x_1 *a minimal generator of* $\mathfrak{m}$ *and* $\mathfrak{m}'$ *minimally generated by* $e-1$ *elements. Then*

$$x_1^{t-1}[\operatorname{ann}_R(\mathfrak{m}') \cap \mathfrak{m}^t] = \mathfrak{m}^s.$$

Theorem 2 does not cover the cases when $c > 1$ and s is even, and $c > 1$ and $s = 3$. These cases remain open.

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Graded Maximal Cohen-Macaulay Modules over Graded One-dimensional Gorenstein Rings

Ragnar Olaf Buchweitz

(joint work with Osamu Iyama, Kota Yamaura)

Let $R = \oplus_{i\geq 0} R_i$ be a one-dimensional Gorenstein ring with $R_0 = k$ a field. With $\mathcal{N} \subset R$ the set of homogeneous nonzerodivisors, set $\mathcal{K} = \mathcal{N}^{-1}R$, the total graded ring of fractions. The a-invariant of R is the largest degree in which $\mathcal{K}/R$ is not zero. Let $p > 0$ be an integer such that $\mathcal{K} \cong \mathcal{K}(p)$. With $\underline{\mathrm{CM}}_0^{\mathbb{Z}}(R)$ the stable category of graded maximal Cohen-Macaulay R-modules that are locally free on teh punctured spectrum of R, we have the following:

Theorem 1. (1) *The triangulated category* $\underline{\mathrm{CM}}_0^{\mathbb{Z}}(R)$ *admits*

$$T = \bigoplus_{i=1}^{a+p} \left(R_{\geq i}\right)(i) = \bigoplus_{i=1}^{a+p} R(i)_{\geq 0}$$

as a tilting object.

(2) *There is an exact equivalence* $\underline{\mathrm{CM}}_0^{\mathbb{Z}}(R) \sim K^\flat(\underline{\mathrm{proj}}(\Lambda))$, *for* $\Lambda = \underline{\mathrm{End}}(\mathrm{gr}_R(T))$, *the endomorphism ring of* T *in* $\underline{\mathrm{CM}}_0^{\mathbb{Z}}(R)$.

(3) Λ *is a finite dimensional Iwanaga-Gorenstein algebra.*

(4)
$$\Lambda = \begin{bmatrix} R_0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ R_1 & R_0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ R_2 & R_1 & R_0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \cdots & \cdots & \cdots & 0 \\ R_{a-1} & R_{a-2} & \cdots & R_1 & R_0 & 0 & \cdots & \cdots & 0 \\ \mathcal{K}_a & \mathcal{K}_{a-1} & \cdots & \mathcal{K}_2 & \mathcal{K}_1 & \mathcal{K}_0 & 0 & \cdots & 0 \\ \vdots & \vdots & & & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & \ddots & \ddots & \vdots \\ \mathcal{K}_{a+p-1} & \mathcal{K}_{a+p-2} & \cdots & \cdots & \cdots & \cdots & \mathcal{K}_2 & \mathcal{K}_1 & \mathcal{K}_0 \end{bmatrix}$$

If R is reduced, then Λ is of finite global dimension

We then explained how this implies that the isomorphism classes of graded maximal Cohen-Macaulay modules without free summands are in bijection with maximal perfect T-complexes:

$$0 \longleftarrow P_m \overset{\partial}{\longleftarrow} P_{m+1} \overset{\partial}{\longleftarrow} \cdots \overset{\partial}{\longleftarrow} P_n \longleftarrow 0,$$

where

- $m, n \in \mathbb{Z}$
- $P_i \in \underline{\mathrm{add}}(T)$
- $\partial^2 = 0$
- The components of each ∂ in $\underline{\mathrm{End}}(T^{\oplus ?})$ are in the radical of Λ.

Linear Syzygies and Hyperbolicity

Matteo Varbaro

(joint work with Alexandru Constantinescu, Thomas Kahle)

Let n be a positive integer, $S = K[x_1, \dots, x_n]$ be a standard graded polynomial ring over a field K, and $I \subseteq S$ be a homogeneous ideal minimally generated by s quadratic forms.

The minimal graded free resolution of S/I has the form:

$$0 \to \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{k,j}} \to \dots \to \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{2,j}} \to \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{1,j}} \to S \to S/I$$

where $k \leq n$ is the projective dimension of S/I, $\beta_{1,j} = 0$ if $j \neq 2$ and $\beta_{1,2} = s$. The *Castelnuovo-Mumford regularity* of S/I is

$$\operatorname{reg} S/I = \max\{j - i : \beta_{i,j} \neq 0\}.$$

In [6], Mayr and Meyer gave examples of quadratic ideals $I \subseteq S$ for which $\operatorname{reg} S/I$ is doubly exponential in n. The ideals of Mayr-Meyer have the property that already the first syzygy module has minimal generators in a very high degree, indeed in their examples $\beta_{2,j} \neq 0$ for a certain $j > 2^{2^{cn}}$ for some $c \in \mathbb{Q}_{>0}$.

Conversely, if the first syzygy module of I is linearly generated (i.e. $\beta_{2,j} = 0$ whenever $j > 3$), no example with $\operatorname{reg} S/I$ big compared to n seems to be known:

Open Question 1. *Is there a family of quadratic ideals $I_m \subseteq K[x_1, \dots, x_m]$ such that the first syzygy modules of I_m are linearly generated and*

$$\lim_{m\to\infty} \frac{\operatorname{reg} I_m}{m} > 0$$

Definition 2. *For a positive integer p, we say that S/I satisfies the property N_p if $\beta_{i,j} = 0$ for all $i \leq p$, $j > i+1$. The Green-Lazarsfeld index of S/I is*

$$\operatorname{index} S/I = \sup\{p \in \mathbb{N} : S/I \textit{ satisfies the property } N_p\}$$

For example, S/I satisfies N_1 just means that I is generated by quadrics, S/I satisfies N_2 means that I is generated by quadrics and has first linear syzygies, S/I satisfies N_3 means that I is generated by quadrics and has first and second linear syzygies, and so on ...

From now on we will focus in the case $I = I_\Delta \subseteq S$ is a square-free monomial ideal. In this case S/I_Δ is denoted by $K[\Delta]$ and called the *Stanley-Reisner ring* of the simplicial complex Δ. The following is a result from [3].

Theorem 3 (Dao, Huneke, Schweigh)**.** *If $K[\Delta]$ satisfies the property N_p for some $p \geq 2$, then*

$$\operatorname{reg} K[\Delta] \leq \log_{\frac{p+3}{2}} \left(\frac{n-1}{p} \right) + 2$$

Corollary 4. *If $I_m \subseteq K[x_1, \dots, x_m]$ is a family of quadratic monomial ideals with linearly generated first syzygy module, then*

$$\lim_{m\to\infty} \frac{\operatorname{reg} I_m}{m} = 0$$

In view of Theorem 3, the following is a natural question for any given $p \geq 2$:

Question (A_p)**.** *Is there a constant $\lambda \in \mathbb{R}$ for which $\operatorname{reg} K[\Delta] \leq \lambda$ whenever $K[\Delta]$ satisfies N_p?*

To deal with the above question we will introduce some concepts in metric group theory: Let Γ be a simple graph on a (possibly infinite) vertex set V. Given two vertices $v, w \in V$, a path e from v to w consists in a subset of vertices

$$\{v = v_0, v_1, v_2, \dots, v_k = w\}$$

such that $\{v_i, v_{i+1}\}$ is an edge for all $i = 0, \dots, k-1$. The length of such a path is $\ell(e) = k$. The distance between v and w is

$$d(v, w) := \inf\{\ell(e) : e \text{ is a path from } v \text{ to } w\}.$$

A path e from v to w is called a geodesic path if $\ell(e) = d(v, w)$. A *geodesic triangle* of vertices v_1, v_2 and v_3 consists in three geodesic paths e_i from v_i to v_{i+1} (mod 3) for $i = 1, 2, 3$. For $\delta \geq 0$, a geodesic triangle e_1, e_2, e_3 is δ-*slim* if $d(v, e_i \cup e_j) \leq \delta$ $\forall\ v \in e_k$ and $\{i, j, k\} = \{1, 2, 3\}$. The graph Γ is δ-hyperbolic if each geodesic triangle of Γ is δ-slim; it is *hyperbolic* if it is δ-hyperbolic for some δ.

Given a group G and a subset $\mathcal{S} \subseteq G$ (not containing the identity) of (distinct) generators of G, the *Cayley graph* $\mathrm{Cay}(G, \mathcal{S})$ is the simple graph with:

(i) G as vertex set;

(ii) as edges, the sets $\{g, gs\}$ where $g \in G$ and $s \in \mathcal{S}$.

The following is a fundamental result of Gromov (see Theorem 12.3.5 in [4]):

Theorem 5 (Gromov)**.** *Given two finite sets of generators $\mathcal{S}$ and $\mathcal{S}'$ of G, the Cayley graph $\mathrm{Cay}(G, \mathcal{S})$ is hyperbolic if and only if $\mathrm{Cay}(G, \mathcal{S}')$ is.*

In view of the theorem above the following is a good notion:

Definition 6. *A group G is hyperbolic if it has a finite set of generators $\mathcal{S}$ such that $\mathrm{Cay}(G, \mathcal{S})$ is hyperbolic.*

The *cohomological dimension* of a group G is defined as:

$$\operatorname{cd} G = \sup\{n \in \mathbb{N} : H^n(G; M) \neq 0 \ \text{ for some } G\text{-module } M\}.$$

If G has nontrivial torsion, then it is well known that $\operatorname{cd} G = \infty$. A group G is *virtually torsion-free* if it has a finite index subgroup which is torsion-free. By a result of Serre, if Γ and Γ' are two finite index torsion-free subgroups of G, then

$$\operatorname{cd} \Gamma = \operatorname{cd} \Gamma'.$$

So it is well-defined the *virtual cohomological dimension* of a virtually torsion-free group G: $\operatorname{vcd} G = \operatorname{cd} \Gamma$ where Γ is a finite index torsion-free subgroup of G.

Finally, recall that a Coxeter group is a pair $(G, \mathcal{S})$ where G is a group with a presentation of the type $\langle \mathcal{S} \mid \mathcal{R} \rangle$ such that:

(i) $\mathcal{S} = \{s_1, s_2, \ldots, s_n\}$ is a system of generators of G;

(ii) the relations $\mathcal{R}$ are of the form $(s_i s_j)^{m_{ij}} = e$ where $m_{ii} = 1$ for all $i = 1, \ldots, n$ and $m_{ij} \in \{2, 3, \ldots\} \cup \{\infty\}$ otherwise.

A Coxeter group is *right-angled* if and only if $m_{ij} \in \{1, 2, \infty\}$. Since a Coxeter group $(G, \mathcal{S})$ can always be embedded in $\mathrm{GL}_n(\mathbb{C})$ (where $n = |\mathcal{S}|$), by Selberg's lemma, a Coxeter group is virtually torsion-free; in particular the virtual cohomological dimension of a Coxeter group is well-defined, and Gromov raised the following:

Question (B)**.** *Is there a global bound for the virtual cohomological dimension of a right-angled hyperbolic Coxeter group?*

A main consequence of the forthcoming joint work with Constantinescu and Kahle [2] is:

Theorem 7. *Questions A_2 and B are equivalent.*

In particular, since the question of Gromov has been negatively answered in 2003 by Januszkiewicz and Świątkowski [5], we get as corollary a previous result achieved in [1]:

Corollary 8. *For any $r \in \mathbb{N}$, there exists a simplicial complex Δ such that $K[\Delta]$ satisfies N_2 and $\operatorname{reg} K[\Delta] \geq r$.*

In [2], we also exploit a method that, in particular, allows us to negatively answer Question A_p in general as follows:

Theorem 9. *For any $p \geq 2$ and any $r \in \mathbb{N}$, there exists a simplicial complex Δ such that $K[\Delta]$ satisfies N_p and $\operatorname{reg} K[\Delta] = r$.*

Unfortunately in our method $K[\Delta]$ is far from being Cohen-Macaulay in general. On the other hand, we already knew that $K[\Delta]$ cannot be Gorenstein by the following result in [1]:

Theorem 10. *If Δ is a simplicial complex such that $K[\Delta]$ satisfies N_2 and is Gorenstein, then*

$$\operatorname{reg} K[\Delta] \leq 4.$$

Open Question 11. *Is there a constant $\lambda \in \mathbb{R}$ for which $\operatorname{reg} K[\Delta] \leq \lambda$ if $K[\Delta]$ satisfies N_2 and is Cohen-Macaulay?*

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