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Harmonic Analysis and the Trace Formula

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Abstract. The purpose of this workshop was to discuss recent results in harmonic analysis that arise in the study of the trace formula. This theme is common to different directions of research on automorphic forms such as representation theory, periods, and families.

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Introduction by the Organisers

The Oberwolfach Workshop took place from May 21st through 27th of 2016, and brought together 50 participants. There were 25 talks of 45 minutes each. A motivation of the subject can be seen from the far-reaching reciprocity and functoriality conjectures of Langlands, according to which all reasonable L-functions arising from Galois representations, arithmetic geometry and harmonic analysis should be automorphic. One of the main tools to study automorphic L-functions is the Arthur-Selberg trace formula. The trace formula is used to globalize local representations into global automorphic forms, and it is essential in the concept of families where one studies an object by deforming it. The trace formula continues to motivate a wide-range of techniques in algebra, representation theory on p-adic, real and adelic groups, in analysis, and in differential and algebraic geometry.

Packets. Gan, Moeglin, Mezo, Cunningham, B. Xu, Kaletha, and W.-W. Li (in the order of their talks) spoke on the Langlands packets (L-packets) and Arthur packets (A-packets) for p-adic, real reductive, and adelic groups. Gan gave an exposition on triality for algebraic groups of type D_4, highlighted by an automorphism θ of order 3 on the root datum of D_4. Then he described three applications: (i) spin functorial lift from $PGSp_6$ to GL_8, (ii) local Langlands correspondence for $PGSO_8$, (iii) adjoint lift from GL_3 to GL_8 and an endoscopic classification for G_2 via the stable twisted trace formula with respect to θ. Moeglin overviewed A-packets, especially in the archimedean case, and addressed the problem of proving multiplicity one for each A-packet in the case of unitary groups. Mezo's talk bridged the gap between the two well-known construction of A-packets for quasi-split orthogonal or symplectic groups, one by Adams-Barbasch-Vogan (ABV) and the other by Arthur. The ABV construction of packets is based on a pairing between representations of a real group G and certain equivariant sheaves on the geometric space of Langlands parameters for G. Cunningham and B. Xu described their joint program with others on adapting the ABV construction from real groups to p-adic groups. Some goals of the program are to give a geometric description of A-packets that is equivalent to the A-packets in the known cases of classical groups, and to be able to compute the packets explicitly via geometry. A central issue is to find a geometric version of the endoscopic pairing between members of a packet and characters of the corresponding centralizer group. Kaletha explained how to use the inspiration from work of Harish-Chandra and Langlands on the local Langlands correspondence (LLC) for real groups to construct supercuspidal L-packets for p-adic groups. The major hurdle for p-adic groups has been that an explicit character formula has been available only in very special cases. Li's talk revolved around the two related topics of stable conjugacy and L-packets for topological covering groups of p-adic reductive groups as defined by Brylinski and Deligne. He pointed out the subtlety in defining stable conjugacy for general covering groups but proposed a working definition for coverings of symplectic groups.

Periods. The restriction of an infinite dimensional representation Π of a real or p-adic Lie group G to a non-compact Lie subgroup $H \subset G$ is typically not a direct sum of irreducible representations. Hence it is natural to consider H-periods, i.e, the H-equivariant continuous linear functionals $\mathrm{Hom}_H(\Pi \otimes \pi, \mathbb{C})$ for a representation π of H. The values of the linear functionals on certain vectors are related to the L-function of the product $\Pi \times \pi$, and if π is one-dimensional, to H-periods of Π. Of particular interest is the dimension (multiplicity) of the space of H-periods. Substantial progress concerning periods of irreducible representations of real Lie groups and in particular orthogonal groups was made in the last couple of years using harmonic analysis. This was the subject of the lecture by T. Kobayashi. In the p-adic case, C. Wan and Beuzart-Plessis discussed some multiplicity one results, obtained using a local trace formula. Murnaghan reported on the progress in constructing supercuspidal representations with nontrivial H-periods. In the special case of tempered representations Π, π of orthogonal or unitary groups, the Gan-Gross-Prasad period is attached to the pair (G, H), where $G = G_n \times G_{n+1}$ and $H = G_n$ is diagonally embedded, where G_n is either a unitary or an orthogonal group. The conjecture of Gan-Gross-Prasad asserts that the following two statements are equivalent: (i) some automorphic representation in the Vogan L-packet of Π has a non-vanishing H-equivariant linear functional; (ii) the central value $L(\frac{1}{2}, \Pi, R)$ does not vanish, where R is a certain representation of the L-group of G. These conjectures have analogues in the local and global cases. An example of the global conjecture was presented by Grobner. In all the cases discussed so far there is exactly one H-period, although there are known examples where multiplicity one fails. D. Prasad presented a conjecture relating, under same assumptions, the multiplicity of H-invariant linear functionals to base change for Langlands parameters of the representation. In another lecture a proof of a special case was presented by Beuzart-Plessis.

Asymptotics. Chenevier, Blomer, Finis, Brumley, and Marshall spoke about various problems concerning the asymptotic behavior of the cuspidal automorphic spectrum of a reductive group over a number field. One set of problems is related to counting automorphic forms, the Weyl law, the limit multiplicity problem, and the study of families. The basic tool to approach these problems is the Arthur–Selberg trace formula. Chenevier spoke about his work with Lannes that provides a complete list of all the unramified cuspidal algebraic automorphic representations of GL_n of motivic weight ≤ 22. He also explained his proof of a finiteness result for cuspidal algebraic automorphic representations of bounded level, motivic weight ≤ 23, and arbitrary n, that uses the explicit formula for the zeros of Rankin–Selberg L-functions and a positivity argument. To deal with the asymptotics of the spectral side in the context of asymptotics, one has to verify two basic properties of the intertwining operators, called (TWN) and (BD), which were formulated by Finis and Lapid. In previous work of Finis, Lapid and Müller it was shown that (TWN) and (BD) hold for SL_n and GL_n. Finis spoke about recent progress concerning the extension to other groups: inner forms of SL_n, classical groups, and partially for G_2. In analogy to counting rational points on algebraic varieties, Sarnak, Shin and Templier have posed the problem to count the number of cuspidal automorphic representations of GL_n of bounded analytic conductor. Brumley spoke about his recent work with Milicevic in which they succeeded to prove an asymptotic formula as $x \to \infty$ for the number of cuspidal automorphic representations with conductor $\leq x$, under a spherical assumption at archimedean places when $n \geq 3$. The proof uses the advances made in recent years to bound the non-discrete and non-central terms of the trace formula. The main term in the asymptotic recalls an analogous result, due to Schanuel, for the counting function for rational points on projective spaces of bounded height. Marshall spoke about lower bounds for Maass forms. This work was initiated by a theorem of Rudnick and Sarnak on lower bounds for Laplace eigenfunctions on compact hyperbolic 3-manifolds. On a negatively curved manifold, the geodesic flow is chaotic and based on the correspondence principle between classical and quantum systems, one expects this to be reflected in the asymptotic behavior of the Laplace eigenfunctions and the spectrum. Blomer spoke about joint work with X. Li and S.D. Miller, in which they develop a reciprocity formula for a spectral sum over central values of L-functions on $\mathrm{GL}_4 \times \mathrm{GL}_2$. As an application it follows that for any self-dual cuspidal automorphic representation Π of GL_4, which is unramified at all finite places, there exist infinitely many Maass forms π of GL_2 such that $L(\frac{1}{2}, \Pi \times \pi) \neq 0$.

Geometry. In the study of the trace formula, geometric questions arise that are of independent interest. Arthur gave a talk on Langlands proposal on beyond endoscopy that seeks to understand using the trace formula the expected behavior of L-functions at $s = 1$, predicted by functoriality. Shahidi explained some of the geometric constructions of Braverman–Kazhdan, and more recently Ngô, that are modeled on the expected properties of local L-functions for any representation of the L-group. He also explained the relationship with other constructions, such as the doubling method for classical groups. Getz described in his talk a global summation formula for the monoid that is attached to the $\mathrm{GL}_2 \times \mathrm{GL}_2$ Rankin–Selberg convolution. G. Zhang explained his work with Fedosova and Rowlett on formulas for the variation along the moduli of Riemann surfaces, of the values of the Selberg zeta function. He discussed examples and counterexamples of positivity properties, based on global geometric inequalities for the Ricci curvature. Y. Zhu presented his recent work on the cohomology of noncompact Shimura varieties associated with $SO(n, 2)$, extending earlier results by Morel for unitary and symplectic groups. He highlighted three ingredients: a comparison of archimedean quantities, proving cancellations in the boundary terms of the trace formula, and a precise comparison of transfer factors. J.-L. Waldspurger described his result towards the following conjecture: in the character expansion of an irreducible representation of a p-adic reductive group, there is a unique nilpotent orbit which contains all the maximal elements in the set of nilpotent orbits with nonzero coefficients. He proved the conjecture for representations of $SO(2n+1)$ which are Aubert involutions of tempered unipotent representations. X. He discussed an algebraic approach to the representation theory of p-adic groups, via establishing decomposition theorems for their cocenter, which is partly modeled on the rich geometry of affine Hecke algebras. Chaudouard explained how the study of the moduli space of Higgs bundles in the recent works of O. Schiffmann, and of Hauzel–Villegas, yields to a description of Arthur's local constants for the unipotent terms in the Lie algebra version of the trace formula.

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Workshop: Harmonic Analysis and the Trace Formula

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Abstracts

Beyond Endoscopy and elliptic terms

James Arthur

Beyond Endoscopy is the proposal [4] of Langlands for applying the stable trace formula to the general principle of functoriality. It will at best be a long term enterprise, calling for the ideas and efforts of many mathematicians. However, there seems also to be a good deal of hidden structure in the problem, which should lead to a number of well defined and perhaps accessible questions. In the lecture, we reviewed the main premises of Beyond Endoscopy. We then discussed the elliptic regular terms in the trace formula for $\mathrm{GL}(n+1)$. Our aim was to give some sense of the hidden structure within these often impenetrable objects.

We restrict our attention to the group

$$G = G(n) = \mathrm{GL}(n+1)$$

over $\mathbb{Q}$, with Langlands dual group equal to

$$\hat{G} = \hat{G}(n) = \mathrm{GL}(n+1, \mathbb{C}).$$

The stable trace formula then reduces to the ordinary trace formula. This amounts to an identity

$$I_{\mathrm{geom}}(f) = I_{\mathrm{spec}}(f), \qquad f \in C_c^\infty(G(\mathbb{A})), \tag{1}$$

between a geometric expansion on the left and a spectral expansion on the right.

While many of the more exotic terms on each side of (1) are likely to have important roles in Beyond Endoscopy, we consider only the approximate identity

$$I_{\mathrm{ell,reg}}(f) \sim I_2(f), \qquad f \in C_c^\infty(G(\mathbb{A})/Z_+), \tag{2}$$

of "primary terms." These are the regular elliptic orbital integrals

$$\begin{aligned} I_{\mathrm{ell,reg}}(f) &= \sum_\gamma \mathrm{vol}(\gamma)\,\mathrm{Orb}(\gamma, f) \\ &= \sum_{\gamma \in \Gamma_{\mathrm{ell,reg}}(G)} \mathrm{vol}(Z_+ G_\gamma(F) \setminus G_\gamma(\mathbb{A}) \int_{G_\gamma(\mathbb{A})\setminus G(\mathbb{A})} f(x^{-1}\gamma x)\, dx) \end{aligned}$$

from the geometric side, and the automorphic characters

$$\begin{aligned} I_2(f) &= \sum_\pi \mathrm{tr}(\pi(f)) \\ &= \sum_{\pi \in \Pi_2(G, Z_+)} \mathrm{tr}(\int_{G(\mathbb{A})/Z_+} f(x)\pi(x)\, dx) \end{aligned}$$

from the spectral side that occur in the discrete spectrum, with Z_+ being the central subgroup

$$Z_+ = Z_G(\mathbb{R})^0 \subset G^+(\mathbb{R}) = \{x_\infty \in G(\mathbb{R}) \ : \ \det(x_\infty) > 0\} \subset G(\mathbb{A})$$

of $G(\mathbb{R})$. These terms are quite challenging enough for now!

For simplicity, we also restrict the test function f. We set

$$f = f_\infty f^\infty = f_\infty f_p^k f^{\infty,p},$$

for a fixed prime p and nonnegative integer k. The archimedean factor f_∞ is in $C_c^\infty(G^+(\mathbb{R})/Z_+)$, while $f^{\infty,p}$ is the characteristic function

$$\chi(\prod_{q\neq p} G(\mathbb{Z}_p))$$

on $G(\mathbb{A}^{\infty,p})$, and f_p^k is the function on $G(\mathbb{Z}_p)$ given by the product

$$p^{-k/(n+1)} \cdot \chi(\{x_p \in \mathfrak{g}(n+1, \mathbb{Q}_p) \ : \ |\det x_p|_p = p^{-k}\}),$$

for the Lie algebra $\mathfrak{g} = \mathrm{gl}(n+1)$ of G. With this function, it is easy to see that the orbital integral $\mathrm{Orb}(\gamma, f)$ in (2) vanishes unless the characteristic polynomial of γ has integral coefficients. More precisely, we can assume that γ is such that

$$\det(\lambda I - \gamma) = \lambda^{n+1} - a_1\lambda + \cdots + (-1)^n a_n \lambda + (-1)^{n+1} a_{n+1} = p_b(\lambda),$$

for $b = (a_1, \ldots, a_n)$ in $\mathbb{Z}^n$, and $a_{n+1} = \det\gamma = p^k$. We then have a bijection $\gamma \to b$ onto the set of $b \in \mathbb{Z}^n$ such that $p_b(\lambda)$ is irreducible.

The mapping $\gamma \to b$ gives a parametrization

$$I_{\mathrm{ell,reg}}(b, f) = \mathrm{vol}(\gamma)\,\mathrm{Orb}(\gamma, f)$$

of the elliptic regular terms by (certain) points $b \in \mathbb{Z}^n$. This parametrization, which was a foundation of Ngo's proof of the fundamental lemma, represents a real change of outlook. It imposes a new structure on the elliptic regular terms in the trace formula. With its focus on irreducible, monic, integral polynomials rather than the roots of these polynomials, it hints at a new role for the theory of equations, which emphasizes Galois resolvents and the explicit determination of Galois groups.

In general, there is typically a parallel structure between geometric and spectral terms in the trace formula. The Galois theoretic properties of the elliptic regular terms on the geometric side would now be analogous to the functorial properties of the discrete, L^2-terms on the spectral side. Whether this is more than just an analogy might depend on the answer to the following question, posed in [3].

Is it possible to apply the Poisson summation formula to the sum

$$I_{\mathrm{ell,reg}}(f) = \sum_{b\in\mathbb{Z}^n} I_{\mathrm{ell,reg}}(b, f) = \sum_{\gamma} \mathrm{vol}(\gamma)\,\mathrm{Orb}(\gamma, f), \quad \gamma \to b,$$

over b in (a subset of) $\mathbb{Z}^n$? In other words, can we write

$$I_{\mathrm{ell,reg}}(f) = \sum_{\xi\in\mathbb{Z}^n} \hat{I}_{\mathrm{ell,reg}}(\xi, f)? \tag{3}$$

The question is very subtle, and clearly fails if it is interpreted literally. For example, the sum over b is taken only over a proper subset of $\mathbb{Z}^n$. Nevertheless, Altug was able to solve the problem in the case $n = 2$ of GL(2). He overcame the analytic difficulties by expanding each $I_{\mathrm{ell,reg}}(b, f)$ into a double sum over integers

l and f, and then taking the original sum over b inside these two supplementary sums. (See [1, Theorem 4.2].) In the lecture, we did not have the time to discuss the difficulties that remain in the case $n > 1$ of higher rank. There has been some progress, but there are also problems that are still to be solved.

In any case, if (3) can be established in general, the summation index ξ could be regarded as an additive spectral variable. It would also index its own monic, integral polynomial $p_\xi(\lambda)$ of degree n. Might there be some concrete relations between the Galois theoretic properties of the polynomials $p_\xi(\lambda)$ on the geometric side and the functorial properties of the representations π on the spectral side. In particular is there any relation between those ξ such that the Galois group of $p_\xi(\lambda)$ is a *proper* subgroup of the symmetric group $S = S_{n+1}$ and the cuspidal automorphic representations π that are *proper* functorial transfers to the general linear group $G = \mathrm{GL}(n+1)$? I have no evidence at all for this, and it is safe to say that any sort of answer lies well in the future. Nevertheless, I find the question intriguing, and worthy of consideration.

The more immediate reason for seeking a Poisson formula (3) is that it seems to be amenable to a geometric description of the spectral contribution of the noncuspidal (residual) representations on the spectral side of (2). In [3] it was conjectured that the trivial one-dimensional representation π of $G(\mathbb{A})$ should correspond to the additive spectral parameter $\xi = 0$ in (3). This problem was solved, again for the case $n = 1$ of GL(2), by Altug [1, Theorem 6.1]. At the end of the lecture, we included a few words on a conjectured extension [2, Section 3] of this result to higher rank that would account for all residual representations π.

References

[1] A. Altuğ. Beyond Endoscopy via the trace formula – I: Poisson summation and isolation of special representations. *Compos. Math.*, 151(10):1791–1820, 2015.

[2] J. Arthur. Functoriality and the Trace Formula. To appear in *Geometric aspects of the Trace Formula*, Simons Symposia Series, Springer.

[3] E. Frenkel, R. Langlands, and B.C. Ngo. Formule des traces et fonctorialité: Le début d'un programme. *Ann. Sci. Math. Québec*, 34:199–243, 2010.

[4] R. Langlands. Beyond Endoscopy. In *Contributions to Automorphic Forms, Geometry, and Number Theory*, pages 611–697. Johns Hopkins Univ. Press, Baltimore, MD, 2004.

Triality and Functoriality

Wee Teck Gan

It is a basic principle in mathematics that an object with more symmetries is more beautiful, or at least easier to study, than a generic one with fewer symmetries. In this talk, we consider this principle in the following situation. Consider the series D_n of simple Lie groups or Lie algebras, so that for example, one may be looking at the linear algebraic groups SO_{2n} (over some field F) and its Lie algebra $\mathfrak{so}_{2n}$. By Lie theory, the outer automorphism group $\mathrm{Out}(D_n)$ of $\mathfrak{so}_{2n}$ is equal to the group of symmetries of its Dynkin diagram. Thus one finds that

$$\mathrm{Out}(D_n) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{if } n \neq 4; \\ \text{the symmetric group } S_3, & \text{if } n = 4. \end{cases}$$

Thus, the Lie algebra of type D_4 has more symmetries than the generic Lie algebra of type D_n. In particular, it has an outer automorphism θ of order 3 and this phenomenon is known as triality. As such, if the above principle were to be true, one would expect that the representation theory of D_4 to be nicer, or more accessible, than that for general D_n. Is this really the case?

Let us be more precise. If one considers the special orthogonal group SO_{2n}, then its outer automorphism group is indeed $\mathbb{Z}/2\mathbb{Z}$ for all n, with an outer automorphism realised by an element in $O_{2n} \setminus SO_{2n}$. This is because the triality automorphism θ of $\mathfrak{so}_8$ cannot be realised on SO_8, but only on the simply-connected or adjoint form of the group, i.e. the group $Spin_8$ or $PGSO_8$ respectively. The center Z_{Spin_8} of $Spin_8$ is the finite group scheme $\mu_2 \times \mu_2$ which has 3 nontrivial subgroups of order 2. The triality automorphism preserves the center, permuting the 3 subgroups of order 2, so that it descends to an automorphism of the adjoint group $PGSO_8$. This implies that one has 3 non-conjugate maps

$$\iota_j : Spin_8 \longrightarrow Spin_8/\mu_2 \cong SO_8,$$

which are cyclically permuted by the triality automorphism, and together gives an embedding

$$\iota : Spin_8 \hookrightarrow SO_8^3.$$

Following [3], we describe a construction of the group $Spin_8$ as a subgroup of SO_8^3, equipped with an outer automorphism of order 3 which is simply the restriction of the cyclic permutation on SO_8^3. This construction is based on the theory of symmetric composition algebras. It gives a realisation of the three 8-dimensional irreducible fundamental representations of $Spin_8$. Likewise, there are 3 non-conjugate maps

$$f_j : SO_8 \longrightarrow PGSO_8$$

which are cyclically permuted by the triality automorphism. The existence of these 3 different maps f_j is a manifestation of the existence of the triality automorphism.

In studying the smooth representation theory of $GSO_{2n}(F)$ (where F is a local field here), it is natural to restrict the representations of $GSO_{2n}(F)$ to the natural subgroup $SO_{2n}(F)$ and then exploit the recent results of Arthur [1] on the local Langlands correspondence for $SO_{2n}(F)$. Such an approach was taken by Bin Xu in his Toronto PhD thesis [5]. Using the stable trace formula, he was able to construct the local L-packets for $PGSO_{2n}(F)$ and establish the endoscopic character identities for these L-packets. However, he was not able to fully associate L-parameters to his L-packets; indeed, it is only apparent how to assign L-parameters up to twists by quadratic characters. When $n = 4$ however, we have

3 different maps (instead of just one)

$$f_j : SO_8(F) \longrightarrow SO_8(F)/\mu_2(F) \hookrightarrow PGSO_8(F),$$

with finite cokernel $\cong F^\times/F^{\times 2}$ and such that the images of any two f_j's generate $PGSO_8(F)$. Thus, one is in a much better position in using this approach to understand the representation theory of $PGSO_8(F)$, as the pullback by each f_j will yield slightly different information. Unfortunately, despite this, we are still unable to deduce the full local Langlands correspondence for $PGSO_8(F)$ from that of $SO_8(F)$. However, in the global setting (i.e. over a number field), we are able to use this idea to define a notion of global A-parameters for $PGSO_8$. Hopefully, this notion is sufficient to parametrize the near equivalence classes in the automorphic discrete spectrum of $PGSO_8(\mathbb{A})$ (at least the part which is invariant under the triality automorphism).

Looking beyond these less-than-definitive results, we are able to exploit the principle of triality to obtain some interesting results towards Langlands functoriality. The basic idea is the following. Suppose one has constructed a functorial lifting of automorphic representations with respect to a morphism of dual groups

$$\iota : H^\vee \longrightarrow PGSO_8^\vee = Spin_8(\mathbb{C}) \xrightarrow{f_1^\vee} SO_8(\mathbb{C}).$$

Now one may compose ι with the triality automorphism θ and obtain another map

$$H^\vee \xrightarrow{\iota} Spin_8(\mathbb{C}) \xrightarrow{\theta} Spin_8(\mathbb{C}) \xrightarrow{f_1^\vee} SO_8(\mathbb{C}).$$

This composite map may give a drastically different instance of Langlands functoriality from H to SO_8 since $\theta \circ \iota$ and ι may not be conjugate in $Spin_8(\mathbb{C})$. The map $f_1^\vee \circ \theta \circ \iota$ can also be simply described using

$$H^\vee \xrightarrow{\iota} Spin_8(\mathbb{C}) \xrightarrow{f_2^\vee} SO_8(\mathbb{C}).$$

The existence of the triality automorphism essentially gives one this new functorial lifting with minimal efforts. This idea has already been observed and exploited in the monograph [2] of Chenevier and Lannes. Indeed, much of the material in this talk will eventually appear in a joint paper with Chenevier.

Let us illustrate this using an example. Consider the split group $PGSp_6$ whose dual group is $Spin_7(\mathbb{C})$. One has the standard representation

$$\mathrm{std} : Spin_7(\mathbb{C}) \longrightarrow SO_7(\mathbb{C})$$

and the associated functorial lifting is simply the restriction of automorphic forms

$$\mathcal{A}(PGSp_6) \xrightarrow{\text{rest.}} \mathcal{A}(Sp_6).$$

On the other hand, one has the faithful Spin representation

$$\mathrm{spin} : Spin_7(\mathbb{C}) \longrightarrow SO_8(\mathbb{C}) \longrightarrow GL_8(\mathbb{C}),$$

so one expects a functorial lifting of automorphic forms

$$\mathcal{A}(PGSp_6) \longrightarrow \mathcal{A}(SO_8) \longrightarrow \mathcal{A}(GL_8).$$

We shall explain how theta correspondence and triality allow one to establish this Spin lifting.

The point is that there are 3 conjugacy classes of embeddings

$$Spin_7(\mathbb{C}) \longrightarrow Spin_8(\mathbb{C})$$

permuted by triality. One may fix such an embedding so that the standard representation of $Spin_7(\mathbb{C})$ is compatible with the map $f_1^\vee$, i.e the center of $Spin_7(\mathbb{C})$ is equal to $\mathrm{Ker}(f_1^\vee)$, so that there is a commutative diagram:

$$\begin{tikzcd} Spin_7(\mathbb{C}) \arrow[r, "\iota"] \arrow[d, "\mathrm{std}"'] & Spin_8(\mathbb{C}) \arrow[d, "\mathrm{f}_1^\vee"] \\ SO_7(\mathbb{C}) \arrow[r] & SO_8(\mathbb{C}) \end{tikzcd}$$

Then the Spin representation of $Spin_7(\mathbb{C})$ is given by the composite map

$$Spin_7(\mathbb{C}) \xrightarrow{\ \iota\ } Spin_8(\mathbb{C}) \xrightarrow{f_2^\vee} SO_8(\mathbb{C}) \xrightarrow{\mathrm{std}} GL_8(\mathbb{C}).$$

By the similitude theta correspondence, one obtains a theta lifting from cuspidal representations from $PGSp_6$ to $PGSO_8$ which realises the functoriality associated to ι. On the other hand, let $\mathcal{A}$ denote the lifting from a classical group to the appropriate GL_N given by Arthur. Then, if $\pi \subset \mathcal{A}_{cusp}(PGSp_6)$ is such that its global theta lift $\Theta(\pi)$ to $PGSO_8$ is nonzero, the above discussion shows that

$$\mathcal{A}(f_1^*(\Theta(\pi))) = 1 \oplus \mathcal{A}(\pi|_{Sp_6})$$

on GL_8: this is not a very exciting functorial lifting. However, using f_2 instead of f_1, one sees that $\mathcal{A}(f_2^*(\Theta(\pi)))$ is the Spin lifting of π, i.e. it is functorial with respect to the map $\mathrm{std} \circ f_2^\vee \circ \iota$. In particular, we have the following theorem with minimal work:

Theorem 1. *Let π be a globally generic cuspidal representation of $PGSp_6$. Then the weak Spin lifting of π to GL_8 exists.*

Finally, we discuss the twisted trace formula associated to $(PGSO_8, \theta)$. The twisted endoscopic groups are the groups G_2, SL_3 and SO_4. For the endoscopic group $H = SL_3$, the map $H^\vee = PGL_3(\mathbb{C}) \longrightarrow Spin_8(\mathbb{C})$ is the unique lifting of the adjoint representation of $PGL_3(\mathbb{C})$. Thus, the stable twisted trace formula of Moeglin and Waldspurger [4], coupled with the transfer of Arthur [1] from SO_8 to GL_8, gives readily the following theorem:

Theorem 2. *The weak adjoint lifting from GL_3 to GL_8 exists.*

We hope that a detailed study of this stable twisted trace formula will yield the Arthur conjecture on the automorphic discrete spectrum of G_2. This is an ongoing project with Zhifeng Peng.

References

[1] J. Arthur, *The endoscopic classification of representations: orthogonal and symplectic groups.* American Mathematical Society Colloquium Publications **61**, American Mathematical Society, Providence, RI, 2013.

[2] G. Chenevier and J. Lannes, *Formes automorphes et réseaux unimodulaires pairs*, preprint (2015), available at https://arxiv.org/pdf/1409.7616.pdf.

[3] M. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol, *The Book of Involutions*, AMS Colloquium Publications, Vol. 44 (1998).

[4] C. Moelgin and J.-L. Waldspurger, *Stabilisation de la formule des traces tordue*, preprint 2015, available at https://webusers.imj-prg.fr/ colette.moeglin/global.pdf.

[5] B. Xu, *L-packets of quasisplit $GSp(2n)$ and $GO(2n)$*, to appear in Math. Annalen (2017).

On level 1 cuspidal algebraic automorphic representations of motivic weight ≤ 23

Gaëtan Chenevier

Let π be a cuspidal automorphic representation of GL_n over $\mathbb{Q}$. We say that π is *algebraic* if the infinitesimal character of its Archimedean component, that we may and shall view as a semisimple conjugacy class in $\mathrm{M}_n(\mathbb{C})$, has integer eigenvalues $k_1 \leq k_2 \leq \cdots \leq k_n$. Those integers are then called the *weights* of π ; up to twisting π by $|\det|^{\mathbb{Z}}$ we may and shall always assume $k_1 = 0$, in which case the biggest weight k_n is also called the *motivic weight* of π, and denoted by $w(\pi)$.

The main interest in those algebraic π's is that they are exactly the ones to which we expect (and know in many cases how) to attach geometric irreducible and pure ℓ-adic representations ρ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (or motives), the weights of π corresponding the Hodge-Tate numbers of ρ, and $w(\pi)$ to Deligne's weight of ρ.

For example, π is of "Artin type" if, and only if, it is algebraic of motivic weight 0. Moreover, if π is generated by a cuspidal elliptic eigenform of weight $k \geq 1$ (so $n = 2$), then π is algebraic with $k_2 = w(\pi) = k - 1$. By Harish-Chandra, there are only finitely many cuspidal algebraic π of given dimension, conductor and weights, but their exact number is unknown already for $n = 3$ and conductor 1.

The following theorem is a joint work with Jean Lannes [CL, Chap. IX Thm. 3.3].

Theorem 1. (Ch.-Lannes) *Let $n \geq 1$ and π be a cuspidal algebraic automorphic representation of GL_n over $\mathbb{Q}$ with $w(\pi) \leq 22$, and such that π_p is unramified for each prime p. Then either :*

(i) *$n = 1$ and π is the trivial representation,*

(ii) *$n = 2$, $w(\pi) \in \{11, 15, 17, 19, 21\}$, and π is generated by the unique normalized cusp form for $\mathrm{SL}_2(\mathbb{Z})$ of weight $w(\pi) + 1$,*

(iii) *$n = 3$, $w(\pi) = 22$ and π is Gelbart-Jacquet's $\mathrm{Sym}^2\Delta$ (suggestive notation),*

(iv) *or* $n = 4$, $w(\pi) \in \{19, 21\}$ *and* π *is the Arthur transfer to* GL_4 *of the cuspidal automorphic representation of* $\mathrm{PGSp}_4(= \mathrm{SO}(3,2))$ *generated by the one-dimensional space of vector-valued Siegel modular cuspforms for* $\mathrm{Sp}_4(\mathbb{Z})$ *of weight* $\mathrm{Sym}^j \otimes \det^k$, *with* $(j,k) = (6,8), (4,10), (12,6)$ *or* $(8,8)$.

In particular, there are exactly 11 *such* π*'s, and none if* $n > 4$.

This theorem was known at least to Serre and Mestre for $w(\pi) < 11$, in which case it implies $\pi = 1$ [Mes, §III Rem. 1]. Observe that modulo the Artin-Langlands reciprocity conjecture, the case $w(\pi) = 0$ is just Minkowski's theorem: any number field different from $\mathbb{Q}$ has a ramified prime. Modulo the Fontaine-Mazur-Langlands conjecture, the case $w(\pi) = 1$ also contains the Abrashkin-Fontaine theorem : there is no nonzero abelian variety over $\mathbb{Z}$. As far as we know, the theorem is already new for $w(\pi) = 11$. See forthcoming work of Taïbi and the author for a version of this theorem for the motivic weights $w = 23$ and 24.

The theorem above has several interesting consequences. It gives for example some hints on the mysterious motive of the moduli spaces $\overline{\mathcal{M}_{g,n}}$ of n-pointed stable curves of genus g for $3g - 3 + n \leq 22$, which confirm numerical computations of Bergström, Faber & Van der Geer. More concretely, when combined with Arthur's classification of automorphic representations of classical groups, it also has the following applications discovered in [CL] :

(a) Niemeier lattices (that application was the main motivation of the theorem and really requires the motivic weight $w(\pi)$ to go up to 22) : new proof that the set X_{24} of isometry classes of Niemeier lattices has 24 elements, simple solution of the *Kneser neighbors problem* for Niemeier lattices, a proof that the genus 12 Siegel theta series map $\mathbb{C}[\mathrm{X}_{24}] \to \mathrm{M}_{12}(\mathrm{Sp}_{24}(\mathbb{Z}))$ is an isomorphism (Eichler's basis problem), and a proof of the conjecture of Nebe-Venkov on higher genus theta series of Niemeier lattices.

(b) Siegel modular forms : computation of $\dim \mathrm{S}_k(\mathrm{Sp}_{2g}(\mathbb{Z}))$ for all $k \leq 12$ and all genus g, except for $(k,g) = (12, 24)$. For instance we prove the following table:

g	1	2	3	4	5	6	7	8	9	10	11	12	> 12
$\dim \mathrm{S}_{12}(\mathrm{Sp}_{2g}(\mathbb{Z}))$	1	1	1	2	2	3	3	4	2	2	1	1	0

(c) Harder congruences : proof of the original "mod 41 example" or Harder's congruence, which concerns the π of dimension 4 with $(j,k) = (4,10)$.

The finiteness of the number of π in Theorem 1 above is already much surprising. It turns out to be special case of a quite more general result, which is our second theorem (to appear in a forthcoming paper of the author) :

Theorem 2. (Ch.) *Let* $N \geq 1$. *There are only finitely many cuspidal automorphic representations* π *of* GL_n *over* $\mathbb{Q}$ *(with* n *varying), whose Artin conductor is* N *and which are algebraic of motivic weight* ≤ 23.

Modulo GRH, this is also true for $w = 24$. We are not aware of any reason why the same result should hold for all w.

Let us now say a word about the proofs of these theorems. We use and develop an analytic argument in the lead of previous works of Stark, Odlyzko, Serre, Mestre, Fermigier and Miller. Roughly, in order to show that a π with given π_∞ does not exist, we show that the Rankin-Selberg L-function $\mathrm{L}(s, \pi \times \pi')$ does not exist for a suitable choice of π' (often $\pi^\vee$ or a known π'). The contradiction comes from the study of the explicit formula "à la Riemann-Weil" applied to $\mathrm{L}(s, \pi \times \pi')$ and to a suitable test function. The analytic properties of those L-functions (such as the meromorphic continuation to $\mathbb{C}$, the functional equation, the determination of the poles, and its boundedness in vertical strips away from the poles) which have been proved by Gelbart, Jacquet, Shalika and Shahidi, play a crucial role. Let us now explain briefly the proof of the finiteness statement (Theorem 2).

We denote by K_∞ the Grothendieck ring of finite dimensional complex representations of the Weil group $\mathrm{W}_\mathbb{R}$ of $\mathbb{R}$ which are trivial on the central subgroup $\mathbb{R}_{>0}$. As an abelian group, K_∞ is freely generated by 1 and by the 2-dimensional representations I_w of $\mathrm{W}_\mathbb{R}$, with $w \geq 0$ an integer, induced from the character $z \mapsto (z/|z|)^w$ of the index 2 subgroup $\mathbb{C}^\times$ of $\mathrm{W}_\mathbb{R}$. If π is a cuspidal algebraic automorphic representation of GL_n of motivic weight w, then the Langlands parameter of $\pi_\infty \otimes |\det|^{-w/2}$ lies in the subgroup $\mathrm{K}_\infty^{\leq w} \subset \mathrm{K}_\infty$ generated by the I_u with $u \leq w$ and $u \equiv w \bmod 2$, and by 1 if w is even.

Let π be as in the statement of Theorem 2. Choose a function $F : \mathbb{R} \to \mathbb{R}_{\geq 0}$ which is even, compactly supported, of class $\mathcal{C}^2$, and whose Fourier transform $\widehat{F}(s)$ has a nonnegative real part on the complex strip $|\mathrm{Im}\, s| \leq \frac{1}{4\pi}$. The explicit formula applied to $\mathrm{L}(s, \pi \times \pi^\vee)$, a Dirichlet series with nonnegative coefficients, shows

$$(1) \qquad \mathrm{J}_F(V \otimes V) \leq \widehat{F}(\frac{i}{4\pi}) + F(0)\, \dim V\, \log N,$$

where V is the Langlands parameter of $\pi_\infty \otimes |\det|^{-w(\pi)/2}$ and $\mathrm{J}_F : \mathrm{K}_\infty \to \mathbb{R}$ is a certain explicit linear form depending on F (see [CL, Chap. IX Proposition-Def. 3.7]). We have also used that the Artin conductor of $\pi \times \pi^\vee$ is bounded above by $N^{2 \dim V}$.

But $V \mapsto \mathrm{J}_F(V \otimes V)$ is a real-valued quadratic form on the free abelian group $\mathrm{K}_\infty^{\leq w}$. An important observation made in [CL, Chap. IX §3] is that for $w \leq 23$, and for a suitable (explicit) choice of test function F, this quadratic form is positive definite. Theorem 2 follows then from (1), as the set of points of an Euclidean lattice whose norm is less that a given affine function is necessarily finite.

This proof is effective, and applied in the special case $N = 1$ it leads to a finite (but still big) list of possiblilities for π_∞. This is just the starting point of the proof of Theorem 1 given in [CL], which is rather long and delicate, and requires a number of new ingredients (and many numerical computations). In particular, we analyse carefully the explicit formula applied to various $\mathrm{L}(s, \pi \times \pi')$, where the

π' are specific known automorphic representations. See [CL, Chap. IX §3] for all the details.

References

[CL] G. Chenevier & J. Lannes, *Automorphic forms and even unimodular lattices*, to appear in Ergebnisse der Math. und ihrer Grenzgebiete.

[Mes] J.-F. Mestre, *Formules explicites et minorations de conducteurs des variétés algébriques*, Compositio Math. 58, 209–232 (1986).

Spectral reciprocity and non-vanishing of automorphic L-functions

Valentin Blomer

(joint work with X. Li, S. D. Miller)

Let $1 \leq m < n$ be two positive integers. Given a cuspidal automorphic representation Π on $\mathrm{GL}(n, \mathbb{Q})\backslash\mathrm{GL}(n, \mathbb{A})$, do there exist (infinitely many?) self-dual cuspidal automorphic representations π on $\mathrm{GL}(m, \mathbb{Q})\backslash\mathrm{GL}(m, \mathbb{A})$ such that $L(\frac{1}{2}, \Pi\times\pi) \neq 0$?

Analytic number theory approaches this problem by trying to prove an asymptotic formula (or a lower bound) for

$$\sum_{\mathrm{cond}(\pi)\leq T} L(1/2, \Pi \times \pi) \tag{1}$$

as $T \to \infty$. The case $m = 1$ (i.e., π corresponds to a quadratic Dirichlet character) is one of the prime applications of the theory of multiple Dirichlet series ([HK]). In the case $m = 2$, $n = 3$, an affirmative answer follows from work of X. Li [Li, Theorem 1.1].

For $n \geq 4$ and any value of m the problem becomes very hard. The most direct approach – asymptotically evaluating (1) – fails, at least with currently available tools, and quickly leads to a "deadlock". Therefore we introduce a different path which we now proceed to describe.

Fix a cuspidal automorphic representation Π on $\mathrm{GL}(4, \mathbb{Q})\backslash\mathrm{GL}(4, \mathbb{A})$, which we assume to be unramified at all finite places. Given a sufficiently nice test function h, we define the following spectral mean value

$$\mathcal{M}_\Pi(h) := \sum_\pi \epsilon_\pi \frac{L(1/2, \Pi \times \pi)}{L(1, \mathrm{Sym}^2\pi)} h(t_\pi) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{L(1/2+it, \Pi)L(1/2-it, \Pi)}{|\zeta(1+2it)|^2} h(t)dt,$$

where π runs over all representations corresponding to Maaß forms for $\mathrm{SL}_2(\mathbb{Z})$, and the parity $\epsilon_\pi \in \{\pm 1\}$ is defined as the root number of the corresponding L-function $L(s, \pi)$. Our first result is a spectral reciprocity identity for $\mathcal{M}_\Pi(h)$ that we state in slightly simplified form. For exact details and hypotheses see [BLM, Theorem 3].

Theorem 1. *We have*

$$\mathcal{M}_{\Pi}(h) = \mathcal{M}_{\tilde{\Pi}}(\tilde{h}) + \text{ two similar terms}$$

where

$$\tilde{h}(r) := \int_{-\infty}^{\infty} \mathcal{K}(t,r)h(t)\,dt$$

for a certain integral kernel $\mathcal{K}(t,r)$ *that can be expressed in terms of hypergeometric* ${}_4F_3$ *functions.*

This formula is motivated by earlier work of Kuznetsov and Motohashi [Mo] who considered the case when Π is replaced by a minimal parabolic Eisenstein series.

A careful analysis shows that for

$$h_T(t) := e^{-(t/T)^2} P_T(t), \quad \text{where} \quad P_T(t) := \left(\prod_{n=1}^{50} t^2 + \left(\frac{2n-1}{2}\right)^2\right)^2 T^{-200}$$

one has

$$\mathcal{M}(h_T) \ll T. \tag{2}$$

On the other hand, lower bound techniques of Rudnick and Soundararajan [RS] yield

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|L(1/2+it,\Pi)|^2}{|\zeta(1+2it)|^2} h_T(t)dt \gg T\log T. \tag{3}$$

Hence if Π is self-dual so that

$$L(\tfrac{1}{2}+it,\Pi)L(\tfrac{1}{2}-it,\Pi) = L(\tfrac{1}{2}+it,\Pi)L(\tfrac{1}{2}-it,\widetilde{\Pi}) = |L(\tfrac{1}{2}+it,\Pi)|^2,$$

we can combine (2) and (3) and return to the nonvanishing question raised at the beginning. This gives our second result:

Theorem 2. *Let* Π *be a cuspidal automorphic representation on* $\mathrm{GL}(4,\mathbb{Q})\backslash\mathrm{GL}(4,\mathbb{A})$ *which is self-dual and unramified at all finite places. Then there exist infinitely many cuspidal automorphic representations* π *associated to Maaß cusp forms for* $\mathrm{SL}(2,\mathbb{Z})$ *such that* $L(\frac{1}{2},\Pi\times\pi)\neq 0$.

The proof of (2) is technically challenging. It uses the GL(4) Voronoi summation formula to manipulate a sum containing the Hecke eigenvalues of Π and Kloosterman sums. It is absolutely crucial to employ the full power of the GL(4) Hecke algebra and use a symmetric version of this formula which includes on both sides the precise Kloosterman sums arising from the Kuznetsov formula. This is captured in the following result (cf. [MZ] and [BLM, Theorem 4] for further details). Let $a_\Pi(n_1,n_2,n_3)$ denote the abelian coefficients of Π normalized such that $a_\Pi(1,1,1)=1$. For $\epsilon\in\mathbb{Z}/2\mathbb{Z}$ denote

$$\mathcal{V}_{\epsilon,\Pi}(s,z) := \sum_{n\in\mathbb{Z}\backslash\{0\}} \sum_{m,c>0} \frac{\mathrm{sgn}(n)^\epsilon a_\Pi(n,m,1)}{(|n|m^2)^z} \frac{S(n,1,c)}{c}\left(\frac{|n|}{c^2}\right)^{-s}.$$

The multiple sum is absolutely convergent in the range $\Re s < -1/4$, $\Re(s+z) > 1$.

Theorem 3. *The function $\mathcal{V}_{\epsilon,\Pi}(s,z)$ has holomorphic continuation to the region*

$$\{(s,z)\in\mathbb{C}^2 \mid \Re(s+2z)>5/4, \Re z>5/4, \text{ and } \Re s<-1/4\}$$

and satisfies the functional equation

$$\mathcal{V}_{\epsilon,\Pi}(s,z)=\mathcal{G}_{\epsilon,\Pi}(1-s-z)\mathcal{V}_{\epsilon,\tilde{\Pi}}(1-s-2z,z),$$

for a certain explicit expression in gamma and trigonometric functions associated to the archimedean representation parameter of Π.

References

[BLM] V. Blomer, X. Li, S. Miller, *A Spectral reciprocity formula and non-vanishing for L-functions on* GL(4) × GL(2), `arXiv:1705.04344`

[HK] J. Hoffstein, A. Kontorovich, *The first non-vanishing quadratic twist of an automorphic L-series*, `arXiv:1008.0839`

[Li] X. Li, *The central value of the Rankin-Selberg L-functions*, Geom. Funct. Anal. **18** (2009), 1660-1695.

[MZ] S. Miller, F. Zhou, *The balanced Voronoi formulas for* GL(n), `arXiv:1612.04886`

[Mo] Y. Motohashi, *A functional equation for the spectral fourth moment of modular Hecke L-functions*, Proceedings of the Session in Analytic Number Theory and Diophantine Equations, 19 pp., Bonner Math. Schriften, **360**, Univ. Bonn, Bonn, 2003.

[RS] Z. Rudnick, K. Soundararajan, *Lower bounds for moments of L-functions*, Proc. Nat. Acad. Sci. **102** (2005), 6837- 6838.

Limit multiplicities and analytic properties of intertwining operators

Tobias Finis

(joint work with Erez Lapid)

In the recent papers [8, 6], a conditional solution to the limit multiplicity problem for the collection of all congruence subgroups of a fixed non-cocompact arithmetic lattice was obtained. To control the contribution of the continuous spectrum, two conditions on the analytic behavior of intertwining operators were imposed: properties (TWN) (or its variant (TWN+)) and (BD). In [8], these properties were verified for the groups GL(n) and SL(n). The talk reported on recent progress for inner forms of these groups, split groups of rank two and quasi-split classical groups [4, 5]. The results might also be relevant to the study of other asymptotic problems with the trace formula.

In [6, Definition 1.2] we defined the limit multiplicity property for a family $\mathcal{K}$ of open compact subgroups K of $G(\mathbb{A}^S)$, where G is a reductive group defined over a number field k and S a finite set of places of k, including the archimedean places. Loosely speaking, this property concerns the convergence of the discrete spectral measures on $G(k_S)^1$ associated to the members of $\mathcal{K}$ to the Plancherel measure of this group, with the necessary modification in the case of a non-trivial center. We also refer to [ibid., Definition 1.3] for the definition of a non-degenerate family of open compact subgroups. Let T be a finite set of non-archimedean places of k that is disjoint to S. We say that a family $\mathcal{K}$ of open compact subgroups of $G(\mathbb{A}^S)$ has *bounded level* at T, if for every $v\in T$ there exists an open compact subgroup L_v

of $G(k_v)$ with $L_v \subset K$ for all $K \in \mathcal{K}$. A trivial modification of the argument of [6] yields then the following result (properties (TWN+) and (BD) will be explained below).

Theorem 1. *Let G be a reductive group defined over a number field k. Suppose that G satisfies property (TWN+) and property (BD) with respect to the set $S_{\mathrm{fin}} - S_0$. Let S be a finite set of places of k, including the archimedean places, and $\mathbf{K}_0^S$ an open compact subgroup of $G(\mathbb{A}^S)$. Then limit multiplicity holds for any non-degenerate family $\mathcal{K}$ of open subgroups of $\mathbf{K}_0^S$ that has bounded level at $S_0 - S$.*

In the case where G is in addition k-simple and simply connected, and $G(k_S)$ is not compact, the strong approximation property allows one to reformulate this result in terms of the discrete spectral measures of the congruence subgroups of the arithmetic lattice $G(\mathbb{A}^S) \cap \mathbf{K}_0^S$ [ibid., Corollary 1.5].

We now explain the definition of property (TWN+). Let M be a proper Levi subgroup of G (defined over k). For each pair $P, Q \in \mathcal{P}(M)$ there is a global intertwining operator

$$M_{Q|P}(\lambda) : \mathcal{A}^2(P) \to \mathcal{A}^2(Q),$$

meromorphic in $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$, where $\mathcal{A}^2(P)$ denotes the space of $\mathbf{K}$- and $\mathfrak{z}$-finite vectors in the induced representation $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}\left(L^2_{\mathrm{disc}}(A_M M(F)\backslash M(\mathbb{A})) \otimes e^{\langle\lambda, H_M(\cdot)\rangle}\right)$, and $\mathcal{A}^2(Q)$ the corresponding space for the group Q.

The study of the operators $M_{Q|P}(\lambda)$ reduces to the case where the parabolic subgroups P and Q are adjacent along a reduced root $\alpha \in \Sigma_M$. Property (TWN+) pertains to the corresponding normalizing factors $n_\alpha(\pi, s)$ introduced by Langlands and Arthur [1]. They are meromorphic functions of a complex variable s depending on $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))$ that are closely connected to certain automorphic L-functions studied by Langlands and Shahidi. They depend only on α and not on the parabolic subgroups $P|^\alpha Q$.

Let U_α be the unipotent subgroup of G corresponding to α. Let $\hat{M}_\alpha$ be the group generated by $U_{\pm\alpha}$. It is a connected k-simple and k-isotropic group. Let $\hat{M}_\alpha^{\mathrm{sc}}$ be the simply connected cover of $\hat{M}_\alpha$, and $p^{\mathrm{sc}} : \hat{M}_\alpha^{\mathrm{sc}} \to \hat{M}_\alpha$ the natural homomorphism.

Definition 1. The group G satisfies property (TWN+) (tempered winding numbers, strong version) if, for any proper Levi subgroup M of G defined over k, and any root $\alpha \in \Sigma_M$ we have the estimate

$$\int_T^{T+1} |n'_\alpha(\pi, \mathrm{i}t)| \; dt \ll \log(|T| + \Lambda(\pi_\infty; p^{\mathrm{sc}}) + \mathrm{level}(\pi; p^{\mathrm{sc}}))$$

for all $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))$ and all real numbers T.

Here we set $\mathrm{level}(\pi; p^{\mathrm{sc}}) = N(\mathfrak{n})$, where $\mathfrak{n}$ is the largest ideal of $\mathfrak{o}_k$ such that

$$\pi^{\mathbf{K}(\mathfrak{n}) \cap p^{\mathrm{sc}}(\hat{M}_\alpha^{\mathrm{sc}}(\mathbb{A}_{\mathrm{fin}}))} \neq 0,$$

with $\mathbf{K}(\mathfrak{n})$ denoting the principal congruence subgroup of level $\mathfrak{n}$ of $G(\mathbb{A}_{\mathrm{fin}})$ (with respect to the usual choices). Furthermore, $\Lambda(\pi_\infty; p^{\mathrm{sc}}) = \Lambda(\pi_\infty|_{\hat{M}_\alpha(k_\infty)}) = 1 +$

$\|\chi(\pi_\infty|_{\hat{M}_\alpha(k_\infty)})\|^2$, where χ denotes the infinitesimal character (the precise dependence on the archimedean component is actually irrelevant to the limit multiplicity problem, but important in other contexts).

We conjecture that all reductive groups G over number fields satisfy property (TWN+). The main result of [4] is the following. Its proof uses the connection of the normalizing factors to automorphic L-functions, and (in the first two cases) functoriality with $\mathrm{GL}(n)$ [2, 3].

Theorem 2. *The following groups satisfy property (TWN+).*

(1) $\mathrm{GL}(n)$ *and its inner forms.*
(2) *Quasi-split classical groups and their similitude groups.*
(3) *The split exceptional group* G_2.

The same holds for any group whose derived group coincides with the derived group of any one of the groups above.

We now turn to property (BD), which is essentially a local property. Let v be a non-archimedean place of k. Set $F = k_v$, let ϖ be a uniformizer of F and q be the cardinality of the residue field of F. Let K_0 be a special maximal compact subgroup of $G(F)$. Let $P = MU$ be a maximal parabolic subgroup of G defined over F, $\bar{P}$ the opposite parabolic subgroup and π a smooth irreducible representation of $M(F)$ on a complex vector space. We consider the family of induced $G(F)$-representations $I_P(\pi, s)$, $s \in \mathbb{C}$, which extend the fixed K_0-representation $I^{K_0}_{P(F)\cap K_0}(\pi|_{M(F)\cap K_0})$, and the associated local intertwining operators

$$M(s) = M_{\bar{P}|P}(\pi, s) : I^{K_0}_{P(F)\cap K_0}(\pi|_{M(F)\cap K_0}) \to I^{K_0}_{\bar{P}(F)\cap K_0}(\pi|_{M(F)\cap K_0}),$$

which we regard as a family of linear maps between vector spaces that are independent of s. For any closed subgroup K of K_0, let $M(s)^K$ be the restriction of $M(s)$ to the space of K-invariant vectors in $I^{K_0}_{P(F)\cap K_0}(\pi|_{M(F)\cap K_0})$.

We recall that the matrix coefficients of the linear operators $M(s)$ are rational functions of q^{-s}, and that the degrees of the denominators are bounded independently of π [12]. For any $n \geq 1$ write K_n for the principal congruence subgroup of K_0 of level ϖ^n with respect to a fixed faithful k-rational representation ρ of G and a suitable lattice Λ_ρ in the space of ρ. Let $\hat{M}$ be the F-simple normal subgroup of G generated by U and $\bar{U}$.

Definition 2. (1) A group G satisfies property (BDmax) at v, if there exists a constant $C > 0$ such that for all maximal parabolic subgroups $P = MU$ defined over $F = k_v$, smooth irreducible representations π of $M(F)$ and all $n \geq 1$, the degrees of the numerators of the matrix coefficients of $M(s)^{K_n \cap \hat{M}(F)}$ are bounded by Cn.
(2) A group satisfies property (BD) (bounded degree) at v, if all its Levi subgroups (defined over $F = k_v$) satisfy property (BDmax).
(3) We say that G satisfies property (BD) for a set S of non-archimedean valuations of k, if it satisfies (BD) at all $v \in S$ with a uniform value of C.

We conjecture that property (BD) is satisfied for all reductive groups over number fields with $S = S_{\mathrm{fin}}$, the set of all non-archimedean valuations.

In [7], property (BD) was established for the groups $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$ with $S = S_{\mathrm{fin}}$. The main result of [5] is that property (BD) for a group G at v is implied by a quantitative bound on the support of supercuspidal matrix coefficients (property (PSC)) for all semisimple normal subgroups of proper Levi subgroups of G defined over $F = k_v$. By [7, Corollary 13], Property (PSC) holds for a reductive group if all its irreducible supercuspidal representations are induced from cuspidal representations of subgroups that are open compact modulo the center. As a consequence of Kim's exhaustion theorem for supercuspidal representations [9], we conclude:

Theorem 3. *Let G be a reductive group defined over a number field k. There exists a finite set S_0 of non-archimedean places of k such that G satisfies property (BD) for the set $S_{\mathrm{fin}} - S_0$.*

Moreover, using additional known results on supercuspidal representations [10, 11], we can be more specific in a number of cases.

Theorem 4.
(1) *Let G be a split group of rank two or an inner form of* $\mathrm{GL}(n)$ *or* $\mathrm{SL}(n)$ *defined over a number field k. Then G satisfies property (BD) with respect to the set S_{fin} of all non-archimedean places.*
(2) *Let G be a symplectic, special orthogonal or unitary group defined over a number field k. Then G satisfies property (BD) with respect to the set $S_{\mathrm{fin}} - \{v \in S_{\mathrm{fin}} : v|2\}$.*

As an application of these results we finally have the following consequence for the limit multiplicity problem.

Theorem 5.
(1) *Let G be a split group of rank two or an inner form of* $\mathrm{GL}(n)$ *or* $\mathrm{SL}(n)$ *defined over a number field k. Then limit multiplicity holds for any non-degenerate family $\mathcal{K}$ of open subgroups of a given open compact subgroup $\mathbf{K}_0^S$ of $G(\mathbb{A}^S)$.*
(2) *Let G be a quasi-split classical group defined over a number field k. Then limit multiplicity holds for any non-degenerate family $\mathcal{K}$ of open subgroups of $\mathbf{K}_0^S$ that has bounded level at $\{v \notin S : v|2\}$.*

References

[1] James Arthur. Intertwining operators and residues. I. Weighted characters. *J. Funct. Anal.*, 84(1):19–84, 1989.

[2] James Arthur. *The endoscopic classification of representations.* Orthogonal and symplectic groups. American Mathematical Society Colloquium Publications, vol. 61, American Mathematical Society, Providence, RI, 2013.

[3] Alexandru Ioan Badulescu. Global Jacquet-Langlands correspondence, multiplicity one and classification of automorphic representations. With an appendix by Neven Grbac. *Invent. Math.*, 172(2):383–438, 2008.

[4] Tobias Finis and Erez Lapid. On the analytic properties of intertwining operators I: global normalizing factors. *Bull. Iranian Math. Soc.*, to appear. arXiv:1603.05475.

[5] Tobias Finis and Erez Lapid. On the analytic properties of intertwining operators II: local degree bounds and limit multiplicities, 2017. arXiv:1705.08191.
[6] Tobias Finis and Erez Lapid. An approximation principle for congruence subgroups II: application to the limit multiplicity problem, 2015. arXiv:1504.04795.
[7] Tobias Finis, Erez Lapid, and Werner Müller. On the degrees of matrix coefficients of intertwining operators. *Pacific J. Math.*, 260(2):433–456, 2012.
[8] Tobias Finis, Erez Lapid, and Werner Müller. Limit multiplicities for principal congruence subgroups of $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$. *J. Inst. Math. Jussieu*, 14(3):589–638, 2015.
[9] Ju-Lee Kim. Supercuspidal representations: an exhaustion theorem. *J. Amer. Math. Soc.*, 20(2):273–320 (electronic), 2007.
[10] V. Sécherre and S. Stevens. Représentations lisses de $\mathrm{GL}_m(D)$. IV. Représentations supercuspidales. *J. Inst. Math. Jussieu*, 7(3):527–574, 2008.
[11] Shaun Stevens. The supercuspidal representations of p-adic classical groups. *Invent. Math.*, 172(2):289–352, 2008.
[12] J.-L. Waldspurger. La formule de Plancherel pour les groupes p-adiques (d'après Harish-Chandra). *J. Inst. Math. Jussieu*, 2(2):235–333, 2003.

Symmetry Breaking Operators for Orthogonal Groups $O(n,1)$

Toshiyuki Kobayashi

Given an irreducible representation π of a group G and a subgroup G', we may think of π as a representation of the subgroup G' (the *restriction* $\pi|_{G'}$). A typical example is the tensor product representation $\pi_1 \otimes \pi_2$ of two representations π_1 and π_2 of a group H, which is obtained by the restriction of the outer tensor product $\pi_1 \boxtimes \pi_2$ of the direct product group $G := H \times H$ to its subgroup $G' := \mathrm{diag}(H)$.

As *branching problems*, we wish to understand how the restriction $\pi|_{G'}$ behaves as a G'-module. For reductive groups, this is a difficult problem, partly because the restriction $\pi|_{G'}$ may not be well under control as a representation of G' even when G' is a maximal subgroup of G. Wild behavior such as infinite multiplicities may occur, for instance, already in the tensor product representation of $SL_3(\mathbb{R})$.

The author proposed in [3] to go on successively, further steps in the study of branching problems via the following three stages:

Stage A: Abstract feature of the restriction $\pi|_{G'}$

Stage B: Branching laws.

Stage C: Construction of symmetry breaking operators (Definition 1).

Here, *branching laws* in Stage B ask an explicit decomposition of the restriction into irreducible representations of the subgroup G' when π is a unitary representation, and also ask the *multiplicity* $m(\pi,\tau) := \dim \mathrm{Hom}_{G'}(\pi|_{G'}, \tau)$ for irreducible representations τ of G'. The latter makes sense even when π and τ are nonunitary. Stage C refines Stage B, by asking an explicit construction of SBOs when π and τ are realized geometrically.

Definition 1. An element in $\mathrm{Hom}_{G'}(\pi|_{G'}, \tau)$ is called a *symmetry breaking operator*, SBO for short.

Stage A includes a basic question whether spectrum is discrete or not, see [1]. Another fundamental question in Stage A is an estimate of multiplicities. In [2, 6], we discovered the following geometric creteria to control multiplicities:

Theorem 1 (geometric criteria for finite/bounded multiplicities)**.**

(1) *The dimension of* $\mathrm{Hom}_{G'}(\pi|_{G'}, \tau)$ *is finite for any irreducible representations* π *of* G *and any* τ *of* G' *iff* $G \times G'/\mathrm{diag}(G')$ *is real spherical.*

(2) *The dimension of* $\mathrm{Hom}_{G'}(\pi|_{G'}, \tau)$ *is uniformly bounded with respect to* π *and* τ *iff* $(G_{\mathbb{C}} \times G'_{\mathbb{C}})/\mathrm{diag}(G'_{\mathbb{C}})$ *is spherical.*

Here we recall

Definition 2. (1) A complex manifold $X_{\mathbb{C}}$ with holomorphic action of a complex reductive group $G_{\mathbb{C}}$ is *spherical* if a Borel subgroup of $G_{\mathbb{C}}$ has an open orbit in $X_{\mathbb{C}}$.

(2) A real manifold X with continuous action of a real reductive group G is *real spherical* if a minimal parabolic subgroup of G has an open orbit in X.

The latter terminology was introduced in [2] in search for a broader framework for global analysis on homogeneous spaces than the usual (*e.g.* group manifolds, symmetric spaces). That is, the function space $C^\infty(G/H)$ (or $L^2(G/H)$ *etc.*) should be under control by representation theory if

$$\dim \mathrm{Hom}_G(\pi, C^\infty(G/H)) < \infty \quad \text{for all } \pi \in \widehat{G}_{\mathrm{adm}}, \tag{1}$$

and hence we could expect to develop global analysis on G/H by using representation theory if (1) holds[2]. We discovered and proved that the geometric property "real spherical" characterizes exactly the representation-theoretic property (1):

Fact 1 ([2, 6])**.** Let $X = G/H$ where $G \supset H$ are algebraic real reductive groups.

(1) $\dim \mathrm{Hom}_G(\pi, C^\infty(X)) < \infty$ ($^\forall \pi \in \widehat{G}_{\mathrm{adm}}$) iff X is real spherical.

(2) $\dim \mathrm{Hom}_G(\pi, C^\infty(X))$ is uniformly bounded iff $X_{\mathbb{C}}$ is spherical.

Theorem 1 follows from Fact 1.

The classification of the real spherical spaces of the form $(G \times G')/\mathrm{diag}(G')$ was accomplished in [5] when (G, G') is a reductive symmetric pair. This *a priori* estimate in Stage A singles out the settings which would be potentially promising for Stages B and C of branching problems. One of such settings arises from a different discipline, namely, from conformal geometry. The first complete solution to Stage C obtained [7] is related to this geometric setting as below.

Given a Riemannian manifold (X, g), we write $G = \mathrm{Conf}(X, g)$ for the group of conformal diffeomorphisms of X. Then there is a natural family of representations π_λ of G on $C^\infty(X)$ for $\lambda \in \mathbb{C}$ given by

$$(\pi_\lambda(h)f)(x) = \Omega(h^{-1}, x)^\lambda f(h^{-1} \cdot x) \quad \text{for } h \in G, x \in X.$$

We can extend this to a family of representations on the space $\mathcal{E}^i(X)$ of differential i-forms, to be denoted by $\pi_\lambda^{(i)}$.

If Y is a submanifold of X, then there is a natural morphism

$$G' := \{h \in G : h \cdot Y \subset Y\} \to \mathrm{Conf}(Y, g|_Y).$$

Then we may compare two families of representations of the group G':

- the restriction $\pi_\lambda^{(i)}|_{G'}$ acting on $\mathcal{E}^{(i)}(X)$,
- the representation ${\pi'}_\nu^{(j)}$ acting on $\mathcal{E}^{(j)}(Y)$.

A conformally covariant SBO on differential forms is a linear map $\mathcal{E}^i(X) \to \mathcal{E}^j(Y)$ that intertwines $\pi_\lambda^{(i)}|_{G'}$ and ${\pi'}_\nu^{(j)}$. Here is a basic question arising from conformal geometry:

Question 1. Let X be a Riemannian manifold X, and Y a hypersurface. Construct and classify conformally covariant SBOs from $\mathcal{E}^i(X)$ to $\mathcal{E}^j(Y)$.

We are interested in "natural operators" D that persist for all pairs (X, Y). The larger $\mathrm{Conf}(X; Y)$ is, the more constrains are on D, and hence, we first focus on the model space with largest symmetries which is given by $(X, Y) = (S^n, S^{n-1})$. In this case the pair (G, G') of conformal groups is locally isomorphic to $(O(n+1, 1), O(n, 1))$. It then turns out that the criterion in Theorem 1 (2) for Stage A is fulfilled. Then Question 1 is regarded as Stages B and C of branching problems. Recently, we have solved completely Question 1 in the model space:

- Continuous SBOs for $i = j = 0$ were constructed and classified in [7].
- Differential SBOs for general i and j were constructed and classified in [4].
- The final classification is announced in [8].

Here is a flavor of the complete classification:

Theorem 2. *If* $\mathrm{Hom}_{G'}(\pi_\lambda^{(i)}|_{G'}, {\pi'}_\nu^{(j)}) \neq \{0\}$ *for some* λ, $\nu \in \mathbb{C}$, *then* $j \in \{i-2, i-1, i, i+1\}$ *or* $i + j \in \{n-2, n-1, n, n+1\}$.

In the talk, I gave briefly the methods of the complete solution [4, 7, 8], some of which are also applicable in a more general setting that Theorem 1 (an *a priori* estimate for Stage A) suggests.

Finally, some applications of these results include:

• an evidence of a conjecture of Gross and Prasad for $O(n, 1)$, see [8];

• periods of irreducible unitary representations with nonzero cohomologies;

• a construction of discrete spectrum of the branching laws of complementary series [7, Chap. 15].

References

[1] T. Kobayashi, Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups and its applications, Invent. Math. **117** (1994), 181–205; Part II, Ann. of Math. (2) **147** (1998), 709–729; Part III, Invent. Math. **131** (1998), 229–256.

[2] T. Kobayashi, Introduction to harmonic analysis on real spherical homogeneous spaces, Proceedings of the 3rd Summer School on Number Theory "Homogeneous Spaces and Automorphic Forms" in Nagano (F. Sato, ed.), 1995, 22–41.

[3] T. Kobayashi, A program for branching problems in the representation theory of real reductive groups, In: Representations of Lie Groups, In Honor of David A. Vogan on his 60th Birthday, Progress in Mathematics **312**, pp. 277–322. Birkhäuser, 2015.

[4] T. Kobayashi, T. Kubo, and M. Pevzner, Conformal Symmetry Breaking Differential Operators for Differential Forms on Spheres, Lecture Notes in Mathematics, vol. **2170**, iv + 192 pages, 2016. ISBN: 978-981-10-2657-7.
[5] T. Kobayashi, T. Matsuki, Classification of finite-multiplicity symmetric pairs, Transform. Groups **19** (2014), 457–493, Special issue in honor of Dynkin for his 90th birthday.
[6] T. Kobayashi and T. Oshima, Finite multiplicity theorems for induction and restriction, Adv. Math. **248** (2013), 921–944.
[7] T. Kobayashi, B. Speh, Symmetry Breaking for Representations of Rank One Orthogonal Groups, Mem. Amer. Math. Soc., vol. **238**, No. 1126, (2015), v+112 pages, ISBN: 978-1-4704-1922-6.
[8] T.Kobayashi, B.Speh, Symmetry breaking for orthogonal groups and a conjecture by B. Gross and D. Prasad, submitted, arXiv:1702.00263.

A Relative Local Langland Correspondence

Dipendra Prasad

For a quadratic extension E/F of local fields, G a reductive algebraic group over F, and π an irreducible admissible representation of $G(E)$, this lecture attempts to understand the multiplicities,

$$m(\pi) = \dim\mathrm{Hom}_{G(F)}(\pi, \mathbb{C}),$$

specially to understand when it is nonzero, and to interpret it in terms of Langlands parameters associated to the representation π.

A basic object in this understanding is a quasi-split reductive group $G^{op}(F)$ called the opposition group such that $G^{op}(E) \cong G(E)$, and $G^{op}(E)$ is obtained by twisting G by the Chevalley involution of $G(\bar{F})$.

The following conjecture was proposed.

Let

$$\widetilde{\Sigma}_G(F) = \mathrm{Hom}(W'_F, {}^L G(\mathbb{C}))$$

be the variety of Langlands parameters, and

$$\Sigma_G(F) = \mathrm{Hom}(W'_F, {}^L G(C))//\widehat{G}(C),$$

the equivalence classes of parameters. It can be seen that $\Sigma_G(F)$ is a countable disjoint union of normal affine varieties. There is the base change map

$$\Sigma_{G^{op}}(F) \xrightarrow{BC} \Sigma_{G^{op}}(E)^{\mathrm{Gal}(E/F)} = \Sigma_G(E)^{\mathrm{Gal}(E/F)},$$

which is a finite map of the varieties involved.

Conjecture: Let π be an irreducible generic representation of $G(E)$ which has a Whittaker model for a character of $N(E)/N(F)$. Then,

$$\Sigma_{\alpha \epsilon H^1(\mathrm{Gal}(E/F), G(E))} \dim\mathrm{Hom}_{G_\alpha(E)}(\pi, \mathbb{C}) = \deg BC \text{ at } \sigma_\pi,$$

which is the number of ways of lifting (counted with multiplicity) of the parameter $\sigma_\pi : W'_E \longrightarrow {}^L G^{op}$ to $W'_F \longrightarrow {}^L G^{op}$.

As a basic example, one has the case of $G(F) = \mathrm{GL}_n(F)$ in which case $G^{op}(F)$ is the quasi-split unitary groups over F. In this case, it is known that $m(\pi) \leq 1$, and also that the basechange map of parameters:

$$BC : \Sigma_{U(n)}(F) \longrightarrow \Sigma_{\mathrm{GL}(n)}(E)$$

is injective.

It is known that for π a generic representation of $\mathrm{GL}_n(E)$,

$$m(\pi) = 1$$

if and only if π is obtained as a basechange of a repersentation of $U_n(F)$.

The lecture ended by discussing higher multiplicities which exist for $G(F) = U_2(F)$, $G(E) = \mathrm{GL}_2(E)$, and interperting them in terms of fibres of the basechange from representations of $\mathrm{GL}_2(F)$ to representation of $\mathrm{GL}_2(E)$.

Distinction of square-integrable representations for Galois pairs and a conjecture of Prasad

Raphaël Beuzart-Plessis

Let F be a p-adic field, E/F be a quadratic extension and H be a connected reductive group over F. Set $G := R_{E/F}H$ where $R_{E/F}$ denotes Weil's restriction of scalars (so that $G(F) = H(E)$). For $\chi : H(F) \to \mathbb{C}^\times$ a continuous character and π a smooth irreducible representation of $G(F)$ we define a multiplicity $m(\pi, \chi)$ by

$$m(\pi, \chi) := \dim Hom_H(\pi, \chi)$$

where $Hom_H(\pi, \chi)$ denotes the space of $(H(F), \chi)$-equivariant linear forms on π. This space is known to always be of finite dimension ([7], Theorem 4.5) and we say that π is (H, χ)-*distinguished* if $m(\pi, \chi) \neq 0$. Recently, Dipendra Prasad has proposed very general conjectures describing this multiplicity in terms of the Langlands parameterization of π ([13]), at least for representations belonging to the so-called 'generic' L-packets. These predictions are in the spirit of what we now call the 'relative' Langlands program whose main local goal is roughly to describe the set of (H, χ)-distinguished representations for very general subgroups H (usually called spherical) of a reductive group G in terms of Langlands dual picture. In this report, we present some results supporting Prasad's very precise conjectures in the particular case of stable (essentially) square-integrable representations (i.e. we will consider the multiplicity $m(\Pi, \chi)$ for Π a discrete L-packet of G). We now state this 'stable-discrete' version of Prasad's conjecture. To this end, we need to restrict slightly the generality by only considering characters χ that are of 'Galois type' i.e. which are in the image of the map constructed by Langlands

$$H^1(W_F, Z(\check{H})) \to Hom_{cont}(H(F), \mathbb{C}^\times)$$

This map is always injective (because F is p-adic) but not always surjective (although it is most of the time, e.g. if H is quasi-split). Prasad associates certain invariants to the situation at hand. First of all, he constructs a quasi-split group

H^{op} over F which is an E/F form of the quasi-split inner form of H as well as a certain quadratic character $\omega_{H,E} : H(F) \to \{\pm 1\}$ of Galois type. We refer the reader to [13]§7-8 for precise constructions of those and content ourself to give three examples:

- If $H = GL_n$, then $H^{op} = U(n)_{qs}$ (quasi-split form) and $\omega_{H,E} = (\eta_{E/F} \circ \det)^{n+1}$ where $\eta_{E/F}$ is the quadratic character associated to E/F;
- If $H = U(n)$ (a unitary group of rank n), then $H^{op} = GL_n$ and $\omega_{H,E} = 1$;
- If $H = SO(2n+1)$ (any odd special orthogonal group), then $H^{op} = SO(2n+1)_{qs}$ (the quasi-split inner form) and $\omega_{H,E} = \eta_{E/F} \circ N_{spin}$ where $N_{spin} : SO(2n+1)(F) \to \{\pm 1\}$ denotes the spin norm.

Secondly, Prasad associates to every character χ of Galois type of $H(F)$ a certain 'Langlands dual group' $\mathcal{H}^{op}_\chi$ which sits in a short exact sequence

$$1 \to \check{H}^{op} \to \mathcal{H}^{op}_\chi \to W_F \to 1 \tag{1}$$

together with a group embedding $\iota : \mathcal{H}^{op}_\chi \hookrightarrow {}^L G$ (where ${}^L G$ denotes the L-group of G) which is compatible with the projections to W_F and algebraic when restricted to $\check{H}^{op}$. Although the short exact sequence (1) always splits, there does not necessarily exists a splitting preserving a pinning of $\check{H}^{op}$ and hence $\mathcal{H}^{op}_\chi$ is not always an L-group. However, in the particular case where $\chi = \omega_{H,E}$, we have $\mathcal{H}^{op}_\chi = {}^L H^{op}$ and ι is the homomorphism of quadratic base-change. We are now ready to state (a slight generalization of) the stable version of Prasad's conjecture for square-integrable representations:

Conjecture 1. *Let $\phi : WD_F \to {}^L G$ be a discrete L-parameter, $\Pi^G(\phi) \subseteq Irr(G)$ the corresponding L-packet and $\Pi_\phi = \sum_{\pi \in \Pi^G(\phi)} d(\pi)\pi$ the stable representation associated to ϕ. Then, we have*

$$m(\Pi_\phi, \chi) = |ker^1(F; H, G)|^{-1} \sum_\psi \frac{|Z(\phi)|}{|Z(\psi)|}$$

where

- *The sum is over the set of 'L-parameters' $\psi : WD_F \to \mathcal{H}^{op}_\chi$ (up to $\check{H}^{op}$-conj) making the following diagram commutes (up to $\check{G}$-conj)*

$$\begin{tikzcd} & \mathcal{H}^{op}_\chi \arrow[d, hook, "\iota"] \\ WD_F \arrow[ur, "\psi"] \arrow[r, "\phi"] & {}^L G \end{tikzcd}$$

- *$ker^1(F; H, G) := Ker\left(H^1(F, H) \to H^1(F, G)\right)$ (corresponds to certain twists of the parameter ψ that become trivial in ${}^L G$);*
- *$Z(\phi) := Cent_{\check{G}}(\phi)/Z(\check{G})^{W_F}$ and $Z(\psi) := Cent_{\check{H}^{op}}(\psi)/Z(\check{H}^{op})^{W_F}$*

As we said, this is only part of Prasad's general conjectures which aim to describe (almost) all the multiplicities $m(\pi,\chi)$ explicitly. This version of the conjecture is known in few particular cases: for $H = GL(n)$ by Kable and Anandavardhanan-Rajan ([9], [1]), for $H = U(n)$ by Feigon-Lapid-Offen ([8]) and for $H = GSp(4)$ by Hengfei Lu ([10]). The following theorems are both formal consequences of conjecture 1 and are the main results presented in this report:

Theorem 1. Let H, H' be inner forms $/F$, $G := R_{E/F}H$, $G' := R_{E/F}H'$ and χ, χ' characters of Galois type of $H(F)$ and $H'(F)$ corresponding to each other (i.e. coming from the same element in $H^1(W_F, Z(\check{H})))$. Let Π, Π' be (essentially) square-integrable representations of $G(F)$ and $G'(F)$ respectively which are stable and transfer of each other (i.e. $\theta_\Pi(x) = \theta_{\Pi'}(y)$ for all stably conjugated strongly regular elements $x \in G_{reg}(F)$ and $y \in G'_{reg}(F)$ with θ_Π, $\theta_{\Pi'}$ the Harish-Chandra characters of Π and Π'). Then, we have

$$m(\Pi,\chi) = m(\Pi',\chi')$$

Theorem 2. For $\pi = St$ the generalized Steinberg representation of $G(F)$ and χ a character of Galois type we have

$$m(St,\chi) = \begin{cases} 1 & \text{if } \chi = \omega_{H,E} \\ 0 & \text{otherwise} \end{cases}$$

Theorem 2 confirms an older conjecture of Prasad ([12], conjecture 3) which was already proved for split groups and tamely ramified extensions by Broussous-Courtès and Courtès ([4], [6]) and for inner forms of GL_n by Matringe ([11]). The proof of Broussous and Courtès is based on geometry of the building whereas Matringe uses some Mackey machinery. Our approach is completely orthogonal to theirs and is based on a certain integral formula computing the multiplicity $m(\pi,\chi)$ in terms of the Harish-Chandra character of π. This formula is reminiscent and inspired by a similar result of Waldspurger in the context of the so-called Gross-Prasad conjecture ([14], [15]). It can also be seen as a 'twisted' version ('twisted' with respect to the non-split extension E/F) of the orthogonality relations between characters of discrete series due Harish-Chandra ([5], Theorem 3). It can be stated as follows (where we assume for simplicity that the group H is semi-simple):

Theorem 3. For π a square-integrable representation of $G(F)$ and χ a continuous character of $H(F)$ we have

$$m(\pi,\chi) = \int_{\Gamma_{ell}(H)} \theta_\pi(x)\chi(x)^{-1}dx$$

where θ_π denotes the Harish-Chandra character of π, $\Gamma_{ell}(H)$ stands for the set of regular elliptic conjugacy classes in $H(F)$ and dx is a certain natural measure on it.

Theorem 1 is an easy consequence of this formula and theorem 2 also follows from it with some extra work (the basic idea being that we know explicitly the character of the Steinberg representation so that we can apply theorem 3 and compute). For its part, theorem 3 is a consequence of a certain simple local trace formula adapted to the situation and which takes its roots in Arthur's local trace formula ([2]) as well as in Waldspurger's work on the Gross-Prasad conjecture for orthogonal groups ([14], [15]). A similar formula has been proved by the author in the context of the Gan-Gross-Prasad conjecture for unitary groups ([3]) and the method of proof is similar.

References

[1] U.K. Anandavardhanan, C.S. Rajan, *Distinguished representations, base change, and reducibility for unitary groups*, Int. Math. Res. Not. 2005, no. 14, 841-854

[2] J. Arthur, *A local trace formula*, Inst. Hautes tudes Sci. Publ. Math. No. 73 (1991), 5-96

[3] R. Beuzart-Plessis, *A local trace formula for the Gan-Gross-Prasad conjecture for unitary groups: the archimedean case*, arXiv preprint 2015, arXiv:1506.01452

[4] P. Broussous, F. Courtès, *Distinction of the Steinberg representation*, to appear in Math. Research Not.

[5] L. Clozel, *Invariant harmonic analysis on the Schwartz space of a reductive p-adic group*, in "Harmonic analysis on reductive groups" (Brunswick, ME, 1989), 101-121, Progr. Math., 101, Birkhuser Boston, Boston, MA, 1991

[6] F. Courtès, *Distinction of the Steinberg representation III: the tamely ramified case*, arXiv preprint 2014, arXiv:1408.6656

[7] P. Delorme, *Constant term of smooth H_ψ-spherical functions on a reductive p-adic group*, Trans. Amer. Math. Soc. 362 (2010), no. 2, 933-955

[8] B. Feigon, E. Lapid, O. Offen, *On representations distinguished by unitary groups*, Publ. Math. Inst. Hautes tudes Sci. 115 (2012), 185-323

[9] A.C. Kable, *Asai L-functions and Jacquet's conjecture*, Amer. J. Math. 126 (2004), no. 4, 789-820

[10] H. Lu, *$GSp(4)$-period problems over a quadratic field extension*, Ph.D Thesis National University of Singapore (2017)

[11] N. Matringe, *Distinction of the Steinberg representation for inner forms of $GL(n)$*, to appear in Math. Z.

[12] D. Prasad, *On a conjecture of Jacquet about distinguished representations of $GL(n)$*, Duke Math. J. 109 (2001), no. 1, 67-78

[13] D. Prasad, *A 'relative' local Langlands correspondence*, arXiv preprint 2015, arXiv:1512.04347

[14] J.-L. Waldspurger, *Une formule intégrale reliée à la conjecture locale de Gross-Prasad*, Compos. Math. 146 (2010), no. 5, 1180-1290

[15] J.-L. Waldspurger, *Une formule intégrale reliée à la conjecture locale de Gross-Prasad, 2e partie: extension aux représentations tempérées*, in "Sur les conjectures de Gross et Prasad I" Astérisque No. 346 (2012), 171-312

Distinguished positive regular representations

Fiona Murnaghan

Let $G = \mathbf{G}(F)$ be the F-rational points of a connected F-group $\mathbf{G}$ that splits over a tamely ramified extension of F, where F is a nonarchimedean local field of odd residual characteristic.

Suppose that θ is an involution of G (that is, an F-automorphism of $\mathbf{G}$ of order two) and $H = G^\theta$ is the group of θ-fixed points in G. A smooth representation π

of G is said to be H-distinguished if the space $\mathrm{Hom}_H(\pi, 1)$ of H-invariant linear forms on the space of π is nonzero. We describe results of [M3] on distinction of particular irreducible admissible (complex) representations of G, which we refer to as positive regular representations.

Suppose that ϕ is a quasicharacter of a tame maximal torus T of G and T_{0+} is the maximal pro-p-subgroup of T. We say that a quasicharacter ϕ of T is G-regular on T_{0+} if ϕ does not agree on T_{0+} with the restriction of any quasicharacter of a twisted Levi subgroup of G that strictly contains T. (Here, a twisted Levi subgroup of G is a subgroup of the form $\mathbf{G}'(F)$, where $\mathbf{G}'$ is a connected, reductive F-subgroup of $\mathbf{G}$ that becomes a Levi subgroup over a finite extension of F.) The equivalence classes of positive regular representations are parametrized by G-conjugacy classes of pairs (T, ϕ), where T is a tamely ramified maximal torus of G and ϕ is a quasicharacter of T that is G-regular on T_{0+}. We denote a representation whose equivalence class corresponds to the G-conjugacy class of a pair (T, ϕ) by $\pi_{(T,\phi)}$.

Under mild restrictions on the residual characteristic of F, we derive necessary conditions for H-distinction of a positive regular representation $\pi_{(T,\phi)}$, expressed in terms of properties of (appropriate conjugates of) T and ϕ relative to the involution θ. In particular, we show that T is θ-stable, T contains G-regular θ-split elements, and the restriction of ϕ to T^{θ}_{0+} is trivial. (Here, an element g of G is θ-split if $\theta(g) = g^{-1}$.)

An H-distinguished admissible representation π is H-relatively supercuspidal ([KT],[L]) if for every λ in $\mathrm{Hom}_H(\pi, 1)$ and every vector v in the space of π, the function $g \mapsto \langle \lambda, \pi(g)v \rangle$ is compactly supported modulo HZ_G, where Z_G is the centre of G. We prove that if $\pi_{(T,\phi)}$ is H-distinguished, T and ϕ have the properties listed above, and T is compact modulo $T^{\theta} Z_G$, then $\pi_{(T,\phi)}$ is H-relatively supercuspidal.

The results discussed here form part of a general program devoted to the study of distinction of tame representations (that is, representations whose inertial support consists of supercuspidal representations arising via Yu's construction). In cases where there are no G-regular θ split elements, there are no H-distinguished positive regular representations. We comment briefly on a more general construction ([M2]) of tame relatively supercuspidal representations that does not require existence of θ-split G-regular elements.

References

[BZ1] I.N. Bernstein and A.V. Zelevinsky, *Representations of the group $GL(n, F)$, where F is a non-archimidean local field*, Russian Math. Surveys **31:3** (1976), 1–68.

[BZ2] I. Bernstein and A.V. Zelevinsky, *Induced representations of reductive p-adic groups. I*, Ann. scient. Éc. Norm. Sup. 4^e série, **10** (1977), 441–472.

[BT] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local, I. Données radicielles valuées*, Inst. Hautes Études Sci. Publ. Math. (1972) **41**, 5–251.

[HM] J. Hakim and F. Murnaghan, *Distinguished tame supercuspidal representations*, Internat. Math. Res. Papers, **2008**, no. 2, 166pp.

[HW] A.G. Helminck and S.P. Wang, *On rationality properties of involutions of reductive groups*, Advances in Math. **99** (1993), 26–97.

[Ka] T. Kaletha, *Regular supercuspidal representations*, preprint.

[KT] S. Kato and T. Takano, *Subrepresentation theorem for p-adic symmetric spaces*, Internat. Math. Res. Notices, **2008**, 40pp.

[KY] J.-L. Kim and J.-K. Yu, *Construction of tame types*, preprint.

[L] N. Lagier, *Terme constant de fonctions sur en espace symétrique réductif p-adique*, J. Funct. Anal. **254** (2008), 1088–1145.

[MP] A. Moy and G. Prasad, *Unrefined minimal K-types for p-adic groups*, Invent. Math. **116** (1994), 393–408.

[M1] F. Murnaghan, *Distinction and tame types*, in preparation.

[M2] F. Murnaghan, *Tame relatively supercuspidal representations*, in preparation.

[M3] F. Murnaghan, *Distinguished positive regular supercuspidal representations*, in special volume of Bull. Iranian Math. Soc. (in honor of the 70th birthday of Freydoon Shahidi), to apper.

[M4] F. Murnaghan, *Parametrization of tame supercuspidal representations*, in "On Certain L-functions: A volume in honor of Freydoon Shahidi on the occasion of his 60th birthday", Clay Math. Proceedings, volume 13, 2011, edited by J. Arthur, J.W. Cogdell, S. Gelbart, D. Goldberg, D. Ramakrishan, pp. 439–470.

[M5] F. Murnaghan, *Regularity and distinction of supercuspidal representations*, in "Harmonic Analysis and Representations of Reductive p-adic Groups", Contemporary Mathematics, volume 543, American Math. Society, 2011, edited by R.S. Doran, P.J. Sally, Jr., L. Spice, pp. 155-183.

[R] A. Roche, *Parabolic induction and the Bernstein decomposition*, Compositio Mathematica **134** (2002), 113–133.

[SS] T.A. Springer and R. Steinberg, *Conjugacy classes*, Seminar on Algebraic Groups and Related Finite Groups (I.A.S., Princeton, N.J., 1968/69), Lecture Notes in Mathematics, volume 131, Springer, Berlin, 1970, pp.167–266.

[T] J. Tits, *Reductive groups over local fields*, in "Automorphic Forms, Representations, and L-Functions", Proceedings of Symposia in Pure Mathematics **33**, part 1, Amer. Math. Soc., Providence RI, 1979, pp. 29–69.

[Y] J.-K. Yu, *Construction of tame supercuspidal representations*, J. Amer. Math. Soc. **14** (2001), 579–622.

Special L-values and the refined Gan-Gross-Prasad conjecture

Harald Grobner

(joint work with Jie Lin)

Motivation

In the algebraic theory of special values of L-functions, Deligne's conjecture for critical L-values of motives is (still) one of the driving forces. Cut down to one line, it asserts that the critical values at $s = m \in \mathbb{Z}$ of the L-function $L(s, \mathbb{M})$ of a motive $\mathbb{M}$ can be described, up to multiplication by elements in the coefficient-field $E(\mathbb{M})$, in terms of certain *geometric* period-invariants $c^{\pm}(\mathbb{M})$ and certain explicit integral powers of $(2\pi i)$, [Del79, Conj. 2.8]:

$$L(m, \mathbb{M}) \sim_{E(\mathbb{M})} (2\pi i)^{d(m)} c^{(-1)^m}(\mathbb{M}). \tag{1}$$

Yielding to the conjectural dictionary between motives $\mathbb{M}/F$ of rank $rk(\mathbb{M}) = n$ and algebraic automorphic representations Π of $\mathrm{GL}_n(\mathbb{A}_F)$, translating $L(s,\mathbb{M})$ into $L(\frac{1-n}{2}+s,\Pi_f)$, one may hope to transfer Deligne's conjecture into the less rigid setup of automorphic representation-theory. Indeed, specializing to cohomological generic representations Π, one may define a *Whittaker period* $p(\Pi) \in \mathbb{C}^\times$ by a fixed comparison of two chosen $\mathbb{Q}(\Pi)$-rational structures, one on the Whittaker model $W(\Pi_f)$ and one on the cohomological realization of Π_f. Deligne's motivic conjecture then translates into a conjectural relation

$$L(\tfrac{1-n}{2}+m,\Pi_f) \sim_{E(\Pi)} p(m,\Pi_\infty)\, p(\Pi), \tag{2}$$

$E(\Pi)$ being a suitable finite extension of the number field $\mathbb{Q}(\Pi)$. In this relation the factor $p(m,\Pi_\infty)$ – essentially the weighted sum of archimedean zeta-integrals at $s = \frac{1-n}{2}+m$, playing the role of $(2\pi i)^{d(m)}$ above – is the most mysterious ingredient: After a line of partial results, B. Sun proved its non-vanishing in great generality; a precise relation to Deligne's all explicit $(2\pi i)^{d(m)}$, however, an indispensable prerequisite in order to retranslate (2) back into (1) is yet to be found.

Rationality for critical values of Rankin-Selberg and Asai L-functions with explicit powers of $(2\pi i)$

In this talk we report on *our solution of this problem* for several L-functions of a large family of automorphic representations over arbitrary CM-fields: More precisely, let

- F any CM-field with maximal totally real subfield F^+, $[F^+:\mathbb{Q}] = d \geq 1$
- Π a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$
- $\Pi' = \Pi_1 + ... + \Pi_k$ an isobaric sum on $\mathrm{GL}_{n-1}(\mathbb{A}_F)$, fully induced from an arbitrary number $k \geq 1$ of cuspidal automorphic representations Π_i,

such that Π_∞ (resp. Π'_∞) is conjugate self-dual and cohomological with respect to E_μ (resp. $E_{\mu'}$). We assume the compatibility-condition between the coefficient modules of Π_∞ and Π'_∞: $\mathrm{Hom}_{\mathrm{GL}_{n-1}(F_\infty\otimes_{\mathbb{Q}}\mathbb{C})}[E_\mu \otimes E_{\mu'},\mathbb{C}] \neq 0$. Then we obtain our two main theorems

Theorem A (Rankin-Selberg L-functions, [Gro-Lin17], Thm. 5.2)**.** For all critical points $s = \frac{1}{2}+m$ of $L(s,\Pi\times\Pi')$,

$$L^S(\tfrac{1}{2}+m,\Pi\times\Pi') \sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\Pi')F^{Gal}} (2\pi i)^{mdn(n-1)-\frac{1}{2}d(n-1)(n-2)}\, p(\Pi)\, p(\Pi')\, \mathcal{G}(\omega_{\Pi'_f}),$$

$\mathcal{G}(\omega_{\Pi'_f})$ being the Gauß-sum of the central character of Π'_f. If $m = 0$ we either assume some auxiliary non-vanishing hypotheses, or sufficient regularity of $\mu\&\mu'$.

Theorem B (Asai L-functions, [Gro-Lin17], Thm. 5.1)**.** Assume that Π' is conjugate self-dual and that μ' is sufficiently regular. Then $L^S(s,\Pi',\mathrm{As}^{(-1)^{n-1}})$ is holomorphic and non-vanishing at $s=1$ and we obtain

$$L^S(1,\Pi',\mathrm{As}^{(-1)^{n-1}}) \sim_{\mathbb{Q}(\Pi')F^{Gal}} (2\pi i)^{d(n-1)}\, p(\Pi').$$

Our two main theorems yield the following corollary.

Corollary 1 ([Gro-Lin17], Thm. 5.6)**.** *If the conditions of both Thm. A & B above are satisfied, then for all critical points* $s = \frac{1}{2} + m$ *of* $L(s, \Pi \times \Pi')$*,*

$$\frac{L^S(\frac{1}{2}+m, \Pi \times \Pi')}{L^S(1, \Pi, \mathrm{As}^{(-1)^n})\ L^S(1, \Pi', \mathrm{As}^{(-1)^{n-1}})} \sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\Pi')F^{Gal}} (2\pi i)^{mdn(n-1)-dn(n+1)/2}.$$

Applications

The refined GGP-conjecture for unitary groups. Our corollary suggests an immediate application to the refined Gan-Gross-Prasad conjecture for unitary groups: Let $G(V)$ and $G(W)$ be unitary groups over F^+ attached to non-degenerate Hermitian spaces of dimension $\dim_F V = n > \dim_F W = n'$. For tempered, decomposable cusp forms $\varphi \in \pi$ and $\varphi' \in \pi'$ the refined GGP-conjecture, cf. [Liu16], asserts that the period-integral $\mathcal{P}(\varphi, \varphi')$ of $\varphi \otimes \varphi'$ satisfies

$$|\mathcal{P}(\varphi, \varphi')|^2 = \frac{\Delta_{G(V)}}{2^a} \frac{L^S(\frac{1}{2}, \pi \times \pi')}{L^S(1, \pi, \mathrm{Ad})\ L^S(1, \pi', \mathrm{Ad})} \prod_{v \in S} \alpha_v(\varphi_v, \varphi'_v).$$

Here, $\alpha_v(\varphi_v, \varphi'_v)$ are local integrals – stabilized and suitably normalized – over certain matrix coefficients, whereas $\Delta_{G(V)}/2^a$ is a rather elementary constant, attached to the (expected) Vogan-Arthur packets of π and π' and the Gross-motives.

An easy observation tells us that $\Delta_{G(V)} \sim_{F^{Gal}} (2\pi i)^{\frac{1}{2}dn(n+1)}$, which equals the inverse of the right-hand-side of the relation of our corollary for $m = 0$. Invoking results of Harris, [MHar13], in the totally definite case we are able to show that

Theorem C ([Gro-Lin17], Thm. 6.8)**.** If V and W are totally definite and if the quadratic base changes $BC(\pi) = \Pi$ and $BC(\pi') = \Pi'$ satisfy both the conditions of Thm. A & B above, then

$$|\mathcal{P}(\varphi, \varphi')|^2 \sim_{E(\pi)E(\pi')} \frac{\Delta_{G(V)}}{2^a} \frac{L^S(\frac{1}{2}, \pi \times \pi')}{L^S(1, \pi, \mathrm{Ad})\ L^S(1, \pi', \mathrm{Ad})} \prod_{v \in S} \alpha_v(\varphi_v, \varphi'_v).$$

A recent theorem of Harder-Raghuram. Independent of our application to the refined GGP-conjecture, we obtain a certain generalization as well as refinement of a result of Harder-Raghuram, cf. [GHar-Rag17], on the ratio of consecutive critical values of the Rankin-Selberg L-function attached to a pair of cuspidal representations (σ, σ') of $\mathrm{GL}_n(\mathbb{A}_{F^+}) \times \mathrm{GL}_{n'}(\mathbb{A}_{F^+})$:

Theorem D ([Gro-Lin17], Thm. 5.5)**.** Assume that the conditions of Thm. A above are satisfied. Let $\frac{1}{2} + m, \frac{1}{2} + \ell$ be two critical points of $L^S(s, \Pi \times \Pi')$. Whenever $L^S(\frac{1}{2} + \ell, \Pi \times \Pi')$ is non-zero (e.g., if $\ell \neq 0$), we obtain

$$\frac{L^S(\frac{1}{2}+m, \Pi \times \Pi')}{L^S(\frac{1}{2}+\ell, \Pi \times \Pi')} \sim_{\mathbb{Q}(\Pi)\mathbb{Q}(\Pi')F^{Gal}} (2\pi i)^{d(m-\ell)n(n-1)}.$$

In particular, the quotient of consecutive critical L-values satisfies

$$(2\pi i)^{dn(n-1)} \frac{L^S(\frac{1}{2}+m, \Pi \times \Pi')}{L^S(\frac{3}{2}+m, \Pi \times \Pi')} \in \mathbb{Q}(\Pi)\mathbb{Q}(\Pi')F^{Gal}.$$

References

[Del79] P. Deligne, *Valeurs de fonctions L et périodes d'intégrales, With an appendix by N. Koblitz and A. Ogus* in: Proc. Sympos. Pure Math., Vol. XXXIII, part II, AMS, Providence, R.I., (1979), pp. 313–346

[Gro-Lin17] *Special values of L-functions and the refined Gan-Gross-Prasad conjecture*, preprint (2017)

[GHar-Rag17] G. Harder, A. Raghuram, *Eisenstein cohomology for $GL(N)$ and ratios of critical values of Rankin-Selberg L-functions. With Appendix 1 by Uwe Weselmann and Appendix 2 by Chandrasheel Bhagwat and A. Raghuram*, preprint (2017)

[MHar13] M. Harris, L-functions and periods of adjoint motives, *Alg. and Number Theory* **7** (2013) 117–155

[Liu16] Y. Liu, Refined global Gan-Gross-Prasad conjecture for Bessel periods, *J. reine angew. Math.* **717** (2016) 133–194

On generalized Fourier transforms in doubling method for classical groups

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We discuss the notion of a generalized Fourier transform introduced by Braverman and Kazhdan on certain space of Schwartz functions on a reductive group, defined by delicately normalized intertwining operators [BK1, BK2]. This is part of their program, now enhanced by ideas of Ngo [N1, N2], to develop a theory of L–functions for any reductive group and any representation of its L–group, generalizing the work of Godement and Jacquet [GJ] in the case of principal L–functions for $GL(n)$.

In the case of standard L–functions for classical groups, one has the work of Piatetski–Shapiro and Rallis [GPSR, PSR, LR], continued by Lapid–Rallis, as well as Jian–Shu Li, S. Yamana and W. T. Gan, which defines and studies these L–functions through their doubling method.

We specialize the Fourier transforms of Braverman–Kazhdan to the case of doubling, showing that they behave the same way as normalized intertwining operators of doubling method which are done by means of "degenerate local coefficients" [PSR, LR]. These local coefficients are defined by degenerate (character) inducing data as opposed to the cases considered in Langlands–Shahidi method [Sh].

Finally, we define the basic function in both setting and show that they are preserved by the Fourier transform. This is the unramified function whose integral against the normalized spherical matrix coefficient of the classical group gives the standard L–function. It requires a shift of s by $-\frac{1}{2}$ in the L–function data which although needed for global reasons, remains mysterious locally and does not produce the shift present in Godement–Jacquet and Ngo [N3]. The geometric aspects of the work involved is presented in a paper by Wen–Wei Li [Li].

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References

[BK1] A. Braverman and D. Kazhdan, γ-functions of representations and lifting. *Geom. Funct. Anal.*, (Special Volume, Part I):237-278, 2000. With an appendix by V. Vologodsky, GAFA 2000 (Tel Aviv, 1999).

[BK2] A. Braverman and D. Kazhdan, Normalized intertwining operators and nilpoent elements in the Langlands dual group, *Moscow Math. J.* 2 (2002), no. 3, 533–553.

[GJ] R. Godement and H. Jacquet, Zeta functions of simple algebras, in *Lecture Notes in Mathematics*, vol. 260 Springer-Verlag, Berlin-New York, 1972.

[GPSR] Stephen Gelbart, Ilya Piatetski–Shapiro, and Stephen Rallis, Explicit constructions of automorphic L–functions, in volume 1254 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, 1987.

[LR] E. M. Lapid and S. Rallis, *On the local factors of representations of classical groups*, in *Automorphic representations, L–functions and Applications: Progress and Prospects*, edited by J. W. Cogdell et al., Ohio State Univ. Math. Res. Inst. Publ. vol. 11, de Gruyter, Berlin, 2005, pp. 309-359.

[Li] Wen-Wei Li, *Zeta integrals, Schwartz spaces and local functional equations*, Preprint, 2015.

[N1] B.C. Ngô, On a certain sum of automorphic L-functions. In *Automorphic Forms and Related Geometry: Assessing the Legacy of I.I. Piatetski-Shapiro*, volume 614 of *Contemp. Math.*, pages 337-343. Amer. Math. Soc., Providence, RI, 2014.

[N2] B.C. Ngô, Semi-group and basic functions, Letter to Sakellaridis.

[N3] B.C. Ngô, Geometry of arc spaces, generalized Hankel transforms and Langlands functoriality, preprint, 2016.

[PSR] I. Piatetski–Shapiro, and S. Rallis, ε factor of representations of classical groups, *Proc. Nat. Acad. Sci. U.S.A.* vol. 83(13): 4589-4593, 1986.

[Sh] F. Shahidi, On certain L-functions, *Amer. J. Math.*, 1981, **103**: 297-355.

A Poisson summation formula for the Rankin-Selberg monoid

Jayce R. Getz

1. Generalizations of Poisson summation

Let F be a number field and let G be a connected reductive group over F. Given a representation $r : {}^LG \to \mathrm{GL}(V_r)$ of its L-group and a cuspidal automorphic representation π of $G(\mathbb{A}_F)$ the associated Langlands L-function $L(s, \pi, \rho)$ is expected to admit a meromorphic continuation to the plane, satisfy a functional equation, and have poles corresponding to fixed vectors of $r \circ \phi(\pi)$, where $\phi(\pi) : \mathcal{L}_F \to {}^LG$ is the (conjectural) Langlands parameter of π.

In the case where r is the standard representation of $G = \mathrm{GL}_n$ these properties were deduced from the Poisson summation formula on the space of $n \times n$-matrices GL_n by Godement and Jacquet. Braverman and Kazhdan introduced an influential conjecture which asserts that this is but the first case of a general phenomenon. More precisely, they conjecture the existence of an r-transform

$$\mathcal{F}_r : \mathcal{S}_r(G(\mathbb{A}_F)) \to \mathcal{S}_r(G(\mathbb{A}_F))$$

from a certain space of conjectural Schwartz functions $\mathcal{S}_r(G(\mathbb{A}_F))$ to itself and the existence of a summation formula generalizing Poisson summation for this transform. There has been quite a bit of work motivated by this conjecture, notably by Bouthier, Cheng, L. Lafforgue, Li, Ngô, and Sakellaridis. I refer to the introduction of my paper [1] for references.

2. The Rankin-Selberg monoid

Motivated by this idea of generalized versions of Poisson summation, in the paper [1] I prove a summation formula for the monoid whose points in an F-algebra R are given by

$$M(R) := \{(X_1, X_2) \in \mathrm{GL}_2^{\oplus 2}(R) : \det X_1 = \det X_2\}.$$

We let G be the unit group of the monoid:

$$G(R) := \{(g_1, g_2) \in \mathrm{GL}_2^{\times 2}(R) : \det g_1 = \det g_2\}.$$

The monoid M is related to the Rankin-Selberg L-function on $\mathrm{GL}_2 \times \mathrm{GL}_2$. We recall that the Rankin-Selberg L-function is the Langlands L-function attached to the exterior tensor product

$$\rho : {}^L\mathrm{GL}_2 \times {}^L\mathrm{GL}_2 \longrightarrow \mathrm{GL}_4(\mathbb{C}).$$

This factors through ${}^LG$.

To state it the summation formula, for $\Phi \in \mathcal{S}(\mathbb{A}_F \times \mathbb{A}_F \times \mathrm{GL}_2^{\oplus 2}(\mathbb{A}_F))$ let

$$\mathcal{I}(\Phi, s)(\gamma) := \int_{\mathbb{A}_F^\times} \left(\int_{\mathrm{GL}_2^{\oplus 2}(\mathbb{A}_F)} \Phi\left(\frac{p(T)}{t}, t, T\right) \psi\left(\frac{\mathrm{tr}\, \gamma T}{t}\right) dT \right) |t|^s dt^\times. \tag{1}$$

This function admits a meromorphic continuation to $\mathrm{Re}(s) > -4$ (see [1]). Let

$$\mathcal{I}(\Phi)(\gamma) = \frac{\mathrm{Res}_{s=-3}\mathcal{I}(\Phi, s)(\gamma)}{\mathrm{Res}_{s=1}\Lambda_F(s)}.$$

Here $\Lambda_F(s)$ is the completed Dedekind zeta function.

Let $h_S \in C_c^\infty((A_{\mathrm{GL}_2}\backslash \mathrm{GL}_2(F_S))^{\times 2})$. Modifying h_S by a central integration which will not be made precise here one arrives at a function $f_S \in C_c^\infty(\mathrm{GL}_2^{\oplus 2}(F_S))$. One then sets

$$\Phi = \Phi_0 \otimes f \in \mathcal{S}(\mathbb{A}_F^2 \times \mathrm{GL}_2^{\oplus 2}(\mathbb{A}_F))$$

where $\Phi_0 \in \mathcal{S}(\mathbb{A}_F^2)$ is any function such that

$$\Phi_0(t, 0) = 0 \text{ for all } t \in \mathbb{A}_F \text{ and } \mathcal{F}_2(\Phi_0)(0, 0) = \Gamma_{F_\infty}(1)$$

where $\mathcal{F}_2$ denotes the Fourier transform in the second variable and $\Gamma_{F_\infty}(s)$ is the archimedean factor of $\Lambda_F(s)$. Let

$$\Phi^{\mathrm{sw}}(x, y, T) = \Phi(y, x, T)$$

(the sw is for "switch"). Finally let

$$\omega : G \longrightarrow \mathbb{G}_m \tag{2}$$

be the character given on points by $(g_1, g_2) \mapsto \det g_1$.

Theorem 1 (Getz). *Assume that the endomorphism of $L^2([\mathrm{GL}_2]^2)$ given by $h_S 1_{\mathrm{GL}_2^{\times 2}(\widehat{\mathcal{O}}_F^S)}$ has cuspidal image. Then*

$$\begin{aligned}&|\omega(g_1g_2^{-1})|^4\sum_{\gamma\in M(F)}\mathcal{I}(\Phi)(g_1\gamma g_2^{-1})\\ =&|\omega(g_1g_2^{-1})|\sum_{\gamma\in M(F)}\mathcal{I}(\Phi^{\mathrm{sw}})(\omega(g_1^{-1}g_2)g_1\gamma g_2^{-1})\\ &+\sum_{\pi=\pi_0\otimes\pi_0^\vee}\frac{\mathrm{Res}_{s=1}L^S(s,\pi,\rho)}{\zeta_F^S(2)}K_{\pi\otimes|\det|(h_S 1_{\mathrm{GL}_2^{\times 2}(\widehat{\mathcal{O}}_F^S)})}(g_1,g_2).\end{aligned}$$

where the sum on π_0 is over cuspidal automorphic representations of $A_{\mathrm{GL}_2}\backslash\mathrm{GL}_2(\mathbb{A}_F)$.

The proof of this theorem uses the expansion of the δ-function developed by Duke, Friedlander and Iwaniec which was cast in the framework of the circle method by Heath-Brown.

3. Avenues for future work

I now explain an application which in fact was my motivation for pursuing this project in the first place. Assume that there is a subfield $k \leq F$ such that F/k is Galois with Galois group $\mathrm{Gal}(F/k) = \langle \iota, \tau\rangle$; that is, $\mathrm{Gal}(F/k)$ is generated by two elements. For example $\mathrm{Gal}(F/k)$ could be any simple nonabelian group.

Let G_0 be the connected reductive k-subgroup of $\mathrm{Res}_{F/k}G$ whose points in a k-algebra R are given by

$$G_0(R) := \{(g_1,g_2)\in G(R\otimes_k F) : \det g_1 = \det g_2 \in R^\times\}.$$

This k-subgroup comes equipped with an action of $\mathrm{Gal}(F/k)^2$, and in particular we have an automorphism θ of G_0 given on points by $\theta(g_1,g_2) := (\iota(g_1),\tau(g_2))$. Set

$$\Sigma(g_1,g_2) := \sum_{\pi=\pi_0\otimes\pi_0^\vee}\frac{\mathrm{Res}_{s=1}L^S(s,\pi,\rho)}{\zeta_F^S(2)}K_{\pi\otimes|\det|(h_S 1_{\widetilde{G}(\widehat{\mathcal{O}}_F^S)})}(g_1,g_2) \tag{3}$$

$$\begin{aligned}&= |\omega(g_1g_2^{-1})|^4\sum_{\gamma\in M(F)}\mathcal{I}(\Phi)(g_1\gamma g_2^{-1}) &(4)\\ &\quad -|\omega(g_1g_2^{-1})|\sum_{\gamma\in M(F)}\mathcal{I}(\Phi^{\mathrm{sw}})(\omega(g_1^{-1}g_2)g_1\gamma g_2^{-1}).\end{aligned}$$

(the second equality is the content of Theorem 1).

Consider

$$\int_{[G_0]}\Sigma(g,\theta(g))dg = \int_{[G_0]}\sum_{\pi=\pi_0\otimes\pi_0^\vee}\frac{\mathrm{Res}_{s=1}L^S(s,\pi,\rho)}{\zeta_F^S(2)}K_{\pi\otimes|\det|(h_S 1_{\widetilde{G}(\widehat{\mathcal{O}}_F^S)})}(g,\theta(g))dg. \tag{5}$$

Only representations π_0 such that all of the Galois conjugates π_0^σ, $\sigma\in\mathrm{Gal}(F/k)$, are isomorphic up to a twist by a character can contribute to this sum. On the other hand, it should certainly be possible (after applying an Arthur-style truncation, perhaps) to evaluate this expression geometrically using the right hand side of (3). This will yield a nonabelian trace formula, that is, a trace formula isolating representations invariant under a nonabelian group of automorphisms.

Making this precise seems tractable, but the end goal would be to compare this nonabelian trace formula to a trace formula over k and establish base change and descent of automorphic representations of GL_2 over nonabelian Galois extensions (see [2, 3]). I have no paradigm or heuristic for executing this comparison. Nevertheless, it should be possible to gain insight. One project that seems more manageable is examining (5) for a family of test functions h giving a count of the number of representations in a tower of $\mathrm{Gal}(F/k)$-invariant congruence subgroups of $G_0(\mathbb{A}_F)$. The goal would be to see if the number, on average, is as expected assuming base change and descent. This would give a numeric version of base change.

References

[1] J. R. Getz. A summation formula for the Rankin-Selberg monoid via the circle method. *ArXiv:1409.2360*, September 2014.

[2] Jayce R. Getz. An approach to nonsolvable base change and descent. *J. Ramanujan Math. Soc.*, 27(2):143–211, 2012.

[3] Jayce R. Getz and Paul Edward Herman. A nonabelian trace formula. *Res. Math. Sci.*, 2:2:14, 2015.

A-packets in the archimedean case

Colette Moeglin

(joint work with David Renard)

The talk given is a report about joint work with David Renard. Let F be a local field of characteristic 0, G_0 be a quasi split reductive group and G a pure inner form of G, both defined over F. We will identify G and G_0 with their F points. We denote by W_F' the Weil-Deligne group of F. If F is archimedean, this is nothing but the Weil group and if F is p-adic, this is the direct product of the Weil group of F with $SL(2,\mathbb{C})$. And we denote by $\hat{G}$ the connected component of ${}^LG$, the L-group of G. We assume that G_0 is a classical group.

First of all, we recall the definition of A-packets following Arthur and specially [2]. Let ψ be a morphism of $W_F' \times SL(2,\mathbb{C}$ in $\hat{G}$. We assume that the restriction of ψ to W_F' is tempered and that the restriction of ψ to $SL(2,\mathbb{C})$ is algebraic. Tempered means essentialy bounded on W_F and algebraic on $SL(2,\mathbb{C})$ if F is p-adic, but this is not general enough if G has a non finite center. We call such a morphism an A-morphism.

All the normalization for the transfer factors and the twisted trace are done using the Whittaker normalization.

To an A-morphism, we consider $A(\psi)$ the group of components:

$$A(\psi) = Cent_{\hat{G}}(\psi)/Cent_{\hat{G}}(\psi)^0,$$

and s_ψ the image of the non trivial element of $SL(2,\mathbb{C})$ by ψ.

To any $s \in Cent_{\hat{G}}(\psi)$ such that $s^2 = 1$, we consider the elliptic endoscopic datum of G, $(s, H, Cent_{\hat{G}}(s)^0\psi(W_F))$, where H is the quasi-split classical group such that ${}^LH$ is isomorphic to $Cent_{\hat{G}}(s)^0\psi(W_F))$ (no auxiliary data are needed in our simple situation). In this way ψ defines naturally an A-morphism for H, denoted ψ_s.

The A-packet associate to ψ is a linear combination with complex coefficients of irreducible representations of $G \times A(\psi)$, denoted $\pi^A_G(\psi)$, satisfying the following transfer relations:

$\pi^A_{G_0}(\psi)(s_\psi)$ is stable and its transfer to the twisted $GL(dim\psi, F)$ is the twisted trace of the representation of $GL(dim\psi, F)$ associated to ψ. Here $dim\psi$ is the dimension of ψ viewed as a representation of $W'_F \times SL(2,\mathbb{C})$ when we compose with the natural representation of the L-group of G.

for any s as above, $e^K(G)\pi^A_G(\psi)(ss_\psi)$ is the transfert of the stable representation $\pi^A_H(\psi_s)(s_\psi)$ of the endoscopic datum associate to s and ψ as above, where $e^K(G)$ is the Kottwitz' sign.

For these properties, we recall that the local spectral transfer is defined in a very great generality as a map sending a stable finite linear combination (with coefficients in $\mathbb{C}$) of irreducible representations of an elliptic endoscopic datum of G to a finite linear combination (with coefficients in $\mathbb{C}$) of irreducible representations of the group G. For the ordinary transfer, this is due to Shelstad ([15]) and Arthur (cf. [3]) and for the twisted transfer (which covers the ordinary transfer) this is [13] IV.3.3 and [14] Appendice.

We say that $\pi^A_G(\psi)$ has multiplicity one for G, if when we write $\pi^A_G(\psi)$ in the basis of irreducible representations of G the coefficients are irreducible characters of $A(\psi)$ instead of a linear combination with coefficient in $\mathbb{C}$ of characters.

Another more familiar way to state the conjecture is the following: first of all, remark that if $Z(\hat{G})$ the center of $\hat{G}$ is non trivial, there exists a character, η_G of this center, character which depends on the inner form G such that for any $s \in A(\psi)$ and any z in this center:

$$\pi^A_G(\psi)(sz) = \eta_G(z)\pi^A_G(\psi)(s).$$

For any character ϵ of $A(\psi)$ with restriction η_G to the center of $\hat{G}$ (if such a center exists) define:

$$\pi^A_G(\psi)_\epsilon := |A(\psi)/Z(\hat{G})|^{-1} \sum_{s \in A(\psi)/Z(\hat{G})} \epsilon(s)\pi^A_G(\psi)(s).$$

Then the multiplicity one conjecture, asserts that for any ϵ as above, $\pi^A_G(\psi)_\epsilon$ is a sum of irreducible representations of G, each occuring with multiplicity one and for $\epsilon' \neq \epsilon$ the representations $\pi^A_G(\psi)_\epsilon$ and $\pi^A_G(\psi)_{\epsilon'}$ are disjoint.

This is now well known in the case where ψ is trivial on $SL(2,\mathbb{C})$, the tempered case, where in fact $\pi^A_G(\psi)$ is irreducible (or zero). This is due to Shestad and Mezo in the archimedean case (see [16] and [7]), to Arthur in [4] for the p-adic case with a method which extends to unitary groups (see for example [5]) and for any G as above (see the explanations in [11]).

The conjecture is that $\pi^A_G(\psi)$ has multiplicity one for G for any ψ with of course a more complicate statement than in the tempered case, because $\pi^A_G(\psi)_\epsilon$ is no more irreducible (or zero) for general ψ.

This is known if F is a p-adic field ([8]), for F a complex field ([10]). For $F = \mathbb{R}$, this is known if the infinitesimal character is integral and regular ([1]) and this is also known for G an orthogonal or symplectic group if the restriction of W_F to the subgroup $\mathbb{C}^*$ of $W_{\mathbb{R}}$ is trivial and all the subrepresentations occuring in ψ are symplectic if $\hat{G}$ is symplectic and orthogonal if $\hat{G}$ is orthogonal ([9])

In the talk we have discuss the case of real unitary groups where it is easy to describe $\pi^A_G(\psi)$ in terms of cohomological and parabolic inductions. Even with this very explicit description the conjecture is not yet totaly proved. And we have sent the audience to [12] for what is known if G is a real orthogonal or symplectic group. In these last cases, we also have an explicit description of $\pi^A_G(\psi)$ but it is much more difficult than in the case of unitary groups to explicitly decompose $\pi^A_G(\psi)$ in irreducible representations and the multiplicity one is not proved. But we have proven that for any ϵ a character of $A(\psi)$, $\pi^A(\psi)_\epsilon$ is a sum with positive integer coefficients of unitary representations of G.

References

[1] N. Arancibia, C. Moeglin, D. Renard, *Paquets d'Arthur des groupes classiques et unitaires, cas cohomologique*, to appear in Annales de la Faculté des sciences de Toulouse, arXiv: 1507.01432

[2] J. Arthur, *Autommorphic representations for GSp4*, in Contribution to Automorphic Forms, Geometry and Number Theory, a volume in honor of J. Shalika, John Hopkins University Press, H. Hida, D. Ramakrishnan, F. Shahidi editors, (2004) 65–82

[3] J. Arthur, *On local character relations*, Selecta **2** (1996), 501–579.

[4] J. Arthur, *The endoscopic classification of representations: orthogonal and symplectic groups* AMS colloquium 61 (2013)

[5] T. Kaletha, A. Minguez, S.W. Shin, P.J. White, *Endoscopic Classification of Representations: Inner Forms of Unitary Groups* , preprint (2014) arXiv:1409.3731

[6] R. Kottwitz and D. Shelstad, *Foundations of twisted endoscopy*, Astérisque **255** (1999)

[7] P. Mezo, *Tempered spectral transfer in the twisted endoscopy of real groups*, http://people.math.carleton.ca/ mezo/research.html.

[8] C. Moeglin, *Multiplicité un dans les paquets d'Arthur* in On certain L-functions, volume in honor of 60th Birthday of F. Shahidi, Clay Math Proceedings **13** (2011), 333–374

[9] C. Moeglin, *Paquets d'arthur spéciaux unipotents aux places archimédiennes et correspondance de Howe*, Progress in Math, Proceedings of the conference in honnor of R. Howe; Cogdell J., Kim J., Zhu C. editors, to appear.

[10] C. Moeglin and D. Renard *Paquets d'arthur des groupes classiques complexes*, *Contemporary Math., to appear.*

[11] C. Moeglin and D. Renard *Sur les paquets d'Arthur des groupes classiques et unitaires non quasi-déployés*, preprint (2017)

[12] C. Moeglin and D. Renard *Sur les paquets d'Arthur des groupes rels*, preprint (2017) arXiv:1703.07226

[13] C. Moeglin and J.L. Waldspurger, *Stabilisation de la formule des traces tordue, vol I* PM 316 Birkhäuser editor (2016)

[14] C. Moeglin and J.L. Waldspurger, *Stabilisation de la formule des traces tordue, vol II* PM 317 Birkhäuser editor (2016)

[15] , D. Shelstad, *Characters and inner forms of a quasi-split group over* $\mathbb{R}$, Compositio **39** (1979) 11–45

[16] ,D. Shelstad *L-indistinguishability for real groups*, *Math. Ann.* **259** (1982), no. 3, p. 385–430.

Arthur Packets from Harmonic Analysis and Sheaf Theory

Paul Mezo

Our talk concerns a comparison of twisted endoscopy from two fundamentally different points of view. The original point of view relies on harmonic analysis. The alternative point of view, relying on sheaf theory, is due to Adams, Barbasch and Vogan ([ABV92]). In order to describe the comparison, we will provide a simplified overview of both perspectives. Experts will realize that we have omitted numerous technicalities in the overview, but our statements will be very nearly true for connected, split and adjoint real algebraic groups G. The actual level of generality is that of a connected reductive algebraic group defined over the real numbers $\mathbb{R}$.

We begin with the original version of endoscopy due to Langlands and Shelstad. Langlands partitioned the irreducible representations of the group $G(\mathbb{R})$ into L-packets Π_ϕ. The index ϕ is an L-parameter, which we will take to be a homomorphism

$$\phi : W_{\mathbb{R}} \to {}^\vee G \tag{1}$$

of the real Weil group into the dual group of G. Let us write the local Langlands Correspondence as

$$\Pi_\phi \longleftrightarrow \phi. \tag{2}$$

If we restrict our attention to tempered representations then every representation $\pi \in \Pi_\phi$ further corresponds to a unique irreducible representation $\tau_\phi(\pi)$ of a finite abelian group which is the component group of the centralizer ${}^\vee G_\phi$. The main point here is that for irreducible tempered representations of $G(\mathbb{R})$ we have a more refined one-to-one correspondence

$$\pi \longleftrightarrow (\phi, \tau_\phi(\pi)). \tag{3}$$

A typical example of standard endoscopy begins by taking an element s in the centralizer ${}^\vee G_\phi$, and setting ${}^\vee H$ equal to the identity component of the fixed-point subgroup of its inner automorphism $\mathrm{Int}(s)$. The group ${}^\vee H$ is the dual group of a reductive algebraic group H– an *endoscopic* group. By construction we have $\phi(W_\mathbb{R}) \subset {}^\vee H \subset {}^\vee G$, so we may define $\phi_H : W_\mathbb{R} \to {}^\vee H$ from (1) by simply replacing the codomain of ϕ with ${}^\vee H$. The facile relationship between ϕ_H and ϕ becomes a deep theorem when comparing the L-packets Π_{ϕ_H} and Π_ϕ. For tempered representations the comparison takes the form of the *spectral transfer* theorem

$$\mathrm{Lift}\left(\sum_{\pi_H \in \Pi_{\phi_H}} \Theta_{\pi_H}\right) = \sum_{\pi \in \Pi_\phi} \tau_\phi(\pi)(\dot{s})\ \Theta_\pi. \tag{4}$$

Here, $\dot{s}$ is the component of $s \in {}^\vee G_\phi$, and Θ_π and Θ_{π_H} are distribution characters. It is in this distributional sense that the spectral transfer theorem is based on harmonic analysis.

One would like to extend the spectral transfer theorem to include nontempered representations. In this direction, Arthur has proposed extensions of L-parameters and L-packets. These extensions are usually called *A-parameters* and *A-packets*. The A-packets are designed to accommodate the nontempered representations which appear as factors of automorphic representations. There is a conjectural extension of (4) for A-packets, which has been proved when G is symplectic or special orthogonal ([Art13]).

The proofs of the known cases of spectral transfer for A-packets rely on an extension of (4) in a different direction. This is the extension to *twisted* endoscopy, which includes endoscopic groups H defined using *outer* automorphisms of ${}^\vee G$.

Example 1. *Take G to be a general linear group. Then ${}^\vee G$ is also a general linear group. Replace the inner automorphism* $\mathrm{Int}(s)$ *above by the outer automorphism of ${}^\vee G$ defined by inverse-transpose. Then ${}^\vee H$ is a special orthogonal group, and the endoscopic group H is either special orthogonal or symplectic.*

A twisted spectral transfer theorem for A-packets was proven in the framework of Example 1 ([Art13]). Its proof relies on the twisted spectral transfer theorem for tempered representations.

We now turn to endoscopy from the sheaf-theoretic perspective, juxtaposing the main points. The first point is the local Langlands Correspondence (3). In [ABV92] the set of L-parameters is replaced by a topological space $X({}^\vee G)$. There is a ${}^\vee G$-action on $X({}^\vee G)$, and the ${}^\vee G$-orbits $S \subset X({}^\vee G)$ are in bijection with (equivalence classes of) L-parameters. Correspondence (2) is now enriched as

$$\Pi_\phi \leftrightarrow \phi \leftrightarrow S.$$

One motivation for introducing $X({}^\vee G)$ is that the closure relations between ${}^\vee G$-orbits imply relationships between the representations of corresponding L-packets. These implications are absent in (2) alone.

Correspondence (3) is also strengthened in [ABV92]. For each ${}^\vee G$-orbit S, an extension of the component group of ${}^\vee G_\phi$ is defined. This extension is still a finite abelian group. Its irreducible representations are in bijection with an *extended* L-packet $\Pi_S \supset \Pi_\phi$. The packets Π_S are extended since they may include representations of real forms of G other than just the given real form $G(\mathbb{R})$. More symbolically, there is a bijection

$$\pi \longleftrightarrow (S, \tau_S(\pi)) \tag{5}$$

where π runs over *all* irreducible representations of some real forms of G including $G(\mathbb{R})$. Correspondence (3) may be recovered from (5) by restricting to tempered representations.

Another striking feature of (5) is that the pair $\xi = (S, \tau_S(\pi))$ determines an irreducible perverse sheaf $P(\xi)$ on $X({}^\vee G)$. Suffice it to say that ξ defines a sheaf (actually a local system) of vector spaces on $S \subset X({}^\vee G)$, and that $P(\xi)$ is some sort of extension of this sheaf to a chain complex of sheaves on $X({}^\vee G)$. Taking this leap for granted, we rewrite (5) as

$$\pi(\xi) \leftrightarrow \xi \leftrightarrow P(\xi). \tag{6}$$

Correspondence (6) expresses a duality between irreducible representations and irreducible perverse sheaves.
The spectral transfer theorem (4) takes on roughly the following form in [ABV92]

$$\mathrm{Lift}\left(\sum_{\pi_H \in \Pi_{S_H}} \pi_H\right) = \sum_{\pi \in \Pi_S} \tau_S(\pi)(\dot{s})\ \pi. \tag{7}$$

(We have suppressed some signs.) The sums here are virtual representations with irreducible tempered summands. The lift in (7) is defined by applying the duality of (6) to the equation

$$\sum_{\xi_H = (S_H, \tau')} P(\xi_H) = \mathrm{Rest}\left(\sum_{\xi = (S,\tau)} \tau(\dot{s})\ P(\xi)\right).$$

On the right, Rest denotes the restriction functor (the inverse image functor induced by the inclusion ${}^\vee H \hookrightarrow {}^\vee G$). This is an identity of virtual perverse sheaves, which has no direct connection to harmonic analysis.

The spectral transfer theorem (7) has a generalization to nontempered representations. However, to properly describe requires the theory of *microlocal geometry.* We limit ourselves to saying that in the nontempered situation, the extended packets Π_{S_H} and Π_S of (7) are replaced by *micro-packets*, and that the extended component groups are replaced by *micro-component* groups. It is furthermore shown that an A-parameter specifies a unique micro-packet and micro-component group. Thus, sheaf-theoretic A-packets are a special class of micro-packets. This leads to the following question:

Are Arthur's A-packets in [Art13] equal to A-packets of [ABV92]?

Recall that Arthur's A-packets were defined using twisted endoscopy as in Example 1. The most direct solution to this question should therefore employ twisted endoscopy using the sheaf-theoretic framework of [ABV92].

References

[ABV92] J. D. Adams, D. Barbasch, and D. A. Vogan. *The Langlands classification and irreducible characters of real reductive groups.* Birkhäuser, 1992.

[Art13] J. Arthur. *The endoscopic classification of representations. Orthogonal and symplectic groups.* American Mathematical Society, 2013.

Arthur packets and Adams-Barbash-Vogan packets for p-adic groups

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(joint work with Andrew Fiori, James Mracek and Ahmed Moussaoui)

1. Overview

Our project concerns Arthur packets, which are objects originating from the classification of automorphic representations. With a view to a local description of Arthur packets, Adams-Barbasch-Vogan [1] initiated the study of the geometry of certain parametrizing spaces of Langlands parameters for connected real reductive algebraic groups and, soon after, Vogan [5] included p-adic groups in this story. Their analysis of the geometry of these parametrizing spaces led to what are now known as ABV-packets and also led to the conjecture that local Arthur packets are ABV-packets. On the other hand, not until a few years ago do the Arthur packets really become known for quasisplit symplectic and orthogonal groups by Arthur [2], and for unitary groups by Mok [4], Kaletha-Minguez-Shin-White [3]. So our main intention in this project is to see how representation theory and geometry interact through local Arthur packets under these two seemingly very different perspectives.

2. Parametrizing space of Langlands parameters

Let F be a p-adic field and G be a quasi-split connected reductive algebraic group over F. We fix an L-group ${}^LG$ for G and equip the complex dual group $\widehat{G}$ with the discrete topology. A Langlands parameter of G is a continuous homomorphism $\phi : L_F \to {}^LG$ from $L_F := W_F \times \mathrm{SL}(2,\mathbb{C})$ to the L-group of G such that: it commutes with projections to W_F on both sides; $\phi|_{\mathrm{SL}(2,\mathbb{C})}$ is algebraic; and $\phi(W_F)$ consists of semisimple elements of ${}^LG$. We denote the set of Langlands parameters by $P({}^LG)$. There is a natural $\widehat{G}$-equivariant fibration $P({}^LG) \to X({}^LG)$, to a complex ind-variety $X({}^LG)$ equipped with an action of $\widehat{G}$ such that the fibre over any $x \in X({}^LG)$ is a principle homogeneous space for the unipotent radical of $\mathrm{Stab}_{\widehat{G}}(x)$. In fact, it is $X({}^LG)$ rather than $P({}^LG)$ that we would like to consider as the parametrizing space of Langlands parameters. The components of the parametrizing space $X({}^LG)$ are complex varieties, each stabilized by the action of $\widehat{G}$.

Our interest lies in studying the category $\mathsf{Perv}_{\widehat{G}}(X({}^LG))$ of $\widehat{G}$-equivariant perverse sheaves on $X({}^LG)$, each component separately. The isomorphism classes of simple objects in this category are in bijection with pairs (C,ρ) with C an $\widehat{G}$-orbit in $X({}^LG)$ and $\rho \in \mathrm{Irrep}(A_C)$, where A_C is the equivariant fundamental group of C, *i.e.*, $A_C \cong \pi_0(\mathrm{Stab}_{\widehat{G}}(x))$ for any $x \in C$. There is a similar bijection on the representation side; in particular, if $\phi \in P({}^LG)$ is a preimage of $x \in X({}^LG)$ then $A_\phi := \pi_0(\mathrm{Stab}_{\widehat{G}}(\phi))$ is naturally isomorphic to $\pi_0(\mathrm{Stab}_{\widehat{G}}(x))$.

3. Local Langlands Correspondence for pure rational forms

A pure rational form δ of G is a 1-cocycle of the absolute Galois group Γ_F in G. It determines an inner form G_δ of G together with an embedding $G_\delta(F) \hookrightarrow G(\bar{F})$. Let $G(\bar{F})$ act by conjugation on pairs of (π,δ) where δ is a pure rational form and π is an isomorphism class of admissible representations of $G_\delta(F)$. The set of $G(\bar{F})$-conjugacy classes will be denoted by $\Pi_{\mathrm{pure}}(G/F)$. The local Langlands correspondence for pure rational forms of G can be stated as follows: *There is a natural bijection between $\Pi_{\mathrm{pure}}(G/F)$ and $\widehat{G}$-conjugacy classes of pairs (ϕ,ρ) with $\phi \in P({}^LG)$ and $\rho \in \mathrm{Irrep}(A_\phi)$.* This conjecture is known for general linear groups and unitary groups, and the case of symplectic and orthogonal groups would essentially follow from the results claimed by Arthur [2, Theorem 9.4.1]. The local Langlands correspondence for pure rational forms of G determines a natural bijection

$$\Pi_{\mathrm{pure}}(G/F) \to \mathsf{Perv}^{\mathrm{simple}}_{\widehat{G}}(X({}^LG))_{/\mathrm{iso}}, \qquad [\pi,\delta] \mapsto \mathcal{P}(\pi,\delta).$$

4. Arthur parameters and Arthur packets

An Arthur parameter for G is a continuous homomorphism $\psi : L_F \times \mathrm{SL}(2,\mathbb{C}) \to {}^LG$ such that: $\psi|_{L_F}$ is a Langlands parameter of G; $\psi|_{\mathrm{SL}(2,\mathbb{C})}$ is algebraic; and the projection of $\psi(W_F)$ to $\widehat{G}$ has compact closure. An Arthur parameter ψ determines a Langlands parameter ϕ_ψ by $\phi_\psi(u) = \psi(u, \mathrm{diag}(|u|^{1/2}, |u|^{-1/2}))$ for $u \in L_F$, where the norm map on L_F is simply the pull-back of that on W_F. This induces an inclusion of the set of $\widehat{G}$-conjugacy classes of Arthur parameters into the set of $\widehat{G}$-conjugacy classes of Langlands parameters.

Arthur's work leads to the following general conjecture on Arthur packets for pure rational forms of G: *Every Arthur parameter ψ naturally determines a finite subset $\Pi_{\mathrm{pure},\psi}(G/F)$ of $\Pi_{\mathrm{pure}}(G/F)$ together with a canonical map*

$$\Pi_{\mathrm{pure},\psi}(G/F) \to \mathrm{Rep}(A_\psi), \qquad [\pi,\delta] \mapsto \langle \cdot, [\pi,\delta]\rangle_\psi,$$

where $A_\psi = \pi_0(Stab_{\widehat{G}}(\psi))$, $\mathrm{Rep}(A_\psi)$ is the set of isomorphism classes of representations of A_ψ and $\langle \cdot, [\pi,\delta]\rangle_\psi$ is the associated character of A_ψ.

We call $\Pi_{\mathrm{pure},\psi}(G/F)$ a pure Arthur packet for G. This conjecture is known for general linear groups and unitary groups; the case of symplectic and orthogonal groups would essentially follow from Arthur's conjecture [2, Conjecture 9.4.2] for inner forms.

5. Vanishing cycles of perverse sheaves

Let ψ be an Arthur parameter for G, and let C_ψ be the $\widehat{G}$-orbit in $X(^LG)$ determined by the image $x_\psi \in X(^LG)$ of ϕ_ψ. The conormal bunde $T^*_{C_\psi}X(^LG)$ has a unique open dense $\widehat{G}$-orbit $T^*_{C_\psi}X(^LG)_{\mathrm{sreg}}$ and the Arthur parameter ψ determines a conormal vector (x_ψ, ξ_ψ) in this orbit. Moreover, the equivariant fundamental group of $T^*_{C_\psi}X(^LG)_{\mathrm{sreg}}$ is canonically isomorphic to A_ψ. Using the vanishing cycle functor, we construct a functor

$$\mathsf{Ev}^0_\psi : \mathsf{Perv}_{\widehat{G}}(X(^LG)) \to \mathsf{Rep}(A_\psi)$$

to the category $\mathsf{Rep}(A_\psi)$ of representations of A_ψ which plays the role of the microlocalization functor in the work of Adams-Barbash-Vogan and Vogan. In particular, the relation between this functor and characteristic cycles is as follows. For any equivariant perverse sheaf $\mathcal{P}$ on $X(^LG)$, the multiplicity of the conormal bundle $[T^*_{C_\psi}X(^LG)]$ in the characteristic cycle of $\mathcal{P}$ is equal to the dimension of $\mathsf{Ev}^0_\psi(\mathcal{P})$.
The conjectural geometric construction of local Arthur packets by Adam-Barbasch-Vogan and Vogan now takes the following form.

Conjecture 1. *For any Arthur parameter ψ of G,*

$$\Pi_{\mathrm{pure},\psi}(G/F) = \{[\pi,\delta] \in \Pi_{\mathrm{pure}}(G/F) | \mathsf{Ev}^0_\psi(\mathcal{P}(\pi,\delta)) \neq 0\}.$$

Moreover, for any $[\pi,\delta] \in \Pi_{\mathrm{pure},\psi}(G/F)$ and any $a \in A_\psi$,

$$\langle a, [\pi,\delta]\rangle_\psi = \mathrm{Trace}\, \mathsf{Ev}^0_\psi(\mathcal{P}(\pi,\delta))(a).$$

We will prove this conjecture in certain cases in a series of papers currently in preparation.

References

[1] Jeffrey Adams, Dan Barbasch and David Vogan, *The Langlands classification and irreducible characters for real reductive groups*, Progress in Mathematics, vol. 104, Birkhäuser Boston, Inc., Boston, MA, (1992), pp. xii + 318.

[2] James Arthur, *The endoscopic classification of representations: orthogonal and symplectic groups*, American Mathematical Society Colloquium Publications, vol. 61, American Mathematical Society, Providence, RI, (2013).

[3] Tasho Kaletha, Alberto Minguez, Sug-Woo Shin, and Paul-James White, *Endoscopic classification of representations: inner forms of unitary groups*, preprint, arXiv:1409.3731 (2014).

[4] Chung-Pang Mok *Endoscopic classification of representations of quasi-split unitary groups*, Memoirs of the American Mathematical Society, vol. 235, no. 1108, (2014), pp. vi + 248.

[5] David Vogan, *The local Langlands conjecture*, Representation theory of groups and algebras, Contemp. Math., vol. 145, Amer. Math. Soc., Providence, RI, (1993), pp. 305–379.

Counting cusp forms by analytic conductor

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(joint work with Djordje Milićević)

Motivation. Let X_0 be a smooth projective genus g absolutely irreducible curve over a finite field $\mathbb{F}_q$ of characteristic p. Let $\mathbb{F} = \overline{\mathbb{F}}_q$ be an algebraic closure. Let $X = X_0 \otimes_{\mathbb{F}_q} \mathbb{F}$ be the base change of X_0 to $\mathbb{F}$. Write Fr for the Frobenius endomorphism of the scheme X. Fix a prime $\ell \neq p$ and consider the set E of isomorphism classes of irreducible rank 2 $\overline{\mathbb{Q}}_\ell$-local systems on X. For every integer $m \geq 1$ the m^{th}-iteration Fr^m acts on E. Let $E_m = E^{\mathrm{Fr}^m}$ denote the set of fixed points of Fr^m.

In 1981, Drinfeld proved the following beautiful

Theorem 1 (Drinfeld)**.** *If $g > 1$ then*

$$\#E_m = q^{(4g-3)m} + \sum_{i=1}^{k} r_i \mu_i^m, \tag{1}$$

where $k \geq 1$, $r_i \in \mathbb{Z}$, and the μ_i are Weil numbers of $\mathrm{wt}/2 < 4g - 3$.

The proof goes as follows: thanks to Drinfeld's Galois to automorphic correspondence over function fields, the set E_m can be identified with the set of everywhere unramified cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_{K_m})$, where K_m is the function field of X_m and $X_m = X_0 \otimes_{\mathbb{F}_q} \mathbb{F}_{q^m}$. (We're neglecting technical issues regarding twisting.) One can then use the Jacquet-Langlands-Selberg trace formula to count the latter set; the identity contribution gives the main term of $q^{(4g-3)m}$.

What is most interesting about this theorem is the following observation, again due to Drinfeld: the formula (1) is reminiscent of the Lefschetz fixed point formula for a smooth projective algebraic variety over $\mathbb{F}_q$ of dimension $4g-3$. One would then like to view E as a space having an appropriate cohomology theory $H^\bullet(E)$ for which $\#E_m$ can be expressed as an alternating trace of Fr^m acting on $H^\bullet(E)$. These ideas have inspired recent work by Deligne and Flicker, but this dream remains far from a reality.

Number field setting. We now move to a number field, but, for lack of an underlying curve, a global Frobenius endomorphism, or the global Arthur L-group, we pass directly to the automorphic side, as in the proof of Drinfeld's theorem.

Let F be a number field and $n \geq 1$ an integer and write $G = \mathrm{GL}_n$ as a group over F. The set of all cuspidal automorphic representations of $G(\mathbb{A}_F)^1$ will be denoted by $\mathcal{A}_0$. Here $G(\mathbb{A}_F)^1 = A_G \backslash G(\mathbb{A}_F)$, where A_G is the connected component of the split part of the $\mathbb{R}$-points of $\mathrm{Res}_{F/\mathbb{Q}}(Z)$. Then $\mathcal{A}_0$ is a discrete subset of the restricted tensor product of the unitary duals $\Pi(G_v)$ over all places v.

The aim here is to truncate $\mathcal{A}_0$ in some natural way and study the asymptotics of resulting exhaustive sequence of finite sets. The hope would then be to see a parallel with an analogous geometric counting problem, as was the case for Drinfeld.

With this in mind, for a real parameter $Q \geq 1$ we define

$$\mathfrak{F}(Q) = \{\pi \in \mathcal{A}_0 : Q_\pi \leq Q\},$$

where Q_π is the analytic conductor of π. Following Sarnak, this is referred to as the *universal family*. As a family of cusp forms, one can study various arithmetic statistics associated with it, such as the symmetry type and Sato-Tate measure. The most basic invariant of this family is its asymptotic size, which is established in the following

Theorem 2 (B.-Milićević).
(A) : Assume $n \leq 2$. There is $C > 0$ such that

$$\#\mathcal{F}(Q) = CQ^{n+1} + O(Q^{n+1}/\log Q).$$

(B) : Let $n \geq 1$ and

$$\mathcal{F}_{\text{Maass}}(Q) = \{\pi \in \mathcal{F}(Q) : \pi_\infty \textit{ is spherical}\}.$$

There is $C_{\text{Maass}} > 0$ such that

$$\#\mathcal{F}_{\text{Maass}}(Q) = C_{\text{Maass}}Q^{n+1} + O(Q^{n+1}/\log Q).$$

Remarks 1. *The implied constants in the error terms (as well as the main terms) depend on n and F: we do not track the uniformity in n and F. Still, one could pose the following interesting questions in the n and F aspects:*

1. *fix n, set $Q = 1 + \epsilon$, and allow the discriminant $D_F \to \infty$; counting such forms would then be similar to Drinfeld's situation.*
2. *fix F, set $Q = 1 + \epsilon$, and allow $n \to \infty$; counting such forms recovers the number field analog of a question of Venkatesh.*

Remarks 2. *The restriction to the Maass spectrum (spherical at infinity) for $n \geq 3$ is purely technical. In the spherical case, one has explicit spectral inversion formulae which facilitate the estimation of archimedean orbital integrals.*

Remarks 3. *We do not obtain power savings in Theorem 2, but we have recently developed a new strategy to do so, and have verified it in the case $n = 1$.*

Leading term constants. We now describe the leading term constants appearing in Theorem 2. They are both of the form $\frac{1}{n+1}$ times the volume of a sort of "Tamagawa measure."

For every place v of F we let $\Pi(G_v)$ denote the unitary dual of G_v, endowed with the Fell topology. Let $\hat{\nu}_v$ be the measure on $\Pi(G_v)$ given by

$$A \mapsto \int_A \frac{1}{q(\pi_v)^{n+1}} \mathrm{d}\hat{\mu}_{\text{pl}}(\pi_v),$$

where $q(\pi_v)$ denotes the local conductor of the irreducible tempered (hence generic) representation π_v, and $\hat{\mu}_{\text{pl}}$ is the (suitably normalized) Plancherel measure. The measure $\hat{\nu}_v$ is finite, and one can evaluate its total volume as $\zeta_v(1)/\zeta_v(n+1)^{n+1}$; one can think of this as the value at $s = n+1$ of the local conductor zeta function.

Let $\Pi(G_{\mathbb{A}_F})$ denote the direct product of $\Pi(G_v)$ over all v, with closed subspace $\Pi(G^1_{\mathbb{A}_F})$. Define a measure $\hat{\nu}(\mathfrak{F})$ on $\Pi(G^1_{\mathbb{A}_F})$ by the following regularized product

$$\hat{\nu}(\mathfrak{F}) = \zeta_F^*(1) \prod_{v<\infty} \zeta_v(1)^{-1} \hat{\nu}_v \cdot \hat{\nu}_\infty.$$

Finally, let μ_{can} be the Gross canonical measure on $[G] = G(F)\backslash G(\mathbb{A}_F)^1$, whose total volume is $D_F^{n^2/2} \xi_F^*(1)\xi_F(2)\cdots\xi_F(n)$. Then the positive constant C appearing in Theorem 2 is given explicitly as

$$C = \frac{1}{n+1} \cdot \|\mu_{\mathrm{can}}\| \cdot \|\hat{\nu}(\mathfrak{F})\|.$$

A similar description is available for C_{Maass}.

One then observes a parallel with the famous theorem of Schanuel, as interpreted by Peyre, giving an asymptotic

$$\#\{x \in \mathbb{P}^n(F) : H(x) \le H\} \sim \frac{1}{n+1} \|\mu_{\mathrm{Tam}}(\mathbb{P}^n)\| H^{n+1}, \qquad H \to \infty,$$

for the number of rational points on projective space over F of bounded exponential Weil height. Here μ_{Tam} is a certain Tamagawa measure on $\mathbb{P}^n$.

Proof of Theorem. The proof of the theorem relies (not surprisingly) upon the Arthur trace formula. Roughly speaking, this is an identity $J_{\mathrm{spec}} = J_{\mathrm{geom}}$ of distributions on $C_c^\infty(G(\mathbb{A}_F)^1)$. In counting problems such as these, the idea is that the most regular part of J_{spec} – the contribution from the discrete spectrum

$$J_{\mathrm{disc}}(\varphi) = \mathrm{tr}\, R_{\mathrm{disc}}(\varphi) = \sum_{\pi \subset L^2([G])} \mathrm{tr}\, \pi(\varphi)$$

– should be governed by the most singular part of J_{geom} – the contribution from the central elements

$$J_{\mathrm{cent}}(\varphi) = \|\mu_{\mathrm{can}}\| \sum_{\gamma \in Z(F)} \varphi(\gamma).$$

We carry this out with test functions φ of the form $\epsilon_{K_1(\mathfrak{q})} \otimes f$, where $\epsilon_{K_1(\mathfrak{q})}$ is the idempotent corresponding to the Hecke congruence subgroup $K_1(\mathfrak{q})$ and f is a Paley-Wiener function of growing exponential type such that $\pi_\infty \mapsto \mathrm{tr}\, \pi_\infty(f)$ approximates a truncation of $\Pi(G^1_\infty)$ by bounded archimedean conductor.

The strategy is then to show that

(1) $J_{\mathrm{disc}} - J_{\mathrm{cent}}$ is small, using recent breakthrough work of Finis, Lapid, Matz, Mueller, Shin, and Templier;
(2) $J_{\mathrm{disc}} - J_{\mathrm{disc}, \pi_\infty\ \mathrm{temp}}$ is small, through a delicate choice of test functions to overcome the growing exponential type of f;
(3) $J_{\mathrm{cent}} - J_1$ is small, which is trivially true for $\mathfrak{q}$ large enough.

The result of these steps is that a smooth sum over the cuspidal spectrum (tempered at infinity) is governed by the identity distribution. One then unsmooths, and uses dimension formulae of Reeder to sieve out for newforms. The logarithmic savings in the error term is then directly read off from taking the exponential type to be logarithmic in the level, a natural threshold.

Second Variation of Selberg Zeta Function and Positivity

Genkai Zhang

(joint work with K. Fedosova, Julie Rowlett)

Let X be a compact Riemann surface of genus $g \geq 2$ equipped with the hyperbolic metric $ds^2 = \rho(z)|dz|^2$. Let $\Delta_m = 4\bar{\partial}_m^* \bar{\partial}_m$ be the Laplace-Beltrami operator on sections of $\mathcal{K}^m$, $m = 0, \pm 1, \cdots$, where $\mathcal{K}$ is the holomorphic co-tangent bundle. The Selberg zeta function $Z(s)$ on X is proportional to the analytic torsion $\det(\Delta_0 + s(s-1))$, up to constants independent of geometry of X. The analytic torsion can be viewed as a function on the Teichmüller space $\mathcal{T}$ of marked complex structures on X. The kernel and co-kernel of $\bar{\partial}_m$ define vector bundle over the Teichmüller space and the Quillen metric is defined on the determinant of the kernel bundles by the quotient of the analytic torsion and the Gram-Schmidt determinant $\det\langle u_i, u_j\rangle)$, where $\{u_i\}$ is a frame of the bundle $H^0(\mathcal{K}^m)$ over the Teichmuller space. The curvature of the Quillen metric is then proportional to the Weil-Petterson metric, namely $\log \frac{\det(\Delta_m)}{\det\langle u_i, u_j\rangle)}$ provides a Kähler potential for the Weil-Petterson metric. The function $\det(\Delta_m)$ is in turn given by the Selberg zeta function $Z(s)$ for $s = m$. We are interested in studying the plurisubharmonicity of the functions $\det(\Delta_0 + s(s-1))$ and $\log\det(\Delta_0 + s(s-1))$; generally the plurisubharmonicity of $\log f$ implies that for f. This amounts to find the second variational formula for $\log\det(\Delta_0 + s(s-1))$. The first variational formula has been studied by [7] for $\Re s > 1$.

Our main results are summarized in the following

(1) The second variation $\bar{\partial}\partial \log Z(s)$, $\Re s > 1$, is given by

$$\bar{\partial}_\mu \partial_\mu \log Z(s) = \sum_{\gamma \in \mathrm{Prim}} \bar{\partial}_\mu \partial_\mu \log \ell(\gamma) A(s) + \sum_{\gamma \in \mathrm{Prim}} |\partial_\mu \log \ell(\gamma)|^2 (A_\gamma(s) + B_\gamma(s)) \tag{1}$$

where

$$A_\gamma(s) = s\frac{d}{ds}\log Z_\gamma(x) + \frac{d}{ds}\log z_\gamma(s)^{-1} = \sum_{k=0}^\infty \frac{(s+k)\ell(\gamma)}{e^{(s+k)\ell(\gamma)} - 1}$$

and

$$B_\gamma(s) = \left(s^2\frac{d^2}{ds^2}\log Z_\gamma(s) + 2s\frac{d^2}{ds^2}\log z_\gamma(s)^{-1} + \frac{d^2}{ds^2}\log \tilde{z}_\gamma(s)^{-1}\right)$$
$$= -\sum_{k=0}^\infty \frac{(s+k)^2 e^{(s+k)\ell(\gamma)}}{(e^{\ell(\gamma)(s+k)} - 1)^2},$$

$Z_\gamma(s)$ and $z_\gamma(s)$ are local zeta functions, and $Prim$ stands for the prime elements in Γ.

(2) The Hessian $\bar{\partial}_\mu \partial_\mu \log Z(s)$ is positive for $s = m$ is sufficiently large and $\lim_{m\to\infty} \frac{1}{m}\bar{\partial}_\mu \partial_\mu \log Z(s)$ exists.

(3) The Hessian $\bar{\partial}_\mu \partial_\mu \log Z'(1)$ (for $s = 1$ the $Z(s)$ should be replaced by $Z'(1)$ since $Z(1) = 0$) is indefinite: For hyperelliptic surfaces there exist μ's such that $\bar{\partial}_\mu \partial_\mu \log Z(s)$ have different signs.

We are interested eventually to find surfaces which are extremal points for $\det(\Delta_0)$. For that purpose we have found $\partial_\mu \log\det(\Delta_0)$ as an integration of μ against an explicit holomorphic quadratic form ϕ given by the geometry of X. Thus the locally extremal points are given by the equation $\phi = 0$. It remains a challenging problem to find these surfaces.

References

[1] Reynir Axelsson and Georg Schumacher, *Geometric approach to the Weil-Petersson symplectic form*, arXiv preprint arXiv:0808.3741 (2008).

[2] ———, *Variation of geodesic length functions in families of Kähler–Einstein manifolds and applications to Teichmüller space*, Ann. Acad. Sci. Fenn. Math **37** (2012), 91–106.

[3] Bo Berndtsson, *Positivity of direct image bundles and convexity on the space of Kähler metrics*, J. Differential Geom. **81** (2009), no. 3, 457–482.

[4] ———, *Strict and nonstrict positivity of direct image bundles*, Mathematische Zeitschrift **269** (2011), no. 3-4, 1201–1218.

[5] Jean-Michel Bismut and Eric Vasserot, *The asymptotics of the ray-singer analytic torsion associated with high powers of a positive line bundle*, Comm. Math. Phys. **125** (1989), no. 2, 355–367.

[6] John Fay, *Kernel functions, analytic torsion, and moduli spaces*, Mem. Amer. Math. Soc. **96** (1992), no. 464, vi+123.

[7] Y Gon, *First variation of Selberg zeta functions and variational trace formulas*, Journal of the Ramanujan Math. Soc. **18** (2003), no. 3, 257–280.

[8] Dennis A Hejhal, *The Selberg trace formula for $PSL(2,\mathbb{R})$*, vol. 1, vol. 2, Springer.

[9] A. McIntyre and L. A. Takhtajan, *Holomorphic factorization of determinants of Laplacians on Riemann surfaces and a higher genus generalization of Kronecker's first limit formula*, Geom. Funct. Anal. **16** (2006), no. 6, 1291–1323. MR 2276541

[10] Peter Sarnak, *Determinants of Laplacians*, Communications in Mathematical Physics **110** (1987), no. 1, 113–120.

[11] L. A. Takhtajan and P. G. Zograf, *A local index theorem for families of $\overline{\partial}$-operators on punctured riemann surfaces and a new Kähler metric on their moduli spaces*, Comm. Math. Phys. **137** (1991), no. 2, 399–426.

[12] P. G. Zograf and L. A. Takhtadzhyan, *A local index theorem for families of $\overline{\partial}$-operators on Riemann surfaces*, Uspekhi Mat. Nauk **42** (1987), no. 6(258), 133–150, 248. MR 933998

Lower bounds for Maass forms

Simon Marshall

(joint work with Farrell Brumley)

Let Y be a closed Riemannian manifold of dimension n and with Laplace operator Δ. Let $\{\psi_i\}$ be an orthonormal basis of Laplace eigenfunctions for $L^2(Y)$, which satisfy $\|\psi_i\|_2 = 1$ and $(\Delta + \lambda_i^2)\psi_i = 0$. We assume that $\{\psi_i\}$ are ordered by eigenvalue, so that $0 = \lambda_1 \leq \lambda_2 \leq \ldots$. It is an important question in harmonic analysis to determine the asymptotic size of ψ_i, i.e. the growth rate of $\|\psi_i\|_\infty$ in terms of λ_i. The basic upper bound for $\|\psi_i\|_\infty$, proved by Avacumović [1] and Levitan [8], is given by

$$\|\psi_i\|_\infty \ll \lambda_i^{(n-1)/2}. \tag{1}$$

This bound is sharp on the round n-sphere. Indeed, the zonal spherical harmonics have peaks of maximal size at the poles of the axis of rotation. More generally, Sogge and Zelditch [17] have shown that the compact Riemannian manifolds saturating (1) necessarily have points which are fixed by an appropriately large number of geodesic returns, in the sense that every geodesic passing through such a point is a loop.

On the other hand, it is reasonable to expect[1] that if Y has strictly negative curvature then the strong bound

$$\|\psi_i\|_\infty \ll_\epsilon \lambda_i^\epsilon \tag{2}$$

holds with density one. This is akin to the Ramanujan conjecture in the theory of automorphic forms [15]: a generic sequence of eigenfunctions is tempered. Any sequence violating (2) will be called *exceptional.*

The theorem we present is the result of an attempt to give sufficient conditions for a negatively curved manifold to support exceptional sequences. Although the question is of interest in this general setting, our techniques are limited to arithmetic locally symmetric spaces. Put succinctly, we show that an arithmetic manifold supports exceptional sequences whenever it has a point with strong Hecke return properties.

0.1. **Comparison to quantum ergodicity.** Before stating our results we would like to discuss the conjectural generic behavior (2), which to our knowledge has not been previously stated in the literature.

One way of approaching this question is via the link between the asymptotics of sequences of eigenfunctions and the dynamical properties of the geodesic flow. Roughly speaking, microlocal analysis suggests that eigenfunctions on Y should exhibit the same degree of chaotic behaviour as the geodesic flow. On S^n, for instance, the geodesic flow is totally integrable, and this is reflected in the fact that one can both write down an explicit basis of eigenfunctions, and find eigenfunctions with large peaks. Conversely, if Y is negatively curved then its geodesic flow is highly chaotic, and one expects this to be reflected in the asymptotics of the eigenfunctions. For example, the quantum ergodicity (QE) theorem of Schnirelman [16], Colin de Verdière [4], and Zelditch [18] states that the L^2-mass of a generic sequence of Laplacian eigenfunctions on a negatively curved manifold equidistributes to the uniform measure. More precisely, every orthonormal basis of eigenfunctions admits a density one subsequence ψ_i such that the measures $|\psi_i|^2 dV$ tend weakly to dV. It would be interesting to see if existing microlocal techniques such as those used in the QE theorem can be used to prove the expected generic behavior (2) of the sup norm, or weaker versions involving power savings off the local bound (1).

Recall now the conjectural strengthening of the quantum ergodicity theorem by Rudnick and Sarnak [13], known as the Quantum Unique Ergodicity (QUE) conjecture. It states that in negative curvature the L^2-mass of *any* sequence of

[1]The Random Wave Model would predict that almost all sequences satisfy this bound. There are in fact known examples of negatively curved Y admitting sequences violating it. In each of these examples, the offending sequences are in fact of zero density.

eigenfunctions equidistributes to the uniform measure. A similar conjecture has been made relative to sup norms for compact hyperbolic surfaces (but unfortunately lacks a catchy name): Iwaniec and Sarnak [6] conjecture that (2) should hold for *all* eigenfunctions of a compact hyperbolic surface. In our terminology, this is saying that hyperbolic surfaces do not support exceptional sequences. This is a very difficult problem, even for arithmetic surfaces (where QUE has actually been proved [9]). In fact, the bound (2) is often referred to as a Lindelöf type bound, as it implies the classical Lindelöf conjecture on the Riemann zeta function in the case of the modular surface. The Iwaniec-Sarnak conjecture is consistent with the Random Wave Model, which itself can be thought of as the eigenfunction analog of the Sato-Tate conjecture in the theory of automorphic forms [15]. Moreover, it is supported by numerical computations as well as a power improvement (in arithmetic settings) over (1) established in [6].

Unlike the setting of the QUE conjecture, there do in fact exist compact manifolds Y of negative curvature which support exceptional sequences, in the sense of violating (2). The first such example was given by Rudnick and Sarnak [13]. They showed the existence of an arithmetic hyperbolic 3-manifold Y and a sequence of L^2-normalised eigenfunctions ψ_i on Y for which $\|\psi_i\|_\infty \gg_\epsilon \lambda_i^{1/2-\epsilon}$. Straining somewhat, one can view this result as being parallel to the early discovery by Piatetski-Shapiro of counter examples to the Ramanujan conjecture for non-generic cusp forms on the symplectic group.

0.2. **Statement of results.** Our main theorem is modelled on a result of Milićević [12], which, building on [13], provides a structural framework for the class of arithmetic hyperbolic 3-manifolds supporting exceptional sequences.

First recall that an arithmetic hyperbolic 3-manifold arises from the following general construction. Let E be a number field having exactly one complex embedding, up to equivalence, and let F be its maximal totally real subfield. For a division quaternion algebra B over E, ramified at all real places of E, denote by G the restriction of scalars of B^1 from E to F. Then any arithmetic hyperbolic 3-manifold is commensurable with a congruence manifold associated with G.

Following [12], an arithmetic hyperbolic 3-manifold as above is said to be of Maclachlan-Reid type if E is quadratic over F and there exists a quaternion division algebra A over F satisfying $B = A \otimes_F E$. The main result of *loc. cit.* is that Maclachlan-Reid type manifolds support exceptional sequences (in fact, satisfying the same lower bounds as the examples of Rudnick-Sarnak).

Notice that when $B = A \otimes_F E$, the following properties hold. Let v_0 be the unique archimedean place of F which ramifies in E. By [10, Theorem 9.5.5] we may assume that A is ramified at v_0. Then

(1) $G_{v_0} = SL_2(\mathbb{C})$ is noncompact, and non-split (as an $\mathbb{R}$-group);
(2) $G_v = \mathbf{H}^1$ (the norm-one Hamiltonian quaternions) is compact for all real $v \neq v_0$;
(3) the global involution $\theta : g \mapsto \sigma(g)$ of G, where σ is the unique non-trivial element in the Galois group of E over F, induces a Cartan involution on

G_{v_0}. Indeed, $G^\theta = A^1$, so that $G^\theta(F_{v_0}) = \mathbf{H}^1$ is the maximal compact $SU(2)$ inside $G_{v_0} = SL_2(\mathbb{C})$.

Our main result is an extension of this to a wide range of compact congruence manifolds.

Theorem 1. *Let F be a totally real number field, and let v_0 be a real place of F. Let G/F be a connected anisotropic semisimple F-group. We make the following additional assumptions on G.*

1. *G_{v_0} is noncompact, not split, and $\mathbb{R}$-almost simple.*
2. *G_v is compact for all real $v \neq v_0$.*
3. *There is an involution θ of G defined over F that induces a Cartan involution of G_{v_0}.*

Let Y be a congruence manifold associated to G. Then there exists $\delta > 0$ and a sequence of linearly independent Laplacian eigenfunctions ψ_i on Y that satisfy

$$\|\psi_i\|_2 = 1, \quad (\Delta + \lambda_i^2)\psi_i = 0, \qquad \textit{and} \qquad \|\psi_i\|_\infty \gg \lambda_i^\delta.$$

0.3. **Remarks on the theorem.** A well-known theorem of Borel [3] addresses the question of whether one can find many groups satisfying the rationality hypothesis (3). One consequence of his theorem is that for any connected, simply-connected, semisimple algebraic $\mathbb{R}$-group G satisfying condition (1), Theorem 1 produces a manifold Y of the form $\Gamma\backslash G/K$ with an exceptional sequence of eigenfunctions.

Theorem 1 goes some distance toward answering the basic question of determining the precise conditions under which one should expect a Lindelöf type bound on a compact congruence negatively curved manifold. The three numbered conditions on the group G are a particularly convenient way of asking that a large enough compact subgroup of G_∞ admits a rational structure, which is a key ingredient in our proof. Although the condition that G_{v_0} is not split should be necessary, we expect that the other conditions can be relaxed somewhat. For example, throughout most of our proof, the condition that G_{v_0} is $\mathbb{R}$-almost simple could be weakened to G being F-almost simple. The stronger form of this condition is only used to simplify the application of a theorem of Blomer and Pohl [2, Theorem 2] and Matz-Templier [11, Proposition 7.2].

Besides the results of Rudnick-Sarnak and Milićević that we already mentioned, both in the context of arithmetic hyperbolic 3-manifolds, there are other results in the literature which provide examples of arithmetic manifolds supporting exceptional sequences. For instance, the techniques of Rudnick-Sarnak were generalised to n-dimensional hyperbolic manifolds for $n \geq 5$ by Donnelly [5]. And Lapid and Offen in [7] discovered a series of arithmetic quotients of $SL(n, \mathbb{C})/SU(n)$ admitting large eigenforms through the link with automorphic L-functions (conditionally on standard conjectures on the size of automorphic L-functions at the edge of the critical strip). Note that Theorem 1 includes the examples of Rudnick-Sarnak, Donnelly, and Milićević, although without explicit exponents. It is unable to reproduce the examples of Lapid-Offen due to the compactness requirement, but – as was indicated above – it can produce compact quotients of $SU(n, \mathbb{C})/SU(n)$ with an exceptional sequence of eigenfunctions. In fact, non-compact quotients should also be amenable to our techniques, via an application of simple trace formulae, but we have not pursued this.

A synthesis of the subject, as well as a general conjecture restricting the possible limiting exponents for exceptional sequences, can be found in the influential letter [14].

References

[1] G. V. Avacumović: *Über die Eigenfunktionen auf geschlossenen Riemannschen Mannigfaltigkeiten*, Math. Z. 65 (1956), 327-344.

[2] V. Blomer, A. Pohl: *The sup-norm problem on the Siegel modular space of rank two*, Amer. J. Math, 138 (2016), 999-1027.

[3] A. Borel: *Compact Clifford-Klein forms of symmetric spaces*, Topology 2 (1963) 111-122.

[4] Y. Colin de Verdière: *Ergodicité et fonctions propres du Laplacien*, Comm. Math. Phys. **102** (1985) 497-502.

[5] H. Donnelly: *Exceptional sequences of eigenfunctions for hyperbolic manifolds*, Proc. AMS 135 no. 5 (2007), 1551-1555.

[6] H. Iwaniec, P. Sarnak: *L^∞ norms of eigenfunctions of arithmetic surfaces*, Ann. of Math. (2) **141** (1995), 301-320.

[7] E. Lapid, O. Offen : *Compact unitary periods*, Compositio Math. 143 (2007), 323-338.

[8] B. M. Levitan: *On the asymptoptic behavior of the spectral function of a self-adjoint differential equation of second order*, Isv. Akad. Nauk SSSR Ser. Mat. 16 (1952), 325-352.

[9] E. Lindenstrauss: *Invariant measures and arithmetic quantum unique ergodicity*, Ann. Math. **163** (2006) 165-219.

[10] C. Maclachlan, A.W. Reid: *The Arithmetic of Hyperbolic 3-Manifolds*, Springer-Verlag, 2003.

[11] J. Matz, N. Templier: *Sato-Tate equidistribution for families of Hecke-Maass forms on $SL(n,\mathbb{R})/SO(n)$*, available at http://arxiv.org/pdf/1505.07285v3.pdf.

[12] D. Milicevic: *Large values of eigenfunctions on arithmetic hyperbolic 3-manifolds*, GAFA 21 no. 6 (2011), 1375-1418.

[13] Z. Rudnick, P. Sarnak: *The behaviour of eigenstates of arithmetic hyperbolic manifolds*, Comm. Math. Phys. 161 (1994), 195-213.

[14] P. Sarnak: *Letter to Morawetz*, available at http://publications.ias.edu/sarnak/paper/480.

[15] P. Sarnak: *Arithmetic quantum chaos*, Israel Math. Soc. Conf Proc., Schur Lectures, Bar-Ilan Univ., Ramat-Gan, Israel, 1995.

[16] A. I. Schnirelman, *Ergodic properties of eigenfunctions*, Usp. Mat. Nauk **29** (1974) 181-182

[17] C. Sogge, S. Zelditch: *Riemannian manifolds with maximal eigenfunction growth*, Duke Math. J. Volume 114, Number 3 (2002), 387-437.

[18] S. Zelditch: *Uniform distribution of eigenfunctions on compact hyperbolic surfaces*, Duke Math. Journal, **55** (1987) 919-941.

On a variant of Arthur's truncation

Pierre-Henri Chaudouard

1. Introduction

In [13] Hausel and Rodriguez-Villegas gave a striking conjectural formula for the Betti numbers of the moduli space of Higgs bundles over a smooth projective curve (in the so-called coprime case). This moduli space is a quasi-projective variety which is smooth over the base field and proper over an affine space. Moreover, its cohomology is pure. When the base field is finite, one can moreover get a conjectural formula for the number of Higgs bundles (cf. [16]). On the other hand, the counting of Higgs bundles is closely related to the nilpotent part of the Lie algebra version of the Arthur-Selberg trace formula for function fields (cf. [9] and [10]). Using the analogy between number fields and function fields, one can state a conjectural explicit formula à la Hausel-Rodriguez-Villegas for the nilpotent part of the trace formula for a specific test function (a Gaussian function ; cf. [6]). In [17], Schiffmann got a closed formula for the counting of Higgs bundles (nonetheless his formula is *a priori* different from Hausel-Rodriguez-Villegas formula). One of his main innovation is to truncate the stack of vector bundles by bounding above the maximal slope of a subvector bundle. In my talk, I drew some consequences of this new truncation on (the Lie algebra version of) the trace formula. In particular, I explained an explicit formula for the nilpotent contribution for a large class of test functions.

2. Stability of vector bundles

For simplicity, the base field is the field $\mathbb{Q}$ of rationals. To explain the new truncation in the adelic setting, it is convenient to use the language of vector bundles over $\mathrm{Spec}(\mathbb{Z})$ ($\mathcal{O}$-lattices in [18] and [12]). A vector bundle (on $\mathrm{Spec}(\mathbb{Z})$) is a pair $\mathcal{E} = (M, b)$ where M is a free $\mathbb{Z}$-module of finite type and b is a scalar product on $M \otimes_{\mathbb{Z}} \mathbb{R}$. One has the notions of subbundles and isomorphisms of vector bundles. Moreover one defines

- the *rank* which is the rank of the underlying $\mathbb{Z}$-module.
- the *degree* which is by definition
$$\deg(\mathcal{E}) = -\log(\mathrm{vol}(M \otimes_{\mathbb{Z}} \mathbb{R}/M))$$
where $M \otimes_{\mathbb{Z}} \mathbb{R}$ is equipped with the Haar measure that gives the covolume 1 to the lattice generated by an orthonormal basis.
- the *slope*
$$\mu(\mathcal{E}) = \frac{\deg(\mathcal{E})}{\mathrm{rank}(\mathcal{E})}$$
provided $\mathrm{rank}(\mathcal{E}) > 0$.
- the *maximal slope*
$$\mu_{\max}(\mathcal{E}) = \max(\mu(\mathcal{F}))$$
where the maximum is taken over the subbundles $\mathcal{F}$ of $\mathcal{E}$ (it is shown that the maximum exists and is finite).
- the *semi-stability*: a vector bundle $\mathcal{E}$ is semi-stable if $\mu_{\max}(\mathcal{E}) \leq \mu(\mathcal{E})$.

Let $n \geq 1$ be an integer and $G = GL(n)$. Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$ and K be the usual maximal compact subgroup of $G(\mathbb{A})$. The quotient
$$G(\mathbb{Q})\backslash G(\mathbb{A})/K$$
is identified, via $g \mapsto \mathcal{E}_g$, with the set of isomorphism class of vector bundles.

Loosely speaking, the trace formula is based on the notion of semistability and the following combinatorial identity for any $g \in G(\mathbb{Q})\backslash G(\mathbb{A})/K$:

$$\sum_P \epsilon_P^G \sum_{\delta\in P(\mathbb{Q})\backslash G(\mathbb{Q})} \hat{\tau}_P(H_P(\delta g)) = \begin{cases} 1 \text{ if } \mathcal{E}_g \text{ is semi-stable} \\ 0 \text{ otherwise.} \end{cases}$$

where the sum is over standard parabolic subgroups P of G. Here $H_P : G(\mathbb{A}) \to a_P = \mathrm{Hom}(X^*(P), \mathbb{R})$ is obtained from the Iwasawa decomposition $G(\mathbb{A}) = P(\mathbb{A})K$ and the natural homomorphism $P(\mathbb{A}) \to a_P$ given by $p \mapsto (\chi \mapsto \log|\chi(p)|_{\mathbb{A}})$. Moreover $\hat{\tau}_P$ is the characteristic function of the open obtuse Weyl chamber in a_P and

$$\epsilon_P^G = (-1)^{\dim(a_P)-\dim(a_G)}.$$

The sum over δ is of finite support.

We will use the new function $E^{G,\leq 0}$ which is the characteristic function of $g \in G(\mathbb{Q})\backslash G(\mathbb{A})/K$ such that $\mu_{\max}(\mathcal{E}_g) \leq 0$. Using the Iwasawa decomposition and a similar condition on each GL-block of the standard Levi factor of a standard parabolic subgroup, one also defines $E^{P,\leq 0}$ on $P(\mathbb{Q})\backslash G(\mathbb{A})/K$.

3. The fine geometric expansion of the Lie algebra version of the Arthur-Selberg trace formula

We will denote Lie algebras by the corresponding gothic letter e.g. $\mathfrak{g}$ is the Lie algebra of G. Let $f \in \mathcal{S}(\mathfrak{g}(\mathbb{A}))$ a Schwartz-Bruhat function. For any standard parabolic subgroup with standard Levi decomposition $P = MN$ (N is the unipotent radical) and any G-orbit $\mathfrak{o}$ in $\mathfrak{g}$ one defines for $g \in G(\mathbb{A})$

$$k_{P,\mathfrak{o}}(f,g) = \sum_{X\in\mathfrak{m}(\mathbb{Q}), I_P^G(X)=\mathfrak{o}} \int_{\mathfrak{n}(\mathbb{A})} f(g^{-1}(X+U)g)\, dU.$$

Here $I_P^G(X)$ is the Lusztig-Spaltenstein induced orbit from the M-orbit of X. Let's define (the sum is over standard parabolic subgroups of G)

$$k_{\mathfrak{o}}(f,g) = \sum_P \epsilon_P^G \sum_{\delta\in P(\mathbb{Q})\backslash G(\mathbb{Q})} \hat{\tau}_P(H_P(\delta g)) k_{P,\mathfrak{o}}(f,\delta g).$$

We have the following refined geometric expansion of the trace formula.

THEOREM (cf. [7])

(1) *The integral*

$$J_{\mathfrak{o}}(f) := \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} k_{\mathfrak{o}}(f,g)\, dg$$

is absolutely convergent where $G(\mathbb{A})^1$ is the subgroup of element $g \in G(\mathbb{A})$ such that $|\det(g)|_{\mathbb{A}} = 1$.

(2) *The sum over all G-orbits $\mathfrak{o}$*

$$\sum_{\mathfrak{o}} J_{\mathfrak{o}}(f)$$

is absolutely convergent.

4. Main result

For any standard Levi subgroup, let $P = MN$ be the attached standard parabolic subgroup. For any $f \in \mathcal{S}(\mathfrak{g}(\mathbb{A}))$ we define its constant term as

$$f_M(X) = \int_K \int_{\mathfrak{n}(\mathbb{A})} f(k^{-1}(X+U)k)\, dU dk \in \mathcal{S}(\mathfrak{m}(\mathbb{A})).$$

One says that f is almost invariant if for any pair (M, L) of standard Levi subgroups and any element w in the Weyl group W such that $L = wMw^{-1}$ one has

$$f_M(X) = f_L(wXw^{-1})$$

for any $X \in \mathfrak{m}(\mathbb{A})$. For example, a bi-$K$-invariant function is almost invariant.

Let $\mathfrak{o}$ be a G-orbit. Let $\mathcal{L}(\mathfrak{o})$ be the set of pair $(M, \mathfrak{o}')$ (up to G-conjugacy) where M is a standard Levi subgroup and $\mathfrak{o}'$ is a M-orbit in $\mathfrak{m}$ such that the induced orbit $I_M^G(\mathfrak{o}')$ is $\mathfrak{o}$. Let $W(M, \mathfrak{o}')$ be the stabilizer of $(M, \mathfrak{o}')$ in the Weyl group W. Let $r_M = (r-1)!$ where r is the number of GL-blocks of M.

From now on, let's assume that $\mathfrak{o}$ is *nilpotent*. Then to any pair $(M, \mathfrak{o}') \in \mathcal{L}(\mathfrak{o})$, one attaches:

- a zeta distribution $Z_{(M,\mathfrak{o}')}(\cdot, s)$ on $\mathcal{S}(\mathfrak{m}(\mathbb{A}))$ defined by an explicit integral which is convergent and holomorphic for $\Re(s) > 0$;
- $\theta_{(M,\mathfrak{o}')}(s)$ which is an explicit truncated integral of an Eisenstein series. This function is convergent and holomorphic for $\Re(s) > 0$.

In fact, $Z_{(M,\mathfrak{o}')}(\cdot, s)$ and $\theta_{(M,\mathfrak{o}')}(s)$ are products over components of $(M, \mathfrak{o}')$ and it suffices to define them in the simple case where $M = G$. Let's remark that $Z_{(G,\mathfrak{o}')}(f, s)$ for $f \in \mathcal{S}(\mathfrak{g}(\mathbb{A}))$ is given by an Eulerian product whose local factors at p are almost all of the following type:

$$\prod_y (1 - p^{-(1+a_y+l_y s)})^{-1}$$

where the product is over boxes y of the Young diagram of $\mathfrak{o}$ such that $l_y > 0$. Here a_y and l_y are the usual arm and leg functions. Here the correspondence between nilpotent orbits and Young diagrams is normalized so that a row corresponds to a zero orbit.

Theorem (cf. [6]) *Let $\mathfrak{o}$ be a nilpotent orbit and $f \in \mathcal{C}_c^\infty(\mathfrak{g}(\mathbb{A}))$ be a compactly supported test function. Let's assume that*

(1) *the support of f is small enough;*
(2) *f is almost invariant.*

Then

$$J_{\mathfrak{o}}(f) = \lim_{s\to 0, \Re(s)>0} s \sum_{(M,\mathfrak{o}')\in\mathcal{L}(\mathfrak{o})} \epsilon_M^G \frac{r_M}{|W(M,\mathfrak{o}')|} Z_{(M,\mathfrak{o}')}(f_M, s) \cdot \theta_{(M,\mathfrak{o}')}(s).$$

Remarks

- The functions $Z_{(M,\mathfrak{o}')}(f_M, s)$ and $\theta_{(M,\mathfrak{o}')}(s)$ may have pole of high degrees at $s = 0$ so the existence of the limit is not obvious *a priori*.

- Let p be a place and $t \in \mathbb{Q}_p$. Then for any $f \in \mathcal{C}_c^\infty(\mathfrak{g}(\mathbb{A}))$, the function $f(t\cdot)$ is of small enough support for $|t| >> 0$. Since the distribution $J_{\mathfrak{o}}(f)$ satisfies some homogeneity properties, the restriction on the support is not too severe.

For any orbit $\mathfrak{o}$ (not necessarily nilpotent), one defines the following variant of $k_{\mathfrak{o}}(f,g)$

$$k_{\mathfrak{o}}^{\leq 0}(f,g) = \sum_P \epsilon_P^G \sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \hat{\tau}_P(H_P(\delta g)) E^{P,\leq 0}(\delta g) k_{P,\mathfrak{o}}(f,\delta g).$$

THEOREM (cf. [6]) *For any $s \in \mathbb{C}$ such that $\Re(s) > 0$, the integral*

$$J_{\mathfrak{o}}^{\leq 0}(f,s) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} k_{\mathfrak{o}}^{\leq 0}(g) |\det(g)|^s \, dg$$

is absolutely convergent and moreover

$$\lim_{s\to 0, \Re(s)>0} s J_{\mathfrak{o}}^{\leq 0}(f,s) = J_{\mathfrak{o}}(f).$$

With the new function $k_{\mathfrak{o}}^{\leq 0}(f,g)$, one can permute the integral and the sum over the parabolic subgroups P. For almost invariant functions, one gets a variant of the main theorem which uses truncated orbital integrals. For nilpotent orbits and functions of small supports, it is possible to separate the truncation from the orbital integrals: this achieves the proof of the main theorem.

References

[1] J. Arthur. A trace formula for reductive groups I. Terms associated to classes in $G(\mathbb{Q})$. *Duke Math. J.*, 45:911–952, 1978.

[2] J. Arthur. A trace formula for reductive groups II. Applications of a truncation operator. *Compositio Math.*, 40:87–121, 1980.

[3] J. Arthur. A measure on the unipotent variety. *Canad. J. Math.*, 37:1237–1274, 1985.

[4] J. Arthur. An introduction to the trace formula. In *Harmonic analysis, the trace formula, and Shimura varieties*, volume 4 of *Clay Math. Proc.*, pages 1–263. Amer. Math. Soc., Providence, RI, 2005.

[5] K. Behrend. Semi-stability of reductive group schemes over curves. *Math. Ann.*, 301(2):281–305, 1995.

[6] P.-H. Chaudouard. Sur une variante des troncatures d'Arthur. To appear.

[7] P.-H. Chaudouard. Sur certaines contributions unipotentes dans la formule des traces d'Arthur. *Amer. J. Math.*, to appear.

[8] P.-H. Chaudouard. Sur la contribution unipotente dans la formule des traces d'Arthur pour les groupes généraux linéaires. *Israel J. Math.*, 218(1):175–271, 2017.

[9] P.-H. Chaudouard. La formule des traces pour les algèbres de Lie. *Math. Ann.*, 322(2):347–382, 2002.

[10] P.-H. Chaudouard. Sur le comptage des fibrés de Hitchin. *Astérisque*, (369):223–284, 2015.

[11] T. Finis and E. Lapid. On the continuity of the geometric side of the trace formula. *Acta Math. Vietnam.*, 41(3):425–455, 2016.

[12] D. Grayson. Reduction theory using semistability. *Comment. Math. Helv.*, 59(4):600–634, 1984.

[13] T. Hausel and F. Rodriguez-Villegas. Mixed Hodge polynomials of character varieties. *Invent. Math.*, 174(3):555–624, 2008. With an appendix by Nicholas M. Katz.

[14] W. Hoffmann. The trace formula and prehomogeneous vector spaces. *ArXiv e-prints*, December 2014.
[15] L. Lafforgue. Chtoucas de Drinfeld et conjecture de Ramanujan-Petersson. *Astérisque*, (243):ii+329, 1997.
[16] S. Mozgovoy. Solutions of the motivic ADHM recursion formula. *Int. Math. Res. Not. IMRN*, (18):4218–4244, 2012.
[17] O. Schiffmann. Indecomposable vector bundles and stable Higgs bundles over smooth projective curves. *Ann. of Math. (2)*, 183(1):297–362, 2016.
[18] U. Stuhler. Eine Bemerkung zur Reduktionstheorie quadratischer Formen. *Arch. Math. (Basel)*, 27(6):604–610, 1976.

Stabilization of the trace formula for orthogonal Shimura varieties

Yihang Zhu

Let (G, X) be a Shimura datum. Let $\mathrm{Sh}_K = \mathrm{Sh}_K(G, X)$ be the associated Shimura variety for a neat open compact subgroup $K \subset G(\mathbb{A}_f)$. It is smooth and quasi-projective over the reflex field E. To simplify notation we assume $E = \mathbb{Q}$ in the following.

We temporarily assume that Sh_K is proper. With the aid of the (conjectural) formalism of Arthur parameters, the local Euler factors of the Hasse-Weil zeta function $\zeta(\mathrm{Sh}_K, s)$ are given conjecturally as follows:

Conjecture A (Kottwitz, cf. [7, §10], [5, Conj. 5.2])**.** For almost all primes p, the following local Euler factors are equal

$$\zeta_p(\mathrm{Sh}_K, s) = \prod_{i=0}^{2\dim \mathrm{Sh}_K} \prod_{\psi} \prod_{\pi_f} \prod_{\nu} L_p\Big(s - \frac{i}{2},\ \psi^i(\nu)\Big)^{(-1)^{i+1} m(\psi,\nu,\pi_f)}.$$

Here ψ runs through the Arthur parameters that are cohomological at infinity, π_f runs through representations of $G(\mathbb{A}_f)$ belonging to ψ, and ν runs through suitable characters of the group S_ψ of self-equivalences of ψ. For given (ψ, π_f, ν), the expression $L_p(\cdot, \psi^i(\nu))$ is the local Euler factor of an automorphic L-function, and $m(\psi, \nu, \pi_f) \in \mathbb{Z}$ is determined by a certain recipe.

To approach Conjecture A, Langlands and Kottwitz have developed the strategy of comparing Grothendieck-Lefschetz trace formulas and stable Arthur-Selberg trace formulas, summarized in Conjecture B below. We drop the assumption that Sh_K is proper. Let $\overline{\mathrm{Sh}_K}$ be the Baily-Borel compactification of Sh_K, which is a normal projective variety over E compactifying Sh_K. Let $\mathbf{IH}^*$ denote the formal alternating sum of the (ℓ-adic) intersection cohomology of $(\overline{\mathrm{Sh}_K})_{\bar{\mathbb{Q}}}$. (For the motivation of considering $\mathbf{IH}^*$ cf. [12, §2] or the introduction of [17])

Conjecture B ([7, §10])**.** Let $p > 2$ be a hyperspecial prime for K, in the sense that $K = K_p K^p$ with $K_p \subset G(\mathbb{Q}_p)$ hyperspecial and $K^p \subset G(\mathbb{A}_f^p)$. Assume $\ell \neq p$ in the definition of $\mathbf{IH}^*$. We have a commuting action of the Hecke algebra $\mathcal{H}(G(\mathbb{A}_f^p)//K^p)$ and $\mathrm{Gal}(\bar{E}/E)$ on $\mathbf{IH}^*$. Let $f^{p,\infty} \in \mathcal{H}(G(\mathbb{A}_f^p)//K^p)$, and let

$\Phi_p \in \mathrm{Gal}(\bar{E}/E)$ be any lift of the geometric Frobenius at p. We have, at least for $j \in \mathbb{N}$ large enough,

$$\mathrm{Tr}(f^{p,\infty} \times \Phi_p^j \mid \mathbf{IH}^*) = \sum_H \iota(G,H) ST^H(f^H), \tag{1}$$

where in the summation H runs through the elliptic endoscopic data of G. For each H the test function $f^H : H(\mathbb{A}) \to \mathbb{C}$ is prescribed in terms of $f^{p,\infty}, (G,X), j$ by a certain recipe [1], and $ST^H(\cdot)$ is the geometric side of the stable trace formula for H. The constants $\iota(G,H)$ are explicitly defined.

Theorem 1 ([17]). *Conjecture B is true for the orthogonal Shimura varieties in Definition 2 below, for almost all p.*

Definition 2. *The* orthogonal Shimura varieties *are associated to Shimura data (G,X) as follows: $G = \mathrm{SO}(V)$ where V is a quadratic space over $\mathbb{Q}$ of signature $(n,2)$, and X is the space of oriented negative definite planes in $V_{\mathbb{R}}$. For simplicity we assume $n \geq 5$.*

Remark 3. *$(n,2)$ is the only possible signature for $G = \mathrm{SO}(V)$ to have a Shimura datum. The orthogonal Shimura varieties are non-compact and not of PEL type. They are examples of abelian type Shimura varieties of types B and D.*

Remark 4. *In the proof of Theorem 1 we have used the simplified formulas for $ST^H(f^H)$ due to Kottwitz. It seems that a proof that they coincide with Arthur's stabilization* [2][1][3] *has not appeared in the published literature. In view of that, Theorem 1 should be regarded as a variant of Conjecture B, with each $ST^H(f^H)$ re-defined by Kottwitz's formulas.*

Remark 5. *The reason that Theorem 1 is only stated for almost all primes is due to a lack of the construction of integral models of the Baily-Borel and toroidal compactifications of abelian type (even at hyperspecial level) in the literature.*

Remark 6. *Theorem 1 is the analogue of Morel's earlier work* [13] [14] *for unitary and Siegel Shimura varieties.*

Remark 7. *In principle, one could deduce Conjecture A from Conjecture B and Arthur's conjectures. For some choices of G in Definition 2, the relevant Arthur's conjectures have been proved by Taïbi* [16], *generalizing Arthur's work on quasi-split classical groups* [4]. *In these cases we will pursue in a future work the proof of a version of Conjecture A based on Theorem 1.*

We now present the idea of the proof of Theorem 1.

The work of Morel ([10] [11] [13], also cf. [12]), together with a theorem of Pink ([15]), allows one to compute the LHS of (1) in terms of compact support cohomology of the strata in $\overline{\mathrm{Sh}_K}$. The contribution from the compact support cohomology of Sh_K itself is computed and stabilized in an ongoing joint work with

[1]The definition of f^H assumes the Langlands-Shelstad transfer and the fundamental lemma (at primes away from p and ∞).

Kisin and Shin [9]. The boundary terms can be organized as a summation over standard proper Levi subgroups $M \subsetneq G_{\mathbb{Q}}$:

$$\text{Boundary terms on the LHS of (1)} = \sum_{M \subsetneq G} \mathrm{Tr}_M.$$

Thus the problem is to stabilize the above expression, i.e. to prove:

$$\sum_{M \subsetneq G} \mathrm{Tr}_M = \sum_H \iota(G,H) \sum_{M' \subsetneq H} ST^H_{M'}(f^H). \tag{2}$$

Here M' runs through the standard proper Levi subgroups of $H_{\mathbb{Q}}$. On the RHS of (2), each term $ST^H_{M'}(f^H)$ has a relatively simple formula due to Kottwitz (Remark 4). On the LHS of (2), each term Tr_M is a mixture of the following:

- Kottwitz's fixed point formula [7][8].
- The topological Lefschetz formula of Goresky-Kottwitz-MacPherson [6].
- *Kostant-Weyl traces.* By this we mean Weyl character formulas for certain algebraic $\tilde{M}$-sub-representations of $\mathbf{H}^*(\mathrm{Lie}\tilde{N}, \mathbb{C})$, where $\tilde{P} = \tilde{M}\tilde{N}$ is a standard parabolic subgroup of G containing M. These sub-representations are defined by truncation in terms of the weights of certain central cocharacters of $\tilde{M}$.

We highlight some key ingredients in the proof of (2).

Archimedean comparison. On the RHS of (2), the archimedean contributions consist of values of stable discrete series characters. On the LHS of (2), the archimedean contributions are understood to be the Kostant-Weyl traces discussed above. After explicit computation we prove identities between these two kinds of objects.

Proving cancellations. We have to prove cancellations among a lot of terms on the RHS of (2). These include the terms with a "bad" factor at p, and also the terms indexed by (H, M') with M' not transferring to G.

Comparing transfer factors. Signs are important to ensure the correct cancellation of terms. One important source of signs comes from comparing different normalizations of transfer factors for real endoscopy. We compute the difference between the normalization $\Delta_{j,B}$ (introduced in e.g. [7, §7]) and the Whittaker normalizations. In case there are more than one Whittaker data, we also compare the resulting different Whittaker normalizations.

References

[1] J. Arthur, *A stable trace formula. II. Global descent*, Invent. Math. **143** (2001), no. 1, 157–220.

[2] J. Arthur, *A stable trace formula. I. General expansions*, J. Inst. Math. Jussieu **1** (2002), no. 2, 175–277.

[3] J. Arthur, *A stable trace formula. III. Proof of the main theorems*, Ann. of Math. (2) **158** (2003), no. 3, 769–873.

[4] J. Arthur, *The endoscopic classification of representations*, American Mathematical Society Colloquium Publications, vol. 61, 2013.

[5] D. Blasius and J. D. Rogawski, *Zeta functions of Shimura varieties*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, 1994, pp. 525–571.

[6] M. Goresky, R. Kottwitz, and R. MacPherson, *Discrete series characters and the Lefschetz formula for Hecke operators*, Duke Math. J. **89** (1997), no. 3, 477–554.
[7] R. E. Kottwitz, *Shimura varieties and λ-adic representations*, Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988), Perspect. Math., vol. 10, 1990, pp. 161–209.
[8] R. E. Kottwitz, *Points on some Shimura varieties over finite fields*, J. Amer. Math. Soc. **5** (1992), no. 2, 373–444.
[9] M. Kisin, S. W. Shin, and Y. Zhu, *The stable trace formula for certain Shimura varieties of abelian type*, in preparation.
[10] S. Morel, *Complexes d'intersection des compactifications de Baily-Borel. Le cas des groupes unitaires sur Q*, PhD Thesis, 2006.
[11] S. Morel, *Complexes pondérés sur les compactifications de Baily-Borel: le cas des variétés de Siegel*, J. Amer. Math. Soc. **21** (2008), no. 1, 23–61.
[12] S. Morel, *The intersection complex as a weight truncation and an application to Shimura varieties*, Proceedings of the International Congress of Mathematicians. Volume II, 2010, pp. 312–334.
[13] S. Morel, *On the cohomology of certain noncompact Shimura varieties*, Annals of Mathematics Studies, vol. 173, 2010, With an appendix by Robert Kottwitz.
[14] S. Morel, *Cohomologie d'intersection des variétés modulaires de Siegel, suite*, Compos. Math. **147** (2011), no. 6, 1671–1740.
[15] R. Pink, *On l-adic sheaves on Shimura varieties and their higher direct images in the Baily-Borel compactification*, Math. Ann. **292** (1992), no. 2, 197–240.
[16] O. Taïbi, *Arthur's multiplicity formula for certain inner forms of special orthogonal and symplectic groups*, ArXiv e-prints (2015).
[17] Y. Zhu, *The stabilization of the Frobenius-Hecke traces on the intersection cohomology of orthogonal Shimura varieties*, PhD Thesis, 2017.

A local trace formula for the Ginzburg-Rallis model and some generalizations

Chen Wan

In this report, we will talk about a local trace formula for the Ginzburg-Rallis model. This trace formula allows us to prove a multiplicity formula for the Ginzburg-Rallis model, which implies the multiplicity one theorem on the Vogan L-packet. Then we will talk about some generalizations of this trace formula to other models.

1. The Ginzburg-Rallis model

Let F be a local field of characteristic zero, and let $G(F) = \mathrm{GL}_6(F)$. Take $P = P_{2,2,2} = MU$ to be the standard parabolic subgroup of G whose Levi part $M(F)$ is isomorphic to $\mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$, and whose unipotent radical $U(F)$ consists of elements of the form

$$u = u(X,Y,Z) := \begin{pmatrix} 1 & X & Z \\ 0 & 1 & Y \\ 0 & 0 & 1 \end{pmatrix},\ X,Y,Z \in M_{2\times 2}(F). \tag{1}$$

We define a character ξ on $U(F)$ by

$$\xi(u(X,Y,Z)) := \psi(tr(X) + tr(Y)) \tag{2}$$

where ψ is a non-trivial additive character on F. It is clear that the stabilizer of ξ is the diagonal embedding of $GL_2(F)$ into $M(F)$, which is denoted by $H_0(F)$. For a given character χ of $F^{\times}$, one induces a one-dimensional representation ω of $H_0(F)$ given by $\omega(h) := \chi(\det(h))$. Combining ω and ξ, we get a character $\omega \otimes \xi$ on $H(F) := H_0(F) \ltimes U(F)$. The pair (G, H) is the Ginzburg-Rallis model introduced by D. Ginzburg and S. Rallis in [GR00]. Let π be an irreducible admissible representation of $G(F)$ with central character χ^2, we would like to study the multiplicity $m(\pi) := \dim(Hom_{H(F)}(\pi, \omega \otimes \xi))$.

In order to consider the formulation of the Vogan L-packets, for $F \neq \mathbb{C}$, let D be the unique quaternion algebra over F. Define $G_D = \mathrm{GL}_3(D)$, and similarly we can define U_D, $H_{0,D}$ and H_D. We can also define the character $\omega_D \otimes \xi_D$ on $H_D(F)$. Then for an irreducible admissible representation π_D of $G_D(F)$ with central character χ^2, we also study the multiplicity $m(\pi_D) := dim(Hom_{H_D(F)}(\pi_D, \omega_D \otimes \xi_D))$.

Let π be a generic representation of $G(F)$, and let π_D be the local Jacquet-Langlands correspondence of π to $G_D(F)$ if it exists; otherwise set $\pi_D = 0$. In particular, if $F = \mathbb{C}$, π_D is always 0. We are interested in the summation $m(\pi) + m(\pi_D)$. **The local conjecture states that**

$$m(\pi) + m(\pi_D) = 1 \tag{3}$$

for all generic representations π. Another aspect of the local conjecture is the so-called *ϵ-dichotomy conjecture*, which relates the multiplicity with the value of the exterior cube epsilon factor. **The conjecture states that**

$$m(\pi) = 1 \Leftrightarrow \epsilon(1/2, (\wedge^3\pi) \otimes \chi^{-1}) = 1;\ m(\pi) = 0 \Leftrightarrow \epsilon(1/2, (\wedge^3\pi) \otimes \chi^{-1}) = -1. \tag{4}$$

In my papers [Wan15], [Wan16a] and [Wan16b], I proved the local conjectures for tempered representations. My results can be stated as follows.

Theorem 1 ([Wan15], [Wan16a], [Wan16b])**.** *Let π be a tempered representation of $G(F)$ and let π_D be the local Jacquet-Langlands correspondence of π to $G_D(F)$. Then the followings hold.*

1. $m(\pi) + m(\pi_D) = 1$.
2. *If F is archimedean, then the ϵ-dichotomy conjecture* (4) *holds. If F is p-adic, then* (4) *holds for all tempered representations π of $G(F) = \mathrm{GL}_6(F)$ that is not a discrete series or the parabolic induction of a discrete series of $\mathrm{GL}_4(F) \times \mathrm{GL}_2(F)$.*

Our proof of Theorem 1 uses the methods developed by Waldspurger and Beuzart-Plessis in their proofs of the local Gan-Gross-Prasad conjecture ([B12], [B15], [W10], [W12]). The main ingredient is to prove a local trace formula

$$I_{geom}(f) = I(f) = I_{spec}(f)$$

for the model. Here for a given strongly cuspidal function $f \in C_c^{\infty}(G(F)/Z_G(F), \chi^{-1})$, we define the distribution $I(f)$ to be

$$I(f) = \int_{H(F)\backslash G(F)} \int_{H(F)/Z_H(F)} f(g^{-1}hg)\omega \otimes \xi(h) dh dg.$$

The spectral side is defined to be

$$I_{spec}(f) = \int_{\Pi_{temp}(G,\chi^2)} \theta_f(\pi) m(\bar{\pi}) d\pi$$

where $\theta_f(\pi)$ is defined via the weighed character. The geometric side is defined via the germs of θ_f, here θ_f is the quasi-character associated to f via the weighted orbital integral. To be specific, $I_{geom}(f)$ is the summation of the germ at the identity element and the integrals of the germs over all maximal elliptic torus of $H_0(F)$. Then the trace formula will gives us the multiplicity formulas:

$$m(\pi) = m_{geom}(\pi),\ m(\pi_D) = m_{geom}(\pi_D).$$

Here $m_{geom}(\pi)$ (resp. $m_{geom}(\pi_D)$) is defined via the germs of the distribution character θ_π (resp. θ_{π_D}). Then by applying the relations between the distribution characters under the local Jacquet-Langlands correspondence, we can show that $m(\pi) + m(\pi_D) = 1$.

2. Some generalizations

Now we would like to generalize the trace formula in the last section to some other models. Let (G, H) be an arbitrary spherical pair. First, we can always use the same formula to define the distribution $I(f)$. However, in general the model (G, H) will not be strongly tempered, hence we can not expect the multiplicity formula holds for all tempered representations. As a result, we will only consider the discrete series. So we define

$$I_{spec}(f) = \int_{\Pi_2(G,\chi^2)} \theta_f(\pi) m(\bar{\pi}) d\pi.$$

And we also require $f \in {}^\circ\mathcal{C}(G(F)/Z_G(F), \chi^{-2})$ which is the dual of all discrete series. Under some mild assumptions for the pair (G, H), we can prove the spectral expansion $I(f) = I_{spec}(f)$ for all $f \in {}^\circ\mathcal{C}(G(F)/Z_G(F), \chi^{-2})$.

Now for the geometric side, it is not clear how to define $I_{geom}(f)$ for general pairs (G, H). So we will focus on some specific models, in particular, the generalized Shalika models. Let F be a p-adic field and let D/F be a division algebra. Let $G(F) = \mathrm{GL}_{2n}(D)$, and let $P = MU$ be the standard parabolic subgroup of type (n, n). Let $H_0(F)$ be the image of the diagonal embedding $\mathrm{GL}_n(D) \to M(F)$. As in the Ginzburg-Rallis model case, we can define a character $\omega \otimes \xi$ on $H(F) := H_0(F) \ltimes U(F)$. The pair (G, H) is called the generalized Shalika model. If $D = F$, this is just the original Shalika model.

For this model, we still define the geometric side $I_{geom}(f)$ in terms of the germs of θ_f. It will be the summation of the integrals of the germs over all maximal elliptic torus of $H_0(F)$. We expect the trace formula

$$I_{geom}(f) = I(f) = I_{spec}(f)$$

holds for all $f \in {}^\circ\mathcal{C}(G(F)/Z_G(F), \chi^{-2})$. If this true, we can then prove a multiplicity formula $m(\pi) = m_{geom}(\pi)$ for discrete series. Based on the multiplicity formula, we can show that the multiplicity is a constant on every discrete Vogan

L-packets. In other word, the multiplicity is invariant under the local Jacquet-Langlands correspondence. At this moment, we only know how to prove the trace formula when $n = 1, 2$, and we are currently trying to prove it for general n ([Wan17a], [Wan17b]).

References

[B12] Beuzart-Plessis, R., *La conjecture locale de Gross-Prasad pour les reprsentations tempérées des groupes unitaires.* Prepublication, 2012.

[B15] Beuzart-Plessis, R., *A local trace formula for the Gan-Gross-Prasad conjecture for unitary groups: the archimedean case.* Preprint, 2015.

[GR00] D. Ginzburg, S. Rallis, *The exterior cube L-function for GL(6).* Compositio Math. 123(2000), no. 3, 243-272

[W10] J.-L. Waldspurger, *Une formule intégrale reliée à la conjecture locale de Gross-Prasad.* Compos. Math. 146(2010), no.5, 1180-1290.

[W12] J.-L. Waldspurger, *Une formule intégrale reliée à la conjecture locale de Gross-Prasad, 2e partie : extension aux représentations tempérées.* in "Sur les conjectures de Gross et Prasad. I" Astérisque No. 346 (2012), 171-312

[Wan15] Chen Wan, *A local relative trace formula for the Ginzburg-Rallis model: the geometric side.* Submited, 2015. 90 pages

[Wan16a] Chen Wan, *Multiplicity One Theorem for the Ginzburg-Rallis Model: the tempered case.* Submitted, 2016. 74 pages

[Wan16b] Chen Wan, *The local Ginzburg-Rallis model over complex field.* Submitted, 2016. 15 pages

[Wan17a] Chen Wan, *A local trace formula for the generalized Shalika model model: the geometric side.* In preparation, 2017.

[Wan17b] Chen Wan, *The Generalized Shalika Model for* $\mathrm{GL}_2(D)$. Preprint, 2017.

Supercuspidal *L*-packets

Tasho Kaletha

This talk presented a few separate but related results on the complex representation theory of reductive p-adic groups:

(1) The definition and classification of a large class of supercuspidal representations, called "regular".
(2) A formula for the Harish-Chandra character of regular supercuspidal representations, evaluated at certain special elements of the group.
(3) An explicit construction of a correspondence between regular supercuspidal representations and Langlands parameters.
(4) An extension of this construction to the setting of general supercuspidal parameters – currently work in progress.

1. Regular supercuspidal representations

Harish-Chandra has classified the discrete series representations of real reductive groups. Given such a group $G/\mathbb{R}$, the topological group $G(\mathbb{R})$ has a discrete series of representations if and only if there exists an elliptic maximal torus $S \subset G$, and in that case the equivalence classes of such representations are in 1-1 correspondence with pairs (B, θ), where $B \subset G \times \mathbb{C}$ is a Borel subgroup containing S and θ :

$S(\mathbb{R}) \to \mathbb{C}^\times$ is a character whose differential is B-dominant. These pairs are taken up to conjugation by the Weyl group $\Omega_\mathbb{R} = N(S,G)(\mathbb{R})/S(\mathbb{R})$. If we restrict attention to those θ whose differential is regular, and let S vary, we obtain a parameterization of most discrete series representations by $G(\mathbb{R})$-conjugacy classes of pairs (S,θ), where $S \subset G$ is an elliptic maximal torus, and $\theta : S(\mathbb{R}) \to \mathbb{C}^\times$ is a character with regular differential. This parameterization can be described as follows: Given (S,θ), there exists a unique discrete series representation whose Harish-Chandra character, evaluated at $\gamma \in S(\mathbb{R})$, is given by the formula

$$(-1)^{q(G)} \sum_{w \in \Omega_\mathbb{R}} \frac{\theta(\gamma^w)}{\prod_{\alpha>0}(1-\alpha(\gamma^w)^{-1})}. \tag{1}$$

We now let F be a non-archimedean local field and G a connected reductive group defined over F. Our goal is to find an analogous classification of a large class of supercuspidal representations of $G(F)$. In the case $G = \mathrm{GL}_N$, assuming the residual characteristic p of F is prime to N, work of Howe and Moy accomplishes that for all supercuspidal representations, and this gives us hope that it may be possible in general. Assuming that G splits over a tame extension of F, J.K. Yu [6] has given a construction of supercuspidal representations, whose initial input is a datum of the form $((G^0 \subset \cdots \subset G^d = G), \pi_{-1}, (\phi_0, \ldots, \phi_d))$, where G^i are tame twisted Levi subgroups of G, $\phi_i : G^i(F) \to \mathbb{C}^\times$ are characters, and π_{-1} is a depth-zero supercuspidal representation of $G^0(F)$. We define a supercuspidal representation to be "regular" when it arises via this construction from a datum where π_{-1} corresponds, via Moy-Prasad theory [5], to a regular Deligne-Lusztig character.

Theorem 1. *Assume that p is not a bad prime for G and does not divide the order of $\pi_1(G_{der})$. Then there is a 1-1 correspondence between $G(F)$-conjugacy classes of tame regular elliptic pairs (S,θ) and equivalence classes of regular supercuspidal representations.*

Definition 2. *A pair (S,θ) consisting of a maximal torus $S \subset G$ and a character $\theta : S(F) \to \mathbb{C}^\times$ is called* tame regular elliptic *if the splitting extension E/F of S is tame, the stabilizer of θ in the Weyl group $N(S,G)(F)/S(F)$ is trivial, and the action of inertia on the subroot system $R_{0+} = \{\alpha \in R(S,G) | \theta \circ N_{E/F} \circ \alpha^\vee(1+\mathfrak{p}_E) = 1\}$ preserves a set of positive roots.*

Here $R(S,G)$ denotes the absolute root system of S, $N_{E/F} : S(E) \to S(F)$ is the norm map, and $\mathfrak{p}_E$ is the maximal ideal in the ring of integers of E.

Remark 3. *The assumption on p in the theorem can be summarized, according to Dynkin types, in the following table.*

A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
$p > n+1$	$p > 2$	$p > 2$	$p > 2$	$p > 3$	$p > 3$	$p > 5$	$p > 3$	$p > 3$

2. Harish-Chandra characters

A recursive formula for the Harish-Chandra character of any supercuspidal representation arising from Yu's construction has been developed by Adler-Spice [1] and refined by DeBacker-Spice [2]. It involves many subtle roots of unity whose definition depends on the fine structure of the p-adic group governed by Bruhat-Tits theory. We are interested in finding a reinterpretation of this formula, in the case of regular supercuspidal representations, that can be formulated in a Lie-theoretic language.

Given a tame regular elliptic pair (S, θ), let $\pi_{(S,\theta)}$ denote the regular supercuspidal representation corresponding to it via Theorem 1. Write $R(S, G)_{\mathrm{sym}}$ for the subset of those roots α whose Galois orbit contains $-\alpha$. Fix a non-trivial character $\Lambda : F \to \mathbb{C}^\times$. For any such α we have its field of definition F_α. Explicit formulas [3, (4.6),(4.9)] determine $a_\alpha \in F_\alpha^\times$ and $\chi_\alpha : F_\alpha^\times \to \mathbb{C}^\times$ and we define the function

$$\Delta_{II}^{\mathrm{abs}}[a,\chi] : S(F) \to \mathbb{C}^\times, \gamma \mapsto \prod_{\alpha \in R(S,G)_{\mathrm{sym}}/\Gamma} \chi_\alpha \left(\frac{\alpha(\gamma) - 1}{a_\alpha} \right).$$

Theorem 4. *For a topologically semi-simple regular $\gamma \in S(F)$, the value of the Harish-Chandra character of $\pi_{(S,\theta)}$ is given by*

$$e(G)\epsilon(\frac{1}{2}, X^*(T_0)_{\mathbb{C}} - X^*(S)_{\mathbb{C}}, \Lambda) \sum_{w \in \Omega_F} \Delta_{II}^{abs}[a,\chi](\gamma^w)\theta'(\gamma^w). \tag{2}$$

Here $e(G)$ is the Kottwitz sign of G, T_0 is the minimal Levi subgroup of the quasi-split inner form of G, and θ' is the product of θ with a certain explicit sign character of $S(F)$, given by a simple formula [3, (4.3)], that is an artifact of the construction of Yu. Following DeBacker-Spice we may reparameterize Yu's construction so that $\theta' = \theta$.

Theorem 5. *Let $F = \mathbb{R}$. Use the same formula to compute a_α and take for χ_α Shelstad's based χ-data. Then* (2) *recovers* (1).

3. Local Langlands Correspondence – regular case

Let ${}^LG$ be the L-group of G. Let W_F the Weil group of F, with inertia subgroup I_F and wild inertia subgroup P_F. Given a disctere parameter $\varphi : W_F \to {}^LG$ such that $\varphi(P_F)$ is contained in a maximal torus and $\mathrm{Cent}(\varphi(I_F), \hat{G})$ is abelian, we can construct, guided by Theorem 5 and Langlands' work [4] for real groups, an L-packet $\Pi_\varphi(G)$ and prove that it has the correct internal structure. Whenever the technical assumptions of the Adler-Spice character formula are satisfied, we can also prove stability and endoscopic transfer identities.

4. Local Langlands Correspondence - general case

This is currently work in progress. We consider a general discrete parameter $\varphi : W_F \to {}^LG$, i.e. we do not impose conditions on $\varphi(P_F)$ and $\varphi(I_F)$. Instead we impose a slightly stricter condition on the residual characteristic p – it is prime to the order of the absolute Weyl group. Again we tabulate this restriction in terms of Dynkin types:

A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
$p > n+1$	$p > n$	$p > n$	$p > n$	$p > 5$	$p > 7$	$p > 7$	$p > 3$	$p > 3$

In this generality, the group $\bar{S}_\varphi = \mathrm{Cent}(\varphi(W_F), \hat{G})/Z(\hat{G})^\Gamma$ may be non-abelian, even for classical Dynkin types. This makes the internal structure of the L-packets more interesting. These L-packets will contain supercuspidal representations that are not necessarily regular. They are obtained by considering pairs (S, θ) where θ satsifies a condition that is weaker than regularity. The $G(F)$-conjugacy class of such a pair gives rise to a finite set of supercuspidal representations that is a torsor for the finite abelian group $\Omega^*_{F,\theta}$. This structure is mirrored by the irreducible representations of $\bar{S}_\varphi$ and its variants.

References

[1] J.D. Adler, L. Spice, *Supercuspidal characters of reductive, p-adic groups.* Amer. J. Math. **131** (2010) no. 4, 1137–1210.

[2] S. DeBacker, L. Spice, *Stability of character sums for positive-depth, supercuspidal representations.* Journal für die reine und angewandte Mathematik, to appear.

[3] T. Kaletha, *Regular supercuspidal representations*, preprint.

[4] R.P. Langlands, *On the classification of irreducible representations of real algebraic groups.* Math. Surveys and Monographs, No. 31, AMS (1988).

[5] A. Moy, G. Prasad, *Jacquet functors and unrefined minimal K-types.* Comment. Math. Helv. **71** (1996), no. 1, 98–121.

[6] J.K. Yu, *Construction of tame supercuspidal representations.* J. Amer. Math. Soc. **14** (2001), no. 3, 579–622.

Examples of wave front sets for some representations of special orthogonal p-adic groups

Jean-Loup Waldspurger

Let F be a finite extension of $\mathbb{Q}_p$, G be a connected reductive group over F, π be an admissible irreducible representation of $G(F)$ (in a complex space), Θ_π be the character of π defined by $\Theta_\pi(f) = \mathit{trace}\,\pi(f)$ for $f \in C_c^\infty(G(F))$, $\mathfrak{g}$ be the Lie algebra of G. We denote $Nil(\mathfrak{g}(F))$ the set of nilpotent conjugacy classes for the adjoint action of $G(F)$ in $\mathfrak{g}(F)$ and, for $N \in Nil(\mathfrak{g}(F))$, we denote I_N the orbital integral over N. Introduce the usual exponential exp, defined in some neighbourhood of 0 in $\mathfrak{g}(F)$, and a Fourier transform $\varphi \mapsto \hat{\varphi}$ in $C_c^\infty(\mathfrak{g}(F))$, with usual properties.

Theorem (Harish-Chandra). There exists some neighbourhood V_π of 1 in $G(F)$, contained in the image of exp, and there exists a family $(a_{\pi,N})_{N\in Nil(\mathfrak{g}(F))}$ of complex numbers, such that, for $f \in C_c^\infty(G(F))$ with $Supp(f) \subset V_\pi$, we have

$$\Theta_\pi(f) = \sum_{N\in Nil(\mathfrak{g}(F))} a_{\pi,N} I_N((f\circ exp)\hat{\ }).$$

We use the order on $Nil(\mathfrak{g}(F))$: $N \leq N'$ if and only if N is included in the topological closure of N'. Denote $Nil_{max}(\pi)$ the maximal elements of the set $\{N \in Nil(\mathfrak{g}(F)); a_{\pi,N} \neq 0\}$ (this set is non-empty). Denote $\bar{F}$ an algebraic closure of F and $Nil(\mathfrak{g}(\bar{F}))$ the set of nilpotent conjugacy classes for the adjoint action of $G(\bar{F})$ in $\mathfrak{g}(\bar{F})$. The know examples suggest the following

Conjecture. There exists an unique $\bar{N}_\pi \in Nil(\mathfrak{g}(\bar{F}))$ such that, for all $N \in Nil_{max}(\pi)$, N is included in $\bar{N}_\pi$.

If this is true, we call $\bar{N}_\pi$ the wave front set of π.

Now, we assume $G = SO(2n+1)$. There is two forms of this group, one split and the other non split and we accept the two forms. We assume p is big relatively to n. Recall the Langlands parametrization of the irreducible admissible representations of $G(F)$. Consider an homomorphism $\psi : W_F \times SL(2,\mathbb{C}) \to \hat{G} = Sp(2n,\mathbb{C})$, with usual properties, where W_F is the Weil group of F. Denote S_ψ the group of connected components of the commutant in $\hat{G}$ of the image of ψ. Denote $\hat{S}_\psi$ its group of characters and $\hat{S}_\psi(G)$ the subset of $\epsilon \in \hat{S}_\psi$ such that $\epsilon(-1_{\hat{G}}) = 1$ if G is split, $\epsilon(-1_{\hat{G}}) = -1$ if G is non-split ($-1_{\hat{G}}$ is the central element -1 in $\hat{G}$). Then the Langlands conjecture says that the irreducible admissible representations of $G(F)$ are parametrized by the conjugaçy classes of pairs (ψ,ϵ) where ψ is as above and $\epsilon \in \hat{S}_\psi(G)$. This parametrization has been obtained by Arthur for the split group G ([1] theorem 2.2.1).

We restrict to the set of tempered representations of unipotent reduction. "Tempered" is equivalent to: $\psi(W_F)$ is relativement compact; "of unipotent reduction" is equivalent to: the restriction of ψ to W_F is unramified. Denote $\Psi_{tunip}(G)$ the set of conjugacy classes of pairs (ψ,ϵ) satisfying these properties. To each $(\psi,\epsilon) \in \Psi_{tunip}(G)$, Lusztig has associated a tempered representation of unipotent reduction $\pi_{\psi,\epsilon}$ of $G(F)$ ([2]). We use this representation constructed by Lusztig. The following theorem ([4] introduction et [5] theorem 1.3) says that it satisfy the expected properties.

Theorem. For $(\psi,\epsilon) \in \Psi_{tunip}(G)$, the representation $\pi_{\psi,\epsilon}$ satisfy the relations relative to endoscopy and twisted endoscopy which determine uniquely this representation.

Let (ψ,ϵ) be an element of $\Psi_{tunip}(G)$. Denote $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(2,\mathbb{C})$. To ψ is associated a nilpotent orbit of the Lie algebra $\mathfrak{sp}(2n,\mathbb{C})$: the orbit of the logarithm of $\psi(1,u)$. The nilpotent orbits in this Lie algebra are parametrized by the set $P^{symp}(2n)$ of symplectic partitions of $2n$ and we denote λ_ψ the partition associated to the preceding nilpotent orbit. Similarly, the nilpotent orbits in $\mathfrak{g}(\bar{F})$ are parametrized by the set $P^{orth}(2n+1)$ of orthogonal partitions of $2n+1$. Following Spaltenstein, we define a "duality" $d : P^{symp}(2n) \to P^{orth}(2n+1)$. For $\lambda \in P^{symp}(2n)$, $d(\lambda)$ is the greatest orthogonal partition of $2n+1$ which is less or equal to ${}^t(\lambda \cup \{1\})$, with usual notations. The image of this duality is the subset of special partitions in $P^{orth}(2n+1)$.

Denote D the Aubert-Zelevinski's involution for the group G. We denote $\delta_{\psi,\epsilon} = D(\pi_{\psi,\epsilon})$.

Theorem. For $(\psi,\epsilon) \in \Psi_{tunip}(G)$, the conjecture above is verified for the representation $\delta_{\psi,\epsilon}$. The wave front set of this representation is parametrized by $d(\lambda_\psi)$

Cf. [6] theorem 3.3. This result was previously obtained by Moeglin ([3] theorem 3.3.5) under different hypothesis. Our proof is different.

References

[1] J. Arthur, *The endoscopic classification of representations: orthogonal and symplectic Groups*, AMS Colloquium Publ. 61 (2013).

[2] G. Lusztig, *Classification of unipotent representations of simple p-adic groups*, Int. Math.Res. Notices (1995), 517–589.

[3] C. Moeglin, *Représentations quadratiques unipotentes des groupes classiques p-adiques*, Duke Math. J. **84** (1996), 267–332.

[4] J.-L.. Waldspurger, *Représentations de réduction unipotente pour $SO(2n+1)$, I: une involution*, arxiv 1611.08249

[5] J.-L.. Waldspurger, *Représentations de réduction unipotente pour $SO(2n+1)$, II: endoscopie*, arxiv 1612.02635

[6] J.-L.. Waldspurger, *Représentations de réduction unipotente pour $SO(2n+1)$, III: exemples de fronts d'onde*, arxiv 1612.06103

Cocenter of p-adic groups

Xuhua He

In this report, we give an overview of the new cocenter project towards some new understanding of complex and mod-l representations of p-adic groups. It is based on the three preprints [5], [6] and [3]. The last one is a joint work with D. Ciubotaru.

0.1. To explain why the study of cocenter (of an algebra) is useful in representation theory, we start with a "toy model".

Let G be a finite group and $\mathbb{C}[G]$ be the group algebra (over complex numbers). Let $\overline{\mathbb{C}[G]} = \mathbb{C}[G]/[\mathbb{C}[G], \mathbb{C}[G]]$ be its cocenter, the algebra modulo its commutator. Let $\mathfrak{R}_{\mathbb{C}}(G)$ be the Grothendieck group of finite dimensional complex representations of G. We have the trace map $Tr_{\mathbb{C}} : \overline{\mathbb{C}[G]} \to \mathfrak{R}_{\mathbb{C}}(G)^*, \quad h \mapsto (V \mapsto Tr(h, V))$. The cocenter $\overline{\mathbb{C}[G]}$ has a standard basis indexed by the conjugacy classes of G and the Grothendieck group $\mathfrak{R}_{\mathbb{C}}(G)$ has a standard basis given by the isomorphism classes of irreducible representations of G. Via these basis, the trace map is represented by the "character table", an invertible square matrix. In particular, the trace map $Tr_{\mathbb{C}}$ is bijective and gives a natural duality between the cocenter and the representations.

For any algebraically closed field R, we may define the trace map Tr_R in a similar way. The map is always surjective, but not injective in general. However, there is a simple and explicit description of the kernel of the map Tr_R.

0.2. Now let F be a nonarchimedean local field with residue characteristic p (i.e. a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_q((\epsilon))$). Let $\mathbb{G}$ be a connected reductive group over F and $G = \mathbb{G}(F)$ be its F-points. We are mainly interested in the smooth admissible representations of G over an algebraically closed field R of characteristic 0 or of positive characteristic l with $l \neq p$. In order to study the smooth admissible representations, we do not use the group algebra as in the finite group case, but use the Hecke algebra instead. To be more precise, the Hecke algebra we use is $H = C_c^\infty(G, \mathbb{Z}[\frac{1}{p}])$, the "integral form" of the usual Hecke algebra over complex numbers. We will describe some structural results on H and its cocenter $\overline{H}$. In order to apply these results to the representation theory (over R), we use the extension of scalar $H_R = H \otimes_{\mathbb{Z}[\frac{1}{p}]} R$ and $\overline{H}_R = \overline{H} \otimes_{\mathbb{Z}[\frac{1}{p}]} R$.

The relation between the cocenter and representations of p-adic groups was first studied by Bernstein, Deligne and Kazhdan in [1] and [8]. They showed that over complex numbers, the cocenter and the representations are dual to each other in the following sense: $Tr_{\mathbb{C}} : \overline{H_{\mathbb{C}}} \cong \mathfrak{R}_{\mathbb{C}}(G)^*_{good}$. Here $\mathfrak{R}_{\mathbb{C}}(G)^*_{good}$ is the space of good linear form on $\mathfrak{R}_{\mathbb{C}}(G)$.

The relation between the cocenter and representations were also studied later by Dat, by Flicker and more recently, by Henniart and Lemaire [4], in which they generalized the duality between the complex cocenter and the complex representations to complex ω-twisted cocenter and complex ω-representations coming from the theory of twisted endoscopy.

However, it is fair to say that the cocenter and its relation to representations is not much studied as it should be. One reason is that the cocenter $\overline{H_{\mathbb{C}}}$ is just a vector space, and does not possess a rich algebraic structure. It took me quite a long time before finally realized that the key point is to get a nice decomposition on the cocenter, which would lead to a nice filtration on the representation side.

0.3. In [5], we introduced the Newton decompositions: $G = \sqcup_\nu G(\nu), H = \oplus_\nu H(\nu)$ and $\overline{H} = \oplus_\nu \overline{H}(\nu)$.

Here $G(\nu)$ is the Newton stratum associated to the given Newton point ν. It is a G-domain defined by $G(\nu) = \cup_{w \in \tilde{W}_{\min}, w \mapsto \nu} {}^G(IwI)$, where I is the Iwahori subgroup, $\tilde{W}$ is the Iwahori-Weyl group and $\tilde{W}_{\min}$ is the subset of $\tilde{W}$ consisting of elements of minimal length in their conjugacy classes. The $\mathbb{Z}[\frac{1}{p}]$-submodule $H(\nu)$ of H by definition consists of functions with support in $G(\nu)$ and $\overline{H}(\nu)$ is the image of $H(\nu)$ in the cocenter $\overline{H}$.

The most technical part of the definition is the minimal length condition. In fact, this is the key idea that makes the decomposition work. The definition of $G(\nu)$ is highly motivated by Kottwitz' work on the σ-isocrystals [9], [10] and by the recent progress in the theory of affine Deligne-Lusztig varieties.

0.4. For applications to representation theory, one would like to understand not only the cocenter of the Hecke algebra H, but also the cocenter of the Hecke algebra $H(G, K)$. Here K is an open compact subgroup of G and $H(G, K)$ is the convolution algebra of K-biinvariant, compactly supported functions on G.

For any Newton point ν, let $H(G, K; \nu) = H(G, K) \cap H(\nu)$ and let $\overline{H}(G, K; \nu)$ be the image of $H(G, K; \nu)$ in $\overline{H}$. Note that for any open compact subgroup K, we have $H(G, K) \neq \oplus_\nu H(G, K; \nu)$. However, we show that for the congruent subgroup I_n of the Iwahori subgroup I of G, we still have the Newton decomposition for $\overline{H}(G, I_n)$. We also give a set of Iwahori-Matsumoto type generators for each

Newton component $H(G, I_n; \nu)$. As an application of it, we give a simple proof of a conjecture of Howe [7] on the invariant distributions.

0.5. In the representation theory of p-adic groups, there are two important functors, the parabolic induction functor i_M and Jacquet functor r_M for any given standard Levi subgroup M. On the dual side, we expect to have the restriction map $\bar{r}_M$ and induction map $\bar{i}_M$. We expect that these maps are compatible with the Newton decompositions of the cocenter of Hecke algebras of G and of M. Also for the purpose of a uniform approach to both complex representations and mod-l representations, it is important to have such map defined over the integral forms, i.e. the Hecke algebras of $\mathbb{Z}[\frac{1}{p}]$-valued functions.

The restriction map $\bar{r}_M$ on the cocenter, which after extension by scalar, is adjunct to the parabolic induction functor on the representations, is given by the Van Dijk's formula. In [3], we showed that the map defined by the Van Dijk's formula has the desired properties.

The induction map $\bar{i}_M$ on the cocenter, is expected to be a map that after extension by scalar is adjunct to Jacquet functor. This is more complicated than the restriction map $\bar{r}_M$. By now, we do not have a complete answer to it. However, in [6], we obtain a partial answer. Namely, we give an explicit formula for the restriction of the induction map $\bar{i}_M$ to the so-called $+$-rigid part of the cocenter of M.

0.6. After we have the Newton decomposition of the cocenter and a fairly good understanding on the induction and restriction map, we are able to explore the representation theory of p-adic groups.

There are two important submodules of the cocenter H: the elliptic cocenter H^{ell} and the rigid cocenter H^{rig}. The elliptic cocenter is the analogous part of the elliptic representations, an important family of irreducible representations. The rigid cocenter, on the other hand, is shown in [6] to be the "building blocks" of the whole cocenter $\overline{H}$. In [3], we establish an important result in the structure of cocenter: the elliptic cocenter is contained in the rigid cocenter.

As an application, we prove the trace Paley-Wiener theorem for mod-l representations under the assumption that l does not divide the order of the relative Weyl group. For complex representations, such result was established in [1].

As another application, we establish the abstract Selberg principle on the relation between smooth projective representations and the orbital integral of compact elements. This was proved in [2] for complex representation. Now we showed that it is also true for mod-l representations.

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References

[1] J. Bernstein, P. Deligne, D. Kazhdan, *Trace Paley-Wiener theorem for reductive p-adic groups*, J. d'Analyse Math. **47** (1986), 180–192.
[2] P. Blanc and J.-L. Brylinski, *Cyclic homology and the Selberg principle*, J. Funct. Anal. **109** (1992), p. 289–330.
[3] D. Ciubotaru and X. He, *Cocenters of p-adic groups, III: Elliptic cocenter and rigid cocenter*, arXiv:1703.00378.
[4] G. Henniart and B. Lemaire, *La transformée de Fourier pour les espaces tordus sur un groupe réductif p-adique*, to appear in Astérisque.
[5] X. He, *Cocenters of p-adic groups, I: Newton decomposition*, 1610.04791.
[6] X. He, *Cocenters of p-adic groups, II: Bernstein-Lusztig presentation*, arXiv:1611.06825.
[7] R. Howe, *Two conjectures about reductive p-adic groups*, Proc. of A.M.S. Symposia in Pure Math. XXVI (1973), 377–380.
[8] D. Kazhdan, *Cuspidal geometry of p-adic groups*, J. Analyse Math. **47** (1986), 1–36.
[9] R. Kottwitz, *Isocrystals with additional structure*, Compositio Math. **56** (1985), 201–220.
[10] R. Kottwitz, *Isocrystals with additional structure. II*, Compositio Math. **109** (1997), 255–339.

Stable conjugacy for the BD-covers of symplectic groups

Wen-Wei Li

1. Overview

The representation theory for coverings of linear reductive groups has spurred some interests in recent years. In order to adapt Langlands' ideas to this setting, the first task is to single out a wide class of coverings with an algebro-geometric origin, which admit classification theorems and behave functorially under various operations. Brylinski and Deligne [1] set up a reasonable framework by considering central extensions of sheaves over the big Zariski site S_{Zar} of the form

$$1 \to \mathbf{K}_2 \to E \to G \to 1,$$

where S is a reasonable base scheme, say $S = \mathrm{Spec}(F)$ in this report for some local field F. Here G is a reductive S-group and $\mathbf{K}_2$ is the sheafification of the usual K_2. For split and simply connected G, such $\mathbf{K}_2$-extensions are known to Matsumoto in the 1960's.

Next, given an integer $m \geq 1$ with

$$m \mid N_F := \begin{cases} |\mu(F)|, & F \neq \mathbb{C} \\ 1, & F = \mathbb{C}, \end{cases}$$

taking F-points and pushing-out by the norm-residue symbol $(\cdot,\cdot)_{F,m} : \mathbf{K}_2(F) \to \mu_m$ yields a topological central extension

$$1 \to \mu_m \to \tilde{G} \to G(F) \to 1.$$

Coverings arising in this way are called *BD-covers*. We refer to [2, 6, 3], among others, for various facets of the Langlands program for BD-covers.

Fix an embedding $\epsilon : \mu_m \to \mathbb{C}^\times$. We study the representations of $\tilde{G}$ on complex vector spaces such that μ_m acts via the character ϵ, called *ϵ-genuine* or simply genuine.

2. Stable conjugacy

One of the main hurdle for Langlands program on BD-covers is the lack of a good notion of stable conjugacy. As far as I know, available theories apply only to

1. the metaplectic groups of Weil (due to Adams), which is based on the oscillator representation, or
2. BD-covers of SL(2) in a unpublished note by Hiraga and Ikeda, based on Flicker's work.

I will propose a definition of stable conjugacy for BD-covers

$$1 \to \mu_m \to \widetilde{\mathrm{Sp}}(2n) \to \mathrm{Sp}(2n, F) \to 1, \quad \mathrm{char}(F) \neq 2$$

arising from Matsumoto's central extensions (without loss of generality!) with arbitrary $m \mid N_F$ such that $\mathrm{char}(F) \nmid m$. When $m = 2$ we recover Weil's metaplectic groups. This will be done by reduction to various $\mathrm{SL}(2, K^\sharp)$, where $K^\sharp$ is some finite separable extension of F. Stable conjugacy for $\mathrm{SL}(2, K^\sharp)$ is realizable by $\mathrm{PGL}(2, K^\sharp)$-action, which lifts uniquely to BD-covers. We also denote it by $\mathrm{Ad}(g)$. In the case $G = \mathrm{SL}(2)$ already, this construction encounters two problems:

1. It is incompatible with Adams' definition when $m = 2$.
2. Let $\delta, \delta' \in G_{\mathrm{reg,ss}}(F)$ be stably conjugate with centralizers T, T'. Stable conjugacy in G yields an isomorphism $\mathrm{Ad}(g) : T \to T'$ that is independent of the choice of $g \in G(\bar{F})$ with $\mathrm{Ad}(g)(\delta) = \delta'$. In contrast, the construction above of $\mathrm{Ad}(g)(\tilde{t})$ depends on the choice of $g \in \mathrm{PGL}(2, F)$.

For $G = \mathrm{SL}(2)$ and $t \in T_{\mathrm{reg}}(F)$, one calibrates $\mathrm{Ad}(g)(\tilde{t})$ by multiplying it by a μ_m-valued factor $\mathbf{C}_m(\cdot, g)$. The new action will be on $\tilde{T}_{Q,m}$, the pullback of $\tilde{G} \to G(F)$ along the isogeny $\iota_{Q,m} : T_{Q,m} \to T$ furnished by general theory [6], or its translate by preimages of -1 when $4 \mid m$; this is expected by the general philosophy of [6]. Specifically, we may identify T with $K^{\mathrm{Nm}=1}$ for an étale F-algebra K of dimension 2, and $\iota_{Q,m} : t_0 \mapsto t_0^{m/\gcd(2,m)}$; for $t_0 \in \iota_{Q,m}^{-1}(t)$, the required calibration factor is simply

$$\mathbf{C}_m(t_0, g) = (N_{K/F}(\omega), \det g)_{F,\gcd(2,m)} \in \mu_m$$

whenever $t_0 = \omega/\bar{\omega}$, for $\omega \in K^\times$.

For general G, some results on the Weil restriction of Brylinski–Deligne coverings will be needed. All in all, we are able to lift stable conjugacy to various $\tilde{T}_{Q,m}$ or their translates by tuples of ± 1 when $4 \mid m$, where $T \subset G$ are maximal tori.

The notion of stable conjugacy might shed some light on the stabilization of trace formula for such coverings. I hope to address these issues in subsequent works.

3. Epipelagic supercuspidal L-packets

Suppose $F \supset \mathbb{Q}_p$ with $p \gg_n 0$. To support the formalism above, one constructs genuine epipelagic supercuspidal L-packets (or some weaker variant when $4 \nmid m$) and establish their *stability* in an appropriate sense. Following Kaletha [4], the epipelagic L-packets are constructed from (i) a stable conjugacy class of embeddings of maximal tori $j : S \hookrightarrow G$, subject to various conditions (eg. tameness, ellipticity); (ii) a character $\theta : S(F) \to \mathbb{C}^\times$ subject to some regularity conditions.

Given (S, θ) and j as above, we first lift θ to a genuine character θ_j of $\widetilde{jS} \subset \tilde{G}$. The maximal torus jS also determines a point $x \in \mathcal{B}(G/F)$. Form the genuine supercuspidal $\text{c-Ind}_{\widetilde{jS} \cdot G(F)_{x,r}}^{\tilde{G}}(\theta_j \epsilon_j)$, the character $\epsilon_j : jS(F) \to \{\pm 1\}$ arises from Kaletha's *toral invariants*, and $r > 0$ is the depth of θ. This yields the required packet by varying j.

The construction of θ_j is rather subtle when $m \equiv 2 \pmod 4$. It involves a splitting of $\mathbb{C}^\times \times^{\mu_m} \widetilde{jS}$, together with a sign that measures the ratio between toral invariants of $\mathrm{Sp}(2n)$ and $\mathrm{SO}(V, q)$ with $\dim V = 2n + 1$, in terms of the moment-map correspondence.

On the other hand, to (S, θ) we may also associate an epipelagic supercuspidal L-parameter for $\tilde{G}$, as in [4]. All these turn out to be compatible with the formalism of L-groups in [6]. When $m = 2$, our L-packet coincides with that obtained from Θ-lifting; this is proven using the results of Loke–Ma–Savin [5].

References

[1] J.-L. Brylinski and P. Deligne. *Central extensions of reductive groups by K_2*. In: Publ. Math. Inst. Hautes tudes Sci. **94** (2001), pp. 5–85.

[2] W. T. Gan and F. Gao. *The Langlands–Weissman Program for Brylinski–Deligne extensions*. `arXiv: 1409.4039`.

[3] D. Gaitsgory and S. Lysenko, *Parameters and duality for the metaplectic geometric Langlands theory*. `arXiv:1608.00284`.

[4] T. Kaletha. *Epipelagic L-packets and rectifying characters*. In: Invent. Math. **202** (2015), pp. 1–89.

[5] H. Y. Loke, J. Ma, and G. Savin. *Local theta correspondences between epipelagic supercuspidal representations*. In: Math. Z. **283** (2016), pp. 169C-196.

[6] M. H. Weissman. *L-groups and parameters for covering groups*. `arXiv:1507.01042`.

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