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Dynamische Systeme

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Abstract. This workshop continued the biannual series at Oberwolfach on Dynamical Systems that started as the “Moser-Zehnder meeting” in 1981. The main themes of the workshop are the new results and developments in the area of dynamical systems, in particular in Hamiltonian systems and symplectic geometry. This year special emphasis where laid on symplectic methods with applications to dynamics. The workshop was dedicated to the memory of John Mather, Jean-Christophe Yoccoz and Krzysztof Wysocki.

Mathematics Subject Classification (2010): 37, 53D, 70F, 70H.

Introduction by the Organisers

The workshop was organized by H. Eliasson (Paris), H. Hofer (Princeton) and V. Kaloshin (Maryland). It was attended by more than 50 participants from 13 countries and displayed a good mixture of young, mid-career and senior people. The workshop covered a large area of dynamical systems centered around classical Hamiltonian dynamics and symplectic methods: closing lemma; Hamiltonian PDE's; Reeb dynamics and contact structures; KAM-theory and diffusion; celestial mechanics. Also other parts of dynamics were represented.

K. Irie presented a smooth closing lemma for Hamiltonian diffeomorphisms on closed surfaces. This result is the peak of a fantastic development in symplectic methods where, in particular, the contributions of M. Hutchings play an important role.

D. Peralta-Salas presented new solutions for the 3-dimensional Navier-Stokes equations with different vortex structures. The proof uses highly oscillatory Beltrami fields and techniques of KAM-type.

L. Buhovsky presented new important works (with V. Humilière and S. Seyfaddini) on the C^0-Arnold conjecture, and S. Seyfaddini presented new works (with F. Le Roux and C. Viterbo) on conjugacy-classes in the group of area-preserving homeomorphisms. Reeb dynamics and contact structures were studied in the talks of A. Abbondandolo, M. Alves, D. Hein, J Nelson, D. Pomerleano and K. Zehmisch. M. Hutchings talked about how to test numerically a conjecture of Viterbo.

Symplectic methods with applications to celestial mechanics and the restricted 3-body problem were presented in the talks of U. Frauenfeld, U. L. Hryniewicz, O. van Koert and P. A. S. Salomão. KAM-theory and diffusion in Hamiltonian systems where discussed in the talks of T. Castan, M. Gidea and M. Saprykina.

Several other topics in dynamics where discussed in different talks. M.-C. Arnaud talked about weak KAM-theory, complete integrability and C^1 Arnold-Liouville theorem. W. Craig discussed Birkhoff normal forms for PDE's. V. Ginzburg talked about pseudo-rotations on projective spaces. S. Hohloch discussed dynamics of vector fields at a focus-focus equilibrium. G. Knieper talked about geodesic flows and zero topological entropy. K. Kuperberg discussed a-periodic flows. S. Tabachnikov introduced symplectic billiards and C. Ulcigrai talked about central limit theorems for certain co-cycles over rotations.

The meeting was held in an informal and stimulating atmosphere. The weather was unstable and the traditional walk to St. Roman, under the leading of Sergei Tabachnikov, took place on Thursday.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Pedro Salomão in the "Simons Visiting Professors" program at the MFO.

Workshop: Dynamische Systeme

Table of Contents

Abstracts

Computing the Ekeland-Hofer Zehnder capacity of four-dimensional convex polytopes

Michael Hutchings
(joint work with Julian Chaidez)

Let X be a compact convex domain in $\mathbb{R}^{2n}$ with smooth boundary such that $0 \in \operatorname{int}(X)$. The *Reeb vector field* R on ∂X is characterized by $d\lambda(R, \cdot) = 0$ and $\lambda(R) = 1$, where

$$\lambda = \frac{1}{2} \sum_{i=1}^{n} (x_i \, dy_i - y_i \, dx_i) \, |_{\partial X}.$$

Define the *Ekeland-Hofer-Zehnder capacity* $c_{\mathrm{EHZ}}(X)$ to be the minimum period of a periodic orbit of the Reeb vector field R on ∂X. A fascinating conjecture of Viterbo [2] implies that

$$c_{\mathrm{EHZ}}(X)^n \leq n! \operatorname{vol}(X).$$

(This inequality is sharp for a ball.)

We introduce numerical methods for testing this conjecture.

More precisely, it is known that c_{EHZ} is a symplectic capacity, i.e. monotone under symplectic embeddings. Consequently, $c_{\mathrm{EHZ}}(X)$ has a unique C^0-continuous extension to a function on arbitrary compact convex domains (which do not necessarily have smooth boundary). We present a combinatorial algorithm for computing $c_{\mathrm{EHZ}}(X)$ when X is a convex polytope in $\mathbb{R}^4$ with no Lagrangian 2-faces. If the polytope X is rational (i.e. its vertices have rational coordinates), then the algorithm computes the exact value of $c_{\mathrm{EHZ}}(X)$ (which in this case is a rational number) in finite time.

As shown in [1], the EHZ capacity $c_{\mathrm{EHZ}}(X)$ is the minimum symplectic action of a piecewise differentiable loop $\gamma : S^1 \to \partial X$ such that $J\gamma'(t) \in N_+X$, where J denotes the standard almost complex structure on $\mathbb{R}^{2n}$ and N_+X denotes the positive normal cone to X. The symplectic action of γ is the symplectic area of a loop bounded by γ. For our polytope X, these loops can be found combinatorially. A key ingredient of the algorithm is to use the Conley-Zehnder index, computed using a quaternionic trivialization, to give an upper bound on the number of 2-faces that an action-minimizing loop can cross.

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Isolated elliptic fixed points for smooth Hamiltonians

Maria Saprykina
(joint work with B. Fayad, R. de la Llave)

KAM theory asserts that generically an elliptic fixed point of a Hamiltonian system is stable in a probabilistic sense, or KAM stable: the fixed point is accumulated by a positive measure set of invariant Lagrangian tori. It was conjectured by M. Herman in his ICM98 lecture that for analytic Hamiltonians, KAM stability holds in a neighborhood of an elliptic fixed point if its frequency vector is assumed to be Diophantine. The conjecture is known to be true in two degrees of freedom, but remains open in general. Partial results in this direction were recently obtained by Eliasson, Fayad and Krikorian.

Below analytic regularity, Herman proved that KAM stability of a Diophantine equilibrium holds without any twist condition in C^∞ in 2 degrees of freedom. In his ICM98 lecture Herman announced that KAM stability of Diophantine equilibria does not hold for smooth Hamiltonians in 4 or more degrees of freedom, without giving any clew about the possible counter-examples. He also wrote that nothing was known about KAM stability of Diophantine equilibria for smooth Hamiltonians in 3 degrees of freedom. In a joint work with Bassam Fayad, we settle this problem by presenting examples of smooth Hamiltonians for any $d \geq 3$ having non KAM stable elliptic equilibria with arbitrary frequency.

In this talk we also present a result on KAM stability for a degenerate case (zero frequency) of real analytic Hamiltonians. Namely, we show that if the Birkhoff normal form of a real-analytic Hamiltonian at an analytic invariant torus is convergent and has a particular form (it is an analytic function of its quadratic part), then there is an analytic canonical transformation—not just a power series—bringing the Hamiltonian into its Birkhoff normal form. The latter result it based on a joint work with Rafael de la Llave.

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Existence of global cross-sections: from Schwartzman cycles to holomorphic curves

Umberto L. Hryniewicz
(joint work with Pedro A. S. Salomão, Krzysztof Wysocki)

We consider a smooth closed 3-manifold M equipped with a smooth flow ϕ^t. It is an important problem to decide whether a given collection of periodic orbits $\gamma_1, \dots, \gamma_N$ bounds a global surface of section. In full generality this problem is extremely challenging. For instance, in [2] Birkhoff conjectures that the retrograde orbit in the (planar circular) restricted three-body problem below the first critical value of the Jacobi constant bounds a disk-like global surface of section. This conjecture was confirmed in [11] when one of the primaries is much heavier than the other and the satellite moves in one of the two bounded Hill regions, but for arbitrary mass ratios the conjecture is wide open.

Recall that a global surface of section is an embedded compact orientable surface $S \hookrightarrow M$ satisfying

(a) $\partial S = \gamma_1 \cup \cdots \cup \gamma_N$,
(b) ϕ^t is transverse to $\mathrm{int}(S)$ and
(c) trajectories not in $\gamma_1 \cup \cdots \cup \gamma_N$ hit $\mathrm{int}(S)$ infinitely many often in the future and in the past.

Of course, such a global section reduces the study of the dynamics to that of the associated return map, allowing powerful two-dimensional methods to come into play. One can relax the above definition and only ask that S is immersed, its interior is properly embedded in $M \setminus L$ and its boundary components multiply cover the γ_i (the order of the covering depends in i).

All this is, of course, a central topic of study in Schwartzman-Fried-Sullivan theory [5, 12, 13]. A valuable source of information is [4]. Let us denote by L the link $\gamma_1, \ldots, \gamma_N$ and by T_i the period of γ_i. Given $f \in H_2(M, L; \mathbb{Z})$ we consider rotation numbers $\rho^f(\gamma_i) \in \mathbb{R}$ defined as follows. Choose tubular neighborhood $N_i \simeq \mathbb{R}/T_i\mathbb{Z} \times \mathbb{D}$ of γ_i with coordinates $(t, re^{i\theta})$, such that $\gamma_i \simeq \mathbb{R}/T_i\mathbb{Z} \times 0$. If $y^f \in H^1(M \setminus L; \mathbb{Z})$ is dual to f (y^f counts algebraic intersection number with f) then $y^f|_N \equiv p(dt/T_i) + q(d\theta/2\pi)$ for some $p, q \in \mathbb{Z}$. One defines

$$\rho^f(\gamma_i) = p + q \lim_{t \to +\infty} \frac{T_i \theta(t)}{2\pi t}$$

where $\theta(t)$ is the infinitesimal argument of the transverse linearized flow along γ_i in the local coordinates. This number is independent of the choice of coordinates. One also needs to consider Borel probability measures in $M \setminus L$ which are ϕ^t-invariant. Denote by $\mathcal{P}_\phi$ the set of such measures. The "real" intersection number $\mu \cdot f \in \mathbb{R}$ is defined for every $\mu \in \mathcal{P}_\phi$ as

$$\mu \cdot f = \int_{M \setminus L} \beta_f(X) d\mu$$

where X is the infinitesimal generator of ϕ^t and the closed 1-form β_f in $M \setminus L$ represents y^f and satisfies suitable properties that make the integrand bounded. This integral does not depend on such a representative.

The following beautiful statement provides sufficient conditions, which are also necessary under very mild non-degeneracy assumptions.

Theorem A. (Schwartzman-Fried-Sullivan) The global surface of section S exists provided that

- $\rho^f(\gamma_i) > 0 \ \forall i$
- $\mu \cdot f > 0 \ \forall \mu \in \mathcal{P}_\phi$

In fact, $L = \partial S$ is the binding of an open book decomposition of M all of whose pages are global surfaces of section. Moreover, C^∞-generically these conditions are also necessary.

This theorem is a formulation (and a generalization) of work of Fried in terms of invariant measures. It gives dynamical criteria to decide if L is a fibered link. Theorem A is so beautiful that it is tempting not to pause and ask about its applicability. Our main result is a version for three-dimensional Reeb flows which does not need linking hypothesis for all invariant measures, but only for those induced by a certain set of periodic orbits. In fact, in many concrete applications, there are no periodic orbits preventing the construction of the global surface of section and their existence follows immediately. We give more details.

Theorem B. (Hryniewicz-Salomão-Wysocki) Let ϕ^t be a Reeb flow on the closed 3-manifold M, and let $\gamma_1, \dots, \gamma_N$ be periodic orbits. Assume that $L = \gamma_1 \cup \cdots \cup \gamma_N$ binds a planar open book decomposition supporting ξ, and consider $f \in H_2(M, L; \mathbb{Z})$ the class of a page. There exists a finite set $J \subset \mathbb{N}$ such that the following holds: if $\rho^f(\gamma_i) > 0$ $\forall i$ and all periodic orbits $\gamma' \subset M \setminus L$ satisfying $\mathrm{CZ}(\gamma') \in J$ algebraically intersect f non-trivially, then L bounds a genus zero global surface of section for ϕ^t which represents the class f. Moreover, C^∞-generically these conditions are also necessary.

This statement allows for applications because the set J can be computed explicitly in many cases. For instance, $J = \{2\}$ in the case of the tight three-sphere and L is equal to an unknotted orbit with self-linking number -1. Hence we get

Corollary. (Hryniewicz-Salomão [10]) Let γ be a periodic orbit of a tight Reeb flow on S^3. Under a non-degeneracy assumption, γ bounds a disk-like global surface of section if, and only if, it is unknotted, has self-linking number -1, satisfies $\mathrm{CZ}(\gamma) \geq 3$ and all periodic orbits with $\mathrm{CZ} = 2$ link with γ.

In the case of a pair of periodic orbits forming a Hopf link for a Reeb flow in the universally tight $\mathbb{R}P^3$ we have $J = \{0, 1, 2\}$. As a consequence we get

Corollary. Consider the planar circular restricted three-body problem with small mass ratio and energy below the first critical value. Assume that the satellite moves in one of the bounded Hill regions. Then any retrograde orbit is a boundary component of an annulus-like global surface of section for the regularized flow.

Corollary. (Birkhoff [1]) For any positively curved Riemannian two-sphere, a simple closed geodesic traversed in both directions spans a annulus-like global surface of section.

These concrete applications are only possible because in Theorem B we need only to consider a very small class of invariant measures. The tools come from pseudo-holomorphic theory as developed by Hofer-Wysocki-Zehnder [6, 7, 8, 9] and the relevant compactness theorem is the SFT-Compactness theorem [3].

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A Central Limit Theorem for certain cocycles over rotations

Corinna Ulcigrai

(joint work with Michael Bromerg)

We present an instance of a temporal Central Limit theorem in entropy zero dynamics. Limit theorems appear often in dynamics as follows. Let $(X, \mathcal{B}, m, T)$ be a measure preserving dynamical system. Let $f : X \to \mathbb{R}$ be a Borel measurable function and set $S_n(x) = S_n(T, f, x) := \sum_{k=0}^{n-1} f \circ T^k(x)$ for the n^{th} Birkhoff sum. If T is ergodic with respect to m and $f \in L^1(X, m)$, the Birkhoff ergodic theorem can be recasted as the Law of Large Numbers (LLN) for the random variables $(X_n)_n$ where $X_n := f \circ T^n$ and x is chosen randomly according to the measure m. For many *hyperbolic* dynamical systems, under suitable assumptions on the regularity of f and the rate of mixing of T, one can study the error term in the LLN by proving a (spatial) *Central Limit Theorem* (CLT) for the r.v.s $(X_n)_n$. On the other hand, in many classical examples of dynamical systems with *zero entropy,* for which $(X_n)_n$ are highly correlated, the CLT, and more in general spatial distributional limit theorems, fail to hold. For example, this is the case when T is an irrational rotation and f is of bounded variation. In this case the CLT can be proved only along subsequences (see [10, 8]).

A different point of view, recently popularized by Dolgopyat and Sarig in [9], is to investigate so called *temporal* limit theorems. Motivated by single orbit dynamics, instead of randomizing space, one considers the Birkhoff sums $S_n(x_0)$ over a *single orbit* of some fixed initial condition $x_0 \in X$ and randomizes time as follows. Define a sequence of *occupation measures* on $\mathbb{R}$ by $\nu_n(F) := \frac{1}{n}\#\{1 \le k \le n : \quad S_k(x_0) \in F\}$ for every Borel measurable $F \subset \mathbb{R}$ and consider a sequence of r.v. Y_n distributed according to ν_n. We say that (T, f) satisfies a *temporal limit theorem* along the orbit of x_0, if there exists a random variable with no atoms Y, and two sequences $A_n \in \mathbb{R}$ and $B_n \to \infty$ such that $(Y_n - A_n)/B_n$ converges in distribution to Y. If the limit Y is a Gaussian random variable, we call this type of behavior a *temporal CLT* along the orbit of x_0. We refer the interested reader to [9] and the reference therein for examples of temporal limit theorems in dynamics.

Perhaps surprisingly, many examples of dynamical systems with zero entropy satisfy a temporal CLT. One example is the following result by Beck. Let us denote by R_α the rotation on the interval $\mathbb{T} = \mathbb{R} \setminus \mathbb{Z}$ by an irrational number $\alpha \in \mathbb{R}$, given by $R_\alpha(x) = x + \alpha \mod 1$. Let $f_\beta(x) := \chi[0,\beta)(x) - \beta$ where χ_I denots the indicator function of the interval I. Beck proved in [3, 4] that if α is a quadratic irrational and β is rational, then the pair (R_α, f_β) satisfies a temporal CLT along the orbit of $x_0 = 0$. More precisely, he shows that there exist constants C_1 and C_2 such that for all $a, b \in \mathbb{R}$, $a < b$,

$$\frac{1}{n}\#\left\{1 \le k \le n : \frac{S_n(R_\alpha, f_\beta, 0) - C_1 \log n}{C_2\sqrt{\log n}} \in [a,b]\right\} \to \frac{1}{\sqrt{2\pi}}\int_a^b e^{-\frac{x^2}{2}}\,dx.$$

In [2], a geometric proof of this result is given for $\beta = \frac{1}{2}$ and, using the same methods, a temporal CLT is proved in [9] for any initial point x.

In [5], we prove the following generalization of Beck's result. Let us write $\alpha \in BA$ if α is *badly approximable* (or *bounded type*), i.e. if there exists a constant $c > 0$ such that $|\alpha - p/q| \ge c/|q|$ for any $p, q \in \mathbb{Z}$, $q \ne 0$. For $\alpha \in (0,1) \setminus \mathbb{Q}$, let us say that β is *badly approximable with respect to* α and write $\beta \in BA(\alpha)$ if there exists a constant $c > 0$ such that $|q\alpha - \beta - p| > \frac{c}{|q|}$ for all $p, q \in \mathbb{Z}$, $q \ne 0$. One can show that given $\alpha \in BA$, the set $BA(\alpha)$ has full Hausdorff dimension.

Theorem 1 (Bromberg-U', [5])**.** *Let $\alpha \in BA$. For every $\beta \in BA(\alpha)$ and every $x \in \mathbb{T}$ there exists a sequence of centralizing constants $A_n := A_n(\alpha, \beta, x)$ and a sequence of normalizing constants $B_n := B_n(\alpha, \beta)$ such that for all $a < b$*

$$\frac{1}{n}\#\left\{1 \le k \le n : \frac{S_n(R_\alpha, f_\beta, x) - A_n}{B_n} \in [a,b]\right\} \to \frac{1}{\sqrt{2\pi}}\int_a^b e^{-\frac{x^2}{2}}\,dx.$$

Thus, for every $\alpha \in BA$ and $\beta \in BA(\alpha)$, the pair (R_α, f) satisfies the temporal CLT along the orbit of any $x \in \mathbb{T}$. Notice that quadratic irrationals belong to BA and, when $\alpha \in BA$, it follows from the definition that any *rational* number β is in $BA(\alpha)$. Thus, our result provides a strict generalization of the results of [3, 4, 9]. Dolgopyat and Sarig informed us that in ongoing work they are also able to prove a temporal CLT for the case in which α *is badly approximable* and β is rational, but their methods do not cover irrational values of β. Furthermore, they can show that the temporal CLT fails for a full Lebesgue measure set of α, thus the assumption $\alpha \in BA$ is crucial. It would be interesting to see whether a temporal CLT holds for a larger class of values of β. In particular, Dolgopyat asked us whether, for every $\alpha \in BA$, one can prove temporal limit theorems for a.e. $\beta \in (0, 1)$. We believe that our methods might allow us to answer this question.

The proof of our result exploits renormalization and a symbolic coding that allows to encode the dynamics via the formalism of adic and Vershik maps [13]. Two recent related results in the context of substitution systems (which correspond to the stationary special case of the Vershik formalism) were proved in [6] and [11]. The renormalization algorithm that we exploit is given by the classical continued fraction algorithm for rotations, with additional data which records the relative position of the break point β of the function f_β under renormalization. This algorithm produces what is known as *Ostrowski expansion* of β with respect to α in the context of non homogeneous Diophantine approximations (see also [1] for the relation with the geodesic flow on the space of affine lattices). Renormalization produces a sequence of Rohlin skyscrapers formed by three towers which represent the dynamics (the two towers of the classical continued fraction, one of which is cut into two by the relative position of β). By recording the sequence of towers to which a given point belongs, one gets a Vershik-type coding. Furthermore, by defining on the symbolic space a sequence of Markov measures, we get a non-homogeneous Markov chain determined by (α, β). In order to study temporal distributions of Birkhoff sums of f_β, we refine this coding further (by coding with respect to subtowers of a renormalization level inside the next) and define a sequence of functions $(\xi_n)_n$ of the Markov chain. The construction is made so that temporal distribution of the Birkhoff sums $S_k(R_\alpha, f_\beta, x)$ where x belongs to the base of a renormalization tower and k runs from 0 to $h-1$ where h is the corresponding tower height, are given by the distribution of the r.v. $\sum_{k=1}^{n} \xi_k$ where n is the level of the renormalization tower (see the key Proposition 2.6 in [5] for a precise statement). Thus, the temporal CLT is reduced to a CLT for non-homogeneous Markov chain (first proved by Dobrushin). We verify the assumptions of a general CLT for φ-mixing triangular arrays of r.v.s by Utev [12]. In particular, growth of the variance is deduced from the function f_β not being a coboundary, while we use the assumption that $\beta \in BA(\alpha)$ to show that the r.v. in the Markov chain are sufficiently independent. It is here that we hope to have room to improve in order to prove a CLT for a.e. β.

Finally, let us point out that the Vershik formalism and Markov chain reduction that we exploit in [5] is quite general and there are other entropy zero dynamical systems for which one can hope to prove temporal limit theorems using similar techniques. For example, the Rauzy-Veech algorithm provides a framework to encode linear flows on translation surfaces and interval exchange transformations through a Vershik adic coding (see [7]). In work in progress, we identify a class of cocycles for which one can prove temporal CLTs for linear flows on infinite translation surfaces and more in general for certain $\mathcal{S}-adic$ systems.

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Systolic inequalities in Reeb dynamics

Alberto Abbondandolo

(joint work with Barney Bramham, Umberto L. Hryniewicz, Pedro A. Salomão)

Classical systolic inequalities. We start by reviewing some of the classical results in systolic geometry, see [8] and reference therein. Given a closed n-dimensional Riemannian manifold (M, g), one defines the quantities

$$\rho(M,g) := \frac{(\text{minimal length of closed geodesic on } (M,g))^n}{\text{vol}(M,g)},$$

$$\rho_{\text{nc}}(M,g) := \frac{(\text{minimal length of non-contractible closed geodesic on } (M,g))^n}{\text{vol}(M,g)}.$$

The second of these numbers, which is defined only for non-simply connected manifolds, coincides with the length of a shortest non-contractible closed curve on

(M, g), as such a curve is necessarily a closed geodesic, and is called the *systolic ratio* of (M, g). Clearly, $\rho \leq \rho_{\rm nc}$.

Both the ratios ρ and $\rho_{\rm nc}$ can be made arbitrarily small by a suitable choice of g. Classical question in systolic geometry are: Are ρ and $\rho_{\rm nc}$ bounded from above on the space of Riemannian metrics on a given manifold M? If so, are there maximizing metrics and how do they look like?

The first results of this kind go back to the fourties, when Loewner and Pu proved that $\rho_{\rm nc}(\mathbb{T}^2, \cdot)$ is maximized by the flat metric given by the lattice in $\mathbb{R}^2$ which is generated by two sides of an equilateral triangle, while $\rho_{\rm nc}(\mathbb{RP}^2, \cdot)$ is maximized by the round metric. Starting from the seventies, Gromov studied these questions systematically and showed that when M is a surface other than S^2, the quantity $\rho_{\rm nc}(M, \cdot)$ has the non-sharp upper bound 2. In general, $\rho_{\rm nc}(M, \cdot)$ is not expected to achieve its maximum on the space of smooth metrics: Indeed, a simple perturbation argument shows that any metric maximizing $\rho_{\rm nc}(M, \cdot)$ would have non-contractible closed geodesics of minimal length through any of its points, and this is not possible for surfaces of high genus. In the early eighties, Gromov also proved a ground breaking result stating that $\rho_{\rm nc}(M, g) \leq C_n$ for all metrics g on an essential manifold M, where the number C_n depends only on the dimension n of M.

The first result about the simply connected case is due to Croke, who at the end of the eighties proved that $\rho(S^2, \cdot)$ is bounded from above. Contrary to what one may expect, the round metric does not maximize $\rho(S^2, \cdot)$. Indeed, the value of ρ at the round metric on S^2 is π, but Calabi and Croke exhibited a sequence of smooth metrics converging to a singular one with value of ρ converging to the number $2\sqrt{3}$, which is slightly larger than π. This value is currently believed to be the supremum of $\rho(S^2, \cdot)$, but this conjecture is open. Here it is interesting to notice that S^2 admits a large class of Zoll metrics, that is smooth metrics all of whose geodesics are closed and have the same length, and that these have all systolic ratio π, as proved by Weinstein. Babenko and Balacheff conjectured that the round metric is a local maximizer of $\rho(S^2, \cdot)$. The presence of a large set of Zoll metrics, which have the same systolic ratio of the round one, makes this question difficult to attack by classical symmetrization techniques. We will see that symplectic techniques allow us to give a positive answer to this conjecture.

Systolic ratio in Reeb dynamics. As Álvarez Paiva and Balacheff showed in [5], it is fruitful to extend the notion of systolic ratio to Reeb dynamics. We recall that a contact form on a $(2n-1)$-dimensional manifold W is a 1-form α such that $\alpha \wedge (d\alpha)^{n-1}$ is a volume form. The kernel of α is a hyperplane distribution on W and is called a contact structure. The non singular vector field R_α which is uniquely defined by the identities $\imath_{R_\alpha} d\alpha = 0$ and $\imath_{R_\alpha} \alpha = 1$ is called the Reeb vector field of α. The systolic ratio of a contact form α on a $(2n-1)$-dimensional closed manifold W can be defined as

$$\rho(W, \alpha) := \frac{(\text{minimal period of closed orbit of } R_\alpha)^n}{\operatorname{vol}(W, \alpha \wedge (d\alpha)^{n-1})}.$$

Here we are assuming that R_α does indeed have periodic orbits, which is believed to be always true (Weinstein conjecture) and is proved in many cases, including all contact forms on closed 3-manifolds.

The unit cotangent bundle S_g^*M of an n-dimensional closed Riemannian manifold (M, g) admits the contact form α_g which is obtained by restricting the canonical Liouville 1-form $p\,dq$ of T^*M to S_g^*M. The Reeb flow of α coincides with the geodesic flow and

$$\rho(S_g^*M, \alpha_g) = \frac{\rho(M, g)}{n\,\omega_n},$$

where ω_n denotes the volume of the unit euclidean n-ball. So the systolic ratio of contact form is a genuine generalization of the metric one. This identity, together with the fact that the contact volume $\mathrm{vol}(W, \alpha \wedge (d\alpha)^{n-1})$ is uniquely determined by the Reeb flow of α, shows that two metrics whose geodesic flows are smoothly conjugate have the same systolic ratio, which is then a dynamical invariant and not just a metric one. For example, all Zoll metrics on S^2 produce smoothly conjugated geodesic flows and have systolic ratio π.

Another important class of contact forms is given by the restriction of following 1-form on $\mathbb{R}^{2n}$

$$\lambda_0 := \frac{1}{2}\sum_{j=1}^{n}(x_j\,dy_j - y_j\,dx_j)$$

to the boundary ∂A of an open bounded domain whose boundary is smooth and transverse to the radial direction. In the particular case in which A is a $2n$-ball, one obtains a Zoll Reeb flow, that is a flow all of its orbits are closed and have the same period, having systolic ratio $\rho = 1$. A weak form of a deep open conjecture of Viterbo [10] states that

$$\rho(\partial C, \lambda_0|_{\partial C}) \leq 1$$

when the domain C is convex, with equality holding if and only if C is symplectomorphic to a ball. The best general result about this conjecture is due to Artstein-Avidan, Milman and Ostrover [7], who proved an upper bound which is independent of the dimension. Moreover, the Viterbo conjecture is known to imply the Mahler conjecture in convex geometry as a very particular case, see [6]. Our first result is the following:

Theorem 1. *If the contact form α on S^3 is C^3-close enough to a Zoll contact form then*

$$\rho(S^3, \alpha) \leq 1,$$

with equality if and only if α is Zoll.

See [2]. The proof uses a reduction to disk maps through global surfaces of section and a fixed point theorem relating the action of fixed points to the Calabi invariant. This results has been very recently generalized to arbitrary closed 3-manifolds by Benedetti and Kang. In higher dimension, the local maximality of Zoll contact forms is still open, but a weaker result in this direction has been proven in [5].

The above theorem has several interesting corollaries. One is that Zoll metrics on S^2 are C^2-local maximizers of the systolic ratio, giving a positive answer to the conjecture of Babenko and Balacheff mentioned above. This result remains true for Finlser metrics (but in the general non-reversible case one has to replace the C^2-topology by the C^3-one). See also [1] for more precise results in the Riemannian case.

Another consequence of Theorem 1 is that the Viterbo conjecture holds true for convex domains in $\mathbb{R}^4$ which are C^3-close to a symplectic ball. The characterization of the equality case uses the fact that if the Reeb flow on the boundary of a starshaped domain $A \subset \mathbb{R}^4$ is Zoll, then A is symplectomorphic to a ball.

Our second result says that the general results of Gromov about boundedness of the systolic ratio in metric geometry do not survive to the generalization to the contact setting:

Theorem 2. *Let ξ be a contact structure on a closed 3-manifold W. Then*

$$\sup\{\rho(W,\alpha) \mid \alpha \text{ contact form on } W \text{ with } \ker\alpha = \xi\} = +\infty.$$

In particular, one can find starshaped domains A in $\mathbb{R}^4$ with volume 1 and such that the all closed Reeb orbits of $\lambda_0|_{\partial A}$ have arbitrarily large period. This particular case is proved in [2], the general one in [3].

Our last result concerns dynamically convex contact forms on S^3. These are contact forms all of whose closed Reeb orbits have Conley-Zehnder index at least three. Hofer, Wysocki and Zehnder [9] showed that the restriction of λ_0 to the boundary of a smooth strictly convex domain in $\mathbb{R}^4$ is dynamically convex, and that many results holding in the convex case extend to the dynamically convex one. The next results shows that the Viterbo conjecture fails for dynamically convex contact forms:

Theorem 3. *For every $\epsilon > 0$ there is a dynamically convex contact form α on S^3 such that*

$$\rho(S^3,\alpha) > 2 - \epsilon.$$

This class of examples, together with more general ones, is constructed in [4]. Actually, at the time of writing there is no known example of dynamically convex contact form which does not come from a convex domain. The example which we construct in the above theorem is either the first example of such contact form or is a counterexample to the Viterbo conjecture. Unfortunately we do not know which of the two possibilities is true.

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A transversal point of view on weak K.A.M solutions

Marie-Claude Arnaud

We are interested in some weak solutions of the Hamilton-Jacobi equations for Hamiltonians that are

- either autonomous $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$;
- or 1-periodic time dependent $H : \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T} \to \mathbb{R}$;

and are uniformly superlinear and C^2-convex in the $\mathbb{R}^n$ direction. We also require that the flow is complete. Such Hamiltonians are said to be *Tonelli*.

A geodesic flow or a mechanical system is defined via such a Hamiltonian. Jürgen Moser (see [3]) proved that any smooth exact symplectic twist map of the 2-dimensional annulus $\mathbb{T} \times \mathbb{R}$ is the time 1 map of a Tonelli time dependent Hamiltonian.

For an unknown function $u_c : \mathbb{T}^n \to \mathbb{R}$ or $u_c : \mathbb{T}^n \times \mathbb{T} \to \mathbb{R}$ and a parameter $c \in \mathbb{R}^n$, the Hamilton-Jacobi equations are:

- stationary in the autonomous setting: $H(q, c + du_c(q)) = \alpha(c)$;
- evolutive in the time-dependent case: $\frac{\partial u_c}{\partial t}(q,t) + H(q, c + \frac{\partial u_c}{\partial q}(q,t), t) = \alpha(c)$.

We are interested in the c-dependence of the weak K.A.M. solutions (that are also called viscosity solutions) of these equations. Guided by what happens in the classical completely integrable case, we raise the following questions:

- can we choose $(q, c) \mapsto u(q, c)$ very regular?
- once we know u, can we say something on the dynamics?

1. The case of exact symplectic twist maps of the 2-dimensional annulus

What is in this section is a joint work with Maxime Zavidovique.

We fix the time $t = 0$ and use the notation $u(q, c) = u_c(q)$ instead of $u_c(q, 0)$. We denote by $\pi_1 : \mathbb{T} \times \mathbb{R} \to \mathbb{T}$ the projection.

Theorem 1. *Let f be a C^1 symplectic twist diffeomorphism of $\mathbb{T}\times\mathbb{R}$. Then there exists a continuous map $u:\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ such that*

- $u(0,c)=0$;
- *each $u_c=u(.,c)$ is a weak K.A.M. solution for the cohomology class c and then:*
- *each $u_c=u(.,c)$ is semi-concave ;*
- *each graph of $c+\frac{\partial u_c}{\partial\theta}$ is backward invariant by f.*

Remarks 1. *(1) The choice of solution we do is not the same than the discounted one given in* [2] *: we built an example for which the choice given in* [2] *is not continuous.*

(2) In our setting, it can even be proved that $c\mapsto \mathrm{graph}(c+du_c)$ is continuous for the Hausdorff distance.

Theorem 2. *With the notations of Theorem 1, we have equivalence of*

(1) f is C^0 integrable;
(2) the map u is C^1.

Moreover, in this case, u is unique and we have

- *the graph of $c+\frac{\partial u_c}{\partial\theta}$ is a leaf of the invariant foliation;*
- *$h_c:\theta\mapsto\theta+\frac{\partial u_c}{\partial c}(\theta)$ is a semi-conjugation between the projected Dynamics $g_c:\theta\mapsto\pi_1\circ f(\theta,c+\frac{\partial u_c}{\partial\theta}(\theta))$ and a rotation R of $\mathbb{T}$, i.e. $h_c\circ g_c=R\circ h_c$.*

Remarks 2. *(1) What is the most surprising in Theorem 2 is the fact that in the C^0 integrable case the semi-conjugation h_c continuously depends on c even at the c where the rotation number is rational. At a irrational rotation number, this is an easy consequence of the unicity of the invariant measure supported on the corresponding leaf. What happens for a rational rotation number is more subtle.*

(2) Observe that in the C^k-integrable case for $k\geq 1$, we can only claim that u and $\frac{\partial u}{\partial\theta}$ are C^k: so in the C^0 case, even the derivability with respect to c is surprising; this surely is related to the 2-dimensional setting in which we work.

An interesting question concerns the restricted Dynamics to the leaves in the C^0-integrable cases. A priori, such a Dynamics can be a Denjoy counter-example (but we have no example for such a phenomenon). With more regularity of the foliation, we obtain the following result.

Theorem 3. *With the notations of Theorem 2, we have equivalence of*

(1) f is Lipschitz integrable;
(2) the map u is C^1 with $\frac{\partial u}{\partial\theta}$ locally Lipschitz continuous and $\frac{\partial u}{\partial c}$ uniformly Lipschitz in the variable θ on any compact set of cs.

In this case, there exists $\Phi:\mathbb{T}\times\mathbb{R}\to\mathbb{T}\times\mathbb{R}$ symplectic homeomorphism that is C^1 in the θ variable such that:

$$\forall(x,c)\in\mathbb{T}\times\mathbb{R},\Phi\circ f\circ\Phi^{-1}(x,c)=(x+\rho(c),c);$$

where $\rho : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism.

Remarks 3. *(1) In this case, all the leaves are C^1 and the foliation is a C^1 lamination;*
(2) the last part of Theorem 3 provides some analogue of the Arnol'd-Liouville coordinates that exist for the completely integrable Hamiltonian systems.

2. The case of autonomous Tonelli Hamiltonians

What is in this section is a joint work with Jinxin Xue (see [1]).

Theorem 4. *Suppose that $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ is a Tonelli Hamiltonian that has a C^1 invariant foliation into C^1 Lagrangian graphs on an open subset $\mathcal{U} \subset \mathbb{T}^n \times \mathbb{R}^n$. Then there exists a neighborhood U of 0 in $\mathbb{R}^n$ and a symplectic homeomorphism $\phi : \mathbb{T}^n \times U \to \mathcal{U}$ that is C^1 in the direction of $\mathbb{T}^n$ such that*

$$\forall\ c \in U, \phi \circ \varphi_t^H \circ \phi^{-1}(x, c) = (x + t\rho(c), c)$$

where $\rho : U \to \mathbb{R}^n$ is a homeomorphism onto $\rho(U)$.

Observe that the conjugacy ϕ that we obtain has the same regularity as the foliation in the direction of the leaves but is just C^0 in the transverse direction. If we replace the C^1-integrability by a Lipschitz integrability, we loose any transverse regularity and we just obtain some results along the leaves.

Theorem 5. *Suppose that the Hamiltonian $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ is Tonelli and has a Lipschitz invariant foliation into Lipschitz Lagrangian graphs on $V \subset \mathbb{T}^n \times \mathbb{R}^n$. Then restricted to each leaf, the Hamiltonian flow has a unique well-defined rotation vector, and is bi-Lipschitz conjugate to a translation flow by the rotation vector on $\mathbb{T}^n$. Moreover, all the leaves are in fact C^1.*

Observe that we do not know if the conjugacies are C^1.

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Birkhoff normal forms for Hamiltonian partial differential equations

Walter Craig

The laws of physics are often expressed in terms of partial differential equations that can be formulated as Hamiltonian systems. This is the fundamental fact that motivates the efforts to describe the basic flow-invariant structures of their phase space $\mathcal{M}$, which is normally a Hilbert or Banach space encompassing the requisite infinite number of degrees of freedom. In this talk I will however address several issues that have to do with the initial question of existence of solutions, and their time of existence before singularity formation. We will study two equations, which we express through their respective Lagrangians; the nonlinear Schrödinger equation

$$\mathcal{L}_{NLS} = \int \mathrm{im}\big(\bar{u}\dot{u}\big) - \tfrac{1}{2}|\nabla u|^2 - F(|u|^2)\,dx\ , \qquad u : x \to \mathbb{C}\ , \tag{1}$$

with $F(|u|^2) \sim \mathcal{O}(|u|^m)$ and the quasilinear wave equations

$$\mathcal{L}_{QW} = \int \tfrac{1}{2}\dot{u})2 - G(\partial_x u, \dot{u})\,dx\ , \qquad (u, \dot{u}) : x \to \mathbb{R}^2\ , \tag{2}$$

with $G(v) \sim \mathcal{O}(|v|^m)$. The resulting Euler - Lagrange equations are the nonlinear Schrödinger equation

$$\partial_t u = -i\big(\tfrac{1}{2}\Delta u - F'(|u|^2)u\big) \tag{3}$$

and respectively the quasilinear wave equation

$$\partial_t^2 u - \Delta u + \sum_{j=1}^{d} \partial_{x_j}(\partial_{u_{x_j}} G) + \partial_t(\partial_{\dot{u}} G) = 0\ . \tag{4}$$

The key point of the talk today is that we will take $x \in \mathbb{R}^d$, $d \geq 1$. In this setting of a noncompact spatial domain, solutions of the linear equations tend to disperse. The existence of small amplitude nonlinear solutions is therefore a competition between this dispersion and the tendency for the nonlinearities to promote singularities. A classical theorem to this effect is the following, which follows from work of S. Klainerman and J. Shatah in the mid 1980s.

Theorem 1. [3][4] *Given initial data $u_0(x)$ for the nonlinear Schrödinger equation* (3) *in an appropriate Sobolev space $u \in B_r(0) \subseteq \mathcal{M}_{NLS} \subseteq L^2(\mathbb{R}^d)$, if*

$$\frac{d}{2}(m-2) > 1$$

and r is sufficiently small then the time of existence $T_r = +\infty$

Given initial data $(u_0(x), \dot{u}_0(x))$ for the quasilinear wave equation (4) *in an appropriate Sobolev space $u \in B_r(0) \subseteq \mathcal{M}_{QW} \subseteq (L^2(\mathbb{R}^d))^2$, if*

$$\frac{d-1}{2}(m-2) > 1$$

and r is sufficiently small then the time of existence $T_r = +\infty$

This sort of result is in contrast to the situation in which $x \in M^d$ a compact manifold (such as $M^d = \mathbb{T}^d$) for which solutions of the linear equations are recurrent and little or no decay is possible. In these settings the emphasis has been on KAM theory, which produces some quasiperiodic solutions for which $T = +\infty$, averaging theories such as Birkhoff normal forms which gives rise to lower bound estimates such as $T \geq r^N$, and Nekhoroshev stability, which gives lower bounds such as $T \geq e^{C/r^\beta}$. In a few cases, constructions of cascade orbits provide upper bounds on growth of action variables and Sobolev norms.

Given the role of the order m of the nonlinear term, it is a motivation to transform these partial differential equations in order to eliminate as much as possible the inessential components of the nonlinearity. Because of their nature as Hamiltonian systems, it is therefore natural to consider Birkhoff normal forms transformations. For this purpose denote a phase space point z in the *NLS* case by $z = u(x)$, and in the case of the wave equation set

$$z = \frac{1}{\sqrt{2}}\big(|D_x|^{1/2}u(x) + \frac{i}{|D_x|^{1/2}}p(x)\big) \ .$$

In both cases we denote the Hamiltonian function that stems from the Legendre transform of the respective Lagrangians by

$$H(z) = H^{(2)} + H^{(3)} + \cdots + H^{(N)} + R^{(N+1)}$$

where we have developed the Taylor expansion with remainder about the elliptic stationary point $z = 0$ up to order $N \geq 3$. We seek a canonical transformation $w = \tau(z)$ that will eliminate to the extent possible the inessential nonlinearities, leaving only the essential terms in the form

$$H_+(w) = H^{(2)} + (Z^{(3)} + \cdots + Z^{(N)}) + R_+^{(N+1)} \ .$$

In finite degree of freedom Hamiltonian systems, and as well from the case of compact domains $x \in M^d$, the obstructions to eliminating nonlinear terms are due to resonance, and the essential nonlinearities $Z^{(3)} + \cdots + Z^{(N)}$ are precisely the resonant terms. To identify these, the frequencies for the Schrödinger equation are given by the dispersion relation $\omega(k) = \frac{1}{2}|k|^2$ for $k \in \mathbb{R}^d$, and for the wave equation by $|k|$. A resonance relation for the term of order m is stated in terms of multiindices P, Q such that $|P| + |Q| = m$

$$\langle \omega | P - Q \rangle = \sum_{P,Q,|P|+|Q|=m} p_k\omega(k) - q_k\omega(k) = 0 \ .$$

However in the case of $x \in \mathbb{R}^d$ the spectrum is continuous, and the effect of a resonance may well be less relevant. This is the conclusion of the following theorem on normal forms for the *NLS*.

Theorem 2 (Craig, Selvitella & Wang 2013 [2])**.** *Consider the nonlinear Schrödinger equation* (3) *with* $F = \pm|u|^4 + \mathcal{O}(|u|^5)$ *defined on the Sobolev space* $\mathcal{M} := H^{s,s}(\mathbb{R}^d)$ *for* $s > d/2$.

(1) The $m = 4$ resonant set $\mathbb{Q}^{3d-1}$ is a $3d-1$ dimensional quadric surface, given by

$$\omega(k_1) + \omega(k_2) - \omega(k_3) - \omega(k_4) = \tfrac{1}{2}(|k_1|^2 + |k_2|^2 - |k_3|^2 - |k_4|^2) = 0 \ ,$$
$$k_1 + k_2 + k_3 + k_4 = 0 \ .$$

(2) Nonetheless for all $r > 0$ sufficiently small there is a canonical transformation $w = \tau(z)$ mapping the ball $B_r(0) \in \mathcal{M}$ into $B_2r(0)$ which eliminates all nonresonant and all resonant terms*, namely $Z^{(4)} = 0$.*

(3) The transformation $\tau = \tau^{(4)}$ is holomorphic.

In principal this normal forms transformation may be iterated for higher values of m. The solution map for equation (3) is holomorphic as well.

Quasilinear wave equations: Quasilinear nonlinearities are more serious. Let $m = 3$ and consider third order resonance conditions

$$\omega(k_1) - \omega(k_2) - \omega(k_3) = 0 \ , \qquad k_1 + k_2 + k_3 = 0 \ ,$$

which has to do with the nonlinear interaction of three principal Fourier modes.

Lemma 3. *All $m = 3$ resonances are co-linear, namely for $(k_1, k_2, k_3) \in \mathbb{R}^{3d}$ if*

$$|k_1| - |k_2| - |k_3| = 0 \ , \qquad k_1 + k_2 + k_3 = 0$$

then k_1 k_2 and k_3 are multiples of some basic $k_0 \in \mathbb{R}^d$.

Klainerman identified a class of nonlinear equations for which the quadratic nonlinear term, namely the term in the equation stemming from the cubic term of the Hamiltonian, has less effect on the time decay. These are termed null forms, and not surprisingly the null condition enters into the considerations of normal forms. Natural phase spaces for the quasilinear wave equation (4) are $\mathcal{M} = H^r(\mathbb{R}^d)$ for $r > (n+2)/2$ the usual Sobolev spaces, and alternatively the scale of invariant norm Sobolev spaces $Z^r(\mathbb{R}^d)$ that will be defined below.

Theorem 4 (Craig, French & Yang 2017 [1])**.** *The Birkhoff normal forms transformation $\tau^{(3)}$ eliminates the cubic terms of the Hamiltonian of a null form wave equation. However:*

(1) The transformation $\tau^{(3)}$ is not holomorphic, it is continuous on $B_\varepsilon(0) \subseteq \mathcal{M}$ but is not in general smooth.

(2) The auxiliary Hamiltonian $K^{(3)}$ that satisfies the cohomological equation

$$\{H^{(2)}, K^{(3)}\} = H^{(3)}$$

whose time-one flow map is designed to give the transformation $\tau^{(3)}$ does not give rise to a bounded vector field in general.

(3) The solution map $\psi_t(z)$ of the vector field $K^{(3)}$ is continuous but is not in general differentiable on $\mathcal{M}$.

(4) The solution map $\varphi_t(z)$ of the quasilinear wave equation is continuous but it also is not smooth on any reasonable phase spaces $\mathcal{M} \subseteq L^2(\mathbb{R}^d)$.

This issue of lack of smoothness of the solution map brings up the issue of what it should mean to be a flow when one is treating partial differential equations as dynamical systems on a phase space which is a Banach space.

We conclude with a definition of the invariant norm Sobolev spaces that are used in this transformation theory. Recall that a phase space point for the quasilinear wave equation, in complex symplectic coordinates is given by

$$z(x) = \frac{1}{\sqrt{2}}\big(|D_x|^{1/2}u(x) + \frac{i}{|D_x|^{1/2}}p(x)\big) \ .$$

The standard energy spaces based on Sobolev norms on $\mathbb{R}^d$ are given by

$$H^r(\mathbb{R}^d) = \{z(x) : \partial_x^\alpha z \in L^2(\mathbb{R}^d) \ , |\alpha| \le r\} \ ,$$

giving the scale of spaces $H^r \subseteq H^{r-1} \cdots \subseteq L^2$.

The invariant norm Sobolev spaces are similar, however based on a larger family of differential operators that are invariant under rotations and dilations as well as translations of $\mathbb{R}^d$. In addition to ∂_x^α these include the angular momentum operators

$$\Omega_{j\ell} = x_j\partial_{x_\ell} - x_\ell\partial_{x_j} \ , \qquad j,\ell = 1,\dots d$$

and the dilation operator

$$\Lambda = \sum_{k=1}^n x_k\partial_{x_k} \ .$$

The scale of invariant norm Sobolev spaces Z^r is defined as

$$Z^r := \{z \ : \ \Lambda^\beta\Omega^\alpha\partial_x^\sigma\sqrt{|D_x|}z \in L^2(\mathbb{R}^n_x) \ , \ |\alpha| + |\beta| + |\sigma| \le r\}$$

The Fourier transform is one of the symplectic transformations that is used in the analysis of the Hamiltonian vector fields $X^K(z)$. Under Fourier transform the angular momentum and dilation operators satisfy

$$\begin{aligned} &\Omega_{j\ell} = \Omega_{j\ell}(x) \mapsto k_j\partial_{k_\ell} - k_\ell\partial_{k_j} = \Omega_{jl}(k) \\ &\Lambda(x) \mapsto -\Lambda(k) - dI \ , \end{aligned}$$

that is, the Lie algebra of angular momentum and dilation operators is invariant under the Fourier transform. Using the above facts, the operators Ω and Λ obey the Leibnitz rule with respect to the integral operators

$$X^{K^{(3)}}(u,v) = \int_{\xi+\xi_1+\xi_2=0} k(\xi,\xi_1,\xi_2)u(\xi_1)v(\xi_2)\,d\xi_1$$

Namely, when operating on the vector field $X^{K^{(3)}}(u,v)$, considered as a bilinear form, then

$$\begin{aligned} \Omega X^{K^{(3)}}(u,v) &= X^{K^{(3)}}(\Omega u,v) + X^{K^{(3)}}(u,\Omega v) \\ \Lambda X^{K^{(3)}}(u,v) &= X^{K^{(3)}}(\Lambda u,v) + X^{K^{(3)}}(u,\Lambda v) \ . \end{aligned}$$

The scale of invariant norm Sobolev spaces is defined similarly to the usual scale of Soblev spaces, namely

$$\begin{aligned} Z^r &:= \{z(x) \,:\, \Lambda^\beta \Omega^\alpha \partial_x^\sigma \sqrt{|D_x|} z \in L^2(\mathbb{R}_x^d) \ , \ |\alpha| + |\beta| + |\sigma| \le r\} \\ &= \{z(x) \,:\, \Lambda^\beta \Omega^\alpha k^\sigma \sqrt{|k|} \hat{z}(k) \in L^2(\mathbb{R}_k^d) \ , \ |\alpha| + |\beta| + |\sigma| \le r\} \\ Z^r &\subseteq Z^{r-1} \subseteq \ldots Z^{r-s} \subseteq \ldots L^2(\mathbb{R}^d) \end{aligned}$$

Finally, we may state the definition of smoothness of the transformation $\tau^{(3)}$ on a scale of spaces, namely that, given $z(x) \in Z^r$ the first and higher Frechet derivatives of the mapping $\tau^{(3)}$ at the point $z(x)$ are continuous, but on larger spaces (of less smooth functions);

$$\begin{aligned} &\tau^{(3)} : B_\rho(0) \subseteq Z^r \to B_{2\rho}(0) \subseteq Z^r \\ &\partial_z \tau^{(3)}(z) - id : Z^{r-1} \to Z^{r-1} \\ &\ldots \\ &\partial_z^\beta \tau^{(3)}(z) : Z^{r-|\beta|} \to Z^{r-|\beta|} \ . \end{aligned}$$

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Holomorphic curves and the three-body problem

Otto van Koert

In this talk we described applications of holomorphic curve techniques due to Hofer, Wysocki and Zehnder, [5, 6, 7, 8, 9], and further developed by Hryniewicz and Salomão, [10, 11] to classical Hamiltonian dynamical systems. These techniques construct foliations of contact 3-manifolds, with some periodic Reeb orbits removed, that are transverse to the Reeb flow. For example, in this way Hofer, Wysocki and Zehnder, [6], proved that a hypersurface bounding a compact, strictly convex set in $\mathbb{R}^4$ always admits a global disk-like surface of section.

In the first part of the talk we explained this background, and described how convexity estimates give an effective way to construct global surfaces of section for several Stark-Zeeman systems. These are magnetic Hamiltonians of the form

$$H = \frac{1}{2}\|p + A(q)\|^2 + V_0(q) + V_1(q),$$

where $V_0(q) = -\frac{1}{|q|}$. These Hamiltonians include many systems from celestial mechanics as well as other well-studied problems, and have many features in common. After regularizing the singularity with the Levi-Civita scheme, we obtained explicit results on the existence of global surfaces of section for the diamagnetic Kepler problem, a well-studied model for chaos in physics, and the planar circular restricted three-body problem, [1, 2, 4]. These results include cases that cannot be proved by perturbative methods.

The construction using holomorphic curves can also be implemented numerically. When done naively, this unfortunately leads to a rather inefficient algorithm. However, the proof by the holomorphic curve techniques provides a good hint on how the global surface of section should behave. In particular, a shooting argument in classical work by Birkhoff, [3], gives a clue on what the binding orbits of the surfaces of section should be. This allows us to come up with an alternative algorithm. We described these numerical results in the second half of our talk. These results illustrate the rich dynamics in these problems. The figure below illustrates one of them, namely return maps for an annulus-type surface of section in the RTBP.

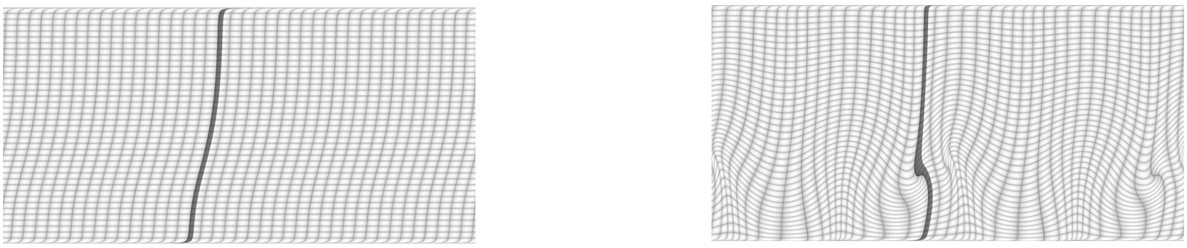

Figure 1. Annulus return maps in RTBP for $\mu = 0$ and $\mu = 0.5325$: the former satisfies the twist condition, the latter does not (at least not in these coordinates)

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Arnold's J^+-invariant and periodic orbits in the restricted three body problem

Urs Frauenfelder

(joint work with Kai Cieliebak, Otto van Koert)

By the Theorem of Whitney-Graustein the rotation number or Whitney index is a complete invariant for homotopy classes of immersed curves in the plane. In a generic homotopy of immersed curves three disasters might occur, triple intersections and direct and inverse self-tangencies. Arnold's J^+-invariant does not change under triple intersections and inverse self-tangencies but is sensible to direct self-tangencies. This is of interest if one considers periodic orbits under a force law. Because the force is proportional to acceleration such an orbit obeys a second order ODE and therefore direct self-tangencies cannot occur.

One can show that Arnold's J^+-invariant does not change under addition of exterior loops, although these affect the Whitney index. In the talk I explain the relation of this fact to periodic orbits in the restricted three body problem. The description of the orbit of the moon using Newton's law of gravitation was for a long time a challenging problem. Indeed, the heavy sun cannot be treated as a small perturbation of the problem. Hill considered the problem in rotating coordinates where the sun and earth both are at rest. The price one has to pay for this transformation is that the moon in rotating coordinates is now subject to four forces, the gravitational force of the sun, the gravitational force of the earth, the Coriolis force and the centrifugal force. This is the content of the restricted three body problem. Hill considered a family of periodic orbits for this problem indexed by the period. This periodic orbit is a model for the orbit of a moon and the period corresponds to a month which is referred to as the "lunarity". Hill observed that while for low lunarity the orbit is immersed for high lunarity at some moments the velocity vanishes where the orbit gets a cusp. He thought that this is the end of the family and referred to this orbit as "the moon of maximal lunarity". However, Adams and Poincaré convinced him that the family can be prolonged further, while the cusps transform to exterior loops. In particular, the Whitney index does not stay constant along this family, but the J^+-invariant does.

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Dynamics near focus-focus singular fibers in semitoric systems

Sonja Hohloch

A *semitoric system* on a connected 4-dimensional symplectic manifold (M, ω) is a completely integrable system $\Phi = (J, H) : (M, \omega) \to \mathbb{R}^2$ such that

- J is proper.
- J induces an effective Hamiltonian $\mathbb{S}^1$-action.
- Φ has only nondegenerate singularities.
- The singularities do not have hyperbolic components.

This narrows the type of critical points down to focus-focus, elliptic-elliptic, and elliptic-regular. The fibers of Φ are either regular (and therefore 2-tori due to the Arnold-Liouville theorem) or singular, meaning they contain singular points. In the latter case they are either 'pinched tori' having a focus-focus point at each pinch or 'circles' consisting of elliptic-regular points or elliptic-elliptic fixed points as sketched in Figure 1. A semitoric system is *simple* if a singular fiber contains at most one focus-focus singular point.

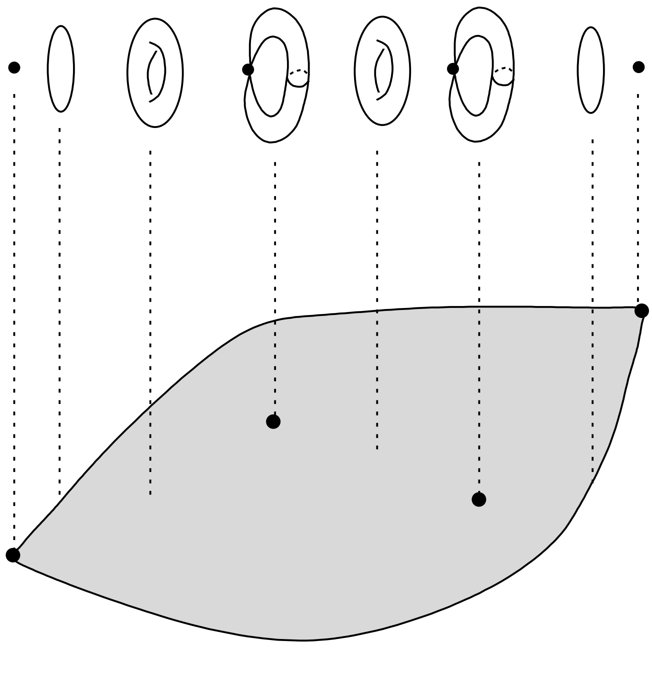

Figure 1. Fibers of a semitoric system with two focus-focus and two elliptic-elliptic critical points.

Simple semitoric systems have been classified by Pelayo & Vũ Ngọc [3, 4] by means of the following five invariants:

1) The number of focus-focus singularities.
2) The *Taylor series invariant* given by the Taylor series expansion of a certain generating function at each focus-focus singular fiber.
3) An equivalence class of *generalized polygons*.
4) The *height invariant* given by the heights of the focus-focus values in the generalized polygons.
5) The *twisting-index invariant* given by integers measuring the 'relative twistedness' near focus-focus singularities.

Two of these invariants, namely the Taylor series invariant and the twisting index, are still somewhat 'mysterious' due to lack of intuition and/or computed examples. We hope to amend this by

(1) Identifying the Taylor series invariant somehow with the Birkhoff normal form and computing it for the coupled angular momenta and the coupled spin oscillators. This is an ongoing project with J. Alonso Fernández and H. Dullin.
(2) Presenting a new compact semitoric system with two distinct focus-focus singular fibers [2]. We suspect that the twisting index is in fact some kind of Dehn twist. This is an ongoing project with J. Palmer.

Moreover, there exists now the new class of *vertical almost-toric systems* — a generalization of semitoric systems 'compatible' with taking subsystems, but enjoying still many pleasant properties of semitoric systems. This is a joint work with S. Sabatini, D. Sepe, and M. Symington [1]. Vertical almost-toric systems are in particular suited for surgeries which is the topic of an ongoing project with the same collaborators.

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Vortex reconnection in the three-dimensional Navier-Stokes equations

Daniel Peralta-Salas
(joint work with Alberto Enciso, Renato Lucà)

A fundamental feature of inviscid incompressible fluids is that *the vorticity is transported along the fluid flow.* More precisely, if $u(x,t)$ is the velocity field of a fluid and $P(x,t)$ is its pressure, they satisfy the *3D Euler equations*:

$$\partial_t u + \nabla_u u = -\nabla P\,, \qquad \operatorname{div} u = 0\,, \qquad u(\cdot,0) = u_0\,,$$

where ∇_u stands for the covariant derivative along u. Then a simple computation shows that the *vorticity* $\omega := \operatorname{curl} u$ evolves according to the transport equation

$$\partial_t \omega = [\omega, u]\,,$$

where $[\cdot,\cdot]$ denotes the Lie bracket of vector fields. This equation ensures that the vorticity at time t can be written in terms of the initial vorticity ω_0 as

$$\omega(\cdot,t) = (\phi_{t,0})_* \, \omega_0\,,$$

that is, as the push-forward of the initial vorticity along the non-autonomous flow $\phi_{t,0}$ generated by the velocity field u. It then follows that, as long as the solution of the Euler equations does not blow up, there are *no changes in the topology of the vortex structures of the fluid*, such as vortex tubes or closed vortex lines.

Definition 1: A closed vortex line (resp. a vortex tube) at time t is defined as a periodic integral curve (resp. a smooth invariant two-dimensional torus) of the vorticity $\omega(\cdot,t)$ frozen at time t. The set of vortex structures at time t is defined as the set of closed vortex lines and vortex tubes of $\omega(\cdot,t)$.

In presence of viscosity, the vorticity is no longer transported along the flow because the diffusion gives rise to a phenomenon known as *vortex reconnection*. Viscous incompressible fluids are described by the *3D Navier–Stokes equations*,

$$\partial_t u + \nabla_u u - \nu\Delta u = -\nabla P\,, \qquad \operatorname{div} u = 0\,, \qquad u(\cdot,0) = u_0\,.$$

where ν is a positive constant denoting the viscosity. We will study solutions to these equations in the torus $\mathbb{T}^3$ with $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$.

Definition 2: A vortex reconnection has occurred at time T if the sets of vortex structures at time T and at time 0 are not diffeomorphic, so there has been a change of topology [5, 1]. For example, a certain vortex tube can break and there can appear vortex tubes or vortex lines that are knotted or linked in a different way as the initial vorticity.

There is overwhelming numerical and physical evidence for vortex reconnection. Particularly relevant for our purposes are the recent experimental results presented in [6, 7], where the authors study how vortex lines and tubes of different knotted topologies reconnect in actual fluids using cleverly designed hydrofoils. Nevertheless, a mathematically rigorous scenario of vortex reconnection has never been constructed so far.

In [4] we filled this gap by providing a rigorous mechanism of vortex reconnection in viscous incompressible fluids. Our proof is indirect, meaning that we prove that there has been a change in the topology of the vortex lines and tubes of the fluid, and even control the initial and final topologies, but we cannot describe the way in which the vortex tubes or lines break. In particular, we do not know if this happens essentially as in the well-known bifurcation model of parallel or anti-parallel reconnection [5, 7].

We will next state a result proved in [4] on rigorous vortex reconnection. In this theorem we will construct *a finite cascade of reconnections* at any sequence of times (fixed a priori)

$$T_1 < T_2 < \cdots < T_n \,,$$

meaning that there is a smooth solution to the Navier–Stokes equations, which one can even assume to be global, such that it has a subset of vortex structures at time T_k (for each odd integer k) that does not have the same topology as any of the subsets of the vortex structures present at the times T_{k-1} or T_{k+1}. Furthermore, the scenario of reconnection that we present is *structurally stable*, which roughly speaking means that the phenomenon occurs for any initial datum that is close enough in $C^{4,\alpha}(\mathbb{T}^3)$ to the initial velocity discussed in the theorem (cf. [4] for a precise definition).

Theorem: Given any constants $0 =: T_0 < T_1 < \cdots < T_n$ and $M > 0$, for each odd integer k in $[1, n]$ let us denote by $\mathcal{S}_k$ any finite collection of closed curves and embedded tori (pairwise disjoint but possibly knotted and linked) that are contained in the unit ball of $\mathbb{T}^3$. Then there is a global smooth solution $u : \mathbb{T}^3 \times [0, \infty) \to \mathbb{R}^3$ of the Navier–Stokes equations, with an initial datum of norm $\|u_0\|_{L^2} = M$ and of zero mean, which, for each odd integer $k \in [1, n]$, exhibits at time T_k a subset of vortex structures diffeomorphic to $\mathcal{S}_k$ that is not diffeomorphic to any subset of the vortex structures of the fluid at time T_{k-1} or T_{k+1}. This scenario of vortex reconnection is structurally stable.

Let us give some heuristic ideas about the proof of this result. The first key observation is that, in order to prove it, the real enemy is not only the fact that the Navier–Stokes equations are notoriously difficult to analyze, but rather the need to prove that a subset of vortex structures originating at time T is not diffeomorphic to any subset of the vortex structures at time 0.

The proof hinges on choosing an initial datum that is the sum of several smooth but *highly oscillatory Beltrami fields* $\mathcal{W}_k$, that is

$$u_0 = M\, \mathcal{W}_0 + \delta_1\, \mathcal{W}_1 + \cdots + \delta_n\, \mathcal{W}_n \,,$$

where δ_k are small enough constants. We recall that a vector field $\mathcal{W}$ in $\mathbb{T}^3$ is a Beltrami field if it satisfies the equation $\operatorname{curl} \mathcal{W} = N \mathcal{W}$ for some eigenvalue N (the frequency of the field) of the curl operator.

The argument involves an interplay between the (very large) frequencies N_k of the fields and their relative sizes that ensures that, at time T_k, the vortex structures of the fluid are somehow related to those of $\mathcal{W}_k$ in the sense that:

$$c_k u(\cdot, T_k) = \mathcal{W}_k + \text{small}$$

for some constant c_k. Key to make this argument work is to find two families of Beltrami fields with arbitrarily large frequencies and such that in the first family one can find a subset of vortex structures diffeomorphic to $\mathcal{S}_k$ (so this family is used to construct $\mathcal{W}_k$ when k is odd) whereas in the second family all the vortex structures are non-contractible.

The Beltrami fields $\mathcal{W}_k$ for k odd are constructed using the tools introduced in [2, 3], while the Beltrami fields $\mathcal{W}_k$ for k even are explicit and read in terms of the Cartesian coordinates (x_1, x_2, x_3) as:

$$\mathcal{W}_k = (2\pi)^{-3/2} (\sin N_k x_3, \cos N_k x_3, 0) .$$

Since the integral curves of these fields lie on the tori $x_3 = \text{const}$ it follows that they are all non-contractible.

An essential property of these families is that they are *"robustly non-equivalent"*, meaning that any (uniformly) small perturbation of a member of the first family is not topologically conjugate to a small perturbation of any member of the second family, and viceversa. This is proved using suitable estimates for Beltrami fields with sharp dependence on the frequency and the KAM theory. It is worth mentioning that the frequencies we need to consider in the proof of the theorem are much larger than $\nu^{-1/2}$, which explains why there is no hope of promoting this scenario of vortex reconnection to the vanishing viscosity limit.

Finally, the global existence of the solutions follows from a suitable *stability theorem for the Navier–Stokes equation* and the fact that our initial data are small perturbations of Beltrami fields.

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Rigidity of conjugacy classes in group of area-preserving homeomorphisms

Sobhan Seyfaddini

(joint work with Frédéric Le Roux, Claude Viterbo)

Let (Σ, ω) be a closed surface equipped with an area form ω and denote by $\overline{Ham}(\Sigma, \omega)$ the C^0 closure of Hamiltonian diffeomorphisms of (Σ, ω). This is often referred to as the group of Hamiltonian homeomorphisms of (Σ, ω). Very little is understood about the algebraic structure of this group. For example, it is not known whether it is simple or not and in the case where $\Sigma = S^2$ we do not know if it admits any homogeneous quasimorphisms. Investigating questions of this nature, F. Béguin, S. Crovisier, and F. Le Roux were lead to the following question:

Does there exist $\theta \in \overline{Ham}(\Sigma, \omega)$ whose conjugacy class is dense in $\overline{Ham}(\Sigma, \omega)$?

The answer to the above question turns out to be negative. In the case of surfaces with non-zero genus this is a consequence of the works of Entov-Polterovich-Py [1] and Gambaudo-Ghys [2]. In the case of S^2, the negative answer was provided by the author in [3]. In each of the above cases the question was answered by constructing continuous conjugacy invariants. One can not associate a dynamical interpretation to these invariants, particularly in the case of S^2, where Hamiltonian Floer theory is used.

In an ongoing joint project with F. Le Roux and C. Viterbo, we show that one can separate closures of conjugacy classes in $\overline{Ham}(\Sigma, \omega)$ by simply counting fixed points with appropriate multiplicities. Here is a more precise statement: Given a Hamiltonian homeomorphism f denote by $\mathrm{Conj}(f)$ its conjgucay class and by $\overline{\mathrm{Conj}}(f)$ its closure. We say f is almost conjugate to g if there exists $h_1, \dots, h_N \in \overline{Ham}(\Sigma, \omega)$ such that $f = h_1, g = h_N$ and $\overline{\mathrm{Conj}}(h_i) \cap \overline{\mathrm{Conj}}(h_{i+1}) \neq \emptyset$. Almost conjugacy is an equivalence relation. Clearly, existence of a dense conjugacy class would imply the triviality of this relation.

Theorem 1. *Let (Σ, ω) denote a closed symplectic surface other than the sphere and suppose that $f, g \in Ham(\Sigma, \omega)$ have finitely many fixed points. If f is almost conjugate to g, then*

$$\sum_{x \in Fix_c(f)} |L(f, x)| = \sum_{x \in Fix_c(g)} |L(g, x)|.$$

We expect that our methods can be modified to prove the above statement in the case $\Sigma = S^2$.

One of the main ingredients of the proof of the above theorem is the theory of barcodes which has recently surfaced in symplectic topology through the works of Polterovich and Shelukhin. In fact, for us the above theorem has served as a good motivation for developing the theory of barcodes for Hamiltonian homeomorphisms of surfaces.

It would of course be very interesting to extend Theorem 1 to the setting where f, g are non-smooth. We have made some progress in this direction which I will outline here: Consider $f \in \overline{Ham}(\Sigma, \omega)$ which has finitely many fixed points. We say that f is *smoothable* if there exists some $g \in Ham(M, \omega)$ which is arbitrarily C^0-close to f and such that $Fix_c(g) = Fix_c(f)$.

Theorem 2 (Le Roux, S., Viterbo)**.** *Theorem 1 is true for smoothable homeomorphisms.*

We can show that many homeomorphisms are smoothable and in fact we conjecture that all homeomorphisms are smoothable. Our main tool for establishing smoothability is Le Calvez's theory of transverse foliations.

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Shape of minimal sets in aperiodic flows

Krystyna Kuperberg

We revisit the aperiodic plugs constructed in 1993 by the speaker to solve the C^∞ Seifert Conjecture [7] in dimension three. All of the known counterexamples to the Seifert Conjecture are plug constructions. A plug was first constructed by F. W. Wilson [12] in 1966, who answered the C^∞ Seifert Conjecture for spheres S^5, S^7, ... A 3-dimensional Wilson-type plug, with two circular orbits, is the starting point for the examples in [7] and [8].

A C^1 aperiodic flow on S^3 was given by Paul A. Schweitzer in 1974 and improved to $C^{2+\delta}$ by Jenny Harrison in 1988. In 1996, Greg Kuperberg, modified Schweitzer's example to be volume-preserving. In 2003, Viktor L. Ginzburg and Başak Z. Gürel, gave a C^2 counterexample to the Hamiltonian version of the Seifert Conjecture in dimension three. These examples have minimal sets of codimension two and cannot be improved to C^3 within the methods used.

In a private communication, William P. Thurston [11] remarked that the method presented in [7] yields real analytic examples as well, and if the example is at least C^1, the flow is not measure preserving. In a joint paper [8], Greg Kuperberg and the author, solve the Modified Seifert Conjecture in the C^ω and PL categories. The respective aperiodic dynamical system posseses only one minimal set, which is of topological (covering) dimension two.

The flows (C^∞ in [7], C^ω and PL in [8]) are constructed by making an aperiodic plug using self-insertions in a Wilson-type plug [12], see Fig. 1 and Fig. 2.[1]

The resulting plug contains a huge minimal set, usually of topological dimension two, see [3] and [8]. PL examples of the self-inserted plug, with the minimal set of dimension one as well as two, are given in [8]. The topological and algebraic properties of the minimal set depend on the formulas of the self-insertions. We are interested in algebraic properties in the category of the Shape Theory developed by Karol Borsuk, Vietoris-Čech Homology, and Étale Homotopy.

An object in the shape theory is a pair (X, M) consisting of a compact metric space X and an absolute retract M in which X is embedded. Thus the shape theory

[1] Figures made by W. Kuperberg in 1993

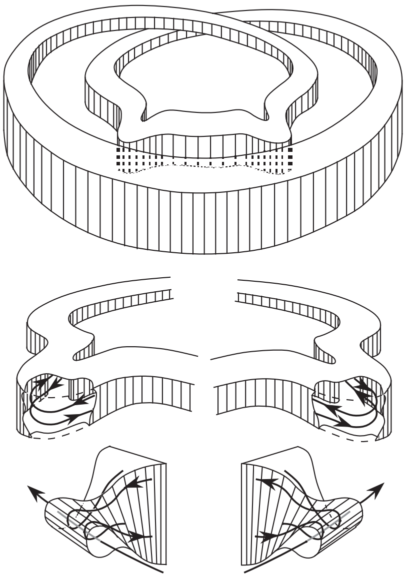

FIGURE 1. Self-insertion preparation

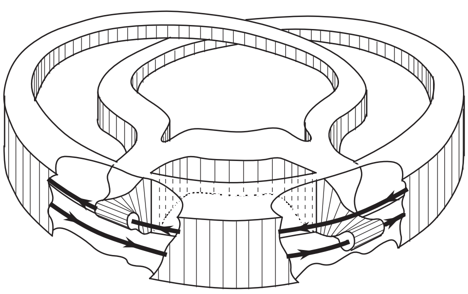

FIGURE 2. Self-inserted aperiodic plug

approach is particularly convenient for flows defined on manifolds as invariant compact sets are often naturally embedded in an absolute retract, a neighborhood homeomorphic to $\mathbb{R}^n$. The shape, movability or UV-properties, and Mittag-Leffler conditions of the minimal set are studied. A tremendous progress in studying the shape properties of the minimal sets described in [7] has been made by Steven Hurder and Ana Rechtman in [5].

Although shape invariants do not depend on the choice of the absolute retract M, embedding the objects with desired properties in a flow depends on M and its dimension, see [10].

The following question remains outstanding: Let ϕ be a flow on $\mathbb{R}^3$ with a compact set A invariant. Does every neighborhood of A contain a movable compact invariant set containing A?

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Two or infinitely many Reeb orbits

Daniel Pomerleano

(joint work with Dan Cristofaro-Gardiner, Michael Hutchings)

The three-dimensional case of the Weinstein conjecture asserts that every contact form on a closed three-manifold has at least one Reeb orbit. This was proved by Taubes in 2007 [Tau1] using Seiberg-Witten theory. This result naturally leads to the following question:

Question 1. *What can one say about the number of simple Reeb orbits of a contact form on a closed three-manifold?*

Without any further assumptions on the contact manifold Y or the contact form λ, a definitive result in this direction was obtained by Cristofaro-Gardiner and Hutchings [CGHut], who showed that there are at least two simple Reeb orbits. The lower bound of two is the best possible without further assumptions, because there exist contact forms on S^3 with exactly two simple Reeb orbits. One can also take quotients of these examples by cyclic group actions to obtain contact forms on lens spaces with exactly two Reeb orbits. In order to obtain stronger results, it is standard to assume that λ is nondegenerate, thereby allowing one to make direct use of the powerful theory of pseudoholomorphic curves in the symplectization $\mathbb{R} \times Y$. An important result in this direction is:

Theorem 2. *(Hofer-Wyoscki-Zehnder* [HWZ2, Cor. 1.10]*) Let λ be a nondegenerate contact form on S^3. Assume that*

(a) $\xi = \mathrm{Ker}(\lambda)$ *is the standard contact structure on S^3.*
(b) The stable and unstable manifolds of all hyperbolic Reeb orbits of λ intersect transversely.

Then λ has either two or infinitely many simple Reeb orbits.

In this talk, we will explain the following generalization of Theorem 2:

Theorem 3. [CGHutPom] *Let Y be a closed connected three-manifold and let λ be a nondegenerate contact form on Y. Assume that $c_1(\xi)$ is torsion in $H^2(Y;\mathbb{Z})$. Then there are either two or infinitely many simple Reeb orbits.*

By combining this with Theorem 1.5 of [HutTau], we obtain:

Corollary 4. *Let Y be a closed connected three-manifold which is not S^3 or a lens space. Then every nondegenerate contact form λ on Y such that $c_1(\xi) \in H^2(Y;\mathbb{Z})$ is torsion has infinitely many simple Reeb orbits.*

The proof of Theorem 3 makes use of embedded contact homology (ECH), a three manifold invariant which is defined using pseudoholomorphic curve theory. While $\mathrm{ECH}(Y, \lambda)$ is defined in terms of symplectic geometry, it is known to be canonically isomorphic to the Seiberg-Witten Floer cohomology of Y by work of Taubes [Tau2, Tau3]. This allows one to import results from gauge theory into symplectic geometry. In particular, using this isomorphism, one is able to produce a certain infinite sequence of Fredholm index two curves in $\mathbb{R} \times Y$. The main step of the proof is to show that, if λ has only finitely many simple orbits and $c_1(\xi)$ is torsion, then at least one of these curves projects to a genus zero global surface of section of the Reeb flow on Y. Following ideas of [HWZ1], the existence of a genus zero global surface of section enables one to transfer dynamical questions about the Reeb flow on Y to questions about periodic points of homeomorphisms of surfaces. From here, Theorem 3 follows from known results.

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$3-2-3$ foliations and Hamiltonian dynamics near critical energy levels

Pedro A. S. Salomão

(joint work with N. de Paulo, U. Hryniewicz)

We study the existence of $3-2-3$ foliations for tight Reeb flows on connected sums $\mathbb{R}P^3\#\mathbb{R}P^3$. These are rational singular foliations of $\mathbb{R}P^3\#\mathbb{R}P^3$ whose singular set is formed by three binding orbits, two of them P_3 and P_3' are non-hyperbolic and 2-unknotted, and P_2 is hyperbolic and unknotted. The regular leaves are transverse to the Reeb vector field and consist of two rigid planes asymptotic to P_2 which separate P_3 and P_3', two rigid cylinders connecting the double covers of P_3 and P_3' to P_2 and two one-parameter families of planes asymptotic to the double covers of P_3 and P_3'. See the Figure 1.

Reeb flows admitting such $3-2-3$ foliations arise naturally when studying certain Hamiltonian flows near critical energy levels. Suppose that the critical level $H^{-1}(0)$ of a Hamiltonian function H has a critical point p_c which is a saddle-center equilibrium point for the Hamiltonian flow (this is a Morse index 1 critical point of H). Suppose also that for energies $E<0$ small, the energy level $H^{-1}(E)$ has two components diffeomorphic to $\mathbb{R}P^3$ which converge, as $E\to 0^-$, to critical subsets $S_0, S_0' \subset H^{-1}(0)$ with $S_0\cap S_0' = \{p_c\}$ as a unique common singularity (both S_0 and S_0' are homeomorphic to $\mathbb{R}P^3$). See Figure 2. We show that if the flow on $S_0\cup S_0'\setminus\{p_c\}$ is dynamically convex then for all energies $E>0$ sufficiently small the energy level $H^{-1}(E)$ contains a component W_E diffeomorphic to the connected sum $\mathbb{R}P^3\#\mathbb{R}P^3$ which admits a $3-2-3$ foliation. These foliations are obtained as projections of finite energy foliations in the symplectization of W_E.

We also discuss the existence of $3-2-3$ foliations in the Euler's problem of two fixed centers on the plane, after a suitable regularization.

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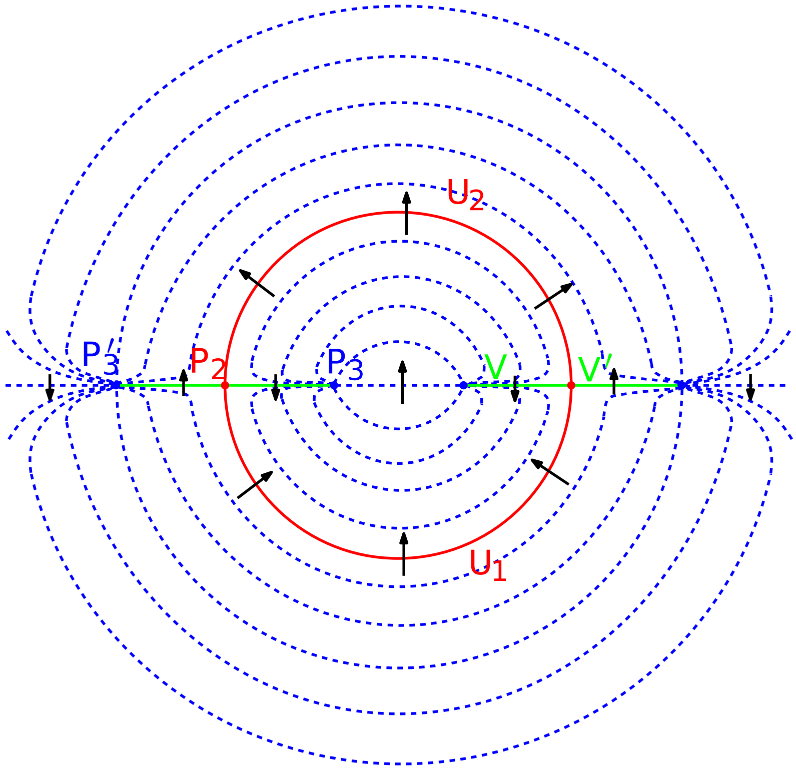

FIGURE 1. This is a slice of a $3-2-3$ foliation. The four dark dots represent the two binding orbits P_3 and P_3'. The other two dots represent the hyperbolic orbit P_2. The one-parameter family of planes asymptotic to the double covers of P_3 and P_3' are represented by dashed curves. The rigid planes U_1 and U_2 and the rigid cylinders V and V' are represented by bold curves.

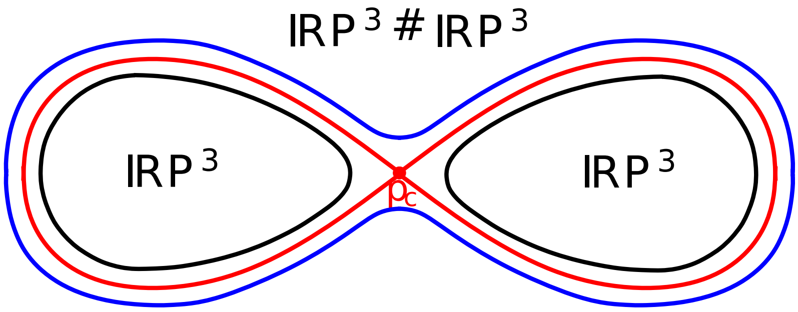

FIGURE 2. If the critical subset $S_0 \cup S_0'$ is dynamically convex then the component W_E diffeomorphic to $\mathbb{R}P^3\#\mathbb{R}P^3$ admits a $3-2-3$ a foliation for small positive energies.

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Construction of diffusing orbits in Hamiltonian systems

Marian Gidea
(joint work with Maciej Capiński, Rafael de la Llave, Jean-Pierre Marco, Tere Seara)

1. Main results

This report is concerned with the *Arnold diffusion problem* for nearly integrable Hamiltonian systems, both the *a priori unstable* case, and the *a priori stable* case.

Geometrically, in the *a priori unstable case* the dynamics is organized by a normally hyperbolic invariant cylinder along a single, simple resonance, while in the *a priori stable* case, the dynamics is organized by a network of normally hyperbolic invariant cylinders along multiple, simple or higher order resonances. In either case, one can distinguish two dynamics: an *inner dynamics*, defined by the restriction of the Hamiltonian flow to the cylinders, and an *outer dynamics*, defined by excursions along homoclinic/heteroclinic trajectories to the cylinders. The outer dynamics can be described via certain geometrically defined *scattering maps*.

We provide a mechanism of diffusion based on algorithmic constructions of diffusing pseudo-orbits – consisting of orbit segments of the inner dynamics interspersed with orbit segments of the outer dynamics –, and on a very general shadowing lemma for normally hyperbolic invariant manifolds. The novelty of our method is that it require very little knowledge on the inner dynamics.

1.1. **Shadowing Lemma.** Assume that $f : \mathscr{M} \to \mathscr{M}$ is a sufficiently smooth map on a compact manifold, and $\mathscr{C} \subseteq \mathscr{M}$ is a compact, normally hyperbolic invariant cylinder with $W^u(\mathscr{C})$ and $W^s(\mathscr{C})$ intersecting transversally. If the homoclinic intersection satisfies some strong transversality conditions, there exists a well defined scattering map $\sigma : \mathscr{U} \to \mathscr{C}$, where $\mathscr{U} \subseteq \mathscr{C}$ is an open set.

Theorem 1 (Shadowing Lemma)**.** *For every $\delta > 0$ there exists $n^* \in \mathbb{N}$, and a family of functions $m_i^* : \mathbb{N}^{2i+1} \to \mathbb{N}$, $i \geq 0$, such that, for every pseudo-orbit $\{y_i\}_{i\geq 0}$ in $\mathscr{C}$ of the form*

$$y_{i+1} = f^{m_i} \circ \sigma \circ f^{n_i}(y_i),\ i \geq 0, \tag{1}$$

with $n_i \geq n^*$ and $m_i \geq m_i^*(n_0, \ldots, n_{i-1}, n_i, m_0, \ldots, m_{i-1})$, there exists an orbit $\{z_i\}_{i\geq 0}$ of f in $\mathscr{M}$ such that, for all $i \geq 0$,

$$z_{i+1} = f^{m_i+n_i}(z_i),\ \text{and } d(z_i, y_i) < \delta.$$

1.2. **Diffusion in the a priori unstable case.** Consider the Hamiltonian system of the form

$$H_\varepsilon(p, q, I, \phi, t) \;=\; h_0(I) + \sum_{i=1}^{n} \pm \left(\frac{1}{2}p_i^2 + V_i(q_i)\right) + \varepsilon H_1(p, q, I, \phi, t; \varepsilon). \tag{2}$$

where $(p, q, I, \phi, t) \in \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{R}^d \times \mathbb{T}^d \times \mathbb{T}^1$.

Theorem 2. *Assume that h_0, V_i, and H_1 are sufficiently smooth, each potential V_i is 1-periodic in q_i and has a non-degenerate global maximum, for $i = 1, \dots, n$, and H_1 satisfies some explicit non-degeneracy conditions that hold for an open and dense set of smooth functions. Then there exists $\varepsilon_0 > 0$, and $\rho > 0$ such that, for each $\varepsilon \in (0, \varepsilon_0)$, there exists a trajectory $(p(t), q(t), I(t), \phi(t))$ of H_ε and $T > 0$ such that*

$$\|I(T) - I(0)\| > \rho.$$

The main idea is that for $\varepsilon > 0$ small the Hamiltonian flow has a normally hyperbolic invariant cylinder $\mathscr{C}$ and a well defined scattering map σ_ε of the form $\sigma_\varepsilon = \mathrm{id} + \varepsilon J\nabla S + O(\varepsilon^2)$, where the function S can be computed in terms of convergent integrals of H_1 along homoclinic trajectories of H_0. For some open and dense set of perturbations H_1, the vector field $J\nabla S$ has an integral curve that moves $O(1)$ in the action direction I. Using Poincaré Recurrence Theorem, we obtain an orbit of the form (1), along which I also changes by $O(1)$. Finally, Theorem 1 implies the existence of a true orbit along which I also changes by $O(1)$. See [2] for details.

1.3. **Diffusion in the a priori stable case.** We consider a Hamiltonian system of the form:

$$H(I, \phi, t) = H_0(I) + H_1(I, \phi), \tag{3}$$

with $(I, \phi) \in \mathbb{R}^3 \times \mathbb{T}^3$. Fix a regular level set of the unperturbed energy $H_0^{-1}(e)$ in the action space $\mathbb{R}^3$, on which the Hamiltonian vector field is complete. In [4], it is shown that, there exists $\varepsilon_0 > 0$, such that, for an open and dense set of perturbations H_1 in inside some generalized ball of radius ε_0, there exists a finite ordered sequence $(\mathscr{C}_k)_{1\leq k\leq k_*}$ of normally hyperbolic invariant cylinders inside $H^{-1}(e)$, with $\mathscr{C}_k \simeq \mathbb{T}^2 \times [a, b]$, such that:

- (i) Each cylinder $\mathscr{C}_k$ admits a global surface of section $\mathscr{A}_k \simeq \mathbb{T}^1 \times [a, b]$, such that the return map ϕ_k to $\mathscr{A}_k$ is a monotone twist map;
- (ii) Each cylinder $\mathscr{C}_k$ admits homoclinic connections, that is, $W^u(\mathscr{C}_k) \cap W^s(\mathscr{C}_k) \neq \emptyset$;
- (iii) Each consecutive pair of cylinders $\mathscr{C}_k$ and $\mathscr{C}_{k+1}$ in the chain, for $1 \leq k \leq k_* - 1$, admits heteroclinic connections, that is, $W^u(\mathscr{C}_k) \cap W^s(\mathscr{C}_{k+1}) \neq \emptyset$.

A chain of cylinders $(\mathscr{C}_k)_{1\leq k\leq k_*}$ satisfying the above conditions, plus some additional technical conditions on the homoclinic and heteroclinic connections as well as on the twist map on each $\mathscr{A}_k$ is called a δ-good chain of cylinders; see [3].

Theorem 3. *For every δ-good chain of cylinders $(\mathscr{C}_k)_{1\leq k\leq k_*}$ contained in $H^{-1}(e)$, there exists a trajectory which intersects the δ-neighborhood in $H^{-1}(e)$ of any essential sub-torus of $\mathscr{C}_k$, $1 \leq k \leq k_*$.*

A key step of the argument is to show that there exists a pseudo-orbit of a certain polysystem, which travels from a δ-neighborhood of $\mathbb{T}^1 \times \{a\}$ to a δ-neighborhood of $\mathbb{T}^1 \times \{b\}$, while it visits a δ-neighborhood of each essential invariant circle in $\mathscr{A}_k$. The aforementioned polysystem consists of the twist map ϕ_k and a family of scattering maps $\{\sigma_k^j\}_j$ induced by the homoclinic connections associated to the cylinder $\mathscr{C}_k$. To produce such a pseudo-orbit, we apply the following algorithmic construction. Start with a δ-neighborhood of $\mathbb{T}^1 \times \{a\}$ and take the union of all forward iterates under ϕ_k. At the boundary, there is an essential invariant circle for ϕ_k. The conditions on the homoclinic connections are so that for every essential invariant circle for ϕ_k there is scattering map σ_k^j that takes points from below that circle onto points above the circle. Repeating the procedure finitely many times yields a pseudo-orbit as in (1) that reaches a δ-neighborhood of $\mathbb{T}^1 \times \{b\}$. Theorem 1 yields a true orbit with similar characteristics.

2. Application to celestial mechanics

We use the methodology outlined in Section 1.2 to show the existence of Arnold diffusion in the planar elliptic restricted three body problem. We view this model as a perturbation of the planar circular restricted three-body problem, with the perturbation parameter being the eccentricity of the orbits of the primaries. We prove that, for all sufficiently small (non-zero) values of the eccentricity, there are orbits that change the energy by a quantity independent of the eccentricity. We provide *explicit constructions of diffusing orbits* and *quantitative estimates on the diffusion time*. Our argument involves *rigorous numerical computations*; see [1].

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Algebraic growth of wrapped Floer homology and contact spheres with positive entropy.

Marcelo R. R. Alves

(joint work with Matthias Meiwes)

The topological entropy is an important measure of the complexity of a dynamical system: it codifies in a single non-negative number the amount of exponential instability of the system. Positivity of the topological entropy is seen as a condition that implies the presence of some chaotic behaviour in the system. In this talk we present the following result of the author and Matthias Meiwes obtained in [4].

Theorem 1.

A) *Let S^{2n-1} be the $(2n-1)$ - dimensional sphere with its standard smooth structure. For $n \geq 4$ there exists a contact structure on S^{2n-1} for which every Reeb flow has positive topological entropy.*

B) *There exists a contact structure on $S^3 \times S^2$ for which every Reeb flow positive topological entropy.*

In order to present this result in the proper context we recall what is known about the topological entropy of Reeb flows and its relation to topological invariants of contact manifolds.

Recall that on a contact manifold (Y, ξ) there is a distinguished class of flows called *Reeb flows.* Although the dynamics of distinct Reeb flows on a given (Y, ξ) can be quite different, there exist dynamical properties that hold for all Reeb flows on (Y, ξ) because they are related to the topology of (Y, ξ).

The first results connecting the behaviour of contact topological invariants to the topological entropy of Reeb flows are due to Frauenfelder, Macarini and Schlenk. Based on the methods developed in [5], it was shown in [6] that if Q is a simply connected rationally hyperbolic manifold or if $\pi_1(Q)$ grows exponentially, then every Reeb flow on the unit cotangent bundle (S^*Q, ξ_{geo}) equipped with the geodesic contact structure has positive topological entropy. This was a generalisation to Reeb flows of a result which was known to hold in the class of geodesic flows: the geodesic flow of any Riemannian metric on Q is a Reeb flow on (S^*Q, ξ_{geo}), and positivity of the topological entropy for geodesic flows on the manifolds Q studied in [6] is the result of the combined works of several mathematicians (see [7]). For simplification we introduce the following terminology.

Definition. *If all Reeb flows on a contact manifold (Y, ξ) have positive topological entropy we say that (Y, ξ) has **positive entropy**.*

In previous work of the author many new examples of contact 3-manifolds with positive entropy were discovered. In [1, 2, 3] it was shown that contact 3-manifolds with positive entropy exist in abundance: there exist hyperbolic contact 3-manifolds and non-fillable contact 3-manifolds with positive entropy, and also 3-manifolds which admit infinitely many non-diffeomorphic contact structures with positive entropy. These results were obtained by studying the exponential growth rate of topological invariants that come from Symplectic Field Theory, and they showed that the class of contact 3-manifolds with positive entropy is much larger than the class of unit cotangent bundles of higher genus surfaces that were studied in [6].

One common feature of all known examples of contact 3-manifolds with positive entropy is that the fundamental group of the underlying smooth 3-manifold has exponential growth. This motivates the following

Conjecture. *If a contact 3-manifold (Y, ξ) has positive entropy, then $\pi_1(Y)$ grows exponentially.*

Theorem 1 shows that, in contrast to what we expect to hold in dimension 3, the phenomenon in higher dimensions is quite flexible from the topological point of view. Basically, there is no restriction on the smooth topology of a high-dimensional contact manifold with positive entropy.

To prove Theorem 1 we introduced in [4] a new measure of growth rate of the wrapped Floer homology $\mathrm{HW}(M, L)$ of an asymptotically conical exact Lagrangian L in a Liouville domain M, which we call **algebraic growth**. Recall that the homology $\mathrm{HW}(M, L)$ comes with a product, the triangle product, which makes $\mathrm{HW}(M, L)$ into an algebra[1]. The algebraic growth measures the exponential complexity of $\mathrm{HW}(M, L)$ as an algebra.

The algebraic growth of $\mathrm{HW}(M, L)$ is related to the "usual" symplectic growth of $\mathrm{HW}(M, L)$ which comes from the action filtration that exists in Floer homology. This relation is a consequence of the spectral triangle inequality for the triangle product of $\mathrm{HW}(M, L)$. Combining this with geometric ideas of [3] one can obtain lower bounds for the topological entropy of Reeb flows from the algebraic growth of $\mathrm{HW}(M, L)$.

We believe that the methods developed in [4] will open way for further investigation on the relations between the rich algebraic structures that exist in symplectic topological invariants and the dynamics of Reeb flows and symplectomorphisms.

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Reeb dynamics and contact homology

Jo Nelson

(joint work with Michael Hutchings)

Contact geometry is the study of certain geometric structures on odd dimensional smooth manifolds. A contact structure is a hyperplane field specified by a one form which satisfies a maximum nondegeneracy condition called complete non-integrability. The associated one form is called a contact form and uniquely determines a vector field called the Reeb vector field on the manifold. I will explain

[1] The homology $\mathrm{HW}(M, L)$ is an algebra when one works with coefficients in $\mathbb{Z}_2$ or any other field. If one uses $\mathbb{Z}$ coefficients the structure one obtains is that of a ring.

how to make use of J-holomorphic curves to obtain a Floer theoretic contact invariant whose chain complex is generated by closed Reeb orbits. In particular, I will explain the pitfalls in defining contact homology and discuss my work (in part joint with Michael Hutchings) which gives a rigorous construction of cylindrical contact homology via geometric methods. I will also discuss some computations and applications of contact homology in the study of Reeb dynamics.

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Introducing symplectic billiards

Serge Tabachnikov

This is a report on a joint work in progress with P. Albers (Heidelberg).

Two types of billiards have been extensively studied: the conventional, inner (Birkhoff) billiards, and the outer billiards. Both systems have variational formulations: the trajectories of Birkhoff billiards are polygons inscribed into the billiard table and having extremal perimeter length, and the trajectories of outer billiards are circumscribed polygons of extremal area. It makes sense to consider two other kinds of billiards: inner ones, extremizing the area, and outer ones, extremizing perimeter length. In this work, we are concerned with the former kind of billiards and their multi-dimensional version.

The definition of the dynamical system is as follows. Let γ be a closed convex planar curve. The symplectic billiard is a transformation T of the space of oriented chords of γ given by the rule $T(xy) = yz$ if the tangent line to γ at point y is parallel to xz. This map has a generating function $\omega(x, y)$ where ω is the standard area form.

One has a similar definition in linear symplectic space $(\mathbf{R}^{2n}, \omega)$: given a smooth convex closed hypersurface M, the map T sends its chord xy to yz if the characteristic direction to M at y is parallel to xz. This is again a discrete Lagrangian system with the generating function $\omega(x, y)$.

The goal of the project it to extend to this setting results known for Birkhoff and outer billiards. One of the main motivations is to investigate an interplay between convex and symplectic geometries. The talk consists of two parts, the planar and multi-dimensional ones. Here are some highlights.

In $\mathbf{R}^2$:

- The map T has a generating function and, as a consequence, an invariant area form Ω. *Theorem*: The total Ω-area of the phase space of the map T equals four times the area of the central symmetrization of the billiard table.
- *Theorem*: If the curvature of γ vanishes at some point, then the symplectic billiard has no caustics. This is a version of Mather's theorem [5].
- On the other hand, if γ is strictly convex and sufficiently smooth, Lazutkin's theorem on the existence of invariant curves [4] applies. The role of Lazutkin's paramer is played by the affine length parameter on γ.
- Consider an n-gon P that is a periodic orbit of the symplectic billiard. The reflection law determines the tangent lines to γ at every vertex of P, and this defines a distribution $\mathcal{D}$ on the space of n-gons.

  Theorem: $\mathcal{D}$ is tangent to the level hypersurfaces of the area function and is totally non-integrable on these level hypersurfaces; namely, the tangent space at every point is generated by the vectors fields tangent to $\mathcal{D}$ and by their first commutators.
- As a consequence, one can construct billiard tables with invariant curves consisting of n-periodic points as horizontal curves of the distribution $\mathcal{D}$. For example, the so-called Radon curves have this property for $n = 4$.
- As another consequence, one has *Theorem*: The set of 3-periodic points of the planar symplectic billiard has empty interior. If γ is strictly convex, then the set of 4-periodic points also has empty interior (these are $n = 3, 4$ cases of a version of Ivrii's conjecture for symplectic billiards). For Birkhoff and outer billiards, see [1, 3, 10, 11].
- The areas of inscribed polygons that are periodic orbits of symplectic billiard constitute the area spectrum of γ. One can apply the Melrose theory of interpolating Hamiltonians [6] to prove that ellipses are recognizable from the area spectrum (this follows from the affine isoperimetric inequality). One can also prove that if the curve is smooth and strictly convex, the symplectic billiard does not have the finite blocking property (cf. [9] for Birkhoff billiards).
- Polygonal symplectic billiards deserve a thorough study. As a first step, we prove that all orbits in (affine-) regular polygons are periodic and describe the periods. We prove that all orbits in trapezoids are periodic as well, and describe the periods (there are three periods, for every trapezoid, making an arithmetic progression with difference 8).

In $\mathbf{R}^{2n}$:

- We discuss two possible continuous limits of the symplectic billiard map as the trajectory gets close to the boundary of the table. In one approach, the limit of such a trajectory is an un-parameterized characteristic curve on M. In the other approach, one considers the limits of odd- and even-numbered points separately, and arrives at two closed parametric curves

$\gamma_1(t)$ and $\gamma_2(t)$ on M, coupled by the relation

$$R(\gamma_1(t)) \parallel \gamma_2'(t), \quad R(\gamma_2(t)) \parallel \gamma_1'(t),$$

where $R(x)$ is the characteristic line at $x \in M$. Such a pair of curves is critical for the functional

$$L(\gamma_0, \gamma_1) = \int \omega(\gamma_0'(t), \gamma_1(t))\ dt.$$

- To study symplectic billiards in ellipsoids, we apply a linear map that takes a symplectic ellipsoid

$$\frac{x_1^2 + y_1^2}{a_1} + \frac{x_2^2 + y_2^2}{a_2} + \ldots + \frac{x_n^2 + y_n^2}{a_n} = 1$$

to the unit sphere and changes the symplectic structure to $\omega = \sum a_j dx_j \wedge dy_j$. The Reeb operator is $R = \operatorname{diag}(ia_1^{-1}, \ldots, ia_n^{-1})$. The map $R^{-1} : \mathbf{C}^n \to \mathbf{C}^n$ takes S^{2n-1} to

$$E = \left\{ (w_1, \ldots, w_n) \mid \frac{|w_1|^2}{a_1^2} + \frac{|w_2|^2}{a_2^2} + \ldots + \frac{|w_n|^2}{a_n^2} = 1 \right\}.$$

Theorem: If $(Z_0, Z_1, Z_2, \ldots)$ is a trajectory of the symplectic billiard in the unit sphere, then $(R^{-1}(Z_0), R^{-1}(Z_2), R^{-1}(Z_4), \ldots)$ is a billiard trajectory in E. Conversely, to a billiard trajectory $(W_0, W_2, W_4, \ldots)$ in E there corresponds a unique symplectic billiard trajectory $(Z_0, Z_1, Z_2, \ldots)$ in S^{2n-1} with $Z_0 = R(W_0), Z_2 = R(W_2), \ldots$

As a consequence, the symplectic billiard in ellipsoid is completely integrable (in fact, superintegrable). We describe the integrals and show that symplectic billiard in ellipsoids are analogs of the discrete Neumann system [7].

- Let $M \subset \mathbf{R}^{2n}$ be a smooth strictly convex closed hypersurface. Concerning periodic orbits of symplectic billiard inside M, we have two results. *Theorem*: For every $k \geq 2$, the symplectic billiard map has a k-periodic trajectory.

This is a weak estimate, and for small periods, we have a stronger one. *Theorem*: The number of 3- and of 4-periodic symplectic billiard orbits inside M is not less than $2n$. The proof is by way of equivariant Morse-Lusternik-Schnirelman theory applied to the symplectic area function on inscribed polygons. For Birkhoff billiards and outer billiards, see [2, 8].

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Local invariant Morse homology in dynamics

Doris Hein
(joint work with U. Hryniewicz, L. Macarini)

In Hamiltonian dynamics, local Floer homology is a useful tool to study periodic orbits. In the contact case, the homology theories are less developed. In particular, relations between local homology of a periodic orbit and its iterations are unclear. In our work, we define an invariant of closed Reeb orbits that can be seen as a local contact homology. It has many features of local Hamiltonian Floer homology and gives hope to be useful in the study of contact dynamics.

As a first step, we consider a tubular neighborhood $U = S^1 \times D^{2n}$ of a closed Reeb orbit γ such that $\gamma = S^1 \times \{0\}$. We assume that γ is 1-periodic and set $S^1 = \mathbb{R}/\mathbb{Z}$. Then we identify the germ of the Reeb flow near γ with the germ of a Hamiltonian diffeomorphism with Hamiltonian H_t. Using generating functions and Chaperon's discrete action functional, we define homology groups $\mathcal{H}_*(H,k)$, which can be seen as a substitute for local contact homology. The most important properties are similar to those of local Floer homology:

- **Non-triviality:** For good, non-degenerate orbits, $\mathcal{H}_*(H,k)$ has one generator in degree of the Conley-Zehnder index.
- **Invariance:** If $\gamma = \eta^k$ is a k-times iterated orbit, $\mathcal{H}_*(H,k)$ is invariant under the $\mathbb{Z}/k\mathbb{Z}$-action by time-shift.
- **Homotopy invariance:** If 0 is uniformly isolated as a k-periodic orbit of φ_H^s along a homotopy H_t^s of germs of 1-periodic Hamiltonians, then $\mathcal{H}_*(H^0,k) \cong \mathcal{H}_*(H^1,k)$.
- **Support:** $\mathcal{H}_*(H,k) \neq 0$ only if $* \in [\Delta(H,k)-n, \Delta(H,k)+n]$, where Δ is the mean Conley-Zehnder index of γ.
- **Persistence:** If $m \in \mathbb{N}$ is admissible and good, then there is an isomorphism $\mathcal{H}_*(H,k) \to \mathcal{H}_{*+s_{k,m}}(H,km)$, where $s_{k,m} = \mathrm{CZ}(\gamma^m) - \mathrm{CZ}(\gamma)$.

The main importance is the persistence result, which has been proved in the Hamiltonian case by Ginzburg and Gürel and has given useful dynamical implications. Moreover, with our definition of $\mathcal{H}_*(H,k)$, we can copy the definition of symplectically degenerate maximal from the Hamiltonian case and hope for equally strong dynamical applications.

The most remarkable feature of $\mathcal{H}_*(H,k)$ is that we can use it by working explicitly with Morse chain complexes. Indeed, $\mathcal{H}_*(H,k)$ is a direct limit of the local invariant Morse homology of Chaperon's discrete action functional letting the discretization become finer to cover the continuous case. As this is a limit over a sequence of isomorphisms, we can work with the limit by working with the individual local invariant Morse homology groups and the corresponding chain complexes and thus get a very hands-on tool to study periodic Reeb orbits.

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Periodic orbits in virtually contact structures

Kai Zehmisch
(joint work with Youngjin Bae, Kevin Wiegand)

Many non-exact magnetic Hamiltonian systems such as on products of closed hyperbolic surfaces admit energy surfaces that are not of contact type, but a certain cover of these are. In order to find closed characteristics one is intended to study finite energy planes in symplectizations of a non-compact contact manifold. For contact forms that admit a uniform bound on all covariant derivatives up to order three existence of periodic orbits can be ensured in many instances.

Let M be a closed connected manifold of dimension $2n-1$ for $n \geq 2$. Let ω be an odd-dimensional symplectic form on M, i.e. a closed 2-form whose kernel is a 1-dimensional distribution. We assume that (M,ω) is virtually contact as in [3]. This means that we can choose a virtually contact structure

$$\big(\pi\colon\ M' \to M, \alpha, \omega, g\big)$$

on (M,ω), where π is a covering of M, α is a contact form on the covering space M'

such that $\pi^*\omega = \mathrm{d}\alpha$, and g is a Riemannian metric on M, whose lift π^*g to M' is denoted by g'. By definition of a virtually contact structure the primitive α is bounded with respect to the norm $|\,.\,|_{(g')^\flat}$ of the dual metric $(g')^\flat$ of g'. Moreover, there exists a constant $c > 0$ such that for all $v \in \ker \mathrm{d}\alpha$ the following lower estimate holds true:

$$|\alpha(v)| > c|v|_{g'}.$$

In addition, we assume that the chosen virtually contact structure is non-trivial, i.e. that ω is not the exterior differential of a contact form on M. We say that the contact form α is C^3-bounded provided that the covariant derivatives $\nabla^k\alpha$ for $k = 0,1,2,3$ are bounded with respect to g'.

Theorem. *([1]) Let $(M', \xi = \ker\alpha)$ be the total space of a virtually contact structure on a closed odd-dimensional symplectic manifold (M,ω). Assume that the contact form α is C^3-bounded. Then the Reeb vector field of α on M' admits* *a contractible periodic orbit provided that one of the following conditions for the* $(2n-1)$*-dimensional contact manifold* (M', ξ) *is satisfied:*

(1) $n = 2$ *and* ξ *is overtwisted,*
(2) $n = 2$ *and* $\pi_2 M' \neq 0$,
(3) $n \geq 3$ *and* (M', ξ) *contains a Legendrian open book with boundary,*
(4) $n \geq 3$ *and* (M', ξ) *contains the upper boundary of the standard symplectic handle of index* $1 \leq k \leq n-1$ *whose belt sphere* $S^{2n-1-k} \subset M'$ *represents a non-trivial element in*
  (a) in $\pi_{2n-2} M'$ *if* $k = 1$,
  (b) in $\pi_3 M'$ *if* $n = 3$ *and* $k = 2$,
  (c) in $\pi_4 M'$ *if* $n = 4$ *and* $k = 3$,
  (d) in the oriented bordism group $\Omega^{SO}_{2n-1-k} M'$ *if* $k \geq 2$,
(5) $n \geq 3$ *and* (M', ξ) *is obtained by covering contact connected sum as introduced in* [3] *such that the underlying connected sum decomposition of* M *is non-trivial and* ω *is not the exterior differential of a contact form on* M.

To handle the non-compactness of M' local compactness properties of holomorphic curves have to be ensured. For holomorphic curves uniformly close to the zero section in the symplectization of (M', α) the required uniform lower and upper bounds on the contact form α suffice to guaranty a tame geometry. This leads to monotonicity type estimates. In order to mimic Hofer's [2] asymptotic analysis we use the deck transformation group of $\mathbb{R} \times M'$ which acts by isometries. The action on the tame structure in $\mathbb{R}$-direction is controlled by the use of the Hofer energy. The action on the contact form α in the direction of M' is controlled by an Arzelà–Ascoli type argumentation that requires higher order bounds on the contact form. The upshot is that the holomorphic analysis developed allows a reduction in the search of periodic orbits to finding bounded primitives of higher order on non-compact covering spaces.

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Stability in the three-body problem

Thibaut Castan

Arnold showed the existence of quasi-periodic solutions in the plane planetary three-body problem, provided that the mass of two of the bodies, the planets, is small compared to the mass of the third one, the Sun. This smallness condition depends in a sensitive way on the analyticity widths of the Hamiltonian of the three-body problem, expressed with the help of some transcendental coordinates.

Hénon gave a minimal ratio of masses (10^{-320}) necessary to the application of Arnold's theorem. The main objective of my work was to determine a sufficient condition on this ratio. A first part of this work was to estimate these analyticity widths, which requires a precise study of the complex Kepler equation, as well as the complex singularities of the disturbing function. A second part consists in reworking the work of Arnold to put the Hamiltonian under a suitable normal form, in order to apply the KAM theorem (KAM standing for Kolmogorov-Arnold-Moser). Working with the secular Hamiltonian, one can then quantify its non-degeneracy, as well as estimate the norm of the perturbation. Finally, it is necessary to derive a quantitative version of the KAM theorem, in order to identify the hypotheses necessary for its application to the plane three-body problem. After this work, it is shown that the KAM theorem can be applied for a ratio of masses that is close to 10^{-85} between the planets and the star.

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C^0 Arnold conjecture via spectral invariants

Lev Buhovsky

(joint work with Vincent Humilière, Sobhan Seyfaddini)

The direct generalisation of the Arnold conjecture to C^0 symplectic geometry fails in dimension greater than 2. I will sketch a proof of this fact, and will state a C^0 formulation of the Arnold conjecture given in terms of spectral invariants, which holds for symplectically aspherical symplectic manifolds.

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Geodesic flows on closed surfaces with vanishing topological entropy

Gerhard Knieper
(joint work with E. Glasmachers, J. P. Schröder)

A classical and well studied invariant measuring the orbit growth of a dynamical system is the topological entropy. Due to a classical result of A. Katok (see [6, 7]) the topological entropy is particularily interesting in low dimension. If $\phi^t : V \to V$ is a flow on a closed 3-dimensional manifold with non-vanishing speed the topological entropy $h_{top}(\phi^t)$ is positive if and only if the flow carries a horseshoe. A similar statement holds for maps on closed surfaces. In particular, in this cases positive topological entropy implies the exponential growth rate of hyperbolic periodic orbits. Hence, it is of great interest to understand the dynamics in case of vanishing topological entropy. The dynamics does not need to be simple as the example of the horocycle flow, which is mixing and uniquely ergodic, shows. In this talk we discussed results and open problems corresponding to the following question. Let (M^2, g) be a closed Riemannian or Finsler surface and $\phi_g^t : SM^2 \to SM^2$ the geodesic flow on the unit tangent bundle. Consider the set

$$EZ(M^2) = \{\text{Riemannian (Finsler) metrics } g \text{ such that } h_{top}(\phi_g^t) = 0\}.$$

Question: What are the dynamical and geometrical properties of this set?

Since by a classical result of Dinaburg this set is, for surfaces of genus at least two, empty one only needs to consider S^2 and T^2.

1. The case of the 2-sphere:

Theorem 1. *(A. Katok [5])There exist non-reversible Finsler metrics on S^2 arbitrarily close to the round metric with*

- *only 2 closed geodesics (in particular, zero topological entropy).*
- *ergodic geodesic flow*

Reversible Finsler metrics close to the round metric having zero topological entropy and ergodic geodesic flow cannot exist as the following result shows.

Theorem 2. *(J. P. Schröder [8]) Let g be a reversible Finsler metric on S^2 with positive flag curvature and zero topological entropy. Then the geodesic flow has no dense orbit and is in particular not ergodic.*

Proof. The positive curvature is used to obtain a global cross-section for the flow based on a simple closed geodesic (Birkhoff annulus). Then the theorem follows from an adaption of the results of Franks and Handel [2]. □

Questions:

- Is the theorem 2 true without curvature assumption?
- Is it possible to extend theorem 2 to non-reversible Finsler metrics provided there are at least three periodic orbits?

- Is it possible to generalize theorem 2 to Reeb flows on S^3 with at least three periodic orbits?

2. The case of the 2-torus:
Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be a torus with a reversible Finsler metric. Of importance is the minmal set

$$\mathcal{M} = \{v \in ST^2 \,|\text{the lift } \tilde{c}_v : \mathbb{R} \to \mathbb{R}^2 \text{ of the geodesic}$$
$$c_v : \mathbb{R} \to T^2 \text{ with initial condition } v \text{ is globally minimal}\}$$

which defines a closed non-empty set invariant under the geodesic flow. By an important result of Hedlund [1] there exists a constant $D > 0$ such that each minimal geodesic $\tilde{c}_v : \mathbb{R} \to \mathbb{R}^2$ stays in the D-tubular neigborhood of a Euclidean line. In particular, each minimal geodesic has an asymptotic direction. In general one defines

Definition 1. *Let $c : \mathbb{R} \to \mathbb{R}^2$ be a geodesic then*

$$\delta(c) := \lim_{t\to\infty} \frac{c(t)}{\|c(t)\|} \in S^1.$$

is called the asymptotic direction. Denote by $\mathcal{M}_\xi \subset \mathcal{M}$ the set of all minimal geodesics with asymptotic direction $\xi \in S^1$.

Theorem 3. *Let g be a reversible Finsler metric on T^2 with zero topological entropy. Then,*
(1) all geodesics have an asymptotic direction. More specifically the function

$$\widehat{\delta} : ST^2 \to S^1$$

with $\widehat{\delta}(v) = \delta(\tilde{c}_v)$ is continuous and its restriction to each fiber is montone and surjective. Furthermore, all geodesics with respect to the lifted metric on $\mathbb{R}^2$ have no self-intersection.
(2) if $\xi = (x, y) \in S^1$ is irrational (i.e. $\frac{y}{x} \in \mathbb{R} \setminus \mathbb{Q}$) then $\widehat{\delta}^{-1}(\xi) = \mathcal{M}_\xi$ is given by a Lipschitz graph $T^2 \hookrightarrow ST^2$.
(3)) if $\xi = (x, y) \in S^1$ is rational (i.e. $\frac{y}{x} \in \mathbb{Q} \cup \{\infty\}$) then either $\widehat{\delta}^{-1}(\xi) = \mathcal{M}_\xi$ is given by a Lipschitz graph $T^2 \hookrightarrow ST^2$ foliated by closed geodesics or the boundary of the set $\widehat{\delta}^{-1}(\xi)$ is given by $\mathcal{M}_\xi = \mathcal{M}_\xi^+ \cup \mathcal{M}_\xi^-$, where $\mathcal{M}_\xi^\pm$ are Lipschitz graphs $T^2 \hookrightarrow ST^2$. Furthermore, the intersection $\mathcal{M}_\xi^{per} = \mathcal{M}_\xi^+ \cap \mathcal{M}_\xi^-$ consists of closed geodesics and $\mathcal{M}_\xi^\pm \setminus \mathcal{M}_\xi^{per}$ are foliated by heteroclinic orbits. The set $E_\xi = \widehat{\delta}^{-1}(\xi) \setminus \mathcal{M}_\xi$ is open and consists of non-minimal geodesics in case E_ξ is non-empty.

Proof. The proof has been given in [3] and [4] for Riemannian and reversible Finsler metrics using the curve-shortening flow and has been extended to non-reversible Finsler metrics [9] using variational methods. □

Remark: Note that $E_\xi = \widehat{\delta}^{-1}(\xi) = \emptyset$ for all rational $\xi \in S^1$ implies that all geodesics are minimal. By a wellknown Theorem of E. Hopf this holds for Riemannian 2-torus if and only if the metric is flat.

Questions:

(1) Is it true that

$$\mu_L\Big(\bigcup_{\xi\in S^1 \text{ rational}} E_\xi\Big) < \mu_L(ST^2)$$

where μ_L is the Liouville measure on ST^2?

(2) Does there exist some rational $\xi \in S^1$ such that $E_\xi = \emptyset$?

(3) Is the set $\{\xi \in S^1 \mid \xi \text{ is rational and } E_\xi \neq \emptyset\}$ finite?

While we believe that the answer to question (1) is yes, the answer to questions (2) or (3) is likely to be no for general Finsler metrics and yes for Riemannian metrics. The only known example of a Riemannian metric on T^2 with zero topological entropy is given by the Liouville metrics

$$ds^2 = (f(x) + g(y))(dx^2 + dy^2)$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are positive 1-periodic smooth functions. In this case only $E_{\pm(1,0)}$ and $E_{\pm(1,0)}$ are non empty. It might be true that the Liouville metrics are the only Riemannian metrics on T^2 with zero topological entropy. This would solve the longstanding open problem of classifying Riemannian metrics on T^2 with integrable geodesic flows. However, as shown in [8], there are Finsler metrics on T^2 with zero topological entropy which have ergodic components of positive Liouville measure located in the sets E_ξ. In particular, the corresponding flows are not integrable.

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A C^∞-closing lemma for Hamiltonian diffeomorphisms of closed surfaces

Kei Irie
(joint work with Masayuki Asaoka)

Let (Y, λ) be a closed contact three-manifold, and $\mathcal{U}(Y, \lambda)$ denote the union of all periodic Reeb orbits of (Y, λ). Our first main result is the following:

Theorem 1. ([5]) For any $h \in C^\infty(Y, \mathbb{R}_{\geq 0}) \setminus \{0\}$, there exists $t \in [0, 1]$ such that $\mathcal{U}(Y, (1 + th)\lambda) \cap \operatorname{supp} h \neq \emptyset$.

Theorem 1 easily implies a C^∞-closing lemma for three-dimensional Reeb flows:

Corollary 1. ([5]) For any nonempty open set U of Y, there exists a sequence $(f_j)_{j\geq 1}$ in $C^\infty(Y, \mathbb{R}_{>0})$ which converges to the constant function 1 in the C^∞-topology, and $\mathcal{U}(Y, f_j\lambda)$ intersects U for every j.

The C^2-version of Corollary 1 follows from the Hamiltonian C^1-closing lemma by Pugh-Robinson [6], thus the point is that we consider the C^∞-topology. Notice that Corollary 1 cannot be generalized to arbitrary autonomous Hamiltonian systems on sympelctic four-manifolds, due to the example by Herman [3].

The proof of Theorem 1 uses quantitative theory of embedded contact homology (ECH). In particular, the key ingredient of the proof is the "volume theorem" by Cristofaro-Gardiner, Hutchings and Ramos [2], which claims that the asymptotics of ECH spectral invariants recover the volume of a contact three-manifold. Theorem 1 easily follows from the volume theorem and other basic properties of ECH spectral invariants (spectrality and C^0-continuity), together with the fact that the set of all finite sums of periods of periodic Reeb orbits is a null set (i.e. Lebesgue measure zero); see [5] or [1] Section 2 for further details.

Our second main result is a C^∞-closing lemma for Hamiltonian surface diffeomorphisms, obtained in joint work with M. Asaoka:

Theorem 2.([1]) Let (S, ω) be a closed symplectic two-manifold, φ be a Hamiltonian diffeomorphism of (S, ω), and U be a nonempty open set of S. Then there exists a sequence $(\varphi_j)_{j\geq 1}$ of Hamiltonian diffeomorphisms of (S, ω), which converges to φ in the C^∞-topology, and each φ_j has a periodic orbit intersecting U.

Let us sketch our proof of Theorem 2. First we take a fixed point q of φ which corresponds to a contractible Hamiltonian loop. Then there exists a perturbation φ' of φ (the perturbation is supported on a neighborhood of q and sufficiently small in the C^∞-topology), such that q is a nondegenerate fixed point of φ', and $\varphi'|_{S\setminus\{q\}}$ is C^∞-conjugate (as area-preserving diffeomorphisms) to a part of a return map of a three-dimensional Reeb flow (associated to a global surface of section). The construction of such a flow uses some classical results about area-preserving maps (convergence of the Birkhoff normal form for hyperbolic fixed points, and KAM theory for elliptic fixed points); see [1] for further details. Since we need a perturbation from φ to φ', which is supported on a neighborhood of q, we cannot assert that $\varphi^{-1} \circ \varphi_j$ is supported on U. Nevertheless, the speaker believes that this is possible:

Conjecture. In Theorem 2, one can take $(\varphi_j)_j$ so that $\varphi^{-1} \circ \varphi_j$ is supported on U for each j.

A possible approach to this conjecture is to develop a quantitative theory of Periodic Floer homology (see [4]), in particular to find an analogue of the "volume theorem" for Periodic Floer homology. The speaker thinks that this is a very interesting problem in itself.

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Pseudo-rotations of Complex Projective Spaces

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(joint work with Başak Z. Gürel)

We consider generalizations, somewhat hypothetical, of pseudo-rotations of the 2-sphere to projective spaces. These are the Hamiltonian diffeomorphisms $\varphi\colon \mathbb{CP}^n \to \mathbb{CP}^n$ with exactly $n+1$ periodic points. At this moment, there are no known examples of such Hamiltonian diffeomorphisms other than true rotations, i.e., Hamiltonian diffeomorphisms generated by quadratic Hamiltonians. However, it is believed that the Anosov–Katok conjugation method, [1, 3], can be used to construct genuine pseudo-rotations at least when $n = 2$. Carrying out this construction is an important open problem.

We extend several results about pseudo-rotations in dimension two ($n = 1$) to higher dimensions. Among these are a theorem on the existence of invariant sets by Le Calvez and Yoccoz, [6], and the rigidity of pseudo-rotations by Bramham, [2]. We also show that, consistently with the conjugation method, every pseudo-rotation φ has a matching rotation, at least when $n = 2$, which is essentially indistinguishable from φ as far the invariants of their periodic orbits are concerned. The proofs utilize the action-index resonance relations and the variant of the Lusternik–Schnirelmann inequality established in [4].

Here are two sample results, the first of which generalizes the aforementioned Le Calvez–Yoccoz theorem and the second is an extension of Bramham's.

Theorem 1. *Let $\varphi\colon \mathbb{CP}^n \to \mathbb{CP}^n$ be a pseudo-rotation. Then for every fixed point x of φ every neighborhood U of x contains an entire orbit $\varphi^k(y)$, $k \in \mathbb{Z}$, different from $\{x\}$, i.e., the orbit x is not isolated as an invariant set.*

The proof of this theorem relies on the method developed in [5] to show that for $\mathbb{CP}^n$ and some other symplectic manifolds a Hamiltonian diffeomorphism with Floer-homologically non-trivial periodic orbit x, isolated as an invariant set, has infinitely many simple periodic orbits. This, in turn, is a consequence of the fact that the energy of a solution of the Floer equation asymptotic to x^k and crossing an isolating neighborhood of x is bounded away from 0 by a constant independent of the order of iteration k.

Theorem 2. *Let $x_0, \dots, x_n$ be the fixed points of a pseudo-rotation $\varphi\colon \mathbb{CP}^n \to \mathbb{CP}^n$. Assume that all iterations φ^k, $k \in \mathbb{N}$, are non-degenerate and the vector of the mean indices*

$$\Delta = \big(\hat{\mu}(x_0), \dots, \hat{\mu}(x_n)\big) \in \mathbb{T} = \mathbb{R}^{n+1}/2(n+1)\mathbb{Z}^{n+1}$$

is exponentially Liouville. Then there exists a sequence of iterations $k_i \to \infty$ such that $\varphi^{k_i} \overset{C^0}{\to} id$.

Here $\Delta \in \mathbb{T}$ is called exponentially Liouville if for every $c > 0$ there exists a sequence $k_i \to \infty$ such that $\|k_i\Delta\| < \exp(-ck_i)$, where $\|\cdot\|$ stands for the distance to the zero in $\mathbb{T}$.

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