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Partial Differential Equations

Organised by
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Abstract. The workshop dealt with nonlinear partial differential equations and some applications in geometry, touching several different topics such as minimal surfaces and geometric measure theory, conformal geometry, geometric flows, metric geometry and structure of Riemannian manifolds.

Mathematics Subject Classification (2010): 28A75, 35C44, 53A30, 53C23.

Introduction by the Organisers

The workshop *Partial differential equations*, organised by Camillo De Lellis (Zürich), Peter Topping (Warwick) and Rick Schoen (Irvine) was held July 30 - August 5, 2015. The meeting was well attended by 52 participants, including 8 females, with broad geographic representation. The program consisted of 21 talks and left sufficient time for discussions.

There were several contributions concerning nonlinear PDE arising in geometric flows. The study of solitons, i.e. self-similar solutions, was a recurring theme. Various contributions covered their existence, their uniqueness (once their asymptotic behaviour has been prescribed) and the elliptic variational problems that they satisfy. In this latter direction, one talk presented the solution of a conjecture by Colding, Ilmanen, Minicozzi and White on hypersurfaces minimizing the entropy that arises most naturally in the study of mean curvature flow.

Several of the talks were related to the issue of understanding singularities in flows, including understanding when flows can be continued uniquely beyond a singular time. New work on Ricci flow and its applications was presented, advancing our understanding of manifolds with lower curvature bounds. Several further applications of flows were given, including to the study of isoperimetric inequalities and to the min-max process. This latter topic was the focus of further talks, alluded to below.

A number of experts in geometric measure theory have attended the workshop. Here new results were presented in the existence and regularity theory of minimal and constant mean curvature surfaces, on the structure of the singular sets and the uniqueness of minimizers of variational problems and on classical measure-theoretic questions. Most notably, two talks addressed the solutions of two long-standing open problems: the existence of closed hypersurfaces with any assigned constant mean curvature in general Riemannian manifolds up to dimension 7 and the characterization of absolutely continuous measures in the Euclidean space through the validity of Rademacher's Theorem.

Two talks were concerned with problems in conformal geometry. One of these considered various sharp properties of eigenvalues of the Laplacian and the structure of nodal sets. In a further talk we were updated on recent progress on the study of eigenvalues and eigenfunctions on closed surfaces, focussing on their extremal cases.

In addition to the variational problems described above, we heard about recent progress on the Willmore functional and the harmonic map functional.

Finally, a group of three talks dealt with metric measure spaces, describing recent results in the theory of calculus on such spaces, curvature bounds and isoperimetry.

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Workshop: Partial Differential Equations

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Abstracts

On the Converse of Rademacher Theorem and the rigidity of mesaures in Lipschitz differentiability spaces.

Guido De Philippis

Rademacher's Theorem asserts that a Lipschitz function $f \in \mathrm{Lip}(\mathbb{R}^d, \mathbb{R}^\ell)$ is differentiable $\mathcal{L}^d$-almost everywhere. A natural question, which has attracted the attention of several researchers, is to understand how sharp is this result. Namely:

Question 1 (Strong converse of Rademacher Theorem)**.** *Given a Lebesgue null set $E \subset \mathbb{R}^d$ is it possible to find some $\ell \geq 1$ and a Lipschitz function $f \in \mathrm{Lip}(\mathbb{R}^d, \mathbb{R}^\ell)$ such that f is not differentiable in any point of E?*

Question 2 (Weak converse of Rademacher Theorem)**.** *Let $\nu \in \mathcal{M}_+(\mathbb{R}^d)$ be a positive Radon measure such that every Lipschitz function is differentiable ν-almost everywhere. Is ν necessarily absolutely continuous with respect to the Lebesgue measure?*

Clearly a positive answer to Question 1 implies a positive answer to Question 2. Let us also stress that in answering Question 1 an important role is played by the dimension ℓ of the target set , see point (2) below, while this does not have any influence for what concern Question 2, see [4, Remark 7.2].

We refer to [1, 2, 4] for a detailed account on the history of these problems and here we simply record the following facts:

(1) For $d = 1$ a positive answer to Question 1 is due to Zahorski [16].
(2) For $d \geq 2$ there exists a null set E such that every Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}^{d-1}$ is differentiable in at least one point of E. This is was proved by Preiss in [13] for $d = 2$ and later extended by Preiss and Speight in [14] to every dimension.
(3) For $d = 2$ a positive answer to Question 1 has been given by Alberti, Csörnyei and Preiss as a consequence of their deep result concerning the structure of null sets in the plane, [1, 2, 3]. Namely they show that for every null set $E \subset \mathbb{R}^2$ there exists a Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that f is not differentiable at every point of E.
(4) For $d \geq 2$ an extension of the result described in point (3) above (i.e. that for every null set $E \subset \mathbb{R}^2$ there exists a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}^d$ such that f is not differentiable at every point of E) has been announced in 2011 by Csörnyei and Jones, [10].

Rademacher Theorem allows to easily prove several rigidity results by "translating" them at a linear level. For example it is an immediate consequence that there is no bi-Lipschitz map from $\mathbb{R}^d$ to $\mathbb{R}^\ell$ if $\ell \neq d$ (this is also a consequence of the much harder to prove "Invariance of the dimension" theorem).

Following this line of reasoning Pansu [12] prove the analogous of Rademacher Theorem between Carnot Groups, endowed with their Carnot Caratheodory metric and Haar measure, with the explicit intention of showing non-imbedding results.

This theory has been further developed by Cheeger [6] which introduced the so called Lipschitz differentiability spaces (see also [11]), a particular class of metric measure spaces on which it is possible to develop a first order calculus and to differentiate Lipschitz functions with respect to suitable charts, more precisely we have the following:

Definition 1. *A metric measure space (X, ν, ρ) is a Lipschitz differentiability space if there a exists a family $\{(U_i, \phi_i)\}_{i\in\mathbb{N}}$ of Borel charts such that*

(i) *$U_i \subset X$ is a Borel set and $X = \bigcup_i U_i$ up to a μ-negligible set,*
(ii) *$\phi_i \colon X \to \mathbb{R}^{d(i)}$ is Lipschitz*
(iii) *For every Lipschitz map $f \colon X \to \mathbb{R}$ at ν-almost every $x_0 \in U_i$ there exists a unique vector $df(x_0) \in \mathbb{R}^{d(i)}$ with*

$$\limsup_{x\to x_0} \frac{|f(x) - f(x_0) - df(x_0)\cdot(\phi(x) - \phi(x_0))|}{\rho(x, x_0)} = 0.$$

In analogy with the converse of Rademacher Theorem in [6] raised he following conjecture:

Question 3. *Let (X, ν, ρ) be a Lipschitz differentiability space and let (U, ϕ_i) be a a chart, then*

$$(\phi_i)_\sharp(\nu \llcorner U_i) \ll \mathcal{L}^{d(i)}.$$

It has been known for several years that a positive answer to question 2 wipuld have also given a positive answer to Cheeger's conjecture 3,see [5, 9, 11, 15] and [7].

The aim of the talk is to provide a positive answer to Question 2, more precisely we have the following theorem:

Theorem 2 (De Philippis-Rindler)**.** *Let $\nu \in \mathcal{M}_+(\mathbb{R}^d)$ be a positive Radon measure such that every Lipschitz function is differentiable ν-almost everywhere, then $\nu \ll \mathcal{L}^d$.*

Quite surprisingly the proof of the above theorem relies on a new structural result concerning the singular part of PDE-constrained measures. This structural theorem is the main result in [8] and beside Theorem 2 it has also other consequences concerning the structure of functions of Bounded Variations and of Bounded Deformation.

The link between the result in [8] and Theorem 2 is due the beautiful theory developed by Alberti and Marchese in [4]. More precisely they show the following:

Theorem 3 (Alberti-Marchese)**.** *. Let $\nu \in \mathcal{M}_+(\mathbb{R}^d)$ be a positive Radon measure, then the following are equivalent*

(i) *Every Lipschitz function is differentiable ν-a.e.*

(ii) *There exists* d $\mathbb{R}^d$*-valued measures* $\mu_1, \dots, \mu_d \in \mathcal{M}(\mathbb{R}^d; \mathbb{R}^d)$ *with measure valued divergence* $\operatorname{div} \mu_i \in \mathcal{M}(\mathbb{R}^d; \mathbb{R})$*, such that* $\nu \ll |\mu_i|$ *for* $1 = 1, \dots, d$ *and*

$$\operatorname{span}\left\{\frac{d\mu_1}{d|\mu_1|}(x), \dots, \frac{d\mu_d}{d|\mu_d|}(x)\right\} = \mathbb{R}^d \qquad \text{for } \nu \text{ a.e. } x. \tag{1}$$

Hence in order to prove Theorem 2 It is enough to show the following Lemma:

Lemma 4. *Let* $\mu_1, \dots, \mu_d \in \mathcal{M}(\mathbb{R}^d; \mathbb{R}^d)$ $\mathbb{R}^d$*-valued measures such that* $\operatorname{div} \mu_i \in \mathcal{M}(\mathbb{R}^d; \mathbb{R})$*, then for* $|\mu_1|^s + \cdots + |\mu_d|^s$ *almost every point*

$$\dim \operatorname{span}\left\{\frac{d\mu_1}{d|\mu_1|}(x), \dots, \frac{d\mu_d}{d|\mu_d|}(x)\right\} \leq (d-1).$$

which is easily follows from the main result in [8], see Corollary 1.13 there.

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Nodal sets and negative eigenvalues in conformal geometry

Dmitry Jakobson

This is an extended abstract of the talk *Nodal sets and negative eigenvalues in conformal geometry* at a conference on PDE in Oberwolfach on July 31, 2017. The talk was based on joint work Yaiza Canzani, Rod Gover, Raphaël Ponge, Asma Hassannezhad and Michael Levitin; the work in graph theory was joint with Thomas Ng, Matt Stevenson and Mashbat Suzuki. The results appeared in the following papers:

- [2] Y. Canzani, D. Jakobson, R. Gover and R. Ponge. *Conformal invariants from nodal sets. I. Negative eigenvalues and curvature prescription.* Int. Math. Res. Not. (9):2356-2400, 2014. With an appendix by R. Gover and A. Malchiodi.
- [3] Y. Canzani, D. Jakobson, R. Gover and R. Ponge. *Nullspaces of Conformally Invariant Operators. Applications to Q_k-curvature.* Electronic Research Announcements in Mathematical Sciences, Vol. 20 (2013), pp. 43–50.
- [5] R. Gover, A. Hassannezhad, D. Jakobson and M. Levitin. *Zero and negative eigenvalues of the conformal Laplacian.* Jour. of Spectral Theory 6 (2016), 793–806.
- [7] D. Jakobson, T. Ng, M. Stevenson and M. Suzuki. *Conformally covariant operators and conformal invariants on weighted graphs.* Geom Dedicata (2015) 174:339?357.

The current abstract is an updated version of an extended abstract of a related talk given at ICMAT in 2013.

Let M be a compact Riemannian manifold of dimension $n \geq 3$, and let g be a Riemannian metric on M. We study eigenfunctions of conformally covariant operators, also called GJMS operators, see [6].

For any positive integer k if n is odd, or for any positive integer $k \leq \frac{n}{2}$ if n is even, there is a covariant, formally self-adjoint, differential operator $P_{k,g}$ of order $2k$ such that

(i) $P_k = \Delta_g^k$ + *lower order terms.*

(ii) If $g_1 = e^{2\omega} g$ is another metric in the conformal class $[g]$, then P_k transforms as follows:

$$P_{k,g_1} = e^{-(\frac{n}{2}+k)\omega} P_{k,g} e^{(\frac{n}{2}-k)\omega} \tag{1}$$

The operator $P_{1,g} := \Delta_g + \frac{n-2}{4(n-1)} R_g$ is called the *Yamabe operator*; here R_g denotes the scalar curvature of g. The operator $P_{2,g}$ is called the *Paneitz operator.*

The *nullspace* $\ker P_{k,g}$ is the subspace of $L^2(M)$ consisting of eigenfunctions u of $P_{k,g}$ with eigenvalue 0: $\{u \in L^2(M) : P_{k,g} u = 0\}$. It follows easily from (1) that

$$\ker P_{k,g_1} = e^{(k-\frac{n}{2})\omega} \ker P_{k,g_0} \tag{2}$$

The following results follow from (2).

Proposition 1. *Let M be a compact manifold of dimension $n \geq 3$, and let g be a Riemannian metric on M.*

- *The dimension* $\dim \ker P_{k,g}$ *is an invariant of a conformal class* $[g]$.

- *The number of negative eigenvalues of* $P_{k,g}$ *is conformally invariant.*
- *If* n *is even and* $k = n/2$, *then* $\ker P_{k,g}$ *itself is conformally invariant.*
- *If* $\dim\ker P_{k,g} \geq 1$, *then the nodal set and nodal domains of any nonzero eigenfunction* $u \in P_{k,g}$ *are invariants of* $[g]$.
- *If* $\dim\ker P_{k,g} \geq 2$, *then (non-empty) intersections of nodal sets of eigenfunctions in* $\ker P_{k,g}$ *are conformally invariant, and hence so are their complements.*

Now assume that $\dim\ker P_{k,g} = m \geq 2$. Let $u_{1,g}, \cdots, u_{m,g}$ be a basis of $\ker P_{k,g}$. Set $\mathcal{N} := \bigcap_{1\leq j\leq m} u_{j,g}^{-1}(0)$ and define $\Phi : M \setminus \mathcal{N} \to \mathbb{RP}^{m-1}$ by

$$\Phi(x) := (u_{1,g}(x) : \cdots : u_{m,g}(x)) \qquad \forall x \in M \setminus \mathcal{N}.$$

Note that the set $\mathcal{N}$ is independent of the choice of the basis $u_{1,g}, \cdots, u_{m,g}$, but Φ depends on the choice of basis only up to the right action of $\mathrm{PGL}_m(\mathbb{R})$.

Proposition 2. *The class of* Φ *modulo the right action of* $\mathrm{PGL}_m(\mathbb{R})$ *is an invariant of the conformal class* $[g]$.

For even n and $k = \frac{n}{2}$, the nullspace of $P_{\frac{n}{2}}$ always contains the constant functions, so we may assume that $u_{1,g}(x) = 1$. The counterpart of Φ in that case can be defined by

$$\Psi(x) := (u_{2,g}(x), \cdots, u_{m,g}(x)) \qquad \forall x \in M,$$

Proposition 3. *The class of* Ψ *modulo the right action of* $\mathbb{R}^{m-1} \ltimes \mathrm{PGL}_{m-1}(\mathbb{R})$ *is an invariant of* $[g]$.

Denote by $dV_g(x)$ the Riemannian measure defined by g.

Proposition 4. *Assume* M *is compact and* $k < \frac{n}{2}$. *Let* $u \in \ker P_{k,g}$ *Then the integral* $\int_M |u_g(x)|^{\frac{2n}{n-2k}} dV_g(x)$ *is an invariant of* $[g]$.

The following result was proved in [5]:

Theorem 5. *Let* M *be a compact manifold of dimension* $n \geq 3$. *For an open and dense subset of metrics* g *on* M, 0 *is* not *an eigenvalue of the conformal Laplacian* $P_{1,g}$ *on* M.

It would be interesting to prove a counterpart for the operators $P_{k,g}, k \geq 2$.

We next discuss metrics g for which $P_{k,g}$ has negative eigenvalues. For $m \in \mathbb{N}_0$, denote by $\mathcal{G}_{k,m}$ the set of metrics g on M such that $P_{k,g}$ has at least m negative eigenvalues (counted with multiplicity). One can show that $\mathcal{G}_{k,m}$ is an open in C^{2k}-topology; and that if $g \in \mathcal{G}_{k,m}$, then $[g] \subset \mathcal{G}_{k,m}$. It follows from this that the number of negative eigenvalues defines a partition of the set of conformal classes. We also observe that by results of Kazdan-Warner [8] $\mathcal{G}_{1,0}$ consists of all metrics that are conformally equivalent to a metric with nonnegative scalar curvature.

The following result can be deduced from Lokhamp [9], and was also proved in [4]:

Theorem 6. *Assume M compact. Then for any m, there is a metric g on M for which the Yamabe operator $P_{1,g}$ has at least m negative eigenvalues.*

It follows that there exist infinitely many conformal classes of metrics on M for which the nullspace of $P_{1,g}$ has dimension ≥ 1, and thus Propositions 1 and 2 all apply.

It would be interesting to obtain similar results for $P_{k,g}, k \geq 2$. For $k = 2$, we can prove the following

Theorem 7. *Assume $M = \Sigma \times \Sigma$, where Σ is a compact surface of genus ≥ 2. Then, for any m, there is a metric g on M for which the Paneitz operator $P_{2,g}$ has at least m negative eigenvalues.*

There is a similar result on compact Heisenberg manifolds.

In addition, as an application of Courant's nodal domain theorem, we obtain

Theorem 8. *Let g be a metric such that the Yamabe operator $P_{1,g}$ has exactly m negative eigenvalues. Then any eigenfunction $u \in \ker P_{1,g}$ has at most $m+1$ nodal domains.*

We next mention an application to the *scalar curvature prescription* problem; we refer to [2, 3] for more details.

Theorem 9. *Let $0 \not\equiv u \in \ker P_{1,g}$ and let Ω be a nodal domain of u. Then, for any metric $g_1 \in [g]$, the scalar curvature R_{g_1} cannot be everywhere nonnegative on Ω.*

Related results appear in [1].

In conclusion, we remark that in [7], the authors defined conformally covariant operators on weighted graphs; the definition of conformally equivalent metrics was given in the Computer Science literature related to image processing. Examples of conformally covariant operators include the adjacency matrix, the incidence matrix (discrete version of the gradient), and the edge Laplacian; the vertex Laplacian is in general *not* conformally covariant. The authors also established graph-theoretic analogues of several results in [2, 3], see [7] for details.

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Mean curvature flow and entropy

Lu Wang
(joint work with Jacob Bernstein)

The *entropy* of a hypersurface, Σ, in $\mathbb{R}^{n+1}$ is

$$\lambda[\Sigma] = \sup_{(x_0,t_0)\in\mathbb{R}^{n+1}\times\mathbb{R}^+} (4\pi t_0)^{\frac{n}{2}} \int_\Sigma e^{-\frac{|x-x_0|^2}{4t_0}}\, d\mathcal{H}^n,$$

where $\mathcal{H}^n$ is the n-dimensional Hausdorff measure. It was introduced by Colding and Minicozzi [6] in their study of generic properties of mean curvature flow. By Huisken's monotonicity formula, entropy is non-increasing under mean curvature flow. It is known that hyperplanes have the lowest entropy among all hypersurfaces. It is a natural question that which hypersurface has the second lowest entropy. Suggested by the dynamical approach of Colding and Minicozzi, they together with Ilmanen and White conjectured, in [5], that

Conjecture 1. *For $2 \le n \le 6$, if Σ is a closed hypersurface in $\mathbb{R}^{n+1}$, then $\lambda[\Sigma] \ge \lambda[\mathbb{S}^n]$, and the equality holds if and only if Σ is a round sphere.*

First, J. Bernstein and myself used a weak mean curvature flow to prove Conjecture 1 – cf. [1]. Recently, J. Zhu [9] has generalized this conjecture to all dimensions. Next, we established a topological rigidity for closed hypersurfaces of small entropy in dimensions 2 and 3 – cf. [2, 3].

Theorem 1. *For $n = 2, 3$, if Σ is a closed hypersurface in $\mathbb{R}^{n+1}$ and $\lambda[\Sigma] \le \lambda[\mathbb{S}^{n-1}]$, then Σ is diffeomorphic to $\mathbb{S}^n$.*

Furthermore, we prove a quantitative Hausdorff stability theorem for round 2-sphere under perturbations of entropy – cf. [4].

Theorem 2. *There is a universal constant C so that if Σ is a closed surface in $\mathbb{R}^3$, there is a $\rho > 0$ and $y \in \mathbb{R}^3$ such that*

$$\mathrm{dist}_H(\Sigma, y + \rho\mathbb{S}^2) < C\rho(\lambda[\Sigma] - \lambda[\mathbb{S}^2])^{1/8}.$$

Here dist_H denotes the Hausdorff distance.

One of the key ingredients in the proof of Theorem 2 is a new local curvature estimate for mean curvature flow of small entropy, which is a variant of Brakke's local regularity theorem of White [8]. Very recently, using a complete different method, S. Wang [7] proved a qualitative Hausdorff stability for round n-sphere under perturbations of entropy for all n.

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Structure of the singular sets of Q-valued functions

Daniele Valtorta

(joint work with Camillo de Lellis, Andrea Marchese and Emanuele Spadaro)

Dirichlet minimizing Q-valued maps have been introduced by Almgren in order to model the branching singularities of minimal surfaces in codimension greater or equal than 2. In order to define them precisely, we introduce $\mathcal{A}_Q(\mathbb{R}^n)$ as the set of unordered Q-tuples of points in $\mathbb{R}^n$. In other words $P \in \mathcal{A}_Q(\mathbb{R}^n)$ can be written as

$$P = \sum_{i=1}^{Q} [\![p_i]\!] \ , \quad p_i \in \mathbb{R}^n \ ,$$

where $[\![p_i]\!]$ is just the Dirac delta measure at p_i. This space has a natural distance function, and one can define the space of $W^{1,2}$ functions with values in $\mathcal{A}_Q(\mathbb{R}^n)$ and so also Dirichlet energy and Dirichlet minimizers.

Regular points of these functions are those points where it is possible to write the function as the sum of Q separate single-valued harmonic functions. In particular, x is regular for u if there exists $r > 0$ such that for all $y \in B_r(x)$

$$u(y) = \sum_{i=1}^{Q} \delta_{u_i(x)} \ ,$$

where u_i are harmonic functions and either $u_i(x) \neq u_j(x)$, or $u_i \equiv u_j$. Here δ_x is the Dirac mass at the point x. The singular set $\mathcal{S}(u)$ is the complement of the regular set, so for all $x \in \mathcal{S}(u)$, at least two valued of the function "collapse" onto each other. An interesting subset of the singular set is the set where *all* the values of the function collapse:

$$\Delta_Q = \{x \in \mathcal{S}(u) \;\; s.t. \;\; u(x) = Q\delta_p \;\; \text{for some } p \in \mathbb{R}^n\} \; .$$

As an example of singularity for such maps, one can consider the two-valued map $u : \mathbb{C} \to \mathbb{C}$, $u(z) = z^{1/2}$. The origin is a singular point for u.

Although singularities are what makes these functions interesting objects (otherwise in some sense they would be simply sums of classical well-behaved harmonic functions), we don't want the singular set to be too "wild". The result we prove in this work is that the singular set $\mathbb{S}(u)$ is $m-2$ rectifiable, and moreover we prove the uniform volume estimates

$$\mathrm{Vol}\Big(B_r\big(\Delta_Q \cap B_1(p)\big)\Big) \leq C(m, Q, N(0,2))r^2 \, .$$

Here $N(x,r)$ denotes Almgren's frequency, which is defined as

$$N(x,r) = \frac{r\int_{B_r(x)} |\nabla u|^2}{\int_{\partial B_r(x)} |u|^2} \, .$$

This object and its properties, along with a Reifenberg-type theorem, are the main tool used to prove our estimate. In particular, N can be used to control how much the map u is close to being homogeneous of a certain degree, in the sense that

$$(1) \qquad \int_{B_2(0)\setminus B_{1/2}(0)} |z\cdot Du(z) - N(0,|z|)u(z)|^2 \, dz \leq C(N(0,4) - N(0,1/4)) \, .$$

Note that if $z\cdot\nabla u(z) = Hu(z)$ for all $z \in \mathbb{R}^m$, then u is H-homogeneous wrt the origin.

This technique can be adapted to prove other singular set estimates in GMT. For example, in [3] Focardi and Spadaro show that the free boundary in thin obstacle problems is rectifiable with uniform volume bounds. Another example is given by the study of liquid crystals carried out by Onur Alper in [1], building on a work by himself, Robert Hardt and Fang-Hua Lin.

Moreover, this technique is based on the quantitative stratification developed in [4] to study the singularities of harmonic maps between Riemannian manifolds. Aside from technical details, the main difference between [4] and this work is that while the tangent maps of harmonic maps are always 0-homogeneous, for Q-valued harmonic functions the tangent maps have variable degree of homogeneity. This extra complication requires a special estimate on the variation of the frequency $N(x,r)$ as a function of x.

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Applications of mean curvature flow

Robert Haslhofer
(joint work with Reto Buzano, Or Hershkovits, Dan Ketover, Mohammad Ivaki)

We discussed some recent topological, geometric and analytic applications of mean curvature flow.

To put the topological application into context, consider the moduli space of embedded n-spheres in $\mathbb{R}^{n+1}$, i.e. the space $\mathcal{M}(S^n) = \mathrm{Emb}(S^n, \mathbb{R}^{n+1})/\mathrm{Diff}(S^n)$ equipped with the smooth topology. By a theorem of Smale [15] the space $\mathcal{M}(S^1)$ is contractible, and by Hatcher's solution of the Smale conjecture $\mathcal{M}(S^2)$ is also contractible. For $n \geq 3$, there are many non-vanishing homotopy groups, see e.g. [3]. In the view of the topological complexity of $\mathcal{M}(S^n)$ for general n, it is an interesting question whether one can still derive some positive results on the space of embedded n-spheres under some curvature conditions. Motivated by the topological classification result from [10], we consider the subspace $\mathcal{M}_{\text{2-conv}}(S^n) \subset \mathcal{M}(S^n)$ of 2-convex embedded n-spheres in $\mathbb{R}^{n+1}$, i.e. we impose the condition that the sum of the two smallest principal curvatures is positive. We proved:

Theorem 1 (Buzano-Haslhofer-Hershkovits [2]). *The moduli space $\mathcal{M}_{\text{2-conv}}(S^n)$ is path-connected in every dimension n.*

Our proof uses mean curvature flow with surgery [8, 10], a connected sum operation for 2-convex hypersurfaces, and a scheme inspired by the work of Marques on the moduli space of positive scalar curvature metrics on the three-sphere [13].

For the geometric application, recall first that by a classical theorem of Lusternik-Schnirelmann [5, 12] every (S^2, g) contains at least 3 simple closed geodesics.

Moving up one dimension, one might hope to prove that any (S^3, g) contains at least 4 embedded minimal two-spheres. The existence of at least 1 embedded minimal two-sphere was established by Simon-Smith [14]. While there are indeed 4 cohomology classes $\alpha, \ldots, \alpha^4$ in the space of embedded two-spheres, the the major difficulty is the phenomenon of *multiplicity* in min-max theory. Namely, it could happen that the min-max spheres associated with the second, third and fourth family, just give the Simon-Smith sphere counted with higher integer multiplicities.

Using combined efforts from min-max theory and mean curvature flow we proved:

Theorem 2 (Haslhofer-Ketover [7]). *Any (S^3, g) equipped with a bumpy metric contains at least 2 embedded minimal two-spheres. More precisely, exactly one of the following alternatives holds:*

(1) (S^3, g) *contains at least* 1 *stable embedded minimal two-sphere, and at least* 2 *embedded minimal two-spheres of index one.*
(2) (S^3, g) *contains no stable embedded minimal two-sphere, at least* 1 *embedded minimal two-sphere* Γ_1 *of index one, and at least* 1 *embedded minimal two-sphere* Γ_2 *of index two. In this case,* $|\Gamma_2| < 2|\Gamma_1|$.

We note that White [16] previously proved the existence of at least 2 minimal two-spheres in the special case that (S^3, g) has positive Ricci curvature.

Illustrative examples for Theorem 2 are dumbbells for case (1) and ellipsoids for case (2). The main way how mean curvature flow enters the proof is via the following theorem (of independent interest) which establishes the existence of smooth mean convex foliations in three-manifolds:

Theorem 3 (Haslhofer-Ketover [7]). *Let* $D \subset (M^3, g)$ *be a smooth three-disc with mean convex boundary. Then exactly one of the following alternatives holds true:*
(1) *There exists an embedded stable minimal two-sphere* $\Gamma \subset Int(D)$.
(2) *There exists a smooth foliation* $\{\Sigma_t\}_{t\in[0,1]}$ *of* D *by mean convex embedded two-spheres.*

Namely, if (S^3, g) contains no stable embedded minimal two-sphere then we can use Theorem 3 to produce a foliation $\{\Sigma_t\}_{t\in[-1,1]}$ of (S^3, g) such that the Simon-Smith sphere sits in the middle as Σ_0 and such that $|\Sigma_t| < |\Sigma_0|$ for all $t \neq 0$. We can then construct a suitable 2-parameter sweepout $\{\Sigma_{s,t}\}$ detecting α^2 with

$$\sup_{s,t} |\Sigma_{s,t}| < 2|\Gamma_1|. \tag{1}$$

Roughly speaking, $\Sigma_{s,t}$ looks like Σ_s connected to Σ_t along a small neck, which we open up near $(s, t) \approx (0, 0)$, using the catenoid estimate from [11]. The estimate (1) ensures that min-max for $\Sigma_{s,t}$ doesn't give Γ_1 with multiplicity two.

The proof of Theorem 3 is again based on mean curvature flow with surgery, refining the methodology from the proof of Theorem 1.

Finally, for the analytic application we consider the Allen-Cahn equation

$$\Delta u = \frac{1}{\epsilon^2} u(1 - u^2) \tag{2}$$

on any bumpy (S^3, g). The recent work of Gaspar-Guaraco [4] establishes the existence of solutions of arbitrarily large index as ϵ becomes smaller. In a different direction, we examine solutions of low index / with a simple interface, and prove:

Theorem 4 (Haslhofer-Ivaki [6]). *The Allen-Cahn equation* (2) *on any bumpy* (S^3, g) *has at least 4 solutions with spherical interface and index at most two.*

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Recent advances about calculus on RCD spaces

Nicola Gigli

Aim of the talk has been to give a survey over recent developments of calculus on RCD spaces. The theory is built upon the concept of Sobolev space $W^{1,2}(\mathrm{X},\mathsf{d},\mathfrak{m})$ of real valued functions on a metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$, as introduced by Cheeger in [6] (see also [17], [1]). Such Sobolev space is always a Banach space and to each $f \in W^{1,2}(\mathrm{X})$ is associated a non-negative function $|\mathrm{D}f| \geq 0$, called *minimal weak upper gradient*, playing the role of the modulus of the distributional differential. Adapting ideas of Weaver ([19]) one can see $|\mathrm{D}f|$ as the 'pointwise norm' of a suitably defined 1-form by introducing the concept of L^2-normed L^∞-module. These are Banach spaces $(M, \|\cdot\|_M)$ which are also modules over the ring of $L^\infty(\mathrm{X},\mathfrak{m})$-functions and for which there is a map $|\cdot| : M \to L^2(\mathrm{X},\mathfrak{m})$, called *pointwise norm*, such that

$$|v| \geq 0 \qquad\qquad |fv| = |f|\,|v| \qquad\qquad \|v\|_M^2 = \int |v|^2\,\mathrm{d}\mathfrak{m}$$

for every $v \in M$, $f \in L^\infty$.

It turns out ([9], [11]) that there exists a unique, up to unique isomorphism, couple $(L^2(T^*\mathrm{X}),\mathrm{d})$ with $L^2(T^*\mathrm{X})$ being a module in the sense just described and $\mathrm{d} : W^{1,2}(\mathrm{X}) \to L^2(T^*\mathrm{X})$ linear and such that

i) $|\mathrm{d}f| = |\mathrm{D}f|$ $\mathfrak{m}$-a.e. for every $f \in W^{1,2}(\mathrm{X})$,
ii) L^∞-linear combinations of $\{\mathrm{d}f : f \in W^{1,2}(\mathrm{X})\}$ are dense in $L^2(T^*\mathrm{X})$.

Such $L^2(T^*\mathrm{X})$ will be called cotangent module and its elements L^2 1-forms, while d is called differential. By duality and in a quite natural way, one can introduce the tangent module $L^2(T\mathrm{X})$ and under reasonable rectifiability assumptions such tangent module is canonically isomorphic to the space of 'L^2-sections of the bundle obtained by considering pointed-Gromov-Hausdorff limits of rescaled spaces' ([12]).

In arbitrary metric measure spaces it seems hard to go beyond the first order theory, but on $\mathsf{RCD}(K,N)$ spaces a reasonably well-developed second-order calculus can be developed. Recall that, leaving aside some technicality, a $\mathsf{RCD}(K,N)$ space can be defined as a metric measure space such that $W^{1,2}(\mathrm{X})$ is Hilbert and the Bochner inequality

$$\Delta\frac{|\nabla f|^2}{2} \geq \frac{(\Delta f)^2}{N} + \langle \nabla f, \nabla\Delta f\rangle + K|\nabla f|^2 \tag{1}$$

holds in the appropriate weak sense ([2], [10], [3], [8], [4]). Starting from this definition and using the self-improvement properties of Bochner's inequality pioneered by Bakry ([5]) it is possible to show that it also holds

$$\Delta\frac{|X|^2}{2} \geq |\nabla X|^2_{\mathsf{HS}} - \langle X, (\Delta_{\mathrm{H}} X^\flat)^\sharp\rangle + K|X|^2 \tag{2}$$

again in the appropriate weak sense. Notice that for $X = \nabla f$, (2) reduces to (1) with the additional non-negative contribution $|\mathrm{Hess} f|^2_{\mathsf{HS}}$ on the right hand side. Here the language of L^2-normed modules provides natural spaces where objects like the Hessian or the covariant derivative belong, and one of the effects of the improved formula (2) is the bound

$$\int |\mathrm{Hess} f|^2_{\mathsf{HS}}\,\mathrm{d}\mathfrak{m} \leq \int (\Delta f)^2 - K|\nabla f|^2\,\mathrm{d}\mathfrak{m} \tag{3}$$

obtained integrating (2) for $X = \nabla f$. Since functions with gradient and Laplacian in L^2 are easy to build using the heat flow, (3) grants that there are 'many' functions with Hessian in L^2. Starting from this, it will not be hard to build a second order calculus and an indication of the novelty of the theory is in the fact that one can prove that the exterior differential is a closed operators on the space of k-forms for any $k \in \mathbb{N}$, whereas previously known results only covered the case $k = 0$ ([6], [19], [7]). In particular, quite natural versions of the de Rham cohomology and of the Hodge theorem can be provided. An example of link between this theory and the geometry of the space is the following result ([9], [13]), which generalizes a classical theorem of Bochner:

Theorem 1. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a* $\mathsf{RCD}(0,N)$ *space. Then the dimension of the first cohomology group is* $\leq N$. *If it is exactly* N, *then the space is a flat torus.*

In a different direction we remark that the Hessian operator, which is introduced by means reminiscent of Γ-calculus, describes second variations along geodesics in the following sense ([14]):

Theorem 2. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a compact* $\mathsf{RCD}(K,N)$ *space,* $N < \infty$, $f \in H^{2,2}(\mathrm{X})$ *and* (μ_t) *a* W_2*-geodesic made of measures with uniformly bounded densities. Then* *the map* $t \mapsto \int f \, \mathrm{d}\mu_t$ *is* $C^2([0,1])$ *and it holds*

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int f \, \mathrm{d}\mu_t{}_{t=0} = \int \mathrm{Hess}(f)(\nabla\varphi, \nabla\varphi) \, \mathrm{d}\mu_0,$$

where φ *is a Kantorovich potential from* μ_0 *to* μ_1.

Finally we notice that these calculus tools allow to define the *Ricci curvature* as:

$$\mathbf{Ric}(X, X) := \Delta \frac{|X|^2}{2} - |\nabla X|^2_{\mathsf{HS}} + \langle X, (\Delta_{\mathrm{H}} X^\flat)^\sharp \rangle.$$

It turns out that $\mathbf{Ric}(X, X)$ is a measure-valued tensor which controls the geometry of the space ([9], [15]) and behaves well under perturbations of the underlying space ([18], [16]).

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Uniqueness of mean curvature flow through (some) singularities

Or Hershkovits
(joint work with Brian White)

It is an old idea in geometric analysis, and PDEs in general, to separate the questions of existence and regularity; one is often led to defining a weak notion of solution, the existence of which can be shown by one set of ideas, while studying its properties may require different methods. In the study of mean curvature flow, one very useful notion of weak solution is that of the level set flow, introduced numerically in [7] and developed rigorously in [2, 1].

Given a closed set $X \subseteq \mathbb{R}^{n+1}$, its level set flow $t \in [0,\infty) \mapsto F_t(X)$ is a one-parameter family of closed sets starting at $F_0(X) = X$ and satisfying the avoidance principle: $F_t(X) \cap M(t) = \emptyset$, provided $t \in [a,b] \mapsto M(t)$ is a smooth mean curvature flow with $[a,b] \subseteq [0,\infty)$ and with $M(a) \cap F_a(X) = \emptyset$. Indeed, the level set flow is fully characterized as the maximal family of sets satisfying the two properties above [6, 5, 11].

Ideally, weak solutions should coincide with smooth solutions whenever the latter exist. In our case, if $t \in [0,T) \mapsto M(t)$ is a smooth mean curvature flow of closed, embedded hypersurfaces in $\mathbb{R}^{n+1}$, then $F_t(M_0) = M(t)$ for every $0 \le t < T$, as was shown in [2, 1].

Although in many regards the level set flow resembles mean curvature flow of smooth surfaces, it was observed already in the original paper [2] that if X is a smooth closed planar curve that crosses itself, then $F_t(X)$ will instantly develop an interior. In general, if the interior of $F_t(X)$ is empty for $t = 0$ and nonempty at some later time, we say that X fattens under the level set flow. Even if the initial hypersurface is smooth and embedded, fattening can occur after the surface becomes singular, as described in [10].

Although the level set flow is unique, the fattening phenomenon is related to non-uniqueness for other weak formulations of mean curvature flow. For example, let $M \subset \mathbb{R}^{n+1}$ be a smooth, closed hypersurface. Let U be the compact region it bounds. Then

$$\begin{aligned} &t \in [0,\infty) \mapsto M_{\text{outer}}(t) := \partial F_t(U),\\ &t \in [0,\infty) \mapsto M_{\text{inner}}(t) := \partial F_t(\overline{U^c}), \text{ and} \\ &t \in [0,\infty) \mapsto F_t(M) \end{aligned} \tag{1}$$

all may be regarded as weak versions of mean curvature flow starting from M. In particular, if the flow $M_{\text{inner}}(\cdot)$ or $M_{\text{outer}}(\cdot)$ is smooth in some region of spacetime, then it is indeed ordinary mean curvature flow in that region. If $F_t(M)$ has interior, then it differs from $M_{\text{inner}}(t)$ and $M_{\text{outer}}(t)$, since neither of those sets has interior.

One can also show, in this case, that $M_{\text{inner}}(t) \neq M_{\text{outer}}(t)$. Thus if M fattens, then $F_t(M)$, $M_{\text{inner}}(t)$ and $M_{\text{outer}}(t)$ are three distinct flows.

In light of the above, it is desirable to find conditions that prevent fattening. We have already mentioned that a smooth hypersurface cannot fatten until after singularities form. Short-time non-fattening for initial sets satisfying a Reifenberg condition with small Reifenberg parameter was established by the first author in [4] (see also [3] for the higher co-dimension surfaces). In that case, the flow immediately becomes smooth (though it may later develop singularities), and the non-fattening follows from short-time existence of smooth flows (with suitable estimates) serving as barriers to the level set flow. In the presence of singularities, two initial conditions are known to imply non-fattening for all time: the star-shapedness of M [8] and mean convexity of M [2]. (See also [9] for a more geometric proof that mean convex sets do not fatten.)

The facts that surfaces can fatten only after they become singular and that mean convex surfaces never fatten suggest the following conjecture:

> An evolving surface cannot fatten unless it has a singularity with no spacetime neighborhood in which the surface is mean convex.

According to the conjecture, to ensure nonfattening, we do not need mean convexity everywhere; it suffices to have it near the singularities.

A major difficulty in concluding non-fattening from local data around singulairiteis is that the effect of the fattening (presumably resulted by the singularity) is present globally instantaneously. This phenomenon is very much related to the strong maximum principle: If $V \subseteq V'$ are two compact domains with smooth,connected boundary with $V \neq V'$ then $F_t(\partial V) \cap F_t(\partial V') = \emptyset$ for every $t > 0$. Thus a key challenge in attacking the problem is to unite the information from the local mean convex structure and the local regular structure.

In this talk, I presented a precise formulation of this conjecture, as well as its proof.

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Min-max theory for constant mean curvature (CMC) hypersurfaces

Xin Zhou
(joint work with Jonathan Zhu)

The mean curvature of a hypersurface Σ^n in a Riemannian manifold M^{n+1} measures the changing rate of the area functional. Σ has constant mean curvature c if and only if it is a critical point of the Lagrange-multiplier functional

$$\mathcal{A}^c = \text{Area} - c\,\text{Vol}\,. \tag{1}$$

CMC hypersurfaces is a classical topic in differential geometry, and play an essential role in many areas, e.g. isoperimetric problems [24], the modeling of interface phenomena [14] and general relativity [11, 4]. Many attempts have been made to construct more CMC hypersurfaces, especially with *prescribed* constant mean curvature, c.f. [9, 10, 29, 30, 12, 31, 7, 22, 26]. However, these works left wide open the question of which values may be prescribed - that is, for which constants c does there exist a closed hypersurface of constant mean curvature c?

We construct, via a min-max approach, nontrivial closed CMC hypersurfaces of *any* prescribed mean curvature, in any smooth closed Riemannian manifold M^{n+1} of dimension at most seven.

Theorem 1. *Let M^{n+1} be a smooth, closed Riemannian manifold of dimension $3 \leq n+1 \leq 7$. Given any $c \in \mathbb{R}$, there exists a nontrivial, smooth, closed, almost embedded hypersurface Σ^n of constant mean curvature c.*

An immersed hypersurface Σ is said to be *almost embedded* if Σ has local decompositions into smoothly embedded components that (pairwise) lie to one side of each other. That is, the sheets may touch but not cross. Almost embedded hypersurfaces are automatically Alexandrov embedded.

We want to compare our result with a classical problem by Arnold [3, page 395] and Novikov [21, Section 5] on the periodic orbits of a charged particle in a magnetic field on a topological two sphere. It is conjectured that there exist closed embedded curves of any prescribed constant geodesic curvature. This conjecture remains open, and we refer to [8, 25] for more backgrounds and some partial results of this conjecture. Our result can be viewed as a complete resolution of the higher dimensional analog of Arnold-Novikov conjecture.

When $c \neq 0$, we can prove that our min-max procedure converges to the constructed hypersurface Σ with multiplicity 1. This is a stark contrast to the minimal ($c = 0$) case, for which the min-max multiplicity 1 conjecture is a fundamental open problem [16].

The existence problem for CMC hypersurfaces has been studied from a number of perspectives. The boundary value problems were substantially developed by Heinz [9], Hildebrandt [10], Struwe [29] using the mapping method, and by Duzaar-Steffen [7] using geometric measure theory, while both approaches can only produce CMC hypersurfaces with some upper bound on the mean curvature. For the case of closed CMC hypersurfaces, the more classical approach is to solve the isoperimetric problem for a given volume. Indeed, for each fixed volume there exists a smooth isoperimetric hypersurface (up to a singular set of codimension 7, c.f. [20]). However, this approach does not yield any control on the value of the mean curvature.

Perturbative methods consist of another class of approaches. One may deform a closed minimal hypersurface to a CMC hypersurface, but only for very small values of the mean curvature. On the other end, there are many attempts to construct foliations by CMC hypersurfaces near minimal submanifolds of strictly lower dimension, c.f. Ye [31], and others [22]. The hypersurfaces produced by this approach necessarily have large mean curvatures, which in fact diverge as the hypersurfaces condense onto the minimal submanifold.

We also mention the gluing procedures by Kapouleas [12] as well as the degree theory developed by Rosenberg-Smith [26]. These provide important examples of CMC hypersurfaces, but the former method is typically restricted by the availability of known solutions, whilst the latter can only produce CMC hypersurfaces of fairly large mean curvature. Finally, we remark that Meeks-Mira-Perez-Ros [18, 19] were able to determine, in the special case of homogeneous ambient 3-manifolds, precisely the values for which there exists a CMC 2-sphere with the specified mean curvature.

Our approach to prove Theorem 1 directly uses the $\mathcal{A}^c$-functional from a variational point of view. It is easy to see that the minimization method does not succeed in detecting a nontrivial critical point for the $\mathcal{A}^c$-functional. In fact, the minimizer of $\mathcal{A}^c$ among domains Ω in M with smooth boundary is always the total manifold M, as $\mathcal{A}^c(M) = -c\,\mathrm{Vol}(M) \leq \mathcal{A}^c(\Omega)$. Therefore, the min-max method becomes the natural way to find nontrivial critical points of $\mathcal{A}^c$.

For finding critical points of the area functional - that is, minimal hypersurfaces - the min-max method has been greatly successful. In [1], Almgren initiated a celebrated program to develop a variational theory for minimal submanifolds in Riemannian manifolds of any dimension and co-dimension using geometric measure theory, namely the min-max theory for minimal submanifolds. He was able to prove the existence of a nontrivial weak solution as stationary integral varifolds [2]. Higher regularity was established in the co-dimension-one case by the seminal work of Pitts [23] (for $2 \leq n \leq 5$) and later extended by Schoen-Simon [37] (for $n \geq 6$). Colding-De Lellis [5] established the corresponding theory using smooth sweepouts based on ideas of Simon-Smith [28]. Indeed, the preceeding body of work completely resolved the $c = 0$ case of Theorem 1.

Very recently, Marques-Neves [15, 17] found surprising applications of the Almgren-Pitts min-max theory to solve a number of longstanding open problems in geometry, including their celebrated proof of the Willmore conjecture. Due to these tremendous successes, there have been a vast number of developments of this program in various contexts, including [6, 16, 13]. In this regard, our work represents a natural extension of the min-max method to the CMC setting.

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Willmore minimizers with prescribed isoperimetric ratio

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(joint work with Yuxiang Li)

For smooth immersed surfaces $f : \mathbb{S}^2 \to \mathbb{R}^3$ we consider the problem of minimizing the Willmore energy $\mathcal{W}(f)$ with prescribed isoperimetric ratio $I(f) = \sigma \in (0,1]$. Here

$$\mathcal{W}(f) = \frac{1}{4}\int_{\mathbb{S}^2} H^2\, d\mu_f, \tag{1}$$

$$I(f) = \sqrt{36\pi}\,\frac{\mathcal{V}(f)}{\mathcal{A}(f)^{3/2}}, \tag{2}$$

where H denotes the mean curvature, μ_f the induced surface measure, $\mathcal{A}(f)$ the area and $\mathcal{V}(f)$ the enclosed volume. Our normalization is such that $\mathcal{W}(\mathbb{S}^2) = 4\pi$ and $I(\mathbb{S}^2) = 1$ for the round sphere. We put

$$\beta(\sigma) = \inf_{f:\mathbb{S}^2\to\mathbb{R}^3,\, I(f)=\sigma} \mathcal{W}(f). \tag{3}$$

In [6] Schygulla proved that the infimum is indeed attained by a smooth embedding of the sphere, for any given ratio $\sigma \in (0,1]$. Moreover he showed that the infimum $\beta(\sigma)$ is strictly decreasing with $\beta(\sigma) \to 8\pi$ as $\sigma \searrow 0$. For surfaces of higher genus an existence result was obtained more recently by Keller, Mondino and Rivière, assuming certain inequalities for the infimum of the energy [3]. The minimizers solve the Euler Lagrange equation

$$\frac{1}{2}\big(\Delta_g H + |A^\circ|^2 H\big) = \Lambda\, \sigma(f)\, \Big(\frac{1}{\mathcal{V}(f)} + \frac{3}{2\mu(f)} H\Big), \tag{4}$$

where $\Lambda \in \mathbb{R}$ is a Lagrange multiplier.

In his paper Schygulla also studies the limit for a sequence of minimizers with isoperimetric ratio converging to zero. He shows that up to translations and dilations, the sequence converges in the varifold sense to a round sphere of multiplicity two [6, Thm. 2]. In our recent work with Yuxiang Li [5] we study this singular limit more precisely, obtaining the following asymptotic results. Here we denote by N, S the north and south pole of $\mathbb{S}^2$ and by π_N, π_S the stereographic projections from the two poles.

Theorem. *Let $f_k : \mathbb{S}^2 \to \mathbb{R}^3$ be conformally parametrized $\mathcal{W}$-minimizers for prescribed isoperimetric ratio $I(f_k) = \sigma_k \to 0$. After conformal reparametrization, scaling and translating, and passing to a subsequence, the following hold:*

(1) $\mathcal{A}(f_k) = 1$,

(2) *f_k converges locally smoothly on $\mathbb{S}^2 \backslash \{N\}$ to a conformal immersion $f^0 : \mathbb{S}^2 \to \mathbb{R}^3$ with $f^0(N) = 0$.*

(3) *There exist $r_k \to 0$ such that the sequence $f_k^1(z) = f_k \circ \pi_S^{-1}(r_k z)$ converges locally smoothly on $\mathbb{R}^2$ to a conformal immersion $f^1 : \mathbb{R}^2 \to \mathbb{R}^3$ with $f^1(\infty) = 0$.*

(4) *Both f^0, f^1 are conformal equivalences to the same round sphere $\mathbb{S}$ of area $1/2$, with opposite orientation.*

(5) *There exist t_k, $\lambda_k \to 0$, such that the sequence $f_k^2(z) = \lambda_k^{-1} f_k \circ \pi_S^{-1}(t_k z)$ converges locally smoothly on $\mathbb{R}^2 \backslash \{0\}$ to a conformally parametrized catenoid $f^2 : \mathbb{R}^2 \backslash \{0\} \to \mathbb{R}^3$, with the origin and the center of $\mathbb{S}$ on the symmetry axis.*

(6) *We have the energy identity*

$$\lim_{k \to +\infty} \int_{\mathbb{S}^2} |A_{f_k}|^2 \, d\mu_{f_k} = \sum_{i=0,1,2} \int |A_{f^i}|^2 \, d\mu_{f^i}$$

(7) *We have $\lim_{k\to\infty} \big(\log t_k \ : \ \log \lambda_k \ : \ \log r_k\big) = (1 : 1 : 2)$, and $\lim_{k\to\infty} \Lambda_k/\lambda_k \in (0,\infty)$ exists.*

The geometric problem which is addressed here is partially motivated by a model for cell membranes due to Helfrich [2]. In the axially symmetric case the Helfrich minimizers are described by ordinary differential equations, which were studied by various authors using numerical approximations, see e.g. [1]. In the case of topological spheres the Helfrich energy reduces to

$$E_C(f) = \frac{1}{4}\int_{\mathbb{S}^2} (H - C)^2 \, d\mu_g,$$

where $C \in \mathbb{R}$ is a parameter called the spontaneous curvature. The literature addresses the minimization of $E_C(f)$ with both area and enclosed volume prescribed. The special case $C = 0$ corresponds exactly to our variational problem, since the two constraints are reduced to the isoperimetric ratio by scaling. In [1] Berndl, Lipowsky and Seifert distinguish three kinds of shapes depending on the isoperimetric ratio σ, which they call the reduced volume: the prolate-dumbbell, the oblate-discocyte and the stomatocyte type. The stomatocyte parameter range is indicated as $0 < \sigma \leq 0.591$, and it is noted that a neck develops as $\sigma \to 0$. Here we provide a rigorous analysis of this neck formation, without assuming axial symmetry.

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Negative lower curvature bounds under Ricci flow

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(joint work with Richard H. Bamler, Burkhard Wilking)

We generalize most of the known Ricci flow invariant non-negative curvature conditions to less restrictive negative bounds that remain sufficiently controlled for a short time. As an illustration of the contents of [1], we prove that metrics whose curvature operator has eigenvalues greater than -1 can be evolved by the Ricci flow for some uniform time such that the eigenvalues of the curvature operator remain greater than $-C$. Here the time of existence and the constant C only depend on the dimension and the degree of non-collapsedness.

More precisely, we denote by Rm_g the curvature operator of a Riemannian n-manifold (M, g) and I represents the curvature operator of the unit round n-sphere. Notice that the bound $\mathrm{Rm}_g \geq -\varepsilon$ can be rephrased by saying that the linear combination $\mathrm{Rm}_g + \varepsilon\,\mathrm{I}$ is non-negative definite. Hereafter we denote curvature conditions by $\mathcal{C}$ and we write $\mathrm{Rm}_g \in \mathcal{C}$ to indicate that Rm_g satisfies the corresponding curvature condition. Using this, we can state our main result:

Theorem 1. *Given $n \in \mathbb{N}$ and a constant $v_0 > 0$, there exist positive constants $C = C(n, v_0) > 0$ and $\tau = \tau(n, v_0) > 0$ such that the following holds. Let (M^n, g) be a complete Riemannian manifold with bounded curvature and consider one of the following curvature conditions $\mathcal{C}$:*

(1) *non-negative curvature operator,*
(2) *2-non-negative curvature operator*
(i.e. the sum of the lowest two eigenvalues is non-negative),
(3) *non-negative complex sectional curvature*
(i.e. weakly PIC_2, meaning that taking the cartesian product with $\mathbb{R}^2$ produces a non-negative isotropic curvature operator),
(4) *weakly PIC_1 (i.e. the cartesian product with $\mathbb{R}$ produces a non-negative isotropic curvature operator),*
(5) *non-negative bisectional curvature, in the case in which (M, g) is Kähler with respect to some complex structure J.*

Assume that

$$\mathrm{vol}_g\big(B_g(p,1)\big) \geq v_0 \quad \text{for all } p \in M \qquad \text{and} \quad \mathrm{Rm}_g + \varepsilon \mathrm{I} \in \mathcal{C},$$

for some $\varepsilon \in [0, 1]$. Then the Ricci flow $g(t)$ with initial metric g exists until time τ, is Kähler if (M, g) is Kähler, and we have the curvature bounds

$$\mathrm{Rm}_{g(t)} + C\varepsilon\,\mathrm{I} \in \mathcal{C} \quad \text{and} \quad |\mathrm{Rm}_{g(t)}| \leq \frac{C}{t} \quad \text{for all } t \in (0, \tau].$$

Finally, let us explain the main idea of the proof of Theorem 1 for the case (1), the other cases will follow similarly, modulo some technical details. Denote by ℓ the negative part of the smallest eigenvalue of Rm. Our hypotheses guarantee that $\ell \leq 1$ at time 0. By standard formulas, ℓ roughly satisfies an evolution inequality of the form

$$\partial_t \ell \leq \Delta\ell + C_1 \mathrm{scal} \cdot \ell + C_2 \ell^2. \tag{1}$$

Traditionally, the invariance of a curvature condition is reduced to a pointwise invariance property via a maximum principle (ODE-PDE comparison). Unfortunately, this strategy only works for specific curvature conditions. Indeed, in dimensions $n \geq 3$, the bound $\mathrm{Rm} \geq \varepsilon$ satisfies this pointwise invariance only if $\varepsilon \geq 0$. This is why we cannot assert strict invariance of the lower bound on Rm.

The degree to which this invariance fails at each point is measured by the reaction term $C_1 \mathrm{scal} \cdot \ell + C_2 \ell^2$ in (1). Our goal will be to show that this failure is compensated by the diffusion of (1). In other words, we will bound the influence of the reaction term on ℓ in an integral sense. For example, if we consider the evolution of the integral of ℓ, then we obtain

$$\begin{aligned}(2)\quad \frac{d}{dt}\int_M \ell d\mu_t &\leq \int_M \big(\Delta\ell + C_1 \mathrm{scal}\cdot\ell + C_2\ell^2 - \ell\cdot\mathrm{scal}\big)d\mu_t \\ &= (C_1-1)\int_M \mathrm{scal}\cdot\ell d\mu_t + C_2\int_M \ell^2 d\mu_t.\end{aligned}$$

A crucial step in our proof will be to show that we can choose $C_1 = 1$ in (1), which implies that the first term on the right-hand side of (2) vanishes. So as long as ℓ remains bounded, its integral cannot grow too fast. We generalize this principle and derive a Gaussian estimate for the heat kernel of the linearization of (1) under certain a priori assumptions. This estimate will enable us to derive *pointwise* estimates for ℓ by localizing (2). The theorem above will then follow via a continuity argument.

As a first application we show that volume non-collapsed closed manifolds that satisfy certain almost non-negative curvature conditions also admit metrics that satisfy the corresponding strict condition.

Corollary 2. *Given $n \in \mathbb{N}$ and positive constants D, v_0, there exists a constant $\varepsilon = \varepsilon(n, v_0, D) > 0$ such that the following holds.*

Let $\mathcal{C}$ be one of the curvature conditions listed in items $(1)-(5)$ of Theorem 1. Then any closed Riemannian manifold (M^n, g) with

$$\mathrm{diam}_g(M) \leq D, \qquad \mathrm{vol}_g(M) \geq v_0 \quad \text{and} \qquad \mathrm{Rm}_g + \varepsilon\,\mathrm{I} \in \mathcal{C}$$

also admits a metric whose curvature operator lies in $\mathcal{C}$.

In [4] J. Lott asks whether each simply connected manifold with almost non-negative curvature operator is diffeomorphic to a torus bundle over a compact symmetric space. Corollary 2 gives an affirmative answer to Lott's question in the non-collapsed case. It will be clear from the proof that the metric whose existence is asserted in Corollary 2 is close to the original metric g in the Gromov-Hausdorff sense. This motivates the following smoothing result for singular limit spaces of sequences of manifolds with lower curvature bounds.

Corollary 3. *Let $\mathcal{C}$ be as in Corollary 2 and (X, d_X) be the Gromov-Hausdorff limit of a sequence $\{(M_i, g_i)\}_{i=1}^{\infty}$ of closed Riemannian manifolds satisfying*

$$\mathrm{vol}_{g_i}(M_i) \geq v_0, \qquad \mathrm{Rm}_{g_i} + \varepsilon_i\,\mathrm{I} \in \mathcal{C}, \qquad \mathrm{diam}_{g_i}(M_i) \leq D.$$

for some sequence $\{\varepsilon_i\} \subset (0,1]$ with $\varepsilon_i \to \varepsilon_\infty$, as $i \to \infty$. Then there exists $\tau = \tau(n, v_0) > 0$, a smooth manifold M_∞ and a smooth solution to the Ricci flow $(M_\infty, g_\infty(t))_{t\in(0,\tau)}$ which satisfies $\mathrm{Rm}_{g_\infty(t)} + \varepsilon_\infty \mathrm{I} \in \mathcal{C}$ and is coming out of the (possibly singular) space (X, d_X) in the sense that

$$\lim_{t\searrow 0} d_{GH}\big((X, d_X), (M_\infty, d_{g_\infty(t)})\big) = 0.$$

In particular, for $\varepsilon_\infty = 0$ the limiting $g_\infty(t)$ satisfies the corresponding non-negative curvature condition $\mathcal{C}$ for all $t \in (0, \tau)$.

Moreover, for any choice of ε_∞, the space X is homeomorphic to the manifold M_∞ and the Riemannian distance $d_{g_\infty(t)}$ converges uniformly to a distance function d_0 on M_∞ as $t \searrow 0$ such that (M_∞, d_0) is isometric to (X, d_X).

By taking convergent sequences of manifolds as above one can generate a large variety of singular spaces that can be smoothed out by the Ricci flow with lower curvature bound. In the case (5), Corollary 3 implies a statement that is similar to a result of Gang Liu (cf. [3]) on the structure of limits of spaces whose bisectional curvature is uniformly bounded from below.

In dimension 3, Theorem 1 and Corollaries 2 and 3 were established by Simon in [5, 6] for the case of almost non-negative and 2-non-negative curvature operator, which in dimension 3 is equivalent to almost non-negative sectional and Ricci curvature, respectively.

We finish by highlighting that in [1] we additionally establish a local version of Theorem 1 in the case of non-negative curvature operator and non-negative complex sectional curvature. By applying this local result to a sequence of larger and larger balls, we obtain a short-time existence result on complete manifolds with possibly unbounded curvature, which generalizes the existence result in [2].

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Isoperimetric inequalities for eigenvalues on Surfaces

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(joint work with Alexei Penskoi)

We discuss some sharp inequalities for the highest eigenvalues on two-dimensional Riemannian manifolds for the Laplacian.

Quantitative Isoperimetry à la Levy-Gromov

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(joint work with Francesco Maggi, Andrea Mondino)

Comparison theorems are an important part of Riemannian Geometry. The typical result asserts that a complete Riemannian manifold with a pointwise curvature bound retains some metric properties of the corresponding simply connected model space. We are interested here in the *Levy-Gromov comparison Theorem*, stating that, under a positive lower bound on the Ricci tensor, the isoperimetric profile of the manifold is bounded from below by the isoperimetric profile of the sphere. More precisely, define the isoperimetric profile of a smooth Riemannian manifold (M,g) by

$$\mathcal{I}_{(M,g)}(v) = \inf\left\{\frac{\mathsf{P}(E)}{\mathrm{vol}_g(M)} : \frac{\mathrm{vol}_g(E)}{\mathrm{vol}_g(M)} = v\right\} \qquad 0 < v < 1\,,$$

where $\mathsf{P}(E)$ denotes the perimeter of a region $E \subset M$. The Levy-Gromov comparison Theorem states that, if $\mathrm{Ric}_g \geq (N-1)g$, where N is the dimension of (M,g), then

$$\mathcal{I}_{(M,g)}(v) \geq \mathcal{I}_{(\mathbb{S}^N, g_{\mathbb{S}^N})}(v) \qquad \forall v \in (0,1)\,, \tag{1}$$

where $g_{\mathbb{S}^N}$ is the round metric on $\mathbb{S}^N$ with unit sectional curvature; moreover, if equality holds in (1) for some $v \in (0,1)$, then $(M,g) \simeq (\mathbb{S}^N, g_{\mathbb{S}^N})$.

Our main result is a quantitative estimate, in terms of the gap in the Levy-Gromov inequality, on the shape of isoperimetric sets in (M,g). We show that isoperimetric sets are close to geodesic balls. Since the classes of isoperimetric sets and geodesic balls *coincide* in the model space $(\mathbb{S}^N, g_{\mathbb{S}^N})$, one can see of our main result as a *quantitative comparison theorem*. In detail, we show that if $\mathrm{Ric}_g \geq (N-1)g$ and $E \subset M$ is an isoperimetric set in M with $\mathrm{vol}_g(E) = v\,\mathrm{vol}_g(M)$, then there exists $x \in M$ such that

$$\frac{\mathrm{vol}_g\big(E \Delta B_{r_N(v)}(x)\big)}{\mathrm{vol}_g(M)} \leq C(N,v)\left(\mathcal{I}_{(M,g)}(v) - \mathcal{I}_{(\mathbb{S}^N, g_{\mathbb{S}^N})}(v)\right)^{\mathrm{O}(1/N)} \tag{2}$$

where $B_r(x)$ denotes the geodesic ball in (M,g) with radius r and center x, and where $r_N(v)$ is the radius of a geodesic ball in $\mathbb{S}^N$ with volume $v\,\mathrm{vol}_{g_{\mathbb{S}^N}}(\mathbb{S}^N)$. More generally the same conclusion holds for every $E \subset M$ with $\mathrm{vol}_g(E) = v\,\mathrm{vol}_g(M)$, provided $\mathcal{I}_{(M,g)}(v)$ on the right-hand side of (2) is replaced by $\mathsf{P}(E)/\mathrm{vol}_g(M)$. In the course of proving (2), we improve on another basic comparison result, namely, *Meyer's Theorem*: if $\mathrm{Ric}_g \geq (N-1)g$, then $\mathrm{diam}(M) \leq \pi$. Indeed, we prove that

$$\pi - \mathrm{diam}(M) \leq \inf_{v \in (0,1)} C(N,v)\Big(\mathcal{I}_{(M,g)}(v) - \mathcal{I}_{(\mathbb{S}^N, g_{\mathbb{S}^N})}(v)\Big)^{1/N}. \tag{3}$$

We approach the proof of (2) and (3) from the synthetic point of view of metric geometry. We regard an N-dimensional Riemannian manifold (M,g) with $\mathrm{Ric}_g \geq (N-1)g$ as a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ satisfying the curvature-dimension condition $\mathsf{CD}(N-1,N)$ of Sturm [3, 4] and Lott–Villani [2] and we actually obtain

(2) in this larger class. It is worth to underline that indeed in [1] the Levy-Gromov comparison Theorem (1) has been proved to hold for on essentially non-branching metric measure spaces verifying the $\mathsf{CD}(N-1,N)$ condition with *any real number* $N>1$.

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Optimal isoperimetric inequalities for surfaces in Cartan-Hadamard manifolds via mean curvature flow

Felix Schulze

The classical isoperimetric inequality in Euclidean space states that for a bounded open set $\Omega\subset\mathbb{R}^{n+1}$ with sufficiently regular boundary, the estimate

$$|\Omega| \le n^{-\frac{n+1}{n}}\omega_{n+1}^{-\frac{1}{n}}|\partial\Omega|^{\frac{n+1}{n}} \tag{1}$$

holds. Here $|\Omega|$ and $|\partial\Omega|$ denote the areas in dimension $n+1$ and n respectively. ω_{n+1} is the measure of the $n+1$-dimensional unit ball. Equality is attained if and only if Ω is a ball.

This was extended to higher codimension by Almgren; loosely formulated as follows:

Theorem 1 (Almgren, [1]**)**: *Corresponding to each m-dimensional closed surface T in $\mathbb{R}^{n+1}$ there is an $(m+1)$-dimensional surface Q having T as boundary such that*

$$|Q| \le \gamma_{m+1}|T|^{\frac{m+1}{m}}$$

with equality if and only if T is a standard round m sphere (of some radius) and Q is the corresponding flat disk.

Again $|Q|$ and $|T|$ denote the areas in dimensions $m+1$ and m respectively, and the constant γ_{m+1} is defined via the required equality.

The proof of this inequality is based on the following area-mean curvature characterisation of standard spheres by Almgren:

Theorem 2 (Almgren, [1]**)**: *Let V be a (sufficiently regular) m-dimensional surface in $\mathbb{R}^{n+1}$ without boundary. Then the following estimate holds*

$$m^m|S_1^m| \le (\sup_V|\vec{H}|)^m|V|\,,$$

where $\vec{H}$ is the mean curvature vector of V and S_1^m the standard unit round m-dimensional sphere. Furthermore, equality holds if and only if V is a standard *round m-dimensional sphere (of some radius).*

It is a natural question to ask in which Riemannian manifolds (M^{n+1}, g) the inequality (1) is still true. We will say that (M^{n+1}, g) is Cartan-Hadamard if it is smooth, complete, simply-connected with non-positive sectional curvatures.

Conjecture (Aubin, Burago-Zalgaller, Gromov-Lafontaine-Pansu): *Let (M^{n+1}, g) be Cartan-Hadamard. Then* (1) *holds.*

The conjecture was proven for $n+1=2$ by Weil [7] in '26, for $n+1=3$ by Kleiner [4] in '92 and for $n+1=4$ by Croke [2] in '84.

In the proof of Kleiner an analogue of Theorem 2 above was used, based on the Gauss-Bonnet formula. Following an idea of Simon [6], using the monotonicity formula, this estimate extends to surfaces of low regularity in Cartan-Hadamard manifolds and to any codimension:

Theorem 3 (Schulze '08/'17, [5]**)**: *Let (M^n, g) be Cartan-Hadamard, $n \geq 3$, and $\Sigma^2 \subset M$ be a compact integer 2-rectifiable varifold with weak mean curvature $\mathbf{H} \in L^2(\mu)$, where μ is the area measure (with multiplicities) on Σ. Then*

$$\int |\mathbf{H}|^2 \, d\mu \geq 16\pi \ . \tag{2}$$

We used this estimate in [5] to give an alternative proof of Kleiner's result. In the present work, we demonstrate that this extends to any codimension:

Theorem 4 (Schulze '17): *Let (M^n, g) be Cartan-Hadamard, $n \geq 3$, and $\Sigma^2 \subset M$ be a smooth, closed, orientable, immersed surface. Let S be an area minimising integer rectifiable 3-current such that $\partial S = \Sigma$. Then*

$$|S| \leq \frac{1}{6\sqrt{\pi}} |\Sigma|^{\frac{3}{2}} \ .$$

The proof, as in [5], is based on the monotonicity of the following isoperimetric difference along a mean curvature flow, starting from Σ. We consider $(\Sigma_t)_{0\leq t<\varepsilon}$ the smooth evolution of $\Sigma_0 = \Sigma$ by mean curvature flow. Assume that there exists a smooth family of area-minimising 3-surfaces $(S_t)_{0\leq t<\varepsilon}$ such that

$$\partial S_t = \Sigma_t \ .$$

Consider the isoperimetric difference

$$I_t = |\Sigma_t|^{\frac{3}{2}} - 6\sqrt{\pi}|S_t| \ .$$

We can estimate, using (2):

$$\begin{aligned}
-\frac{d}{dt}|S_t| &= -\int_{S_t} \mathrm{div}_{S_t}(X)\, d\mathcal{H}^3 = \int_{\Sigma_t} \langle \vec{H}, \vec{n}\rangle\, d\mathcal{H}^2 \le \int_{\Sigma_t} |\vec{H}|\, d\mathcal{H}^2 \\
&\le \left(\int_{\Sigma_t} |\vec{H}|^2\, d\mathcal{H}^2\right)^{1/2} |\Sigma_t|^{1/2} \cdot \frac{1}{4\sqrt{\pi}} \left(\int_{\Sigma_t} |\vec{H}|^2\, d\mathcal{H}^2\right)^{1/2} \\
&\le \frac{1}{4\sqrt{\pi}}|\Sigma_t|^{1/2} \int_{\Sigma_t} |\vec{H}|^2\, d\mathcal{H}^2 = -\frac{1}{6\sqrt{\pi}}\frac{d}{dt}|\Sigma_t|^{3/2}\,,
\end{aligned}$$

where X is the variation vectorfield along the family $(S_t)_{0\le t<\varepsilon}$ and $\vec{n}$ the unit outer conormal of S_t along Σ_t . This implies $\frac{d}{dt}I_t \le 0$. If one would have that for a maximal time $T > 0$ it holds that $\lim_{t\to T}|S_t| = 0$, then this would imply $I_0 \ge 0$, which is the desired isoperimetric inequality.

Note that this computation is only valid as long as the flow and the family $(S_t)_{0\le t<\varepsilon}$ remains smooth. To overcome this difficulty, we work with a weak Brakke flow solution, starting from Σ, constructed via the elliptic regularisation scheme of Ilmanen [3]. This solution is defined past singularities and one can show that it exists only up to a finite time $T > 0$ where the spanning area goes to zero. To extend the above monotonicity calculation to this weak setting, we work with the approximating flows of the elliptic regularisation scheme in one dimension higher and an almost monotonicity of a suitable approximate isoperimetric difference.

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On the uniqueness of minimisers of Ginzburg-Landau functionals

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(joint work with R. Ignat, V. Slastikov, A. Zarnescu)

Abstract. We provide necessary and sufficient conditions for the uniqueness of minimisers of the Ginzburg-Landau functional for $\mathbb{R}^n$-valued maps under a suitable convexity assumption on the potential and for $H^{1/2} \cap L^\infty$ boundary data that is non-negative in a fixed direction $e \in \mathbb{S}^{n-1}$. Furthermore, we show that, when minimisers are not unique, the set of minimisers is generated from any of its elements using appropriate orthogonal transformations of $\mathbb{R}^n$. We also prove corresponding results for harmonic maps.

We consider the following Ginzburg-Landau type energy functional

$$E_\varepsilon(u) = \int_\Omega \Big[\frac{1}{2}|\nabla u|^2 + \frac{1}{2\varepsilon^2} W(1-|u|^2)\Big]\, dx, \tag{1}$$

where $\varepsilon > 0$, $\Omega \subset \mathbb{R}^m$ $(m \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$ and the potential $W \in C^1((-\infty, 1], \mathbb{R})$ satisfies

$$W(0) = 0,\ W(t) > 0 \text{ for all } t \in (-\infty, 1] \setminus \{0\},\ W \text{ is strictly convex.} \tag{2}$$

We investigate the minimisers of the energy E_ε over the following set

$$\mathcal{A} := \{u \in H^1(\Omega; \mathbb{R}^n) : u = u_{bd} \text{ on } \partial\Omega\}, \quad n \geq 1, \tag{3}$$

consisting of H^1 maps with a given boundary data $u_{bd} \in H^{1/2} \cap L^\infty(\partial\Omega; \mathbb{R}^n)$.

There exists a large literature using various methods that addresses the question of uniqueness of minimisers in the framework of Ginzburg-Landau type models and the related harmonic map problem. See e.g. Béthuel, Brezis and Hélein [1], Mironescu [6], Pacard and Rivière [7], Ye and Zhou [10], Farina and Mironescu [2], Millot and Pisante [5], Jäger and Kaul [3, 4], Sandier and Shafrir [8, 9] and the references therein.

We begin with a simple result on the uniqueness and symmetry of minimisers, which will be subsequently extended in a much more general setting. Assume $\Omega \subset \mathbb{R}^2$ is the unit disk, $u : \Omega \to \mathbb{R}^3$, and the boundary data carries a given winding number $k \in \mathbb{Z} \setminus \{0\}$ on $\partial\Omega$, namely[1]

$$u_{bd}(\cos\varphi, \sin\varphi) = (\cos(k\varphi), \sin(k\varphi), 0) \in \mathbb{S}^1 \times \{0\} \subset \mathbb{R}^3, \quad \forall \varphi \in [0, 2\pi). \tag{4}$$

By restricting E_ε to a subset of $\mathcal{A}$ consisting of suitable rotationally symmetric maps, it is not hard to see that E_ε admits critical points of the form

$$u_\varepsilon(r\cos\varphi, r\sin\varphi) := f_\varepsilon(r)(\cos(k\varphi), \sin(k\varphi), 0) \pm g_\varepsilon(r)(0,0,1), \tag{5}$$

where $r \in (0,1)$, $\varphi \in [0, 2\pi)$.

[1] We note that u_{bd} is non-negative in e_3-direction.

Theorem 1. *Let $\Omega = \{x \in \mathbb{R}^2 : |x| < 1\}$ be the unit disk in $\mathbb{R}^2$, $u : \Omega \to \mathbb{R}^3$, $W \in C^1((-\infty, 1], \mathbb{R})$ satisfy (2), and boundary data u_{bd} be given by (4) where $k \in \mathbb{Z} \setminus \{0\}$ is a given integer. Then we have for every $\varepsilon > 0$:*

(1) *Any minimiser u_ε of E_ε in the set $\mathcal{A}$ (with $n = 3$) has the representation (5) where $f_\varepsilon > 0$, $g_\varepsilon \geq 0$ in $(0, 1)$.*
(2) *In the set of critical points of E_ε in $\mathcal{A}$ and of the form (5), there is at most one critical point satisfying $g_\varepsilon > 0$ and there is exactly one critical point satisfying $\tilde{f}_\varepsilon > 0$, $\tilde{g}_\varepsilon \equiv 0$.*
(3) *If a critical point of E_ε of the form (5) and satisfying $g_\varepsilon > 0$ exists, then E_ε has exactly two minimisers that are given by $(f_\varepsilon, \pm g_\varepsilon)$ via (5).*

Let us now return to the general case. One main restriction in our treatment is the assumption that the boundary data is non-negative in a (fixed) direction $e \in \mathbb{S}^{n-1}$:

$$u_{bd} \cdot e \geq 0 \quad \mathcal{H}^{m-1}\text{-a.e. in } \partial\Omega. \tag{6}$$

For any integrable $\mathbb{R}^n$-valued map u on a measurable set ω we denote the essential image of u on ω by $u(\omega) = \{u(x) : x \in \omega \text{ is a Lebesgue point of } u\}$.

The following result establishes the minimising property and uniqueness (up to certain isometries) for *critical points* of E_ε that are *positive* in a fixed direction $e \in \mathbb{S}^{n-1}$ inside the domain $\Omega \subset \mathbb{R}^m$.

Theorem 2. *Let $m, n \geq 1$, $\Omega \subset \mathbb{R}^m$ be a bounded domain with smooth boundary, potential $W \in C^1((-\infty, 1], \mathbb{R})$ satisfying (2), and a boundary data $u_{bd} \in H^{1/2} \cap L^\infty(\partial\Omega; \mathbb{R}^n)$ satisfying (6) in a fixed direction $e \in \mathbb{S}^{n-1}$. Fix any $\varepsilon > 0$ and let $u_\varepsilon \in H^1 \cap L^\infty(\Omega, \mathbb{R}^n)$ be a critical point of E_ε in $\mathcal{A}$ such that*

$$u_\varepsilon \cdot e > 0 \ \text{a.e. in } \Omega. \tag{7}$$

Then u_ε is a minimiser of E_ε in $\mathcal{A}$ and we have the following dichotomy:

(1) *If there exists a Lebesgue point $x_0 \in \partial\Omega$ of u_{bd} such that $u_{bd}(x_0) \cdot e > 0$ then u_ε is the unique minimiser of E_ε in the set $\mathcal{A}$.*
(2) *If $u_{bd}(x) \cdot e = 0$ for $\mathcal{H}^{m-1}$-a.e. $x \in \partial\Omega$, then all minimisers of E_ε in $\mathcal{A}$ are given by Ru_ε where $R \in O(n)$ is an orthogonal transformation of $\mathbb{R}^n$ satisfying $Rx = x$ for all $x \in \mathrm{Span}\, u_{bd}(\partial\Omega)$.*

Using the above theorem, we prove the following result which completely characterises uniqueness and its failure for minimisers of the energy E_ε.

Theorem 3. *Let m, n, Ω, W, u_{bd} and $e \in \mathbb{S}^{n-1}$ be as in Theorem 2 and $V \equiv \mathrm{Span}\, u_{bd}(\partial\Omega)$. Then for every $\varepsilon > 0$ there exists a minimiser u_ε of E_ε on $\mathcal{A}$ and this minimiser is unique unless both following conditions hold:*

(1) *$u_{bd}(x) \cdot e = 0$ $\mathcal{H}^{m-1}$-a.e. $x \in \partial\Omega$,*
(2) *the functional E_ε restricted to the set*

$$\mathcal{A}_{res} := \{u \in \mathcal{A} : u(x) \in \mathrm{Span}(V \cup \{e\}) \ \text{a.e. in } \Omega\}$$

has a minimiser $\tilde{u}_\varepsilon$ with $\tilde{u}_\varepsilon(\Omega) \not\subset V$.

Moreover, if uniqueness of minimisers of E_ε *in* $\mathcal{A}$ *does not hold, then all minimisers of* E_ε *in* $\mathcal{A}$ *are given by* Ru_ε *where* $R \in O(n)$ *is an orthogonal transformation of* $\mathbb{R}^n$ *satisfying* $Rx = x$ *for all* $x \in V$*.*

Harmonic map problem. We note that Theorem 3 is similar to a well-known result of Sandier and Shafrir [8] on the uniqueness of minimising harmonic maps into a closed hemisphere. In fact, our proof of Theorem 3 can be adapted to give an alternative proof of their result. Our argument does not assume the smoothness of boundary data and does not use the regularity theory of minimising harmonic maps, which appears to play a role in the argument of [8].

Theorem 4. *Let* $m \geq 1$, $n \geq 2$, $\Omega \subset \mathbb{R}^m$ *be a bounded domain with smooth boundary,* $u_{bd} \in H^{1/2}(\partial\Omega; \mathbb{S}^{n-1})$ *be a boundary data satisfying* (6) *in a direction* $e \in \mathbb{S}^{n-1}$*, and* $V \equiv Span\, u_{bd}(\partial\Omega)$*. Then there exists a minimising harmonic map* $u \in \mathcal{A} \cap H^1(\Omega; \mathbb{S}^{n-1})$ *and this minimising harmonic map is unique unless both following conditions hold:*

(1) $u_{bd}(x) \cdot e = 0$ $\mathcal{H}^{m-1}$*-a.e.* $x \in \partial\Omega$*,*

(2) *the Dirichlet energy* $E(w) = \frac{1}{2}\int_\Omega |\nabla w|^2\, dx$ *restricted to the set*

$$\mathcal{A}^*_{res} := \{w \in \mathcal{A} \cap H^1(\Omega; \mathbb{S}^{n-1}) : w(x) \in Span(V \cup \{e\}) \text{ a.e. in } \Omega\}$$

has a minimiser $\tilde{u}$ *with* $\tilde{u}(\Omega) \not\subset V$*.*

Moreover, if u *is not the unique minimising harmonic map in* $\mathcal{A} \cap H^1(\Omega; \mathbb{S}^{n-1})$*, then all minimising harmonic maps in* $\mathcal{A} \cap H^1(\Omega; \mathbb{S}^{n-1})$ *are given by* Ru *where* $R \in O(n)$ *satisfies* $Rx = x$ *for all* $x \in V$*.*

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Structure of Noncollapsing Ricci limit spaces

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(joint work with Jeff Cheeger, Aaron Naber)

Let $(M_i^n, g_i, p_i) \to (X, d, p)$ satisfy $Ric_i \geq -(n-1)g_i$ and $Vol(B_1(p_i)) \geq v > 0$. The singular set of X is defined by

$$S = \{x \in X : \text{ no tangent cone at } x \text{ is isometric to } \mathbb{R}^n \}.$$

For any $0 \leq k \leq n$, the stratification S^k is given by

$$S^k = \{x \in X : \text{ no tangent cone at } x \text{ splits an } \mathbb{R}^{k+1} \text{ factor}\}.$$

It was proved by Cheeger-Colding [2, 3, 1] that

$$S = S^{n-2} \text{ and } \dim S^k \leq k.$$

Our main result is the following theorem:

Theorem 1 ([4])**.** *Let $(M_i^n, g_i, p_i) \to (X, d, p)$ satisfy $Ric_i \geq -(n-1)g_i$ and $Vol(B_1(p_i)) \geq v > 0$. For each $0 \leq k \leq n$ we have that S^k is k-rectifiable.*

We introduced the quantitative singular set S^k_ϵ to study S^k. Before giving the precise definition of S^k_ϵ, we constructed an example to give an intuition of such quantitative set. We constructed a 2-dim Alexandrov space Y with infinite singular points through gluing infinite 2-dim flat cones to a flat disc. Since each 2-dim flat cone has a definite L^1 curvature depending on the cone angle upper bound$< 2\pi$, by the global L^1 curvature bound from Gauss-Bonnet formula, we have for any $\epsilon > 0$ that

$$\#\{x \in Y : \text{ cone angle at } x \leq 2\pi - \epsilon \} \leq C(\epsilon). \tag{1}$$

Basing on this observation, we introduced as [5] the quantitative singular set S^k_ϵ for any $\epsilon > 0$:

$$S^k_\epsilon(X) = \{x \in X : d_{GH}(B_r(x), B_r(\bar{x})) \geq \epsilon r \text{ for any } r \leq 1$$
$$\text{and any } B_r(\bar{x}) \subset \mathbb{R}^{k+1} \times C(Z) \text{ with cone vertex } \bar{x}.\}$$

Roughly, the above Alexandrov space Y satisfies

$$\{x \in Y : \text{ cone angle at } x \leq 2\pi - \epsilon \} = S^0_\epsilon(Y)$$

From the definition of S^k_ϵ we see that $S^k_\epsilon \subset S^k$ and $S^k_{\epsilon'} \subset S^k_\epsilon$ for any $\epsilon \leq \epsilon'$. Furthermore we have

$$S^k = \bigcup_{\epsilon > 0} S^k_\epsilon.$$

Cheeger-Naber [5] proved for any $\delta, \epsilon > 0$ and $r \leq 1$ that

$$Vol(B_r(S^k_\epsilon) \cap B_1(p)) \leq C(\epsilon, \delta, n, v) r^{n-k-\delta}.$$

In this talk, we mainly focused on the quantitative set S^k_ϵ and we showed for any $\epsilon > 0$ that S^k_ϵ is k-rectifiable and S^k_ϵ has finite volume estimate $Vol(B_r(S^k_\epsilon) \cap B_1(p)) \leq C(\epsilon, n, v) r^{n-k}$ which is sharp by the example (1). Furthermore, as a direct application, we have that S^k is k-rectifiable which gives the proof of Theorem

1. Applying the structure results of S^k_ϵ to bounded Ricci curvature limit, we gave a new proof of the main result in [7]:

Theorem 2 ([7])**.** *Let $(M^n_i, g_i, p_i) \to (X, d, p)$ satisfy $|Ric_i| \leq n-1$ and $Vol(B_1(p_i)) \geq v > 0$. Then*

(1) *S is $n-4$-rectifiable.*
(2) *For any $0 < r \leq 1$, we have $Vol(B_r(S) \cap B_1(p)) \leq C(n,v)r^4$.*

The main point here for bounded Ricci curvature limit is that $S^{n-4}_\epsilon = S$ for some $\epsilon(n,v) > 0$ by an ϵ-regularity in [6].

One key notion in our proof is the so called (k,δ,η)-neck region for given $\delta,\eta > 0$. We introduced the so called (k,δ,η)-neck region for $\delta,\eta > 0$ to study the quantitative singular set S^k_ϵ. Instead of writing down the precise definition of the neck region, we presented two examples to give an intuition of this abstract notion. The definition of the neck region in [4] is just to make these two examples precise in our setting.

Let $(M^n_i, g_i, p_i) \to (X, d, p)$ satisfy $Ric_i \geq -(n-1)g_i$ and $Vol(B_1(p_i)) \geq v > 0$. Given $\delta,\eta > 0$ and $0 \leq k \leq n$, a neck region $\mathcal{N} \subset B_2(p) = B_2(p) \setminus \cup_{x\in\mathcal{C}} \bar{B}_{r_x}(x)$ is an open subset of $B_2(p) \subset X$ associated with a closed set $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_+ \subset \bar{B}_2(p)$ and radius function $r_x : \mathcal{C} \to \mathbb{R}_+$ such that $r_x > 0$ for $x \in \mathcal{C}_+$ and $r_x \equiv 0$ on $\mathcal{C}_0$. The radius function r_x and the closed set $\mathcal{C}$ satisfy several conditions. Let us present two examples to show that what the (k,δ,η) neck region $\mathcal{N}$ looks like.

Example 1: Let $X = \mathbb{R}^k \times C(Y)$ with $p = (0^k, \bar{y})$ a cone vertex, where $C(Y)$ is not η-close to any splitting cone $\mathbb{R} \times C(Z)$. For any $0 < r < 1$ choose a minimal $r/10$-dense subset $\{x_i\}$ of $B_2(p) \cap \mathbb{R}^k \times \{\bar{y}\}$. Let $\mathcal{C} = \mathcal{C}_+ = \{x_i\}$, $\mathcal{C}_0 = \emptyset$ and $r_x = r$ for all $x \in \mathcal{C}$. The open set $\mathcal{N}_r = B_2(p) \setminus \cup_{x\in\mathcal{C}} \bar{B}_r(x)$ is a (k,δ,η)-neck region.

For $r = 0$, we can define $\mathcal{C} = \mathcal{C}_0 = \bar{B}_2(p) \cap \mathbb{R}^k \times \{\bar{y}\}$. Then the open set $\mathcal{N}_0 = B_2(p) \setminus \mathcal{C}$ is a (k,δ,η)-neck region. Moreover $S^k_\eta \cap B_2(p) \subset \mathcal{C}_0$. We will see in the below decomposition theorem that S^k_η indeed contains in the set $\mathcal{C}_0$ associated with a neck region.

For $X = \mathbb{R}^k \times C(Y)$, one can define more general neck regions by combining the cases of $r > 0$ and $r = 0$ under the restrictions that $\{B_{r_x}(x), x \in \mathcal{C}\}$ satisfies Vitali condition and $B_2(p) \cap \mathbb{R}^k \times \{\bar{y}\}$ is covered by $\cup_{x\in\mathcal{C}} \bar{B}_{r_x/5}(x)$. □

Example 2: Let $X = \mathbb{R}^k \times Y$ and $0 < r < 1$ with $p = (0^k, \bar{y}) \in \{0^k\} \times Y$ such that $B_s(\bar{y})$ is $\delta^2 s$-close to a cone $C(Z_s)$ but not ηs-close to any splitting cone $\mathbb{R} \times C(W)$ for all $r \leq s \leq 1$. It is worth pointing out that the cone $C(Z_s)$ may not be the same for different scales s. Let us choose a minimal $r/10$-dense subset $\{x_i\}$ of $B_2(p) \cap \mathbb{R}^k \times \{\bar{y}\}$. Let $\mathcal{C} = \mathcal{C}_+ = \{x_i\}$, $\mathcal{C}_0 = \emptyset$ and $r_x = r$ for all $x \in \mathcal{C}$. The open set $\mathcal{N} = B_2(p) \setminus \cup_{x\in\mathcal{C}} \bar{B}_r(x)$ is a (k,δ,η)-neck region. □

For each (k, δ, η)-neck region $\mathcal{N} \subset B_2(p) = B_2(p) \setminus \cup_{x \in \mathcal{C}} \bar{B}_{r_x}(x)$ we can define the packing measure

$$\mu_{\mathcal{N}} = \sum_{x \in \mathcal{C}_+} r_x^k \delta_x + \mathcal{H}^k|_{\mathcal{C}_0},$$

where δ_x is the Delta measure of x and $\mathcal{H}^k$ is the k-Hausdorff measure. One main challenge of the whole proof is the following structure theorem on neck regions:

Theorem 3 (Neck Structure Theorem). *Let $(M_j^n, g_j, p_j) \to (X, d, p)$ be a limit with $Vol(B_1(p_j)) > v > 0$ and $\eta > 0$. Then for $\delta \leq \delta(n, v, \eta)$ if $\mathcal{N} = B_2(p) \setminus \bar{B}_{r_x}(\mathcal{C})$ is a (k, δ, η)-neck region, then the following hold:*

(1) *For each $x \in \mathcal{C}$ and $B_{2r}(x) \subset B_2(p)$ the induced packing measure $\mu_{\mathcal{N}}$ satisfies the Ahlfors regularity condition $A(n)^{-1} r^k < \mu_{\mathcal{N}}(B_r(x)) < A(n) r^k$.*
(2) *$\mathcal{C}_0$ is k-rectifiable.*

The proof of Theorem 3 relies on a bilipschitz estimate for splitting harmonic functions. The bilipschitz estimate depends on a transformation argument developed in [6], telescoping estimate of harmonic functions and hessian L^2 decay estimate of harmonic functions. See [4] for more details.

Once we proved the neck structure theorem 3, we can prove the following decomposition theorem which is the key goal toward the rectifiability of S^k.

Theorem 4 (Neck Decomposition). *Let $(M_i^n, g_i, p_i) \to (X, d, p)$ satisfy $Vol(B_1(p_i)) > v > 0$ and $Ric_i \geq -(n-1)$. Then for each $\eta > 0$ and $\delta \leq \delta(n, v, \eta)$ we can write*

$$B_1(p) \subseteq \bigcup_a \left(\mathcal{N}_a \cap B_{r_a}\right) \cup \bigcup_b B_{r_b}(x_b) \cup S^{k,\delta,\eta}(X),$$

$$S^{k,\delta,\eta}(X) \cap B_1(p) \subseteq \bigcup_a \left(\mathcal{C}_{0,a} \cap B_{r_a}\right) \cup \tilde{S}^{k,\delta,\eta}(X)$$

such that

(1) *$\mathcal{N}_a \subseteq B_{2r_a}(x_a)$ are (k, δ, η)-neck regions.*
(2) *$B_{2r_b}(x_b)$ are ηr_b-close to a cone $\mathbb{R}^{k+1} \times C(Z)$.*
(3) *$\sum_a r_a^k + \sum_b r_b^k + \mathcal{H}^k\left(S^{k,\delta,\eta}(X)\right) \leq C(n, v, \delta, \eta)$.*
(4) *$\mathcal{C}_{0,a} \subseteq B_{2r_a}(x_a)$ is the k-singular set associated to $\mathcal{N}_a$.*
(5) *$\tilde{S}^{k,\delta,\eta}(X)$ satisfies $\mathcal{H}^k\left(\tilde{S}^{k,\delta,\eta}(X)\right) = 0$.*
(6) *$S^{k,\delta,\eta}(X)$ is k-rectifiable.*
(7) *For any ϵ if $\eta \leq \eta(n, v, \epsilon)$ and $\delta \leq \delta(n, v, \eta, \epsilon)$ we have $S_\epsilon^k(X) \subset S^{k,\delta,\eta}(X)$.*

From (6) and (7) we see that S_ϵ^k is k-rectifiable. By a covering argument, we can use (3) to prove the volume estimate of S_ϵ^k. See [4] for more details.

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Minimal hypersurfaces in manifolds of finite volume

Yevgeny Liokumovich
(joint work with Gregory R. Chambers)

By a result of Bangert and Thorbergsson (see [Th] and [Ba]) every complete surface of finite area contains a closed geodesic of finite length. We generalize this result to higher dimensions.

Theorem 1. *Every complete non-compact Riemannian manifold M^{n+1} of finite volume contains a (possibly non-compact) embedded minimal hypersurface of finite volume. The hypersurface is smooth in the complement of a closed set of Hausdorff dimension $n-7$.*

In fact, we prove this result not only for manifolds of finite volume, but for all complete manifolds with sublinear volume growth.

Our result follows from a more general statement for which we will need to introduce some technical definitions.

Let M^{n+1} be a complete Riemannian manifold of dimension $n+1$. For an open set $U \subset M$ define the relative width of U, denoted by $W_\partial(U)$, to be the supremum over all real numbers ω, such that every Morse function $f : U \to [0,1]$ has a fiber of volume at least ω.

Theorem 2. *Let M^{n+1} be a complete Riemannian manifold of dimension $n+1$. Suppose M contains a bounded open set U with smooth boundary, such that $Vol_n(\partial U) \leq \frac{W_\partial(U)}{10}$. Then M contains a complete embedded minimal hypersurface Σ of finite volume. The hypersurface is smooth in the complement of a closed set of Hausdorff dimension $n-7$.*

The proof is based on Almgren-Pitts min-max theory [Pi]. We consider a sequence of sweepouts of U and extract a sequence of hypersurfaces of almost maximal volume that converges to a minimal hypersurface. The main difficulty is to rule out the possibility that the sequence completely escapes into the "ends" of the manifold. Our key tool is a Proposition which allows us to rule out this possibility. Loosely speaking, the content of the Proposition is that we can replace an arbitrary family of hypersurfaces with a nested family of hypersurfaces which are level sets of a Morse function, while increasing the maximal area by at most ε in the process. The ideas used in the proof of the Proposition are similar to the ideas used to attack in analogous problem for curves on surfaces in [CR]. We use this Proposition together with some hands on geometric constructions to show that there exists a sequence of hypersurfaces that converges to a minimal hypersurface and the volume of their intersection with a small neighbourhood of U is bounded away from 0.

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Conformal Gap Theorems on S^4 and CP^2

Sun-Yung Alice Chang
(joint work with Matt Gursky, Siyi Zhang)

There are many "Sphere Theorems", where mathematicians work to characterize up to homeomorphic or diffeomorphic type the spheres S^n with the surface measure g_c. Famous result includes the "quarter-pinching" theorem solved by Brendle-Schoen in 2012 (which works for all $n \geq 4$). In the special dimension when $n = 4$, there is also a sharp pinching theorem by C. Margerin ('98); in which he defined the "weak pinching condition" on compact, closed dimension four manifold (M^4, g) as

$$(WP)_g :=: \frac{|W|^2 + 2|E|^2}{R^2},$$

where W denotes the Weyl curvature, E the traceless Ricci curvature and R the scalar curvature of the metric g

Theorem 1 ([1])**.** *On (M^4, g), if $R_g > 0$ and $(WP)_g < \frac{1}{6}$, then (M^4, g) is diffeomorphic to either (S^4, g_c) or (P^4, g_c). Moreover this weak pinching condition is sharp. The spaces (CP^2, g_{FS}) (when $\int_{M^4} |W|^2$ does not vanish) and $(S^1 \times S^3, g_{prod.})$ (when Weyl vanishes) are the only space with $WP \equiv \frac{1}{6}$.*

We remark that there are other pointwise curvature pinching theorems on the sphere by C. Margerin ('84) and G. Huisken ('85) which hold for manifolds of dimension $n \geq 4$, but the pinching constants are not sharp. Motivated by the Gauss-Bonnet formula for 4-manifolds, we consider the Schouten tensor $P_g :=: \frac{1}{n-2}(Ric_g - \frac{1}{2(n-1)} Rg)$, and its second elementary symmetric function $\sigma_2 :=: \sigma_2(P_g)$; on 4-manifold, $\sigma_2 = \frac{1}{96}R^2 - \frac{1}{8}|E|^2$; and we observe that on 4-manifolds,

$$WP < \frac{1}{6} \quad iff \quad \frac{1}{4}|W|^2 < 4\sigma_2.$$

It is based on this observation that in around 2002, Chang-Gursky-Yang extended the above theorem of Margerin to its integral form:

Theorem 2 ([2]). *On (M^4, g), assume the Yamabe constant $Y(g) > 0$ and $\int_{M^4} \sigma_2 > 0$; then*
(i) If

$$\frac{1}{4}\int_{M^4} |W|^2 < 4\int_{M^4} \sigma_2,$$

then M is diffeomorphic to either S^4 or RP^4.
(ii) If the inequality in (i) becomes equality and (M^4, g) is not diffeomorphic to either S^4 or RP^4, then $M^4, g)$ is conformal equivalent to CP^2 with the Fubini-Study metric g_{FS}.

One remark that a crucial step in the proof of case (ii) in Theorem 2 above is the observation that under the conditions of (ii), the metric g becomes the critical point of the functional $\int_{M^4} |W|^2$, hence is Bach flat. That is $0 = B_{ij} = \nabla^k \nabla^l W_{kijl} + \frac{1}{2} R^{kl} W_{kijl}$.

In this talk we discuss some "gap" theorems which can be viewed as extensions of the theorem 2 above. First we recall an earlier result of gap theorem on (S^4, g_c).

Theorem 3 ((Chang-Qing-Yang, '07; Li-Qing-Shi, '15)). *There exists some $\epsilon > 0$, so that if a Bach flat metric g on (M^4, g), with $Y(g) > 0$, satisfying $\int_{M^4} \sigma_2 > 4\pi^2(1 - \epsilon)$, then (M^4, g) is conformally equivalent to (S^4, g_c).*

The above theorem was first established by Chang-Qing-Yang under the additional assumption that, there exists some constant C with $\int_{M^4} |W|^2 \leq C$; this assumption was later dropped in the work of Li-Qing-Shi, in which they apply the more recent work of Cheeger-Naber on the codimension 4 conjecture.

The two gap theorems we report on the workshop are:

Theorem 4. *If M^4 is topologically S^4, then the gap in Theorem 3 above is $\epsilon = \frac{1}{2}$.*

Theorem 5. *There exists some $\epsilon_1 > O$, such that on (M^4, g) with $b_2^+(M) = 1$, if g is Bach flat with $Y(g) > 0$, satisfying*

$$0 < 4\int_{M^4} \sigma_2 \leq \int_{M^4} |W|^2 \leq 4(1 + \epsilon_1)\int_{M^4} \sigma_2$$

then (M^4, g) is conformally equivalent to (CP^2, g_{FS}).

In the rest of the talk, the speaker presents the proof of above two theorems.

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Asymptotic rigidity of shrinking gradient Ricci solitons

Brett Kotschwar

We describe some results obtained by reframing various problems in the classification of noncompact shrinking Ricci solitons as problems of unique continuation for appropriate parabolic systems of PDE. Shrinking Ricci solitons (shrinkers) are generalized fixed points of the Ricci flow equation and models for the geometry of a solution near a developing singularity. Their classification is critical to the the understanding of the long-time behavior of the flow. Growing evidence suggests that the class of complete noncompact shrinkers may be rigid enough to admit a structural classification in terms of their geometries at infinity. At present, all known examples of such solitons are either asymptotic to a regular cone at infinity or locally reducible as products, and recent work of O. Munteanu and J. Wang [3, 4] supports the conjecture that, at least in four dimensions, these are the only possibilities. Our interest is in the associated question of uniqueness, namely: *to what extent is a noncompact shrinking soliton determined by its asymptotic geometry?*

In previous joint work with L. Wang [8], we have addressed this question in the asymptotically conical case, showing that any two shrinkers which are asymptotic to the same regular cone along some end of each must actually be isometric to each other near infinity on those ends. This result, an analog of a theorem of Wang for self-shrinkers to the mean curvature flow [10], uses the formulation in [5] to reduce the problem to one of backward uniqueness for solutions to a certain coupled system of mixed differential inequalities. This problem is then amenable to the application of adapted versions of the Carleman estimates proven in [1] for solutions to backward parabolic inequalities on the complement of a ball in $\mathbb{R}^n$.

The result in [8] has the specific consequence that any isometry of the cone will be reflected in an isometry of the corresponding end of the asymptotic shrinker and it raises the question of what other geometric properties of the cone are necessarily inherited by the shrinker. In this direction, we show in [7] that, if a complete shrinker is asymptotic to a Kähler cone along an end, then the shrinker must itself be Kähler. In this case, the problem reduces to that of the backward propagation of the Kähler property along the Ricci flow, a problem we have previously considered in [6] for smooth complete solutions of bounded curvature. In [7], we combine the formulation in [6] with the Carleman estimates in [8] to treat the (incomplete) asymptotically conical solutions associated to the shrinkers in our setting.

We have also considered the uniqueness in the asymptotically cylindrical case. In recent work [9] with L. Wang, we prove that a shrinker which agrees to infinite order at infinity with a generalized cylinder $S^k \times \mathbb{R}^{n-k}$ for $k \geq 2$ along some end must be isometric to the cylinder on that end. This is again an analog of a theorem of Wang [11] for self-shrinkers to the mean curvature flow, and, as in that case, does not require the manifold to be complete nor place any a priori restriction on the number of its ends. Examples constructed in [11] in the mean curvature flow setting suggest that the assumption of infinite order-decay may be necessary in this generality. The analysis of the associated problem of backward uniqueness here is more involved than in the conical case, since the model solution (the shrinking cylinder) becomes singular along the spherical factor at the terminal time. To track the percolation of this singularity through the problem, we make use of a prolonged system which is somewhat more elaborate than that considered in [5, 8], and develop corresponding anisotropic Carleman inequalities in the spirit of [2, 11].

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Regularity and compactness for stable codimension 1 CMC varifolds

Neshan Wickramasekera
(joint work with Costante Bellettini)

1. introduction

The talk reported on the recent joint work [5] of Costante Bellettini and the speaker. The work considers codimension 1 integral n-varifolds V on an open subset $U \subset \mathbb{R}^{n+1}$ that have generalised mean curvature locally in L^p for some $p > n$ and that are, on any orientable piece away from the singular set and relative to a choice of orientation, stationary and stable with respect to the area functional (with multiplicity) for "volume preserving" deformations. The main result (Theorem 1 below) gives two structural conditions (hypotheses (a) and (b) of Theorem 1) on such a varifold V that imply that its support, away from a closed set of Hausdorff dimension at most $n-7$, is locally either a single smoothly embedded constant-mean-curvature (CMC) disk or precisely two smoothly embedded CMC disks intersecting tangentially, with the value of the scalar mean curvature the same constant everywhere. Simple examples show that neither of the two structural conditions can be dropped; see remarks (1) and (3) following Theorem 1. The work also establishes an associated compactness theorem (Theorem 2 below).

This work is to be regarded an an extension to CMC hypersurfaces of the regularity theory [9] (that if an n-dimensional stationary codimension 1 integral varifold has stable regular set and satisfies the structural condition (a) of Theorem 1, i.e. has no "classical singularities" (see the definition below in Section 3) then it is regular except on a set of Hausdorff dimension $n-7$, together with the corresponding compactness theory and various applications). Some significant additional technical difficulties needed to be accounted for in the CMC setting, and in addition there were surprising subtleties in formulating the optimal set of hypotheses.

One motivating factor for the work [5] is its potential applicability to the question of existence of a CMC hypersurface in a compact Riemannian manifold with a prescribed value of the mean curvature. In the case of zero mean curvature, the affirmative answer to this question is a long known theorem resulting from the combined work of Almgren, Pitts and Schoen–Simon. Based on a result of Y. Tonegawa and the speaker ([8]) that establishes regularity of minimal hypersurfaces (limit interfaces) arising from sequences of stable solutions to the elliptic Allen–Cahn equations with perturbation parameter $\epsilon \to 0^+$, and standard PDE min-max principles for semilinear equations, M. Guaraco ([6]) has recently given a simple, elegant new proof of the existence, in any smooth compact Riemannian manifold, of a minimal hypersurface that is smooth away from a codimension 7 singular set. This new proof avoids the intricate Almgren–Pitts geometric min-max theory on the space of varifolds that was needed in the original proof, but for the regularity conclusions it relies on the work [9] through its dependence on [8].

2. Variational hypotheses in the smooth setting

Since in Theorems 1 and 2 the variational hypotheses only concern orientable portions of the regular part of the varifold, let us first consider the classical (i.e. C^2) setting, where V corresponds to an embedded, oriented C^2 hypersurface $M \subset U$ with $\partial M \cap U = \emptyset$ and with ν a continuous choice of unit normal on M. For an open set $\mathcal{O} \subset\subset U$, write

$$\mathcal{A}_{\mathcal{O}}(M) = \mathcal{H}^n(M \cap \mathcal{O}) \text{ and,}$$

$$vol_{\mathcal{O}}(M) = \frac{1}{n+1} \int_{M \cap \mathcal{O}} x \cdot \nu(x) \, d\mathcal{H}^n(x).$$

Note that $vol_{\mathcal{O}}(M)$ is the volume enclosed by M in case M is the boundary of a bounded open set $\Omega \subset \mathcal{O}$ and ν is the outward pointing unit normal to M.
Definition: M is *stationary in U with respect to the area functional for volume preserving deformations* if $\frac{d}{dt}\big|_{t=0} \mathcal{A}_{\mathcal{O}}(\varphi_t(M)) = 0$ for each open $\mathcal{O} \subset\subset U$ and each smooth map $\varphi : U \times (-\epsilon, \epsilon) \to U$, $\epsilon > 0$, with: (i) $\varphi_t = \varphi(\cdot, t) : U \to U$ a diffeomorphism for each $t \in (-\epsilon, \epsilon)$, (ii) $\varphi_0 = identity$, (iii) $\varphi_t|_{U \setminus \overline{\mathcal{O}}} = identity|_{U \setminus \overline{\mathcal{O}}}$ for each $t \in (-\epsilon, \epsilon)$ and (iv) $vol_{\mathcal{O}}(\varphi_t(M)) = vol_{\mathcal{O}}(M)$ for each $t \in (-\epsilon, \epsilon)$.

Write H_M for the mean curvature vector of M. Let $\lambda \in \mathbb{R}$ be a constant, and let

$$\mathcal{J}_{\mathcal{O}}(M) = \mathcal{A}_{\mathcal{O}}(M) + \lambda \, vol_{\mathcal{O}}(M).$$

It is well-known, and is straightforward to verify, that the following statements are equivalent (see e.g. [4]):

(a) M is CMC with $H_M \cdot \nu = \lambda$.
(b) $\lambda = \frac{1}{\mathcal{A}_{\mathcal{O}}(M)} \int_{M \cap \mathcal{O}} H_M \cdot \nu \, d\mathcal{H}^n$ fr some open $\mathcal{O} \subset\subset U$, and M is stationary in U with respect to area for volume preserving deformations.
(c) For each open $\mathcal{O} \subset\subset U$, M is stationary with respect to $\mathcal{J}_{\mathcal{O}}(\cdot)$ for arbitrary deformations (i.e. for φ_t as above but not necessarily with $vol_{\mathcal{O}}(\varphi_t(M)) = vol_{\mathcal{O}}(M) \ \forall t$).

Definition: An embedded CMC hypersurface M in U is stable if for each open $\mathcal{O} \subset\subset U$, $\frac{d^2}{dt^2}\Big|_{t=0} \mathcal{A}_{\mathcal{O}}(\varphi_t(M)) \geq 0$ for all $vol_{\mathcal{O}}(\cdot)$ preserving φ_t as in the definition above. Stability of M is equivalent to the fact that

$$\int_M |A|^2 \zeta^2 \, d\mathcal{H}^n \leq \int_M |\nabla \zeta|^2 \, d\mathcal{H}^n$$

for each $\zeta \in C^\infty_c(M)$ with $\int_M \zeta \, d\mathcal{H}^n = 0$, where A is the second fundamental form of M and ∇ is the gradient operator on M (see [4]).

3. The varifold setting and two special types of singularities

Now consider an integral n-varifold $V = (M, \theta)$ in U with generalized mean curvature vector H_V and associated weight measure $\|V\|$ (notation as in [7], except for $\|V\|$ which is denoted μ_V in [7]). This means that M is an $\mathcal{H}^n$ measurable, countably n-rectifiable subset of U, $\theta : M \to \mathbb{N}$ is a positive integer valued $\mathcal{H}^n$

measurable function on M, $\|V\| = \mathcal{H}^n \llcorner \widetilde{\theta}$ where $\widetilde{\theta} = \theta$ on M and $\widetilde{\theta} = 0$ in $U \setminus M$, $H_V \in L^1_{loc}(\|V\|)$ in U, and the formula

$$\int_M \operatorname{div}_M X \, d\|V\| = -\int_M H_V \cdot X \, d\|V\|$$

holds for every $X \in C^\infty_c(U; \mathbb{R}^{n+1})$. Here $\operatorname{div}_M X(x) = \sum_{j=1}^n \tau_j \cdot D_{\tau_j} X(x)$ where $\{\tau_1, \ldots, \tau_n\}$ is any orthonormal basis for the approximate tangent space $T_x M$ and D_τ denotes the directional derivative in the direction τ. Note that when $V = (M, 1)$ with M an oriented C^2 hypersurface, the validity of this formula with H_V equal to the classical mean curvature vector of M follows from the divergence theorem. In the varifold setting (where M is merely countably n-rectifiable), this is the defining formula for the generalized mean curvature vector H_V.

To have any hope of regularity of $\operatorname{spt}\|V\| \cap U$, we need $H_V \in L^p_{loc}(\|V\|)$ for some $p \geq n$. Else $\operatorname{spt}\|V\| \cap U$ need not even be n-dimensional as can be seen by taking the union of suitable countably many concentric spheres around rational points in $\mathbb{R}^3$. If on the other hand $H_V \in L^p_{loc}(\|V\|)$ in U for some $p > n$, then $\operatorname{spt}\|V\| \cap U$ is n-rectifiable, $\mathcal{H}^n(((\operatorname{spt}\|V\| \setminus M) \cup (M \setminus \operatorname{spt}\|V\|)) \cap U) = 0$ and the C^1 embedded part $\operatorname{reg}_1 V$ of $\operatorname{spt}\|V\| \cap U$ is a relatively open, dense subset of $\operatorname{spt}\|V\| \cap U$. In fact $\operatorname{reg}_1 V$ is of class $C^{1,1-\frac{n}{p}}$ if $n < p < \infty$. The condition $H_V \in L^p_{\mathrm{loc}}(\|V\|)$ also implies (via the well-known approximate monotonicity formula for the area ratio) that the area density $\Theta(\|V\|, p) = \lim_{\rho \to 0} \frac{\|V\|(B^{n+1}_\rho(p))}{\omega_n \rho^n}$ exists for every $p \in U$, and that $\operatorname{spt}\|V\| \cap U = \{p \in U \,:\, \Theta(\|V\|, p) \geq 1\}$. Here ω_n denotes the volume of the unit ball in $\mathbb{R}^n$, and $B^{n+1}_\rho(p)$ is the open ball in $\mathbb{R}^{n+1}$ with centre p and radius ρ. These facts were all established in the landmark work of Allard ([1]) that extended earlier fundamental work of Almgren ([2]).

Definition: For an integral n-varifold V as above, the singular set $\operatorname{sing} V$ is defined by $\operatorname{sing} V = (\operatorname{spt}\|V\| \setminus \operatorname{reg}_1 V) \cap U$, where $\operatorname{reg}_1 V$ is the set of points $p \in \operatorname{spt}\|V\| \cap U$ near which $\operatorname{spt}\|V\|$ is a C^1 embedded hypersurface.

Note that a.e. regularity (i.e. the fact that $\mathcal{H}^n(\operatorname{sing} V) = 0$) does not follow from the assumption $H_V \in L^p_{\mathrm{loc}}$ for some $p > n$; a construction due to Brakke ([3]) gives an integral 2-varifold V in $\mathbb{R}^3$ with $H_V \in L^\infty_{\mathrm{loc}}$ such that $\operatorname{sing} V$ has positive $\mathcal{H}^2$ measure.

Let us now introduce two special types of singularities that will play a key role in the main theorems given in the next section.

Definition: A point $p \in \operatorname{spt}\|V\| \cap U$ is a *classical singularity* if there are $\alpha \in (0,1)$ and $\sigma > 0$ such that $\operatorname{spt}\|V\| \cap B^{n+1}_\sigma(p)$ is the union of three or more embedded $C^{1,\alpha}$ hypersurfaces-with-boundary having common boundary S containing p, meeting pairwise only along S, and with at least two of the hypersurfaces-with-boundary meeting transversely.

Let $\operatorname{sing}_C V$ be the set of classical singularities of V.

Definition: A point $p \in \operatorname{spt}\|V\| \cap U$ is a *touching singularity* of V if $p \notin \operatorname{sing}_C V \cup \operatorname{reg}_1 V$ and there are $\sigma > 0$, an affine hyperplane L containing p, $\alpha \in (0,1)$ and two $C^{1,\alpha}$ functions $u_1, u_2 \,:\, L \to L^\perp$ such that $\operatorname{spt}\|V\| \cap B^{n+1}_\sigma(p) = (\operatorname{graph} u_1 \cup$

graph $u_2) \cap B_\sigma^{n+1}(p)$. (Note that it follows from the definition that $u_1(p) = u_2(p)$, $Du_1(p) = Du_2(p)$.)

Let $\operatorname{sing}_T V$ be the set of touching singularities of V.

4. MAIN THEOREMS

Theorem 1 (**CMC REGULARITY THEOREM**). *Let V be an integral n-varifold on an open subset $U \subset \mathbb{R}^{n+1}$, $n \geq 2$, with $H_V \in L^p_{loc}(\|V\|)$ for some $p > n$. Suppose that V satisfies the following:*
Structural Hypotheses:

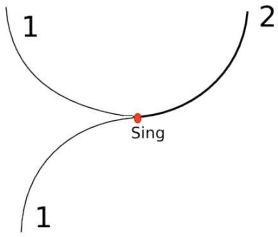
(a) $\operatorname{sing}_C V = \emptyset$;
(b) *For each $p \in \operatorname{sing}_T V$, there is $\rho > 0$ such that $\mathcal{H}^n(\{y : \Theta(\|V\|, y) = \Theta(\|V\|, p)\} \cap B_\rho^{n+1}(p)) = 0$;*

Variational Hypotheses:

(c) *stationarity: for each open $\mathcal{O} \subset\subset U \setminus \operatorname{sing} V$ such that $\operatorname{reg}_1 V \cap \mathcal{O}$ is orientable, the varifold $W \equiv V \llcorner \mathcal{O}$ is stationary in $\mathcal{O}$ with respect to the area functional $\mathcal{A}_\mathcal{O}(W) = \|W\|(\mathcal{O})$ for ambient deformations that leave the region outside a compact subset of $\mathcal{O}$ fixed and preserve $vol_\mathcal{O}(W) = \frac{1}{n+1}\int x \cdot \nu \, d\|W\|$, where ν is a choice of continuous unit normal on $\operatorname{reg}_1 V \cap \mathcal{O}$;*
(d) *stability: the C^2 immersed part M of $\operatorname{spt}\|V\|$ (which, by virtue of the hypotheses (a), (c) and the C^2 assumption, contains $\operatorname{reg}_1 V$ and is a CMC hypersurface consisting locally of a single embedded disk or precisely two embedded disks intersecting tangentially) is stable, as an immersion, with respect to (multiplicity 1) area for volume preserving deformations, or equivalently, the stability inequality $\int_M |A|^2\zeta^2 \leq \int_M |\nabla \zeta|^2$ holds for each $\zeta \in C_c^\infty(M)$ with $\int_M \zeta = 0$.*

Then there exists a closed set $\Sigma \subset \operatorname{spt}\|V\|$ with $\dim_\mathcal{H}(\Sigma) \leq n - 7$ such that $\operatorname{spt}\|V\| \setminus \Sigma$ locally near each point is either a smoothly embedded disk or the union of precisely two smoothly embedded disks intersecting tangentially along a set contained in a smooth $(n-1)$-dimensional submanifold; moreover, if $H_V \neq 0$ identically, then $\operatorname{spt}\|V\| \setminus \Sigma$ is an orientable smooth immersion and there is a constant $\lambda \in \mathbb{R} \setminus \{0\}$ such that $H_V = \lambda\nu$ on $\operatorname{spt}\|V\| \setminus \Sigma$ where ν is a choice of orientation on $\operatorname{spt}\|V\| \setminus \Sigma$.

Remarks: (1) Hypothesis (a) cannot be dropped, as shown by a piece of two intersecting unit spheres or cylinders.
(2) If hypothesis (b) is dropped, then C^2 regularity cannot be guaranteed, as shown by the example $\Gamma \times \mathbb{R}^{n-1}$ where Γ is the following 1-dimensional varifold in a suitable open subset of $\mathbb{R}^2$:



In this picture, each arc is a piece of a unit circle, and the numbers 1, 2 denote the multiplicity on an arc. The singular point is a touching singularity (not a classical singularity, since no pair of arcs meet transversely). The only hypothesis of the theorem not satisfied by this example is (b). The support of the varifold is not the union of two C^2 graphs (it is however the union of a C^2 graph and a $C^{1,1}$ graph).

Theorem 2 (**CMC COMPACTNESS THEOREM**). *Let (V_j) be a sequence of integral n-varifolds in open $U \subset \mathbb{R}^{n+1}$, $n \geq 2$, satisfying $H_{V_j} \in L^{p_j}_{loc}(\|V_j\|)$ for some $p_j > n$ and (a)-(d) (as in the above theorem) with $V = V_j$. If $\limsup_{j\to\infty} \|V_j\|(K) < \infty$ for each compact $K \subset U$ and $\limsup_{j\to\infty} |H_{V_j}| < \infty$ (note that $|H_{V_j}|$ is constant for each j by the above theorem), then there is an integral n-varifold V in U satisfying the conclusions of Theorem 1, and a subsequence $\{j'\}$ such that $V_{j'} \to V$ as varifolds in U.*

In view of De Giorgi's structure theorem for Caccioppoli sets, the following is an immediate consequence of Theroem 1:

Corollary 3. *If V is the multiplicity 1 varifold associated with the boundary of a Caccioppoli set, and if the hypotheses (a), (c), (d) of Theroem 1 hold, then V satisfies the conclusions of the Theorem 1.*

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Expanding solutions of the harmonic map flow

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(joint work with Tobias Lamm)

In this short note, we consider the Cauchy problem for the harmonic map flow of maps $(u(t))_{t\geq 0}$ from $\mathbb{R}^n$, $n\geq 3$ to a Wickramasekera sphere $\mathbb{S}^{m-1}\subset\mathbb{R}^m$, $m\geq 2$:

$$\begin{cases} \partial_t u = \Delta u + |\nabla u|^2 u, & \text{on } \mathbb{R}^n\times\mathbb{R}_+, \\ u|_{t=0} = u_0, \end{cases} \tag{1}$$

for a given map $u_0 : \mathbb{R}^n \to \mathbb{S}^{m-1}\subset\mathbb{R}^m$. Remark that equation (1) is equivalent to $\partial_t u - \Delta u \perp T_u\mathbb{S}^{m-1}$ for a family of maps $(u(t))_{t\geq 0}$ with values in $\mathbb{S}^{m-1}$. Recall that this evolution equation is invariant under the scaling:

$$(u_0)_\lambda(x) := u_0(\lambda x), \quad x\in\mathbb{R}^n, \tag{2}$$

$$u_\lambda(x,t) = u(\lambda x, \lambda^2 t), \quad \lambda>0, \quad (x,t)\in\mathbb{R}^n\times\mathbb{R}_+. \tag{3}$$

If u_0 is invariant under scaling, i.e. if u_0 is 0-homogeneous, solutions of the harmonic map flow which are invariant under scaling are potentially well-suited for smoothing u_0 out instantaneously. Such solutions are called expanding solutions. In this setting, it turns out that (1) is equivalent to a static equation, i.e. that does not depend on time anymore. Indeed, if u is an expanding solution in the previous sense then the map $U(x) := u(x,1)$ for $x\in\mathbb{R}^n$, satisfies:

$$\begin{cases} \Delta U + \frac{r}{2}\partial_r U + |\nabla U|^2 U = 0, & \text{on } \mathbb{R}^n, \\ \lim_{|x|\to+\infty} U(x) = u_0(x/|x|). \end{cases} \tag{4}$$

Conversely, if U is a solution to (4) then the map $u(x,t) := U(x/\sqrt{t})$, for $(x,t)\in\mathbb{R}^n\times\mathbb{R}_+$ is a solution to (1). Because of this equivalence, u_0 can be interpreted either as an initial condition or a boundary data at infinity.

The interest of expanding solutions is twofold. On one hand, these scale invariant solutions are important with respect to the continuation of a weak harmonic map flow between two closed Riemannian manifolds. Indeed, by the work of Chen and Struwe [3], there always exists a weak harmonic map flow starting from a smooth map between two closed Riemannian manifolds. It turns out that such a flow is not always smooth and the appearance of singularities is caused by either non-constant 0-homogeneous harmonic maps defined from $\mathbb{R}^n$ to $\mathbb{S}^{m-1}$ called tangent maps, or shrinking solutions (also called quasi-harmonic spheres) that are ancient solutions invariant under scalings. Expanders can create an ambiguity in the continuation of the flow after it reached a singularity by a gluing process. On the other hand, one might be interested in using the smoothing effect of the harmonic map flow. More precisely, it is tempting to attach a canonical map to any map between (stratified) manifolds with prescribed singularities: it turns out that 0-homogeneous maps are the building blocks of such singularities and expanding solutions are likely to be the best candidates to do this job.

We investigate the question of existence of expanding solutions coming out of u_0 in case there is no topological obstruction, i.e. if u_0 is homotopic to a constant. Our main result is:

Theorem 1. *Let $n \geq 3$ and $m \geq 2$ be two integers and let $u_0 : \mathbb{R}^n \to \mathbb{S}^{m-1} \subset \mathbb{R}^m$ be a Lipschitz 0-homogeneous map homotopic to a constant.*

Then there exists a weak expander $u(\cdot, 1) =: U(\cdot)$ of the harmonic map flow coming out of u_0 weakly. Moreover,

$$\|\nabla u(t)\|_{L^2(B(x_0,1))} \leq C(n, m, \|\nabla u_0\|_{L^2_{loc}(\mathbb{R}^n)}, t)\|\nabla u_0\|_{L^2(B(x_0,1))}, \quad \forall x_0 \in \mathbb{R}^n,$$
$$\|\partial_t u\|_{L^2((0,t),L^2_{loc}(\mathbb{R}^n))} \leq C(n, m, t)\|\nabla u_0\|_{L^2_{loc}(\mathbb{R}^n)},$$

where $\lim_{t\to 0} C(n, m, \|\nabla u_0\|_{L^2_{loc}(\mathbb{R}^n)}, t) = \lim_{t\to 0} C(n, m, t) = 0$.

In particular, $u(\cdot, t)$ tends to u_0 as t goes to 0 in the $H^1_{loc}(\mathbb{R}^n)$ sense and if u_0 is not harmonic then $u(\cdot, t)$ is not constant in time.

Let us mention a few remarks about this theorem. The energy estimates of Theorem 1 are reminiscent of and based on the fundamental work of Chen [2]. Theorem 1 and its proof provide the existence of a non constant in time (or equivalently non radial) expanding solution in case the initial condition is not harmonic. Since the initial condition u_0 is allowed to have large local-in-space energy, it is likely that uniqueness will fail. In particular, the authors do not know if the solution produced by Theorem 1 coming out of a 0-homogeneous harmonic map will stay harmonic.

Now a few words about the proof of Theorem 1. A direct perturbative approach does not seem appropriate without imposing either any further symmetry on the initial condition u_0 or any smallness of the $L^2_{loc}(\mathbb{R}^n)$ energy of u_0. Indeed, the nonlinearity of the target manifold and the formation of singularities are the two main obstacles. Therefore, we follow Chen's penalization procedure [2]. As a preliminary result, we prove the existence of expanding solutions of the homogeneous Ginzburg-Landau flow with parameter $K > 0$:

$$\begin{cases} \partial_t u = \Delta u + \dfrac{K}{t}(1 - |u|^2)u, & \text{on } \mathbb{R}^n \times \mathbb{R}_+, \\ u|_{t=0} = u_0. \end{cases} \tag{5}$$

The reason why we introduce the factor t^{-1} in front of the term $K(1-|u|^2)u$ is to make the Ginzburg-Landau flow invariant under the same scalings (2) and (3). The next step consists in taking a limit of a sequence of expanding solutions of (5) as K goes to $+\infty$. Despite its physical relevance, the homogeneous Ginzburg-Landau flow does not seem to give a precise estimate on the singular set as noticed by Chen and Struwe [3]: this comes essentially from a lack of a good Bochner formula which in turn is caused by the difficulty of controlling the vanishing set of an expanding solution a priori. Therefore, we plan to prove the existence of expanding solutions starting from the same initial condition u_0 of a so called homogeneous Chen-Struwe flow with parameter K.

We would like to relate our work to previous articles on this subject. To our knowledge, most of the literature concerns maps from $\mathbb{R}^n$ to an hemisphere of a rotationally symmetric target manifold: see Germain and Rupflin [8], Biernat and Bizón [1] and the more recent work due to Germain, Ghoul and Miura [7]. In particular, our setting includes theirs in the case the target is a sphere since a map from $\mathbb{R}^n$ with values in an hemisphere is homotopic to a constant. Of course, since (1) reduces to an ODE in such a corotational setting, the above mentioned works obtain more quantitative results even if the question of regularity is not really addressed.

There are at least two equations that motivates this work. Jia and Šverák [9] proved the existence of smooth expanding solutions of the Navier-Stokes equation. In this case, the homogeneity is of degree -1. To prove Theorem 1, we proceed similarly to their work by using Leray-Schauder degree theory. To do so, one needs a path of initial conditions $(u_0^\sigma)_{0\leq\sigma\leq1)} : \mathbb{S}^{n-1} \to \mathbb{S}^{m-1}$ connecting the restriction u_0^0 of u_0 to $\mathbb{S}^{m-1}$ to a simpler map u_0^1. By simpler, we mean here that there is an obvious solution coming out of u_0^1 for which there is a uniqueness-in-the-small result for a suitable function space. In the case of the Navier-Stokes equation, the path is given for free: it suffices to contract the initial vector field to zero. In our case, the path is given by assumption. There is also a deep analogy with the Ricci flow that exhibits the same scale invariance. In the setting of the Ricci flow, u_0 is replaced by a metric cone $C(M)$ over a closed Riemannian manifold (M, g) endowed with its Euclidean cone metric $dr^2 + r^2 g$ and the topological assumption on M similar to the triviality of the homotopy class of u_0 is that it is null cobordant. See [6] and [5] in the case (M, g) is a Riemmanian manifold with curvature operator larger than 1 or [4] in a more algebraic context.

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Synthetic upper Ricci bounds and rigidity of metric measure cones

Karl-Theodor Sturm
(joint work with Matthias Erbar)

We present a synthetic notion of upper Ricci curvature bounds in terms of the L^2-Kantorovich-Wasserstein distance between heat flows. And we prove that every Ricci-flat metric measure space which can be written as a metric cone is isomorphic to the Euclidean space.

Recall from [1] that a metric measure space (X, d, m) has Ricci curvature bounded from below by K iff for all $t \geq 0$ and all probability measures μ and ν on X

$$-\partial_t\big|_{t=0} \log W_2(P_t\mu, P_t\nu) \geq K$$

where P_t denotes the dual heat semigroup on X acting on probability measures. We say that (X, d, m) has Ricci curvature bounded from above by L iff for all $x \in X$

$$-\limsup_{y\to x} \partial_t\big|_{t=0} \log W_2\big(P_t\delta_x, P_t\delta_y\big) \leq L.$$

For weighted Riemannian spaces, these conditions are equivalent to lower or upper bounds on the Bakry-Emery-Ricci tensor.

Assume now that the metric measure space (X', d', m') is a N-cone of another metric measure space, say of (X, d, m). By the work of Ketterer, [2], we know that the N-cone X' has Ricci curvature ≥ 0 iff the base space X satisfies a curvature-dimension condition $\mathrm{CD}(N-1, N)$ in the sense of Lott-Sturm-Villani.

Theorem 1. *Assume that $Ric_{X'} \geq 0$. Then the following are equivalent:*

- $Ric_{X'} < \infty$
- $\int_X \int_X \cos d(x,y)\, dm(x)\, dm(y) \leq 0$
- $N \in \mathbb{N}$, $X = \mathbb{S}^N$, $X' = \mathbb{R}^{N+1}$.

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