Mini-Workshop: MASAs and Automorphisms of C*-Algebras

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ABSTRACT. The main aim of this workshop was to study maximal abelian ∗-subalgebras of C*-algebras from various points of view. A chief motivation is the UCT problem, which asks whether all separable nuclear C*-algebras satisfy the universal coefficient theorem of Rosenberg and Schochet. The connection, in terms of existence of invariant Cartan MASAs for certain ∗-automorphisms of the Cuntz algebra, has been brought up only very recently; it opens up a line of new perspectives on pressing questions in the structure and classification theory of simple nuclear C*-algebras and their automorphism groups, which has made giant leaps forward in the past five years. Connections to other areas, in particular von Neumann algebras and coarse geometry, have been explored as well.

Introduction by the Organisers

This workshop dealt with maximal abelian ∗-subalgebras (MASAs) and ∗-automorphisms of C*-algebras (i.e., norm-closed self-adojoint algebras of bounded linear operators on Hilbert spaces). Both are natural mathematical objects associated with C*-algebras, and the workshop's goal was to foster the study of their interaction.

Understanding MASAs in general is very ambitious, as they exist in abundance for abstract reasons (by Zorn's Lemma). The workshop's main focus therefore was on Cartan subalgebras, a class of relatively well-behaved and yet rather common MASAs introduced by Renault. Cartan subalgebras have a geometric flavour to them, as they turn out to be isomorphic (as sub-C*-algebras) to distinguished

MASAs of (twisted) groupoid C*-algebras. This makes their study often more tractable than that of arbitrary MASAs. However, it should be mentioned that more exotic MASAs, like constructions of MASAs with connected spectrum inside the CAR algebra, have been discussed as well.

A chief motivation of this workshop was the UCT problem, which asks whether all separable nuclear C*-algebras satisfy the universal coefficient theorem of Rosenberg and Schochet. It is a pressing open question, arguably the most important structural question on nuclear C*-algebras, which is receiving increasing attention due to the dramatic progress in the structure and classification theory of simple nuclear C*-algebras and their automorphism groups. Very recenlty, Barlak and Li gave a new characterization of the UCT problem in terms of existence of invariant Cartan subalgebras for certain *-automorphisms of a specific C*-algebra, the Cuntz algebra $\mathcal{O}_2$. In other words, this perspective on the UCT problem incorporates the interplay of the main objects of study of this workshop. Other similar viewpoints on the UCT problem have been presented as well, one of which builds on finite group actions by *-automorphisms on the Razak-Jacelon algebra $\mathcal{W}$, in spirit a stably projectionless analogue of the Cuntz algebra.

Another related problem that fueled this workshop was the question which simple nuclear C*-algebras that are classifiable in the sense of the Elliott program have Cartan subalgebras. Loosely speaking, this asks which classifiable C*-algebras arise from well-behaved groupoids. Another related and interesting problem is classifiability of Cartan subalgebras in classifiable C*-algebras. Questions of this type are to some extent reminiscent of existence and uniqueness results for Cartan subalgebras in von Neumann algebra factors arising in Popa's deformation/rigidity theory, where the intertwining-by-bimodules technique is being applied very successfully. On these grounds, it is natural to sound out possible connections and transfer of techniques from von Neumann algebra theory. It has been made an effort at this workshop to pave the grounds for future developments in this direction.

Coarse geometry, studying the "large scale behaviour" of metric spaces, was a further topic the workshop touched upon. Uniqueness results for certain Cartan subalgebras of the uniform Roe algebra, a C*-algebra naturally associated to a metric space that captures its coarse geometry, have been presented. Furthermore, connections between Cartan MASAs and other regularity properties for nuclear C*-algebras, like finite nuclear dimension, have been explored as well.

The workshop featured 19 talks (each 45 minutes) with extended breaks in between to allow for discussions. In addition, two problem sessions (each 45 minutes) were held. In the first one, open problems around the workshop's topic have been collected, whereas in the second one selected problems have been discussed in rounds with all participants. These problems are made available in this report in form of an extended abstract. As customary with Oberwolfach meetings, there was also plenty of time reserved for interaction outside the regular program.

The workshop was a great opportunity to bring together experts and push forward this exciting topic. We would like to thank the Mathematisches Forschungsinstitut Oberwolfach for giving us this opportunity and providing the great environment to having such a meeting organized so smoothly. We are very grateful to the staff of MFO for all their work and help. This enabled a very inspiring and productive yet relaxed atmosphere.

It is also a pleasure to the organizers to thank all the participants for their contributions in lectures held at the workshop and stimulating discussions. Finally, we would like to thank the reporters for collecting the extended abstracts and putting together this report.

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Abstracts

Approaching the UCT problem via crossed products

GÁBOR SZABÓ

(joint work with Selçuk Barlak)

One of the major open problems in the structure theory of C*-algebras is commonly referred to as the UCT problem:

Problem. *Do all separable, nuclear C*-algebras satisfy the universal coefficient theorem of Rosenberg–Schochet* [9]*?*

In my two talks I explained the main ideas towards the following characterization of the UCT problem. This perspective is the basis for some of the results presented in the talk given by Xin Li.

Theorem. *The following statements are equivalent:*

   (1) All separable, nuclear C*-algebras satisfy the UCT;
   (2) For $p \in \{2, 3\}$ and every action $\alpha : \mathbb{Z}_p \curvearrowright \mathcal{O}_2$, the crossed product $\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_p$ satisfies the UCT;
   (3) For $p \in \{2, 3\}$ and every action $\alpha : \mathbb{Z}_p \curvearrowright \mathcal{W}$, the crossed product $\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_p$ satisfies the UCT.

In the above, $\mathcal{W}$ denotes the Razak–Jacelon algebra [4].

The equivalence (1)⇔(2) was proved in [1] a few years ago. Its proof involves the known charactization of the UCT problem in terms of Kirchberg algebras, and some classification theory [5, 8, 6, 7]. This was the main focus of the first talk.

The equivalence (1)⇒(3) was the focus of the second talk. This result has yet to be published, and is based on a very similar idea, but instead utilizes a less prominent characterization of the UCT problem in terms of certain TAF algebras [2], together with more recent results in classification theory [10, 3].

Let us briefly sketch the key part of the proof that is common to both equivalences. For notational convenience we restrict to the case $p = 2$.

Proposition. *There exists*

   (1) an action $\gamma : \mathbb{Z}_2 \curvearrowright \mathcal{O}_2$ such that $\mathcal{O}_2 \rtimes_\gamma \mathbb{Z}_2$ is KK-equivalent to M_{2^∞};
   (2) an action $\gamma : \mathbb{Z}_2 \curvearrowright \mathcal{W}$ such that $\mathcal{W} \rtimes_\gamma \mathbb{Z}_2$ is KK-equivalent to M_{2^∞}.

Both of these model actions are constructed in exactly the same manner, using classification theory, as fairly specific inductive limit actions.

The further ingredients from the aforementioned literature is:

Theorem. *Let A be a separable, unital, nuclear, simple C*-algebra. Then $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$. If additionally A has a unique tracial state, then $A \otimes \mathcal{W} \cong \mathcal{W}$.*

Sketch of proof for the main theorem. (1)⇔(2): Assume that (1) fails. Then the UCT must fail for a unital Kirchberg algebra A. Since we always have an extension

$$0 \longrightarrow \mathcal{C}_0(0,1) \otimes M_6^\infty \otimes A \longrightarrow Z_{2^\infty,3^\infty} \otimes A \longrightarrow (M_{2^\infty} \oplus M_{3^\infty}) \otimes A \longrightarrow 0$$

and $Z_{2^\infty,3^\infty} \sim_{KK} \mathbb{C}$, we may assume that $A \cong A \otimes M_{p^\infty}$ for $p = 2$ or $p = 3$. For notational convenience we only consider $p = 2$. Let $\gamma : \mathbb{Z}_2 \curvearrowright \mathcal{O}_2$ be the model action from above. Then

$$A \cong A \otimes M_{2^\infty} \sim_{KK} A \otimes (\mathcal{O}_2 \rtimes_\gamma \mathbb{Z}_2) \cong (A \otimes \mathcal{O}_2) \rtimes_{\mathrm{id}_A \otimes \gamma} \mathbb{Z}_2.$$

Since $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$, the action $\mathrm{id}_A \otimes \gamma$ can be identified with some action α on $\mathcal{O}_2$. This yields a counterexample to (2).

(1)⇔(3): Assume that (1) fails. Then the UCT must fail (see [2]) for a separable, unital, simple, nuclear C*-algebra A with a unique trace. Consider the model action $\gamma : \mathbb{Z}_2 \curvearrowright \mathcal{W}$ as above, and proceed exactly as before. By using $A \otimes \mathcal{W} \cong \mathcal{W}$, one obtains a counterexample to (3). □

It remains open whether it suffices to exclusively consider $p = 2$ (or another single prime number) in the main theorem.

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Almost finiteness and dynamical comparison

DAVID KERR

We formulate a notion of almost finiteness for actions of amenable groups on compact metrizable spaces as a topological analogue of hyperfiniteness for finite von Neumann algebras and probability-measure-preserving equivalence relations. This extends Matui's concept of the same name from the zero-dimensional setting [1] by incorporating the idea of a topologically small remainder for a disjoint collection of towers whose levels are open and whose shapes are Følner sets.

Almost finiteness can be viewed as a dynamical analogue of the conjunction of nuclearity and $\mathcal{Z}$-stability for C*-algebras. Indeed we show that it plays the role of $\mathcal{Z}$-stability in a dynamical version of the Toms–Winter conjecture by proving that it implies (dynamical) comparison, and that it is also a consequence of comparison when the set of ergodic probability measures is finite, paralleling results in the theory of C*-algebras due to Rørdam [3] and Matui–Sato [2] respectively. Combining this with work of Szabó, Wu, and Zacharias [4], one can deduce almost finiteness for every free minimal action of a finitely generated nilpotent group on the Cantor set for which the set of ergodic probability measures is finite.

Further strengthening the connection to the C*-algebra side, we prove that the crossed product of an almost finite free minimal action is $\mathcal{Z}$-stable. As a consequence one can produce new examples of monotracial crossed products which are classifiable, particularly in cases where dynamical techniques of a dimensional nature do not apply due to the fact that the asymptotic dimension of the acting group is infinite.

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Noncommutative dimensions and topological dynamics

JIANCHAO WU

(joint work with Ilan Hirshberg)

This talk consists of a summary of existing results on the problem of bounding nuclear dimension for crossed products as well as some newly attained results.

The motivation of the problem comes from the fact that finite nuclear dimension ([WZ10]) provides an important regularity property for C*-algebras that plays a

crucial role in the Elliott classification program of simple separable nuclear C*-algebras. This is testified by the following groundbreaking classification result that combines the work of many people in the field over several decades.

Theorem 1 ([EGLN15, TWW17]). *The class of simple separable unital C*-algebras with finite nuclear dimension and satisfying the UCT is classified by the Elliott invariant.*

Since a prominent source of C*-algebras is provided by the crossed product construction, in particular, with commutative coefficient algebras, we ask:

Question 2. *When $C_0(X) \rtimes G$ has finite nuclear dimension for a continuous G-action on a locally compact Hausdorff space X?*

There are essentially two strategies so far. The first aims to show $\mathcal{Z}$-stability first and then resorts to the solution of the Toms-Winter conjecture to deduce finite nuclear dimension. Such an approach has the ability to deal with the case where X is infinite-dimensional but has zero mean dimension with regard to the G-action ([EN14]). However, minimality of the action appears to be a crucial condition due to the use of the Toms-Winter conjecture. On the other hand, the second strategy, which makes use of Rokhlin dimension ([HWZ15]) or other related dimensions defined for topological dynamical systems, does not require minimality but needs X to have finite covering dimension.

Following the second strategy and given X has finite covering dimension, we know $C_0(X) \rtimes G$ has finite nuclear dimension in the following situations:

(1) $G = \mathbb{Z}$, X is compact and metrizable, and $G \curvearrowright X$ minimally ([TW13]);
(2) $G = \mathbb{Z}^d$, X is compact and metrizable, and $G \curvearrowright X$ freely ([Sza15]);
(3) G is finitely generated and virtually nilpotent, X is compact and metrizable, and $G \curvearrowright X$ freely ([SWZ14]);
(4) $G = \mathbb{R}$, X is metrizable, and $G \curvearrowright X$ freely ([HSWW17]);
(5) $G = \mathbb{Z}$ ([HW17]).

The last result, in particular, provides examples of non-virtually-nilpotent polycyclic groups whose group C*-algebras have finite nuclear dimension but infinite decomposition rank. The new results announced in this talk extend the above list to the following general situations:

(6) $G = \mathbb{R}$;
(7) G is finitely generated and virtually nilpotent.

Both results are from joint work with Ilan Hirshberg.

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Uniqueness of Cartan subalgebras in II$_1$ factors: a survey and key methods

STEFAAN VAES

The existence and uniqueness of Cartan subalgebras in C*-algebras is one of the main focuses of this mini-workshop. Compared to the state-of-the-art in the theory of von Neumann algebras, very little is known on Cartan subalgebras of C*-algebras. The main goal of this lecture is to give a survey of the von Neumann algebraic results that were obtained in the last 10 years and to highlight the key methods and techniques that are used to prove these results. This may serve as an inspiration for the future developments in C*-algebra theory, much in the same way as the work of Connes, Haagerup and Popa on the classification of injective factors has driven the stunning recent progress on the classification of simple nuclear C*-algebras.

The first uniqueness theorem for Cartan subalgebras in any specific II$_1$ factor was established by Ozawa and Popa in [2]. They proved that a crossed product II$_1$ factor $M = L^\infty(X) \rtimes \mathbb{F}_n$ given by an essentially free, ergodic, probability measure preserving (pmp), *profinite* action of a free group has $L^\infty(X)$ as its unique Cartan subalgebra up to unitary conjugacy. This result was first generalized to crossed products by a much wider class of groups Γ, most notably in [1], where the same result was shown to hold for arbitrary non-elementary hyperbolic groups. Finally, in [3, 4], Popa and I proved the uniqueness of the Cartan subalgebra for crossed products by *arbitrary* free ergodic pmp actions of non-elementary hyperbolic groups Γ.

All these uniqueness theorems for Cartan subalgebras in II$_1$ factors are proved in the framework of Sorin Popa's deformation/rigidity theory. The first key method in that theory is Popa's intertwining-by-bimodules, providing in particular a verifiable criterion for the unitary conjugacy of two Cartan subalgebras. One of the main challenges will be to find an appropriate C*-algebra analogue of this concept. Secondly, the key properties of the groups Γ mentioned above are *weak amenability*

and the fact that they belong to *Ozawa's class $\mathcal{S}$*. I explained the deep and subtle idea of [2] deducing from weak amenability an asymptotic invariance property, which is one of the main ingredient for all of the proofs in [1, 2, 3, 4].

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Cartan subalgebras in uniform Roe algebras

RUFUS WILLETT

(joint work with Stuart White)

A Cartan subalgebra B in a unital C*-algebra A is a maximal abelian self-adjoint subalgebra (MASA) equipped with a faithful conditional expectation $E : A \to B$, and with the property that the *normalizer* of B in A

$$N_A(B) := \{a \in A \mid aBa^* \cup a^*Ba \subseteq A\}$$

generates A as a C*-algebra. This notion was introduced by Renault [3], who showed that to a pair $B \subseteq A$ of a Cartan subalgebra in a C*-algebra there is a canonically associated (twisted) groupoid such that A becomes the associated (twisted) groupoid C*-algebra. Roughly, this says that A arises from a dynamical system in some sense. It then becomes interesting to ask to what extent this Cartan subalgebra is unique; roughly this asks to what extent the underlying dynamics is unique. This is a question that has been very well-studied in the von Neumann algebra context, but much less is known for C*-algebras: see [2] for a survey.

The goal of my talk was to discuss uniqueness of Cartan subalgebras for a particular class of examples. To describe these, let X be a discrete metric space, which we assume has bounded geometry (this means that for each r, there is a uniform bound on the cardinalities of all r-balls). For example, X might be a discrete group Γ equipped with a choice of word metric. Let $\mathbb{C}_u[X]$ denote the collection of all X-by-X indexed matrices $a = (a_{xy})$ with uniformly bounded entries and such that the *propagation*

$$\text{prop}(a) := \sup\{d(x,y) \mid a_{xy} \neq 0\}$$

is finite. This is a $*$-algebra when equipped with the usual matrix operations; completing in the norm arising from its natural representation on $\ell^2(X)$ gives the *uniform Roe algebra $C_u^*(X)$*; several variants of this C*-algebra were originally introduced for applications to index theory. If $X = \Gamma$ is a discrete group, then $C_u^*(X)$ is canonically $*$-isomorphic to $\ell^\infty(\Gamma) \rtimes_r \Gamma$.

Now, the uniform Roe algebra $C_u^*(X)$ contains the multiplication operators $\ell^\infty(X)$ as a canonical Cartan subalgebra. In this talk, I discussed examples showing that it also contains 'exotic' Cartan subalgebras that are not even $*$-isomorphic to $\ell^\infty(X)$. On the other hand, if one assumes that X is has *finite decomposition complexity*[1] [1], then any Cartan subalgebra of $C_u^*(X)$ that is $*$-isomorphic to $\ell^\infty(X)$ and satisfying a mild separability condition is automatically unitarily conjugate to $\ell^\infty(X)$.

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Cartan subalgebras, crossed products, and the UCT

Xin Li

(joint work with S. Barlak)

In my talk, I explained the connection between the UCT problem and Cartan subalgebras, and I presented a reformulation of the UCT problem in terms of the behaviour of Cartan subalgebras under actions of finite cyclic groups.

I started by explaining the result (see [1]) that if a separable nuclear C*-algebra has a Cartan subalgebra, then it satisfies the UCT.

The proof combines Renault's result (see [3]) and work of Tu on the UCT for groupoids (see [4]). Renault's result says that if a separable C*-algebra A has a Cartan subalgebra B, then there is a twisted groupoid (G, Σ), where G is étale, Hausdorff, and topologically principal, such that $(A, B) \cong (C_r^*(G, \Sigma), C_0(G^{(0)}))$. Tu proved that if G is an étale, Hausdorff groupoid which is amenable, then $C_r^*(G)$ satisfies the UCT. Here, topologically principal is not needed. I then explained in more detail the ingredients needed to prove Tu's theorem.

Next I presented the following theorem (see [2]), which says: Let p be prime, $\alpha : \mathbb{Z}_p \curvearrowright \mathcal{O}_2$ an outer strongly approximately inner action. Then $\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_p$ satisfies the UCT if and only if there exists a Cartan subalgebra $C \subseteq \mathcal{O}_2$ with $\alpha(C) = C$.

I finished by explaining the key ideas of the proof of this theorem.

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Ample groupoid algebras

Lisa Orloff Clark

An ample groupoid is an étale groupoid in which the unit space is totally disconnected. In this talk, I describe the Steinberg algebra associated to an ample Hausdorff groupoid as introduced (independently) in [4] and [3]. The class of Steinberg algebras includes Leavitt path algebras, Kumjian-Pask algebras, group algebras and discrete inverse semigroup algebras.

Each Steinberg algebra sits as a dense $*$-subalgebra inside a groupoid C*-algebra and recent results have shown this purely algebraic subalgebra can be used to gain insight into the bigger analytic C*-algebra. For example, we show in [1] that a Steinberg algebra is simple (in that it has no nontrivial ideals) if and only if the corresponding C*-algebra is simple (in that it has no nontrivial closed ideals). This result is unexpected and it shows that the connection between these two structures is deeper than one might expect. Indeed, the techniques developed in [1] offer a new strategy for proving results about groupoid C*-algebras: first, consider the corresponding property in the Steinberg algebra and use insights gained to prove the C*-algebra result.

The class of ample groupoid C*-algebras is broad. For example, all Kirchberg algebras in UCT can be realised as ample groupoid C*-algebras. A long-standing open question relating to this is the following: What are the necessary and sufficient conditions on the groupoid to ensure the corresponding C*-algebra is purely infinite simple? We hope the Steinberg algebra can help to answer this. We make some progress in [2] where we show that the Steinberg algebra being (algebraically) purely infinite simple implies that the larger C*-algebra is purely infinite simple as well. Whether the converse is true is not known.

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Regular C*-algebras for semigroups associated to toric varieties
Joachim Cuntz

Let S be a finitely generated subsemigroup of $\mathbb{Z}^n$. Then its monoid algebra $\mathbb{C}S$ is a finitely generated commutative $\mathbb{C}$-algebra with no non-zero nilpotent elements. It is therefore the coordinate ring of an affine variety over $\mathbb{C}$ - a so-called called affine toric variety. Here we study the left regular semigroup C*-algebra $C^*_\lambda S$. The case $n = 1$ is without interest: for a non-trivial subsemigroup S of $\mathbb{Z}$ the C*-algebra $C^*_\lambda S$ is in fact always isomorphic (non-canonically) to the ordinary Toeplitz algebra $C^*_\lambda \mathbb{N}$. Our objective here is the computation of the K-theory of $C^*_\lambda S$ for a finitely generated subsemigroup S of $\mathbb{Z}^2$. In [2], [1] we had determined the K-theory of a large class of semigroup C*-algebras using the independence condition introduced by Xin Li. The interesting feature of the semigroups $S \subset \mathbb{Z}^n$ however is that they do not satisfy this condition except in trivial cases, so that the method of [2], [1] cannot be used. Another consequence is that the K-theory for $C^*_\lambda S$ contains a torsion part. But we are able to show that the K-theory of $C^*_\lambda S$ for $S \subset \mathbb{Z}^2$ is always described by a simple formula involving only the 'faces' of S. A face in the semigroup $S \subset \mathbb{Z}^n$ is a subsemigroup with the property that $a + b \in F$ for $a, b \in S$ implies that $a, b \in F$. From now on we assume that S is a subsemigroup of $\mathbb{Z}^2$ that generates $\mathbb{Z}^2$ as a group and that S contains no invertible elements (for addition) besides 0. Such a semigroup has exactly two one-dimensional faces F_1, F_2. Denote by $F = F_1 + F_2$ the subsemigroup generated by F_1 and F_2. It is then easy to see that the semigroup quotient S/F is equal to the quotient of the group $\mathbb{Z}^2$, generated by S, by the group $F - F$ generated by F, and that this quotient is a finite abelian group. We prove the following theorem

Theorem. *Let S be a finitely generated subsemigroup of $\mathbb{Z}^2$ as above. The K-theory of $C^*_\lambda S$ is determined by the formula $K_0(C^*_\lambda S) = S/F \oplus \mathbb{Z}$ $K_1(C^*_\lambda S) = 0$.*

The proof of this theorem is based on the long exact sequences in K-theory associated to natural ideals in $C^*_\lambda S$. Explicit formulas for the maps occuring in these sequences are obtained from an analysis of the algebraic structure of the semigroup S and its quotients S/F_1, S/F_2. We mention that, in [4], on the basis of previous results in three papers by different authors, a formula for the K-theory of $C^*_\lambda S$ (which looks different from ours, but gives the same result) had been established in the important special case of a 'saturated' finitely generated subsemigroup of $\mathbb{Z}^2$. In the saturated case our computation is somewhat more direct than the one in [4]. But much of the analysis in our paper is really concerned with the non-saturated case.

The results described in this extended abstract appear in [3]

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$\mathcal{Z}$-stability, Property Γ, Partitions of Unity and Nuclear Dimension

AARON TIKUISIS, STUART WHITE

(joint work with Jorge Castillejos, Samuel Evington and Wilhelm Winter)

Following the complete classification of non-elementary, simple, separable and unital C*-algebras of finite nuclear dimension by the Elliott invariant, it is a major task to identify which simple nuclear C*-algebras have finite nuclear dimension.

Conjecture (Toms-Winter). *Let A be a non-elementary, simple, separable, unital and nuclear C*-algebra. The following are equivalent:*

(1) A has finite nuclear dimension;
(2) A absorbs the Jiang-Su algebra $\mathcal{Z}$ tensorially;
(3) A has strict comparison of positive elements.

A stronger form of the conjecture also predicts that when A is stably finite (1) can be replaced by the stronger condition:

(1′) A has finite decomposition rank.

This conjecture has seen substantial work: the implications (1)⇒(2)⇒(3) hold in general (due to Winter and Rørdam respectively). The directions (3)⇒(2)⇒(1) would represent the full force of Connes' characterisations of injectivity. Partial results are known either assuming A has particular internal approximations, or under assumptions on the trace space of A.

Recent developments in the structure of crossed product C*-algebras, show the importance of being able to access classification from Jiang-Su stability. Despite the massive progress that has been made in obtaining finite nuclear dimension for crossed products through dynamical notions of dimension (such as the Rohklin dimension), it now seems likely that the class of groups for which this approach will succeed will be relatively limited. In contrast Kerr has identified a dynamical condition (almost finiteness), which gives $\mathcal{Z}$-stability for the crossed product (see his talk in this workshop), and, among things, with Conley, Jackson, Marks, Seward and Tucker-Drob, he has shown that this holds for generic free minimal actions of elementary amenable groups.

Murray and von Neumann's Property Γ

In their foundational work on II_1 factors, Murray and von Neumann introduced *property* Γ in order to distinguish the free group factor(s) from the hyperfinite II_1 factor. In today's language a separably acting II_1 factor $\mathcal{M}$ has property Γ if and only if the central sequence algebra $\mathcal{M}^\omega \cap \mathcal{M}'$ is non-trivial. In contrast, Akemann and Pedersen showed that the only separable C*-algebras with no non-trivial *norm* approximately central sequences are of continuous trace. In order to make a useful definition for simple C*-algebras, we use Diximer's characterisation: a II_1 factor $\mathcal{M}$ has property Γ if for each (or equivalently for some) $k \geqslant 2$, there are k pairwise orthogonal projections in $\mathcal{M}^\omega \cap \mathcal{M}'$ each of trace $1/k$. This formulation has been used by Christensen and Pisier to establishing the Kadison's similarity property in the presence of property Γ, and Ge and Popa use it to show these factors are singly generated.

In order to state our definition, recall that if A is a separable C*-algebra with ultrapower A_ω, then the *limit traces*, denoted $T_\omega(A_\omega)$, on A_ω are those traces[2] of the form $\tau((x_n)_{n=1}^\infty) = \lim_{n \to \omega} \tau_n(x_n)$, for some sequence $(\tau_n)_{n=1}^\infty$ of traces on A.

Definition. We say that a simple unital C*-algebra A has *property* Γ if and only if, for each $k \geqslant 2$, there exist pairwise orthogonal positive contractions $e_1, \ldots, e_k$ in $A_\omega \cap A'$ such that

$$(1) \qquad \tau(e_i a) = \frac{1}{k}\tau(a), \quad a \in A, \ \tau \in T_\omega(A_\omega), i = 1, \ldots, k.$$

As with II_1 factors, it suffices to verify property Γ for some $k \geqslant 2$. Also, when $\partial T_e(A)$ is compact, it suffices to verify (1) with $a = 1_A$.

It is open whether all simple, separable, unital and nuclear C*-algebras have property Γ. For our purposes, there are two important classes of examples:

- All $\mathcal{Z}$-stable separable unital C*-algebras have property Γ.
- All separable unital nuclear C*-algebras with no finite dimensional representations whose tracial boundary is compact and of finite covering dimension have property Γ.

Our main result shows that property Γ is precisely the condition needed to establish the Toms-Winter conjecture.

Theorem 1. *The Toms-Winter conjecture holds under the additional assumption of property* Γ. *In particular* (2)⇒(1) *holds in general. Furthermore, when A is stably finite,* (1′) *is additionally equivalent to conditions* (1) *and* (2) *precisely when all traces on A are quasidiagonal.*

Partitions of unity

In our earlier work [1] with Brown, Bosa, Sato and Winter, we established (2)⇒(1) in the Toms-Winter conjecture under the hypothesis that the tracial boundary of A is compact. One major ingredient in this work, and the fundamental reason

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[2]For us, at least here, traces are always states.

we required this compactness, is the ability to use $\mathcal{Z}$-stability to 'glue' together elements which have good tracial behaviour locally across the entire trace space. In [1] this was achieved using Ozawa's theory of W*-bundles, which relies on a compact tracial boundary. In this setting $\mathcal{Z}$-stability gives rise to a trivial bundle, and then the required gluing can be performed using a partition of unity argument. Abstracting a version of this argument leads to the following definition.

Definition. Let A be a separable C*-algebra. Say that A has *complemented partitions of unity* if for any finite family $a_1, \ldots, a_n$ of positive contractions in A, and $\delta > 0$ such that

$$\sup_{\tau \in T(A)} \min_{i=1,\ldots,n} \tau(a_i) < \delta,$$

there exist pairwise orthogonal positive contractions $e_1, \ldots, e_n \in A_\omega \cap A'$ such that:

(1) $\tau(\sum_{i=1}^n e_i) = 1$ for all $\tau \in T_\omega(A_\omega)$;
(2) $\tau(a_i e_i) \leqslant \delta \tau(e_i)$ for all $\tau \in T_\omega(A_\omega)$.

The first property says that the e_i form a tracial partition of unity; the second is that, in a suitable sense, they complement the a_i (thought of as being "subordinate to the complement of the support of $(a_i - \delta)_+$," as might be the case for a classical partition of unity on $C(X)$). In applications the orthogonality and asymptotic centrality of the e_i are vital. For example, suppose A is nuclear and has no finite dimensional representations. Then one can use compactness of $T(A)$ to obtain finitely many order zero maps $\phi_1, \ldots, \phi_n : M_2 \to A$ such that for each $\tau \in T(A)$, there is some i with $\tau(\phi_i(1))$ is large. Taking $a_i = 1_A - \phi_i(1)$, we can use a complemented partition of unity (e_i) to define an order zero map $\phi(\cdot) = \sum_{i=1}^n e_i^{1/2} \phi_i(\cdot) e_i^{1/2}$, so that $\tau(\phi(1))$ is globally large. The pairwise orthogonality of the e_i and asymptotic centrality ensure that this really is an order zero map.

Our main technical result is that property Γ gives rise to these complemented partitions of unity, at least when A is nuclear.[3] The main theorem is then obtained using the complemented partitions of unity to replace the gluing arguments of [1].

<div align="center">OBTAINING PARTITIONS OF UNITY</div>

We end this note with a brief description of our main technical result:

Theorem 2. *Let A be a separable nuclear C*-algebra with property Γ. Then A has complemented partitions of unity.*

We fix $a_1, \ldots, a_n$ and δ as in the definition of complemented partitions of unity. The first stage is to use a refined version of the completely positive approximation property, due to Brown, Carrión and SW. We can then push a partition of unity obtained at the level of the approximation back to A in a fashion compatible with the multiplication. This produces e_i as in the definition of complemented partitions of unity, except for the fact that the e_i need not be pairwise orthogonal.

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[3]When the tracial boundary of A is compact, we do not need a nuclearity hypothesis.

The second step is to use property Γ to convert the e_i in the first step into tracial projections, and then place these underneath pairwise orthogonal approximately central elements (as in the definition of property Γ). This ensures that the resulting e_i are pairwise orthogonal, but it comes at a cost: they only have $\tau(\sum_{i=1}^n e_i) = \frac{1}{n}$ for all $\tau \in T_\omega(A_\omega)$. The argument is then repeated underneath the tracial projection $1 - \sum_{i=1}^n e_i$, obtaining another $\frac{1}{n}$ of the remaining trace. Carrying on in this way gives the required partition of unity: the point being that the geometric series $\frac{1}{n} + \left(1 - \frac{1}{n}\right)\frac{1}{n} + \left(1 - \left(1 - \frac{1}{n}\right)\frac{1}{n}\right)\frac{1}{n} + \cdots = 1$.

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Nuclear Dimension for Cartan MASAs

WILHELM WINTER

(joint work with Kang Li, Hung-Chang Liao)

Operator algebras associated to dynamical systems often store partial – or even complete – information about the underlying system. This talk addresses the problem of how to extract dimension type invariants from amenable crossed product C*-algebras.

The key concept is a notion of covering dimension for the crossed product, modeled after nuclear dimension as introduced by myself and Joachim Zacharias. The idea behind nuclear dimension is it to use completely positive finite dimensional approximations in order to describe "coloured" approximations of a (possibly non-commutative) space. The concept has many applications, and arises naturally in abundance. However, for crossed products coming from classical dynamical systems, one in fact has approximations which also keep track of the canonical Cartan MASAs, and the same holds for uniform Roe algebras associated to coarse metric spaces with finite asymptotic dimension. Li, Liao and I have turned this phenomenon into an abstract definition: the result is a version of nuclear dimension which keeps track of a prescribed commutative C*-algebra. It turns out a posteriori that the subalgebra has to be a Cartan MASA (in fact, even a diagonal, as observed during the workshop by Selcuk Barlak and Xin Li). For a finitely generated group with word length metric, the subalgebra dimension of the uniform Roe algebra then precisely agrees with the asymptotic dimension of the group as a coarse metric space. For integer (or $\mathbb{Z}^d$) actions on compact spaces, subalgebra dimension of the crossed product is closely related to dynamic dimension (we do not know yet whether the two are equal). Our subalgebra dimension of course has nice permanence properties and can be estimated (sometimes even computed) for many stock-in-trade examples.

Finding elements in C^*-algebras using mapping telescopes

KRISTIN COURTNEY

(joint work with Tatiana Shulman)

In this talk we demonstrate a technique developed in [2] which uses AF mapping telescopes to find elements with prescribed properties in certain C^*-algebras. An AF mapping telescope is a C^*-subalgebra of the mapping cone over an AF algebra that turns out to be projective ([4]) and to map nontrivially into some nonzero (sub)quotient of any given C^*-algebra ([2]). These together allow us to build a nonzero $*$-homomorphism lift from an AF mapping telescope into any C^*-algebra. Of course, something similar can be said of many projective C^*-algebras, but AF telescopes are particularly nice because they consist of functions on a subset of the real line, which often makes building elements with certain prescribed properties rather straightforward. Once we build such elements in the AF telescope, we use projectivity to lift them to our target C*-algebra. If the properties are preserved by the lift, we have found our desired elements in our chosen C^*-algebra.

To illustrate, we answer a question inspired by [3, Lemma 2.7], which shows (using Choi's trick from [1, Theorem 7]) that every element in the canonical dense subset $\mathbb{C}\mathbb{F}_n \subseteq C^*(\mathbb{F}_n)$ achieves its universal norm under some finite-dimensional representation of $C^*(\mathbb{F}_n)$. If a C^*-algebra has a dense subset of elements that attain their norm under a finite-dimensional representation, then the C^*-algebra has a separating family of finite-dimensional representations, i.e. it is RFD. Is the converse true? Moreover, how can we characterize the C^*-algebras for which this subset is the whole space? In [2, Theorem 3.2], we answer the first in the affirmative using standard techniques. In [2, Theorem 4.4], we use AF mapping telescopes to prove that a C^*-algebra contains an element that does not attain its norm under any finite-dimensional representation iff it has an infinite-dimensional irreducible representation iff it has a simple infinite-dimensional AF subquotient.

We conclude with a brief survey of some other results from [2] and [5] that can be obtained using this technique. In particular, it is used in [2] to show that the set of elements in an RFD C^*-algebra that attain their norm under some finite-dimensional representation is a subspace iff it is the whole C^*-algebra. It was used in [5] to show that the spectral radius is continuous only on type I C^*-algebras.

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Diagonal quasi-free automorphisms of Cuntz-Krieger algebras

SELÇUK BARLAK

(joint work with Gábor Szabó)

Let $A \in M_n(\{0,1\})$ be a $\{0,1\}$-matrix with no zero rows or columns. Consider the associated Cuntz-Krieger algebra $\mathcal{O}_A$, [2], and denote by $T_1, \ldots, T_n \in \mathcal{O}_A$ its canonical generators. Following [3] and [7], an automorphism $\alpha \in \mathrm{Aut}(\mathcal{O}_A)$ is said to be diagonal quasi-free if $\mathrm{span}\{T_i : i = 1, \ldots, n\}$ is invariant under α and $\alpha(T_i T_i^*) = T_i T_i^*$ for all $i = 1, \ldots, n$.

Using Kirchberg-Phillips classification, [5, 6], one can show that diagonal quasi-free automorphisms on simple, purely infinite Cuntz-Krieger algebras are always approximately inner. One may therefore ask the following question.

Question 1. *Is every finite, abelian group action $\alpha : G \curvearrowright \mathcal{O}_A$ by diagonal quasi-free automorphisms on a simple, purely infinite Cuntz-Krieger algebra strongly approximately inner in the sense of Izumi (see [4])? That is, is each α_g a point-norm limit of inner automorphisms by unitaries in the fixed point algebra $\mathcal{O}_A^\alpha$?*

Izumi proved in [4] that for Cuntz algebras (in their standard presentation) this is indeed true. The main result presented in this talk extends this and shows that Question 1 has an affirmative answer for Cuntz-Krieger algebras associated with aperiodic matrices (which are known to be simple and purely infinite by [2]).

Theorem 2 (cf. [1]). *Let $n \geqslant 2$ and $A \in M_n(\{0,1\})$ be an aperiodic matrix, that is, there exists some $k \geqslant 1$ such that each entry of A^k is strictly positive. Let $\alpha : G \curvearrowright \mathcal{O}_A$ be a finite, abelian group action by diagonal quasi-free automorphisms. If α is outer, then it is strongly approximately inner.*

As a consequence, Izumi's classification result [4] applies to outer actions $\alpha : \mathbb{Z}_m \curvearrowright \mathcal{O}_A$ and $\beta : \mathbb{Z}_m \curvearrowright \mathcal{O}_B$ as in Theorem 2 with m a prime power, provided that $\mathcal{O}_A$ and $\mathcal{O}_B$ are (possibly non-canonically) isomorphic to $\mathcal{O}_2$. In particular, if $m = 2$, then in this situation α and β are (cocycle) conjugate if and only if $\mathcal{O}_A^\alpha$ and $\mathcal{O}_B^\beta$ are (stably) isomorphic.

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A dynamical version of the Cuntz semigroup

Joachim Zacharias

(joint work with Joan Bosa, Francesc Perera and Jianchao Wu)

The Cuntz semigroup is an invariant for C^*-algebras combining K-theoretical and tracial properties of the algebra in question, carrying important information but being notoriously difficult to determine. It might feature in a refined invariant in future developments of the classification programme. For dynamical systems we consider a dynamical version of the Cuntz semigroup which we hope to be easier to determine than the Cuntz semigroup of the crossed product and which might make it more accessible. Following an idea of Wilhelm Winter we consider for an action α of a discrete group G on a compact metric space X tuples of open sets $(U_1, \ldots, U_n)$ in X and define $(U_1, \ldots, U_n) \preceq (V_1, \ldots, V_m)$ if for all $K_i \subset U_i$ compact there are pairwise disjoint open subsets $W_{ijk} \subset V_j$ and $g_{ijk} \in G$ (k an auxiliary index) such that $K_i \subset \bigcup_{j,k} g_{ijk} W_{ijk}$ for all i. Tuples are added by concatenation i.e. $(U_1, \ldots, U_n) + (V_1, \ldots, V_m) = (U_1, \ldots, U_n, V_1, \ldots, V_m)$. The dynamical Cuntz semigroup is defined as the semigroup of subequivalence classes of such tuples. It should be thought of as modelling the subsemigroup of the Cuntz semigroup of the crossed product $C(X) \times_\alpha G$ generated by elements from $C(X)$. The concept of strict comparison can be defined for the dynamical Cuntz semigroup and if the action α satisfies a certain Rokhlin tower decomposition property which David Kerr [2] recently introduced under the name almost finiteness, then the dynamical Cuntz semigroup indeed satisfies strict comparison and it models the subsemigroup generated by $C(X)$ inside $C(X) \times_\alpha G$. In good cases the subsemigroup coincides with that of the whole crossed product.

We generalize the definition of our dynamical dynamical semigroup to the setting of general crossed products. We even just require the situation, where G is acting on a semigroup and generalise the definition of almost finiteness allowing us to prove similar results as in the commutative case. This not only provides information on the Cuntz semigroup of the crossed product but also a route to classification via the Toms-Winter conjecture.

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Strange MASAs in UHF algebras

N. Christopher Phillips

(joint work with Simon Wassermann)

We describe only the basic result.

Theorem 1. *There exist uncountably many maximal abelian subalgebras of the* 2^∞ *UHF algebra, each isomorphic to* $C([0,1])$*, and no two of which are conjugate by automorphisms of the* 2^∞ *UHF algebra.*

None of them is a Cartan maximal abelian subalgebra in the sense of Definition 4.13 in Chapter II of [4], or a diagonal in the sense of Definitions 1 and 3 in Section 1 of [2].

The standard Cartan maximal abelian subalgebra of the 2^∞ UHF algebra is isomorphic to $C(K)$ with K being the Cantor set. Blackadar's paper [1] gives a Cartan maximal abelian subalgebra of the 2^∞ UHF algebra which is isomorphic to $C(K \times S^1)$. Kumjiam [3] has given an example of a maximal abelian subalgebra of a simple AF algebra which is isomorphic to $C(S^1)$. However, there were no known examples of maximal abelian subalgebras of UHF algebras (Cartan or not) which are isomorphic to $C(X)$ with X connected.

The basic idea is the following construction of a maximal abelian subalgebra $D \subset C([0,1], M_2)$ which is isomorphic to $C([0,1])$. Let $S \subset U(M_2)$ be a path connected closed subset of the unitary group M_2 such that $1 \in S$. Let $v \colon [0,1) \to U(M_2)$ be a continuous function such that the range of $v|_{[1-\varepsilon,\, 1)}$ is dense in S for every $\varepsilon > 0$. Then set

$$D = \big\{f \in C([0,1], M_2) \colon v^*(t)f(t)v(t) \text{ is diagonal for all } t \in [0,1)\big\}.$$

If $1 \in \mathrm{int}(S)$, then $f \in D$ implies that $f(1)$ is diagonal. It is not hard to show that

$$D \cong \big\{f \in C([0,1], M_2) \colon f(t) \text{ is diagonal for all } t \in [0,1) \text{ and } f(1) \in \mathbb{C} \cdot 1\big\},$$

which is isomorphic to $C([0,1])$. A suitable more complicated iteration of this construction, for which we omit details, provides maps

$$C([0,1], M_2) \longrightarrow C([0,1], M_4) \longrightarrow C([0,1], M_8) \longrightarrow \cdots$$

such that the image of D in each of these algebras is maximal abelian, and such that classification results imply that the direct limit is the 2^∞ UHF algebra A.

To prove that the image of D really is maximal abelian in A, we need the following definition.

Definition 2. *Let B be a unital C*-algebra, and let $D \subset B$ be a commutative subalgebra. For $a \in B$, we define*

$$\Lambda(D, a) = \sup\big(\{\|xa - ax\| \colon x \in D \text{ satisfies } \|x\| \leqslant 1\}\big).$$

We define the commutation constant $\Gamma_B(D)$ *to be*

$$\Gamma_B(D) = \inf\big(\{M \in [0, \infty) \colon \mathrm{dist}(a, D) \leqslant M\Lambda(D, a) \text{ for all } a \in B\}\big).$$

If $\Gamma_B(D) < \infty$, then D is maximal abelian. The commutation constant behaves well in direct limits, making it possible to prove that, with suitable choices, D as above is in fact maximal abelian in the 2^∞ UHF algebra A. We get many nonconjugate maximal abelian subalgebras because we can vary $\Gamma_B(D)$ by varying the choice of the set S used above.

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The Rieffel projection via groupoids

GEORGE ELLIOTT

(joint work with Dickson Wong)

The Rieffel projection is both not quite canonical and also, perhaps, somewhat hard to understand. The Rieffel Hilbert module, from which it derives, is perhaps canonical, but is, on the face of it, especially when one considers the algebra-valued inner product, also somewhat hard to understand.

It turns out that an extension of the well-known groupoid underlying the rotation algebra, which itself does not yield a non-trivial projection in the algebra since its object space is connected, has a disconnected space of objects. One of its clopen components corresponds to the unit of the rotation algebra, and its complement corresponds to a projection in a larger, Morita equivalent, C*-algebra with the same K_0-class as the Rieffel projection.

This groupoid, consisting of a disconnected version of the cut-down of the groupoid of the Kronecker flow to the figure eight consisting of the union of the two generating circles on the torus, can also be used to construct the Rieffel module directly. One notices that it has a (clopen) copy of the real line on which the cut-down subgroupoid corresponding to the rotation algebra acts, and then the module action of the algebra of continuous functions of compact support on the groupoid on the space of continuous functions of compact support on the line, arising from (finite sum) convolution, constitutes on completion the Rieffel Hilbert module over the rotation C*-algebra. (This construction yields in a natural way that the module is finitely generated and projective.)

On Classification of MASAs in Graph C^*-Algebras

Wojciech Szymański

(joint work with Tomohiro Hayashi and Jeong Hee Hong)

Maximal abelian subalgebras (MASAs) have played very important role in the study of von Neumann algebras from the very beginning, and their theory is quite well developed by now. In particular, classification of Cartan subalgebras plays a central role in Popa's deformation-rigidity theory. Theory of MASAs of C^*-algebras is somewhat less advanced, several nice attempts in this direction notwithstanding. Our particular interest lies in classification of MASAs in purely infinite simple C^*-algebras, and especially in graph C^*-algebras. In addition to its intrinsic interest, better understanding of MASAs could have significant consequences for the still very much open classification of automorphisms and group actions. In this context, we would like to single out the recent work of Barlak and Li, [1], where a connection between the outstanding UCT problem for crossed products and existence of invariant Cartan subalgebras is investigated.

Let $C^*(E)$ be the C^*-algbra of a finite graph, purely infinite and simple, and let $\mathcal{D}_E$ be its diagonal MASA, a Cartan subalgebra of $C^*(E)$. Much better understanding of the automorphism group $\mathrm{Aut}(C^*(E))$ could be achieved through: (i) analyzing the subgroup $\mathrm{Aut}(C^*(E), \mathcal{D}_E)$ of diagonal-preserving automorphisms, and (ii) classifying Cartan subalgebras of $C^*(E)$ that are (outer) conjugate to $\mathcal{D}_E$. A good progress towards (i) has been obtained in [2], but question (ii) remains wide open. If one is interested in the structure of the outer automorphism group $\mathrm{Out}(C^*(E))$ instead, then in relation to (ii) the relevant problem is to decide for which outer automorphisms α of $C^*(E)$ the two MASAs $\mathcal{D}_E$ and $\alpha(\mathcal{D}_E)$ are inner conjugate. Building on the work in [2] on the Cuntz algebras, progress towards resolving this problem for certain classes of automorphisms of $C^*(E)$ has been made in [4]. In particular, $\mathcal{D}_E$ and $\alpha(\mathcal{D}_E)$ are not inner conjugate for every quasi-free automorphism α such that $\alpha(\mathcal{D}_E) \neq \mathcal{D}_E$. The same holds true for certain (non quasi-free) localized automorphisms (or even proper endomorphisms). A definite answer to this question for all localized automorphisms is still out of reach, but perhaps should be possible.

References

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Problem sessions

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There have been two 45 minute problem sessions. In the first one, various problems around the subject of the mini-workshop have been collected, which are listed below.

Problem 1. Does the Razak–Jacelon algebra $\mathcal{W}$ have a Cartan subalgebra?

Problem 2. Do all simple nuclear classifiable C*-algebras (within the scope of the Elliott program) have a Cartan subalgebra?

Problem 3. Does the Jiang-Su algebra $\mathcal{Z}$ have any distinguished (one-dimensional) Cartan subalgebras?

Problem 4. Is there sometimes a classification of Cartan subalgebras, e.g. in the case of classifiable C*-algebras in terms of the spectrum and if necessary some information on the isotropy groups of the underlying topologically principal groupoid?

Problem 5. What happens if one relaxes the defining conditions of a Cartan subalgebra, e.g. faithfulness of the conditional expectation or existence of an approximate unit?

Problem 6. Is there a notion for strongly self-absorbing Cartan pairs? If so, what about existence and uniqueness results for such Cartan pairs?

Problem 7. Characterize C*-algebras for which all Cartan subalgebras are C*-diagonals in the sense of Kumjian, that is, which have the unique extension property. Do all simple infinite dimensional C*-algebras with Cartan subalgebras also admit a Cartan subalgebra that is not a C*-diagonal?

Problem 8. What can be said about the induced map on K-theory of the natural inclusion of a Cartan subalgebra?

Problem 9. Is there an appropriate equivalence relation on Cartan subalgebras à la Popa's intertwining by bi-modules " $\prec$ " for von Neumann algebras?

Problem 10. What is the connection between Cartan subalgebras and the completely positive approximation property?

Problem 11. Which homeomorphism types of compact Hausdorff spaces do appear as MASAs of the CAR algebra M_{2^∞}? Does the CAR algebra have a Cartan subalgebra with connected spectrum?

Problem 12. Given an action $G \curvearrowright X$ of a locally compact, second countable group on a compact Hausdorff space. When does there exist a G-equivariant embedding of $C(X)$ into M_{2^∞} such that the image is a MASA?

Problem 13. Is every order two automorphism of $\mathcal{O}_2$ strongly approximately inner in the sense of Izumi?

Problem 14. Let A be an AF algebra and let $\alpha : \mathbb{Z}_n \curvearrowright A$ be an action of a finite cyclic group. Does $A \rtimes_\alpha \mathbb{Z}_n$ always satisfy the UCT? Is it possible to reduce the UCT problem to such crossed products? What if we assume that A is a UHF algebra?

Problem 15. Which groups (or groupoids) are UCT-preserving in the following sense: whenever A is a separable C*-algebra satisfying the UCT and $\alpha : G \curvearrowright A$ is a continuous action, then $A \rtimes_{\alpha,\mathrm{r}} G$ satisfies the UCT?

Problem 16. Is the hyperfinite II_1-factor $\mathcal{R}$ quasidiagonal? Is it MF?

The aim of the second session was to discuss selected problems in rounds with all participants. Much of this session revolved mainly around the first problem, being by far the most accessible one — a participant initially responded to the question *How can it not?* — while still being interesting in its own right. The discussions were very vivid and carried over to the evening and the next day. This resulted in three separate (and a priori different) constructions of Cartan subalgebras inside $\mathcal{W}$ within less than 24 hours of the problem session, as well as related questions for further research, showcasing the timelyness of the mini-workshop.

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