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New Trends in Teichmüller Theory and Mapping Class Groups

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Abstract. In this workshop, various topics in Teichmüller theory and mapping class groups were discussed. Twenty-three talks dealing with classical topics and new directions in this field were given. A problem session was organised on Thursday, and we compiled in this report the problems posed there.

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Introduction by the Organisers

The workshop *New Trends in Teichmüller Theory and Mapping Class Groups*, organised by Ken'ichi Ohshika (Osaka), Athanase Papadopoulos (Strasbourg), Robert Penner (Bures-sur-Yvette) and Anna Wienhard (Heidelberg) was attended by 50 participants, including a number of young researchers, with broad geographic representation from Europe, Asia and the USA. During the five days of the workshop, 23 talks were given, and on Thursday evening, a problem session was organised.

Teichmüller theory originates in the work of Teichmüller on quasi-conformal maps in the 1930s, and the study of mapping class groups was started by Dehn and Nielsen in the 1920s. The subjects are closely interrelated, since the mapping class group is the automorphism group of Teichmüller space with respect to its canonical complex structure and with respect to various metrics. Classically, Teichmüller theory is the study of moduli for complex structures on surfaces, but the expression has now a broader sense as the study of geometric structures on surfaces with several applications, and the various aspects of the theory are at the intersection of the fields of low-dimensional topology, algebraic topology, hyperbolic geometry, representations of discrete groups in Lie groups, symplectic geometry, topological quantum field theory, string theory, mathematical physics and others. All these interactions originate from the fact that Teichmüller space can be seen from various angles: as a space of equivalence classes of marked hyperbolic metrics, as a space of equivalence classes of complex algebraic curves, as a space of equivalence classes of marked conformal structures, as a space of equivalence classes of representations of the fundamental group of a surface into a Lie group $\mathrm{SL}(2,\mathbb{R})$ and as a component of the moduli space of flat G-connections on a fixed surface where G is the Möbius group.

Since the works of Thurston starting in the 1970s, Teichmüller theory has a great impact on on low-dimensional geometry and topology. It has absorbed new techniques and viewpoints coming from complex analysis, combinatorial group theory, low-dimensional topology among others. Recently Teichmüller theory became a wider theory through its ramification and development into higher Teichmüller theory which extends this theory to representations of the surface into appropriate Lie groups G. There is also a well-developed quantum Teichmüller theory, and a super Teichmüller theory which is beginning to grow. In fact, there was a substantive breakthrough during the workshop which produced the N=1 super McShane identity, the super version of an a priori constraint on length spectra that has found wide application in the classical theory. This super version has been long-sought in the physics community. The study of mapping class group is now closely related with thriving field of geometric group theory. All these developments were represented among the participants of the workshop.

The talks given in this workshop cover at the same time the new trends and the classical topics in a well balanced way. A relatively large amount of time for discussion was left.

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Workshop: New Trends in Teichmüller Theory and Mapping Class Groups

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Diffeomorphism groups of critical regularity I

Thomas Koberda
(joint work with Sang-hyun Kim)

In this talk, we give an overview of the main result of [6], which is to construct finitely generated groups of critical regularity for each $1 \leq \alpha < \infty$. Given such an α, we write $\alpha = k + \epsilon$, where $k = \lfloor \alpha \rfloor$. A diffeomorphism f of a manifold M is of class C^α if f is a C^k diffeomorphism such that $f^{(k)}$ and $(f^{-1})^{(k)}$ are ϵ-Hölder continuous.

We concentrate on the case where $M \in \{I, S^1\}$, and we write $\mathcal{G}^\alpha$ for set of isomorphism classes of countable subgroups of $\mathrm{Homeo}^+(M)$ (that is, orientation preserving homeomorphisms of M) which admit faithful actions by C^α diffeomorphisms. The main result of [6] is that

$$\mathcal{G}^\alpha \setminus \bigcup_{\beta > \alpha} \mathcal{G}^\beta$$

and

$$\bigcap_{\beta < \alpha} \mathcal{G}^\beta \setminus \mathcal{G}^\alpha$$

both contain a finitely generated group and a simple group. This result illustrates an interplay between group theory (i.e. algebra of finitely generated groups), dynamics (i.e. groups acting on manifolds), and analysis (i.e. regularity of diffeomorphisms).

Thus, if one defines the *critical regularity* of a countable group $G < \mathrm{Homeo}^+(M)$ to be

$$\sup\{\alpha \geq 1 \mid G \in \mathcal{G}^\alpha\},$$

then the main result implies that there exist groups of all finite critical regularities, as well as examples where the critical regularity is achieved and where it is not achieved.

The reason for restricting the critical regularity to lie in $[1, \infty]$ is due to a result of Deroin–Kleptsyn–Navas [3], which asserts that a countable subgroup of $\mathrm{Homeo}^+(M)$ is topologically conjugate to a group of Lipschitz homeomorphisms. Thus, a countable group of homeomorphisms of M in some reasonable sense has regularity at least one.

Applications of the main result include the existence of C^α codimension-1 foliations on all closed, orientable 3-manifolds which have nontrivial second integral homology. The main result also implies that all (abstract) homomorphisms $\mathrm{Diff}^\alpha(\mathbb{R}) \longrightarrow \mathrm{Diff}^\beta(\mathbb{R})$ for $\alpha < \beta$ have trivial image, with the possible exception of $\alpha = 2$, in which case any such homomorphism would necessarily have abelian image.

The motivations for studying critical regularities of finitely generated groups arises from the following very general question: let M be an arbitrary smooth manifold with charts of regularity C^α. To what degree do isomorphism classes of finitely generated subgroups of $\mathrm{Diff}^\alpha(M)$ determine M and α?

For full groups of diffeomorphisms, an isomorphism between $\mathrm{Diff}^{k_1}(M)$ and $\mathrm{Diff}^{k_2}(N)$ for smooth manifolds M and N and positive integers k_1 and k_2 implies that $k_1 = k_2 = k$ and M and N are C^k-diffeomorphic, by results of Takens [11] and Filipkiewicz [4].

From the point of view of finite dimensional continuous groups, one has the famous rigidity results of Weil, Mostow, and Margulis, which assert (with various suitable hypotheses) that homomorphisms from lattices Lie groups into other Lie groups extend to the ambient Lie group of the lattice [12, 8, 7]. The connection to our main result is that if $\alpha < \beta$ and if $H \in \mathcal{G}^\beta$, then we naturally have $H \in \mathcal{G}^\alpha$ by including $\mathrm{Diff}^\beta(M) \longrightarrow \mathrm{Diff}^\alpha(M)$. On the other hand, there exist nonabelian examples $G \in \mathcal{G}^\alpha$ such that any homomorphism $G \longrightarrow \mathrm{Diff}^\beta(M)$ has abelian image.

Another main motivation for considering critical regularity of groups arises from studying which groups (especially lattices in Lie groups) can act on which compact manifolds, and preserving what sorts of structure (such as smooth atlases, volume forms, measures, etc.). Taken together, this general body of questions is known as the Zimmer Program. In dimension one, Witte [13], Burger-Monod [2], and Ghys [5] show that C^1 actions on the circle of lattices in higher rank simple Lie groups factor through finite groups. Navas [9] and Bader-Furman-Gelander-Monod [1] prove that any infinite property (T) group acting faithfully on the circle has critical regularity at most 1.5. It is therefore interesting to ask if the critical regularity of property (T) groups can be improved at all, and conjecturally no infinite property (T) groups can act faithfully on the circle.

A final motivation for studying critical regularity is a poorly understood connection to geometric group theory, which is illustrated by a result of Navas [10] and which asserts that a finitely generated group of critical regularity greater than one cannot have intermediate growth.

The purpose of this talk is mostly expository, intending to give the audience an overview of the main result and the motivations behind it. Details about the methods which go into the proof are the subject of S. Kim's talk at this same workshop.

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Diffeomorphism groups of critical regularity II

Sang-hyun Kim
(joint work with Thomas Koberda)

Let M be a smooth manifold, and let $r \geq 1$ be a real number. We say a map $f\colon M \longrightarrow M$ is C^r if f is C^k where $k = \lfloor r \rfloor$, and if the following holds in each local chart:

$$\sup_{x \neq y} \frac{|D^k f(x) - D^k f(y)|}{|x-y|^{r-k}} < \infty.$$

We denote by $\mathrm{Diff}^r_+(M)$ the group of all orientation–preserving C^r–diffeomorphisms on M.

This talk is the second in the series of two talks on the following theorem, which is a joint work with Thomas Koberda. For two groups A and B, we write $A \not\hookrightarrow B$ if there does not exist an embedding from A into B.

Theorem 1 ([1])**.** *Let M be a compact connected one–manifold, and let $r \geq 1$ be a real number. Then there exist finitely generated groups G_r and H_r satisfying the following.*

(i) $G_r \leq \mathrm{Diff}^r_+(M)$ *and* $G_r \not\hookrightarrow \bigcup_{s>r} \mathrm{Diff}^s_+(M)$*;*
(ii) $H_r \leq \bigcap_{s<r} \mathrm{Diff}^s_+(M)$ *and* $H_r \not\hookrightarrow \mathrm{Diff}^r_+(M)$*.*

The proof relies on two main ideas. These ideas can be succinctly stated as the following philosophical statements.

(A) Smoother diffeomorphisms are slower.
(B) Slower representations have more commuting relations.

The idea (A) connects the analysis (*smoother*) to the dynamics (*slower*). The idea (B) relates the dynamics to the group theory. To elaborate on this point, let us begin with a collection of open intervals $\mathcal{V}$ in $I = [0,1]$. For two points $x < y$ in I, we define their $\mathcal{V}$*–covering distance* as

$$d_{\mathcal{V}}(x,y) = \inf \{\ell \mid [x,y] \subseteq U_1 \cup \cdots \cup U_\ell \text{ for some } U_i \in \mathcal{V}\}.$$

Note that $d_{\mathcal{V}}(x,y) = \infty$ is allowed.

A concrete statement for the idea (A) is the following lemma.

Lemma 2 (Slow Progress Lemma)**.** *Let $s \geq 1$ be a real number, and let G be a group with a finite generating set V. Suppose we have a representation*

$$\psi\colon G \longrightarrow \mathrm{Diff}^s_+(I),$$

and set $\mathcal{V} = \bigcup_{v\in V} \pi_0 \operatorname{supt}\psi(v)$. Assume we have sequences

- *$v_1, v_2, \ldots \in V$ such that the natural density of*
$$\{i \in \mathbb{N} \mid v_i = v\}$$
is well-defined for each $v \in V$;
- *$N_1, N_2, \ldots \in \mathbb{N}$ such that $N_i = O(i^{s-1})$.*

Then for each $x \in I$, we have the following:

$$\lim_{i\longrightarrow\infty} \left(i - d_{\mathcal{V}}\left(x, \psi\left(v_i^{N_i}\cdots v_2^{N_2}v_1^{N_1}\right)x\right)\right) = \infty.$$

Note that $d_{\mathcal{V}}$ is a notion that is invariant under C^0–conjugacy of the action. Hence, the C^0–topological dynamics imposes an obstruction for the analytic condition of being a C^s–action, or even more strongly, being C^0–conjugate to a C^s–action.

Moreover $d_{\mathcal{V}}(x, y)$ coincides with the minimal syllable length of $g \in G$ such that $\psi(g)x \geq y$. Hence, a condition on the covering length is necessarily related to a group theoretic restriction. More precisely, the idea (B) can be materialized as follows.

Lemma 3. *Let G be a group with a finite generating set V. Suppose we have two representations*

$$\phi, \psi\colon G \longrightarrow \mathrm{Homeo}_+(I)$$

such that the following hold:

(i) *for each compact interval $J \subseteq \operatorname{supt}\phi$, the restriction of $\phi(G)$ onto J is C^2;*

(ii) *there exists $u_0 \in G \setminus (\ker\phi \cup \ker\psi)$ satisfying*
$$\overline{\operatorname{supt}\phi(u_0)} \subseteq \operatorname{supt}\phi, \quad \overline{\operatorname{supt}\psi(u_0)} \subseteq \operatorname{supt}\psi;$$

(iii) *for each pair of compact intervals*
$$J \subseteq \operatorname{supt}\phi, \quad K \subseteq \operatorname{supt}\psi,$$
there exists some $w = w(J, K) \in G$ satisfying
$$\ell_{\phi(V)}(\phi(w).\partial J) > \ell_{\psi(V)}(\psi(w).\partial K) + 1.$$

Then, we have that

$$\ker\psi \setminus \ker\phi \neq \emptyset.$$

Actually, one can further require that

$$(\ker\psi \setminus \ker\phi) \cap [G, G] \cap \langle\!\langle u_0 \rangle\!\rangle \neq \emptyset.$$

Combining the above two lemmas, we can prove the following theorem. We let $\mathrm{BS}(1, 2)$ denote the $(1, 2)$–type Baumslag–Solitar group

$$\langle a, e \mid aea^{-1} = e^2\rangle.$$

We fix $G^* = (\mathbb{Z} \times \mathrm{BS}(1,2)) * F_2$.

Theorem 4. *For each real number $r \geq 1$, there exists a representation $\phi_r : G^* \longrightarrow \mathrm{Diff}^r_+(I)$ such that for all $s > r$ and for all representation $\psi : G^* \longrightarrow \mathrm{Diff}^s_+(I)$, we have that*

$$\ker \psi \setminus \ker \phi \neq \emptyset.$$

A certain generalization of similar ideas will imply the main theorem, as is detailed in the talk.

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Pseudoconvex domains fibered by open Riemann surfaces of the same topological type

Sachiko Hamano

(joint work with Hiroshi Yamaguchi)

Let R be an open Riemann surface of genus g ($1 \leq g < \infty$) and $\chi = \{A_k, B_k\}_{k=1}^g$ be a canonical homology basis of R modulo dividing cycles C_j ($j = 1, \ldots, \nu$). Consider a triplet $(S, \widetilde{\chi}, \iota)$ consisting of a closed Riemann surface S of the same genus g, a canonical homology basis $\widetilde{\chi} = \{\widetilde{A}_k, \widetilde{B}_k\}_{k=1}^g$ of S, and a conformal embedding ι of R into S such that $\iota(A_k)$ (resp. $\iota(B_k)$) are homologous to the cycles $\widetilde{A}_k$ (resp. $\widetilde{B}_k$) on S for all $k = 1, \ldots, g$. We say that two such triplets $(S, \widetilde{\chi}, \iota)$ and $(S', \widetilde{\chi}', \iota')$ are equivalent if there is a conformal mapping f of S onto S' with $f \circ \iota = \iota'$ on R. Each equivalence class is denoted by $[S, \widetilde{\chi}, \iota]$, which is called a *closing* of the marked open Riemann surface (R, χ). Let $\mathcal{C}(R, \chi)$ denote the set of all closings of (R, χ). Since a closing $[S, \widetilde{\chi}, \iota]$ of (R, χ) induces the closed Riemann surface S of genus g, there uniquely exist normal differentials ω_j ($j = 1, \ldots, g$). Setting $\tau_{jk} := \int_{\widetilde{B}_k} \omega_j$ ($1 \leq j, k \leq g$), we have the *Riemann period matrix* T of S:

$$\mathcal{C}(R, \chi) \ni [S, \widetilde{\chi}, \iota] \quad \mapsto \quad T = T[S, \widetilde{\chi}, \iota] := \begin{pmatrix} \tau_{11} & \cdots & \tau_{1g} \\ \vdots & \ddots & \vdots \\ \tau_{g1} & \cdots & \tau_{gg} \end{pmatrix} \in \mathfrak{S}_g,$$

where $\mathfrak{S}_g$ is the Siegel upper half space of degree g. To characterize $\mathcal{C}(R, \chi)$, we shall study $\mathfrak{M}(R, \chi) := \{T[S, \widetilde{\chi}, \iota] \in \mathfrak{S}_g \mid [S, \widetilde{\chi}, \iota] \in \mathcal{C}(R, \chi)\}$.

In the case of genus one, Shiba [11] showed that $\mathfrak{M}(R, \chi) = \{\tau[S, \widetilde{\chi}, \iota] := \int_{\widetilde{B}} \omega \in \mathbb{H} \mid [S, \widetilde{\chi}, \iota] \in \mathcal{C}(R, \chi)\}$ is a closed disk in the upper half plane $\mathbb{H}$, namely, there exist $\tau^* \in \mathbb{H}$ and $\rho \in \mathbb{R}$ with $0 \leq \rho < \Im \tau^*$ such that

$$\mathfrak{M}(R, \chi) = \{\tau \in \mathbb{H} \mid |\tau - \tau^*| \leq \rho\}.$$

The Euclidean diameter 2ρ of $\mathfrak{M}(R, \chi)$ represents the size of the ideal boundary of the open torus R. On the other hand, ρ depends on R and χ, so we use the hyperbolic geometry of $\mathfrak{M}(R, \chi)$ instead of the Euclidean geometry. More precisely, the upper half plane carries the Poincaré metric, and the set $\mathfrak{M}(R,\chi)$ is again a closed disk with respect to this metric. Hence it makes sense to refer to the hyperbolic diameter σ of $\mathfrak{M}(R,\chi)$, which is invariant under any change of χ.

In higher genus case, Schmieder-Shiba [10] proved that the restriction $\mathfrak{M}_j(R,\chi)$ of $\mathfrak{M}(R,\chi)$ to each diagonal element τ_{jj} $(1 \le j \le g)$ is a closed disk in $\mathbb{H}$, i.e.,

$$\mathfrak{M}_j(R,\chi) := \{\tau_{jj}[S,\widetilde{\chi},\iota] \in \mathbb{H} \mid [S,\widetilde{\chi},\iota] \in \mathcal{C}(R,\chi)\}$$

is a closed disk in $\mathbb{H}$.

To state our main results on the behavior of non-diagonal elements, we define the $\mathbf{a}$-span for (R,χ): Let $\mathbf{a} = (a_1 \cdots a_g) \in \mathbb{R}^g$ be a $1 \times g$ matrix, and let ${}^t\mathbf{a}$ be the transpose matrix of $\mathbf{a}$. For an arbitrarily fixed matrix $\mathbf{a} \in \mathbb{R}^g \setminus \{\mathbf{0}\}$, we define the $\mathbf{a}$-*span for* (R,χ) by the complex number $\tau_{\mathbf{a}} = \tau_{\mathbf{a}}[S,\widetilde{\chi},\iota] := \mathbf{a}\ T[S,\widetilde{\chi},\iota]\ {}^t\mathbf{a} \in \mathbb{H}$. Then we can show that

$$\mathfrak{M}_{\mathbf{a}}(R,\chi) = \{\tau_{\mathbf{a}} = \tau_{\mathbf{a}}[S,\widetilde{\chi},\iota] \in \mathbb{H} \mid [S,\widetilde{\chi},\iota] \in \mathcal{C}(R,\chi)\}$$

is a closed disk in $\mathbb{H}$, namely, there exist $\tau^*_{\mathbf{a}} \in \mathbb{H}$ and $\rho_{\mathbf{a}} \in \mathbb{R}$ with $0 \le \rho_{\mathbf{a}} < \Im\tau^*_{\mathbf{a}}$ such that

$$\mathfrak{M}_{\mathbf{a}}(R,\chi) = \{\tau_{\mathbf{a}} \in \mathbb{H} \mid |\tau_{\mathbf{a}} - \tau^*_{\mathbf{a}}| \le \rho_{\mathbf{a}}\}.$$

Here, the center $\tau^*_{\mathbf{a}}$ and the radius $\rho_{\mathbf{a}}$ depend on $\mathbf{a}$ and (R,χ).

We shall study how $\mathfrak{M}_{\mathbf{a}}(R,\chi)$ varies when $R(t)$ deforms with complex parameter t from the point of view of several complex variables. Let $\widetilde{\mathcal{R}}$ be a two-dimensional complex manifold, and let $\Delta = \{t \in \mathbb{C}_t \mid |t| < r\}$ be a disk. Let $\pi : \widetilde{\mathcal{R}} \longrightarrow \Delta$ be a holomorphic submersion such that $\widetilde{R}(t) := \pi^{-1}(t)$, $t \in \Delta$, is noncompact and irreducible. We consider a two-dimensional complex submanifold $\mathcal{R}$ of $\widetilde{\mathcal{R}}$ such that for $t \in \Delta$, $R(t) := (\pi|_{\mathcal{R}})^{-1}(t)$ is a relatively compact of $\widetilde{R}(t)$; $R(t)$ is an open Riemann surface of finite genus g $(1 \le g < \infty)$; $\partial R(t)$ consists of C^ω smooth closed curves $C_j(t)$ $(j = 1, \dots, \nu)$ in $\widetilde{R}(t)$. We remark that g and ν do not depend on $t \in \Delta$. We identify $\mathcal{R}$ with the smooth deformation of open Riemann surfaces $R(t)$ of genus g. We may take a canonical homology basis $\chi(t) = \{A_k(t), B_k(t)\}_{k=1}^g$ of $R(t)$ modulo $C_j(t)$ $(j = 1, \dots, \nu)$ which moves continuously in $\mathcal{R}$ with $t \in \Delta$. Then each marked open Riemann surface $(R(t),\chi(t))$, $t \in \Delta$, induces the closed disk.

The main statements are as follows:

Theorem 1 ($g = 1$; joint with Shiba and Yamaguchi [5])**.** *Assume that $\mathcal{R}$ is a pseudoconvex domain in $\widetilde{\mathcal{R}}$. Then*

(1) *the hyperbolic diameter $\sigma(t)$ of $\mathfrak{M}(R(t),\chi(t))$ is subharmonic on Δ,*
(2) *$\sigma(t)$ is harmonic on Δ if and only if $\mathcal{R}$ is a trivial variation $\Delta \times R(0)$.*

Theorem 2 ($2 \le g < \infty$; joint with Yamaguchi [6])**.** *Assume that $\mathcal{R}$ is a pseudoconvex domain in $\widetilde{\mathcal{R}}$. Fix $\mathbf{a} = (a_1 \cdots a_g) \in \mathbb{R}^g \setminus \{\mathbf{0}\}$. Then*

(1) *the Euclidean diameter $2\rho_{\mathbf{a}}(t)$ of $\mathfrak{M}_{\mathbf{a}}(R(t),\chi(t))$ is subharmonic on Δ,*
(2) *if $\rho_{\mathbf{a}}(t)$ is harmonic on Δ, then $\rho_{\mathbf{a}}(t) = \rho_{\mathbf{a}}(0)$ for all $t \in \Delta$ and the center $\tau^*_{\mathbf{a}}(t)$ of $\mathfrak{M}_{\mathbf{a}}(R(t),\chi(t))$ is holomorphic on Δ.*

Theorem 3 ($2 \le g < \infty$; joint with Yamaguchi [6]). *Assume that $\mathcal{R}$ is a pseudoconvex domain in $\widetilde{\mathcal{R}}$. Fix $\mathbf{a}_j = (0 \cdots 0\ 1\ 0 \cdots 0) \in \mathbb{R}^g$ where 1 is the j-th element. Then*

(1) *the hyperbolic diameter $\sigma_{\mathbf{a}_j}(t)$ of $\mathfrak{M}_{\mathbf{a}_j}(R(t), \chi(t))$ is subharmonic on Δ,*
(2) *if $\sigma_{\mathbf{a}_j}(t)$ is harmonic on Δ, then $\mathfrak{M}_{\mathbf{a}_j}(R(t), \chi(t)) = \mathfrak{M}_{\mathbf{a}_j}(R(0), \chi(0))$ for all $t \in \Delta$. Namely, $\mathfrak{M}_{\mathbf{a}_j}(R(t), \chi(t))$ does not depend on $t \in \Delta$.*

Our proof of Theorem 1 is based on the variational formulas of the second order for L_1- and L_0-differentials as follows. Thanks to Kusunoki [7], on each open Riemann surface $R(t)$, $t \in \Delta$, we have the L_1-(resp. L_0-) differential $\phi_1(t,z)$ (resp. $\phi_0(t,z)$) for $(R(t), \chi(t))$, so that $\Re\phi_1 = 0$ (resp. $\Im\phi_0 = 0$) on $C_j(t)$ and $\int_{A(t)} \phi_1(t,z) = \int_{A(t)} \phi_0(t,z) = 1$. The ideal boundary of $R(t)$ is realized on parallel slits of a torus by ϕ_s. We set $\tau_s(t) = \int_{B(t)} \phi_s(t,z)$, $(s = 1, 0)$. Then we established the following variational formulas for $\Im\tau_1(t)$ and $\Im\tau_0(t)$.

Lemma 4. *Let $\phi_s(t,z) = f_s(t,z)dz$ $(s = 1, 0)$ by use the local parameter z of $\overline{R(t)}$. For each $t \in \Delta$, we have*

$$\frac{\partial^2 \Im\tau_1(t)}{\partial t \partial \bar{t}} = \frac{1}{2} \int_{\partial R(t)} \kappa(t,z) |f_1(t,z)|^2 |dz| + \left\| \frac{\partial \phi_1(t,z)}{\partial \bar{t}} \right\|^2_{R(t)},$$

$$\frac{\partial^2 \Im\tau_0(t)}{\partial t \partial \bar{t}} = -\left(\frac{1}{2} \int_{\partial R(t)} \kappa(t,z) |f_0(t,z)|^2 |dz| + \left\| \frac{\partial \phi_0(t,z)}{\partial \bar{t}} \right\|^2_{R(t)} \right).$$

Here the Dirichlet norm of a holomorphic differential ϕ on R is denoted by $\|\phi\|^2_R$, and $\kappa(t,z)$ is the following Levi curvature due to [8, (1.2)] *for a C^2-smooth defining function $\varphi(t,z)$ of $\partial\mathcal{R}$ in $\widetilde{\mathcal{R}}$:*

$$\kappa(t,z) := \left(\frac{\partial^2\varphi}{\partial t \partial \bar{t}} \left| \frac{\partial\varphi}{\partial z} \right|^2 - 2\mathrm{Re}\left\{ \frac{\partial^2\varphi}{\partial t \partial z} \frac{\partial\varphi}{\partial t} \frac{\partial\varphi}{\partial \bar{z}} \right\} + \left| \frac{\partial\varphi}{\partial t} \right|^2 \frac{\partial^2\varphi}{\partial z \partial \bar{z}} \right) \left| \frac{\partial\varphi}{\partial z} \right|^{-3} \quad \text{on } \partial\mathcal{R}.$$

Remark. The radius $\rho = \Im\tau_1 - \Im\tau_0$ of $\mathfrak{M}(R, \chi)$ gives a generalization of the Schiffer span [9] for a planar Riemann surface (cf. [1]). In the case of the deforming **planar** bordered Riemann surface $R(t)$, we established the variational formula for the Schiffer span $s(t)$ in [3]. If $\mathcal{R}$ is pseudoconvex and each $R(t)$ is planar, then $s(t)$ is *logarithmically* subharmonic (cf. [4]) . As an application, we showed the simultaneous uniformization of moving planar Riemann surfaces of class O_{AD}, which led to the extension of [2].

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Super-Teichmüller spaces and related structures

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(joint work with Ivan C.-H. Ip, Robert C. Penner)

Introduction. The study of the superstring theory has drawn the attention to the very important generalizations of Riemann surfaces, known as super-Riemann surfaces (SRS) that could be viewed as $(1|N)$-dimensional complex supermanifolds with extra structures [1], [2]. Moduli spaces of $N = 1$ SRS are of special importance (see e.g. [11] for a review). One of the ways to look at the corresponding Teichmüller spaces $S\mathcal{T}(F)$ (here F is the underlying Riemann surface of genus g with s punctures), is the view through the prism of the higher Teichmüller theory. They are defined in a manner analogous to that for the standard pure even case, $S\mathcal{T}(F) = \mathrm{Hom}'(\pi_1(F) \longrightarrow G)/G$, where instead of $PSL(2,\mathbb{R})$, G is a *supergroup* $OSp(1|2)$. Here $\pi_1(F)$ is the fundamental group of the underlying Riemann surface with punctures, and Hom' stands for the homomorphisms that map the elements of $\pi_1(F)$, corresponding to small loops around the punctures, to parabolic elements of $OSp(1|2)$, which means that their natural projections to $PSL(2,\mathbb{R})$ are parabolic elements. The image of the fundamental group under Hom' produces a generalization of the standard Fuchsian group Γ, which acts on a super-analogue of the upper half-plane H^+ producing $N = 1$ super-Riemann surfaces as a factor H^+/Γ [1]. It is necessary to build the super analogues of known objects in Teichmüller theory for the successful study of such spaces. In this note I will give an overview of the results from [9], [5], [6] concerning the generalization of the Penner coordinates [7], [8] for the super-Teichmüller space. The Penner coordinates are the coordinates on $\mathbb{R}^s_+$-bundle $\tilde{\mathcal{T}}(F) = \mathbb{R}^s_+ \times \mathcal{T}(F)$ over the Teichmüller space of $\mathcal{T}(F)$ of s-punctured surfaces with negative Euler characteristics. The construction is based on the *ideal triangulation* of F (i.e. vertices of triangualtion are the punctures) and the assigninment of a positive number to every edge of the triangulation. An important feature of these coordinates is that under the elementary changes of triangulation, known as Whitehead moves, or *flips* generating the mapping class group, the change of coordinates is rational, described by the so-called *Ptolemy relations*. Therefore, it is making the mapping class group action rational. The difficulty in constructing an analogue of these coordinates for $S\tilde{\mathcal{T}}(F) = \mathbb{R}_+^s \times S\mathcal{T}(F)$ is that $S\mathcal{T}(F)$ has many connected components enumerated by spin structures. Thus, to proceed further it is necessary to have a suitable combinatorial description of the spin structures.

Fatgraphs and spin structures. Consider the trivalent *fatgraph* τ, corresponding to an s-punctured ($s > 0$) Riemann surface F of the negative Euler characteristic (i.e. a graph with trivalent vertices), which is homotopically equivalent to F, with cyclic orderings on half-edges for every vertex [8] induced by the orientation of the surface. There is a one-to-one correspondence between the ideal triangulations and trivalent fatgraphs. Let ω be an orientation on the edges of τ. As in [9], we define a *fatgraph reflection* at a vertex v of (τ, ω) as a reversal of the orientations of ω on every edge of τ incident to v. Let us define the $\mathcal{O}(\tau)$ to be the equivalence classes of orientations on a trivalent fatgraph τ of F, where $\omega_1 \sim \omega_2$ if and only if ω_1 and ω_2 differ by a finite number of fatgraph reflections. In [9],[5], we identified such classes of orientations on fatgraphs with the spin structures on F. The paths corresponding to the boundary cycles on the fatgraph (i.e. the punctures of F) are divided into two classes depending on the parity of number k – the number of edges with orientation opposite to the canonical orientation of γ. The punctures are called Ramond (R) when k is even, and Neveu-Schwarz (NS) [11] when k is odd. In [9], we have also proved that under the flip transformations the orientations change in the generic situation as in Figure 1, where ϵ_i stand for

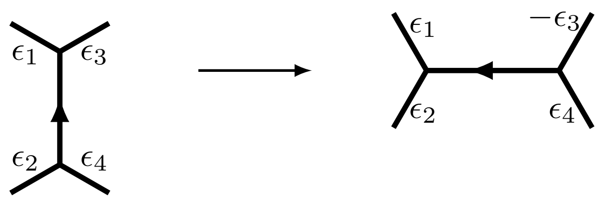

FIGURE 1. Spin graph evolution in the generic situation

orientations on edges, and extra minus sign stands for the orientation reversal.

Main Result. In [9], we have proved the following Theorem.

Theorem 1. (1) *The components of $S\tilde{T}(F)$ are determined by the space of spin structures on F. For each component C of $S\tilde{T}(F)$, there are global affine coordinates on C given by assigning to a triangulation Δ of F,*

- *one even coordinate called l-length for each edge;*
- *one odd coordinate called μ-invariant for each triangle, taken modulo an overall change of sign.*

In particular we have a real-analytic homeomorphism:

$$C \longrightarrow \mathbb{R}_{>0}^{6g-6+3s|4g-4+2s}/\mathbb{Z}_2.$$

(2) *The super Ptolemy transformations* [9] *provide the analytic relations between coordinates assigned to different choice of triangulation Δ' of F, namely upon flip transformation. Explicitly (see Figure 2), when all a, b, c, d are different edges of the triangulations of F, the Ptolemy transformations are as follows:*

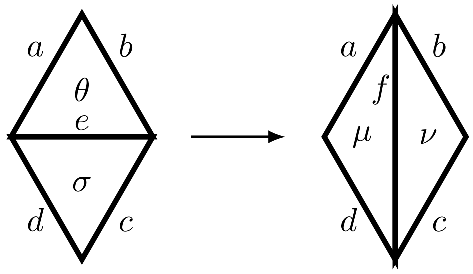

FIGURE 2. Generic flip transformation

$$ef = (ac + bd)\Big(1 + \frac{\sigma\theta\sqrt{\chi}}{1+\chi}\Big), \quad \nu = \frac{\sigma - \theta\sqrt{\chi}}{\sqrt{1+\chi}}, \quad \mu = \frac{\theta + \sigma\sqrt{\chi}}{\sqrt{1+\chi}},$$

where $\chi = \frac{ac}{bd}$, so that the evolution of arrows is as in Figure 1.

To every edge e (see the picture above) we can associate the *shear coordinate* $z_e = \log(\frac{ac}{bd})$. These parameters satisfy a linear relation for every puncture, and together with odd variables they form a set of coordinates on the $ST(F)$, thus producing the removal of the decoration, as it was in the pure even case. Here we mention that there is a physically and algebro-geometrically interesting refinement of $ST(F)$ studied in [6], corresponding to the removal of certain odd degrees of freedom associated with R-punctures.

N = 2 Super-Teichmüller space and beyond. Replacing $OSp(1|2)$ in the definition of $ST(F)$ by $OSP(2|2)$, one obtains the super-Teichmüller space of punctured $N = 2$ SRS. It has been investigated in [5], and the analogue of Penner coordinates was constructed there. Unlike $N = 1$ case, the resulting $N = 2$ SRS, obtained by uniformization, correspond to a certain subspace in the moduli space of $N = 2$ SRS [2]. We also mention that according to [2] $N = 2$ SRS are in one-to-one correspondence with $(1|1)$-dimensional supermanifolds.

An important problem [10] will be to see explicitly how to glue $N = 1$ and $N = 2$ super-Riemann surfaces using the fatgraph data following the analogue with the Strebel theory.

Another important task is to understand the (complexified) version of the results of [9], [5] in the context of spectral networks and the abelianization construction of Gaiotto, Moore and Neitzke [3]. In the super case it looks that only quasi-abelianization seems to work: one should be able to describe our constructions via the moduli space of $GL(1|1)$ local systems.

Finally, the super-Ptolemy transformations from the Theorem above, discovered in [9] (see also [5] for $N = 2$ case) should lead to new interesting generalizations of cluster algebras.

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Affine actions with Hitchin linear part

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(joint work with Tengren Zhang)

This project is about an application of some rapidly developing tools from higher Teichmüller-Thurston theory, most prominently Anosov representations, to the study of properly discontinuous group actions in affine geometry, flat pseudo-Riemannian geometry, and also pseudo-Riemannian hyperbolic geometry.

An *affine manifold* is a manifold M equipped with a flat, torsion-free affine connection ∇. If the geodesic flow of ∇ is complete, then M is called a *complete affine manifold*. Equivalently, a complete affine manifold is the quotient $M = \Gamma\backslash\mathbb{R}^d$ of a *proper affine action*, i.e. a properly discontinuous action of a group Γ by affine automorphisms of $\mathbb{R}^d$. Here the group Γ, which identifies with the fundamental group $\pi_1 M$, is required to be torsion free (otherwise the quotient is an orbifold rather than a manifold). Complete affine manifolds are generalizations of complete Euclidean manifolds, for which the connection ∇ is the Levi-Cevita connection of a complete flat Riemannian metric or equivalently the action by Γ preserves the Euclidean metric on $\mathbb{R}^d$. However, by contrast to the setting of Euclidean geometry, in which Bieberbach's theorem greatly restrict the behavior, the general picture of what complete affine manifolds M can look like is much more mysterious. The Auslander conjecture [Au, AMS3] gives a conjectural analogue of Beiberbach's theorem for the case that M is compact. However, in the non-compact case, it is unclear what restrictions the presence of a complete affine structure puts on the topology of M. Indeed in 1983, Margulis [Ma1, Ma2] found examples of proper affine actions by non-abelian free groups, destroying the natural intuition that a complete flat affine structure ought to obstruct word hyperbolicity in the fundamental group.

Recently, Danciger-Guéritaud-Kassel [DGK3] found examples of proper affine actions for *any* right-angled Coxeter group, and consequently any subgroup of such a group. Belonging to that very large and rich class, is the sub-class of *surface groups*, i.e. the fundamental groups $\pi_1 S_g$ of a closed orientable surface S_g of genus $g \geq 2$. In this case, the construction of [DGK3] gives examples of proper affine actions in dimension as low as $d = 6$. This project takes up the problem of classifying proper affine actions by surface groups $\Gamma = \pi_1 S_g$, or equivalently complete affine manifolds which are homotopy equivalent to a surface S_g.

The group of affine automorphisms $\mathsf{Aff}(\mathbb{R}^d) = \mathsf{GL}(d, \mathbb{R}) \ltimes \mathbb{R}^d$ decomposes as the semi-direct product of the linear automorphisms $\mathsf{GL}(d, \mathbb{R})$ with the translation subgroup $\mathbb{R}^d$. Hence an affine action of the group Γ consists of two pieces of data

$$(\rho, u) : \Gamma \longrightarrow \mathsf{Aff}(\mathbb{R}^d) = \mathsf{GL}(d, \mathbb{R}) \ltimes \mathbb{R}^d$$

where here $\rho : \Gamma \longrightarrow \mathsf{GL}(d, \mathbb{R})$, a homomorphism, is called the *linear part*, and $u : \Gamma \longrightarrow \mathbb{R}^d$, a cocycle twisted by ρ, is called the *translational part*. The main theorem is:

Theorem 1. *Suppose that $(\rho, u) : \Gamma \longrightarrow \mathsf{Aff}(\mathbb{R}^d) = \mathsf{GL}(d, \mathbb{R}) \ltimes \mathbb{R}^d$ is a proper affine action. Then the linear part ρ does* not *lie in the Hitchin component.*

Hitchin's famous work [Hi] found that the space $\mathsf{Hom}(\pi_1 S_g, \mathsf{PSL}(d, \mathbb{R}))$ of representations of a surface group in $\mathsf{PSL}(d, \mathbb{R})$ has a small number of connected components when $d \geq 3$, namely three in the case d odd, and six in the case d even. Hitchin singled out one (in fact two components, if d even) of those connected components for its connection to Teichmüller theory, and nowadays that component is called the *Hitchin component*. The Hitchin component is the connected component of $\mathsf{Hom}(\pi_1 S_g, \mathsf{PSL}(d, \mathbb{R}))$ containing the *Fuchsian* representations, namely those obtained as the composition $\tau_d \circ \sigma$ of the irreducible representation $\tau_d : \mathsf{PSL}(2, \mathbb{R}) \longrightarrow \mathsf{PSL}(d, \mathbb{R})$ with a discrete faithful representation $\sigma : \pi_1 S_g \longrightarrow \mathsf{PSL}(2, \mathbb{R})$ corresponding to a hyperbolic structure on S_g. Hitchin proved that this component is a ball of dimension $(2g-2) \dim \mathsf{PSL}(d, \mathbb{R})$. In Theorem 1, we say that a representation into $\mathsf{GL}(d, \mathbb{R})$ is in the Hitchin component if it is a lift of a representation in the Hitchin component for $\mathsf{PSL}(d, \mathbb{R})$.

Many tools have been developed in order to study the Hitchin component. Indeed, Labourie [L2] invented the notion of Anosov representation, now central in higher Teichmüller-Thurston theory, for this purpose. The notion was then generalized by Guichard–Wienhard [GW3] to the setting of representations of any word hyperbolic group into a semi-simple Lie group G. Anosov representations, and their recent characterizations due to Guichard–Guéritaud–Kassel–Wienhard [GGKW] and Kapovich–Leeb–Porti [KLPa, KLPb], are an essential tool used in the proof of Theorem 1.

In the case $d = 3$, the key case of Theorem 1 follows from Mess [Mes] and Goldman-Margulis [GM], and in general dimension Labourie [L1] proved that the linear part ρ of a proper affine action as in Theorem 1 can not be (a lift of) a Fuchsian representation.

Flat pseudo-Riemannian geometry in signature $(n, n-1)$. Using the basic observation that all elements of the Zariksi closure of the linear part $\rho(\Gamma)$ must have one as an eigenvalue, together with a characterization of the possible Zariski closures of Hitchin representations announced by Guichard, we may reduce to the case that $d = 2n-1$ is odd, and that the linear part $\rho(\Gamma) \subset \mathsf{SO}(n, n-1)$ in Theorem 1. Here $\mathsf{SO}(n, n-1)$ denotes the special orthogonal group of the standard indefinite quadratic form $\langle \cdot, \cdot \rangle_{n,n-1}$ of signature $(n, n-1)$. In this case the affine action (ρ, u) preserves the standard flat pseudo-Riemannian metric on $\mathbb{R}^d$ of signature $(n, n-1)$, and we use the notation $\mathbb{R}^{n,n-1}$ to denote $\mathbb{R}^d$ equipped with this flat metric. Theorem 1 is a corollary of:

Theorem 2. *Suppose $(\rho, u) : \pi_1 S_g \longrightarrow \mathrm{Isom}^+(\mathbb{R}^{n,n-1}) = \mathsf{SO}(n, n-1) \ltimes \mathbb{R}^{2n-1}$ is an action by isometries of $\mathbb{R}^{n,n-1}$ with linear part ρ a Hitchin representation in $\mathsf{SO}(n, n-1)$. Then the action is not proper.*

Note that if n is odd, Theorem 1 follows from an observation of Abels-Margulis-Soifer [AMS1], so we need only treat the case that n is even.

The strategy for Theorem 1 follows the key point of view in the work of Danciger-Guéritaud-Kassel [DGK1, DGK2] on proper actions by free groups in $\mathbb{R}^{2,1}$ and their quotients, called Margulis spacetimes. In that context, Margulis spacetimes were studied as limits of the their negative curvature counterparts, namely three-dimensional AdS spacetimes which are quotients of anti de Sitter space $\mathrm{AdS}^3 = \mathbb{H}^{2,1}$. Similarly, here we will study the above isometric actions on $\mathbb{R}^{n,n-1}$ by thinking of these as infinitesimal versions of isometric actions on the pseudo-Riemannian hyperbolic space $\mathbb{H}^{n,n-1}$.

Deforming into hyperbolic geometry of signature $(n, n-1)$. The pseudo-Riemannian hyperbolic space $\mathbb{H}^{n,n-1}$ is the model for constant negative curvature in signature $(n, n-1)$. The projective model for $\mathbb{H}^{n,n-1}$ is:

$$\mathbb{H}^{n,n-1} = \mathbb{P}\left\{x \in \mathbb{R}^{n,n} \smallsetminus \{0\} : \langle x, x \rangle_{n,n} < 0\right\} \subset \mathbb{P}(\mathbb{R}^{n,n}).$$

The projective orthogonal group $\mathsf{PSO}(n, n)$ acts on $\mathbb{H}^{n,n-1}$ as the orientation preserving isometry group of a complete metric of constant negative curvature with signature $(n, n-1)$. With coordinates respecting the orthogonal splitting $\mathbb{R}^{n,n} = \mathbb{R}^{n,n-1} \oplus \mathbb{R}^{0,1}$, the stabilizer of the basepoint $\mathcal{O} = [0 : \cdots : 0, 1]$ is precisely the orthogonal group $\mathsf{O}(n, n-1)$ acting on the $\mathbb{R}^{n,n-1}$ factor in the standard way and acting trivially on the $\mathbb{R}^{0,1}$ factor.

Now consider a Hitchin representation $\rho : \pi_1 S_g \longrightarrow \mathsf{SO}(n, n-1)$. Let $i_{n,n} : \mathsf{SO}(n, n-1) \hookrightarrow \mathsf{PSO}(n, n)$ be the natural inclusion for the orthogonal splitting $\mathbb{R}^{n,n} = \mathbb{R}^{n,n-1} \oplus \mathbb{R}^{0,1}$ above. Then $i_{n,n} \circ \rho$ stabilizes the basepoint $\mathcal{O} = [0 : \cdots : 0, 1] \in \mathbb{H}^{n,n-1}$ and acts on the tangent space of that point, a copy of $\mathbb{R}^{n,n-1}$, in the standard way by linear isometries. Now consider a deformation path $\rho_\varepsilon : \pi_1 S_g \longrightarrow \mathsf{PSO}(n, n)$ based at $\rho_0 = i_{n,n} \circ \rho$. Any such continuous deformation ρ_ε (not just small deformations) is called a $\mathsf{PSO}(n, n)$-*Hitchin representation* and such representations make up the so-called $\mathsf{PSO}(n, n)$ Hitchin component. (In general for G real split, and adjoint, the G-Hitchin component is the component of the image of the Teichmuller component by the principle representation of $\mathsf{PSL}(2, \mathbb{R})$). The derivative of ρ_ε at time $\varepsilon = 0$ is naturally a cocycle $v : \pi_1 S_g \longrightarrow \mathfrak{pso}(n, n)$ twisted by the adjoint action of ρ_0, under which $\mathfrak{pso}(n, n)$ splits as an invariant orthogonal sum $\mathfrak{pso}(n, n) = \mathfrak{so}(\mathfrak{n}, \mathfrak{n} - \mathfrak{1}) \oplus \mathbb{R}^{n,n-1}$, where the action in the second factor is by the standard representation. Hence the projection u to the $\mathbb{R}^{n,n-1}$ factor of the infinitesimal deformation v gives a cocycle of translational parts for an affine action (ρ, u) on $\mathbb{R}^{n,n-1}$. The geometric way to think of this fact is as follows: As $\varepsilon \longrightarrow 0$, the action of each $\rho_\varepsilon(\gamma)$ moves the basepoint $\mathcal{O}$ less and less, but by zooming in on the basepoint at just the right rate and taking a limit (in an appropriate sense), the action converges to an affine action (ρ, u) whose action on the basepoint (the origin in $\mathbb{R}^{n,n-1}$) is simply the derivative at $\varepsilon = 0$ to the path $\rho_\varepsilon(\gamma)\mathcal{O}$, thought of as a vector in $T_{\mathcal{O}}\mathbb{H}^{n,n-1} = \mathbb{R}^{n,n-1}$.

Now, since $i_{n,n} \circ \rho$ has trivial centralizer in $\mathsf{PSO}(n, n)$, Goldman's work [Gol] on representation varieties of surface groups implies that any ρ-cocycle $u : \pi_1 S_g \longrightarrow \mathbb{R}^{n,n-1}$ is realized as the $\mathbb{R}^{n,n-1}$ part of the derivative of a smooth deformation path ρ_ε as above (and the $\mathfrak{so}(n, n-1)$ part may be taken to be zero). We prove a key lemma that connects a criterion [GLM, GhT] for properness of the affine action (ρ, u) on $\mathbb{R}^{n,n-1}$ with the first order behavior of the two middle eigenvalues of elements $\rho_\varepsilon(\gamma)$, an inverse pair $\lambda_n, \lambda_n^{-1}$ which converges to the two one-eigenvalues of $i_{n,n} \circ \rho$ as $\varepsilon \longrightarrow 0$. From this eigenvalue behavior, we use [GGKW, KLPa, KLPb] to prove that the representations ρ_ε satisfy an unexpected Anosov condition, specifically that for $\varepsilon > 0$ small enough, $i_{2n} \circ \rho_\varepsilon$ is Anosov with respect to the stabilizer P_n of an n-plane in $\mathbb{R}^{2n}$, where here $i_{2n} : \mathsf{PSO}(n, n) \longrightarrow \mathsf{PSL}(2n, \mathbb{R})$ is the inclusion.

Remark 3. *We acknowledge that Sourav Ghosh has also announced independent work showing that a proper action in $\mathbb{R}^{n,n-1}$ corresponds to a deformation path into $\mathsf{PSO}(n, n)$ for which the inclusions into $\mathsf{PSL}(2n)$ are P_n-Anosov.*

Theorem 2 then follows from the next theorem, which is the main technical result of the project:

Theorem 4. *If $\varrho : \pi_1 S_g \longrightarrow \mathsf{PSO}(n, n)$ is a $\mathsf{PSO}(n, n)$-Hitchin representation, then $i_{2n} \circ \varrho : \pi_1 S_g \longrightarrow \mathsf{PSL}(2n, \mathbb{R})$ is not P_n-Anosov .*

Like $\mathsf{PSL}(d, \mathbb{R})$-Hitchin representations, $\mathsf{PSO}(n, n)$-Hitchin representations also enjoy all possible forms of Anosovness available in $\mathsf{PSO}(n, n)$. However, a $\mathsf{PSO}(n, n)$-Hitchin representation has no reason to be P_n-Anosov in the larger group $\mathsf{PSL}(2n, \mathbb{R})$, and the representations of the form above $i_{n,n} \circ \rho$ obviously fail this condition. Theorem 4 says that P_n-Anosovness in $\mathsf{PSL}(2n,\mathbb{R})$ never happens, even by accident, to the inclusion of a $\mathsf{PSO}(n,n)$-Hitchin representation.

The proof of Theorem 4 uses more than just Anosovness of $\mathsf{PSO}(n,n)$-Hitchin representations, specifically it uses that $\mathsf{PSO}(n,n)$-Hitchin representations satisfy Fock-Goncharov positivity [FG]. However, we remark that the proof of Theorem 2 outlined above only requires Theorem 4 for $\mathsf{PSO}(n,n)$-Hitchin representations which are small deformations of $\mathsf{SO}(n,n-1)$ Hitchin representations. A proof Theorem 4 in that case can be achieved without the full strength of Fock-Goncharov positivity, using in its place some special properties of Hitchin representations in $\mathsf{SO}(n,n-1)$ (Labourie's Property (H) from [L2]) and an argument about persistence of such properties under small deformation into $\mathsf{PSO}(n,n)$.

We also use Theorem 4 to show the negative curvature analogue of Theorem 2.

Theorem 5. *A $\mathsf{PSO}(n,n)$-Hitchin representation $\varrho : \pi_1 S_g \longrightarrow \mathsf{PSO}(n,n)$ does not act properly on $\mathbb{H}^{n,n-1}$.*

We note that proper actions by surface groups on $\mathbb{H}^{n,n-1}$ do exist when n is even (Okuda [Ok]), but not when n is odd (Benoist [B1]). In the case $n = 2$, Theorem 5 follows from work of Mess [Mes] or of Guéritaud–Kassel [GK]. In that case $\mathbb{H}^{n,n-1} = \mathbb{H}^{2,1}$ is the three-dimensional anti-de Sitters space, whose study is greatly simplified by the accidental isomorphism between $\mathsf{PSO}_0(2,2)$ and $\mathsf{PSL}(2,\mathbb{R}) \times \mathsf{PSL}(2,\mathbb{R})$. The $n = 2$ case of the proof given here of Theorem 5, through Theorem 4, is fundamentally different. Indeed, the work of Mess and of Guéritaud–Kassel does not naturally generalize to higher $\mathbb{H}^{n,n-1}$, though it does generalize to the setting of some other homogeneous spaces whose structure group is a product.

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Holomorphic Rigidity of Teichmüller space

George Daskalopoulos

I discussed the proof the *holomorphic rigidity conjecture of Teichmüller space* which loosely speaking states that the action of mapping class group uniquely determines the Teichmüller space as a complex manifold. The method of proof is through harmonic maps. The main result states that the singular set of a harmonic map from a smooth n-dimensional Riemannian domain to the Weil-Petersson completion $\overline{\mathcal{T}}$ of Teichmüller space has Hausdorff dimension at most $n-2$, and moreover, u has certain decay near the singular set. Combined with the earlier work of Schumacher and Jost-Yau, this provides a proof of the holomorphic rigidity of Teichmüller space. In addition, as I discussed, our results provide as a byproduct a harmonic maps proof of both the high rank and the rank one superrigidity of the mapping class group proved via other methods by Farb-Masur and Yeung. The main results of the talk state as follows:

Theorem 1. *Let M be a complete, finite volume Kähler manifold with universal cover $\tilde{M}$, Γ the mapping class group of an oriented surface S of genus g and p*

marked points such that $k = 3g - 3 + p > 0$*,* $\overline{\mathcal{T}}$ *the Weil-Petersson completion of the Teichmüller space* $\mathcal{T}$ *of* S *and* $\rho : \pi_1(M) \to \Gamma$ *a homomorphism. If there exists a finite energy* ρ*-equivariant harmonic map* $u : \tilde{M} \to \overline{\mathcal{T}}$*, then there exists a stratum* $\mathcal{T}'$ *of* $\overline{\mathcal{T}}$ *such that* u *defines a pluriharmonic map into* $\mathcal{T}'$*. Furthermore,*

$$\sum_{i,j,k,l} R_{ijkl} d''u_i \wedge d'u_j \wedge d'u_k \wedge d''u_l \equiv 0$$

where R_{ijkl} *denotes the Weil-Petersson curvature tensor. In particular, if additionally the (real) rank of* u *is* ≥ 3 *at some point, then* u *is holomorphic or conjugate holomorphic.*

Corollary 2 (**Holomorphic Rigidity of Teichmüller Space**)**.** *Let* Γ *denote the mapping class group of an oriented surface* S *of genus* g *and* p *marked points such that* $k = 3g - 3 + p > 0$*. Assume that* Γ *acts (as a discrete automorphism group) on a contractible Kähler manifold* $\tilde{M}$ *such that there is a finite index subgroup* Γ' *of* Γ *satisfying the properties:*

(i) $M := \tilde{M}/\Gamma'$ *is a smooth quasiprojective variety that is properly homotopy equivalent to* $\mathcal{T}/\Gamma'$ *where* $\mathcal{T}$ *is the Teichmüller space of* S *and*
(ii) M *admits a compactification* $\overline{M}$ *as an algebraic variety such that the codimension of* $\overline{M} \backslash M$ *is* ≥ 3*.*

Then $\tilde{M}$ *is equivariantly biholomorphic or conjugate biholomorphic to the Teichmüller space* $\mathcal{T}$ *of* S *and* Γ *acts like the mapping class group.*

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Curve graphs for infinite-type surfaces

Federica Fanoni

(joint work with Matthew Durham and Nicholas Vlamis)

The *mapping class group* of a surface S, denoted $\mathrm{MCG}(S)$, is the group of orientation preserving homeomorphisms of the surface, up to homotopy. If S is of finite type, i.e. its fundamental group is finitely generated, $\mathrm{MCG}(S)$ is a finitely generated group which has attracted a lot of attention, in particular because of its connections with Teichmüller theory and the study of three-manifolds.

Many authors have studied actions of $\mathrm{MCG}(S)$ on combinatorial objects associated to a surface, such as the curve graph or the arc graph. The *curve graph* $\mathcal{C}(S)$

is a graph whose set of vertices is the set of homotopy classes of essential simple closed curves. Two classes are adjacent in $\mathcal{C}(S)$ if they admit disjoint representatives. Similarly, for a surface S with at least a puncture, the *arc graph* $\mathcal{A}(S)$ is a graph whose vertices are simple arcs joining punctures, up to homotopy relative to the punctures. Edges correspond again to disjointness.

If the surface has finite topological type, both graphs have many interesting properties and the mapping class group acts "nicely" on them. Moreover, studying these actions has proven very useful to understand properties of the mapping class group, such as its coarse geometry and its cohomological properties.

In our work we are interested in *infinite-type surfaces*, i.e. those whose fundamental group is not finitely generated. Work of Kerékjártó and Richards [12] shows that these surfaces are topologically classified by the pair of topological spaces $(\mathrm{Ends}(S), \mathrm{Ends}_g(S))$ (the space of ends and its subspace of non-planar ends) and the genus $g \in \{0, 1, 2 \dots\} \cup \{\infty\}$.

Motivation to study mapping class group of infinite-type surfaces comes from studying group actions on finite-type surfaces (see [4, 5]), constructing foliations of 3-manifolds (see [6]), and from the Artinization of automorphism groups of trees and stable properties of mapping class groups (see [11, 9, 10, 8]).

Note that mapping class groups of infinite-type surfaces are uncountable, and hence not finitely generated. Moreover, while we can define curve and arc graphs as in the finite-type case, these graphs are often uninteresting from the viewpoint of coarse geometry: indeed, it is easy to see that the curve graph of an infinite-type surface has diameter two, and in many cases so does the arc graph.

Recently, a sizeable amount of work has been devoted to build interesting graphs with a mapping class group action – see the work of Bavard [3], Aramayona, Fossas and Parlier [1], and Aramayona and Valdez [2]. This body of work was concentrated on the study of *subgraphs* of curve and arc graphs. Instead, Matthew Durham, Nicholas Vlamis and myself were interested the following motivating question:

Question 1. *Given a surface S, is there a* good *graph* made of curves*?*

Here by *good* graph we mean a connected graph with an action of $\mathrm{MCG}(S)$ with unbounded orbits. A graph is *made of curves* if its set of vertices is a subset of the set of homotopy classes of essential simple closed curves. Note that we do not have any requirement on the edge relation (besides being mapping class group invariant).

To answer this question, we defined a topological invariant $\mathfrak{f}$ of surfaces as follows:

- $\mathfrak{f}(S) = \infty$ if $0 < g < \infty$;
- $\mathfrak{f}(S) \geq n$ if there exists a collection $\mathcal{P}$ of pairwise disjoint proper closed subsets of $\mathrm{Ends}(S)$, with $|\mathcal{P}| = n$, which is $\mathrm{MCG}(S)$-invariant, that is: for every $P \in \mathcal{P}$, for every $\phi \in \mathrm{MCG}(S)$, there is $Q \in \mathcal{P}$ such that $\phi(P) = Q$;
- $\mathfrak{f}(S) = 0$ otherwise.

Our main result is:

Theorem 2 ([7]). *If $\mathfrak{f}(S) = 0$, the surface S does not admit a good graph made of curves, while it does if $\mathfrak{f}(S) \geq 4$. In the second case, if there is a collection $\mathcal{P}$ as above made of singletons, then there is a (uniformly) Gromov hyperbolic good graph made of curves.*

We also classifies surfaces with $\mathfrak{f}(S) = 0$: these are the plane minus the Cantor set (the *Cantor tree*), the *Loch Ness monster* (the surface without punctures and one end accumulated by genus) and the *blooming Cantor tree* (the surface without punctures and with a Cantor set of ends).

We conjecture that for $\mathfrak{f}(S) = 1$ the answer to the motivating question is no, while for $\mathfrak{f}(S) = 2$ or 3 we have examples of surfaces for which the answer is positive or negative.

In ongoing work, Lanier and Loving constructed a large class of surfaces with $\mathfrak{f}(S) = 1$ for which the answer is no. For a large family of surfaces with $\mathfrak{f}(S) \in \{2, 3\}$ they gave a criterion to distinguish which surfaces admit a good graph made of curves and which ones don't.

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Regularity of limit sets of Anosov representations.

Tengren Zhang

(joint work with Andrew Zimmer)

Let Γ be a hyperbolic group and G be a non-compact semisimple Lie group. An *Anosov representation* $\rho : \Gamma \longrightarrow G$ is a representation that is "geometrically well-behaved". In the case when G is Lie group of real rank 1, Anosov representations agree exactly with the more classical notion of a convex cocompact representation. These representations were first introduced by Labourie [1] to study Hitchin representations, and were later generalized by Guichard-Wienhard [2] to the current generality we work with today.

In the case when $G = \mathrm{PGL}(d, \mathbb{R})$, there are several types of Anosov representations, which are indexed by an integer $k = 1, \dots, d-1$. Any k-Anosov representation $\rho : \Gamma \longrightarrow \mathrm{PGL}(d, \mathbb{R})$ admits a unique pair of continuous, ρ-equivariant, dynamics preserving, and transverse *limit maps*

$$\xi^{(k)} : \partial\Gamma \longrightarrow \mathrm{Gr}_k(\mathbb{R}^d) \quad \text{and} \quad \xi^{(d-k)} : \partial\Gamma \longrightarrow \mathrm{Gr}_{d-k}(\mathbb{R}^d).$$

We will refer to $\xi^{(k)}(\partial\Gamma) \subset \mathrm{Gr}_k(\mathbb{R}^d)$ as the *k-limit set* of ρ. This generalizes the classical notion of a limit set for convex cocompact representations.

The behavior of the k-limit set of a k-Anosov representation can vary a lot depending on the nature of the representation. There are examples of 1-Anosov representations (such as non-Fuchsian, quasi-Fuchsian representations) where the 1-limit set in $\mathbb{P}(\mathbb{R}^d)$ is fractal. On the other hand, there are also examples of 1-Anosov representations (such as Hitchin representations or representations arising from convex divisible domains in $\mathbb{P}(\mathbb{R}^d)$) whose 1-limit set is always a $C^{1,\alpha}$-submanifold of $\mathbb{P}(\mathbb{R}^d)$ for some $\alpha > 0$. In this work, we investigate what causes this difference.

Henceforth, let $\Gamma \subset \mathrm{PO}(m, 1)$ be a cocompact lattice, and let $\rho : \Gamma \longrightarrow \mathrm{PGL}(d, \mathbb{R})$ be a 1-Anosov representation. Our first result gives a sufficient condition for the 1-limit set of ρ to be a $C^{1,\alpha}$-submanifold of $\mathbb{P}(\mathbb{R}^d)$ for some $\alpha > 0$. Sepcifically, if the condition

$$(\star) : \begin{cases} \rho \text{ is } m\text{-Anosov, and } \xi^{(1)}(x) + \xi^{(1)}(y) + \xi^{(d-m)}(z) \subset \mathbb{R}^d \\ \text{is a direct sum for all pairwise distinct } x, y, z \in \partial\Gamma \end{cases}$$

holds, then the condition

$$(\dagger) : \begin{cases} \text{there is some } \alpha > 0 \text{ so that the 1-limit set} \\ \text{of } \rho \text{ is a } C^{1,\alpha}\text{-submanifold of } \mathbb{P}(\mathbb{R}^d) \end{cases}$$

also holds. Furthermore, we also show that if ρ has "sufficient irreducibility", then the sufficient condition above is in fact necessary. More precisely, if

$$\bigwedge^m \rho : \Gamma \longrightarrow \mathrm{PGL}\left(\bigwedge^m \mathbb{R}^d\right)$$

is irreducible, or if Γ is a surface group and ρ is irreducible, then $(\star)$ is equivalent to $(\dagger)$. Finally, we prove that if ρ is irreducible, one can in fact give the optimal

α in terms of the eigenvalue data of ρ. In other words, if ρ is irreducible and ($\star$) holds, then

$$\sup\left\{\alpha: \begin{array}{c}\text{the 1-limit set of } \rho \text{ is a} \\ C^{1,\alpha}\text{-submanifold of } \mathbb{P}(\mathbb{R}^d)\end{array}\right\} = \inf\left\{\frac{\log\frac{\lambda_m(\rho(\gamma))}{\lambda_{m+1}(\rho(\gamma))}}{\log\frac{\lambda_1(\rho(\gamma))}{\lambda_m(\rho(\gamma))}} : \gamma \in \Gamma \setminus \{\mathrm{id}\}\right\},$$

where $\lambda_1(g) \geq \cdots \geq \lambda_d(g)$ denote the absolute values of the (generalized) eigenvalues of $g \in \mathrm{PGL}(d, \mathbb{R})$. This generalizes earlier work due to Guichard [3].

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Harmonic surfaces in 3-manifolds and the Simple Loop Theorem

Vladimir Markovic

Denote by $\mathfrak{M}(\Sigma)$ the space of hyperbolic metrics on a closed, orientable surface Σ and by $\mathfrak{M}(M)$ the space of negatively curved Riemannian metrics on a closed, orientable 3-manifold M. We show that the set of metrics for which the corresponding harmonic map is in Whitney's general position is an open, dense, and connected subset of $\mathfrak{M}(\Sigma) \times \mathfrak{M}(M)$. The main application of this result is the proof of the Simple Loop Theorem.

Throughout the paper Σ denotes a closed, orientable, and smooth surface of genus ≥ 2, and M a closed, orientable 3-manifold M. Let $\mathfrak{M}(\Sigma)$ denote an open and connected subset of the space of *hyperbolic* Riemannian metrics on Σ. We let $\mathfrak{M}(\Sigma)$ denote an open and connected subset of the space of smooth *negatively curved* Riemannian metrics on M.

Let $\mathbf{f}$ be a homotopy class of mappings from Σ into M. We say that $\mathbf{f}$ is admissible if $\mathbf{f}_*(\pi_1(\Sigma_0))$ is not Abelian. Unless otherwise stated, $\mathbf{f}$ denotes an admissible homotopy class. Then, for each $\mu \in \mathfrak{M}(\Sigma)$, and $\nu \in \mathfrak{M}(M)$, there exists the unique harmonic map $f_{\mu,\nu} : (\Sigma, \mu) \longrightarrow (M, \nu)$ in the homotopy class $\mathbf{f}$. Set

$$\mathfrak{M} = \mathfrak{M}(\Sigma) \times \mathfrak{M}(M),$$

and define

$$\mathfrak{M}^W = \{(\mu, \nu) \in \mathfrak{M} : \ f_{\mu,\nu} \text{ is in Whitney's general position}\}.$$

Our first main result is the following theorem.

Theorem 1. *Let* $\mathbf{f}$ *be an admissible homotopy class of mappings from a closed orientable surface* Σ *of genus at least seven into a closed, orientable 3-manifold* M*. Then* $\mathfrak{M}^W$ *is open, dense, and connected subset of* $\mathfrak{M}$*.*

The proof of Theorem 1 goes through (word for word) assuming that M is a convex-cocompact hyperbolic 3-manifold.

The main application of Theorem 1 is the proof of the Simple Loop Theorem. A map $f : X \longrightarrow Y$ from a closed surface X into a compact manifold Y is essential if the induced map $f_* : \pi_1(X) \longrightarrow \pi_1(Y)$ is injective. We say that f is incompressible if there exists an essential simple closed loop on X which lies in the kernel of f_*.

Theorem 2. *Let* $f : S \longrightarrow M$ *be an incompressible homotopy class of maps from a closed orientable surface* S *into a closed orientable negatively curved 3-manifold* M*. Then* f *is essential.*

The proof of Theorem 2 goes through when M is a convex cocompact hyperbolic 3-manifold.

Dynamical systems arising from classifying geometric structures

William Goldman

The theory of locally homogeneous geometric structures (flat Cartan connections) was first enunciated by Charles Ehresmann in [2]. It was later rejuvenated by Thurston in the 1970's as a context for his *Geometrization Program* for 3-manifolds. Given a fixed topology Σ, one can ask can the geometry of a manifold X with a transitive action of a Lie group G preserving the "geometry" be transplanted locally to Σ. That is, we seek an atlas of local coordinates on Σ taking values in X such that the coordinate changes locally lie in G. For example when Σ is a sphere, there are no structures modeled on Euclidean geometry (here X is Euclidean space and G its group of isometries). On the other hand, when Σ is a torus T^2, the Euclidean structures form a space with rich geometry.

Similar to the *classification* of Riemann surfaces by *Riemann's moduli space,* one would like a space $\mathfrak{M}$ whose points parametrize the (G, X)-structures on Σ. However, for many reasons, it is better to encode additional structure (a "topological coordinate system") to give a more tractable answer to the classification problem. Namely a *marked* (G, X)*-manifold* is a homeomorphism $\Sigma \longrightarrow M$ from the fixed topology Σ to a (G, X)-manifold. The space of isotopy classes of marked (G, X)-structures forms a *deformation space* $\mathfrak{D}$ analogous to the Teichmüller space $\mathfrak{T}$ in the classification of Riemann surfaces. Changing the marking amounts to an action of the *mapping class group* Γ of Σ on $\mathfrak{D}$; the quotient $\mathfrak{D}/\Gamma$ is the analog of the Riemann moduli space $\mathfrak{M} = \mathfrak{T}/\Gamma$.

Unlike $\mathfrak{M}$, where Γ acts properly on $\mathfrak{T}$, in general the Γ-action is not proper and the quotient $\mathfrak{D}/\Gamma$ is intractable. A particularly striking example is the theorem of Oliver Baues that the deformation space of *complete affine structures* on T^2 is $\mathbb{R}^2$

and the action of $\Gamma = \mathsf{GL}(2,\mathbb{Z})$ is the usual *linear* action. (See [1] for an exposition of Baues's result.)

This action fixes the origin (corresponding to Euclidean structures), preserves area, and has infinitely many discrete orbits. Yet the action is ergodic with respect to the Lebesgue measure class, so almost every orbit is dense.

In general the action displays both trivial dynamics and chaotic dynamics. The *character variety* $\mathfrak{X}$ is the GIT quotient of $\mathsf{Hom}\big(\pi_1(\Sigma), G\big)$ by $\mathsf{Inn}(G)$. It supports an action of Γ using the natural homomorphism $\Gamma \longrightarrow \mathsf{Out}\big(\pi_1(\Sigma)\big)$. In many cases of interest) *holonomy* describes a Γ-equivariant *local homeomorphsm* $\mathfrak{D} \longrightarrow \mathfrak{X}$ (this was first noted by Thurston in unpublished lecture notes in the late 1970's).

My earlier work [3] describes the dynamical decomposition for the $\mathbb{R}$-points of the $\mathsf{SL}(2)$-character variety in terms of geometric structures (possibly with singularities) and relates the geometry to the dynamics of the Γ-action.

My lecture described these results and more recent developments.

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Formality of the Goldman bracket and the Turaev cobracket

Nariya Kawazumi

(joint work with A. Alekseev, Y. Kuno and F. Naef)

In this talk we confine ourselves to consider a compact connected oriented C^∞ surface of genus $g \geq 1$ with one boundary component, $\Sigma = \Sigma_{g,1}$. We choose a base point $* \in \partial\Sigma$, and consider the fundamental group $\pi := \pi_1(\Sigma, *)$. Let $\mathbb{K}$ be a field of characteristic 0. Since the group π is a free group of finite rank, the formality problem for the group ring $\mathbb{K}\pi$ as a Hopf algebra with respect to the augmentation ideal is solved through a group-like expansion θ. As was proved by Kawazumi-Kuno [4] for $\Sigma_{g,1}$ and generalized to any compact connected oriented surface with non-empty boundary by Massuyeau-Turaev [9] and Kawazumi-Kuno [5] independently , if θ is conjugate to a symplectic expansion in the sense of Massuyeau [7], the map $|\theta|$ induced on the completion of the trace space $|\mathbb{K}\pi| = \mathbb{K}\pi/[\mathbb{K}\pi, \mathbb{K}\pi]$ is a Lie algebra isomorphism with respect to the Goldman bracket. Recently Alekseev-Kawazumi-Kuno-Naef [2] proved the converse. Moreover, as is proved by Alekseev-Kawazumi-Kuno-Naef [1], there is a symplectic expansion whose induced map $|\theta|$ is a Lie coalgebra isomorphism with respect to the framed Turaev cobracket if and only if $g \geq 2$ or ($g = 1$ and the framing satisfies some condition). Such a symplectic expansion is a solution of a higher analogue of the Kashiwara-Vergne problem [6] [3]. The formality problem of the Turaev cobracket for genus 0 surfaces was previously solved by Massuyeau [8] through the Kontsevich integral.

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Toward Complex analysis with Thurston's theory

Hideki Miyachi

Motivation. Let $\mathcal{T}_{g,m}$ be the Teichmüller space of Riemann surfaces of genus g with m marked points ($2g-2+m>0$). Teichmüller space $\mathcal{T}_{g,m}$ has a canonical complex structure, and is the universal space of holomorphic families in the sense that any holomorphic mapping "into" $\mathcal{T}_{g,m}$ is associated to a geometric object, a holomorphic family of Riemann surfaces of analytically finite type (g,m). A fundamental question befind this research is to find or formulate reasonable geometric objects associated to holomorphic functions and mappings "from" Teichmüller space.

Backgrounds. Let $\Sigma_{g,m}$ be an orientable reference surface of genus g and m-marked points ($2g-2+m>0$). For a homotopy class α of a simple closed curve on $\Sigma_{g,m}$, let $\mathrm{Ext}_x(\alpha)$ be the *extremal length* of α on the marked Riemann surface $x\in\mathcal{T}_{g,m}$. The extremal length function Ext_x extends continuously to the space $\mathcal{MF}$ of measured foliations (cf. [5]). The Teichmüller space admits a canonical complete distance d_T called the *Teichmüller distance* on $\mathcal{T}_{g,m}$.

Fix $x_0\in\mathcal{T}_{g,m}$. Let Γ_0 be the Fuchsian group acting on the upper-half plane $\mathbb{H}$ with $\mathbb{H}/\Gamma_0$ is conformally equivalent to the underlying surface of x_0. Let $A^2(\Gamma_0)$ be the space of bounded autormorphic forms of weight -4 on the lower-half plane $\mathbb{H}^-$. For $\varphi\in A^2(\Gamma_0)$, let W_φ be a locally univalent function on $\mathbb{H}^-$ whose Schwarzian derivative coincides with φ. The *holonomy representation* ρ_φ *of* φ is a homomorphism $\rho_\varphi\colon\Gamma_0\longrightarrow\mathrm{PSL}_2(\mathbb{C})$ such that $\rho_\varphi(\gamma)\circ W_\varphi=W_\varphi\circ\gamma$ for $\gamma\in\Gamma_0$. The

Bers slice $\mathcal{T}_{x_0}^B$ is a subset of $A^2(\Gamma_0)$ consisting of $\varphi \in \Gamma_0$ such that W_φ admits a quasiconformal extension to the Riemann sphere. Through the identification $\pi_1(\Sigma_{g,m}) \cong \Gamma_0$ via the marking of x_0, $W_\varphi(\mathbb{H})/\rho_\varphi(\Gamma_0)$ is recognized as a marked Riemann surface. This recognition induces a biholomorphic mapping from $\mathcal{T}_{x_0}^B$ to $\mathcal{T}_{g,m}$. The Bers slice $\mathcal{T}_{x_0}^B$ is a bounded domain. The closure $\overline{\mathcal{T}_{x_0}^B}$ and the boundary $\partial\mathcal{T}_{x_0}^B$ are called the *Bers compactification* and the *Bers boundary.*

Toward Complex analysis with Thurston theory.

Demailly's theory. A bounded domain $\Omega \subset \mathbb{C}^N$ is said to be *hyperconvex* if it admits a negative continuous plurisubharmonic exhaustion. Demailly [2] observed that for any hyperconvex domain Ω in $\mathbb{C}^N$ and $w \in \Omega$, there is a unique plurisubharmonic function $g_{\Omega,w}\colon \Omega \longrightarrow [-\infty, 0)$ such that (1) $(dd^c g_{\Omega,w})^N$ is the Dirac measure with support at w; and (2) $g_{\Omega,w}(z) = \sup_v\{v(z)\}$ where the supremum runs over all non-positive plurisubharmonic function v on Ω with $v(z) \leq \log\|z - w\| + O(1)$ around $z = w$ (cf. [2, Théorème (4.3)]). The function $g_\Omega(w,z) = g_{\Omega,w}(z)$ is called the *pluricomplex Green function* of Ω. For $w \in \Omega$, Demailly also found a Borel measure $\mu_{\Omega,w}$ on $\partial\Omega$ satisfying

$$V(w) = \int_{\partial\Omega} V(\zeta) d\mu_w(z) - \frac{1}{(2\pi)^N}\int_\Omega dd^c V \wedge |g_{\Omega,z}|(dd^c u_w)^{N-1}$$

for all plurisubharmonic function V on Ω which is continuous on $\overline{\Omega}$. The measure $\mu_{\Omega,w}$ is called the *pluriharmonic measure* of $w \in \Omega$ (cf. [2, Définition (5.2)]).

Poisson integral formula. Krushkal [7] showed that the Bers slice is hyperconvex. Furthermore, he also observed that

$$g_{\mathcal{T}_{g,m}}(x,y) = g_{\mathcal{T}_{x_0}^B}(x,y) = \log\tanh d_T(x,y)$$

for $x, y \in \mathcal{T}_{g,m}$ ([8]. See also [9]).

Let $\partial^{ue}\mathcal{T}_{x_0}^B \subset \partial\mathcal{T}_{x_0}^B$ the subset consisting of totally degenerate groups whose ending laminations are the supports of minimal and uniquely ergodic measured foliations. We define a function $\mathcal{T}_{g,m} \times \mathcal{T}_{g,m} \times \partial\mathcal{T}_{x_0}^B$ by

$$\mathbb{P}(x,y,\varphi) = \begin{cases} \left(\dfrac{\mathrm{Ext}_x(F_\varphi)}{\mathrm{Ext}_y(F_\varphi)}\right)^{3g-3+m} & (\varphi \in \partial^{ue}\mathcal{T}_{x_0}^B) \\ 1 & (\text{otherwise}) \end{cases} \tag{1}$$

where for $\varphi \in \partial^{ue}\mathcal{T}_{x_0}^B$, F_φ is the measured foliation whose support is the ending lamination of the Kleinian manifold associated with φ.

Theorem 1 (Poisson integral formula [10])**.** *Let V be a continuous function on the Bers compactification $\overline{\mathcal{T}_{x_0}^B}$ which is pluriharmonic on $\mathcal{T}_{x_0}^B \cong \mathcal{T}_{g,m}$. Then*

$$V(x) = \int_{\partial\mathcal{T}_{x_0}^B} V(\varphi)\mathbb{P}(x_0,x,\varphi)\boldsymbol{\mu}_{x_0}^B(\varphi) \tag{2}$$

where $\boldsymbol{\mu}_{x_0}^B$ is a probability measure on $\partial\mathcal{T}_{x_0}^B$.

Namely, $\mathbb{P}(x_0, x, \varphi)\boldsymbol{\mu}^B_{x_0}(\varphi)$ is Demailly's pluriharmonic measure of $x \in \mathcal{T}^B_{x_0} \cong \mathcal{T}_{g,m}$, and the function (1) is nothing but the Poisson kernel for pluriharmonic functions on Teichmüller space.

The probability measure $\boldsymbol{\mu}^B_{x_0}$ is defined as follows: For $x \in \mathcal{T}_{g,m}$, we set $\mathcal{SMF}_x = \{F \in \mathcal{MF} \mid \mathrm{Ext}_x(F) = 1\}$. The projection $\mathcal{MF} - \{0\} \longrightarrow \mathcal{PMF}$ induces a homeomorphism $\mathcal{SMF}_x \longrightarrow \mathcal{PMF}$. We define

$$\hat{\mu}^x_{Th}(E) = \frac{1}{\mu_{Th}(\{\mathrm{Ext}_x(F) \leq 1\})}\mu_{Th}(\{tF \mid F \in E, 0 \leq t \leq 1\}),$$

for $E \subset \mathcal{PMF} \cong \mathcal{SMF}_x$, where μ_{Th} is the Thurston measure on $\mathcal{MF}$.

Let $\mathcal{PMF}^{min} \subset \mathcal{PMF}$ be the set of minimal projective measured foliations in the sense that the projective class of a measured foliation F is said to be *minimal* if $i(F, \alpha) > 0$ for all non-trivial and non-peripheral simple closed curves. By the Klarrarich theorem [6] (see also [4]) and the ending lamination theorem [1], there is an injective continuous map $\Xi_x \colon \mathcal{PMF}^{min} \longrightarrow \partial\mathcal{T}^B_{x_0}$ such that to $[F] \in \mathcal{PMF}^{min}$, we assign the Kleinian surface group whose ending lamination is the support of F. The measure $\boldsymbol{\mu}^B_x$ is defined to be the push-forward measure of $\hat{\mu}^x_{Th}$ by Ξ_x.

When we consider $V \equiv 1$ in (2), we deduce

$$\mu_{Th}(\{\mathrm{Ext}_x(F) \leq 1\}) = \mu_{Th}(\{\mathrm{Ext}_y(F) \leq 1\}) \tag{3}$$

for $x, y \in \mathcal{T}_{g,m}$. Equation (3) is first proved by Dumas and Mirzakhani (cf. [3]).

A conjecture. The Poission integral formula (2) asserts that any holomorphic function V on the Bers compactification $\overline{\mathcal{T}^B_{x_0}}$ satisfies

$$\int_{\partial\mathcal{T}^B_{x_0}} V(\varphi)(\overline{\partial}\mathbb{P})(x_0, \cdot, \varphi)\boldsymbol{\mu}^B_{x_0}(\varphi) = 0 \tag{4}$$

on $\mathcal{T}_{g,m}$. It is conjectured that when a complex valued continuous function V on $\partial\mathcal{T}^B_{x_0}$ satisfies the condition (4), the Poisson integral (2) defines a continuous function on $\overline{\mathcal{T}^B_{x_0}}$ which is holomorphic on the Bers slice. Even if this conjecture is not true, it is interesting to find any extra condition for characterizing the boundary functions of holomorphic functions.

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Counting lariats on hyperbolic surfaces with cusps

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On a hyperbolic surface a *lariat* is a bi infinite simple geodesic with both ends at a common cusp. The reduced length of a lariat is defined by choosing a horocycle for truncation. We discuss adapting Mirzakhani's method from "Growth of the number of simple closed geodesics..." to count by length the lariats at a given cusp. The method combines counting and estimating the integral lattice points in $\mathcal{MGL}$, the space of measured geodesic laminations; Masur's ergodicity for the mapping class group acting on $\mathcal{MGL}$ and integration over the moduli space of hyperbolic surfaces to evaluate constants. Two components of Mirzakhani's method particularly requiring adaption are the zig-zag argument for estimating lariat length in terms of Dehn-Thurston coordinates for multi curves, and the relation giving the count by length in terms of the Thurston volume of the unit length ball in $\mathcal{MGL}$. The overall result is that lariats counted by length have the same growth rate as simple closed geodesics and as in Mirzakhani's result, we expect that the leading constant in the count is given by characteristic classes on the moduli space. We discuss applying the method to counting Markoff forms, special binary indefinite quadratic forms.

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Generalized stretch lines for surfaces with boundary

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(joint work with Daniele Alessandrini)

Introduction. In the paper [1] we study various asymmetric metrics on the Teichmüller space of a surface S with boundary. Our goal is to extend the results obtained by Thurston for closed surfaces in his foundational paper [14]. Indeed, Thurston defines two natural distances between hyperbolic structures defined on the same topological surface, which somehow mimics the Teichmuller distance in the hyperbolic setting. The first one is the *length spectrum distance*, defined via the lengths of the simple closed curves on the surface, and the other one is the *Lipschitz distance*, defined via the Lipschitz constants of the homeomorphisms isotopic to the identity. Thurston proves that the two distances coincide by defining a preferred family of paths in the Teichmüller space, called *stretch lines*, which are geodesics for both distances. A survey with a complete exposition of Thurston's metric for closed surfaces can be found in [10]. The detailed case of the once-punctured torus was studied very recently by Dumas-Lenzhen-Rafi-Tao [4]. The convergence of the stretch lines on the boundary of the Teichmüller space was studied in various papers by Papadopoulos [9], Théret [13], Walsh [15]. Other results about the coarse geometry of the stretch lines can be found in works by Lenzhen-Rafi-Tao [6] and [7]. The analogies between Thurston's distance and Teichmüller distance were studied by Choi-Rafi [3].

While it is well-known that Thurston's stretch lines are not the only geodesics, the properties of the geodesic flow of the Thurston metric are still rather mysterious. In the case of closed surfaces the length spectrum distance is the key ingredient in the construction of the stretch lines. In the case of surfaces with boundary, Parlier [12] proved that Thurston's length spectrum distance is no longer a distance as it can be negative. Guéritaud-Kassel [5] studied this functional in detail in the case when the surface has funnels. They also give beautiful applications of their work to the affine actions on $\mathbb{R}^3$ and Margulis space-times. When the surface has compact boundary, there are many asymmetric distances that mimic Thurston's distance for closed surfaces: the *arc distance* d_A, defined via the lengths of simple arcs and curves on S (first defined by Liu-Papadopoulos-Su-Théret [8]); the *Lipschitz distance* d_{Lh}, defined via the Lipschitz constants of the homeomorphisms isotopic to the identity (as defined by Thurston [14] for closed surfaces); the *Lipschitz map distance* $d_{L\partial}$, defined via the Lipschitz constants of the maps isotopic to the identity (we will define it here). It is immediate to see that $d_A \leq d_{L\partial} \leq d_{Lh}$.

Question 1. *Do Thurston's results still extend to the case of surfaces with boundary for some (or all) the distances above? Do the distances $d_A, d_{L\partial}$ and d_{Lh} coincide?*

Denote by S^d the *double* of S, that is, the surface obtained glueing two copies of S the boundary. Furthermore, denote by X^d the hyperbolic structure on S^d

obtained doubling the hyperbolic structure X on S. There is a natural embedding:

$$\mathrm{Teich}(S) \ni X \hookrightarrow X^d \in \mathrm{Teich}(S^d).$$

Question 2. *Is the natural embedding* $(\mathrm{Teich}(S), d_?) \hookrightarrow (\mathrm{Teich}(S^d), d_{Th})$ *a geodesic embedding for* $d_? = d_A, d_{L\partial}$ *or* d_{Lh}*?*

In our paper [1] we investigate these questions for d_A and d_{Lh}.

Our results. We will first construct a large family of paths in Teich(S) that we will call *generalized stretch lines* following Thurston [14]. We will prove that our generalized stretch lines are also geodesics for the two distances d_A and $d_{L\partial}$. As in Thurston [14], for any two points on the same generalized stretch line, there is a special Lipschitz map between them having optimal Lipschitz constant. We will call this map a *generalized stretch map.*

Theorem 3 (Existence of generalized stretch maps)**.** *Let S be a surface with non-empty boundary and let X be a hyperbolic structure on S. For every maximal lamination λ on S and for every $t \geq 0$ there exists a hyperbolic structure X^t_λ and a Lipschitz map $\Phi^t : (S, X) \longrightarrow (S, X^t_\lambda)$ with the following properties:*

(1) $\Phi^t(\partial S) = \partial S$;
(2) Φ^t *stretches the arc length of the leaves of λ by the factor e^t;*
(3) *for every geometric piece $\mathcal{G}$ in $S \setminus \lambda$ the map Φ^t restricts to a generalized stretch map $\phi^t : \mathcal{G} \longrightarrow \mathcal{G}_t$;*
(4) $\mathrm{Lip}(\Phi^t) = e^t$

Furthermore, if λ contains a non-empty measurable sublamination we have:

$$\mathrm{Lip}(\Phi^t) = \min\{\mathrm{Lip}(\psi) \mid \psi \in \mathrm{Lip}_0(X, X^t_\lambda), \psi(\partial S) \subset \partial S\}.$$

In the case of [14], any maximal lamination decomposes a closed surface into ideal triangles and Thurston uses the horocycle foliation to construct explicitely the stretch map between two ideal triangles. In the case of surfaces with boundary, new problems arise. Indeed, a maximal lamination decomposes a surface with boundary in geometric pieces of four different types: ideal triangles; right-angled quadrilaterals with two consecutive ideal vertices; right-angled pentagons with one ideal vertex, and right-angled hexagons. Unlike Thurston [14], we will not construct the maps explicitely.

Corollary 4 (Existence of generalized stretch lines)**.** *The path*

$$s_\lambda : \mathbb{R}_{\geq 0} \longrightarrow \mathrm{Teich}(S)$$
$$t \mapsto X^t_\lambda$$

is a geodesic path parametrized by arc-length for both d_A and $d_{L\partial}$.

We will call the path s_λ a *generalized stretch line.*

Corollary 5. *For any two hyperbolic structures X, Y on S, there exists a continuous map $\phi \in \mathrm{Lip}_0(X, Y)$, with $\phi(\partial S) \subset \partial S$ such that*

$$\log(\mathrm{Lip}(\phi)) = d_A(X, Y).$$

This map has an optimal Lipschitz constant. Therefore, $d_A = d_{L\partial}$.

We use Theorem 3 to prove an analogue of Thurston's theorem.

Theorem 6. *The space* $(\mathrm{Teich}(S), d_A)$ *is a geodesic metric space. Furthermore, any two points* $X, Y \in \mathrm{Teich}(S)$ *can be joined by a geodesic for* d_A *and* $d_{L\partial}$*, which is a finite concatenation of generalized stretch lines.*

Corollary 7. *There exists a Finsler metric on* $\mathrm{Teich}(S)$ *whose induced distance is* d_A*.*

Our results also have applications in the case of closed surfaces. Indeed, we can answer positively to Question 2. Our proof relies on a result by Liu-Papadopoulos-Théret-Su [8].

Corollary 8. *The embedding* $\jmath : (\mathrm{Teich}(S), d_A) \hookrightarrow (\mathrm{Teich}(S^d), d_{Th})$ *is a geodesic embedding, i.e. every two points in the image are joined by a geodesic in the image.*

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Computations on Johnson homomorphisms

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(joint work with Shigeyuki Morita, Takuya Sakasai)

Let $\mathcal{M}_g$ be the mapping class group of a closed oriented surface Σ_g of genus g and let $\mathcal{I}_g \subset \mathcal{M}_g$ be the Torelli subgroup. Namely, it is the subgroup of $\mathcal{M}_g$ consisting of all the elements which act on the homology $H_1(\Sigma_g;\mathbb{Z})$ trivially.

There exist two filtrations of the Torelli group. One is the lower central series which we denote by $\mathcal{I}_g(k)$ $(k = 1, 2, \ldots)$ where $\mathcal{I}_g(1) = \mathcal{I}_g$ and $\mathcal{I}_g(k+1) = [\mathcal{I}_g(k), \mathcal{I}_g]$ for $k \geq 1$. The other is called the Johnson filtration $\mathcal{M}_g(k)$ $(k = 1, 2, \ldots)$ of the mapping class group where $\mathcal{M}_g(k)$ is defined to be the kernel of the natural homomorphism

$$\rho_k : \mathcal{M}_g \to \mathrm{Out}(N_k(\pi_1\Sigma_g)).$$

Here $N_k(\pi_1\Sigma_g)$ denotes the k-th nilpotent quotient of the fundamental group of Σ_g and $\mathrm{Out}(N_k(\pi_1\Sigma_g))$ denotes its outer automorphism group. $\mathcal{M}_g(1)$ is nothing other than the Torelli group $\mathcal{I}_g$ so that $\mathcal{M}_g(k)$ $(k = 1, 2, \ldots)$ is a filtration of $\mathcal{I}_g$. This filtration was originally introduced by Johnson [7] for the case of a genus g surface with one boundary component. The above is the one adapted to the case of a closed surface. It can be shown that $\mathcal{I}_g(k) \subset \mathcal{M}_g(k)$ for all $k \geq 1$. Johnson showed in [9] that $\mathcal{I}_g(2)$ is a finite index subgroup of $\mathcal{M}_g(2)$ and asked whether this will continue to hold for the pair $\mathcal{I}_g(k) \subset \mathcal{M}_g(k)$ $(k \geq 3)$. He also showed in [8] that $\mathcal{M}_g(2)$ is equal to the subgroup $\mathcal{K}_g$, which is called the Johnson subgroup or Johnson kernel, consisting of all the Dehn twists along separating simple closed curves on Σ_g.

The above question was answered negatively in [10]. More precisely, a homomorphism $d_1 : \mathcal{K}_g \to \mathbb{Z}$ was constructed which is non-trivial on $\mathcal{M}_g(3)$ while it vanishes on $\mathcal{I}_g(3)$ so that the index of the pair $\mathcal{I}_g(3) \subset \mathcal{M}_g(3)$ was proved to be infinite. Furthermore it was shown in [11] that there exists an isomorphism

$$H^1(\mathcal{K}_g;\mathbb{Z})^{\mathcal{M}_g} \cong \mathbb{Z} \quad (g \geq 2)$$

where the homomorphism d_1 serves as a rational generator. It is characterized by the fact that its value on a separating simple closed curve on Σ_g of type $(h, g-h)$ is $h(g-h)$ up to non-zero constants. This homomorphism was defined as the secondary characteristic class associated with the fact that the first MMM class, which is an element of $H^2(\mathcal{M}_g;\mathbb{Z})$, vanishes in $H^2(\mathcal{I}_g;\mathbb{Z})$.

Now let us consider the following two graded Lie algebras

$$\operatorname{Gr}\mathfrak{t}_g = \bigoplus_{k=1}^{\infty} \mathfrak{t}_g(k), \quad \mathfrak{t}_g(k) = (\mathcal{I}_g(k)/\mathcal{I}_g(k+1)) \otimes \mathbb{Q}$$

$$\mathfrak{m}_g = \bigoplus_{k=1}^{\infty} \mathfrak{m}_g(k), \quad \mathfrak{m}_g(k) = (\mathcal{M}_g(k)/\mathcal{M}_g(k+1)) \otimes \mathbb{Q}$$

associated with the above two filtrations of the Torelli group. Hain [6] gave an explicit finite presentation of $\operatorname{Gr}\mathfrak{t}_g \otimes \mathbb{C}$ which implies that $\operatorname{Ker}(\mathfrak{t}_g(2) \to \mathfrak{m}_g(2)) \cong \mathbb{Q}$. Furthermore, he proved that the natural homomorphism

$$\mathfrak{t}_g(k) \to \mathfrak{m}_g(k)$$

is surjective for any k and also that the index of the pair $\mathcal{I}_g(k) \subset \mathcal{M}_g(k)$ remains infinite for any $k \geq 4$ extending the above mentioned result.

On the other hand, Ohtsuki [14] defined a series of invariants λ_k for homology 3-spheres the first one being the Casson invariant λ. He also initiated a theory of finite type invariants for homology 3-spheres in [15]. Then Garoufalidis and Levine [5] studied the relation between this theory and the structure of the Torelli group extending the case of the Casson invariant mentioned above extensively.

In these situations, it would be natural to ask whether there exists any other difference between the two filtrations of the Torelli group than the Casson invariant, in particular whether any finite type rational invariant of homology 3-spheres of degree greater than 2 appears there or not. This is equivalent to asking whether the natural homomorphism $\mathfrak{t}_g(k) \to \mathfrak{m}_g(k)$ is an isomorphism for $k = 3, 4, \ldots$ or not.

Now it was proved in [12] that $\mathfrak{t}_g(3) \cong \mathfrak{m}_g(3)$. The main theorem is the following.

Theorem 1. *For any $k = 4, 5, 6$, we have*

$$\mathfrak{t}_g(k) \cong \mathfrak{m}_g(k).$$

As a corollary to the above theorem, we obtain the cases $k = 5, 6, 7$ of the following result. The case $k = 3$ follows from Hain's theorem [6] combined with a result of [10] and the case $k = 4$ follows from a result of [12] mentioned above.

Corollary 2. *For any $k = 3, 4, 5, 6, 7$, the k-th group $\mathcal{I}_g(k)$ in the lower central series of the Torelli group is a finite index subgroup of the kernel of the non-trivial homomorphism*

$$d_1 : \mathcal{M}_g(k) \to \mathbb{Z}.$$

Next, we give the explicit form of the rational abelianization $H_1(\mathcal{K}_g; \mathbb{Q})$ of the Johnson subgroup as an application of Theorem 1. Dimca and Papadima [3] proved that $H_1(\mathcal{K}_g; \mathbb{Q})$ is finite dimensional for $g \geq 4$. Then Dimca, Hain and Papadima [2] gave a description of it. However, they did not give the final explicit form. Here we compute their description by combining the case $k = 4$ of Theorem 1 and former results concerning the Johnson homomorphisms to obtain the following result.

Theorem 3. *The secondary class d_1 together with the refinement of the second Johnson homomorphism gives the following isomorphism for $g \geq 6$.*

$$H_1(\mathcal{K}_g; \mathbb{Q}) \cong \mathbb{Q} \oplus [2^2] \oplus [31^2].$$

Here for a given Young diagram $\lambda = [\lambda_1 \cdots \lambda_h]$, we denote the irreducible representation of $\mathrm{Sp}(2g, \mathbb{Q})$ *corresponding to λ simply by $[\lambda_1 \cdots \lambda_h]$.*

By making use of recent remarkable results of Ershov-He [4] and Church-Ershov-Putman [1], we obtain the following.

Corollary 4. (i) *Two subgroups $[\mathcal{K}_g, \mathcal{K}_g]$ and $\mathcal{I}_g(4)$ of the Torelli group $\mathcal{I}_g$ are commensurable for $g \geq 6$.*
(ii) *$[\mathcal{K}_g, \mathcal{K}_g]$ is finitely generated for $g \geq 7$.*

Finally, we would like to propose the following conjecture.

Conjecture 5. *For any $k \neq 2$, we have*

$$\mathfrak{t}_g(k) \cong \mathfrak{m}_g(k).$$

In this report, whenever we mention groups like $\mathcal{M}_g, \mathcal{I}_g, \mathcal{M}_g(k), \mathcal{I}_g(k)$ and modules like $\mathfrak{t}_g(k), \mathfrak{m}_g(k)$, which depend on the genus g, we always assume that it is sufficiently large, more precisely in a stable range with respect to the property we consider, unless we describe the range of g explicitly.

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Normal generators for mapping class groups are abundant

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(joint work with Dan Margalit)

Let S_g be the closed orientable surface of genus g, and let $\mathrm{Mod}(S_g)$ be its mapping class group. The Nielsen–Thurston classification says that a mapping class is either periodic or reducible or both, or else it is pseudo-Anosov. In the last case the mapping class has a stretch factor as an invariant.

We give an abundance of new examples of mapping classes whose normal closure equals $\mathrm{Mod}(S_g)$—mapping classes that *normally generate* [3]. While our methods can be used to give new examples of reducible normal generators, our results focus on periodic and pseudo-Anosov mapping classes. We completely solve the problem for periodic elements.

Theorem 1 (Lanier–Margalit). *For $g \geq 3$, every nontrivial periodic mapping class that is not a hyperelliptic involution normally generates* $\mathrm{Mod}(S_g)$.

We also compute the normal closures of all periodic elements when g is 1 or 2.

Turning to pseudo-Anosovs, we give first examples of pseudo-Anosov normal generators for all $g \geq 2$, answering a question of D. D. Long from 1986 [2]. Even more, we show that every pseudo-Anosov with small stretch factor normally generates.

Theorem 2 (Lanier–Margalit). *If a pseudo-Anosov element of* $\mathrm{Mod}(S_g)$ *has stretch factor less than $\sqrt{2}$ then it normally generates* $\mathrm{Mod}(S_g)$.

We additionally show how to construct pseudo-Anosov normal generators with arbitrarily large stretch factor, as well as arbitrarily large asymptotic translation length on the curve complex. These examples disprove Ivanov's conjecture that normal closures of such pseudo-Anosovs should be free groups [1].

Our main tools for proving these theorems involve analyzing how elements act on individual simple closed curves. Whenever a curve c and its image under a mapping class f have a sufficiently simple intersection pattern, f is a normal generator. For example, whenever c is a nonseparating curve and c and $f(c)$ intersect exactly once, then f is a normal generator for $\mathrm{Mod}(S_g)$. We call this collection of results *well-suited curve criteria*.

Our result on periodic normal generators can be used to derive a number of consequences. Some are easy proofs of familiar facts, such as the fact that the Torelli group is torsion free. Others include extensions of known results and resolutions of open problems. For instance, Mann and Wolff recently used Theorem 1 in their proof that when $g \geq 3$, every representation $\rho : \mathrm{Mod}(S_{g,1}) \to \mathrm{Homeo}_+(S^1)$ is either trivial or semi-conjugate to the standard Gromov boundary action [4].

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Cubulating surface-by-free groups

Mahan Mj

(joint work with Jason Manning and Michah Sageev)

Cubulable groups, by which we mean finitely presented groups acting freely, properly discontinuously and cocompactly by isometries (cellular isomorphisms) on a CAT(0) cube complex, have been objects of much attention over the last few years particularly due to path-breaking work of Agol and Wise. Wise's quasiconvex hierarchy theorem for hyperbolic cubulable groups states:

Theorem 1. *Let G be a hyperbolic group with a quasiconvex hierarchy. Then G is virtually special cubulable.*

We make a first attempt at relaxing the quasiconvexity hypothesis in the above Theorem by addressing the following.

Question 2. *Let $G = A *_C B$ (resp. $G = A*_C$) be a free-product with amalgamation such that G, A, B, C are hyperbolic. If A, B are cubulable and (hence special by Agol's Theorem), is G cubulable?*

Question 2 therefore asks for a combination theorem for cubulated groups along **non-quasiconvex** subgroups. We specialize Question 2 to a case where a lot is known about A, B. Let A (resp. B) be the fundamental group of a closed hyperbolic 3-manifold M_1 (resp. M_2) fibering over the circle with fiber a closed surface S of genus at least 2. Let $C = \pi_1(S)$ be the fundamental group of the fiber and we ask if Question 2 has an affirmative answer in this case.

Question 2 can then be reformulated. Let $1 \longrightarrow H \longrightarrow G \longrightarrow Q \longrightarrow 1$ be an exact sequence where $H = \pi_1(S)$ is the fundamental group of a closed surface S of genus greater than one, G is hyperbolic and Q is finitely generated free. We give sufficient conditions to prove that G is cubulable, i.e. G acts freely cocompactly and properly by isometries on a CAT(0) cube complex. By results of Agol and Wise, it follows that G has good separability properties, in particular that it is linear. The main result may be thought of as a combination theorem for virtually special hyperbolic groups when the amalgamating subgroup is not quasiconvex. Ingredients include the theory of tracks, the quasiconvex hierarchy theorem of Wise, the distance estimates in the mapping class group from subsurface projections due to Masur-Minsky et al and the model for doubly degenerate Kleinian surface groups used in the proof of the ending lamination theorem.

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McShane identities for convex projective surfaces and beyond

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(joint work with Zhe Sun)

In [4], Birman and Series showed that for any finite area complete hyperbolic surface Σ, the *Birman-Series* set, defined as the set of points which lie on (complete) simple geodesics, occupies zero area. As a part of his doctoral dissertation [10], McShane undertook to understand the Birman-Series set around the cusp of an arbitrary 1-cusped torus $\Sigma_{1,1}$. He showed that the restriction of the Birman-Series set to the cuspidal neighborhood $\mathbb{S}^1 \times \mathbb{R}_{>0}$ capped off by the length 1 horocycle is homeomorphic to $(\mathfrak{C} \cup \mathcal{A}) \times \mathbb{R}_{>0}$, where:

- $\mathfrak{C} \subset \mathbb{S}^1$ is a Cantor set and
- $\mathfrak{A} \subset \mathbb{S}^1$ is countable.

The Birman-Series theorem ensures that the measure of $\mathfrak{C}$ is 0, and therefore the horocyclic lengths of the open intervals constituting $\mathbb{S}^1 \backslash \mathfrak{C}$ must sum to 1. Concretely computing these horocyclic lengths, one obtains the McShane identity:

$$\sum_{\gamma \in \mathcal{C}(\Sigma_{1,1})} \frac{2}{e^{\ell(\gamma)} + 1} = 1, \tag{1}$$

where $\mathcal{C}(\Sigma_{1,1})$ is the collection of simple closed geodesics on $\Sigma_{1,1}$. A plethora of related identities soon followed McShane's archetype, including identities for

- hyperbolic surfaces of general topological type [11, 14], with geodesic [12, 16] or small cone-point boundary [16] or a marked point [7] or crowns [8];
- complex deformations of Fuchsian representations [1, 2, 3]
- a very general class of cross-ratios [9].

In some sense, Labourie and McShane's identities for the aforementoned cross-ratios is more of a powerful framework for obtaining McShane-type identities; the summands in these identities are expressed in terms of cross-ratios or character variety coordinates – clean and succinct but lacking the same level of geometric clarity as their classical counterparts. For example, the classical $\Sigma_{1,1}$ identity can be used

- to show the discreteness of its simple length spectrum on $\Sigma_{1,1}$, or
- to establish a collar lemma for geodeiscs in $\mathcal{C}(\Sigma_{1,1})$, or
- to show that Thurston's asymmetric curve-length ratio metric is positive on the space $\mathcal{T}_{1,1}$, or
- to bound the Teichmüller distances in $\mathcal{T}_{1,1}$ [13], or
- to compute the Weil-Petersson volume of the moduli space $\mathcal{M}_{1,1}$ [12];

but none of these applications are natural to the language of cross-ratios.

Our work is motivated by the desire to produce McShane identities for *positive representations* $\rho : \pi_1(S_{g,n}) \to \mathrm{PSL}(n, \mathbb{R})$, in the sense of Fock–Goncharov's Higher Teichmüller theory, that admit either parabolic (unipotent) or hyperbolic (loxodromic) boundary monodromy. These representations naturally generalize Fuchsian representations, and their most concrete (non-classical) geometric manifestation arises in the $\mathrm{PSL}(3, \mathbb{R})$ case, where the theory of positive representations coincides with the theory of *marked convex real projective surfaces* [5].

Convex real projective surfaces are a naturally generalize hyperbolic surfaces, but admit richer geometry, even at the level of ideal triangles. This is seen from the fact that the *triple ratio* for hyperbolic ideal triangles is always equal to 1, whereas they may vary over $\mathbb{R}_{>0}$ for ideal triangles in general convex real projective surfaces. In fact, we show that this is a characterizing condition for hyperbolicity: a convex real projective surface is hyperbolic iff. all of its embedded ideal triangles have triple ratio equal to 1. However, the behavior of triple ratios is not unconstrained, as we also show that the spectrum of triple ratios of embedded triples on a convex real projective surface is universally bounded within some closed interval in $\mathbb{R}_{>0}$.

Our McShane identity for convex projective surfaces is most cleanly stated (as with the classical case), for 1-cusped tori: let $\Sigma_{1,1}$ now be a finite-area convex real projective 1-cusped torus, then

$$\sum_{\vec{\gamma} \in \vec{\mathcal{C}}(\Sigma_{1,1})} \frac{1}{T(\vec{\gamma}) \cdot e^{\ell_1(\vec{\gamma})} + 1} = 1, \tag{2}$$

where

- $\vec{\mathcal{C}}(\Sigma_{1,1})$ denote that set of *oriented* simple closed geodesics on $\Sigma_{1,1}$,

- $T(\overrightarrow{\gamma})$ denotes the triple ratio of either of the two embedded ideal triangles on $\Sigma_{1,1}$ with one side given by the unique ideal geodesic disjoint from γ and the other two sides asymptotically spiraling towards $\overrightarrow{\gamma}$ so in the direction of its orientation, and
- $\ell_1(\overrightarrow{\gamma})$ denotes the *simple root length* of $\overrightarrow{\gamma}$, obtained by taking the logarithm of the largest eigenvalue divided by the second largest eigenvalue of the monodromy matrix representing $\overrightarrow{\gamma}$.

Paralleling the classical strategy, we prove a generalization of the Birman-Series theorem for finite-area convex real projective surfaces. We then repurpose the Goncharov-Shen potential [6] (whose classical analogue is horocycle length) as a measure C^1-compatible to the Hilbert Finsler length measure of a horocycle η on $\Sigma_{1,1}$. To compute the contributing measure of each horocyclic gap interval complementary to the Birman-Series set on η, we employ the technology of Fock and Goncharov's A-coordinates — much like how one might compute [8] the classical gap measure using Penner's λ-lengths [15].

We obtain McShane-type identities not just for finite-area convex real projective surfaces (i.e.: positive representations of surface groups into $\mathrm{PSL}(3,\mathbb{R})$ with parabolic or hyperbolic boundary monodromy), but also for general positive representations into $\mathrm{PSL}(d,\mathbb{R})$ with hyperbolic boundary monodromy. When the representation has parabolic boundary monodromy, we are able to obtain a (non-strict) McShane-type inequality. The summands that we obtain enable us to demonstrate the discreteness of various simple "length" spectra associated to these positive representations. None of this is novel in the hyperbolic case, where the geodesic flow is Anosov, but seems to be novel in the parabolic boundary case. When $d = 3$, we obtain a collar lemma and reverse engineer the identity to define a Thurston-type (mapping class group invariant) asymmetric metric on the space of finite-area marked convex real projective 1-cusped tori. Note that our McShane identity proof in the higher rank hyperbolic boundary context relies on a very mild generalization of Labourie-McShane's scheme for deriving McShane identities

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Harmonic map, symmetric space and Einstein equation

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(joint work with Marcus Khuri, Gilbert Weinstein)

Hermann Weyl, in 1918, in his book "Space, Time, Matter" had written down the then-newly discovered Schwarzschild solution to the Einstein equation using cylindrical coordinates, and discovered that it is completely determined by a harmonic function. Over the next hundred years since, the Einstein equation has been associated to elliptic variational problems, often called the σ-model approach. We constructed a new set of 5-dimensional vacuum stationary Einstein metrics with a set of axial symmetries, and with 3-dimensional blackhole horizons which are not necessarily spherical.

First recall the basic setups. Spacetime is a Lorentzian n-manifold $(\mathcal{N}^{n+1}, \mathbf{g})$ satisfying the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R_{\mathbf{g}}\, \mathbf{g}_{\mu\nu} = T_{\mu\nu}$$

where $R_{\mu\nu}$ is the Ricci curvature of the metric $\mathbf{g}$, $R_{\mathbf{g}}$ the scalar curvature of the Lorentzian metric $\mathbf{g}$, and $T_{\mu\nu}$ is the energy-momentum-stress tensor of the matter fields. When vacuum; $T_{\mu\nu} = 0$, taking the trace of the Einstein vacuum equations we get $R_{\mathbf{g}} = 0$, and hence the VEE is equivalent to $R_{\mu\nu} = 0$..

We introduce the so-called Ernst reduction scheme in $3 + 1$ dimension. Let $(\mathcal{N}^{3+1}, \mathbf{g})$ be asymptotically flat, stationary and axially symmetric, satisfying the vacuum Einstein equation, and let G be the symmetry group ($\cong \mathbb{R} \times U(1)$). We further define X: the (Lorentzian) norm (> 0) of the Killing field generator $\xi = \partial/\partial\phi$, and Y: the potential of the *twist* $*(\xi \wedge d\xi)$. Then the two quantities $\varphi = (X, Y)\colon \mathcal{M}/G \longrightarrow \mathbb{H}^2$ form a (weighted) harmonic map, i.e. a critical point of the energy

$$\int \left(\frac{|\nabla X|^2 + |\nabla Y|^2}{X^2} \right) \rho \, d\mu_g,$$

where g is the metric on $\mathbb{R}^2 = \mathcal{M}/G$ (thus $d\mu_g = d\rho dz$) and ρ is the area element of the $U(1)$-orbits of symmetry. The map φ is identified with an θ-axisymmetric harmonic map (B. Carter)

$$\varphi : \mathbb{R}^3 \setminus \Gamma \to \mathbb{H}^2 \ni (X, Y).$$

with $\varphi(\rho, \theta, z) = (X, Y)$. (Note $d\mu_{\mathbb{R}^3} = [\rho d\rho dz]d\theta$ with θ a dummy variable.) Furthermore, Y is constant on any component of Γ, and the difference between constants is the angular momentum induced by the existence of the horizon between the two components. The *data* consists of Γ together with a constant for each component of Γ.

We now seek stationary solutions in the higher dimeisnion, namely find spacetimes $(\mathcal{N}^{4+1}, \mathbf{g})$ of the Einstein vacuum equations with three mutually commuting Killing fields $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial \phi_1}$, $\frac{\partial}{\partial \phi_2}$, the generators of an isometry group $\mathbb{R} \times U(1)^2$ acting on N. If $(\mathcal{N}^{4+1}, \mathbf{g})$ has an event horizon, its connected components are diffeomorphic to 3-manifolds Σ^3 of positive Yamabe type, i.e. each component admits a metric of positive scalar curvature (Galloway-Schoen [2]). Under the additional symmetry condition above, the list of topological types (Hollands-Yazadjiev [3]) is further restricted to S^3, $S^1 \times S^2$, $\mathbb{R}P^3$, $L(n, m)$. Examples of solutions include

- No horizon: Minkowski space: $\mathbb{R}^{4,1}$ with its metric
- S^3: Myers-Perry ([8])— 4+1 analog of Kerr solution.
- $S^2 \times S^1$: Emperan-Reall ([1]) Pomeransky-Senkov([9])— *Black Ring.*

Following the $3+1$-dimensional cases, solutions $\mathbf{g}$ on $(\mathcal{N}^{4+1} \setminus \{\text{axis}\})$ can be modeled by the Weyl-Papapetrou Coordinates $\big(\mathbb{R} \times U(1)^2\big) \times \{(\rho, z) \colon z \in \mathbb{R}, \rho > 0\}$ with

$$\mathbf{g} = G_{ij}\, dx^i\, dx^j + e^{2\nu}(d\rho^2 + dz^2) \quad (y^1 = \phi^1, y^2 = \phi^2, y^3 = t)$$

and

$$\rho = \sqrt{|\det G_{ij}|}, \quad \nu = \nu(\rho, z), \quad G_{ij} = G_{ij}(\rho, z).$$

Note: ρ is harmonic, and thus a coordinate function, i.e. $\nabla \rho \neq 0$, and Let z be its conjugate harmonic function.

The Einstein vacuum equations now reduce to a *harmonic map* ([7])

$$\varphi\colon \mathbb{R}^3 \setminus A \longrightarrow SL(3, \mathbb{R})/SO(3) {:=} X, \text{ with}$$

$$G_{ij}dx^i dx^j = -\frac{\rho^2}{f}dt^2 + \sum_{1 \le i,j, \le 2}^{2} f_{ij}(d\phi^i + \omega^i dt)(d\phi^j + \omega^j dt),$$

where $f = \det(f_{ij})$, define a positive definite, symmetric matrix with $\det = 1$:

$$\Phi = \frac{1}{f}\begin{pmatrix} 1 & -v_1 & -v_2 \\ -v_1 & f\, f_{11} + (v_1)^2 & f\, f_{12} + v_1 v_2 \\ -v_2 & f\, f_{21} + v_2 v_1 & f\, f_{22} + (v_2)^2 \end{pmatrix}$$

representing a point in $SL(3, \mathbb{R})/SO(3)$. Here v_i is the Ernst potential for the Killing field $\frac{\partial}{\partial \phi^i}$, a generalization of Y above.

On the z-axis $\{\rho = \sqrt{|\det G|} = 0\}$, we have $\dim \ker(G(0, z)) = 1$ except at isolated values $\{p_i\}_{i=1}^N$ on the z-axis (corners).

Let $(0, z)$ be a non-corner point, and let $0 \neq V \in \ker G$. There are two cases:

- *V null* $\Rightarrow$ the rod $p_i \leq z \leq p_{i+1}$ corresponds to *a horizon rod.*
- *V spacelike* $\Rightarrow$ the rod $p_j \leq z \leq p_{j+1}$ corresponds to *an axis rod.*

In the second case, one can scale V so that $V = k\frac{\partial}{\partial t} + n\frac{\partial}{\partial \phi^1} + m\frac{\partial}{\partial \phi^2}$ and $n, m \in \mathbb{Z}$ with $gcd(n, m) = 1$.

Definition. The *rod structure* consists of the set of axis rods $\{\Gamma_i\}_{i=1}^N$ together with the integers (n_i, m_i) for each axis rod, plus the values of Ernst potentials v_1, v_2 for each axis rod Γ_i.

The rod structure encodes the topological types of the horizons, as the three manifolds on our lists have singular torus foliations whose singular fibers are S^1's, appearing at one of the axis rods. Analytically the rod structure specifies the blow-ups of the harmonic map φnear the axis rods: observe $f \to 0$ (or $X \to 0$ in $3+1$ case) as $\rho \to 0$.

We present our main theorem [4, 5]. Denote by Γ the set of axis rods.

Theorem 1 (Khuri-Weinstein-Yamada 2017)**.** *Given any rod structure $\Gamma = \cup\Gamma_i$, and any set of axis rod constants $v_1 = a_i, v_2 = b_i$ for Γ_i, there exists* a unique harmonic map $\varphi\colon \mathbb{R}^3 \setminus A \longrightarrow \Gamma$ *with singularities on* Γ *whose asymptotic behavior near* Γ *and spacetime infinity corresponds to the correct rod structure and to asymptotic flatness respectively.*

Furthermore, in [6] with the help of Y. Matsumoto, we have classified the topological types of the domain of outer communications arising as consequences of the main theorem, utilizing the language of *plumbing construction* from 4-dimensional topology, where the topological data is specified by the rod structures of the spacetime.

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Harmonic maps between ideal hyperbolic complexes

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(joint work with Victòria Gras Andreu)

The spaces under consideration are 2-dimensional simplicial complexes endowed with ideal hyperbolic metrics as introduced by Charitos and Papadopoulos in [1]. Our 2-dimensional complexes are assumed to be finite (meaning there are a finite number of vertices, edges, and faces), without boundary (meaning each edge is incident to at least two faces) and locally 1-connected. This last condition means that the link of each vertex is connected, or equivalently that the complex can be built from a disjoint union of triangles by identifications along edges.

On a complex X, let S denote the set of vertices. The metrics introduced in [1] are defined on $X\backslash S$ as follows. Endow each triangle (with its vertices removed) with the metric of an ideal hyperbolic triangle, and when two triangles are identified along an edge, identify their respective edges by an isometry. After these identifications, this endows each face of X with the structure of an ideal hyperbolic triangle, and defines a distance metric on the whole of X by considering lengths of paths.

Each edge of an ideal hyperbolic triangle is isometric to the real line, and the isometries of $\mathbb{R}$ form a 1-dimensional family, so each identification can be identified by a single parameter. In the case of a triangulated surface these parameters are the usual shear coordinates. In order for the identifications along a single edge of X to be consistent, if k faces meet at an edge there are $k-1$ independent shear parameters at that edge.

In order for an ideal hyperbolic structure on X to be complete, each punctured vertex of X must look like a cusp. For any simple closed loop in the link of a vertex v, a neighborhood of v intersected with the corresponding faces of X is a hyperbolic surface which must have a cusp at v in order to be complete. This imposes one linear relation among the shear parameters for each closed loop in the link of a vertex. The total number of relations imposed by completeness is the sum of the ranks of the fundamental groups of the links of the vertices of X.

The Teichmüller space $\mathcal{T}(X)$ is defined to be the collection of complete ideal hyperbolic structures on $X\backslash S$. By considering the shear parameters discussed above, the dimension of $\mathcal{T}(X)$ is $E-V$, where E and V are the number of edges and vertices of X. In the case where $X\backslash S$ is homeomorphic to a punctured surface $S_{g,V}$, this dimension count reduces to the familiar $6g-6+2V$.

Fix $\sigma,\tau\in\mathcal{T}(X)$. Let Y denote the class of maps $u:(X\backslash S,\sigma)\longrightarrow(X\backslash S,\tau)$ such that the restriction $u|_e$ of u to an edge e of X maps e to itself, the restriction $u|_T$ to a face T of X maps T to itself, and $u|_T\in W^{1,2}(T;T)$ is a finite energy map on each face T. Define the energy of a map $u \in Y$ as the sum over all the faces of X:

$$E(u) = \sum_T \int_T |\nabla u|^2 d\mu.$$

Let $\mathcal{D}$ denote the class of maps $u \in Y$ such that $u|_T$ maps T diffeomorphically onto T for each face T of X. We prove the existence of energy minimizing mappings in each of these classes.

Theorem 1. *There exist energy minimizing maps $u \in Y$ and $\tilde{u} \in \mathcal{D}$. That is, for any $v \in Y$ and $\tilde{v} \in \mathcal{D}$, $E(u) \leq E(v)$ and $E(\tilde{u}) \leq E(\tilde{v})$. Both maps are proper, degree 1, and locally Lipschitz continuous. The map u is analytic on the closure of each face. The map v is a harmonic diffeomorphism on the interior of each face.*

The first step in the proof is to construct a finite energy map. In a neighborhood of each punctured vertex we can write down an explicit map, and we fill in the compact remainder of $X \backslash S$ by solving a Dirichlet problem. To find an energy minimizing map, we consider a sequence of maps whose energies approach the target energy. Along this sequence we perform harmonic replacement on a sequence of compact sets exhausting $X \backslash S$ in order to get uniform Lipschitz bounds and apply the Arzela-Ascoli theorem.

Our constructions are motivated by the variational principle of Gerstenhaber and Rauch [2], where they propose characterizing Teichmüller mappings by a minimax principle. In particular they suggest that given two conformal structures on a closed surface, the minimum dilitation K^* of any diffeomorphism isotopic to the identity can be calculated via

$$\sup_\rho \inf_f E_\rho(f) = \frac{1}{2}\left(K^* + \frac{1}{K^*}\right).$$

Here ρ is taken from the class of conformal metrics on the target with fixed area and at most conical singularities, f is taken from the homotopy class of the identity, and $E_\rho(f)$ denotes the energy of f with respect to the metric ρ.

The Gerstenhaber-Rauch princple was investigated by Reich [7] and Reich-Strebel [8] in the case of the unit disc, and by Kuwert [5], who proved that the Teichmüller map along with a singular flat metric associated to its terminal differential realizes the minimax. Mese [6] carried out the minimax procedure, proving from this point of view the existence of Teichmüller mappings.

The minimization part of the minimax program involves finding harmonic diffeomorphisms. With this application in mind, we are currently working on generalizing our construction to more general metrics on the target complex, in parallel with trying to prove that our harmonic maps are in fact diffeomorphisms.

For harmonic maps between discs, Jost [3] proves that the solution to the Dirichlet problem is a diffeomorphism when the boundary values are a homeomorphism. Jost and Schoen [4] use this result to minimize over the class of diffeomorphisms to prove that every diffeomorphism between surfaces is isotopic to a harmonic diffeomorphism. Schoen and Yau [9], by a continuity argument, prove the same result as in [4].

The techniques of [4] motivated our construction of the minimizing mapping $v \in \mathcal{D}$, but the analysis along the edges of X required to show that this map also minimizes energy in Y is subtle. Similarly, the analysis of [9] does not generalize well to the singular edges and non-compactness in our present case. Nonetheless we phrase this as a conjecture.

Conjecture 2. *The minimizing maps in Y and in $\mathcal{D}$ are identical. In other words, the minimizing map $u \in Y$ is a diffeomorphism when restricted to each face of X.*

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Partitioning Teichmueller space in contractible pieces

Norbert A'Campo

(joint work with Athanase Papadopoulos, Sumio Yamada)

Let $P_d = \{f = z^d + a_{d-1}z^{d-1} + \cdots + a_0\}$ be the space of monic polynomials of degree $d \geq 1$. Let Γ_z be the union of the real and imaginary axis in the Gaussian plane of complex numbers $\mathbb{C}$. Think Γ_z as a collared graph, the real axis being green, the imaginary blue. Moreover orient this graph from $-\infty$ to $+\infty$, from $-i\infty$ to $+i\infty$ and label the quadrants by A, B, C, D counter clockwise. The *picture* of $f \in P_d$ is the preimage $\Gamma_f = f^{-1}(\Gamma_z)$ with induced coloring, orientation of the edges and labeling of the connected components of its complement. As first main properties of Γ_f notice that Γ_f is a forest. The non bounded edges are in number $4d$, all asymptotic to a ray from $0 \in \mathbb{C}$ with direction a $4d^{\text{th}}$-root of unity. The forest Γ_f has no terminal vertices. All vertices have even valencies ≥ 4. The orientations of the non bounded edges alternate in between ingoing/outgoing near infinity, also the colors alternate between green/blue near infinity. The edges incident with a vertex can be all green or all blue, but then the orientations alternate in between going away or towards the vertex. It edges of both colors green and blue are incident with a vertex, then orientations and colorings alternate. Moreover near a vertex the graph is diffeomorphic to finitely many real lines in $\mathbb{C}$ through the origin such that all sectors have equal angle. Last but least, going from a D-region to a A.region is possible by crossing a right handed oriented green edge, from A- to B-region by crossing a right handed oriented blue edge, etc. An by orientation, coloring and region labeling enhanced graph is called admissible if above combinatorial properties are satisfyed. Let Γ be an admissible graph with $4d$ asymtotic terminal edges. Let $P_{d,\Gamma}$ be the subset in P_d of polynomials f such that Γ and P_f are isotopic preserving asymptotics.

Main result in [1] is: *The family* $(P_{d,\Gamma})_{\Gamma\,\text{admissible}}$ *is a partition of* P_d *with contractible real sub-algebraic pieces.*

A polynomial f belongs to the subspace P_d^* of polynomials having d distinct roots if and only if all vertices in Γ_f incident with edges of both colors are of valency 4. It follows that the partition of P_d induces a partition of P_d^* and of the discriminant locus $\Delta_d = P_d \setminus P_d^*$. The partition in contractible real sub-algebraic pieces provides an explicite finite complex for computing the co-homology of the braid group, since P_d^* is a classifying space.

The aim of our joint work is the construction of an explicite partition in contractible pieces of Teichmueller space T_g that is invariant by the action of the mapping class group. Our construction depends on $g = 0$, $g = 1$, $g \geq 2$. The case $g = 0$ is for free since T_0 consists of one point. The case $g = 1$ needs a special treatment, but let us first consider the general case $g \geq 2$.

Let S be a differentiable oriented surface of genus $g \geq 2$. Let $\Sigma \in T_g$ be realized by a conformal structure σ on S. For a pair (p, q) of distinct points on S let $f_{p,q,\sigma}$ be the σ-harmonic map $f_{p,q,\sigma} : S \longrightarrow \mathbb{R}$ such that the meromorphic differential $df - idf \circ J_\sigma$ has only simple poles at p, q with residues $+1, -1$ respectively. Here J_σ is the almost complex structure defined by σ. The map $f_{p,q,\sigma}$ is well defined up to a constant. Moreover, if σ' also represents Σ the maps $f_{p,q,\sigma}$ and $f_{p,q,\sigma'}$ are isotopic.

For generic pairs (p, q) on the Riemann surface (S, σ) the fonction $f_{p,q,\sigma}$ is a Morse function with $2g$ critical points by the Riemann-Roch Theorem. Moreover, $f_{p,q,\sigma}$ has $2g$ critical values. Each critical level consists of a closed loop with one σ-orthogonal self intersection. Every system consisting of $2g$ simple loops that separate the singular levels provide a pant decomposition for the twice punctured surface (S, p, q).

For special pairs (p, q), the critical points of $f_{p,q,\sigma}$ are modelled on the critical point of the function $\text{Re}(z^k)$, $k \geq 2$ from which $2k$ descending sectors start. In each such decending sector consider the gradient line segment (with respect to any Riemannian metric in the conformal class σ) that departs along the bisector and stops at next singular level or at the point q. Moreover, connected components of critical levels consist of loops with equi-σ-angular transversal self intersections.

The *picture* $\Gamma_{p,q,\sigma}$ of $f_{p,q,\sigma}$ is defined as the union of all singular levels of $f_{p,q,\sigma}$ and all bisectorial decending gradient segments. Consider two pictures as equivalent if isotopic as graphs in S. For fixed conformal structure σ let $\Pi(\sigma)$ be the partition of $S \times S \setminus \mathrm{Diag}_S$ induced by the equivalence relation on pictures $\Gamma_{p,q,\sigma}$. Label each class by the equivalence class of its typical picture.

Finally consider two conformal structures $\Sigma_1, \Sigma_2 \in T_g$ on S as equivalent, if the partitions Π_{σ_1}, $\sigma_1 \in \Sigma_1$, and Π_{σ_2}, $\sigma_2 \in \Sigma_2$, together with the labeling are isotopic by a diagonal product isotopy of $S \times S$.

Main result: *The induced partition on Teichmueller space T_g has real sub-analytic contractible pieces. This partition is mapping class invariant and induces a finite real-sub-analytic partition with contractible pieces on the moduly space M_g.*

The genus $g = 1$ case differs, since genus 1 surfaces (S, σ) admit a transitive system of holomorphic symmetries. Choose first $p \in S$ and consider the holomorphic group law on S with p as neutral element. Let q_1, q_2, q_3 be the three 2-division points. Let $f_{p,q_j,\sigma}$, $j = 1, 2, 3$, be the harmonic functions as above. For general conformal structure σ all three functions have 2 critical values and the corresponding pictures do not connect the critical points. In least special case only one function will have 1 critical value. This last condition defines the 3-valent Farey graph in Teichmueller space. The midpoint of edges in the Farey graph correspond to next special case where two functions have 1 critical value. At the triple points of the Farey graph all three functions have 1 critical value. Last property caracterizes elliptic curves with maximal amount of symmetry.

A similar caracterization holds for the Bolza surface: a genus 2 surface (S, σ) admitting three pairs of points (p_j, q_j), $j = 1, 2, 3$, consisting of 6 distinct points in total, such that the functions $f_{p_j,q_j,\sigma}$ have 1 critical value is bi-holomorphic to the Bolza surface.

Remarks. The topology in between two distinct points p, q on a surface can be described by a ribbon graph construction, see [4], or by a Morse function having no minima or maxima as used in present paper. Bob Penner asks how both approaches in constructing cell decompositions of moduli spaces $M_{g,2}$ are related.

In [2] the authors J. Harer and D. Zagier and in [3] the author R.C. Penner have computed the orbifold Euler caracteristic of the moduli spaces M_g and $M_{g,1}$. It would be very interesting to compute the number of cells of presents paper cell decomposition of M_g and deduce the Euler caracteristic of M_g. In [2] and [3] a cell decomposition of the space $M_{g,1}$ is constructed and used in an essential way.

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Problem Session compiled by Robert Penner

Robert Penner et al.

Here is a list of open problems on Teichmüller theory posed on the evening of 6 September, 2018, at Oberwolfach. Norbert A' Campo as master of ceremonies began with the rule: "You should not know the solution already, but you should want to know a solution".

Problem 1: Scott Wolpert

Consider the Weil-Petersson completion of the Teichmüller space of a surface, and show that some Dirichlet fundamental domain for the action of the mapping class group is finite-sided. It is a Dirichlet domain so automatically convex. The difficulty is controlling geodesics near infinity. In particular, geodesics could start spiraling tangentially to the boundary in the directions where the boundary is not locally compact. Near any compact set, there are only a finite number of faces that are in play, but this number may increase as you tend towards infinity. Since the mapping class group is known to be finitely presented, perhaps you can use this to control the asymptotic geometry. It is not clear how the desired fundamental domain is meant to be presented, that is, in which coordinates.

Problem 2: Marc Burger

Take a compact hyperbolic surface S with the orthonormal system $\phi_0, \phi_1, \ldots$ of eigenfunctions of the Laplacian with corresponding eigenvalues $\lambda_0, \lambda_1, \ldots$ taken with multiplicities. Show that

$$\frac{\sum \int_C \phi_n \; ds}{\#\{C : \ell(C) \leq T\}} = O(\lambda_n^{-\alpha} T^{-\beta})$$

for some α, β, where the sum is over all simple closed geodesics C in S of length $\ell(C)$ at most T and ds denotes arc length on S.

Problem 3: Norbert A'Campo

Construct a McShane-type identity with the eigenvalues of the Laplacian. In light of work by Heinz Huber that the eigenvalues determine the simple length spectrum, it is reasonable to expect such an identity.

Problem 4: Mahan Mj

One knows from work of Kapovich-Leek-Porti and Guéritaud-Guichard-Kassel-Wienhard that an Anosov subgroup H of a Lie group G is uniformly regular undistorted, i.e., quasi-isometrically embedded in G/P for the word metric and equivalently described as asymptotically embedded in G/P. The problem is to relax the condition of undistorted and impose just that H is Gromov hyperbolic and ask if there is asymptotically a continuous map from the Gromov boundary of H to G/P. In rank 1, we are asking for asymptotic embeddedness of non convex co-compact subgroups. The answer is YES for $G = PO(3, 1)$.

Problem 5: Anton Zeitlin

Two questions about super business:

(1) What is the correct notion of cluster super algebra? There was some progress by Ovsienko. The main difficulty is how to include spin structure, and a key question is to explore the Laurent phenomenon.

(2) What is the role of spectral networks in higher super Teichmüller theory? Spectral networks provide a method of abelianization, and you can actually deduce higher Teichmüller structure from abelian connections by a procedure developed by Gaiotto-Moore-Neitzke. The instrumental tool was spectral networks, and we ask if there is an analogous version in the super case? For $OSp(2|2)$, the "abelianization" is not quite abelianization but rather another super group $GL(1|1)$ whose Lie algebra is the Clifford algebra, which is not commutative but nearly so.

Problem 6: Jeff Danciger

This might be related to Problem 4 above. Consider two representations

$$\rho_1 : \pi_1 \longrightarrow SO(3,1) \text{ quasi} - \text{Fuchsian}$$

$$\rho_2 : \pi_1 \longrightarrow SO(3,1) \text{ geometrically infinite}$$

of the fundamental group π_1 of a closed surface of genus $g \geq 2$ so that there exists $\gamma \in \pi_1$ so that the ρ_2-length of γ exceeds its ρ_1-length. The direct sum $\rho_1 \oplus \rho_2$ maps into $SO(3,1) \oplus SO(3,1)$ which we may regard as a subgroup of $G = SO(6,2), SO(4,4)$ or $SL(8)$. Our question is whether $\rho_1 \oplus \rho_2$ is stable in G in the sense that any small *deformation in G* is still discrete faithful. Because of the existence of γ this is not an Anosov representation, but has some Anosov-like properties it would seem. More generally, how do you find discrete faithful representations of hyperbolic groups that are stable but not Anosov?

Problem 7: Thomas Koberda

Suppose that Γ is a finite graph and let $A(\Gamma)$ be the associated right-angled Artin group. What is the smallest n so that $A(\Gamma)$ sits as a subgroup of $SL_n(\mathbb{R})$? There are the following special cases:

(1) Insist that for each vertex of Γ the corresponding generator of $A(\Gamma)$ is unipotent. This case has "moot" connections to non-linearity of the mapping class group.

(2) Insist that for each vertex of Γ the corresponding generator is an axial isometry, i.e., there are 2 real eigenvalues, one greater and one less than unity, with the others complex.

(3) If Γ is the pentagon, then is $A(\Gamma)$ a subgroup of $SL(3,\mathbb{R})$?

It seems likely that to answer questions like these will require something beyond discrete methods, at least to find the minimal embedding. And using tools like Baire category to find faithful linear representations, then you have to use a field like the reals. Over the rationals it is hard to say very much. And over the integers there are very few right-angled Artin groups; if $A(\Gamma)$ is a subgroup of $SL(3,\mathbb{Z})$, for example, then it has to be either a free group or it might be a $\mathbb{Z}$ direct sum free product or a free product of such groups, although no such examples are known.

Ideally one would want to know more than just the minimal n but also know the images or control the types of images that could occur.

Now formulate the same questions over other fields or rings and ask how the answer is different for $\mathbb{R}, \mathbb{Q}, \mathbb{Z}$. For example, in $SL(3, \mathbb{R})$ you can find a copy of a free group times $\mathbb{Z}$ but you cannot do that over $\mathbb{Z}$. Most interesting is $\mathbb{R}$ with the motivation coming from mapping class group linearity.

Problem 8: Valentina Disarlo

Let S be a compact finite-type surface of genus g with $r > 0$ geodesic boundary components and let $T(S)$ denote its Teichmüller space. There are three natural generalizations of Thurston's asymmetric metric on $T(S)$:

$$d_A(X, Y) \; = \; \log \sup \frac{\ell_Y(\gamma)}{\ell_X(\gamma)},$$

where ℓ denotes the geodesic length and the supremum is over all γ in the union $\{$properly embedded arcs$\} \cup \{$boundary components$\} \cup \{$simple closed curves$\}$,

$$\begin{aligned} d_{Lh}(X, Y) \; &= \; \log \inf \; \{Lip(\phi) : \phi \in Lip_0(X, Y) \text{ homeomorphism}\}, \\ d_{L\partial}(X, Y) \; &= \; \log \inf \; \{Lip(\phi) : \phi \in Lip_0(X, Y) \text{ and } \phi(\partial S) \subset \partial S\}, \end{aligned}$$

where $Lip(\phi)$ denotes the Lipschitz constant of a map $\phi \in Lip_0(X, Y)$, namely those Lipschitz maps homotopic to the identity. All three are asymmetric metrics on $T(S)$. The conjecture is that all three metrics coincide (and we have proved already that $d_A = d_{L\partial}$).

Problem 9: Norbert A' Campo

Consider two essential simple closed curves meeting transversely in a single point in some surface with negative Euler characteristic, so that respective right Dehn twists a, b upon them satisfy the relation $aba = bab$ in the mapping class group. Can you lift to respective homeomorphisms A, B of the surface satisfying $ABA = BAB$? To once-continuously differentiable homeomorphisms? To r-times continuously differentiable homeomorphisms? A negative answer to this would reprove a theorem of Vlad Markovic, and a positive answer would be a surprise.

Problem 10: Tengren Zhang

Suppose that Γ is the fundamental group of closed orientable surface of genus $g \geq 2$. Does there exist a 1-Anosov representation of Γ in $PGL(d, \mathbb{R})$ so that the limit set $\xi(\partial\Gamma) \subset \mathbb{P}(\mathcal{R}^d)$, which is a circle, is rectifiable but not $\mathcal{C}^{1,\alpha}$? That is, does rectifiable imply $\mathcal{C}^{1,\alpha}$?

Problem 11: Justin Lanier

What are the proper normal subgroups of a mapping class group that are NOT contained in any congruence subgroup, or alternatively in their union? Sub-normal subgroups of mapping class groups have been intensively studied such as the Torelli group or the other groups in the Johnson filtration and likewise the congruence subgroups themselves. But all of these live in a congruence subgroup. I do not ask if there are any since I know of a construction of one class that involves playing high powers of pseudo-Anosovs against one another so they do not lie in a congruence subgroup; that will be in a forthcoming work, but this is the only construction I know. These groups just seem to be under-studied, and I would like to know more about them, for example their classification.

Problem 12: Dmitri Gekhtman

Fix a finite cover $f : S_h \longrightarrow S_g$ of a genus g surface S_g by a genus h surface S_h, where $h > g \geq 2$. Is there always a sphere minus four disks H with essential boundary components in S_g so that the interior of $f^{-1}(H)$ is disconnected? What about for H a sphere minus three disks? And consider only regular covers if you want. You can ask any number of simple questions of this form about finite covers that are hard to answer. My motivation comes from the induced map $T(F_g) \longrightarrow T(F_h)$ of Teichmüller spaces, and I have a result about this embedding that currently only works under this technical assumption, namely, the smaller Teichmüller space is not a holomorphic retract of the bigger one if there is such an H. The existence of such an H may just be an unnecessary requirement of the current proof but seems an interesting question in its own right.

Problem 13: Weixu Su

The problem involves counting simple closed geodesics in the moduli space. Fix points x, y in Teichmüller space, consider the number of points in the mapping class orbit of y that occur in a ball of radius R about x in the Thurston metric and compute the asymptotics as R tends to infinity. This is known for the Teichmüller metric by works of Eskin, Mirzakhani and others. My problem is related to counting extremal length instead of hyperbolic length functions, and the question is how do these relate or how can they be unified?

Problem 14: Daniele Alessandrini

Suppose that M is a closed manifold and consider the mapping given by the holonomy $\mathrm{Def}_{(G,X)}\, M \longrightarrow \mathrm{Rep}(\pi_1(M), G)$ from the deformation space of (G, X)-structures on M to the representation variety, which wants to be a local homeomorphism but is NOT in some very special cases. It is however an open map with discrete fiber. Is this always a branched local homeomorphism? (Probably YES always.) Find conditions so that it is a local homeomorphism. (It "usually" is says Bill Goldman.). These are interesting questions aimed at a better understanding of deformation spaces.

Problem 15: Marc Burger

Hitchin representations give finite-dimensional families of $\mathcal{C}^1$ curves. There should be an ordinary differential equation describing at least some of them. Find this ODE.

Problem 16: Dragomir Sáric

Let $T = T(\mathbb{H}^2)$ be the universal Teichmüller space with its Thurston boundary $PML_{bdd} = PML_{bdd}(\mathbb{H}^2)$ of projectivized bounded measured laminations. Meanwhile, the universal Weil-Petersson Teichmüller space $T_{WP} = T_{WP}(\mathcal{H}^2)$ as described by Sullivan, namely the locus where the universal WP form converges, is a proper subspace of T. In fact, Takhtajan-Teo have shown that T_{WP} is a complex Hilbert manifold with geodesics and is complete, and Yuliang has explicitly described this WP slice. The problem is to find the limit points of WP-geodsics in PML_{bdd} and more modestly just give examples. For instance, if α is a bounded measured lamination then the corresponding earthquake path $\frac{1}{t}E^{t\alpha}$ tends to α as t tend to infinity; give similar examples for WP.

Reporter: Valentina Disarlo

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