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MiniWorkshop: Asymptotic Invariants of Homogeneous Ideals

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Abstract. Recent decades have witnessed a shift in interest from isolated objects to families of objects and their limit behavior, both in algebraic geometry and in commutative algebra. A series of various invariants have been introduced in order to measure and capture asymptotic properties of various algebraic objects motivated by geometrical ideas. The major goals of this workshop were to refine these asymptotic ideas, to articulate unifying themes, and to identify the most promising new directions for study in the near future. We expect the ideas discussed and originated during this workshop to be poised to have a broad impact beyond the areas of algebraic geometry and commutative algebra.

Mathematics Subject Classification (2010): primary 14C20, 13C05; secondary 13D02, 14N05.

Introduction by the Organisers

The miniworkshop *Asymptotic invariants of homogeneous ideals*, organised by Thomas Bauer (Marburg), Susan Cooper (Manitoba), Brian Harbourne (Lincoln) and Justyna Szpond (Kraków) was attended by 17 participants from Europe and North America. There was a diversity in experience level ranging from early postdocs to established, internationally recognized professors. Thanks to this diversity we were able to achieve considerable progress on topics highlighted at the workshop and to provide excellent training for early career participants. It is also worthwhile highlighting the fact that the majority of workshop participants were female researchers. Workshop activities were divided between 14 half hour talks and group research collaborations which took place mostly in the afternoons. Activities commenced on the first day with an in-depth discussion of problems to be studied. There were also two formal progress report sessions, apart from informal discussions held throughout the workshop.

The research groups focused their efforts on three main problems, labeled A–C and described below.

A. Ideals with extremal behavior with respect to the Containment Problem. Ein–Lazarsfeld–Smith [4] and Hochster–Huneke [7] proved that the containment

$$I^{(m)} \subset I^r \tag{1}$$

holds for an arbitrary nontrivial homogeneous ideal $I \subset \mathbb{K}[x_0, \dots, x_n]$ and all $m \geq rn$. The proof by Ein, Lazarsfeld, and Smith was inspired by work of Swanson on ideal topologies and uses mainly the techniques of asymptotic multiplier ideals in characteristic zero. The work of Hochster and Huneke uses mainly a tight closure approach in finite characteristics. It is natural to wonder to what extent the bound $m \geq rn$ is optimal. This question, known as the Containment Problem, has recently attracted a lot of attention, see [1] and [9] for detailed surveys of results and open problems considered in the last decade.

In the simplest situation of ideals I describing points in $\mathbb{P}^2$, the containment $I^{(4)} \subset I^2$ is guaranteed by (1), but it is not completely understood when the containment $I^{(3)} \subset I^2$ fails. In fact this containment was proved in many cases and it came as a surprise when the first non-containment example was discovered in 2013. The ideals for which the containment $I^{(3)} \subset I^2$ fails seems to be rare. This working group focused on constructions of such ideals related to the group law on a nodal cubic. Surprisingly, it was discovered that this construction provides an alternative and more uniform approach to a series of examples studied in combinatorics by Böröczky.

B. Symbolic defect of some classes of geometrically motivated ideals. Studying the Containment Problem is one way of comparing regular and symbolic powers of ideals. The difference between the regular power I^m and the symbolic power $I^{(m)}$ can be measured in an alternative way introduced recently in [5]. The *m-th symbolic defect* of an ideal I is the minimal number of generators of the module $I^{(m)}/I^m$. The properties of this invariant are to a large extent unexplored. The main result of [5] is the classification of sets of general points in the projective plane with vanishing second symbolic defect.

This working group focused on Fermat point configurations in the plane. These are defined by almost complete intersection ideals I_n of the form

$$\langle x(y^n - z^n), y(z^n - x^n), z(x^n - y^n) \rangle,$$

with $n \geq 3$. The interest in these ideals is motivated by their extremal behavior from the point of view of the Containment Problem (see, for example, [3] and [8]). A sample result obtained in this direction is the computation of the n-th symbolic defect of I_n and a conjectural formula for the kn-th symbolic defects. These results are the first instance where an asymptotic version of the symbolic defect appears.

C. Seshadri constants on blow-ups of projective spaces. Seshadri constants have turned out to be a fundamental tool in the study of positivity questions in algebraic geometry. A lot of research is currently focused on problems related to Seshadri constants. One such open problem is whether Seshadri constants can be irrational. Even in the case of algebraic surfaces the exact values of Seshadri constants are very hard to compute. Recent results in [2] and [6] show, somewhat surprisingly, that the existence of irrational one point Seshadri constants on blow-ups of the projective plane follows from the Segre-Harbourne-Gimigliano-Hirschowitz (SHGH) Conjecture. There are also many different approaches to studying Seshadri constants. For this working group the focus was on the largest and the smallest values of the Seshadri constants $\varepsilon(X; L, x)$ of a fixed ample line bundle L as a single point x varies over a fixed surface X. These values behave very differently. The largest value $\varepsilon(X; L, 1)$ is achieved for x very general. The smallest value $\varepsilon(X; L)$ is usually attained at special points. We show, again assuming the SHGH Conjecture, that there exist line bundles on a blow-up of the projective plane with irrational Seshadri constant $\varepsilon(X; L)$. This suggests a negative answer to a conjecture raised around 2000 by Szemberg.

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MiniWorkshop: Asymptotic Invariants of Homogeneous Ideals

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Abstracts

Negative curves on symmetric blowups of $\mathbb{P}^2$ and resurgence

Jack Huizenga
(joint work with Thomas Bauer, Sandra Di Rocco, Brian Harbourne, Alexandra Seceleanu, and Thomas Szemberg)

The Klein and Wiman configurations are highly symmetric configurations of lines in the projective plane arising from complex reflection groups. One noteworthy property of these configurations is that all the singularities of the configuration have multiplicity at least three. In this talk we study the surface X obtained by blowing up $\mathbb{P}^2$ in the singular points of one of these line configurations. We study invariant curves on X in detail, with a particular emphasis on curves of negative self-intersection. We use the representation theory of the stabilizers of the singular points to discover several invariant curves of negative self-intersection on X, and use these curves to study Nagata-type questions for linear series on X.

The homogeneous ideal I of the collection of points in the configuration is an example of an ideal where the symbolic cube of the ideal is not contained in the square of the ideal; ideals with this property are seemingly quite rare. The resurgence and asymptotic resurgence are invariants which were introduced to measure such failures of containment. We use our knowledge of negative curves on X to compute the resurgence of I exactly. We also compute the asymptotic resurgence and Waldschmidt constant exactly in the case of the Wiman configuration of lines, and provide estimates on both for the Klein configuration.

Let us discuss the case of the Klein configuration in more detail. The automorphism group $\mathrm{PGL}_3(\mathbb{C})$ of $\mathbb{P}^2$ has a subgroup G isomorphic to $\mathrm{PSL}(2,7)$, the finite simple group of order 168. This group contains 21 involutions, and each of them fix a line in $\mathbb{P}^2$. These lines intersect precisely in 28 triple points and 21 quadruple points; let $\mathcal{K}$ be these 49 points, and let $I_\mathcal{K}$ be their homogeneous ideal. Then we show that $I_\mathcal{K}^{(3)} \not\subset I_\mathcal{K}^2$; more specifically, the product of the lines is an element of $I_\mathcal{K}^{(3)}$ which is not contained in $I_\mathcal{K}^2$. The key is to show that $I_\mathcal{K}^2$ does not contain any G-invariant forms of degree 21.

The concepts of resurgence and asymptotic resurgence are closely related to Waldschmidt constants, which are limits

$$\widehat{\alpha}(I) = \lim_{n\to\infty} \frac{\alpha(I^{(n)})}{n},$$

where $\alpha(J)$ is the initial degree of J. Upper bounds on the Waldschmidt constant $\widehat{\alpha}(I_\mathcal{K})$ can be obtained by exhibiting forms (ideally of low degree) in $I_\mathcal{K}^{(n)}$. This is not hard to do: we can use the (highly singular) line configuration to get highly singular curves of low degree, and then add on an additional curve which balances the multiplicities. In this way, we can construct for any $k \geq 1$ an element of $I_{\mathcal{K}}^{(14k)}$ of degree $91k + 2$, and in the limit we find $\widehat{\alpha}(I_{\mathcal{K}}) \leq 6.5$.

Lower bounds on Waldschmidt constants are more interesting. If the Waldschmidt constant were too small, it would give a curve C on the blowup $X_{\mathcal{K}} = \mathrm{Bl}_{\mathcal{K}}\mathbb{P}^2$ that is very singular for its degree. If we have a nef divisor N on $X_{\mathcal{K}}$, then $C \cdot N \geq 0$. Since singularities of C will typically make $C \cdot N$ smaller, if C is too singular then $C \cdot N$ will become negative. Thus, constructing interesting nef divisors produces interesting lower bounds on $\widehat{\alpha}(I_{\mathcal{K}})$. By using representation theory we construct negative curves on $X_{\mathcal{K}}$ and use these to construct nef divisors on $X_{\mathcal{K}}$. These divisors establish the next result.

Theorem 1 ([1])**.** *The Waldschmidt constants for the Klein and Wiman configurations of points satisfy*

$$6.44 \leq \widehat{\alpha}(I_{\mathcal{K}}) \leq 6.5 \qquad\qquad \widehat{\alpha}(I_{\mathcal{W}}) = 13.5.$$

The resurgence $\rho(I)$ and asymptotic resurgence $\widehat{\rho}(I)$ are defined by

$$\rho(I) = \sup\left\{\frac{m}{n} : I^{(m)} \not\subset I^n\right\}$$
$$\widehat{\rho}(I) = \sup\left\{\frac{m}{n} : I^{(tm)} \not\subset I^{tn} \text{ for all } t \gg 0\right\}.$$

Corollary 2. *We have $\rho(I_{\mathcal{K}}) = \rho(I_{\mathcal{W}}) = 3/2$. The asymptotic resurgence satisfies*

$$\frac{16}{13} \leq \widehat{\rho}(I_{\mathcal{K}}) \leq 1.234 \qquad\qquad \widehat{\rho}(I_{\mathcal{W}}) = \frac{32}{27}.$$

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Waldschmidt constants of points in $\mathbb{P}^N$

Justyna Szpond
(joint work with Grzegorz Malara and Tomasz Szemberg)

In the 70's, motivated by problems in complex analysis and diophantine approximation, the following invariant was first defined.

Definition 1 (Waldschmidt constant)**.** *Let $Z \subset \mathbb{C}^N$ be a finite set of points. The* Waldschmidt constant *of Z is the real number*

$$\widehat{\alpha}(Z) = \lim_{m\to\infty} \frac{\alpha(mZ)}{m}.$$

The existence of the limit has been showed by Chudnovsky [1, Lemma 1]. It is well known that $\widehat{\alpha}(Z) = \inf_{m\geq 1} \frac{\alpha(mZ)}{m}$. Chudnovsky established also the following fundamental fact, see [1, Theorem 1].

Theorem 2. *Let $Z \subset \mathbb{C}^N$ be a finite set of points. Then*

$$\widehat{\alpha}(Z) \geq \frac{\alpha(Z)}{N}. \tag{1}$$

The bound in (1) can now be easily derived from the seminal results of Ein, Lazarsfeld and Smith [4]. Chudnovsky expected that the bound in (1) is not optimal and raised the following Conjecture, see [1, Problem 1].

Conjecture 3 (Chudnovsky). *Let $Z \subset \mathbb{C}^N$ be a finite set of points. Then*

$$\widehat{\alpha}(Z) \geq \frac{\alpha(Z) + N - 1}{N}.$$

This has been subsequently generalized by Demailly, see [2, p. 101].

Conjecture 4 (Demailly). *Let $Z \subset \mathbb{C}^N$ be a finite set of points. Then for all $m \geq 1$*

$$\widehat{\alpha}(Z) \geq \frac{\alpha(mZ) + N - 1}{m + N - 1}.$$

Of course, for $m = 1$ Demailly's Conjecture reduces to that of Chudnovsky.

The main result presented in the talk is the following.

Theorem 5. *Demailly's Conjecture holds for $s \geq (m+1)^N$ very general points in $\mathbb{P}^N$.*

In particular, for $m = 1$ we recover the a result from [3] to the effect that the Chudnovsky Conjecture holds for $s \geq 2^N$ very general points in $\mathbb{P}^N$.

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The containment problem and Harbourne's conjecture for very general points in $\mathbb{P}^N_{\mathbb{C}}$

Yu Xie

Let $R = \mathbb{C}[x_0, x_1, \ldots, x_N]$ be a polynomial ring over $\mathbb{C}$. Let $I = \cap_{i=1}^s I(p_i)$ be the ideal of s points in $\mathbb{P}^N_{\mathbb{C}}$. Recall the m-th symbolic power of I is $I^{(m)} = \cap_{i=1}^s I(p_i)^m$. It is clear that $I^m \subseteq I^{(m)}$, but $I^{(m)}$ is not contained in I^m in general. The containment problem consists of determining all the values of a and b for which $I^{(a)} \subseteq I^b$ holds. The resurgence $\rho(I)$ is then defined to be $\sup\{\frac{a}{b} \mid I^{(a)}$ is not contained in $I^b\}$. A fundamental result of Ein-Lazarsfeld-Smith [5] and Hochster-Huneke [6] proved that $I^{(Nm)} \subseteq I^m$ for $m \geq 1$. Harbourne conjectured whether the containment can be improved to $I^{(Nm-N+1)} \subseteq I^m$ for $m \geq 1$ [1]. Harbourne's conjecture was proved for ideals of finite sets of general points when $N = 2, 3$ (See [2] and [3]). The first counterexample to this conjecture was found by Dumnicki, Szemberg, and Tutaj-Gasińska [4] who proved that the ideal of a certain configuration of twelve points in $\mathbb{P}^2_{\mathbb{C}}$ fails to have $I^{(3)} \subseteq I^2$.

We prove that Harbourne's conjecture holds for $I^{(t)}$ whenever $t \geq 2$, and any finite set of very general points in $\mathbb{P}^N_{\mathbb{C}}$.

Theorem 1. *Let $I = \cap_{i=1}^s I(p_i)$ be the ideal of s points in $\mathbb{P}^N_{\mathbb{C}}$. Then*

(1) $\left(I^{(t)}\right)^{(Nm-N+1)} \subseteq \left(I^{(t)}\right)^m$ for $t \geq 2$ and $m \geq 1$.
(2) $\rho\left(I^{(m)}\right) \leq \frac{m+N-1}{m}$ for $m \geq 1$.
(3) $\lim_{m\to\infty} \rho\left(I^{(m)}\right) = 1$.

Theorem 2. *Let $I = \cap_{i=1}^s I(p_i)$ be the ideal of s very general points in $\mathbb{P}^N_{\mathbb{C}}$. Then*

$$I^{(Nm-N+1)} \subseteq I^m \quad \text{for } m \geq 1.$$

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Local positivity on projective spaces

Tomasz Szemberg
(joint work with Marcin Dumnicki, Justyna Szpond)

Waldschmidt constants appeared in the 70's in the realms of complex analysis in works of Chudnovsky, Moreau, Skoda and Waldschmidt. They have been introduced to commutative algebra only recently by Dumnicki and Harbourne.

Definition 1. *Let $I \subset \mathbb{K}[x_0, \dots, x_N]$ be a homogeneous ideal. The* initial degree $\alpha(I)$ *is the least integer t such that the graded part $(I)_t$ is non-zero. The* Waldschmidt constant $\widehat{\alpha}(I)$ *is defined asymptotically as*

$$\widehat{\alpha}(I) = \lim_{m \to \infty} \frac{\alpha(I^{(m)})}{m} = \inf_{m \geq 1} \left\{ \frac{\alpha(I^{(m)})}{m} \right\},$$

where $J^{(k)}$ is the k^{th} symbolic power of J.

These invariants are very hard to compute in general. If I is a radical ideal of a finite set of s points in the projective space $\mathbb{P}^N$, then there is always

$$\widehat{\alpha}(I) \leq \sqrt[N]{s},$$

and it expected that the equality holds for sufficiently many points s in general position. This expectation, in the complex projective plane, is equivalent to the celebrated Nagata Conjecture, and hence seems out of reach at the moment. However there is an intriguing question raised by Demailly relating the Waldschmidt constant of an ideal of *arbitrary* points I to the initial degree of symbolic powers of I.

Conjecture 2 (Demailly)**.** *Let I be a saturated ideal of a finite set of points in $\mathbb{P}^N$. Then*

$$\widehat{\alpha}(I) \geq \frac{\alpha(I^{(m)}) + N - 1}{m + N - 1} \tag{1}$$

holds for all $m \geq 1$.

This conjecture reappears as a question in the recent work by Harbourne and Huneke [2, Question 4.2.1].

In my talk, based on a joint work with Dumnicki and Szpond [1], I presented a new approach to bounding Waldschmidt constants from below.

Theorem 3. *Let $H_1, \dots, H_s$ be $s \geq 2$ mutually distinct hyperplanes in $\mathbb{P}^N$. Let $a_1, \dots, a_s \geq 1$ be real numbers such that*

$$1 - \sum_{j=1}^{s-1} \frac{1}{a_j} > 0 \text{ and } 1 - \sum_{j=1}^{s} \frac{1}{a_j} \leq 0.$$

Let $Z_i = \{P_{i,1}, \dots, P_{i,r_i}\} \in H_i \setminus \bigcup_{j \neq i} H_j$ *be the set of* r_i *points such that* $\widehat{\alpha}(H_i; Z_i) \geq a_i$ *and let* $Z = \bigcup_{i=1}^{s} Z_i$. *Then*

$$\widehat{\alpha}(\mathbb{P}^N; Z) \geq \left(1 - \sum_{j=1}^{s-1} \frac{1}{a_j}\right) \cdot a_s + s - 1.$$

This result, accompanied by a symbolic algebra script, allows a recursive computation of very efficient lower bounds on sets of points in projective spaces. In particular, we obtain the following improvement of [1, Main Theorem].

Corollary 4. *The inequality in* (1) *holds for* $s \geq m^N$ *general points in* $\mathbb{P}^N$.

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Configurations with triple points by a rational cubic

Halszka Tutaj-Gasińska

(joint work with Thomas Bauer, Łucja Farnik, Brian Harbourne)

The talk presents a work in progress. It concerns a way to construct some configurations of lines with many triple points by means of a singular cubic. The interest in such configurations come from the containment problem: determine all m, r such that $\mathcal{I}^{(m)} \subset \mathcal{I}^r$. By the results of [2] and [4], we know that for a homogeneous radical $\mathcal{I}$ the containment $\mathcal{I}^{(nr)} \subset \mathcal{I}^r$ holds for $r \geq 0$. Thus, in $\mathbb{P}^2$, we know that $\mathcal{I}^{(4)} \subset \mathcal{I}^2$ and it is easy to give an example of an ideal where $\mathcal{I}^{(2)} \not\subset \mathcal{I}^2$. Thus, the question (stated eg in [3]) was: for $\mathcal{I}$, a homogeneous radical ideal of points in the projective plane, is there $\mathcal{I}^{(3)} \subset \mathcal{I}^2$?

The first example of the ideal which does not satisfy this containment appeared in [1]. The ideal for which the containment fails is there the ideal of triple points in dual Hesse configurations, so the dual to the configuration of twelve lines passing through triples of 3-torsion points of a smooth cubic.

In this talk we present a way to construct configurations with many triple points, and such, that the ideal of the triple points of these configurations does not satisfy the containment $\mathcal{I}^{(3)} \subset \mathcal{I}^2$. More precisely, assume we have a nodal cubic with s-torsion points. If s is not divisible by 3 we construct a set of s lines with $s-1$ double and $\frac{s^2-3s+2}{6}$ triple points, and if s is divisible by 3, then the configuration has $s-3$ double and $\frac{s^2-3s+6}{6}$ triple points. By computer-aided computations we checked, that ideals of triple points of these configurations give an example of noncontainment in $\mathcal{I}^{(3)} \subset \mathcal{I}^2$, for $12 \leq s \leq 22$, and that for $s \leq 11$ the containment holds.

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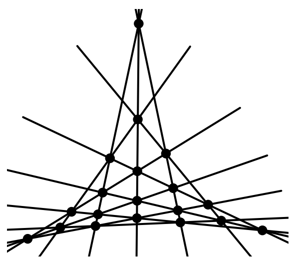
Containment Problem and Combinatorics

Łucja Farnik

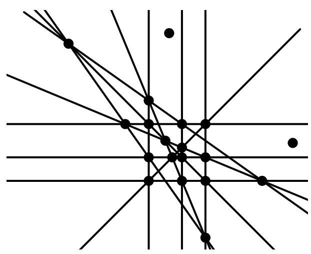
(joint work with Jakub Kabat, Magdalena Lampa-Baczyńska, Halszka Tutaj-Gasińska)

The Containment Problem asks for which values of m and r there is the containment of the m-th symbolic power of a homogeneous ideal I in $\mathbb{C}[\mathbb{P}^N]$ in its r-th ordinary power. General results in [3] and [5] show that $m \geq Nr$ implies $I^{(m)} \subset I^r$. Thus in $\mathbb{C}[\mathbb{P}^2]$, it is always $I^{(4)} \subset I^2$ and it is natural to wonder if $I^{(3)} \subset I^2$ holds. The first counterexample was given in [2]. Such non-containments are rare and therefore it is tempting to understand their nature.

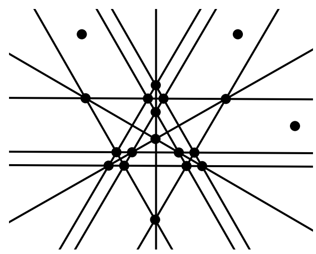
I will consider the ideals of nineteen triple points of three special arrangements of twelve lines, namely the Böröczky configuration of 12 lines and two configurations described in [1], i.e., the configuration C_2 and the configuration C_7.



The Böröczky configuration of 12 lines



The configuration C_2



The configuration C_7

Every configuration above has the same arrangemental combinatorial features, which means that in all three arrangements of lines there are nine of twelve lines having five triple points and one double point, and three lines having four triple points and three double points.

If I denotes the ideal of triple points of a configuration then, surprisingly, for one of the configurations, the containment $I^{(3)} \subset I^2$ holds, while for the others it does not. Hence, I will conclude that for ideals of points defined by arrangements of lines the (non)containment of the third symbolic power in the second ordinary power is not determined alone by arrangemental combinatorial features (see [4] for details).

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A stable version of Harbourne's Conjecture

Eloísa Grifo

Given a radical ideal I in a regular ring R, the n-th symbolic power of I is given by

$$I^{(n)} = \bigcap_{P \in \mathrm{Min}(I)} I^n R_P \cap R.$$

While symbolic powers have good geometric properties, they can be very difficult to compute; on the other hand, ordinary powers are easily computable, but do not enjoy good geometric properties. In general, $I^n \neq I^{(n)}$. The Containment Problem tries to compare I^n and $I^{(n)}$ by asking when $I^{(a)} \subseteq I^b$.

Theorem 1 ([3, 8, 9])**.** *Let R be a regular ring and I be a radical ideal of big height h. For all $n \geqslant 1$, $I^{(hn)} \subseteq I^n$.*

In characteristic p, this can be improved; one has $I^{(hq-h+1)} \subseteq I^q$ for all $q = p^e$.

Conjecture 2 (Harbourne, 2008)**.** *Let R be a regular ring and I be a radical ideal of big height h. Then for all $n \geqslant 1$, $I^{(hn-h+1)} \subseteq I^n$.*

Conjecture 2 holds for nice classes such as ideals defining general points in $\mathbb{P}^2$ [6] and $\mathbb{P}^3$ [1] or ideals defining F-pure rings [5]. Despite the existence of counterexamples to Harbourne's Conjecture [2, 7], the following remains open:

Conjecture 3. *Under the conditions above, $I^{(hn-h+1)} \subseteq I^n$ for all $n \gg 0$.*

In [4], this and other related questions are discussed, including some partial results that serve as evidence towards Conjecture 3.

Theorem 4 ([4])**.** *Let R be a regular ring containing a field and I be a radical ideal of big height h. If $I^{(hk-h)} \subseteq I^k$ for some k, then for all $n \gg 0$ we have*

$$I^{(hn-h)} \subseteq I^n.$$

Does $I^{(hk-C)} \subseteq I^k$ for some k imply $I^{(hn-C)} \subseteq I^n$ for all $n \gg 0$? A sufficient condition [4] is that $I^{(n+h)} \subseteq II^{(n)}$ for all $n \geqslant 1$. While not all ideals verify this condition even asymptotically, as an Example by Seceleanu [4, Example 2.12] shows, there are indeed classes of ideals verifying this condition:

Theorem 5 ([4])**.** *Let R be a regular ring of characteristic p and let I be a radical ideal of big height h. If R/I is F-pure, then for all $n \geqslant 1$*

$$I^{(n+h)} \subseteq II^{(n)}.$$

In particular $I^{(hn-C)} \subseteq I^n$ for all $n \gg 0$ as long as $I^{(hk-C)} \subseteq I^k$ for some k.

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Noncontainments between symbolic and ordinary powers in reflection arrangements

Alexandra Seceleanu

(joint work with Benjamin Drabkin)

Symbolic and ordinary powers of ideals define cofinal topologies. Because of this, it is natural to compare the two topologies by means of containments of symbolic powers in ordinary powers and vice-versa. The containment of the m-th ordinary power of an ideal in the m-th symbolic power $I^m \subseteq I^{(m)}$ is immediate from the definition $I^{(m)} = \bigcap_{P \in \mathrm{Ass(I)}} I^m R_P \cap R$. The containment of symbolic powers in ordinary powers is the subject of the following important theorem:

Theorem 1 (Ein-Lazarsfeld-Smith [4], Hochster-Huneke[6], Ma-Schwede[8])**.** *For any ideal I in a regular ring R (which is additionally required to be excellent if R has unequal characteristic) the following containment holds*

$$I^{(m)} \subseteq I^r, \forall r \geq 1 \text{ and } m \geq \mathrm{bigheight}(I) \cdot r,$$

where $\mathrm{bigheight}(I) = \max\{\mathrm{ht(P)} \mid \mathrm{P} \in \mathrm{Ass(I)}\}$.

In the early 2000s Craig Huneke asked whether the containment above could be improved uniformly in that case when $\mathrm{bigheight}(I) = 2$. In the remainder of this note the ideals considered in relation to this question are radical and equidimensional. In this context, Huneke's question could be stated equivalently as asking for examples of codimension two ideals where $I^{(3)} \not\subseteq I^2$. Several examples satisfying this property have arisen [3, 5, 1, 8]. These examples are obtained according to the following recipe:

- pick a highly symmetric hyperplane arrangement $\mathcal{A} = V(\prod_{i=1}^n \ell_i)$
- let X to be the set of points lying on at least 3 of the hyperplanes in $\mathcal{A}$
- let $I = I_X$ and show that $I_X^{(3)} \not\subseteq I_X^2$ by proving $\prod_{i=1}^n \ell_i \in I^{(3)} \setminus I^2$.

In this note we address the question

Question 2. *Which hyperplane arrangements lead to noncontainments $I_X^{(3)} \not\subseteq I_X^2$ following the process above?*

We restrict our question to reflection arrangements in light of the fact that the examples of [3, 5, 1, 8] arise from reflection groups, as explained below. A reflection is a linear transformation of finite order which fixes a hyperplane pointwise. A reflection group is a finite subgroup of $\mathrm{GL}_n(k)$ generated by reflections over hyperplanes in k^n. The hyperplane arrangement $\mathcal{A}(G)$ determined by a reflection group G is the set of reflecting hyperplanes for the elements of G which are reflections.

An important piece of information in the case $k = \mathbb{C}$ is given by the following

Theorem 3 (Shephard-Todd). *The irreducible complex reflection groups belong to 3 infinite families: the symmetric groups, the cyclic groups, and the monomial groups $G(m, r, n)$, and 34 sporadic groups numbered G_4 through G_{37}.*

In light of this, we restrict our question to irreducible reflection arrangements.

Question 4. *For which irreducible complex reflection groups do the reflection arrangements lead to noncontainments $I_X^{(3)} \not\subseteq I_X^2$?*

The arrangements which are currently known to behave in this way are

Name of the ideal	Complex reflection group
Fermat [3, 5, 8]	$G(m, m, n)$
Klein [1]	G_{24}
Wiman [1]	G_{27}
new example [2]	G_{29}
new example [2]	G_{33}
new example [2]	G_{34}

The new examples listed in the table are consequences of the following result

Theorem 5 (Drabkin [2]). *If G is a reflection group with reflection arrangement $\mathcal{A}(G)$, X is a subvariety of the triple locus of $\mathcal{A}$ and H a reflection subgroup of G that fixes X pointwise, setting I to be the ideal defining the points of $\mathcal{A}(G)$ which are contained in at least three hyperplanes and J to be the ideal defining the* *points of $\mathcal{A}(H)$ which are contained in at least three hyperplanes, if $J^{(3)} \nsubseteq J^2$ then $I^{(3)} \nsubseteq I^2$.*

As a corollary of this theorem, the noncontainment $I^{(3)} \nsubseteq I^2$ is verified in the case of codimension two flats in monomial arrangements by induction on the embedding dimension since $G(m,m,n) \leq G(m,m,n+1)$, recovering the result of [8], and also in the case of the new examples listed above by means of using $G(4,4,3) \leq G_{29}$ and $G(3,3,4) \leq G_{33} \leq G_{34}$.

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Asymptotic syzygies of Stanley-Reisner rings of iterated subdivisions

Martina Juhnke-Kubitzke

(joint work with Aldo Conca, Volkmar Welker)

In a series of articles, Ein, Lazarsfeld and Erman [3, 4, 5] and Zhou [6] study the asymptotic behavior of syzygies of algebraic varieties under high Veronese embeddings. In particular, for the syzygies of r^{th} Veronese embeddings $v_r(\mathbb{P}^n)$ of projective space $\mathbb{P}^n$, they prove that for large r the syzygies of $v_r(\mathbb{P}^n)$ are non-zero for most of the homological positions and internal degrees that are allowed by the restrictions imposed by the projective dimension and by the Castelnuovo-Mumford regularity. Similar results, but with less precise bounds, are obtained for arithmetically Cohen-Macaulay varieties and are conjectured in general. The obtained bounds are used to show that if $A = \bigoplus_{i\geq 0} A_i$ is an arbitrary Cohen-Macaulay algebra, then for every $1 \leq j \leq \dim A - 1$ one has:

$$\lim_{r\to\infty} \frac{\#\{i \;:\; \beta_{i,i+j}(A^{\langle r\rangle}) \neq 0\}}{\operatorname{pdim}(A^{\langle r\rangle})} = 1. \tag{1}$$

Here and in the following, we let $A^{\langle r\rangle} = \bigoplus_{i\geq 0} A_{ir}$ denote the r^{th} Veronese algebra of A. For $j = \dim A$, one notes that the previous limit is 0 since the number of non-zero syzygies in that strand is bounded independently of d, see [4, Cor. 5.2].

Inspired by those results, we investigate the asymptotic behavior of graded Betti numbers of Stanley-Reisner rings of iterated barycentric subdivisions and edgewise subdivisions of a simplicial complex. Given a simplicial complex Δ we denote with $\mathbb{K}[\Delta]$ its Stanley-Reisner ring over a field $\mathbb{K}$. It was shown by Brun and Römer [1] that the Stanley-Reisner ring of the r^{th} edgewise subdivision $\Delta^{\langle r\rangle}$ of Δ is a Gröbner deformation of $\mathbb{K}[\Delta]^{\langle r\rangle}$. In particular, we have $\beta_{i,i+j}(\mathbb{K}[\Delta^{\langle r\rangle}]) \geq \beta_{i,i+j}(\mathbb{K}[\Delta]^{\langle r\rangle})$ and the results above apply if Δ is a Cohen-Macaulay complex.

Our main results, that appear in [2], can be summarized as follows

Theorem 1. *Let Δ be a simplicial complex of dimension $d-1>0$. Let $\Delta(r)$ be either the r^{th} iterated barycentric subdivision or the r^{th} edgewise subdivision of Δ. Then for large r the Castelnuovo-Mumford regularity of $\mathbb{K}[\Delta(r)]$ is given by:*

$$\operatorname{reg}(\mathbb{K}[\Delta(r)]) = \begin{cases} d-1, & \text{if } \widetilde{H}_{d-1}(\Delta;\mathbb{K}) = 0 \\ d, & \text{if } \widetilde{H}_{d-1}(\Delta;\mathbb{K}) \neq 0. \end{cases}$$

Furthermore:

(1) *For every $1 \leq j \leq d-1$ one has that $\#\{i \ : \ \beta_{i,i+j}(\mathbb{K}[\Delta(r)]) = 0\}$ is bounded above in terms of d, j (and independently of r). In particular:*
$$\lim_{r\to\infty} \frac{\#\{i \ : \ \beta_{i,i+j}(\mathbb{K}[\Delta(r)]) \neq 0\}}{\operatorname{pdim}(\mathbb{K}[\Delta(r)])} = 1.$$

(2) *If $\widetilde{H}_{d-1}(\Delta;\mathbb{K}) \neq 0$, then*
$$\lim_{r\to\infty} \frac{\#\{i \ : \ \beta_{i,i+d}(\mathbb{K}[\Delta(r)]) \neq 0\}}{\operatorname{pdim}(\mathbb{K}[\Delta(r)])}$$
is a rational number in the interval $[0,1)$ that can be described in terms of the minimal $(d-1)$-cycles of Δ.

We remark that the limit in (2) does not depend on whether one takes iterated barycentric subdivision or edgewise subdivision. We provide an example for part (2) of the previous theorem.

Example 2. *Let Δ_1 be a triangulation of a $(d-1)$-sphere with p facets and let Δ_2 be a union of q isolated $(d-1)$-simplices. Then Δ_1 is the minimal $(d-1)$-cycle of the union $\Delta = \Delta_1 \cup \Delta_2$ and the limit in (2) equals*
$$1 - \frac{p}{p+q} = \frac{q}{p+q}.$$

The previous example can be used to show that for any d any rational number in the interval $[0,1)$ can apper as the limit in (2).

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Star Configurations and Symbolic Powers

Ştefan O. Tohăneanu

Let $\mathcal{H} := \{H_1, \dots, H_s\}$ be a hyperplane arrangement in $\mathbb{P}^n$, with $s \geq n+1$. Let $H_i = V(\ell_i), i = 1, \dots, s$, where $\ell_1, \dots, \ell_s$ are linear forms in $R := \mathbb{K}[x_0, \dots, x_n]$; $\mathbb{K}$ is any field of characteristic 0.

Suppose $\mathcal{H}$ is generic, meaning that any $n+1$ of the ℓ_i's are linearly independent.

Let $1 \leq c \leq n$ be an integer. The *codimension c star configuration with skeleton* $\mathcal{H}$ is

$$V_c(\mathcal{H}) := \bigcup_{1 \leq i_1 < \cdots < i_c \leq s} H_{i_1} \cap \cdots \cap H_{i_c}.$$

The defining ideal of $V_c(\mathcal{H})$ is

$$I_{V_c(\mathcal{H})} := \bigcap_{1 \leq i_1 < \cdots < i_c \leq s} \langle \ell_{i_1}, \dots, \ell_{i_c} \rangle,$$

and for $k \geq 1$, the *k-th symbolic power* of $I_{V_c(\mathcal{H})}$ becomes

$$I^{(k)}_{V_c(\mathcal{H})} = \bigcap_{1 \leq i_1 < \cdots < i_c \leq s} \langle \ell_{i_1}, \dots, \ell_{i_c} \rangle^k.$$

In [5, Conjecture 4.1], the following is conjectured about the k-th ordinary power of $I_{V_c(\mathcal{H})}$:

Conjecture 1. *For any $k \geq 1$,*

$$I^k_{V_c(\mathcal{H})} = I^{(k)}_{V_c(\mathcal{H})} \cap I^{(2k)}_{V_{c+1}(\mathcal{H})} \cap \cdots \cap I^{((n-c+1)k)}_{V_n(\mathcal{H})} \cap \mathfrak{m}^{(s-c+1)k},$$

where $\mathfrak{m} := \langle x_0, \dots, x_n \rangle$.

The conjecture is known for

- $k = 1$; see [5, Remark 4.2].
- $c = 1$; immediate since $I_{V_1(\mathcal{H})} = \langle \ell_1 \ell_2 \cdots \ell_s \rangle$, a principal ideal.
- $c = n$; see [2, Lemmas 2.3.3(c) and 2.4.2].
- $s = n+1$ (i.e., $\mathcal{H}$ is the Boolean arrangement); see [5, Theorem 4.8].

Remark 2. By localization, [5, Corollary 4.9] shows that the saturation of $I^k_{V_c(\mathcal{H})}$ is

$$(I^k_{V_c(\mathcal{H})})^{sat} = I^{(k)}_{V_c(\mathcal{H})} \cap I^{(2k)}_{V_{c+1}(\mathcal{H})} \cap \cdots \cap I^{((n-c+1)k)}_{V_n(\mathcal{H})}.$$

So, by [4], coupled with [3, Corollaries 4.5 and 4.4], Conjecture 1 is equivalent to showing that $I^k_{V_c(\mathcal{H})}$ has linear graded free resolution.

Let $\Sigma := (\ell_1, \ldots, \ell_m)$ be a collection of m linear forms in R, some possibly proportional. Let $1 \leq a \leq m$ be an integer, and consider the ideal

$$I_a(\Sigma) := \langle \{\ell_{i_1} \cdots \ell_{i_a} | 1 \leq i_1 < \cdots < i_a \leq m\} \rangle.$$

In [1, Conjecture 1] it is conjectured the following:

Conjecture 3. *For any Σ and any a, the ideal $I_a(\Sigma)$ has linear graded free resolution.*

Remark 4. For any collection $\Sigma = (\ell_1, \ldots, \ell_m)$, and for any $k \geq 1$, the *k-fattening of* Σ is $\Sigma(k) := (\underbrace{\ell_1, \ldots, \ell_1}_{k}, \ldots, \underbrace{\ell_m, \ldots, \ell_m}_{k})$. It is not difficult to see that $I_a(\Sigma)^k = I_{ka}(\Sigma(k))$.

Also, by modifying just a bit the proof of [5, Proposition 2.9(4)], one has $I_{V_c(\mathcal{H})} = I_{s-c+1}(\mathcal{H})$.

Put together, Remarks 2 and 4 give that Conjecture 1 is a particular case of Conjecture 3.

Just recently (in an updated version of [6]), Conjecture 3 is proven for any Σ, and $a = m - 2$. Then, with $\Sigma = \mathcal{H}(2)$, and $m = 2s$, we have:

Corollary 5. *Conjecture 1 is true when $c = k = 2$.*

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On the integrality of Seshadri constants of abelian surfaces

Thomas Bauer

(joint work with Felix Fritz Grimm, Maximilian Schmidt)

For an ample line bundle L on a smooth projective variety X, the *Seshadri constant* of L at a point $x \in X$ is by definition the real number

$$\varepsilon(L, x) = \inf \left\{ \frac{L \cdot C}{\mathrm{mult}_x(C)} \;\middle|\; C \text{ irreducible curve through } x \right\}.$$

On abelian varieties, thanks to homogeneity, these invariants are independent of the chosen point x. The real numbers $\varepsilon(L)$ attached in this way to polarized abelian varieties (X, L) have been the focus of a great deal of attention (see [6, 4, 5, 1]).

The motivation for the research reported here comes from work by Schulz and the first author [2], where Seshadri constants on self-products $E \times E$ of elliptic curves were studied. If E does not have complex multiplication, then these results imply in particular that the Seshadri constants $\varepsilon(L)$ of all ample line bundles L on $E \times E$ are integers. The same holds when E admits an automorphism different from ± 1. Geometrically, this behavior is explained by the fact that all Seshadri constants in these situations are computed by elliptic curves. Our expectation at that point was that integrality of Seshadri constants should hold for *all* surfaces $E \times E$, where E is any elliptic curve. Rather surprisingly, we found that quite the opposite is true: Fractional Seshadri constants do occur on all self-products $E \times E$ except for the ones considered so far. Our result provides the complete picture:

Theorem 1. *Let E be an elliptic curve with complex multiplication. Then the Seshadri constants $\varepsilon(L)$ of all ample line bundles L on $E \times E$ are integers, if and only if E admits an automorphism different from ± 1.*

The theorem raises a more general question: How is integrality of Seshadri constants (on abelian surfaces in general) related to elliptic curves? Clearly, if on a given abelian surface all Seshadri constants are computed by elliptic curves, then certainly these numbers are integers. The question is whether the converse implication holds true as well. We found that this is almost the case, but not quite:

Theorem 2. *Let X be an abelian surface. The following conditions are equivalent:*

(i) *For every ample line bundle L on X, the Seshadri constant $\varepsilon(L)$ is an integer.*

(ii) *For every ample line bundle L on X, one has $\varepsilon(L) = \sqrt{L^2}$ and $\sqrt{L^2}$ is an integer, or $\varepsilon(L)$ is computed by an elliptic curve, i.e., there exists an elliptic curve $E \subset X$ such that $\varepsilon(L) = L \cdot E$.*

As an application of our methods, we are able to extend a result by Hayashida and Nishi [3]. They studied the problem of determining under which conditions a product $E \times E$ of elliptic curves is a Jacobian. Equivalently, the question is: On which of these surfaces do smooth curves of genus 2 exist? They show: *Let E be an elliptic curve such that* $\mathrm{End}(E)$ *is isomorphic to the maximal order of* $\mathbb{Q}(\sqrt{-m})$, *where $m > 0$ is a squarefree integer. Then there exist smooth curves of genus 2 on $E \times E$ if and only if $m \notin \{1, 3, 7, 15\}$.* We extend their result in the following way:

Proposition 3. *Suppose E is an elliptic curve with* $\mathrm{End}(E) = \mathbb{Z}[\sqrt{-e}]$ *for some integer $e > 0$ satisfying $e \equiv 2, 3 \pmod 4$. Then there exist smooth curves of genus 2 on $E \times E$.*

By way of example, consider an elliptic curve E with $\mathrm{End}(E) = \mathbb{Z}[f\sqrt{-2}]$ for some $f \in \mathbb{N}$. If $f = 1$, then $\mathrm{End}(E)$ is the *maximal* order in $\mathbb{Q}(\sqrt{-2})$. So the result of Hayashida and Nishi applies and shows that there is a smooth curve of genus 2 on $E \times E$. On the other hand, if $f > 1$, then $\mathrm{End}(E)$ is a *non-maximal* order in $\mathbb{Q}(\sqrt{-2})$. In that case Proposition 3 applies when f is odd and shows the existence of the desired curve.

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Seshadri constants on rational surfaces

Krishna Hanumanthu

(joint work with Brian Harbourne)

Let X be a complex projective variety and let L be an ample line bundle on X. The *Seshadri constant of L at a point $x \in X$* is defined as

$$\varepsilon(X, L, x) := \inf_{x \in C} \frac{L \cdot C}{\mathrm{mult}_x C},$$

where the infimum is taken over all irreducible and reduced curves passing through x. If the dimension of X is n, we always have $0 < \varepsilon(X, L, x) \leq \sqrt[n]{L^n}$.

A longstanding open question asks if Seshadri constants can be irrational. In this talk we discuss a recent joint work [2] with Brian Harbourne in which we exhibit irrational Seshadri constants on blow ups of $\mathbb{P}^2$ assuming the following conjecture is true.

(-1)**-curves Conjecture:** Let X_r be a blow up of $\mathbb{P}^2$ at $r \geq 0$ general points. If C is an irreducible and reduced curve on X_r such that $C^2 < 0$, then C is a *(-1)-curve*, i.e., $C^2 = -1$ and C is a smooth rational curve.

Our main result is the following.

Theorem 1. *Let $r \geq 9$. If the (-1)-curves conjecture is true for X_{r+1} then there exists an ample line bundle L on X_r such that $\varepsilon(X_r, L, x) \notin \mathbb{Q}$ for a very general point $x \in X_r$.*

The motivation for this result came from [1] where the authors exhibit ample line bundles on X_r with very general irrational Seshadri constants assuming that the *SHGH Conjecture* is true for X_{r+1}. The SHGH Conjecture is known to imply the (-1)-curves Conjecture.

Let X_r be a blow up of $\mathbb{P}^2$ at $r \geq 0$ general points. Let H denote the pull-back of $\mathcal{O}_{\mathbb{P}^1}(1)$ and let E_i denote the exceptional divisors. A line bundle $L = dH - m_1E_1 - \ldots - m_rE_r$ is said to be in *standard form* if $d \geq m_1 \geq m_2 \geq \ldots \geq m_r \geq 0$ and $d \geq m_1 + m_2 + m_3$.

The SHGH Conjecture is the following statement.

SHGH Conjecture: Let X_r be a blow up of $\mathbb{P}^2$ at $r \geq 0$ general points. If L is a line bundle on X_r in standard form, then L is *non-special*, which means that $h^0(L) = \max\{0, \frac{L^2 - K_{X_r} \cdot L}{2} + 1\}$.

The basic idea in proving Theorem 1 is the following observation which is easy to prove: if a line bundle L on X_{r+1} is in standard form then $L \cdot C \geq 0$ for every (-1)-curve C on X_{r+1}.

Now let L be an ample line bundle on X_r such that L^2 is not a perfect square. If $\varepsilon(L, x) \in \mathbb{Q}$, then there exists a curve C on X_r which computes the Seshadri constant $\varepsilon(L, x)$. If $\pi : X_{r+1} \to X_r$ denotes the blow up of X_r at x, then the strict transform $\tilde{C}$ of C has negative self-intersection and $\pi^\star(L) \cdot \tilde{C} < 0$. If the (-1)-curves Conjecture is true for X_{r+1} then $\tilde{C}$ is a (-1)-curve and we will obtain a contradiction if we choose L carefully to ensure that $\pi^\star(L)$ is in standard form. We show that this is possible to do for every $r \geq 9$. In fact, we show that for a suitable positive integer d (depending on r), the line bundle $L = dH - E_1 - \ldots - E_r$ does the job.

It is well-known that SHGH and (-1)-curves conjectures both imply the famous Nagata Conjecture.

Nagata Conjecture: Let X_r be a blow up of $\mathbb{P}^2$ at $r \geq 9$ general points. Let $C = dH - m_1E_1 - \ldots - m_rE_r$ be the class of an irreducible and reduced curve on X with $d > 0$. Then $d \geq \frac{m_1 + \ldots + m_r}{\sqrt{r}}$.

Finally we ask if the conclusion of our theorem holds only assuming that the Nagata Conjecture is true for X_r.

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A primer on unexpected varieties in projective space

Brian Harbourne

(joint work with Juan Migliore, Uwe Nagel, Zach Teitler)

1. Introduction

Let $R = \mathbb{C}[\mathbb{P}^n]$ be the homogeneous coordinate ring of $\mathbb{P}^n$ over the field $\mathbb{C}$ of complex numbers, so $R = \mathbb{C}[x_0, \dots, x_n]$. Denote by R_d the $\mathbb{C}$-vector space of all forms of degree d on $\mathbb{P}^n$. There is no point $p \in \mathbb{P}^n$ such that all $F \in R_d$ vanish at p. The subset of forms in R_d vanishing at p is a vector subspace of R_d of codimension 1; i.e., vanishing at p imposes 1 condition on forms in R_d. More generally, vanishing to order m at p (i.e., vanishing on mp) imposes $\max(R_d, \binom{m+n-1}{n})$ conditions.

Suppose we replace p by a linear space L. In order that two such spaces can be disjoint, we will specify that $\delta = \dim L < \frac{n}{2}$. Then vanishing on mL imposes some conditions on R_d; we will denote the number of conditions by $c(m, d, n, \delta)$.

Let $X = m_1L_1 + \cdots + m_rL_r$, where $L_1, \dots, L_r$ are general linear subspaces of dimension $\delta < \frac{n}{2}$. Let Z be a fixed variety in $\mathbb{P}^n$. Then vanishing on X (i.e., vanishing on each L_i to order m_i) would naively impose $\max(R_d, \sum_i c(m_i, d, n, \delta))$ conditions on R_d, and it would naively impose $\max(R_d, \sum_i c(m_i, d, n, \delta))$ conditions on the dth homogeneous component $I(Z)_d$ of the ideal of Z

2. Unexpectedness

The following definition comes from [2].
Definition: We say (d, X, Z, n) is *unexpected* if

$$\dim(I(X) \cap I(Z))_d > \max\left(0, \dim I(Z)_d - \sum_i c(m_i, d, n, \delta)\right).$$

Example 1. The well known SHGH Conjecture classifies all unexpected (d, X, Z, n) for $n = 2$ and $\delta = 0$ with $Z = \emptyset$.

Example 2. Let $n = 3, d = 12, \delta = 1, X = 4L_1 + 3L_2 + \cdots + 3L_6$ with $Z = \emptyset$; then $c(m, d, n, \delta) = (1/6)(m+1)m(3d+5-2m)$ (see [5]) and (d, X, Z, n) is unexpected. Numerically one expects there is no dodectic vanishing on X, but in fact there is one (which one can show by applying results of [8]).

Example 3. The first case with $Z \neq \emptyset$ comes from [4]. In this case $(d = 4, X = 3p, Z, n = 2)$ is unexpected when Z consists of a certain set of 9 points in $\mathbb{P}^2$ which (using results of [3]) impose independent conditions on forms of degree 4. Thus we expect there not to be quartic curve containing Z with a general triple point $X = 3p$ but in fact there is one, and moreover Z is the unique point set (over $\mathbb{C}$) having an unexpected curve of degree $d \leq 4$ with a general triple point [6].

It is an open problem to understand where such examples come from. For point sets Z in the plane, there seems to be an intimate but not well understood connection between line arrangements and unexpected $(d, X = (d-1)p, Z, n = 2)$.

Recent work [7] suggests that the example from [4] mentioned above is one of an infinite family. The 9 points in that example are the projectivizations of the 18 roots of the B_3 roots system. More generally, consider the $2(n+1)^2$ roots of the B_{n+1} root system. These are the $2(n+1)^2$ vectors $(a_1, \ldots, a_{n+1}) \in \mathbb{R}^{n+1}$ of the form $1 \leq a_1^2 + \cdots + a_{n+1}^2 \leq 2$ where each a_i is an integer. In $\mathbb{P}^n_{\mathbb{R}}$ these vectors give $(n+1)^2$ points. Taking these points for Z, computer calculations suggest that $(X = 4p, Z, d = 4, n \geq 4)$ is unexpected.

The example from [4] also displays an interesting duality, first noticed by [1] and defined and extended by [7]. Above it was mentioned that there is a connection between unexpected $(d, X = (d-1)p, Z, n = 2)$ and line arrangements. The new duality raises the question of the extent to which freeness of the line arrangements leading to unexpected $(d, X = (d-1)p, Z, n = 2)$ matters. See [7] for discussion and more results.

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On exterior powers of the tangent bundle on smooth toric varieties

David Schmitz

The study of consequences of the positivity of the tangent bundle $\mathcal{T}_X$ of a projective manifold X and of related bundles has been long and fruitful. The most famous instance is Mori's result ([5, Theorem 8]) stating that ampleness of $\mathcal{T}_X$ forces an n-dimensional projective manifold X to be isomorphic to $\mathbb{P}^n$. Campana and Peternell in [1] weakened the assumption of ampleness to nefness of $\mathcal{T}_X$, meaning that the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{T}_X)}(1)$ is nef on $\mathbb{P}(\mathcal{T}_X)$. They classified the 3-folds with this property and formulated their well-known conjecture predicting that any Fano manifold with nef tangent bundle should be rational homogeneous.

Instead of the tangent bundle itself, it is natural to study its exterior powers. The leading example being $\Lambda^n \mathcal{T}_X = -K_X$, whose nefness forces the Kodaira dimension of X to be at most 0 but yields very little in terms of classification. On the other hand, projective 3-folds with $\Lambda^2\mathcal{T}_X$ nef have been classified by Campana and Peternell in [2].

In this talk, we investigate positivity of arbitrary exterior powers of the tangent sheaf in the case of toric varieties. On the one hand, the focus on toric varieties is a substantial restriction of the subject matter. For example, in the above theorem, none of the 3-folds of case b) are toric. On the other hand, the theory of toric varieties provides us with sufficient machinery to handle arbitrary dimensions.

A particularly pleasant feature of (complete) toric varieties for our investigation is the fact that nefness and ampleness of an equivariant vector bundle $\mathcal{F}$ test on torus-invariant curves as has been shown in [4]. As such a curve is isomorphic to $\mathbb{P}^1$, the restriction of $\mathcal{F}$ decomposes as a sum of line bundles, and the positivity of $\mathcal{F}$ is determined by the splitting types

$$\mathcal{F}|_C \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_r)$$

for invariant curves C. By utilising the *Klyachko filtration* of $\mathcal{F}$, these splitting types can in principal be determined. The splitting type of $\mathcal{T}_X$ restricted to an invariant curve C turns out to be given by the coefficients of the *extremal relation* $b_1\mathbf{v}_1 + \cdots + b_{n-1}\mathbf{v}_{n-1} + \mathbf{v}_n + \mathbf{v}_{n+1} = 0$ corresponding to C. Here $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}$ are the primitive generators of the rays spanning the wall $\tau \in \Sigma(n-1)$ corresponding to C and $\mathbf{v}_1$ and $\mathbf{v}_2$ are the primitive generators of those rays which together with τ respectively span those two maximal cones $\sigma, \sigma' \in \Sigma(n)$, which intersect in τ. My first main result is the following criterion for positivity.

Theorem 1. *Let X_Σ be a smooth toric variety of dimension n. Then for $1 \leq m \leq n-1$ the exterior power $\Lambda^m\mathcal{T}_X$ is ample (nef) if and only if for any $\tau \in \Sigma(n-1)$ with extremal relation $b_1\mathbf{v}_1 + \cdots + b_{n-1}\mathbf{v}_{n-1} + \mathbf{v}_n + \mathbf{v}_{n+1} = 0$ the inequalities*

$$b_{i_1} + b_{i_2} + \cdots + b_{i_m} \underset{(=)}{>} 0$$
$$2 + b_{j_1} + b_{j_2} + \cdots + b_{j_{m-1}} \underset{(=)}{>} 0$$

hold for each $1 \leq i_1 < \cdots < i_m \leq n$ and $1 \leq j_1 < \cdots < i_{m-1} \leq n$.
Similarly, $\Lambda^n\mathcal{T}_X = -K_X$ is ample (nef) if and only if for any $\tau \in \Sigma(n-1)$ the coefficients in the corresponding extremal relation satisfy

$$2 + b_1 + b_2 + \cdots + b_{n-1} \underset{(=)}{>} 0.$$

Corollary 2. *Let X be a smooth toric variety of dimension n. If $\Lambda^m\mathcal{T}_X$ is nef for some $1 \leq m < n$, then X is Fano.*

Remembering that the signs of the b_i in an extremal relation corresponding to an invariant curve C determine the type of contraction given by the extremal ray containing the class C, the inequalities in the above theorem restrict the possible types of contractions, thus enabling us to use an inductive approach, at least in the case of small numbers m. For example, we readily see that the tangent bundle $\mathcal{T}_X$ itself is nef if and only if all b_i in the relations corresponding to any invariant curve are positive, which means that all contractions of X are of fibre type. Similarly, the nefness of $\Lambda^2\mathcal{T}_X$ excludes flipping contractions as well as divisorial contractions for which the image of the exceptional divisor is positive dimensional. In fact, by using upper bounds on the intersections $-K_X \cdot C$ for extremal invariant curves C recently established by Fujino and Sato ([3]), we can also exclude divisorial contractions with singular image. Similar arguments work for the case when $\Lambda^3 \mathcal{T}_X$ is ample. By the above analysis of possible contractions, we are enabled to inductively proving the following classification results.

Theorem 3. *Let X be a smooth toric variety of dimension $n \geq 3$ with $\Lambda^2 \mathcal{T}_X$ nef. Then either $\mathcal{T}_X$ is nef, or X is the blowup of P^n in a point.*

Theorem 4. *Let X be a smooth toric variety of dimension $n \geq 4$ with $\Lambda^3 \mathcal{T}_X$ ample. Then either $\mathcal{T}_X$ is nef, or X is the blowup of P^n in a point.*

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