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Arbeitsgemeinschaft: Rigidity of Stationary Measure

Organised by
Yves Benoist, Paris
Jean-Francois Quint, Bordeaux

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Abstract. The aim of this Arbeitsgemeinschaft was to understand the classification of stationary measures for semisimple random walks on a finite volume homogeneous space and its applications as described in the two papers [1] and [2].

Mathematics Subject Classification (2010): 22E30, 22E40, 28C10, 28D05, 37C40, 37C85, 60B15, 60J05.

Introduction by the Organisers

The Arbeitsgemeinschaft on the Rigidity of Stationary Measures was attended by 56 participants. One third of the participants were PhD students, one third were PostDoc and the last third were mathematicians with a permanent position. Most of the participants were working on a topic related to the conference: Ergodic Theory, Dynamical Systems, Fractals, Random walks, Group Theory or Number theory. They came from various countries: Germany, France, England, Greece, Poland, Norway, Switzerland, United States, Mexico, Israel, Iran, Russia, China, India, Korea,... It is our pleasure to thank the Oberwolfach Institute for providing us wonderful working and living conditions, to thank the speakers for the precision of their talks, and to thank the participants for making this week so lively.

We first recall the main theme of this Arbeitsgemeinschaft as explained in the scheduled program. Stationary probability measures ν are useful when one wants to understand the dynamics of the action of a non-abelian group or semigroup Γ on a compact space X. The reason is that in this situation, there might not exist any Γ-invariant probability measures on X. To overcome this issue, one chooses a probability measure μ on Γ and one defines the stationary measures as the probability measures ν on X which are invariant by convolution by μ. These measures control the asymptotic distributions of the associated random walk on X. Since they exist on the closure of any Γ-orbit, they are useful to describe the closure of the Γ-orbits on X. They are also useful to describe equidistribution properties of a sequence of finite Γ-orbits. In order to clarify the ideas we mainly focused on the case where $X = \mathbb{T}^d$ is the torus and also on the case where $X = \mathrm{SL}(d, \mathbb{R})/\mathrm{SL}(d, \mathbb{Z})$ is the space of unimodular lattices in $\mathbb{R}^d$.

The lectures followed carefully the scheduled program. We added two evening discussion sessions were participants could ask and answer basic questions.

In the first lecture (by Seonhee Lim) the main results of [1] and [2] were stated precisely.

Lectures 2 to 6 (by Catherine Bruce, Anthony Sanchez, Asaf Katz, Bajpai Jitendra and Shreyasi Datta) discussed classical examples of dynamical systems and the classification of their invariant probability measures. They also discussed classical Markov chains and the classification of their stationary measures.

Lectures 7 to 9 (by Oliver Sargent, Weikun He and Maxim Kirsebom) were a short introduction to the linear random walks. These random walks play a crucial role for controling the drift in the exponential drift argument. These lectures focused on Furstenberg stationary measure, on the positivity of the first Lyapunov exponent and on Guivarch-Lepage Central Limit Theorem.

Lectures 10 to 12 (by Vladimir Finkelshtein, Seul Bee Lee and Tom Kempton) focused on various recurrence properties for random walks. These properties are crucial, both in the proof of the rigidity of stationary measures but also in the applications of these rigidity results.

Lectures 13 to 17 (by Laurent Dufloux, Nicolas de Saxcé, Timothée Benard, Ilya Khayutin, and Homin Lee) formed the most technical part of the conference. We explained in detail the exponential drift argument. This argument, based on the martingale theorem, is reminiscent of the polynomial drift argument in the proof of Ratner measure rigidity theorem which was based on Birkhoff ergodic theorem. We explained how the conditional measures along a group action are used and how one controls the drift. We also explained in detail the following two technical facts: the equidistribution of pieces of fibers and the law of the angle.

The four last lectures (by Felipe Ramirez, Irving Calderon, René Ruhr and Jonathan de Witt) presented without proof recent rigidity results for stationary measures and various applications based on the exponential drift argument. Results by D. Simmons and B. Weiss on diophantine approximation on fractals, results by O. Sargent and U. Shapira on the space of 2-lattices in the 3-space, results by A. Eskin, M. Mirzakhani and A. Mohammadi on the $\mathrm{SL}(2, \mathbb{R})$-action on the moduli space of flat surfaces, and results by A. Brown and F. Rodriguez Hertz classifying stationary measures for a random dynamics on surfaces.

References

[1] Y. Benoist, J.-F. Quint, *Stationary measures and invariant subsets of homogeneous spaces (II)*, Journal American Math. Soc. **2**6 (2013) 659–734.
[2] Y. Benoist, J.-F. Quint, *Stationary measures and invariant subsets of homogeneous spaces (III)*, Annals of Math. **1**78 (2011), 1017–1059.

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Workshop: Arbeitsgemeinschaft: Rigidity of Stationary Measure

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Abstracts

Stationary measures on tori and applications

Seonhee Lim

1. Introduction

As the first lecture of the week, the aim of this introductory lecture is to state precisely the main results of [1] for the torus and the space of unimodular latttices. Let us first look at the problems we are interested in.

Let $X = \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2$ be the 2-dimensional torus. Since $SL_2(\mathbb{R})$ acts on $\mathbb{R}^2$ linearly and preserves $\mathbb{Z}^2$, it also acts on X. Let

$$a_0 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad a_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

Note that a_0, a_1 do not have a common eigenvector, and moreover, unions of eigenspaces of a_0 are not a_1-invariant and vice versa.

Set $\mu = \frac{1}{2}\delta_{a_0} + \delta_{a_1}$. The support of μ is $A = \{a_0, a_1\}$. Let Γ be the semigroup generated by A, which is the set of words with alphabet A.

Now for a given point $x \in X$, let $x_1 = g_1 x_0, x_2 = g_2 x_1, \ldots$. The first question we ask ourselves is the behavior of Γx_0, for example whether it is dense or not, and if it is not, what would be the closure of Γx_0. A more quantitative question can be described using the emperical measures: for a given sequence $g_i \in A$, independently chosen accoording to the law μ, the emperical measure is defined by

$$\nu_n = \frac{1}{n}(\delta_{x_0} + \cdots + \delta_{x_{n-1}}).$$

Now the question is whether ν_n converges weakly to some finite measure ν on X. Breiman law of large numbers says that if ν_n converges to ν, then ν is μ-stationary:

$$\mu * \nu = \nu.$$

Note that X is compact here, thus the limit measure ν will be a probability measure again. The next question would be to describe all the μ-stationary μ-ergodic measures on X and describe the topology of the set.

2. Tori

Let $X = \mathbb{R}^d/\mathbb{Z}^d = \mathbb{T}^d$ be the d-dimensional torus. As in the 2-dimensional case, $SL_d(\mathbb{Z})$ acts linearly on $\mathbb{R}^d$ and preserves $\mathbb{Z}^d$. Let $A = \{a_0, \cdots, a_k : a_i \in SL_d(\mathbb{Z})\}$ be a finite subset of $SL_d(\mathbb{Z})$ and let

$$\mu = \sum p_i \delta_{a_i}.$$

The support of μ is A. Let Γ be the subsemigroup of $SL_d(\mathbb{Z})$ generated by A, acting strongly irreducibly, i.e. no finite union of proper subspaces of $\mathbb{R}^d$ is Γ-invariant.

We call $x_0 \in X$ rational if $x_0 \in \mathbb{Q}^d/\mathbb{Z}^d$, and irrational otherwise. Set $d\nu_X = d_{x_1} \cdots d_{x_d}$ to be the volume measure (=Haar measure = Lebesgue measure) on $\mathbb{T}^d$.

Theorem 1. *Let x_0 be an irrational vector.*

1. *Γx_0 is dense in X.*
2. *For $\mu^{\otimes \mathbb{N}}$-almost every sequence $g_i \in A$, the trajectory $x_n = g_n \cdots g_1 x_0$ equidistributes in X. In other words, $\nu_n \to \nu_X$.*
3. *$\frac{1}{n}\sum_{k=0}^{n-1} \mu^{*k} * \delta_{x_0} \to \nu_X$, i.e. the average of the law of the first n points converges weakly to ν_X.*
4. *If ν is atom-free, μ-stationary probability measure, then $\nu = \nu_X$.*
5. *Any sequence ν_{Y_n} of finite orbits Y_n converges weakly to ν_X.*

Remarks.

1. Part (1) is by Guivarch-Starkov (2004) and Muchnik (2005). Part (b) is by Benoist-Quint (2013). Parts (c),(d),(e) are by Bourgain-Furman-Lindenstrauss-Mozes (2011) for special case of proximal actions and Benoist-Quint for the general case.
2. Note that if x_0 is rational, then $x_0 \in \left(\frac{1}{n}\mathbb{Z}\right)^d/\mathbb{Z}^d$ for some n, and so is Γx_0. Thus Γ-orbit of x_0 is finite and ν has atoms.
3. Let G be the semidirect product of $SL(d,\mathbb{Z})$ with $\mathbb{T}^d$. The group G acts transitively on $\mathbb{T}^d$ and for $\Lambda = Stab_G(\{0\}) = Stab_G(\mathbb{Z}^d) = SL(d,\mathbb{Z})$, we have $X = G/\Lambda$.

3. Space of lattices

In this section, let X be the space of lattices $\Delta \subset \mathbb{R}^d$ of covolume 1, i.e.,

$$X = \{\mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \cdots \oplus \mathbb{Z}e_d : e_i \in \mathbb{R}^d, det(e_1, \cdots, e_d) = 1.\}$$

The group $G = SL(d,\mathbb{R})$ acts transitively on X with stabiliizer $Stab_G(\mathbb{Z}^d) = SL(d,\mathbb{Z})$. Thus

$$X = G/\Lambda = SL(d,\mathbb{R})/SL(d,\mathbb{Z}).$$

Let Γ be the subsemigroup of $SL(d,\mathbb{Z})$ which is Zariski dense in $SL(d,\mathbb{R})$, i.e. $Ad(\Gamma)$ acts irreducibly on the Lie algebra $\mathfrak{g} = sl(d,\mathbb{R})$). For example, consider $d = 2$, $\mu = \frac{1}{2}(\delta_{a_0} + \delta_{a_1})$ and Γ the subsemigroup generated by

$$a_0 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad a_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

as before. The adjoint action $Ad(a_0)$ sends a matrix $X \in sl(2,\mathbb{R})$ to $a_0 X a_0^{-1}$;

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x & y \\ z & -x \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 3x+z-2y & -4x+4y-z \\ 2x+z-y & -3x+2y-z \end{pmatrix}$$

Thus for the basis $\beta = \{e_{11}, e_{12}, e_{21}\}$, $[a_0]_\beta = \begin{pmatrix} 3 & -2 & 1 \\ -4 & 4 & -1 \\ 2 & -1 & 1 \end{pmatrix}$. The characteristic polynomial of a_0 is $-x^3 + 2x^2 + 22x - 17$ with three distinct real eigenvalues.

Similarly, the matrix representation of a_1 is $[a_1]_\beta = \begin{pmatrix} 3 & -1 & 2 \\ -2 & 1 & -1 \\ 4 & -1 & 4 \end{pmatrix}$, which is a conjugate of $[a_0]$ by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$. Thus it is easy to see that the adjoint action is indeed irreducible on $\mathfrak{g}$.

In this setting, x_0 is rational if $x_0 \subset \lambda\mathbb{Q}^d$, for some $\lambda > 0$ and irrational if not. Note that if x_0 is rational, then $x_0 = \mathbb{Z}e_1 \oplus \cdots \mathbb{Z}e_d$ where $e_i \in (\lambda\frac{1}{q}\mathbb{Z})^d$. Then for any $g \in SL(d,\mathbb{Z})$, $ge_i \in (\lambda\frac{1}{q}\mathbb{Z})^d$. Thus $gx_0 \subset (\lambda\frac{1}{q}\mathbb{Z})^d$. Since there are only finitely many lattices of covolume 1 in $(\lambda\frac{1}{q}\mathbb{Z})^d$, it follows that Γx_0 is finite.

The questions asked in the introduction are still valid. Unlike the torus case, X is not compact, thus there is another question of whether a weak limit of emperical measures ν_n is again a probability measure: for $\forall\epsilon > 0$, is there a compact set $K_\epsilon \subset X$ such that $\nu(K_\epsilon) \geq 1-\epsilon$ or equivalently $\nu_n(K_\epsilon) \geq 1-\epsilon$ for all n?

Theorem 2. *Let x_0 be irrational.*

(1) *Γx_0 is dense in X.*
(2) *For $\mu^{\otimes\mathbb{N}}$-almost every sequence $g_i \in A$, the trajectory $x_n = g_n \cdots g_1 x_0$ equidistributes in X. In other words, $\nu_n \to \nu_X$.*
(3) *$\frac{1}{n}\sum_{k=0}^{n-1} \mu^{*k} * \delta_{x_0} \to \nu_X$, i.e. the average of the law of the first n points converges weakly to ν_X.*
(4) *If ν is atom-free, μ-stationary probability measure, then $\nu = \nu_X$.*
(5) *Any sequence ν_{Y_n} of finite orbits Y_n coverges weakly to ν_X.*

Remark. These two theorems are special cases of a general statement for G a real Lie group, $\Lambda \subset G$ a lattice, $X = G/\Lambda$, Γ a compactly generated closed subsemigroup of G and when the Zariski closure of $Ad(\Gamma) \subset GL(\mathfrak{g})$ is semisimple with no compact factor.

References

[1] Y. Benoist, J.-F. Quint, *Stationary measures and invariant subsets of homogeneous spaces (II)*, Journal American Math. Soc. **26** (2013) 659–734.

Examples of invariant measures

Catherine Bruce

We discuss the special case of the general setting of the workshop when the dynamics are deterministic. This is when we are dealing with a single continuous transformation g on a locally compact metric space X. Here, we are interested in the closure of orbits of the form $(g^n x)_{n\geq 1}$ for $x \in X$, the empirical measures are of the form $\frac{1}{n}\sum_{k=0}^{n-1} \delta_{g^k x}$, and the stationary measures are exactly the g-invariant measures. We look at some examples of classical dynamical systems and their invariant measures. We see that even in this simple setting some interesting behaviour occurs.

1. Two simple examples

We consider two well known deterministic dynamical systems. The first is the doubling map, i.e. the transformation

$$m_2 : x \to 2x \text{ on } X = \mathbb{T}.$$

In order to find invariant measures for this map we code the dynamics using a full one sided 2-shift accompanied by the left shift map, and a coding map $\xi : \{0,1\}^{\mathbb{N}} \to X$. By doing this, we can find shift invariant measures on the symbolic space β_p, which are defined as products of $(p, 1-p)$ Bernoulli measures on $\{0,1\}$, then project them under ξ to find invariant measures for m_2. Using this method we can construct an uncountable number of m_2-invariant measures, $\mu_p := \xi(\beta_p)$ for each $p \in (0,1)$. We also note that the empirical measures $\sum_{k=0}^{n-1} \delta_{2^k x}$ converge to μ_p for μ_p-a.e. $x \in X$ by ergodicity. The second example we consider is the cat map, i.e. the transformation

$$a_0 : (x_1, x_2) \to (2x_1 + x_2, x_1 + x_2) \text{ on } X = \mathbb{T}^2.$$

We again use a coding to find invariant measures. Here we do not code the full dynamics but first construct an a_0-invariant subset using a Smale horseshoe construction, then code the dynamics restricted to this subset. By doing this we achieve a full two sided 2-shift and again construct an uncountable number of a_0-invariant measures using the same method as with the doubling map.

2. Linear and affine maps on the torus

We denote by ν_X the Haar measure on $X = \mathbb{T}^d$. We note that this is equivalent to Lebesgue measure.

Theorem 1 (Auslander)**.** *Let $g \in SL(d, \mathbb{Z})$ be a matrix with no eigenvalue being a root of unity. Then for ν_X-almost any $x \in X$, The sequence $(g^n x)_{n \geq 1}$ is dense in X. More precisely this sequence equidistributes towards ν_X.*

This result is a consequence of Birkhoff's ergodic theorem. We prove that g is ergodic with respect to Haar measure using Fourier coefficients. Similar techniques are then used to prove the following result on an affine map of the torus. We consider rotations on the circle, i.e. a transformation of the form $\tau_\alpha : x \to x + \alpha$ on $X = \mathbb{T}$. Then we have

When $\alpha \notin \mathbb{Q}$, τ_α is uniquely ergodic with respect to the Haar measure ν_X.

The second example we consider is the transformation $\tau'_\alpha : (x_1, x_2) \to (x_1 + \alpha, x_1 + x_2)$ on $X = \mathbb{T}^2$. Then we have a corresponding result.

When $\alpha \notin \mathbb{Q}$, τ'_α is uniquely ergodic with respect to ν_X.

The proof of this result requires techniques from Furstenberg's group extension theory, using the fact that τ_α is uniquely ergodic.

References

[1] Y. Benoist, J.-F Quint Introduction to random walks on homogeneous spaces. Tenth Takagi Lecture in Kyoto, *Japan J. Math.* (7), 135–166, 2012.

Dynamics of Heisenberg Nilmanifolds

Anthony Sanchez

In this talk we provide a criterion for unique ergodicity of rotations on quotients of the 3-dimensional Heisenberg group H. We follow [1].

Recall, that the 3-dimensional Heisenberg group is simply

$$H = \left\{ h_{a,b,c} := \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}.$$

For dynamical reasons, we are interested in the quotient of H by the discrete subgroup

$$\Gamma = \{h_{a,b,c} : a, b, c \in \mathbb{Z}\}.$$

Notice that coset multiplication on H/Γ is commutative in the coordinates a and b. Thus, by factoring out the last coordinate c, we can view H/Γ as a bundle over the 2-torus $\mathbb{T}^2$ with fibers equal to $\mathbb{T}$.

Let $\tau = h_{\alpha,\beta,\gamma}$ for some $\alpha, \beta, \gamma \in \mathbb{R}$. Then we can define rotation by τ on H/Γ by multiplying on the left by τ which we call S. Notice that the factor on the dynamical system $(H/\Gamma, S)$ obtained by projecting onto the first two coordinates is then rotating on the 2-torus by the element (α, β). Call this dynamical system $(\mathbb{T}^2, T)$. The dynamics of rotation on the 2-torus are well understood and provide us with an avenue to understand the dynamics on H/Γ.

The main theorem of the talk is

Theorem 1. *The map S is uniquely ergodic on H/Γ if and only if T is uniquely ergodic on $\mathbb{T}^2$.*

Notice that if we consider the trivial bundle over $\mathbb{T}^2$ with fibers the circle (i.e. the 3-torus), the above theorem is false. Thus, we must use the twisted multiplication on H/Γ in an important way.

The main idea of the backwards direction has similarities with the proof of Ratner's powerful theorems on unipotent flows. Namely, it considers how nearby points diverge from each other under S and uses this to produce additional invariance of an invariant measure. We are then able to connect this additional invariance of a measure on H/Γ to measures on the factor $(\mathbb{T}^2, T)$ where we have unique ergodicity.

We are ultimately interested in the classification of invariant measures because the information this yields on the dynamics of the space. In the concrete case of rotations of quotients of the Heisenberg group, we have only one such measure under the above criterion.

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Introduction to Ratner's theorems

Asaf Katz

A major problem in the field of dynamics is to classify the ergodic probability measures which are invariant under a given map $T : X \to X$. In general, this set is rather large, and the structure of each such measure may be complicated to describe. In the specialized case of homogeneous dynamics, where the given map T is a translation by an Ad-unipotent element, usually such classification is possible with the most general form due to M. Ratner [6].

Given a connected real Lie group G and a lattice Γ, we define the homogeneous space $X = G/\Gamma$, where G (and therefore each subgroup of) acts by left translations. For any probability measure μ defined over X, we may attach the *stability subgroup* G_μ defined as

$$G_\mu = \{g \in G \mid g.\mu = \mu\}.$$

A probability measure μ is called *homogeneous* if its stability subgroup G_μ acts *transitively* over its support.

Let $H \leq G$ be a subgroup generated by Ad-unipotent elements, let μ be an H-invariant and ergodic probability measure defined over X, then Ratner's measure classification theorem asserts the μ must be homogeneous.

The most basic case, namely translations over compact abelian groups was known long before. The next case, dealing with G being a nilpotent Lie group, is due to Lesigne [4]. The case where G is semisimple is due to Ratner [5], ending with the general case in [6].

In the lecture we surveyed the proof of this theorem for the case of $G = SL_2(\mathbb{R})$, which was proven earlier by H. Furstenberg [2] in the case where Γ is a uniform lattice, and by S.G. Dani [1] in the general case.

Let $G = SL_2(\mathbb{R})$, and let $\Gamma \leq G$ be a lattice. Denote by $U \leq G$ the subgroup of upper-triangular unipotent matrices. Let μ be a U-invariant and ergodic probability measure. Furstenberg's measure classification asserts that in the case where Γ is uniform, μ must be equal to the Haar measure over X. Dani's measure classification theorem asserts that in the case where Γ is non-uniform, either μ equals the Haar measure, or μ is supported on a single closed U-periodic orbit of the form $U.x\Gamma \subset X$.

The proof is based over the so-called "shearing argument" of Ratner, which proves that unless μ is U-periodic, the stability subgroup G_μ is strictly larger than μ (which in this particular case, actually it will be equal to the normalizer subgroup of U, $N_G(U)$), and then showing that each such measure must be invariant under the whole group G.

The shearing argument is based on the study of divergence properties of two nearby generic points, $x, y \in G$. Writing $\varepsilon \in G$ in a small neighborhood of the identity such that $y = \varepsilon.x$, the divergence of the trajectories $u_t.x$ and $u_t.y$ is governed by the expression $u_t.\varepsilon.u_{-t}$, which can be shown to be dominant in the U-direction, by explicit calculation. Choosing an appropriate time change $s = s(t)$. given by an algebraic function, one may cancel this U-divergence by studying $u_t.\varepsilon.u_{-s(t)}$ so that $u_t.\varepsilon.u_{-s(t)}$ lies in the lower-triangular matrices, which in turn can be shown to have the diagonal direction as the dominant divergence direction, in the case where the U-orbit is not contained in a fixed parabolic subgroup. Carefully renormalizing the times, one may deduce (using the fact that trajectories along generic points approximate the measure) additional invariance of the measure μ.

One can then invoke an entropy based technique, building upon the Ledrappier-Young formula [3] to conclude that the measure μ is a *measure of maximal entropy* with respect to the diagonal action, therefore it must be the Haar measure.

References

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Examples of Stationary Measures

Jitendra Bajpai

Let X be a locally compact metric space, G be a locally compact group and μ be a probability measure defined on G. Further we assume that G acts on X continuously. Let A be the support of μ and Γ denotes the closure of the semigroup generated by A. Then we call a probability measure ν on X μ-stationary if one has

$$\mu * \nu = \nu,$$

where convolution is the average of translates, i.e.

$$\mu * \nu = \int_A g_* \nu d\mu(g).$$

We quickly recall some notations and definitions to be used in what follows. For $x_0 \in X$, the Γ-orbit of x_0 is the set $\Gamma x_0 = \{gx_0 | g \in \Gamma\}$. Starting from the point x_0, we consider the trajectory

$$x_1 = g_1 x_0, x_2 = g_2 x_1, \ldots, x_n = g_n x_{n-1}, \ldots$$

where the g_i's are chosen independently from A. The empirical measures are the probability measures

$$\nu_n = \frac{1}{n}(\delta_{x_0} + \delta_{x_1} + \cdots + \delta_{x_{n-1}}).$$

In the talk, we focus on stationary measures. We discussed few classical properties and examples of μ-stationary probability measures on X, briefly described below. For details see section 1 and 3 of [2] and chapter 2 of [3]. A few topic covered are their existence on compact set, maximum principle for countable spaces, Choquet-Deny theorem for abelian groups and an example of μ-stationary measure which is not Γ-invariant, defined shortly.

1. Five Questions

As the reader may have realized that one of the goal in the subject is to make an investigation around following five questions.

(1) Can one describe all the orbit closures $\overline{\Gamma x_0}$ in X?
(2) Do the empirical measures ν_n converge? If converge then what is the limit?
(3) Prove that there is no escape of mass for the empirical measures ν_n?
(4) Describe all the μ-stationary measures ν on X.
(5) Describe the topology of the set of μ-ergodic μ-stationary measures on X.

2. Existence of Stationary Measures

Theorem 1 (Kakutani). *If X is compact then there exists a μ-stationary probability measure ν on X.*

We discuss the behaviour of certain μ-stationary measure ν such that the support of ν, denoted by $Supp(\nu)$, in X is a countable set. More precisely, the following

Theorem 2. *Let ν be any μ-ergodic and μ-stationary measure. If $Supp(\nu)$ is countable then ν is Γ-invariant. Moreover, it is supported by a finite set.*

Recall that we say ν is Γ-invariant if $g_*\nu = \nu$ for μ-almost every $g \in \Gamma$.

3. Choquet-Deny Theorem

Observe that every Γ-invariant probability measure ν on X is μ-stationary. This follows immediately from the definition. However, when the semigroup Γ is abelian then the converse is also true. More precisely, we discuss in detail the following theorem.

Theorem 3 (Choquet, Deny). *When Γ is abelian, every μ-stationary probability measure ν on X is Γ-invariant.*

To prove the above theorem, we make use of the properties of one-sided Bernoulli dynamical system denoted by (B, β, T) with alphabet (A, μ). This means $B = A^{\mathbb{N}}$ is the space of trajectories $b = (b_i)_{i=1}^{\infty}$ with $b_i \in A$. Here $\beta = \mu^{\otimes \mathbb{N}}$ and $T : B \longrightarrow B$ is the shift defined by $Tb = (b_2, \ldots, b_{n+1}, \ldots)$. The following result of Furstenberg was a key ingredient in the proof of the theorem 3 which tells us that the data of a stationary measure ν is equivalent to the data of an equivariant family $b \mapsto \nu_b$ of probability measures.

Theorem 4 (Furstenberg). *Let ν be a μ-stationary measure on X. Then the limit $\nu_b := lim_{n\to\infty}(b_1 \dots b_n)_*\nu$ exists, for β-almost every $b \in B$, and satisfies the equivariance property $\nu_b = (b_1)_*\nu_{Tb}$, and one can recover ν as the average $\nu = \int_B \nu_b d\beta(b)$.*

4. An intricate example

Following a result of Guivarc'h and Raugi [1] we have the following

Theorem 5 (Guivarc'h-Raugi). *If Γ is discrete nilpotent then all μ-stationary measures ν are Γ-invariant.*

However, when Γ is solvable then its not true in general. To explain this, let us discuss the following interesting example. Let $X = [0, 1]$ be the closed unit interval and c_0, c_1 be the two contractions on X defined as follows

$$c_0 : x \mapsto \frac{x}{3}, \qquad \text{and} \qquad c_1 : x \mapsto \frac{x+2}{3}.$$

Let Γ be the semigroup generated by c_0 and c_1 equipped with measure

$$\mu = \frac{1}{2}(\delta_0 + \delta_1)$$

Note that support of μ in this case is simply $\{c_0, c_1\}$. Consider the Cantor set

$$K = \left\{ x = \sum_{i=1}^{\infty} \frac{2b_i}{3} \;\middle|\; b_i = 0 \quad \text{or} \quad 1 \right\} = \bigcap_{n=1}^{\infty} \Bigg(\bigcup_{\ell(w)=n} I_w \Bigg),$$

where I_w denotes the image of X by a word of length $\ell(w) = n$. Observe that for every n, there will be exactly 2^n intervals I_w of length n. Following this we immediately realize that K is closed and Γ-invariant. To see that K is Γ-invariant, it is enough to show that $c_i K \subseteq K$ for $i = 0, 1$. Let

$$B = \{0, 1\}^{\mathbb{N}} := \left\{ b = \{b_i\}_{i=1}^{\infty} \middle| b_i = 0 \quad \text{or} \quad 1 \right\},$$

and define a coding map $\xi : B \longrightarrow K$ by $\xi(b) = \sum_{i=1}^{\infty} \frac{2b_i}{3^i}$. Let $\beta = \mu^{\otimes\mathbb{N}}$ be the Bernoulli measure on the space B and define the probability measure $\nu_X := \xi_*(\beta)$ on X. Then we show the following

Lemma. *ν_X is μ-stationary but not Γ-invariant.*

This follows quickly by the definition $\mu * \nu_X = \nu_X$ and $g_*\nu_X \neq \nu_X$ for some $g \in \Gamma$, in particular for $g = c_0$ or c_1 both will provide the desired aforementioned identity.

An important take away of this example is that even though ν_X is not Γ-invariant, we can still answer the above five questions mentioned in section 1, and the answers are summarized in the form of following

Theorem 6.

(a) *For all* $x \in X$*, the orbit closure* $\overline{\Gamma x}$ *is equal to* $\Gamma x \cup K$*.*

(b) *The only* μ*-stationary measure* ν *on* X *is* ν_X*.*

(c) *For all* $x_0 \in X$*, for* β*-almost all* $b \in B$*, the empirical measure* $\nu_n = \frac{1}{n}\big(\delta_{x_0} + \delta_{x_1} + \cdots + \delta_{x_{n-1}}\big)$ *converges to* ν_X*, where* $x_n = b_n \cdots b_1 x_0$*.*

References

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Empirical Measure

Shreyasi Datta

In this talk we discussed weak$\star$ limits of Empirical measures. Throughout the talk, X was assumed to be a locally compact, second countable, Hausdorff, σ-compact, metrizable space. Let $\mathcal{X}$ be the Borel σ-algebra on X. Denote $\Omega = X^{\mathbb{N}\cup\{0\}}$, the forward trajectories with the product σ-algebra $\mathcal{X}^{\mathbb{N}\cup\{0\}}$.
Empirical measures are the probability measures

$$\frac{1}{n}(\delta_{x_0} + \delta_{x_1} + \cdots + \delta_{x_{n-1}})$$

where $(x_0, x_1, \cdots, x_n, \cdots) \in \Omega$ and $n \in \mathbb{N}$. Heuristically, this sequence of measures tells us where the trajectories spend a positive proportion of their time. In the lecture we proved Breiman's law of large numbers in the context of Markov chains and continuous semigroup actions, which tells us about weak$\star$ limits of empirical measures. We introduced Markov chains for the aforementioned space X and Markov-Feller chains in particular for compact spaces. Then existence of markov measurs $\mathbb{P}_x$ on Ω corresponding to a Markov chain $P = \{P_x\}$ on X was discussed. We now can state the first Theorem (ref. Corollary 3.4 in [3]) we proved, which is due to Breiman.

Theorem 1 (**Breiman**)**.** *Let* X *be a compact metric space and let* P *be a Markov-Feller chain on* X*. Then for any* $x \in X$*, for* $\mathbb{P}_x$ *a.e* $\omega \in \Omega$ *every weak-*$\star$ *limit of*

$$\frac{1}{n}(\delta_{w_0} + \delta_{w_1} + \cdots + \delta_{w_{n-1}})$$

is P*-invariant.*

We first proved the Lemma 3.3 from [3] and showed Theorem 1 as a Corollary of that. We then discussed continuous semigroup actions. Let G be a locally compact second countable semigroup and μ be a Borel probablity measure on G. Suppose G acts on X continuously. A borel probability measure ν on X is called μ-stationary if $\mu * \nu = \nu$. The following law of large numbers is due to Breiman:

Theorem 2 (**Breiman**). *Let X be a compact metric space. Let G acts continuously on X and μ is a borel probability measure on G. Then for all $x \in X$, $\mu^{\mathbb{N}}$ a.e $b \in G^{\mathbb{N}}$, weak-$\star$ limits of the empirical measures*

$$\frac{1}{n}(\delta_{x_0} + \delta_{x_1} + \cdots + \delta_{x_{n-1}}),\ x_n = b_n.\cdots b_1 x$$

are μ-stationary.

We showed that Theorem 1 implies Theorem 2. For reference see Proposition 2.4 in [2].
We say that the action of G on X is uniquely ergodic with respect to μ if there exists only one μ -stationary borel proababilty measure on X. As an immediate corollary of Theorem 2 the following (ref. Proposition 3.8 in [1])was deduced:

Corollary 1. *Let X be a compact space metric space. If a continuous action of G on X is uniquely ergodic with respect to μ with ν as the unique μ-stationary measure on X. Then for all $x \in X$, $\mu^{\mathbb{N}}$ a.e $b \in G^{\mathbb{N}}$, one has convergence of the empirical measures $\frac{1}{n}(\delta_{x_0} + \delta_{x_1} + \cdots + \delta_{x_{n-1}}) \xrightarrow{n\to\infty} \nu$, where $x_n = b_n.\cdots b_1 x$.*

Next we discussed Raugi's Theorem for equicontinuous Markov-Feller operator (ref. Proposition 2.9 and 2.2 in [2]):

Theorem 3 (**Raugi**). *Let X be a compact metric space and P be a equicontinuous Markov-Feller chain. For every $x \in X$, $\mathbb{P}_x$ a.e $\omega \in \Omega$*

$$\nu_{\omega,n} = \frac{1}{n}\sum_{k=0}^{n-1} \delta_{w_k} \xrightarrow{n\to\infty} \nu_\omega.$$

Moreover ν_ω turns out to be P-invariant and also P-ergodic measure.

We proved this Theorem using Breiman's law of large number for Markov chains and Von Neumann functional ergodic Theorem.

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Proximal Actions

Oliver Sargent

In this talk our main goal was to prove the following theorem due to H. Furstenberg.

Theorem 1. *Let V be finite dimensional vector space over a field $\mathbb{K}$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $\mu \in \mathcal{P}(\mathrm{GL}(V))$ be a Borel probability measure on $\mathrm{GL}(V)$ and Γ_μ be the subsemigroup of $\mathrm{GL}(V)$ generated by the support of μ. Suppose that the action $\Gamma_\mu \curvearrowright V$ is strongly irreducible and proximal. Then there exists a unique μ-stationary probability measure on $\nu \in \mathcal{P}(\mathbb{P}V)$. Moreover, ν is μ-proximal.*

The talk was based on [1, §2.5 + §4.2].

The main tool in this proof is a collection of measures $\{\nu_b\}_{b \in B}$ associated to a μ-sationary measure ν. Elements ν_b of this collection of measures are called the *limit measures* of ν. In the talk we defined these measures in the following situation. Let G be a locally compact second countable group and $\mu \in \mathcal{P}(G)$ be a Borel probability measure. Suppose that $G \curvearrowright X$, where X is a compact metrisable space and that $\nu \in \mathcal{P}(X)$ is μ-stationary. Let $B = G^{\otimes \mathbb{N}}$ and $\beta = \mu^{\otimes \mathbb{N}}$ and then we showed that for β-almost every $b \in B$ the limit

$$\lim_{n \to \infty} b_1 \dots b_n \nu \tag{A}$$

exists. Whenever it exists, we denote the limit by $\nu_b \in \mathcal{P}(X)$ and call the collection of ν_b's the collection of limit measures of ν. In addition, we showed that the limit measures ν_b satisfy two important properties. The first is that it is possible to recover ν from the ν_b's as an average. In other words we have that

$$\textstyle\int_B \nu_b \mathrm{d}\beta(b) = \nu.$$

The second important property is equivariance. It says that

$$\nu_b = b_1 \nu_{Tb} \quad \text{for} \quad \beta\text{-a.e. } b \in B,$$

where $T\colon B \to B$ denotes the left shift. Moreover, we also claimed that for $\beta \otimes \mu^{*m}$-a.e. $(b, g) \in B \times G$ one has that

$$\lim_{n \to \infty} b_1 \dots b_n g \nu = \nu_b. \tag{B}$$

In the second half of the talk first we showed that the strong irreducibility assumption in Theorem 1 implies that

$$\nu(\mathbb{P}W) = 0 \text{ for all proper } W \subseteq V. \tag{C}$$

Finally, we completed the proof of Theorem 1 by showing that if $\nu \in \mathcal{P}(X)$ is an arbitrary μ-stationary measure, then one obtains a map $\xi\colon B \to X$ such that for β-a.e. $b \in B$ and any nonzero limit point $f \in \mathrm{End}(V)$ of $\lambda_n b_1 \dots b_n$, ($\{\lambda_i\}_{i \in \mathbb{N}} \subset \mathbb{K}$) has rank-1 and admits $\xi(b)$ as its image. Moreover, we showed that $\nu_b = \delta_{\xi(b)}$. The map ξ is defined to be the smallest subspace which contains the support of ν_b. The strong irreducibility is used via (C) to show that $\nu(\mathbb{P}\ker(fg)) = 0$ and hence the map $fg \in \mathrm{End}\, V$ induces a continuous map from $\mathrm{supp}\, \nu \to \mathrm{End}(V)$. Then, by using (A) and (B) we concluded that for β-a.e. $b \in B$ and all $g \in \Gamma_\mu \cup \{\mathrm{Id}\}$ one has

$$(fg)\nu = \nu_b. \tag{D}$$

This allows us to conclude that $\xi(b) = Imf$. Moreover, the proximality assumption is used to show that there exists a nonzero limit point $p \in \mathrm{End}(V)$ of rank-1 and using (D) again together with the fact that $\Gamma_\mu \curvearrowright V$ is irreducible one can show that this is in fact the generic situation.

References

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Positivity of the first Lyapunov exponent

Weikun He

Let $V = \mathbb{R}^d$ be a finite dimensional Eulidean space. Let μ be a Borel probability measure on the group $G = \mathrm{GL}(V)$. Let Γ_μ denote the closed subsemigroup generated by the support of μ. We associate a random walk to the probability measure μ. More precisely, let $g_1, g_2, \ldots$ be an i.i.d. sequence of random variables distributed according to μ. We are interested in the generic asymptotic behaviour of $g_n \cdots g_1$ as n goes to infinity. The aim of this talk is to present two fundamental results due to Furstenberg [1].

The first one is a law of large number for the norm. It allows us to define $\lambda_{1,\mu}$, the first Lyapunov exponent of μ.

Theorem 1. *Let μ and $(g_n)_{n\geq 1}$ be as above. There is a number $\lambda_{1,\mu} \in \mathbb{R}$ such that,*

$$\lim_{n\to+\infty} \frac{1}{n} \log\|g_n \cdots g_1\| = \lambda_{1,\mu}$$

almost surely and in L^1. If moreover the group Γ_μ acts strongly irreducibly on V then for every nonzero vector $v \in V$,

$$\lim_{n\to+\infty} \frac{1}{n} \log\|g_n \cdots g_1 v\| = \lambda_{1,\mu}$$

almost surely and in L^1. If ν is a μ-stationary measure on $\mathbb{P}(V)$, then

$$\lambda_{1,\mu} = \iint_{G\times\mathbb{P}(V)} \log \frac{\|gv\|}{\|v\|}\, \mathrm{d}\mu(g)\, \mathrm{d}\nu(\mathbb{R}v).$$

The second result is a characterisation for the first Lyapunov exponent to be nontrivial (positive in the case of $\mathrm{SL}(V)$).

Theorem 2. *If* Γ_μ *acts strongly irreducibly on* V *and its projection to* PGL(V) *is unbounded, then*

$$\lambda_{1,\mu} > \frac{1}{d}\int_G \log|\det(g)|\ \mathrm{d}\mu(g).$$

In particular, if Γ_μ *is included in* $\mathrm{SL}(V)$*, acts strongly irreducibly on* V *and is unbounded, then*

$$\lambda_{1,\mu} > 0.$$

The proof for both theorem use Birkhoff's ergodic theorem applied to the forward system. In the proof of Theorem 1 we also use Kolmogorov's law of large number presented in Lecture 6. In the proof of Theorem 2 we also use properties of the Furstenberg boundary map presented in Lecture 7 and an ergodic lemma about the divergence of Brikhoff sums.

References

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Central Limit Theorem

Maxim Kirsebom

In this talk we discussed the Central Limit Theorem (CLT) for cocycles which is due to Guivarc'h and LePage along with a brief mention of the Local Limit Theorem (LLT) by the same authors.

From a probabilistic point of view, these theorems provide more precise information about the convergence in the Law of Large Numbers (LLN). That is, for a probability space (X,μ) and a sequence of real i.i.d. random variables $\{\xi_i\}$ the LLN tells us that

$$\frac{1}{n}S_n = \frac{1}{n}\sum_{i=1}^{n}\xi_i \to \mathbb{E}(\xi_i) =: \lambda \quad \mu-\text{a.s}$$

Writing this convergence as $\frac{1}{n}S_n - \lambda \to 0$ we may ask about the distribution of $\frac{1}{n}S_n - \lambda$ around 0 relative to the size of n. If μ is a non-degenerate Borel probability measure and the variance σ^2 is finite, then the CLT tells us that under rescaling by $\sqrt{n}$ we get a Gaussian law in the limit, that is

$$\mu\left(\left\{\frac{1}{n}S_n - \lambda \le \frac{r}{\sqrt{n}}\right\}\right) \to \Phi\left(\frac{r}{\sigma}\right)$$

where Φ is the standard normal cumulative distribution function.

If we instead consider $S_n - n\lambda$, the LLN tells us that this difference may get arbitrarily big, but not too fast. Hence we are interested in the distribution of this quantity as n grows. If μ is a non-degenerate Borel probability measure which is not supported on an arithmetic progression[1] and the variance σ^2 is finite, then the LLT tells us that the probability of $S_n - n\lambda$ belonging to a fixed interval decays

[1] The statement only changes slightly if μ is supported on an arithmetic progression.

as the length of the interval divided by $\sqrt{n}$. More precisely, for any real numbers $a_1 < a_2$ we have

$$\sqrt{n}\mu\left(S_n - n\lambda \in [a_1, a_2]\right) \to \frac{a_2 - a_1}{\sqrt{2\pi}\sigma}.$$

Guivarc'h and LePage proved these theorems for cocycles on the projective space of $\mathbb{R}^d$ which may in turn be used to describe the same statistical properties for products of random matrices. The theorem discussed in the talk may be found in an abstract version in [1] (Theorem 12.1) or in a more concrete setting in [2] (Theorem 5.1). We sketched the proof of [1] (Theorem 12.1) in the simplified setting of [1] (Theorem 1.7). The setting and statement is as follows.

Let $V = \mathbb{R}^d$ and let $\|\cdot\|$ denote the Euclidian norm on V. Let $G = \mathrm{GL}(d, \mathbb{R})$ and let μ be a Borel probability measure on G. Set $A := \mathrm{supp}(\mu)$ and let Γ_μ be the closed subsemigroup of G spanned by A. Let

$$B := A^{\mathbb{N}} = \{b = (g_1, \dots, g_n, \dots) : g_i \in G\}$$

and let $\beta := \mu^{\otimes \mathbb{N}}$ be the natural measure on B. We write the n'th convolution of μ as

$$\mu^{*n} := \mu * \cdots * \mu.$$

Then an element $b \in B$ chosen with law β is an i.i.d. sequence g_k of elements of G with law μ. Also, the product $g_1 \cdots g_n$ has law μ^{*n}. Let $X = \mathbb{P}(V)$, let ν be the unique μ-stationary probability measure on X. We define the log–cocycle $\sigma : G \times X \to \mathbb{R}$ by

$$\sigma(g, v) = \log\left(\frac{\|gv\|}{\|v\|}\right).$$

We say that μ has **finite exponential moment** if

$$\int_G \|g\|^\alpha \, d\mu(g) < \infty \quad \text{for some} \quad \alpha > 0.$$

We say that Γ_μ **acts strongly irreducibly** on V if there is no proper finite union of vector subspaces of V which is Γ_μ–invariant. We say that ν **is μ-proximal** if the measure $\nu_b = \lim_{n\to\infty}(g_1, \dots, g_n)_*\nu$ is a Dirac mass β–a.s. (see [1] Chapter 2 for details).

Set

$$\lambda_1 := \lim_{n\to\infty} \frac{1}{n} \int_G \log\|g\| \, d\mu^{*n}(g)$$

and

$$\Phi := \lim_{n\to\infty} \frac{1}{n} \int_G (\log\|g\| - n\lambda_1)^2 \, d\mu^{*n}(g).$$

In the given setting it is known that these limits exist and that $\lambda_1 > 0$ and $\Phi > 0$.

Theorem 1 (CLT for the log–cocycle)**.** *Assume that μ has finite exponential moment, that Γ_μ acts strongly irreducibly on V and that ν is μ-proximal. Then for any bounded continuous function ψ on $X \times \mathbb{R}$, uniformly for $x \in X$ we have*

$$\text{(1)} \qquad \int_G \psi\left(gx, \frac{\sigma(g,x) - n\lambda_1}{\sqrt{n}}\right) d\mu^{*n}(g) \to \int_{X\times\mathbb{R}} \psi(y, s) \frac{e^{-\frac{s^2}{2\Phi}}}{\sqrt{2\pi\Phi}} \, ds \, d\nu$$

as $n \to \infty$.

The proof of the theorem may be sketched as follows.

First we simplify the task at hand. First we are allowed to restrict ourselves to cocycles of zero average by using the so-called recentering trick. Secondly we are allowed to restrict ourselves to functions ψ which satisfy $\psi(y,s) = \phi(y)\rho(s)$ where ρ is bounded and continuous on $\mathbb{R}$ and ϕ is γ–Hölder continuous on X (see [1] Chapter 11 for definition). This simplification may be justified by standard approximation arguments.

We are then left to prove that

$$\int_G \phi(gx)\rho\left(\frac{\sigma(g,x)}{\sqrt{n}}\right) d\mu^{*n}(g) \to \int_X \phi\, d\nu \int_{\mathbb{R}} dN_\mu$$

where $dN_\mu := \frac{e^{-\frac{s^2}{2\Phi}}}{\sqrt{2\pi\Phi}}\, ds$. We may define measures $\mu^\phi_{n,x}$ by rewriting the left hand side as

$$\int_{\mathbb{R}} \rho\, d\mu^\phi_{n,x} := \int_G \phi(gx)\rho\left(\frac{\sigma(g,x)}{\sqrt{n}}\right) d\mu^{*n}(g)$$

which allows us to reformulate the objective as proving that, uniformly in x, $\mu^\phi_{n,x} \to \left(\int_X \phi\, d\nu\right) N_\mu$ for $n \to \infty$.

To verify this convergence the characteristic function of a measure is introduced. For ν a finite Borel measure on $\mathbb{R}$, this is defined by

$$\widehat{\nu}(\theta) = \int_{\mathbb{R}} e^{i\theta x}\, d\nu(x).$$

Importantly, it may be shown that $\widehat{N_\mu}(\theta) = e^{-\frac{1}{2}\Phi(\theta)}$. Characteristic functions are important due to the **Levy Continuity Method** which states that finite Borel measures ν_n and ν_∞ on $\mathbb{R}$ with the property that

$$\widehat{\nu}_n(\theta) \to \widehat{\nu}_\infty(\theta)$$

as $n \to \infty$ for all $\theta \in \mathbb{R}$, also satisfy

$$\nu_n(\psi) \to \nu_\infty(\psi)$$

for any continuous bounded function ψ on $\mathbb{R}$.

Hence the desired convergence can be shown by proving the analog convergence for the respective characteristic functions. The crucial step of the proof now relies on the observation that $\widehat{\mu}^\phi_{n,x}(\theta)$ may be rewritten using the so-called complex transfer operator. Let $\theta \in \mathbb{C}$ with real part less than α (this is the α from the exponential moment assumption). For ϕ continuous on X and $x \in X$ we define **the complex transfer operator** P_θ by

$$P_\theta\phi(x) = \int_G e^{\theta\sigma(g,x)}\phi(gx)\, d\mu^{*n}(g).$$

Then, for $\theta \in \mathbb{R}$

$$\widehat{\mu}^\phi_{n,x}(\theta) = P^n_{\frac{i\theta}{\sqrt{n}}}\phi(x).$$

The challenge is now to understand the behaviour of the iterates of the complex transfer operator. However, in the given setting this operator possesses a strong spectral gap property which, though technically complicated, ensures the desired convergence.

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Recurrence in law

Vladimir Finkelshtein

Let Γ be an irreducible lattice in a semisimple Lie group G. Let $X = G/\Gamma$ and consider the left action by G. Dani showed that there exists a compact subset of X in which any orbit of unipotent flow spends 99% of the time. We discuss an analogue of this theorem for random walks on X. Let μ be a probability measure on G with exponential moment. Assume that the support of μ generates a Zariski dense subsemigroup in G. Let $\epsilon > 0$. Eskin and Margulis proved that there exists a compact set $K \subset X$, such that for any $x \in X$ and for n large enough we have $\mu^{*n} * \delta_x(K) > 1 - \epsilon$. In other words, random walk on X with law μ finds itself after n steps in the compact set K with high probability, regardless of where the walk originated. We call it recurrence in law of the random walk.

We present the proof of Eskin and Margulis. First we reduce the proof of the theorem to showing that the Markov operator P_μ associated with the random walk satisfies the uniform contraction hypothesis. The hypothesis states that there exists a proper function $f : X \to [0, \infty)$ and $a < 1, b \geq 0$ such that $P_\mu f \leq af + b$. We give simple examples of Markov operators for which the hypothesis holds and examples for which it does not.

We first prove recurrence in law for the case of $SL(d, \mathbb{Z})$ acting on a punctured torus $X = \mathbb{R}^d/\mathbb{Z}^d \setminus \{0\}$. In this setting, we can choose the function ensuring the hypothesis to be the distance to the singularity raised to a small negative power. The contraction is guaranteed by the positivity of Lyapunov exponent for linear random walks and by the observation that close to the singularity, the random walk establishes "linear" behavior.

Inspired by this example, we then give the proof that the hypothesis is satisfied for random walks on $X = SL(2, \mathbb{R})/SL(2, \mathbb{Z})$. The proof is very similar, while now the distance from the cusp is replaced by the systole function. We then proceed to the case of $X = SL(d, \mathbb{R})/SL(d, \mathbb{Z})$ for arbitrary $d \geq 3$ and explain how to combine higher dimensional systoles together to construct the desired function f.

We mention few applications. One corollary is an alternative proof (due to Margulis) of the Theorem of Borel and Harish-Chandra, which says that $\mathbb{G}(\mathbb{Z}) < \mathbb{G}(\mathbb{R})$ is a lattice. We also briefly discuss the two applications that motivate us. First is the existence of μ-stationary measures on noncompact X as weak-star limits of $\nu_N = \frac{1}{N}\sum_{i=0}^{N-1}\mu^{*n} * \delta_x$. Second is that for $SL(d,\mathbb{Z})$ action on the torus the limiting measures of ν_n do not have an atom at 0.

Orbit closures on the torus $\mathbb{T}^d$

Seul Bee Lee

In the first lecture, we discuss some general questions about the orbit closures and equidistributed measures on random trajectories of a locally compact metric space. Benoist and Quint answered these questions on homogeneous spaces (see [1], [3], [4]). For a first concrete example of the questions, we consider the d-dimensional torus. In this lecture, we show that $^{(R)}$the rigidity of stationary measures implies $^{(1)}$the denseness of orbit closures of irrational points and $^{(2)}$the equidistribution of random trajectories on the d-dimensional torus. It is based on the lecture note [2] of Benoist and Quint (see Section 4.1 and 4.2 in [2]).

Let X be the d-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. Consider a subsemigroup Γ of $\mathrm{SL}_d(\mathbb{Z})$ generated by a finite subset A of $\mathrm{SL}_d(\mathbb{Z})$. Assume that Γ is *strongly irreducible*, i.e., there is no Γ-invariant union of proper subspaces of $\mathbb{R}^d$. Let μ be a probability measure on Γ supported by A. This semigroup Γ acts continuously on X. For a point $x_0 \in X$ and a sequence $(b_1, b_2, \cdots)$ of elements in $\mathrm{SL}_d(\mathbb{Z})$, we consider a trajectory $\{x_0, x_1, \cdots\}$ such that $x_k = b_k x_{k-1}$. On the trajectory, define measures $\nu_n = \frac{1}{n}\sum_{i=0}^{n-1}\delta_{x_i}$. We call ν_n the *empirical measures*. Let $B = A^{\mathbb{N}}$ and $\beta = \mu^{\otimes\mathbb{N}}$ be the product measure on B. Let ν_X be the Haar measure of X. For a measure ν, we define the convolution of μ and ν as $\mu * \nu = \int_\Gamma g_*\nu \mathrm{d}\mu(g)$. We say that a measure ν on X is *μ-stationary* if $\mu * \nu = \nu$. A point $x \in X$ is called an *irrational point* if $x \notin \mathbb{Q}^d/\mathbb{Z}^d$. Precise statements of (R), (1) and (2) are as follows:

- (R) A μ-stationary measure is a convex combination of the Haar measure on $\mathbb{T}^d$ and an atomic measures which is supported by rational points.
- (1) For any irrational point x_0, the Γ orbit of x_0 is dense in X.
- (2) The sequence of the empirical measures ν_n converges to ν_X in weak-* sense for any irrational point x_0 and β-almost every sequence b in B.

We readily see that (2) implies that (1). By Breiman's law of large numbers, any weak-* limit ν_∞ is μ-stationary. Thus, we only need to show that ν_∞ is atom-free. The following proposition gives us $\nu_\infty(\{0\}) = 0$. The remaining part is how to prove $\nu_\infty(\{x\}) = 0$ for all rational points $x \in X$. We can check it by using a map $k : X \to X$, $k \in \mathbb{N}$ which is defined as $k(x) = kx \bmod 1$.

Proposition 1. *For $\varepsilon_0 > 0$, there is $r > 0$ such that for all $x_0 \in \mathbb{T}^d\backslash\{0\}$, for β almost every $b \in B$,*

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\delta_{x_k}(\overline{\mathbf{B}(0,r)}) \le \varepsilon_0$$

where $\mathbf{B}(0,r)$ is the ball of radius r centered at 0.

This proposition means that for any starting point, for almost every random trajectory, we can control the "time" which the points of a trajectory stay near 0. For fixed $r_0 > 0$, let $Y = X - \mathbf{B}(0, r_0)$. For $x \in X$ and $b \in B$, the *nth return time to Y* $\tau_{Y,n}(b, x)$ is defined as

$$\tau_{Y,n}(b, x) = \inf\{k > \tau_{Y,n-1}(b, x) : x_k \in Y\}$$

with $\tau_{Y,0} = 0$. The *nth excursion time outside Y* $\sigma_{Y,n}(b, x)$ is defined as

$$\sigma_{Y,n}(b, x) = \tau_{Y,n} - \tau_{Y,n-1}.$$

If a point on a trajectory is near 0, then the excursion time on Y is long enough. More precisely, if a point x_0 is in $\mathbf{B}(0, r)$ for a small radius r, then $\sigma_{Y,1}(b, x_0)$ is larger than $T := M^{-1} \log \frac{r_0}{r}$ where $M := \max_{g \in A}\{\log ||g||, \log ||g^{-1}||\}$. We define

$$\sigma^T_{Y,n} := \sigma_{Y,n} \cdot 1_{\{\sigma_{Y,n} \geq T\}}, \quad \tau^T_{Y,n} := \sum_{p=1}^{n} \sigma^T_{Y,p}.$$

Then it is enough to show that for any ε_0,

$$(**) \qquad \limsup_{n\to\infty} \frac{1}{n} \sum_{p=1}^{n} \sigma^T_{Y,p} \leq \varepsilon_0.$$

By Furstenberg's positivity of Lyapunov exponent, there is $N > 0$ such that for all $n_0 > N$, for all $x \in X$,

$$(1) \qquad \frac{1}{n_0} \int_\Gamma \log \frac{||gx||}{||x||} d\mu^{*n_0}(g) > \lambda > 0.$$

Although a torus is compact without singular points, as a viewpoint of the random walk, we can consider the underlying space as a punctured torus since 0 is fixed by the action of Γ. A function $u(x) = \mathbf{d}(x, 0)^{-\delta}$ is defined on the punctured torus $X - \{0\}$ where $\mathbf{d}(x, y)$ is the distance between x and y. By the equation (1), the following lemma holds. The following lemma means that for large $C > 0$, most points are recurrent to a compact set

$$\{x : u(x) \leq C\}$$

with respect to μ^{*n_0}.

Lemma. *There are $n_0 > 0$, $\delta > 0$, $0 < a < 1$, $C > 0$ such that*

$$\int_G u(gx) d\mu^{*n_0}(g) \leq a u(x), \ x \notin Y, \qquad \int_G u(gx) d\mu^{*n_0}(g) \leq C, \ x \in Y.$$

A more general result of a locally compact second countable group is in Proposition 6.3 of [3]. By the above lemma, an upper bound of the probability of $\{\tau_{Y,1} > n\}$ exists as

$$\mathbb{P}_{x_0}(\{\tau_{Y,1} > n\}) \leq r_0^\delta \mathbb{E}_{x_0}(u(x_n) 1_{\{\tau_{Y,1} > n\}}) \leq r_0^\delta a^{n-1} C.$$

Thus, we get the following lemma. The following lemma means that the first return time has a uniform finite exponential moment.

Lemma. *There is $\alpha > 0$ such that $\mathbb{E}_x(e^{\alpha \tau_{Y,1}})$ is uniformly bounded above.*

By Markov property, $\sigma^T_{Y,1}, \sigma^T_{Y,2}, \cdots, \sigma^T_{Y,n}$ are independent, thus for any $\varepsilon > 0$,

$$\mathbb{P}_{x_0}(\{\tau^T_{Y,n} \geq n(\varepsilon_0 + \varepsilon)\}) \leq e^{-\alpha n\varepsilon/2}\mathbb{E}_{x_0}(e^{\alpha\sigma^T_{Y,1}}).$$

Thus,

$$\sum_{n=0}^{\infty} \mathbb{P}_{x_0}(\{\tau^T_{Y,n} > n\varepsilon_0\}) < \infty.$$

By Borel-Cantelli Lemma,

$$\mathbb{P}_{x_0}(\limsup_{n\to\infty} \frac{1}{n}\sum_{p=1}^{n} \sigma^T_{Y,p} > \varepsilon_0) = 0.$$

We conclude that $(**)$ holds for any ε_0.

References

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[2] Y. Benoist, J-F. Quint, *Introduction to random walks on homogeneous spaces Thenth Takagi Lecture in Kyoto*, Japanese Journal of Mathematics **7**(2) (2012), 135–166.
[3] Y. Benoist, J-F. Quint, *Stationary measures and invariant subsets of homogeneous spaces (II)*, Journal of the American Mathematical Society **26**(3) (2013), 659–734.
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Non-Degeneracy of the Limit Measure ν_b

Tom Kempton

The main point of this talk is to prove that, under the assumption that a stationary measure ν on $\mathbb{T}^d$ is non-atomic, th e corresponding limit measures ν_b give zero mass to any stable leaf. Formally stated, we proved the following theorem.

Theorem 1. *Let X be the d-dimensional torus and let Γ be a subsemigroup of $SL(d, \mathbb{Z})$ whose action on $\mathbb{R}^d$ is strongly irreducible. Let μ be a probability measure on Γ whose support is a finite set A and which spans Γ. Then for any atom-free μ-stationary measure ν on X, almost-all of the corresponding limit measures ν_b give zero mass to any stable leaf.*

This allows the exponential drift argument of the following lectures to be developed, finally allowing the conclusion of the above theorem to be replaced by 'Then any atom-free μ-stationary measure ν on X is the Haar measure'.

There are three key ideas that go in to the proof.

Part 1: Firstly we consider the random walk on $X \times X$ associated to μ, that is, the random walk which maps pairs $(x, x') \in X \times X$ to (gx, gx'), where $g \in \Gamma$ is chosen according to μ. We need a technical condition called 'positive μ-instability of the diagonal', this essentially says that for any (x, x') with $x \neq x'$, for small enough $\epsilon > 0$ the random walk applied to (x, x') does not spend too much time within ϵ of the diagonal, i.e. in the set

$$D_\epsilon = \{(x, x') \in X \times X : |x - x'| < \epsilon\}.$$

Positive μ-instability of the diagonal can be proved in a manner similar to the recurrence in law away from 0 of the previous lecture.

Part 2: In this part we proved, using part 1, that if ν is atom free then the limit measures ν_b are also atom free. We began by assuming that for typical b the measure ν_b consisted of a single atom at $f(b) \in X$. The function f is crucial as it gives us an opportunity to apply the Chacon-Ornstein ergodic theorem, and derive a contradiction with positive μ-instability.

If typical ν_b were not atom free, but did not consist of single atoms, one can decompose ν_b into a finite number of atoms plus a non-atomic part. Then considering the atomic part, one is again able to define a function f and use the Chacon-Ornstein ergodic theorem.

Part 3: The third part of the talk aimed to extend the statement 'ν_b is atom free for typical b' to 'ν_b gives no mass to any stable leaf, for typical b'. In fact, once the correct space has been defined, this is a simple consequence of the Poincaré recurrence theorem. First one builds a larger dynamical system and invariant measure out of the ν_b. Then one supposes that typical b give positive mass to stable leaves, but not to any single point, and observe that the dynamics on stable leaves contracts everything, creating a contradiction with the Poincaré recurrence theorem.

This talk relied on [1][Lemma 4.5], which in turn relies on [2][Prop 6.17 and Cor. 6.26].

References

[1] Y. Benoist, J-F. Quint, *Introduction to random walks on homogeneous spaces Thenth Takagi Lecture in Kyoto*, Japanese Journal of Mathematics **7**(2) (2012), 135–166.

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Conditional measures along a group operation

Laurent Dufloux

The aim of this talk was to introduce conditional measures along a group operation with discrete stabilizers, and also to recall the disintegration of measures in the usual sense. The following preliminary result was stated: if $p : X \to Y$ is a Borel mapping between standard Borel spaces, and λ is a σ-finite Borel measure on X, then there is a finite measure μ on Y (called a pseudo-image of λ) and a family $(\lambda_y)_{y \in Y}$ of σ-finite Borel measures on X, such that

(1) For μ-almost every y, λ_y is supported on $p^{-1}(y)$;

(2) For any Borel subset A of X,

$$\lambda(A) = \int \mathrm{d}\mu(y) \lambda_y(A)$$

The family of measures $(\lambda_y)_{y\in Y}$, along with μ, is a *disintegration* of λ along p. It is unique in the following sense: if μ' and $(\lambda'_y)_{y\in Y}$ satisfy the same properties, then μ' is equivalent to μ and, for almost every $y \in Y$ (with respect to either μ or μ'), λ'_y is a scalar multiple of λ_y. This disintegration Theorem follows from the classical Radon-Nikodym Theorem along with a countability argument.

Now for the setting of the main result. Let X be a standard Borel space where a locally compact second countable topological group G acts in a Borel way and with discrete stabilizers. For any Borel probability measure λ on X, we define a Borel mapping σ from X into the space $\mathcal{M}^1(G)$ of projective Radon measures on G (a projective measure $[\mu]$ is a class of measure modulo normalization, *i.e.* $[\mu] = [s\mu]$ for any $s > 0$)

$$\sigma : X \to \mathcal{M}^1(G)$$

where, for any x, $\sigma(x)$ can be interpreted as the conditional measure of λ along the G-orbit at x. This is interesting only in the case when the operation of G on X is not *tame*, that is, the quotient space X/G with the quotient Borel structure is *not* a standard Borel space. A good example to keep in mind is an irrational flow of the 2-torus.

The construction of σ is as follows. By virtue of a Theorem of A. Kechris, [1], we can find a Borel subset $\Sigma \subset X$ that enjoys the following properties:

(1) $G\Sigma = X$ ("completeness")
(2) There exists a relatively compact neighbourhood of the identity U in G such that, for any $x \in \Sigma$, the only element $g \in U$ that maps x into Σ is the identity element. ("lacunarity")

We say that Σ is a complete lacunary section (for the operation of G on X). The mapping $a : G \times \Sigma \to X$ defined by $a(g, x) = gx$ has countable fibers because G has discrete stabilizers. Hence λ can be lifted through a; this yields a σ-finite measure $a^*\lambda$ on $G \times \Sigma$ defined by

$$a^*\lambda(A) = \int \mathrm{d}\lambda(x) \sum_{(g,x')\in a^{-1}(x)} \mathbf{1}_A(g, x')$$

(where $\mathbf{1}_A$ is the indicator function of A).

Now disintegrate $a^*\lambda$ along the projection $G \times \Sigma \to \Sigma$ onto the second coordinate:

$$a^*\lambda = \int \mathrm{d}\lambda_\Sigma(x')\sigma_\Sigma(x') \times \delta_{x'}$$

where λ_Σ is a finite measure on Σ, $\sigma_\Sigma(x')$ is a σ-finite measure on G (and in fact it is Radon for almost every x') and $\delta_{x'}$ is the Dirac mass at x'.

The main result can now be stated:

Proposition 1 ([2]). *There is a Borel mapping $\sigma : X \to \mathcal{M}^1(G)$ such that, for any complete lacunary section Σ as above, there is a Borel subset X' of X of full measure (with respect to λ), in which σ can be defined by the following relation:*

$$\sigma(x) = [(R_g)_*\sigma_\Sigma(gx')]$$

for any $(g, x') \in G \times \Sigma$ satisfying $gx' = x$.

The mapping σ is G-equivariant in the following sense: there is a subset X_0 of X of full measure such that if x and gx belong to X_0 (where $g \in G$), then

$$\sigma(gx) = (R_g)_* \sigma(x)$$

Here, R_g is the right translation $h \mapsto hg^{-1}$ in G, and $(R_g)_*$ denotes, as usual, the push-forward by R_g.

The existence of σ will be put to use in Nicolas de Saxcé's talk with the help of the following result:

Proposition 2. *Keeping the previous notations and assumptions, for λ to be invariant with respect to a closed subgroup H of G, it is necessary and sufficient that, for almost every x, $\sigma(x)$ be H-invariant (in the sense that any measure in the projective class $\sigma(x)$ is H-invariant). In particular if $G = H$ this is equivalent to the statement that $\sigma(x)$ is almost surely the (projective class of the) Haar measure on G.*

References

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[2] Y. Benoist, J.-F. Quint, *Mesures stationnaires et fermés invariants des espaces homogènes. (French. English, French summary) [Stationary measures and invariant subsets of homogeneous spaces]*, Ann. of Math. (2) 174 (2011), no. 2, 1111–1162.

The exponential drift, work of Y. Benoist and J.-F. Quint

Nicolas de Saxcé

The goal of theses notes is to present the ideas used in the proof of the following result.

Theorem 1. *Let $G = \mathrm{SL}_d(\mathbb{Z})$, let μ be a Borel probability measure on G such that the semi-group Γ_μ generated by the support of μ acts strongly irreducibly on $\mathbb{R}^d$. Let $X = \mathbb{T}^d$ be the d-dimensional torus and m_X be the Haar probability measure on X. Then m_X is the unique atom-free μ-stationary measure on X.*

1. Strategy of proof

Taking ν to be any atom-free μ-stationary probability measure on X, we want to show that $\nu = m_X$. We shall make use of the Bernoulli space $B = G^{\mathbb{N}}$ endowed with the product measure $\beta = \mu^{\otimes \mathbb{N}}$. Recall the following results of Furstenberg:

(1) For β-a.e. $b = (b_i)_{i \geq 1}$, the limit $b_1 \dots b_n \nu = \nu_b$ exists in the space $\mathcal{P}(X)$ of probability measures on X.

(2) For β-a.e. $b = (b_i)_{i \geq 1}$, if π_b is any limit point of $\frac{b_1 \dots b_n}{\|b_1 \dots b_n\|}$ write $V_b = \Im \pi_b$; this does not depend on the choice of the limit point π_b.

To make things simpler, we shall from now on assume that the action of Γ_μ on $\mathbb{R}^d$ is proximal, i.e. that almost surely, $\dim V_b = 1$. We then have the following proposition.

Proposition 1. *For β-a.e. $b \in B$, the measure ν_b is V_b-invariant.*

Let $B^X = B \times X$, endowed with the probability measure $\beta^X = \int \delta_b \otimes \nu_b \, \mathrm{d}\beta(b)$. Consider the action of $\mathbb{R}$ on B^X given by

$$\begin{array}{rccc} \Phi_t : & B^X & \to & B^X \\ & (b,x) & \mapsto & (b, x + tv_b) \end{array}$$

From the previous talk on conditional measures along group actions, we may define, for β^X-almost every (b,x), the conditional measure $\sigma(b,x)$, which is an element in the space of Radon measures on $\mathbb{R}$, modulo multiplication by a non-zero scalar. We also know that the above proposition can be restated as follows:

Proposition 2. *For β-a.e. $b \in B$, $\sigma(b,x)$ is translation invariant:*

$$\forall \varepsilon_0 > 0, \ \exists t \in]0, \varepsilon_0[: \ (\tau_t)_* \sigma(b,x) = \sigma(b,x),$$

where $\tau_t : \mathbb{R} \to \mathbb{R}$ denotes the map $x \mapsto x + t$.

The proof of this proposition is based on the exponential drift argument, which we now explain.

2. The exponential drift argument

Given a generic point (b,x) in B^X, the idea will be to construct another point (b', x') and a small vector D, the drift, such that $\|D\| = t \asymp \varepsilon_0$ and:

- $\sigma(b,x) = \sigma(b', x')$;
- $\sigma(b,x) = \sigma(b', x' + D) = (\tau_t)_* \sigma(b', x')$.

This will allow us to conclude that $\sigma(b,x) = (\tau_t)_* \sigma(b,x)$, which is the desired invariance statement.

Let $T_X : B^X \to B^X$ be the map $(b,x) \mapsto (Tb, b_1^{-1} x)$. The drift argument uses the following lemma, which relates the conditional measures of two points on the same fiber of T_X^n.

Lemma. *Let $T : B \to B$ be the shift, and for β-a.e. b in B, write v_b for a unit vector generating the line V_b. For a.e. $b \in B$ and $n \geq 1$, define $\theta_n(b) = \|b_1 \dots b_n v_{T^n b}\|$. Then, for all $(a_1, \dots, a_n) \in G^n$, for β^X-a.e. (b,x) in B^X,*

$$\sigma(aT^n b, a_1 \dots a_n b_n^{-1} \dots b_1^{-1} x) = \theta_n(aT^n b) \theta_n(b)^{-1} \sigma(b,x),$$

where $aT^n b = (a_1, \dots, a_n, b_{n+1}, b_{n+2}, \dots)$.

To construct the drift D, we start from a point $(b, x+v)$ close to (b,x), and let n be the minimal integer (depending on v and going to infinity as $v \to 0$) such that

$$\theta_n(b) \|b_n^{-1} \dots b_1^{-1} v\| \geq \varepsilon_0.$$

The integer n is well defined provided $v \notin V_b$ because V_b is contracted at speed $\theta_n(b)$ by $b_n^{-1} \dots b_1^{-1}$ and any vector outside V_b is exponentially less contracted.

Once n is fixed, we shall choose $a = (a_1, \dots, a_n)$ inside the intersection of the following three subsets of G^n (for well-chosen $C > 0$ and $\eta_n \to_{n \to \infty} 0$):

(1) (dilation control) $A_{n,b}^{\theta} = \{(a_1, \dots, a_n) \mid \frac{1}{2} \leq \theta_n(aT^n b)\theta_n(b)^{-1} \leq 2\}$;

(2) (drift control) $A_{n,b}^{drift} = \{(a_1, \dots, a_n) \mid \|D_v\| \in [\frac{\varepsilon_0}{C}, C\varepsilon_0]$ and $d([D_v], V_{b^v}) \leq \eta_n\}$;
(3) (continuity control)

$$A_{n,b,x,v}^K = \{(a_1, \dots, a_n) \mid (aT^n b, a_1 \dots a_n b_n^{-1} \dots b_1^{-1} x),$$

$$(aT^n b, a_1 \dots a_n b_n^{-1} \dots b_1^{-1}(x+v)) \in K\},$$

where $K \subset B^X$ is a compact subset on which $(b, x) \mapsto (V_b, \sigma(b, x))$ is continuous.

We now prove Proposition 2 assuming we can show that $A_{n,b}^{\theta} \cap A_{n,b}^{drift} \cap A_{n,b,x,v}^K \neq \varnothing$.

Proof of Proposition 2. By Lusin's theorem, fix a compact set $K \subset B^X$ of large measure on which the map $(b, x) \mapsto (V_b, \sigma(b, x))$ is continuous. For β^X-a.e. $(b, x) \in K$, since $\nu_b(x + V_b) = 0$ (see talk on non-degeneracy of the limit measures), we can find $v \in \mathbb{R}^d$ arbitrarily small such that $(b, x + v) \in K$ and $v \notin V_b$. As explained above, we choose n such that $\theta_n(b)\|b_n^{-1} \dots b_1^{-1} v\| \in [\varepsilon_0, C\varepsilon_0]$, and $(a_1, \dots, a_n) \in A_{n,b}^{\theta} \cap A_{n,b}^{drift} \cap A_{n,b,x,v}^K$. Let $(b^v, x^v) = (aT^n b, a_1 \dots a_n b_n^{-1} \dots b_1^{-1} x)$ and $(b^v, x^v + D_v) = (aT^n b, a_1 \dots a_n b_n^{-1} \dots b_1^{-1}(x + v))$. From the lemma above,

$$\sigma(b^v, x^v) = (\theta_n(aT^n b)\theta_n(b)^{-1})_* \sigma(b, x)$$

and

$$\sigma(b^v, x^v + D_v) = (\theta_n(aT^n b)\theta_n(b)^{-1})_* \sigma(b, x + v).$$

Letting $v \to 0$ and extracting a subsequence to ensure convergence, we obtain $\theta_n(aT^n b)\theta_n(b)^{-1} \to \theta \in [\frac{1}{2}, 2]$ (dilation control), $(b^v, x^v) \to (b', x') \in K$ and $(b^v, x^v + D_v) \to (b', x' + D) \in K$ (continuity control), with $\|D\| \asymp \varepsilon_0$ and $D \in V_b$ (drift control). By continuity of $(b, x) \mapsto (V_b, \sigma(b, x))$ on K, we may take limits in the above two equalities to get

$$\sigma(b', x') = \theta_* \sigma(b, x) \text{ and } \sigma(b', x' + D) = \theta_* \sigma(b, x).$$

Now, since $D \in V_{b'}$ and $\sigma(b', x')$ is the conditional of $\nu_{b'}$ along the translation along $V_{b'}$, we get $\sigma(b', x' + D) = (\tau_t)_* \sigma(b', x')$, and therefore

$$\sigma(b, x) = (\tau_t)_* \sigma(b, x).$$

This holds for almost every (b, x) in the compact set K, but by Lusin's theorem, we may take $\beta^X(K)$ arbitrarily close to 1, so that we get the desired invariance for β^X-almost every (b, x). ☐

3. Other important parts of the proof

Of course, it is crucial in the above sketch of the drift argument that we can ensure that the intersection $A_{n,b}^{\theta} \cap A_{n,b}^{drift} \cap A_{n,b,x,v}^K$ is non-empty. It is in fact the case that this intersection contains an arbitrarily large proportion of $A_{n,b}$. Let us mention the tools needed to show this.

(1) (dilation control) By the law of the iterated logarithm, we have almost surely $\theta_n(b) - n\lambda_1 = O((\log\log n)\sqrt{n})$. The local limit theorem with moderate deviations $(\varepsilon(\log n)\sqrt{n})$ for $\theta_n(aT^n b)$ then shows that $\mu^{\otimes n}(A_{n,b}) \gg n^{-C}$ for some constant $C \geq 0$.
(2) (drift control) To show that $A_{n,b}^{drift}$ intersects $A_{n,b}$ in a large proportion, we use the law of the angles, which will be proved in a later talk, and whose proof is based on the previous observation that $\mu^{\otimes n}(A_{n,b}) \gg n^{-C}$.
(3) (continuity control) This part is based on the equidistribution of pieces of fibers, which is the fact that for β^X-a.e. $(b,x) \in B^X$, the limit $\psi_{b,x} = \lim_{n\to\infty} \frac{1}{|A_{n,b}|} \sum_{a\in A_{n,b}} \mathbf{1}_K(aT^n b, a_1 \dots a_n b_n^{-1} \dots b_1^{-1} x)$ exists and satisfies $\int_{B^X} \psi_{b,x} \, \mathrm{d}\beta^X(b,x) = \beta^X(K)$. As will be explained in the next talk, this is a consequence of the inverse martingale theorem.

Finally, once it has been shown that ν_b is almost surely V_b-invariant, one has to upgrade this invariance to the equality $\nu_b = m_X$, almost surely. For this, one considers the map

$$\begin{array}{rccl} S: & B & \to & \mathcal{F} \\ & b & \mapsto & S_b = \operatorname{Stab}\nu_b \end{array}$$

where $\mathcal{F}$ is the set of closed subgroups of $X = \mathbb{T}^d$. Since $S_b = b_1 S_{Tb}$, the measure $S_*\beta$ is μ-stationary and μ-ergodic on the countable set $\mathcal{F}$. It follows that the support of $S_*\beta$ is finite and Γ-invariant, which forces $S_b = \mathbb{T}^d$ almost surely, by strong irreducibility of the action of Γ on $\mathbb{R}^d$. Therefore $\nu_b = m_X$ almost surely and hence $\nu = \int_B \nu_b \, \mathrm{d}\beta(b) = m_X$.

Equidistribution of pieces of fibers

Timothée Bénard

To prove the rigidity of meaures in the case of a random walk on the torus, Benoist and Quint introduce a σ-finite dynamical system $(B^{\theta,X}, \beta^{\theta,X}, T^{\theta,X})$ and a map $\sigma : B^{\theta,X} \to \mathcal{M}^{\sigma}_{\mathrm{proj}}(\mathbb{R})$ (which is a conditionnal measure along some $\mathbb{R}$-action). The point is to show that this map is $\beta^{\theta,X}$-almost surely constant. To this end, for $n \in \mathbb{N}$, $c \in B^{\theta,X}$, writing $F_{n,c} = \{c' \in B^{\theta,X},\ (T^{\theta,X})^n(c') = (T^{\theta,X})^n(c)\}$ the n-fiber going through the point c, Benoist and Quint prove that σ is constant on $F_{n,c}$. They deduce from this that σ is constant thanks to their exponential drift argument. It relies on some weak equidistribution phenomenon for the fibers and an explicit description of their behavior, that we present in this text.

Distribution of fibers. Let $(E, \mathcal{E}, m, T)$ be a probability dynamical system, or in other words a probability space together with a probability-preserving transformation. For $c \in E$, $n \geq 0$, write

$$F_{n,c} := \{c' \in E,\ T^n(c') = T^n(c)\}$$

the n-fiber going through c.

One wants to understand the distribution of the points of $F_{n,c}$ in E when n is very large and c is fixed. To this end, denote $\mathcal{Q}_n := T^{-n}(\mathcal{E})$ the sub σ-algebra of $\mathcal{E}$, whose atom containing c is exactly $F_{n,c}$. Then, for every measurable subset $A \in \mathcal{E}$, the conditionnal expectation $\mathbb{E}_m(1_A|\mathcal{Q}_n)(c)$ describes the proportion (with respect to m) of elements in $F_{n,c}$ which are actually in A. One can control the the distribution of fibers with the following theorem :

Theorem 1. *Let $\psi \in L^1(E, \mathcal{E}, m)$. Almost everywhere, one has the convergence:*

$$\mathbb{E}_m(\psi|\mathcal{Q}_n) \to \mathbb{E}_m(\psi|\mathcal{Q}_\infty)$$

where $\mathcal{Q}_\infty := \cap_{n\geq 0}\mathcal{Q}_n$.

Proof. For every $n \geq 0$, one has $\mathcal{Q}_{n+1} \subseteq \mathcal{Q}_n$. Apply a convergence theorem for martingals with respect to a decreasing sequence of sub σ-algebras. □

Distribution of pieces of fibers. When the dynamical system is of infinite measure, one can study the distribution of fibers in the whole space through their distribution in finite-measure windows.

We use the notations of the preceding paragraph, except that that m is only assumed to be σ-finite, not necessarily finite. Set $U \in \mathcal{E}$ a finite-measure subset, interpreted as a window through which we observe the fibers.

One gets a measure space $(U, \mathcal{E}_{|U}, m_{|U})$ and some trace sub σ-algerbas $\mathcal{Q}_{U,n} := \mathcal{Q}_{n|U}$, $\mathcal{Q}_{U,\infty} := \mathcal{Q}_{\infty|U}$. If $A \in \mathcal{E}_{|U}$, the conditionnal expectation $\mathbb{E}_{m_{|U}}(1_A|\mathcal{Q}_{U,n})(c)$ describes the proportion of elements of $F_{n,c}$ which are in A among those which are in U.

Theorem 2. *Let $\psi \in (U, \mathcal{E}_{|U}, m_{|U})$. Almost everywhere, one has the convergence :*

$$\mathbb{E}_{m_{|U}}(\psi|\mathcal{Q}_{U,n}) \to \mathbb{E}_{m_{|U}}(\psi|\mathcal{Q}_{U,\infty})$$

Proof. Same proof as in the case of fibers.

□

How do you use these theorems? First of all, if the sub σ-algebra $\mathcal{Q}_{U,\infty}$ is $m_{|U}$-trivial, one has that for every measurable subset $A \subseteq U$, and almost every $c \in U$:

$$\mathbb{E}_{m_{|U}}(1_A|\mathcal{Q}_{U,n})(c) \to \frac{m(A)}{m(U)}$$

So, for n large enough, the n-fiber going through c and restricted to U equidistributes in U with respect to m. In this case, we have an equidistribution of pieces of fibers.

In the proof of the exponential drift, one does not assume $\mathcal{Q}_{U,\infty}$ to be trivial. However, the previous theorem implies a weak form of equidistribution stating that fibers can note accumulate in subsets of small measure, and this is enough to make te proof work. More precisely, let $A \subseteq U$ be a measurable susbet such that $m(A) < \epsilon^2$. Then $\mathbb{E}(1_A|\mathcal{Q}_{U,\infty})$ is a function taking its values in $[0, 1]$ and whose integral against $m_{|U}$ is less than ϵ^2. One can deduce that there exists some

$U' \subseteq U$ of measure $m(U') > m(U) - \epsilon$ such that $\mathbb{E}(1_A|\mathcal{Q}_{U,\infty}) < \epsilon$ on U'. Thanks to Egoroff theorem, one can assume that the convergence in the preceding theorem is uniform on U'. One gets the existence of a rank $n_0 \geq 0$ such that for $n \geq n_0$, for every $c \in U'$, $\mathbb{E}_{m_{|U}}(1_A|\mathcal{Q}_{U,n})(c) < 2\epsilon$, in other words the n-fibers of points in U' do not accumulate in A as soon as n is big enough.

Computation of conditionnal measures. A key argument to prove the exponential drift is the law of the angles, that describes the drift between the n-fibers of two points $c, c' \in B^{\theta,X}$ which are close enough. To prove this law, one describes explicitely the n-fiber of a point and how it distributes with respect to the measure $\beta^{\theta,X}$. As the reasoning will involve a point c fixed once and for all, we shall prefer the point of view of conditionnal measures to the point of view of conditionnal expectations (even if they are more or less the same).

Proposition 1 (Conditionnal measures)**.** *Let $(E, \mathcal{E})$ be a standard borel space, $\mathcal{Q} \subseteq \mathcal{E}$ a sub σ-algebra, m a positive measure on $(E, \mathcal{E})$ such that $m_{|\mathcal{Q}}$ is σ-finite. There exists a $\mathcal{Q}$-mesurable map $E \to \mathcal{M}(E, \mathcal{E}), c \mapsto m_c$ such that :*

- $m = \int_E m_c \, dm(c)$
- *For $c \in E$, one has m_c concentrated on the atom of $\mathcal{Q}$ containing c.*

  Moreover, the family $(m_c)_{c \in C}$ is unique outside of zero-measure subset and it satisfies the following : For any function $\psi \in L^1(E, \mathcal{E}, m)$, one has ψ integrable against m_c for almost every c and the map $c \mapsto m_c(\psi)$ is a version of the conditionnal expectation $\mathbb{E}_m(\psi|\mathcal{Q})$.

  One calls m_c the conditionnal measure of m with respect to the sub σ-algebra $\mathcal{Q}$ at the point c.

Description of the fibers in the setting of Benoist-Quint. Fix $d \in \mathbb{N}_{\geq 2}$, denote $X = \mathbb{T}^d$, $G = SL_d(\mathbb{Z})$. Let μ be a probability measure on G. It induces a random walk on the torus X. Let ν be a probability measure on X which is μ-stationnary, μ-ergodic. Set $B = G^{\mathbb{N}}$, $\beta = \mu^{\otimes \mathbb{N}}$, $T : B \to B$ the shift. One wants to show that if μ satisfies some properties (finite support generating a strongly irreducible proximal subgroup of G), then ν has to be the Harr probability or the uniform probability on a finite orbit of Γ_μ (closure of the group generated by the support of μ). One comes to study the following dynamical system :

- $B^X := B \times X$
- $\beta^X := \int_B \delta_b \otimes \nu_b \, d\beta(b)$
- $T^X : B^X \to B^X, \ (b, x) \mapsto (Tb, b_1^{-1}.x)$

Here the measure β^X is a probability. Let us describe precisely the n-fibers. Fix $c \in B^X$. For $a \in G^n$, one can observe that $(aT^n b, \, a_1 \dots a_n b_n^{-1} \dots b_1^{-1}.x) \in F_{n,c}$. It is easy to check that all elements in $F_{n,c}$ have this form. Thus, the n-fiber going through c is parametrized by G^n. But in our setting, G^n has a natural measure, which is $\mu^{\otimes n}$, and one can ask if the measure β^X conditionned with respect to the n-fiber going through c is the same as $\mu^{\otimes n}$. The answer is yes :

Lemma. *Write* $\mathcal{Q}_n^X := (T^X)^{-n}(\mathcal{B}(B^X))$, $(\beta_{n,c}^X)_{c\in B^X}$ *a disintegration of* β^X *into conditionnal measures with respect to* $\mathcal{Q}_n^X$. *For* $c=(b,z)\in B^{\theta,X}$, $n\geq 0$, *define the bijection :*

$$h_{n,c}: G^n \to F_{n,c},\ a=(a_1,\dots,a_n)\to (aT^nb,\ a_1\dots a_n b_n^{-1}\dots b_1^{-1}.x)$$

Then, for almost every $c\in B^{\theta,X}$, *one has*

$$\beta_{n,c}^X = h_{n,c_\star}\mu^{\otimes n}$$

Proof. The map $B^X \to \mathcal{P}(B^X)$, $c\mapsto h_{n,c_\star}\mu^{\otimes n}$ is $\mathcal{Q}_n^X$-measurable, each image $h_{n,c_\star}\mu^{\otimes n}$ is concentrated on $F_{n,c}$. It remains to check that

$$\beta^X = \int_{B^X} h_{n,c_\star}\mu^{\otimes n}\ d\beta^X(c)$$

which is a straightforward computation.

□

It turns out the study of B^X is not enough. One has to add a third parameter with real values and consider the σ-finite dynamical system $B^{\theta,X}$ of the previous presentations. Recall its definition. Let $\theta : B\to\mathbb{R}$ denote some real-valued map on B and set :

- $B^{\theta,X} := B\times\mathbb{R}\times X$
- $\beta^{\theta,X} := \int_B \delta_{b,z}\otimes\nu_b\ d(\beta\otimes \mathrm{leb})(b,z)$
- $T^{\theta,X}: B^{\theta,X}\to B^X,\ (Tb, z-\theta(b), b_1^{-1}.x)$

This dynamical system is σ-finite but not finite. One describes the behavior of the fibers thanks to the distribution of pieces of fibers. To this end, consider $U\subseteq\mathbb{R}$ a bounded open subset of positive lebesgue measure and consider the restrictions :

- $B_U^{\theta,X} := B\times U\times X$
- $\beta_U^{\theta,X} := \beta^{\theta,X}_{|B_U^{\theta,X}}$

Let us describe the trace of n-fibers in the window $B_U^{\theta,X}$. Fix $c=(b,z,x)\in B_U^{\theta,X}$. Just as before, $F_{n,c}$ is parametrized by G^n. More precisely, $F_{n,c} := \{(aT^nb,\ z+\theta_n(aT^nb)-\theta_n(b),\ a_1\dots a_nb_n^{-1}\dots b_1^{-1}.x), a\in G^n\}$. Thus, $F_{n,c}\cap B_U^{\theta,X}$ identifies with $\{a\in G^n,\ \theta_n(aT^nb)\in U-z+\theta_n(b)\}$. This subset of G^n has a natural probability measure, which is $\mu^{\otimes n}$ once normalized. One can show that this normalized measure is well defined and is actually the conditional measure on the piece of fiber $F_{n,c}$:

Lemma. *Write* $\mathcal{Q}_{U,n}^{\theta,X} := (\mathcal{Q}_n^{\theta,X})_{|B_U^{\theta,X}}$, $(\beta_{U,n,c}^{\theta,X})_{c\in B_U^{\theta,X}}$ *a disintegration of* $\beta_U^{\theta,X}$ *into conditionnal measures with respect to* $\mathcal{Q}_{U,n}^{\theta,X}$.

For $c=(b,z,x)\in B^{\theta,X}$, $n\geq 0$, *write* $Q_{n,c} := \{a\in G^n,\ \theta_n(aT^nb)\in U-z+\theta_n(b)\}$ *and set :*

$$h_{n,c}: Q_{n,c}\to F_{n,c}\cap B_U^{\theta,X},$$
$$a=(a_1,\dots,a_n)\to(aT^nb,\ z+\theta_n(aT^nb)-\theta_n(b),\ a_1\dots a_nb_n^{-1}\dots b_1^{-1}.x)$$

Then, for almost every $c \in B^{\theta,X}$, one has $\mu^{\otimes n}(Q_{n,c}) > 0$ et

$$\beta^{\theta,X}_{U,n,c} = h_{n,c_\star} \frac{\mu^{\otimes n}}{\mu^{\otimes n}(Q_{n,c})}$$

Proof. The formula for $h_{n,c}$ makes sens for every $a \in G^n$. Thus, one can see $h_{n,c} : G^n \to B^{\theta,X}$. Just as in the previous case of B^X, the fiber $F_{n,c}$ identifies with G^n via $h_{n,c}$ and for almost every c, one has $h_{n,c_\star}\mu^{\otimes n} = \beta^{\theta,X}_{n,c}$. Then check that $\beta^{\theta,X}_{n,c}(B^{\theta,X}_U)$ is positive for almost every $c \in B^{\theta,X}$ and that one has the following (general) link between conditionnal measures before and after restriction to the window : for any measurable function $\psi : B^{\theta,X} \to [0,+\infty]$,

$$\beta^{\theta,X}_{U,n,c}(\psi) = \frac{\beta^{\theta,X}_{n,c}(\psi 1_{B^{\theta,X}_U})}{\beta^{\theta,X}_{n,c}(B^{\theta,X}_U)}$$

□

The law of the angles

Ilya Khayutin

1. Random Walks oh Homogeneous Spaces

In this talk we have discussed the law of the angles which is an important tool to control the magnitude and direction of the drift. Let G be a locally compact group, Γ a lattice in G and μ a Borel probability measure. Assume G is the product of p-adic and real algebraic groups. Our goal is to study the random walk on the homogeneous space ${}_{\Gamma}\backslash^{G}$ with law μ, where μ is a Borel probability measure on G. We denote by

$$\Gamma^+_\mu = \overline{\langle \operatorname{supp} \mu \rangle^+}$$
$$\Gamma_\mu = \overline{\langle \operatorname{supp} \mu \rangle}$$

the closed semi-group and the closed group generated by the support of the measure μ. Let H be the Zariski closure of Γ_μ. The ultimate objective of the law of the angles is to understand the drift, with respect to the random walk generated by μ, of a vector in an algebraic representation of H, conditioned on the growth of its norm. In particular, the space ${}_{\Gamma}\backslash^{G}$ plays no role in this talk.

For simplicity, we restrict to the case that $H = \mathbf{SL}(d, \mathbb{R})$, although the results hold mutatis mutandis for any finite product of semi-simple groups over characteristic 0 local fields. Additional difficulties arise when H is not split or when it has non-archimedean factors.

2. The Cartan Decomposition

Write the Cartan decomposition of H as KA^+K, where $K = \mathbf{SO}_d(\mathbb{R})$ is the maximal compact, A is the group of unimodular diagonal matrices and A^+ is the positive Weyl chamber

$$A^+ = \left\{ \operatorname{diag}(\lambda_1, \dots, \lambda_d) \mid \lambda_1 \geq \lambda_2 \geq \dots \lambda_n , \prod_{i=1}^{n} \lambda_i = 1 \right\} .$$

Set $\mathfrak{a} := \operatorname{Hom}(A, \mathbb{R})^*$ and denote by $\omega \colon A \to \mathfrak{a}$ the evaluation map. In this setting, the vector space $\mathfrak{a}$ can be identified with the Lie algebra of A and ω is the logarithm map. Denote by $\mathfrak{a}^+ \subset \mathfrak{a}$ the closed positive cone, such that $\omega(A^+) = \mathfrak{a}^+$. We define $\kappa \colon H \to \mathfrak{a}^+$ using the Cartan decomposition. If $H \ni h = k_h z_h l_h$ with $k_h, l_h \in K$, $z_h \in A^+$ then

$$\kappa(h) = \omega(z_h) .$$

We assume henceforth that μ has a finite exponential moment, i.e. $\exists \tau > 0$ such that

$$\int_H exp(\tau \|\kappa(h)\|) \, \mathrm{d}h < \infty .$$

3. Flag Varieties and Density Points

Let $P < H$ be a maximal parabolic corresponding to A^+, i.e. P is the group of unimodular upper triangular matrices. Set $\mathcal{P} = {}^{H}\!/_{P}$ to be the corresponding flag variety. There is an obvious action of H on $\mathcal{P}$, this action is proximal and there is a unique μ-stationary Borel probability measure that we denote by ν_μ. An important notion is that of the density point in $\mathcal{P}$ attached to $h \in H$.

Definition. *For any $h \in H$ write the Cartan decomposition $h = k_h z_h l_h$ with $k_h, l_h \in K$ and $z_h \in A^+$. Define the (stable)* density point *of h as*

$$\xi_h^M = k_h P \in {}^{H}\!/_{P} = \mathcal{P}$$

Notice that ξ_h^M does not depend on the representative of k_h in $K/(K \cap A)$ as $A < P$.

There is also a similar notion of an unstable density point in the dual flag variety $\check{\mathcal{P}} = {}^{H}\!/_{P^t}$.

The importance of the density point is that there is an open subset of $\mathcal{P}$ which under the action of h is contracted towards ξ_h^M. The complement of the contracted space is a proper Schubert sub-variety. Let V be an irreducible algebraic real representation of H equipped with a norm $\| \cdot \|$ satisfying standard compatibility conditions. Let $\chi \colon \mathfrak{a} \to \mathbb{R}$ be the (logarithm of) highest weight of the representation V. Using the the theory of the highest weight we can see that the growth of the norm $\|h.v\|/\|v\|$ for $v \in V$ is approximately $e^{\chi(\kappa(h))}$, unless v is very close to a proper hyperspace that can be defined using the unstable density point of h.

Moreover, the Large Deviations Principle implies several quantitative results about the drift of the random walk on $\mathcal{P}$ generated in μ. In particular, except for an exponentially small in n set of $\eta \in \mathcal{P}$, the random walk $h = h_0 h_1 \cdot \ldots \cdot h_{n-1}.\eta$ of length n will be very close to the stable density point $\xi_h^M \in \mathcal{P}$.

4. The Multi-norm Cocycle

We introduce the shift dynamical Bernoulli system attached to the random walk. The symbol space is $B = H^{\mathbb{N}}$ with the σ-algebra $\mathrm{Borel}(H)^{\otimes\mathbb{N}}$ and the product measure $\mu^{\otimes\mathbb{N}}$. The left-shift transformation is denoted by T. We need the measurable map

$$\xi\colon B \to \mathcal{P}$$

which is defined uniquely β-a.e. by the property

$$\xi_b = b_0 \xi_{Tb}$$

for $b = (b_0, b_1, \ldots) \in B$. Recall that the the probability measures $b_0 \cdot b_1 \cdot \ldots \cdot b_n - 1.\nu_\mu$ converge weak-$*$ for a.e. $b \in B$ to a Dirac δ-measure on $\mathcal{P}$. This point mass is supported on ξ_b.

The definition of the cocycle requires us to fix a Borel section $s\colon {}^{H}\!/_{U} \to \mathcal{P} = H/P$, where U is the group of upper triangular unipotent matrices ($P = AU$). We require the section to satisfy the relation

$$s(kP) \in kU$$

for all $k \in K$. Such a section exists because of the Iwasawa decomposition $H = KAU$.

We can now define the multi-norm cocycle $\sigma\colon H \times \mathcal{P} \to A$ using the property

$$hs(\eta) = s(h\eta)\sigma(h, \eta)$$

for all $h \in H$ and $\eta \in \mathcal{P}$. This cocycle generalizes the norm cocycle of a single representation of H. Its utility lies in the fact that it allows to control the norm growth of vectors in any nice representation of H. Let V be an irreducible algebraic representation of H with a K-invariant inner-product, such that all A weight spaces are orthogonal. Then for every vector v in the highest-weight subspace of H the ratio $\|h.v\|/\|v\|$ can be computed using σ.

5. The Skew-product Dynamical System

Define the map $\theta\colon B \to A$ using

$$\theta(b) = \sigma(b_0, \xi_{Tb})$$

Define also $\theta^n(b) \coloneqq \prod_{0 \le k < n} \theta(T^k b) = \sigma(b_0 \cdot \ldots \cdot b_{n-1}, \xi_{T^n b})$ for all $n \in \mathbb{N}$. We define the skew-product $B^\theta = B \times A$ with the product σ-algebra $\mathcal{B}^\theta$ (the σ-algebra on A is the Borel one). The measure β^θ on B^θ is the product of β and a fixed Haar measure on A. Notice that this is not a probability measure. At last we define the skew-shift $T^\theta(b, z) = (Tb, \theta(b)^{-1} z)$. The measure β^θ is invariant under the skew-shift.

6. The Law of the Angles

We are ready to consider the σ-algebras with respect to which we need to condition in the law of the angles. The usefulness of the law of the angles is that it establishes equidistribution of the density points on $\mathcal{P}$ even if we only consider those walks whose multi-norm cocycle is close to the multi-norm cocycle of a fixed generic walk.

We set $Q_n^\theta = \left(T^\theta\right)^{-n}(\mathcal{B}^\theta)$. A Q_n^θ-atom of a point $(b,z) \in B \times A$ is the set of all walks (a,z') such that $a_i = b_i$ for all $i \geq n$. Fix a bounded convex set $C \subset \mathfrak{a}$ and set $\widetilde{C} = \omega^{-1}(C) \subset A$. Define $B^C = B \times \widetilde{C}$ and $Q_n^C = Q_n^\theta \cap B \times \widetilde{C}$. An atom of $(b,z) \in B^C$ for the σ-algebra Q_n^C is all the walks $(a,z') \in B^C$ such that $a_i = b_i$ for all $i \geq n$ and $\theta_n(b)^{-1} z \in \widetilde{C}$. Literary, the atom is the set of all walks that shadow b after time n and whose cocycle is C-close to the cocycle of b at time n.

We can finally state the law of the angles. The proof is a delicate combination of several effective equidistribution theorems. A crucial feature is that the rate of the drift of θ_n for a typical walk is bounded by the law of the iterated logarithm. This drift is not too large so that one can apply a local limit theorem to understand the probability density of β^θ in a vicinity of the typical point.

Theorem 1 (Law of the Angles)**.** *Consider the probability measure*

$$\beta^C = \frac{1}{\beta^\theta(B^C)} \beta^\theta \restriction_{B^C} .$$

For β^C-a.e. $c = (b,z) \in B^C$ denote by $\beta^C_{c,n}$ the conditional measure of β^C at the point c with respect to the σ-algebra Q_n^C. Then for any continuous function $\varphi \colon \mathcal{P} \to \mathbb{C}$ and for β^C-a.e. $(b,z) \in B$

$$\int_{B^C} \varphi(\xi^M_{\beta'_{n-1} \cdots \cdot b'_0}) \, \mathrm{d}\beta^C_{n,c}(b',z') \to_{n\to\infty} \int_{\mathcal{P}} \varphi \, \mathrm{d}\nu .$$

An analogous statement holds for the unstable density points.

Stationary measures on the space of lattices

Homin Lee

Before this talk, we proved Theorem 1 in the first lecture, which states classification of stationary measures, orbit closure, and equidistribution results on $X = \mathbb{T}^d$ torus case.

In this talk, we will prove following theorem on

$$X = \mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$$

using same strategy on Torus case. During the proof, we can realize two issues arise in $X = \mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$ case. Before going further, let me state the theorem again. We will assume that Γ is generated by compact supported probability measure μ on $SL(n,\mathbb{R})$ and Γ is Zariski dense in $SL(n,\mathbb{R})$. Let ν_X be the Haar measure on X.

Theorem 1 ([1],[2]). *For any $x_0 \in X$,*

(1) *Γx_0 is dense in X or finite.*
(2) *When Γx_0 is dense in X, for $\mu^{\otimes \mathbb{N}}$-almost every sequence $g_i \in A$, the trajectory $x_n = g_n \cdots g_1 x_0$ equidistributes in X. In other words, $\nu_n \to \nu_X$.*
(3) *When Γx_0 is dense in X, $\frac{1}{n}\sum_{k=0}^{n-1} \mu^{*k} * \delta_{x_0} \to \nu_X$, i.e. the average of the law of the first n points converges weakly to ν_X.*
(4) *If ν is atom-free, μ-stationary probability measure, then $\nu = \nu_X$.*

Recall the strategy for the proof in Torus case. Almost every strategy, including exponential drift argument, still available in this case. However, there are 2 main differences for $X = \mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$ cases. First of all, due to absence of compactness, there may be "Escape of mass". So we may not be able to get probability measure of *weak** limit of sequence of probability measure. On the other hand, in the torus case, we use the fact that there are only countable subtori in torus. We need to find similar argument for $X = \mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$ case.

1. Non-Escape of mass

First we want to prove (1),(2) and (3) assume (4) in the theorem. More precisely, we want to prove (2) and (3) assume (4). In the torus case, we can ignore the issue about escape of mass. Since Torus is compact, any *weak**-limit of sequence of probability measure is probability measure. Now in $X = \mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$ case, we have to show that any $weak^* - limit$ is still probability measure.

For this issue, we can use the argument in Lecture "Recurrence in Law". We constructed function $f : X \to [0,\infty]$ which holds contraction hypothesis on previous lecture. More precisely f satisfies $P_\mu f \le af + b$ for the Markov operator P_μ. As explained in the previous "Recurrence in Law" lecture, this guarantees that the sequence of the probability measures in the statement (2) and (3) are tight. So that we can prove that any *weak**-limit of that sequence is indeed probability measure. After resolve this difference, the remaining part of the proof is almost same in the Torus case.

Remark Indeed, similar argument using function which satisfies contraction hypothesis is also used in the proof of positive μ-instability of diagonal which is explained in previous Nondegeneracy of ν_b lecture. In order to prove positive μ-instability of diagonal in $X = \mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$ case, we can construct appropriate function for that purpose.

2. Countability

Let me briefly recall that where and how can we use the fact that there are only countable subtori in torus. In the proof of torus case, we can prove following key step using draft argument. As previous talks, let μ be a finitely supported probability measure on $\mathrm{SL}(d,\mathbb{Z})$, Γ is semigroup generated by support of μ and further assume Γ is strong irreducible.

Theorem 2 (Key step). *Under same notation as previous lectures, ν_b is V_b-invariant and V_b is non trivial.*

We want to prove that the only non-atomic μ stationary measure on $\mathbb{T}^d$ is Haar measure from [Key step]. First we can construct μ-stationary measure on the space F of non-trivial subtori,

$$F = \{S : S \text{ is non zero subtori on } \mathbb{T}^d\}$$

We know that F is countable, and we use following fact.

Lemma. *If δ is μ-stationary measure on countable set then δ is indeed Γ invariant and finitely supported.*

So we can conclude that Γ stables finite subtori, in other words, Γ stables finite subspaces of $\mathbb{R}^d$. Using strong irreduciblility of Γ, one can show that the only non-atomic μ-stationary measure is Haar measure.

Now lets return to the $X = \mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$ case. Lets $G = \mathrm{SL}(n,\mathbb{R})$. Using almost same argument in torus case, we can prove following key step.

Theorem 3 (Key step)**.** *Using same notations in the previous lectures, ν_b is $exp(V_b)$-invariant and V_b is non trivial.*

Since we are now in $X = \mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$ case, we have to think action of V_b on X through exponential map. The key observation is that $exp(V_b)$ is indeed unipotent subgroup of $\mathrm{SL}(n,\mathbb{R})$ so that we can use Ratner's theorem.

For instance $n = 2$ case, there are only 2 possibilities. If ν_b is $G = \mathrm{SL}(2,\mathbb{R})$-invariant for β-a.e. b, then ν is Haar measure so we are done. Otherwise, thanks to Ratner's theorem, β almost surely, ν_b is the average of ν_Y for closed unipotent orbits $Y \subset X$. The group $G = \mathrm{SL}(2,\mathbb{R})$ acts on the space F of closed unipotent orbits transitively. So,

$$F = \{\text{Closed unipotent orbits } Y \subset X\} \simeq G/U$$

for standard one parameter unipotent subgroup U. Also we know that $G/U \simeq \mathbb{R}^2 - \{0\}$. We can construct μ-stationary measure on F as before. However, this gives contradiction due to the fact that the only μ-stationary probability measure on $\mathbb{R}^2$ is δ_0.

For general n, lets define

$$F = \{\alpha \in Prob(X); S_\alpha \neq \{1\}\,, \alpha \text{ is } S_{\alpha,u} \text{ ergodic }, \ \alpha \text{ is supported on } S_\alpha \text{ orbit}\}$$

Where S_α is connected component of the identity in the stabilizer of α in G, with respect to the action by translations on X and $S_{\alpha,u}$ is to be the subgroup of S_α generated by the one-parameter unipotent subgroups of S_α. Then by [3], we know that there are only countably many orbits of G action on F. So using similar argument on the space of orbits F/G and the fact that $\mathrm{SL}(n,\mathbb{R})$ is connected almost simple, we can deduce that the only non atomic μ-stationary measure is Haar measure on X.

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Diophantine Approximation on Cantor Sets

Felipe A. Ramírez

Let G be a real semisimple Lie group, $\Lambda \subset G$ a lattice. Fix elements $h_1, \dots, h_t \in G$ and probability measure μ on $E := \{1, \dots, t\}$. Yves Benoist and Jean-François Quint proved that if the the sub-semigroup $\Gamma \subset G$ spanned by $\{h_1, \dots, h_t\}$ is Zariski dense in G, then for any $x \in X := G/\Lambda$ and $\mu^{\otimes \mathbb{N}}$-almost every $(i_1, i_2, \dots) \in E^{\mathbb{N}}$, the sequence

$$\{h_{i_n} \dots h_{i_1} x : n \in \mathbb{N}\} \tag{1}$$

is equidistributed in X with respect to m, the probability measure induced by Haar measure of G (see [1, Theorem 1.3]). Adapting the methods of Benoist and Quint, David Simmons and Barak Weiss obtained the same conclusion in the absence of the Zariski density assumption, but in the presence of certain other assumptions regarding the adjoint action of of the h_i on the Lie algebra of G. In particular, they proved this in the case where $G = \mathrm{SL}_{d+1}(\mathbb{R})$ and $\Lambda = \mathrm{SL}_{d+1}(\mathbb{Z})$, and each h_i has the form

$$h_i = \begin{bmatrix} c_i O_i & \mathbf{y}_i \\ 0 & c_i^{-d} \end{bmatrix} \in G \quad (i = 1, \dots, t), \tag{2}$$

where $c_i > 1$, $\mathbf{y}_i \in \mathbb{R}^d$, and $O_i \in \mathrm{SO}_d(\mathbb{R})$ (see [4, Theorem 1.1]); this last result has a consequence for Diophantine approximation on fractals, as we shall presently discuss.

First, we recall some background from Diophantine approximation. The most basic theorem in this field is due to Dirichlet (1840s); it implies that for any $\mathbf{x} \in \mathbb{R}^d$ there are infinitely many $(\mathbf{p}, q) \in \mathbb{Z}^d \times \mathbb{N}$ such that $\|q\mathbf{x} - \mathbf{p}\|_\infty < q^{-1/d}$. Replacing $q^{-1/d}$ with some other shrinking function of q invariably results in a statement that is true for Lebesgue-almost every $\mathbf{x} \in \mathbb{R}^d$ or Lebesgue-almost no $\mathbf{x} \in \mathbb{R}^d$. For example, consider the set of badly approximable vectors:

$$\mathrm{BA}_d = \{\mathbf{x} \in \mathbb{R}^d : (\exists c := c(x))(\forall (\mathbf{p}, q) \in \mathbb{Z}^d \times \mathbb{N})\, \|q\mathbf{x} - \mathbf{p}\|_\infty \geq c q^{-1/d}\}.$$

It is well-known that BA_d is non-empty, and that it has Lebesgue measure 0 (but full Hausdorff dimension!). For another example, consider the set of very well approximable vectors:

$$\mathrm{VWA}_d = \{\mathbf{x} \in \mathbb{R}^d : (\exists \epsilon > 0)(\exists^\infty (\mathbf{p}, q) \in \mathbb{Z}^d \times \mathbb{N})\, \|q\mathbf{x} - \mathbf{p}\|_\infty < q^{-1/d - \epsilon}\}.$$

Like BA_d, the set VWA_d is non-empty and has Lebesgue measure 0.

There is a tradition in Diophantine approximation of investigating whether generic properties of $\mathbb{R}^d$ are inherited by subsets. For example, it is well-known that if $M \subset \mathbb{R}^d$ is a non-degenerate submanifold supporting a natural measure μ_M, then $\mu_M(\mathrm{VWA}_d) = 0$. This was proved by Kleinbock and Margulis in answer to a long-standing conjecture of Sprindzuk, using methods from homogeneous dynamics [3]. Non-degenerate (and degenerate) submanifolds of $\mathbb{R}^d$ have been studied a great deal and, indeed, it is also known that one always has $\mu_M(\mathrm{BA}_d) = 0$, as well as many other generic Diophantine properties. Another family of subsets of great current interest is fractals, and here there are still many omissions in the literature, notably the question of the s-dimensional Hausdorff measure of the set of badly approximable vectors lying on an s-dimensional fractal. Previous to the work of Simmons and Weiss, it was known that the $(\dim_H C_n)$-dimensional Hausdorff measure of $\mathrm{BA}_1 \cap C_n$ is 0, where $n \in \mathbb{N}$ and C_n is the middle-nth Cantor set [2]. Their methods relied heavily on the $\times n$-invariance of C_n, and so did not extend to many other simple self-similar sets, like "middle-ϵ" Cantor sets, translates of Cantor sets, or self-similar sets living in higher dimensional Euclidean spaces, like the Koch snowflake or the Sierpinski gasket. All these, and many others are now covered by the following theorem of Simmons and Weiss.

Theorem 1 ([4, Theorem 1.2])**.** *Let $\mathcal{K} \subseteq \mathbb{R}^d$ be the limit set of an irreducible finite system of contracting similarity maps satisfying the open set condition, let $s = \dim_H(\mathcal{K})$, and let $\mu_{\mathcal{K}}$ denote the restriction to $\mathcal{K}$ of s-dimensional Hausdorff measure. Then $\mu_{\mathcal{K}}$-a.e. $\mathbf{x} \in \mathcal{K}$ is not badly approximable, and is moreover of generic type.*

The open set condition commonly appears in results of fractal geometry. In particular, it guarantees that the measure $\mu_{\mathcal{K}}$ is the push forward of some measure $\mu^{\otimes\mathbb{N}}$ on an appropriate sequence space $E^{\mathbb{N}}$ under the fractal's coding map $\pi : E^{\mathbb{N}} \to \mathcal{K}$. That is, the measure is Bernoulli. The alphabet E is simply the indexing set for the system of contracting similarities $\Phi = \{\phi_1, \dots, \phi_t\}$, and the coding map is

$$\pi(i_1, i_2, \dots) = \lim_{k\to\infty} \phi_{i_1} \circ \cdots \circ \phi_{i_k}(0).$$

For example, the familiar coding of the standard Cantor set takes $(0, 1, 1, 0, \dots)$ to the point in $[0, 1]$ with ternary expansion $0.0220\dots$. In this case $\phi_0(x) = x/3$ and $\phi_1(x) = (x + 2)/3$.

The point of contact between Diophantine approximation and homogeneous dynamics is "Dani's Correspondence," which says that $\mathbf{x}$ is a badly approximable vector in $\mathbb{R}^d$ if and only if the trajectory $\{a_t u_{\mathbf{x}} \Lambda\}_{t\geq 0}$ is bounded in X, where

$$a_t := \begin{bmatrix} e^t I_d & 0 \\ 0 & e^{-dt} \end{bmatrix} \qquad \text{and} \qquad u_{\mathbf{x}} := \begin{bmatrix} 1 & \mathbf{x} \\ 0 & 1 \end{bmatrix}.$$

Therefore, in proving Theorem 1, the goal should be to show that for $\mu^{\otimes\mathbb{N}}$-almost every $b = (i_1, i_2, \dots) \in E^{\mathbb{N}}$, the trajectory $\left\{a_t u_{\pi(b)} \Lambda\right\}_{t\geq 0}$ is *unbounded.* But it is shown that this is equivalent to the unboundedness of the random trajectory (1), where the h_i are as in (2) and act on $\mathbb{R}^d$ as ϕ_i^{-1}, and one takes x to be the identity coset in $X := G/\Lambda$. From here, the equidistribution theorem of Simmons and Weiss mentioned at the beginning of this discussion implies that for $\mu_{\mathcal{K}}$-almost every point $\mathbf{x} \in \mathcal{K}$, the trajectory $\{a_t u_{\mathbf{x}} \Lambda\}_{t \geq 0}$ is unbounded. In fact, with more work, one can show it is also *equidistributed*, for which the terminology is that $\mathbf{x}$ is of "generic type." This proves Theorem 1.

It should be noted that the work of Simmons and Weiss is more extensive than what has been discussed here. Their results on random walks apply in more generality than can be reasonably explained in a short discussion, and also have consequences for Diophantine approximation on fractals other then self-similar ones, for example limit sets of systems of Möbius transformations.

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Space of rank two discret subgroups in $\mathbb{R}^3$

Irving Calderon

Def: A 2-lattice of $\mathbb{R}^3$ is a rank 2 discrete subgroup of $\mathbb{R}^3$.

$$X := \{\text{homothecy classes of 2-lattices of } \mathbb{R}^3\} \tag{1}$$

$$= \mathbb{R}^\times \backslash \{\text{2-lattices of } \mathbb{R}^3\} \tag{2}$$

For any 2-lattice Λ we denote by $[\Lambda]$ its class in X.

The group $G = SL(3, \mathbb{R})$ acts transitively on X. We endow X with its topology of homogeneous G-space.

Objective: Study the actions of subsemigroups of G on X.

Consider the map

$$\pi : X \to \{\text{2-dim subspaces of } \mathbb{R}^3\} =: \mathbb{P}^*$$

that sends $[\Lambda]$ to the plane containing Λ. Observe that X is a fiber bundle over $\mathbb{P}^*$ with fiber $SL(2, \mathbb{R})/SL(2, \mathbb{Z})$.

Theorem 1. *Let Γ be a compactly generated Zariski dense subsemigroup of G. There is a non-empty closed subset K of $\mathbb{P}^*$ such that*

$$\bigcap_{x \in X} \overline{\Gamma x} = \pi^{-1}(K).$$

Example: $\Gamma = SL(3,\mathbb{Z})$ is a lattice in G, so it is Zariski dense by Borel Density Theorem.

This result is proved by studying random walks on X.

Let $\mu \in \mathcal{P}(G)$ be a compactly supported probability measure on G, let Γ_μ be the closed semigroup generated by the support of μ, and let H_μ be the Zariski closure of Γ_μ.

OBJECTIVE': Describe all the μ-stationary Borel probability measures on X.

The article considers two cases:

(1) CASE 1: $H_\mu = G$.
(2) CASE 2: $H_\mu = SO(2,1)$.

Since the space X is not compact, the existence of μ-stationary measures is not immediate. Let us exhibit one: Let $\nu_{\mathbb{P}}^*$ be a μ-stationary probability measure on $\mathbb{P}^*$. In fact there is only one such probability measure since in both cases the action of Γ_μ on $(\mathbb{R}^3)^*$ is strongly irreducible and proximal because this is so for G and $SO(2,1)$, and it is μ-proximal. For any plane $P \in \mathbb{P}^*$, let η_P be the unique $SL(P)$-invariant probability measure on the fiber $\pi^{-1}(P)$.

Then

$$\nu_X = \int_{\mathbb{P}^*} \eta_P \mathrm{d}\nu_{\mathbb{P}}^*(P)$$

is a μ-stationary probability measure on X. Indeed, since $g_*\eta_P = \eta_{gP}$ for $g \in G$ and any $P \in \mathbb{P}^*$, then:

$$\mu * \nu_X = \int_{\mathbb{P}^*} \nu_P \mathrm{d}\mu * \nu_{\mathbb{P}}^*(P) = \int_{\mathbb{P}^*} \nu_P \mathrm{d}\nu_{\mathbb{P}}^*(P) = \nu_X.$$

[Sargent-Shapira] Let μ be a compactly supported B.p.m. on G.

- Case I: ν_X is the only μ-stationary B.p.m. on X.
- Case II: There might be others...

Example. *Let Q be the quadratic form*

$$Q(x,y,z) = 2xz - y^2.$$

Consider the set

$$C := \{\textit{planes} \quad Q-\textit{orthogonal to} \quad Q-\textit{isotropic lines}\}$$

Recall the following notation:

$$B = G^{\mathbb{N}}, \beta = \mu^{\otimes \mathcal{N}}, S : B \to B \quad \text{the shift map.}$$

Theorem 2. *Let μ be a compactly supported B.p.m. on G such that case I or II holds. Consider a μ-stationary and μ-ergodic B.p.m. ν on X.*

- *If ν_b has no atoms for β-almost any $b \in B$, then $\nu = \nu_X$.*
- *In case I, ν_b has no atoms β-almost surely.*
- *Case II+ technical conditions $\Rightarrow$ ν is either ν_X, or there exists an integer k such that for β-almost any $b \in B$, ν_b is a uniform probability measure on a set of size k.*

- *In both cases, for any $x \in X$, any cluster point of the sequence*

$$\left(\frac{1}{n}\sum_{j=1}^{n}\mu^{*j} * \delta_x\right)_n$$

is a μ-stationary B.p.m. on X.

Rigidity of SL(2)-orbits in the moduli spaces of flat surfaces after Eskin, Mirzakhani and Mohammadi

Rene Rühr

We will start by giving the definition of a translation surface as it was introduced in [1]. Fix a closed topological surface S of genus g and a finite subset $\Sigma \subset S$. A flat chart on S with a conical singularity is a triple (U, φ, ψ) where U is an open set in S, φ is a homeomorphism from U to an open subset of $\mathbb{C}$ containing 0 and $\psi : U \to \mathbb{C}$ is the map $x \mapsto \varphi(x)^{\alpha+1}$, where α is an integer ≥ 0, uniquely determined by φ and ψ. The set Σ defines precisely those charts for which $\alpha > 0$. Now define a maximal atlas with respect to a compatibility condition, which asks (U_1, φ_1, ψ_1) and (U_2, φ_2, ψ_2) to be compatible if there exists $c \in \mathbb{C}$ for which, for all $x \in U_1 \cap U_2$, one has $\psi_2(x) = \psi_1(x) + c$.

This structure in particular induces a holomorphic atlas, giving a Riemann surface structure, and a holomorphic differential by pullback of dz.

There are two natural actions on the collection of these translation atlases of the same singularity data: Postcomposition by $\mathrm{GL}_2^+(\mathbb{R})$ and precomposition by self-homormorphism of S. Modding out the latter, we obtain a stratum $\mathcal{H}$ of the moduli space of translation surfaces. It has an orbifold structure, with charts locally identified with the relative cohomology space $H^1(S, \Sigma, \mathbb{R}^2)$, whose $\mathbb{Z}$-structure induces a natural volume notion on $\mathcal{H}$ which is invariant under $\mathrm{SL}_2(\mathbb{R})$.

An orbit closure classification is obtained by Eskin-Mirzakhani-Mohammadi [3]. Its metric variant, classifying all $\mathrm{SL}_2(\mathbb{R})$-invariant finite measures when restricted to area 1 translation surfaces is obtained in [2].

This *magic wand* is an analogue of Ratner's theorems on homogeneous spaces that classifies all orbit closures of a semi-simple group H inside a bigger ambient homogeneous space $X = G/\Gamma$ to be algebraic, that is, $\overline{Hx} = Lx$ for some group L generated by unipotent elements, and every H-invariant finite measure μ is Haar for some closed orbit Lx.

For moduli space of translation surfaces, the linear sequence of subspaces $\mathfrak{h} < \mathfrak{l} < \mathfrak{g}$ implied from the discussion in the previous paragraph is replaced by subspaces of $H^1(S, \Sigma, \mathbb{R}^2)$ on which the moduli space is modeled on. The theorem of Eskin-Mirzakhani states that any $\mathrm{SL}_2(\mathbb{R})$-invariant finite measure is the restriction of Lebesgue to a subspace of $H^1(S, \Sigma, \mathbb{R}^2)$.

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Stationary measures for group actions on surfaces

Jonathan DeWitt

In this talk, we describe some recent work by Aaron Brown and Federico Rodriguez Hertz [1] that provides an analog of the results of Benoist and Quint in the smooth setting. Let M be a two dimensional closed surface and consider a Borel measure μ on $\mathrm{Diff}^2_{\mathrm{vol}}(M)$ subject to a mild moment condition. The measure μ drives the dynamics of the associated Markov process on M. We let ν be a stationary measure for this associated process.

It is convenient to view the random dynamical system as a skew product $F\colon (\mathrm{Diff}^2(M))^{\mathbb{Z}} \times M \to (\mathrm{Diff}^2(M))^{\mathbb{Z}} \times M$. We then write f^n_ω for the dynamics of the nth iterate of the system on M given the sequence of diffeomorphisms $\omega \in (\mathrm{Diff}^2(M))^{\mathbb{Z}}$.

We introduce the Lyapunov exponents of this random dynamical system and the associated sequence of Lyapunov subspaces. We say that the stationary measure ν is *hyperbolic* if the Lyapunov exponents with respect to μ are non-zero. This assumption corresponds to the conclusion of lecture 8: that the first Lyapunov exponent is positive. Given that the measure is hyperbolic, we write $E^s_\omega(x)$ for the stable subspace at the point x. In the volume preserving case, the main result is the following:

Theorem 1. *Let $\Gamma \subset \mathit{Diff}^2(M)$ be a subgroup and assume Γ preserves a probability measure m equivalent to the Riemannian volume on M. Let μ be a probability measure on $\mathit{Diff}^2(M)$ with $\mu(\Gamma) = 1$ and satisfying a moment condition. Let ν be an ergodic, hyperbolic, μ stationary Borel probability measure on M. Then at least one of the following holds:*

(1) *ν has finite support,*
(2) *the stable distribution $E^s_\omega(x)$ is non-random, or*
(3) *ν is absolutely continuous and is μ-a.s. invariant.*

Further, in the third case, up to scaling, ν is the restriction of volume to a positive measure subset.

In the case that one may eliminate the first two possibilities above, then one obtains a precise description of the invariant measures. One way of doing this is via the following condition. We say that a measure μ on $\mathrm{Diff}(M)$ is *uniformly* *expanding* if there exist c and N such that for all $x \in M$ and all $v \in T_x^1 M$,

$$\int \log \|D_x f^n v\| \, d\mu^{(n)} > c > 0.$$

The results in the non-volume preserving case are necessarily more difficult to state as one no longer has volume as an obvious candidate for an invariant measure. If a system does not preserve volume, then the next natural candidate for an invariant measure is an SRB measure. The analog of an SRB measure in this case will be a measure that is fiberwise SRB in the following sense. We may define the limit measures ν_ω exactly as in the homogeneous case as described in lecture 12. Then for a limit measure ν_ω, we may further disintegrate this measure along the unstable manifolds for the word ω. We say that ν is *fiberwise SRB* if almost surely the disintegrated measures on unstable manifolds are absolutely continuous with respect to volume. The result is then:

Theorem 2. *Let $\Gamma \subset \mathrm{Diff}^2(M)$ be a closed surface and let μ be a Borel probability measure on $\mathrm{Diff}^2(M)$ satisfying a moment condition. Let ν be an ergodic, hyperbolic, μ-stationary Borel probability measure on M. Then at least one of the following holds:*

(1) *ν is finitely supported,*
(2) *the stable disribution $E^s_\omega(x)$ is non-random,*
(3) *ν is fiberwise SRB.*

The third condition is motivated well by the work of Ledrappier and Young on SRB measures, see [2] and [3].

Finally, let us comment on how it is possible to run an exponential drift argument in this setting. To do this, one parametrizes the unstable manifolds $W^u_\omega(x)$ by maps $H^u_\omega \colon \mathbb{R} \to W^u_\omega(x)$. Using this family of maps, one pulls back the disintegration of ν_ξ along $W^u_\omega(x)$ to $\mathbb{R}$. One then uses these parametrizations to run an exponential drift argument to show that the pulled back measure on $\mathbb{R}$ is Haar. One reason this is feasible is that the parameterizations H^u_ω linearize the dynamics on the entire unstable leaf.

There are many other differences between the smooth and homogeneous settings. One important tool that is missing in the smooth setting is the "Law of the Angles," which is avoided by using a much cruder result about the randomness of the stable and unstable distributions. Another important tool used by Brown and Rodriguez Hertz, which did not appear in earlier talks is entropy.

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