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Global Solutions to Stochastic Wave Equations with Superlinear Coefficients

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GLOBAL SOLUTIONS TO STOCHASTIC WAVE EQUATIONS WITH SUPERLINEAR COEFFICIENTS

ANNIE MILLET AND MARTA SANZ-SOLÉ

Abstract. We prove existence and uniqueness of a random field solution $(u(t,x);(t,x)\in[0,T]\times\mathbb{R}^d)$ to a stochastic wave equation in dimensions $d=1,2,3$ with diffusion and drift coefficients of the form $|x|\big(\ln_+(|x|)\big)^a$ for some $a>0$. The proof relies on a sharp analysis of moment estimates of time and space increments of the corresponding stochastic wave equation with globally Lipschitz coefficients. We give examples of spatially correlated Gaussian driving noises where the results apply.

1. Introduction

In this paper, we study the stochastic wave equation in spatial dimension $d\in\{1,2,3\}$, with a multiplicative noise W,

$$\begin{aligned}&\frac{\partial^2}{\partial t^2}u(t,x)-\Delta_x u(t,x)=b(u(t,x))+\sigma(u(t,x))\dot W(t,x),\ t\in(0,T],\\ &u(0,x)=u_0(x),\quad \frac{\partial}{\partial t}u(0,x)=v_0(x),\end{aligned}\tag{1.1}$$

$x\in\mathbb{R}^d$.

The choice of $\dot W$ depends on the dimension d. First, we consider the case $d=1$ with space-time white noise. Then, we consider simultaneously the dimensions $d=2,3$ with a noise white in time and coloured in space, that is, with a non trivial spatial covariance. The initial conditions u_0 and v_0 are real-valued functions. The coefficients $b,\sigma:\mathbb{R}\to\mathbb{R}$ are locally Lipschitz functions such that, for $|x|\to\infty$,

$$|b(x)|\le\theta_1+\theta_2|x|\,(\ln|x|)^\delta,\ \ |\sigma(x)|\le\sigma_1+\sigma_2|x|\,(\ln|x|)^a,\tag{1.2}$$

where $\theta_i,\sigma_i\in\mathbb{R}_+,\ i=1,2,\ \theta_2,\sigma_2>0,\ \delta,a>0$.

We are interested in proving global existence of a random field solution to (1.1), which means the existence of a stochastic process $\big(u(t,x),(t,x)\in[0,T]\times\mathbb{R}^d\big)$ satisfying

$$\begin{aligned}u(t,x)=&[G(t)*v_0](x)+\frac{\partial}{\partial t}\big[G(t)*u_0\big](x)+\int_0^t ds\,[G(s)*b(u(t-s,\cdot))](x)\\ &+\int_0^t\int_{\mathbb{R}^d}G(t-s,x-y)\sigma(u(s,y))W(ds,dy),\end{aligned}\tag{1.3}$$

for all $(t,x)\in[0,T]\times\mathbb{R}^d$ a.s. Here, $G(t)$, $t>0$, is the fundamental solution to the partial differential operator $\frac{\partial^2}{\partial t^2}-\Delta_x$, the notation "$*$" denotes the convolution with respect to the space variable, and the last integral on the right-hand side is the stochastic convolution (or infinite dimensional stochastic integral) defined for example in [6].

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When σ and b are globally Lipschitz functions, therefore having linear growth, results on global existence of random field solutions to (1.1) have been established for different type of noises (see e.g. [29], [6]). However, for superlinear coefficients blow-up may occur. This is a well-known and extensively studied phenomenon in PDEs (see for instance the survey paper [13] for parabolic equations, and [3], [15], [24][Section X.13, p. 293] for hyperbolic equations).

Our research is motivated by [11]. This paper studies the parabolic SPDE

$$\frac{\partial}{\partial t}u(t,x) - \frac{\partial^2}{\partial x^2}u(t,x) = b(u(t,x)) + \sigma(u(t,x))\dot{W}(t,x),\ t \in (0,T],\ x \in (0,1),$$
$$u(0,x) = u_0(x),\ x \in [0,1], \tag{1.4}$$

with Dirichlet boundary conditions $u(t,0) = u(t,1) = 0$, and locally Lipschitz coefficients such that, as $|x| \to \infty$,

$$|b(x)| = O(|x|(\ln|x|)),\ |\sigma(x)| = o\left(|x|(\ln|x|)^{1/4}\right). \tag{1.5}$$

Assuming that the initial value u_0 is Hölder continuous, one of the main results in [11] is the existence of a unique global random field solution to (1.4) on $\mathcal{C}(\mathbb{R}_+ \times [0,1])$. This solution satisfies

$$\sup_{(t,x)\in[0,T]\times[0,1]} |u(t,x)| < \infty,\ a.s., \quad \text{for any } T > 0.$$

If in equation (1.4), σ is constant and $|b(x)| \ge |x|(\ln|x|)^{1+\varepsilon}$ when $|x| \to \infty$, with ε arbitrarily close to zero, Bonder and Groisman ([2]) prove that blow-up occurs in finite time $t > 0$. From [11], one concludes that this condition on b is sharp. Notice that the assumption on b is related to the classical *Osgood's condition* in ordinary differential equations.

There are many papers devoted to the study of blow-up phenomena for parabolic SPDEs. We refer the reader to references given in [11] for a representative sample. There are however less results for stochastic wave equations. To the best of our knowledge, existence or absence of blow-up has been studied so far in the setting of functional-valued solutions, rather than for random field solutions. Some important contributions to the problem are given in [5] (see also other papers by P.-L. Chow) and [23]. These are for wave equations whose coefficients have polynomial growth and for noises white in time and with strong conditions on the space covariance. More general noises are considered in [21], where a stochastic wave equation with $d = 2$ and $b(x) = -|x|^\rho x$, for some value of $\rho > 0$, is shown to have a global weak functional-valued solution. Observe that the minus sign in b has the effect of bringing the solution back to the origin, rather than pushing it away to infinity, as may happen with positive nonlinearities. A quite general setting is considered in [19], where, for a bounded spatial domain, the authors study a wave equation with a maximal monotone graph b driven by a martingale noise, and existence (but not uniqueness) of functional-valued global solution is proved.

The main results of this paper are Theorem 3.4 and Theorem 4.14, relative to the two noise type scenarios considered in the paper. Briefly stated, we prove that, if the initial conditions satisfy some Hölder properties, and the coefficients are such that a superlinear growth as described in (1.2) holds (see condition **(Cs)** in Section 3), then, if b dominates σ (see conditions **(C1)**, **(Cd)** in Sections 3 and 4, respectively), then a global random field solution to (1.3) exists.

Our approach follows a L^∞ method, as in [11], except that we do not use comparison theorems, since they do not hold for the wave equation. The main work consists in establishing qualitative sharp upper bounds on $E\left(\sup_{(t,x)\in K}|u(t,x)|^p\right)$, for some $p\geq 1$, where $(u(t,x))_{(t,x)}$ is the random field solution to (1.3) with *globally Lipschitz* coefficients, and K is a compact subset of $\mathbb{R}_+\times\mathbb{R}^d$. Such upper bounds are expressed in terms of the value at the origin and the Lipschitz constants of the coefficients b and σ (see Propositions 3.3 and 4.13, and the notation (2.12) below). They are obtained from L^p estimates of increments in time and in space of the process $(u(t,x))_{(t,x)}$ (see Propositions 3.2 and 4.12) via a version of Kolmogorov's theorem ([10][Theorem A.3.1]). Why is this important? Existence of solution to differential equations with locally Lipschitz coefficients is often proved by transforming the coefficients into globally Lipschitz functions, by means of truncation. With a classical sequential stopping argument, involving an increasing sequence of stopping times $(\tau_N)_N$, if $\tau_N\uparrow\infty$ a.s., then existence of global solution follows. In our case, a sufficient condition for $\tau_N\uparrow\infty$ to hold (a.s.) is

$$E\Big(\sup_{(t,x)\in K}|u_N(t,x)|^p\Big)=o(N^p), \tag{1.6}$$

where u_N denotes the random field solution to (1.3) with the truncated (by N) coefficients b^N, σ^N (see (3.26)).

We prove (1.6) for equation (1.3) in two different situations, thereby deducing absence of blow-up. This is done throughout the sections that we now describe. In Section 3, we consider the case $d=1$ and space-time white noise. The simple setting allows to better highlight the details of the method. In Section 4, we consider equation (1.3) with $d=2,3$. Since we are interested in random field solutions, in contrast with the case $d=1$, we cannot take a space-time white noise. Instead, we consider a class of Gaussian noises white in time and with a spatial covariance measure absolutely continuous with respect to the Lebesgue measure for which a well developed stochastic integral theory in any dimension d exists (see e.g. [6], [10]). In comparison with Section 3, the arguments and computations are more difficult; they are inspired by the approach to sample path regularity of the random field solution of (1.3) for $d=3$ given in [8] and [14]. The statements of Section 4 introduce several conditions on the spatial covariance density. Section 5 is devoted to prove that they are satisfied on several examples namely, the Bessel and Riesz kernels and densities of fractional type.

We end this introduction with some remarks.

Consider the case where b and σ are globally Lipschitz functions. From the first statement of Proposition 4.12 (see (4.78)), we deduce the existence of a version of the process $(u(t,x))_{(t,x)}$ with locally Lipschitz continuous sample paths, jointly in (t,x), with exponents ν_1, ν_2, respectively. Thus, for the class of spatial covariances considered in Section 4, this provides a unified approach to sample path regularity of the stochastic wave equation when $d=2,3$. Related results are in [20] for $d=2$, and [8], [14] for $d=3$.

Without much additional technical effort, the results of this paper can be extended to the stochastic wave equation (1.3) with coefficients b and σ depending also on (t,x) (with suitable assumptions), that is, replacing $b(u(t,x))$ and $\sigma(u(t,x))$ by $b(t,x,u(t,x))$ and $\sigma(t,x,u(t,x))$ respectively.

2. Preliminaries and notations

We recall that for $d = 1, 2$ and for any fixed $t > 0$, the fundamental solution $G(t)$ to the partial differential operator $\frac{\partial^2}{\partial t^2} - \Delta_x$, is a function. More precisely,

$$G(t,x) = \begin{cases} \frac{1}{2} 1_{\{|x|<t\}}, & x \in \mathbb{R}, \\ \frac{1}{2\pi} \frac{1}{\sqrt{t^2-|x|^2}} 1_{\{|x|<t\}}, & x \in \mathbb{R}^2, \end{cases} \tag{2.1}$$

while for $d = 3$,

$$G(t,dx) = \frac{1}{4\pi t}\ \sigma_t(dx),\ \ x \in \mathbb{R}^3, \tag{2.2}$$

where $\sigma_t(dx)$ denotes the uniform surface measure on the sphere centred at zero and with radius t, (see e.g. [12][Ch. 5]). By integration, we see that

$$\int_{\mathbb{R}^d} G(t,dx) = t, \tag{2.3}$$

where, for $d = 1, 2$, $G(t,dx) := G(t,x)dx$. Observe that $G(t)$ is symmetric in x.

The fundamental solution $G(t)$ satisfies the scaling property

$$G(t,dx) = t\ G(1,dz),\ \ t \ge 0. \tag{2.4}$$

This follows easily by changing the variable x into tz in (2.1) and (2.2). Also, for any continuous function f defined on $\mathbb{R}^d$, $d = 1, 2, 3$, and any $s, t > 0$,

$$\int_{\mathbb{R}^d} G(s,dx)\ f(x) = \frac{s}{t} \int_{\mathbb{R}^d} G(t,dx)\ f\left(\frac{s}{t}x\right), \tag{2.5}$$

as can be checked by applying the change of variable $u \mapsto \frac{t}{s}u$. This property will be used in the proof of Proposition 4.6.

We recall also that, for any $d \ge 1$, the Fourier transform of $G(t,\cdot)$ is

$$\mathcal{F}G(t,\cdot)(\zeta) = \int_{\mathbb{R}^d} e^{-ix\cdot\zeta} \varphi(x)\ dx = \frac{\sin(t|\zeta|)}{|\zeta|} \tag{2.6}$$

(see [28][p. 49]).

Throughout the paper, we will write $G(t, x - dy)$ to denote the translation by $-x$ of the measure $G(t,dy)$ in the distribution sense (see e.g. [25][p. 55]).

Let $B(0;\rho)$ denote the Euclidean ball centred at 0 and with radius $\rho \ge 0$. Because of the special form of the support of the fundamental solution G, if the initial values u_0, v_0 have compact support included in $B(0;\rho)$ for some $\rho > 0$, the support of the solution $(u(t,x); (t,x) \in [0,T] \times \mathbb{R}^d)$ is included in $[0,T] \times B(0;\rho + T)$. This fact will be used in several computations.

Throughout the article, we will often write (1.3) in the compact form

$$u(t,x) = \sum_{i=0}^{2} I_i(t,x), \qquad t \in [0,T],\ x \in \mathbb{R}^d, \tag{2.7}$$

where

$$I_0(t,x) = [G(t) * v_0](x) + \frac{\partial}{\partial t}\big[G(t) * u_0\big](x),$$

$$I_1(t,x) = \int_0^t ds\, [G(s) * b(u(t-s,\cdot))](x),$$

$$I_2(t,x) = \int_0^t \int_{\mathbb{R}^d} G(t-s, x-dy)\sigma(u(s,y))W(ds,dy). \tag{2.8}$$

In some proofs, the following facts will be used.
For any $\gamma > 0$, $k \geq 1$,

$$\sup_{t\geq 0} \left(t^k \, e^{-\gamma t} \right) = k^k \gamma^{-k} e^{-k}. \tag{2.9}$$

Since the function $t \longrightarrow 1 - e^{-\gamma t}(1+\gamma t)$ is increasing,

$$\sup_{t\geq 0} \left[1 - e^{-\gamma t}(1+\gamma t) \right] = \lim_{t\to\infty} \left[1 - e^{-\gamma t}(1+\gamma t) \right] = 1.$$

This implies

$$\sup_{t\geq 0} \left(\int_0^t s \, e^{-\gamma s} ds \right) = \gamma^{-2} \sup_{t\geq 0} \left[1 - e^{-\gamma t}(1+\gamma t) \right] = \gamma^{-2}. \tag{2.10}$$

Integrating by parts twice, we obtain,

$$\int_0^t s^2 e^{-\gamma s} ds = 2\,\gamma^{-3} \left[1 - e^{-\gamma t}(1+\gamma t + \gamma^2 t^2/2) \right]. \tag{2.11}$$

Notations

For a Lipschitz continuous function $g : \mathbb{R} \to \mathbb{R}$, we set $c(g) = |g(0)|$ and denote by $L(g)$ its Lipschitz constant. Thus,

$$|g(x)| \leq c(g) + L(g)|x|, \;\; x \in \mathbb{R}^d. \tag{2.12}$$

For fixed $\alpha > 0$, $p \in [2, \infty)$, and any jointly measurable random field

$$\Phi : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R},$$

we define the family of seminorms

$$\mathcal{N}_{\alpha,p}(\Phi) := \sup_{t\geq 0} \sup_{x\in\mathbb{R}^d} \; e^{-\alpha t} \|\Phi(t,x)\|_p, \tag{2.13}$$

where $\| \cdot \|_p$ denotes the norm in $L^p(\Omega)$.

For $\phi : \mathbb{R} \to \mathbb{R}$, set $\|\phi\|_\infty = \sup_{x\in\mathbb{R}} |\phi(x)|$ and, for $R \geq 0$, $\|\phi\|_{\infty,R} = \sup_{|x|\leq R} |\phi(x)|$. For $\gamma \in (0,1)$, we define

$$\|\phi\|_\gamma = \sup_{x\neq y} \frac{|\phi(x) - \phi(y)|}{|x-y|^\gamma}, \tag{2.14}$$

We denote by $\mathcal{C}_0^\infty(\mathbb{R}^k; \mathbb{R})$ the space of real-valued, infinitely differentiable functions with compact support. As usual, except if specified otherwise, $C, \bar{C}, \tilde{C}, c, \ldots$ are positive constants that may change throughout the paper, and $C(a), \bar{C}(a)$, etc., denote positive constants depending on the parameter a.

3. The stochastic wave equation in dimension one with space-time white noise

In this section, we consider the stochastic wave equation (1.3) for $d = 1$, with a space-time white noise W, and coefficients satisfying the superlinear growth condition (1.2). The goal is to prove the existence of a global random field solution. The study goes through several steps developed in the next subsections.

3.1. **Qualitative moment estimates.** We assume that the coefficients of (1.3), b and σ, are globally Lipschitz continuous functions. According to (2.12), we have

$$|b(x)| \le c(b) + L(b)|x|, \quad |\sigma(x)| \le c(\sigma) + L(\sigma)|x|, \; x \in \mathbb{R}. \tag{3.1}$$

The goal is to obtain upper bounds on $\sup_{x\in\mathbb{R}} \|u(t,x)\|_p$ in terms of the constants $c(b)$, $c(\sigma)$, $L(b)$, $L(\sigma)$ for some range of values of p. This will be done using the approach of [16][Chapter 5] for the stochastic heat equation (see also [11]).

Using the definition of $I_0(t,x)$ and G given in (2.8) and (2.1), respectively, we have

$$I_0(t,x) = \frac{1}{2}\int_{x-t}^{x+t} v_0(y)dy + \frac{1}{2}\big(u_0(x-t) + u_0(x+t)\big). \tag{3.2}$$

From this equality, we deduce

$$\sup_{x\in\mathbb{R}} |I_0(t,x)| \le t\|v_0\|_\infty + \|u_0\|_\infty. \tag{3.3}$$

Clearly, if u_0, v_0 are bounded functions, the right-hand side of (3.3) is finite. Assuming this fact, from Proposition II.3 in [4] we know that (1.3) has a unique random field solution $\big(u(t,x); (t,x) \in [0,T]\times\mathbb{R}\big)$, and for any $p \in [1,\infty)$, this solution satisfies

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \|u(t,x)\|_p < \infty.$$

Proposition 3.1. *Let u_0 and v_0 be Borel functions satisfying $\|u_0\|_\infty + \|v_0\|_\infty < \infty$. Suppose that $L(b) \ge 8L(\sigma)^2$. Then, there exists a universal constant $C>0$ such that, for any $p \in \left[2, \frac{L(b)}{4L(\sigma)^2}\right]$,*

$$\mathcal{N}_{2\sqrt{L(b)},p}(u) \le \mathcal{T}_0 + C\Big[\frac{c(b)}{L(b)} + \frac{c(\sigma)}{L(\sigma)}\Big], \tag{3.4}$$

where

$$\mathcal{T}_0 = \frac{e^{-1}\|v_0\|_\infty}{\sqrt{L(b)}} + 2\|u_0\|_\infty. \tag{3.5}$$

Thus,

$$\sup_{x\in\mathbb{R}} E(|u(t,x)|^p) \le e^{2pt\sqrt{L(b)}}\left\{\mathcal{T}_0 + C\left[\frac{c(b)}{L(b)} + \frac{c(\sigma)}{L(\sigma)}\right]\right\}^p, \; t\in[0,T]. \tag{3.6}$$

Proof. Fix $\alpha > 0$ and $p \in [2,+\infty)$. From (3.3) and (2.9) with $k=1$, we obtain

$$\mathcal{N}_{\alpha,p}(I_0) \le \frac{e^{-1}}{\alpha}\|v_0\|_\infty + \|u_0\|_\infty. \tag{3.7}$$

By applying Minkowski's inequality, and then (3.1), we have

$$\begin{aligned}\|I_1(t,x)\|_p &\le \int_0^t ds\int_{\mathbb{R}} dy\; G(t-s,x-y)\|b(u(s,y))\|_p\\ &\le \int_0^t ds\int_{\mathbb{R}} dy\; G(t-s,x-y)\left[c(b) + L(b)\|u(s,y)\|_p\right].\end{aligned}$$

Using (2.3), along with (2.10) and (2.9) with $k=2$, we deduce

$$\begin{aligned}\mathcal{N}_{\alpha,p}(I_1) &= \sup_{t\ge 0}\sup_{x\in\mathbb{R}} e^{-\alpha t}\|I_1(t,x)\|_p\\ &\le c(b)\sup_{t\ge 0}\left(\frac{t^2}{2}e^{-\alpha t}\right) + L(b)\,\mathcal{N}_{\alpha,p}(u)\sup_{t\ge 0}\int_0^t (t-s)e^{-\alpha(t-s)}ds\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{2e^{-2}}{\alpha^2} c(b) + \frac{1}{\alpha^2} L(b)\, \mathcal{N}_{\alpha,p}(u) \\
&\leq \frac{1}{\alpha^2} \left[c(b) + L(b) \mathcal{N}_{\alpha,p}(u) \right]. \qquad (3.8)
\end{aligned}
$$

Using first the version of Burkholder-Davies-Gundy's inequality given in [16][Theorem B1, p. 97], and then Minkowski's inequality, we obtain

$$
\begin{aligned}
\|I_2(t,x)\|_p^2 &= \Big\| \int_{\mathbb{R}} G(t-s,x-y)\sigma(u(s,y)) W(ds,dy) \Big\|_p^2 \\
&\leq 4p \left\{ E\Big(\Big| \int_0^t \int_{\mathbb{R}} G^2(t-s,x-y)\sigma^2(u(s,y)) ds dy \Big|^{\frac{p}{2}} \Big) \right\}^{\frac{2}{p}} \\
&= 4p \, \Big\| \int_0^t \int_{\mathbb{R}} G^2(t-s,x-y)\sigma^2(u(s,y)) ds dy \Big\|_{\frac{p}{2}} \\
&\leq 4p \, \int_0^t \int_{\mathbb{R}} G^2(t-s,x-y) \|\sigma^2(u(s,y))\|_{\frac{p}{2}} ds dy.
\end{aligned}
$$

By (3.1), this is bounded above by

$$
8p \left\{ \int_0^t ds \int_{\mathbb{R}} dy \; G^2(t-s,x-y) \left[c(\sigma)^2 + L(\sigma)^2 \|u(s,y))\|_p^2 \right] \right\}.
$$

Since $G^2(t,x) = \frac{1}{2} G(t,x)$, the inequalities (2.9) with $k=1$ and (2.10) imply

$$
\begin{aligned}
\mathcal{N}_{\alpha,p}(I_2) &= \sup_{t\geq 0} \sup_{x\in\mathbb{R}} e^{-\alpha t} \|I_2(t,x)\|_p \\
&\leq \sup_{t\geq 0} \sup_{x\in\mathbb{R}} e^{-\alpha t} \Big\{ 2pc(\sigma)^2 t^2 \\
&\quad + 8pL(\sigma)^2 \int_0^t ds \int_{\mathbb{R}} dy \; G^2(t-s,x-y) \|u(s,y))\|_p^2 \Big\}^{1/2} \\
&\leq \sqrt{2p} c(\sigma) \sup_{t\geq 0} \left(t e^{-\alpha t} \right) + \sqrt{8p} L(\sigma) \mathcal{N}_{\alpha,p}(u) \\
&\quad \times \left(\int_0^t ds \int_{\mathbb{R}} dy \; G^2(t-s,x-y) e^{-2\alpha(t-s)} \right)^{1/2} \\
&\leq \sqrt{2p}\, \frac{e^{-1}}{\alpha} c(\sigma) + \sqrt{8p} L(\sigma) \mathcal{N}_{\alpha,p}(u) \Big(\int_0^t \frac{1}{2} s e^{-2\alpha s} ds \Big)^{\frac{1}{2}} \\
&\leq \sqrt{2p}\, \frac{e^{-1}}{\alpha} c(\sigma) + \sqrt{p}\, \frac{1}{\alpha} \; L(\sigma) \mathcal{N}_{\alpha,p}(u) \\
&\leq \frac{\sqrt{p}}{\alpha} \left[c(\sigma) + L(\sigma) \mathcal{N}_{\alpha,p}(u) \right]. \qquad (3.9)
\end{aligned}
$$

The inequalities (3.7), (3.8) and (3.9) imply

$$
\begin{aligned}
\mathcal{N}_{\alpha,p}(u) \leq \frac{e^{-1}}{\alpha} \|v_0\|_\infty + \|u_0\|_\infty + \frac{1}{\alpha^2} c(b) + \frac{\sqrt{p}}{\alpha} c(\sigma) \\
+ 2 \max \Big(\frac{L(b)}{\alpha^2}, \frac{\sqrt{p} L(\sigma)}{\alpha} \Big) \mathcal{N}_{\alpha,p}(u). \qquad (3.10)
\end{aligned}
$$

Fix $\alpha^2 = 4L(b)$; since $L(b) \geq 8L(\sigma)^2$, the interval $\Big[2, \frac{L(b)}{4L(\sigma)^2}\Big]$ is nonempty. Observe that for any $p \in \Big[2, \frac{L(b)}{4L(\sigma)^2}\Big]$, $\sqrt{p} L(\sigma) \leq \frac{\sqrt{L(b)}}{2} = \frac{\alpha}{4}$, and that the choice of α implies

$\max\left(\frac{L(b)}{\alpha^2}, \frac{\sqrt{p}L(\sigma)}{\alpha}\right) = \frac{1}{4}$, and $\frac{\sqrt{p}}{\alpha} \le \frac{1}{4L(\sigma)}$. Hence, from (3.10) we deduce (3.4). The estimate (3.6) is an immediate consequence of the definition of $\mathcal{N}_{\alpha,p}(u)$ for $\alpha = 2\sqrt{L(b)}$. □

Remarks on Proposition 3.1

(1) Assume that $L(b) \ge \kappa L(\sigma^2)$, for some $\kappa > 4$. Then, with the same proof as above, we obtain

$$\mathcal{N}_{\sqrt{\frac{\kappa L(b)}{2}},p}(u) \le \mathcal{T}_0 + C\Big[\frac{c(b)}{L(b)} + \frac{c(\sigma)}{L(\sigma)}\Big], \tag{3.11}$$

and thus,

$$\sup_{x\in\mathbb{R}} E(|u(t,x)|^p) \le e^{t\sqrt{\frac{\kappa L(b)}{2}}}\left\{\mathcal{T}_0 + C\left[\frac{c(b)}{L(b)} + \frac{c(\sigma)}{L(\sigma)}\right]\right\}^p, \ t\in[0,T], \tag{3.12}$$

for any $p \in \left[2, \frac{2L(b)}{\kappa L(\sigma)^2}\right]$.

Observe that, in comparison with Proposition 3.1, the constraint on the Lipschitz constants $L(b), L(\sigma)$ is weaker and the set of values of p for which the uniform L^p estimate (3.12) holds is larger. However, for the use of Proposition 3.1 we make in this article, this improvement does not seem to be relevant.

(2) The proof of Proposition 3.1 uses the inequalities (3.1) on the coefficients but not really the fact that $L(b)$, $L(\sigma)$ are the Lipschitz constants of b and σ, respectively. Thus, the assumption $L(b) \ge 8L(\sigma)^2$ could be removed by replacing $L(b)$ by some other constant $\bar{L}(b)$ satisfying $\bar{L}(b) \ge \max(L(b), 8L(\sigma)^2)$. By doing so, the estimates (3.4), (3.6) should be changed accordingly. If, for example, $\bar{L}(b)$ is a constant multiple of $L(b)$, this will not have any consequence in the subsequent results of the paper.

3.2. **Uniform bounds on moments.** In this section, we still assume that the coefficients of (1.3) are globally Lipschitz continuous functions, thereby satisfying (3.1). We prove an upper bound for

$$E\left(\sup_{t\in[0,T]}\sup_{|x|\le R}|u(t,x)|^p\right), \tag{3.13}$$

for any $R > 0$, and for a specific range of values of p that depend on the initial values u_0, v_0, and the constants $c(b)$, $c(\sigma)$, $L(b)$, $L(\sigma)$. This will be a consequence of the following proposition relative to estimates on moments of space and time increments of the random field $\big(u(t,x); (t,x) \in [0,T]\times\mathbb{R}^d\big)$.

Proposition 3.2. *We assume that the initial condition u_0 is Hölder continuous with exponent $\gamma_1 \in (0,1]$ and v_0 is continuous. Set $\gamma = \gamma_1 \wedge \frac{1}{2}$, and fix $T, R \ge 0$. Then, for any $p \in [2,\infty)$, there exists a positive constant $C(p,T,R)$ such that, for any $t, \bar{t} \in [0,T]$, $x, \bar{x} \in [-R,R]$ and $\alpha > 0$,*

$$\frac{\|u(t,x) - u(\bar{t},\bar{x})\|_p}{(|t-\bar{t}| + |x-\bar{x}|)^\gamma} \le C(p,T,R)\left[\mathcal{M}_1 + \mathcal{M}_2 e^{\alpha T}\mathcal{N}_{\alpha,p}(u)\right], \tag{3.14}$$

where

$$\begin{aligned}\mathcal{M}_1 &= \|u_0\|_{\gamma_1} + \|v_0\|_{\infty,R+T} + c(b) + \sqrt{p}\ c(\sigma),\\ \mathcal{M}_2 &= L(b) + \sqrt{p}\ L(\sigma).\end{aligned} \tag{3.15}$$

Moreover, if $L(b) \geq 8L(\sigma)^2$ *then for any* $p \in \left[2, \frac{L(b)}{4L(\sigma)^2}\right]$,

$$\frac{\|u(t,x) - u(\bar{t},\bar{x})\|_p}{(|t-\bar{t}| + |x-\bar{x}|)^\gamma} \leq C(p,T,R)\left[\mathcal{M}_1 + \mathcal{M}_2 e^{2\sqrt{L(b)}T}\left(\mathcal{T}_0 + \frac{c(b)}{L(b)} + \frac{c(\sigma)}{L(\sigma)}\right)\right], \tag{3.16}$$

with $\mathcal{T}_0$ *given in* (3.5).

Proof. Since u_0 is γ_1-Hölder continuous, we clearly have

$$|u_0(x-t) + u_0(x+t) - (u_0(\bar{x}-\bar{t}) + u_0(\bar{x}+\bar{t}))| \leq 2\|u_0\|_{\gamma_1}\left(|x-\bar{x}|^{\gamma_1} + |t-\bar{t}|^{\gamma_1}\right).$$

The function $V_0(z) = \int_0^z v_0(y)\,dy$ is continuously differentiable; hence,

$$\left|\int_{\bar{x}-\bar{t}}^{\bar{x}+\bar{t}} v_0(y)\,dy - \int_{x-t}^{x+t} v_0(y)\,dy\right| \leq 2\,\|v_0\|_{\infty,R+T}\left(|x-\bar{x}| + |t-\bar{t}|\right).$$

Consequently, from the expression (3.2) we obtain,

$$|I_0(t,x) - I_0(\bar{t},\bar{x})| \leq C(T,R)\left(\|u_0\|_{\gamma_1} + \|v_0\|_{\infty,R+T}\right)\left(|x-\bar{x}|^{\gamma_1} + |t-\bar{t}|^{\gamma_1}\right). \tag{3.17}$$

for some $C(T,R) > 0$.

In the next arguments, we will use the following properties on increments of the fundamental solution to the one-dimensional wave equation, whose proofs are straightforward. For all $0 \leq \bar{t}, t \leq T$, $x, \bar{x} \in \mathbb{R}$, there exists a constant $C(T)$ such that

$$\begin{aligned}
&\int_0^T ds \int_{\mathbb{R}} dy\ |G(t-s,x-y) - G(\bar{t}-s,\bar{x}-y)| \\
&\qquad = 2\int_0^T ds \int_{\mathbb{R}} dy\ |G(t-s,x-y) - G(\bar{t}-s,\bar{x}-y)|^2 \\
&\qquad \leq C(T)\left(|t-\bar{t}| + |x-\bar{x}|\right).
\end{aligned} \tag{3.18}$$

As in the proof of Proposition 3.1, Minkovski's inequality along with (3.1) imply

$$\begin{aligned}
\|I_1(t,x) - I_1(\bar{t},\bar{x})\|_p &\leq \int_0^T ds \int_{\mathbb{R}} dy\ |G(t-s,x-y) - G(\bar{t}-s,\bar{x}-y)|\|b(u(s,y))\|_p \\
&\leq \int_0^T ds \int_{\mathbb{R}} dy\ |G(t-s,x-y) - G(\bar{t}-s,\bar{x}-y)|\left[c(b) + L(b)\|u(s,y)\|_p\right] \\
&\leq C(T)\left[c(b) + L(b)e^{\alpha T}\mathcal{N}_{\alpha,p}(u)\right]\left(|t-\bar{t}| + |x-\bar{x}|\right),
\end{aligned} \tag{3.19}$$

for any $\alpha > 0$.

Bounds from above of increments of I_2, are also obtained following the arguments in the proof of Proposition 3.1, based on the Burkholder-Davies-Gundy and Minkowski inequalities. More precisely,

$$\begin{aligned}
\|I_2(t,x) - I_2(\bar{t},\bar{x})\|_p^2 &\leq 4p\left\|\int_0^T ds \int_{\mathbb{R}} dy\ |G(t-s,x-y) - G(\bar{t}-s,\bar{x}-y)|^2\sigma^2(u(s,y))\right\|_{\frac{p}{2}} \\
&\leq 4p\int_0^T ds \int_{\mathbb{R}} dy\ |G(t-s,x-y) - G(\bar{t}-s,\bar{x}-y)|^2\|\sigma^2(u(s,y))\|_{\frac{p}{2}} \\
&\leq 8p\int_0^T ds \int_{\mathbb{R}} dy\ |G(t-s,x-y) - G(\bar{t}-s,\bar{x}-y)|^2\left\|\left[c(\sigma)^2 + L(\sigma)^2\|u(s,y)\|_p^2\right]\right. \\
&\leq 8p\ C(T)\left[c(\sigma)^2 + L(\sigma)^2 e^{2\alpha T}\mathcal{N}_{\alpha,p}(u)^2\right]\left(|t-\bar{t}| + |x-\bar{x}|\right),
\end{aligned}$$

for any $\alpha > 0$. Consequently,

$$\|I_2(t,x) - I_2(\bar{t},\bar{x})\|_p \le 2\sqrt{2C(T)}\sqrt{p}\left[c(\sigma) + L(\sigma)e^{\alpha T}\mathcal{N}_{\alpha,p}(u)\right](|t-\bar{t}| + |x-\bar{x}|)^{\frac{1}{2}}. \quad (3.20)$$

Let $\gamma = \gamma_1 \wedge \frac{1}{2}$; the inequalities (3.17), (3.19) and (3.20) imply (3.14).

Let $\alpha = 2\sqrt{L(b)}$ and $p \in \left[2, \frac{L(b)}{4L(\sigma)^2}\right]$; from (3.4) we obtain (3.16). The proof of the proposition is complete. □

Combining Proposition 3.2 and the version of Kolmogorov's continuity lemma given in [10][Theorem A.3.1] (see also [16][Theorem C.6]) leads to an upper bound for uniform L^p moments in (3.13). The precise statement is as follows.

Proposition 3.3. *Let the initial values u_0, v_0 be as in Proposition 3.2. Let $\gamma = \gamma_1 \wedge \frac{1}{2}$ and suppose that $L(b) > \frac{8}{\gamma}L(\sigma)^2$. Then u has a Hölder continuous version, jointly in (t,x), still denoted be u, with exponent $\eta \in (0,\gamma)$, and for any $p \in \left(\frac{2}{\gamma}, \frac{L(b)}{4L(\sigma)^2}\right]$, there exists a constant $C(p,T,R)$ such that,*

$$E\left(\sup_{t\in[0,T]}\sup_{|x|\le R}|u(t,x)|^p\right) \le 2^{p-1}\|u_0\|^p_{\infty,R} + C(p,T,R)\left[\mathcal{M}_1^p + \mathcal{M}_2^p\mathcal{M}_3^p e^{2pT\sqrt{L(b)}}\right], \quad (3.21)$$

where $\mathcal{M}_1$, $\mathcal{M}_2$ are defined in (3.15)*, and*

$$\mathcal{M}_3 = \frac{e^{-1}\|v_0\|_{\infty,R}}{\sqrt{L(b)}} + 2\|u_0\|_{\infty,R} + C\left[\frac{c(b)}{L(b)} + \frac{c(\sigma)}{L(\sigma)}\right],$$

with C the universal constant in the right-hand side of (3.4)*.*

Proof. For any $s,t \in [0,T]$, $x,y \in [-R,R]$, set

$$\Delta(t,x;s,y) = |t-s|^\gamma + |x-y|^\gamma.$$

Proposition 3.2 implies that

$$E(|u(t,x) - u(s,y)|^p) \le K(\Delta(t,x;s,y))^p,$$

with

$$K := C(p,T,R)\left[\mathcal{M}_1^p + \mathcal{M}_2^p e^{\alpha pT}\mathcal{N}_{\alpha,p}(u)^p\right] \quad (3.22)$$

for any $\alpha > 0$.

Apply [10][Theorem A.3.1] with $k = 1$, $\alpha_1 = \alpha_2 = \gamma$, $I = [0,T]$, $J = [-R,R]$, $p \in \left(\frac{2}{\gamma}, \infty\right)$, to infer the existence of a version of u (that we still denote by u) with jointly Hölder continuous sample paths of exponent $\eta \in (0,\gamma)$. Moreover, since by (1.1),

$$C_1 := E\left(\sup_{|x|\le R}|u(0,x)|^p\right) = \|u_0\|^p_{\infty,R},$$

we deduce from [10][Equation(2.8.50)],

$$E\left(\sup_{t\in[0,T]}\sup_{|x|\le R}|u(t,x)|^p\right) \le 2^{p-1}\|u_0\|^p_{\infty,R} + C(p,T,R)K, \quad (3.23)$$

with K is defined in (3.22). Observe that K depends on α.

Choose $\alpha = 2\sqrt{L(b)}$. Then (3.23) and (3.22) yield

$$E\left(\sup_{t\in[0,T]}\sup_{|x|\le R}|u(t,x)|^p\right) \le 2^{p-1}\|u_0\|^p_{\infty,R}$$

$$+ C(p,T,R)\left[\mathcal{M}_1^p + \mathcal{M}_2^p e^{2pT\sqrt{L(b)}}\mathcal{N}_{2\sqrt{L(b)},p}(u)^p\right]. \quad (3.24)$$

Notice that, since $\gamma \le 1/2$, the condition $L(b) > \frac{8}{\gamma}L(\sigma)^2$ implies that the hypotheses of Proposition 3.1 are satisfied. Hence, using (3.4) to upper estimate $\mathcal{N}_{2\sqrt{L(b)},p}(u)$ on the right-hand side of (3.24), and since we are considering $|x| \le R$, we obtain (3.21). □

3.3. **Existence and uniqueness of global solution.** In this section, we consider the equation (1.3) with coefficients having superlinear growth. More precisely, we introduce the following hypothesis that implies (1.2).

(Cs) The functions $b, \sigma : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz and are such that:

(1) $|b(0)| \le \theta_1$ and $|\sigma(0)| \le \sigma_1$, for some $\theta_1, \sigma_1 \in \mathbb{R}_+$;

(2) as $|x|, |y| \to \infty$,

$$|b(x) - b(y)| \le \theta_2 |x-y| \left[\ln_+(|x-y|)\right]^{\delta}, \ |\sigma(x) - \sigma(y)| \le \sigma_2 |x-y| \left[\ln_+(|x-y|)\right]^{a},$$

where $\theta_2, \sigma_2, \in (0,\infty)$, $\delta, a > 0$, and $\ln_+(z) = \ln(z \vee e)$ for $z \ge 0$.

We also assume that the coefficient b *dominates* σ, in the way formulated by the following assumption.

(C1) The parameters δ, a in (1.2) satisfy one of the properties:

(1) $\delta > 2a$,

(2) $\delta = 2a$ and the constants θ_2 and σ_2 are such that $\theta_2 > \bar{\gamma}\sigma_2^2$, for some $\bar{\gamma} > 0$.

The next theorem proves existence and uniqueness of global random field solution to Equation (1.3).

Theorem 3.4. *Assume that the initial conditions u_0, v_0 are functions with compact support included in $[-\rho, \rho]$, for some $\rho > 0$, that u_0 is Hölder continuous with exponent γ_1 and v_0 is continuous. Set $\gamma = \gamma_1 \wedge \frac{1}{2}$; let the coefficients b and σ satisfy* **(Cs)** *and* **(C1)** *with $\delta < 2$ and $\bar{\gamma} = 8\gamma^{-1}$.*

Then, there exists a random field solution $\big(u(t,x); (t,x) \in [0,T] \times \mathbb{R}\big)$ to (1.3). This solution is unique and satisfies

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} |u(t,x)| < \infty, \ a.s. \quad (3.25)$$

Proof. We notice that because of the properties of the initial values, if a solution to (1.3) exists, it should have its support on $[0,T] \times [-R,R]$, where $R = \rho + T$. Hence, (3.25) is equivalent to

$$\sup_{(t,x)\in[0,T]\times[-(\rho+T),\rho+T]} |u(t,x)| < \infty, \ a.s.$$

Solution for truncated Lipschitz continuous coefficients.

For a locally Lipschitz function $g : \mathbb{R} \to \mathbb{R}$ and $N \ge 1$, we define a globally Lipschitz function g_N by

$$g_N(x) = g(x)1_{\{|x|\le N\}} + g(N)1_{\{x>N\}} + g(-N)1_{\{x<-N\}}. \quad (3.26)$$

Using this definition for σ and b, we consider (1.1) with coefficients σ_N, b_N, and denote by $u_N := (u_N(t,x); (t,x) \in [0,T] \times \mathbb{R})$ its unique random field solution (see the first part of Section 3.1 for details). From **(Cs)** we see that if $N \geq 2$, σ_N, b_N satisfy the conditions (3.1) with

$$c(b_N) = \theta_1, \quad c(\sigma_N) = \sigma_1, \quad L(b_N) = \theta_2(\ln(2N))^\delta, \quad L(\sigma_N) = \sigma_2(\ln(2N))^a. \tag{3.27}$$

Therefore, Proposition 3.2 applies and by Kolmogorov's continuity criterion, there is a version of u_N with jointly Hölder continuous sample paths of exponent $\eta \in (0, \gamma)$ in both variables. In the sequel we will consider this version that we will still denote by u_N.

Bounds for L^p moments of u_N.

Assume that condition **(C1)** (1) holds. Then, for N large enough, we have $L(b_N) > \frac{8}{\gamma} L(\sigma_N)^2$. On the other hand, if condition **(C1)** (2) is satisfied, then $L(b_N) > \frac{8}{\gamma} L(\sigma_N)^2$ holds for any $N \geq 2$. We can therefore apply Proposition 3.3 to see that for any $p \in \left(\frac{2}{\gamma}, \frac{\theta_2(\ln(2N))^\delta}{4\sigma_2^2(\ln(2N))^{2a}}\right]$, $R > 0$, N large enough (if necessary).

$$E\Big(\sup_{t\in[0,T]} \sup_{|x|\leq R} |u_N(t,x)|^p\Big) \leq 2^{p-1} \|u_0\|_{\infty,R}^p + C(p,T,R)\left[\mathcal{M}_1^p + \mathcal{M}_2^p(N)\, \mathcal{M}_3^p(N)\, e^{2pT\sqrt{L(b_N)}}\right], \tag{3.28}$$

where

$$\begin{aligned} \mathcal{M}_1 =& \|u_0\|_{\gamma_1} + \|v_0\|_{\infty,R+T} + \theta_1 + \sqrt{p}\ \sigma_1, \\ \mathcal{M}_2(N) =& L(b_N) + \sqrt{p}\ L(\sigma_N), \\ \mathcal{M}_3(N) =& \frac{e^{-1}\|v_0\|_{\infty,R}}{\sqrt{L(b_N)}} + 2\|u_0\|_{\infty,R} + C\left[\frac{\theta_1}{L(b_N)} + \frac{\sigma_1}{L(\sigma_N)}\right]. \end{aligned} \tag{3.29}$$

Existence and uniqueness of global solution

For any $N \geq 2$, set

$$\tau_N := \inf\Big\{t > 0 \ :\ \sup_{|x|\leq R} |u_N(t,x)| \geq N\Big\} \wedge T. \tag{3.30}$$

The uniqueness of the solution and the local property of stochastic integrals imply that $u_N(t,x) = u_{N+1}(t,x)$ a.s. for $t \leq \tau_N$. Hence, almost surely, $(\tau_N)_{N\geq 2}$ is an increasing sequence, bounded by T.

Assume that $\sup_N \tau_N = T$, a.s., and thus $\{t \leq \tau_N\} \uparrow \Omega$, a.s. On $\{t \leq \tau_N\}$, define $(u(t,x), (t,x) \in [0,T) \times \mathbb{R})$ by $u(t,x) = u_N(t,x)$; then $u(t,x) = u_M(t,x)$, for every $M \geq N$. The random variable $u(t,x)$ is well-defined and moreover, for any (t,x),(1.3) holds a.s. Indeed, on $\{t \leq \tau_N\}$, by definition,

$$\begin{aligned} u(t,x) = I_0(t,x) &+ \int_0^t ds \int_{\mathbb{R}} dy\ G(t-s,x-y) b_N(u_N(s,y)) \\ &+ \int_0^t \int_{\mathbb{R}} G(t-s,x-y)\sigma_N(u_N(s,y)) W(ds,dy). \end{aligned}$$

But on $\{t \leq \tau_N\}$, $b_N(u_N(s,y)) = b(u_N(s,y)) = b(u(s,y))$ and $\sigma_N(u_N(s,y)) = \sigma(u_N(s,y)) = \sigma(u(s,y))$. Since $\{t \leq \tau_N\} \uparrow \Omega$ a.s., we conclude that $(u(t,x), (t,x) \in [0,T) \times \mathbb{R})$ satisfies (1.3). Notice that, in this case, the stochastic integral in (1.3) is not defined in $L^2(\Omega)$, but using instead an extension defined a.s. (see e.g. [10]).

The last part of the proof is devoted to check that indeed, $\sup_N \tau_N = T$ a.s. This will follow from the property

$$\lim_{N\to\infty} P(\tau_N < T) = 0, \tag{3.31}$$

that we now establish.

Let $C(p,T,R,N)$ denote the right-hand side of (3.28). To emphasise the terms that depend on N, we write

$$C(p,T,R,N) = C_1(p,T,R) + C_2(p,T,R,N), \tag{3.32}$$

with

$$C_1(p,T,R) = 2^{p-1}\|u_0\|_{\infty,R}^p + C(p,T,R)\mathcal{M}_1^p,$$
$$C_2(p,T,R,N) = C_2(p,T,R)\mathcal{M}_2^p(N)\,\mathcal{M}_3^p(N)\,e^{2pT\sqrt{L(b_N)}}.$$

Fix $p \in \left(\frac{2}{\gamma}, \frac{\theta_2(\ln(2N))^\delta}{4\sigma_2^2(\ln(2N))^{2a}}\right]$. Applying Chebychev's inequality and then (3.28), we have

$$\begin{aligned} P\left(\tau_N < T\right) &\le P\left(\sup_{t\in[0,T]}\sup_{|x|\le R}|u_N(t,x)| \ge N\right) \\ &\le N^{-p}E\left(\sup_{t\in[0,T]}\sup_{|x|\le R}|u_N(t,x)|^p\right) \\ &\le N^{-p}C(p,T,R,N) = N^{-p}\left[C_1(p,T,R) + C_2(p,T,R,N)\right]. \end{aligned} \tag{3.33}$$

Assume that

$$C_2(p,T,R,N) = \mathrm{o}(N^p). \tag{3.34}$$

Then, from (3.33), we clearly obtain (3.31).

For the proof of (3.34), we first write the expressions of $\mathcal{M}_2(N)$ and $\mathcal{M}_3(N)$ in (3.29), substituting $L(b_N)$ and $L(\sigma_N)$ by their respective values given in (3.27). Because of the property $\sup_{N\ge 2}\mathcal{M}_3(N) \le C$, we obtain

$$C_2(p,T,R,N) = \tilde{C}_2(p,T,R)\exp\left(p(\delta\vee a)\ln[\ln(2N)] + 2pT\theta_2^{1/2}[\ln(2N)]^{\delta/2}\right).$$

This implies (3.34), because $\delta < 2$. The proof of the theorem is complete. □

4. The stochastic wave equation in dimensions $d = 2,3$ with coloured noise

The aim of this section is to discuss the same questions as in Section 3 in the setting of a noise W white in time and spatially correlated, with $d = 2,3$. It is well-known that for dimensions $d \ge 2$, if W is a space-time white noise, the stochastic convolution in (1.3) fails to be a well-defined random variable in $L^2(\Omega)$, for almost any $(t,x) \in [0,T]\times\mathbb{R}^d$. This is the case even if σ is constant. However, we can still obtain a random field solution of (1.3) by taking a smoother noise in the spatial variable (see e.g. [29]). This leads to the introduction in the next subsection 4.1 of a new class of Gaussian noises that are white in time and correlated in space.

4.1. **Spatially homogeneous Gaussian noise and stochastic integrals.** Let Λ be a non-negative definite distribution in $\mathcal{S}'(\mathbb{R}^d)$. By the Bochner-Schwartz theorem (see e.g. [25][Chap. VII, Thoerem XVIII]), Λ is the Fourier transform of a non-negative, tempered, symmetric measure μ on $\mathbb{R}^d$ called the spectral measure of Λ. In particular, Λ is also a tempered distribution.

On a complete probability space $(\Omega, \mathcal{A}, P)$, we consider a Gaussian process $\{W(\varphi), \varphi \in \mathcal{C}_0(\mathbb{R}^{d+1}\}$, indexed by the set of Schwartz test functions, with mean zero and covariance

$$E\left(W(\varphi)W(\psi)\right) = \int_0^\infty dt \int_{\mathbb{R}^d} \Lambda(dx)\ \left(\varphi(t) * \tilde{\psi}(t)\right)(x), \tag{4.1}$$

where "$*$" denotes the convolution operator in the spatial variable and $\tilde{\psi}$ means reflection in the spatial variable too.

We will consider spatial covariances Λ satisfying the following hypothesis ([6]):

(h0) The spectral measure $\mu = \mathcal{F}^{-1}\Lambda$ is such that

$$\int_{\mathbb{R}^d} \frac{\mu(d\zeta)}{1+|\zeta|^2} < \infty. \tag{4.2}$$

Using (2.6), this is seen to be equivalent to

$$\int_0^T dt \int_{\mathbb{R}^d} \mu(d\zeta)\, |\mathcal{F}G(t)(\zeta)|^2 < \infty.$$

Consider a jointly measurable adapted process $Z = \left(Z(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\right)$ such that $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(|Z(t,x|^p) < \infty$, for some $p \in [2,\infty)$, and assume **(h0)**. Then, the stochastic integral

$$((GZ)\cdot W)(t,x) := \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) Z(s,y)\ W(ds,dy)$$

is a well-defined random variable. Moreover, for any $x \in \mathbb{R}^d$, the process $((GZ)\cdot W)(t,x), t \in [0,T])$ is a martingale with respect to the natural filtration generated by W and, by Burholder's inequality ([16][Theorem B.1]), the moment estimate

$$\begin{aligned} \|((GZ)\cdot W)(t,x)\|_p^p &\le (2\sqrt{p})^p \left(\int_0^t ds \int_{\mathbb{R}^d} \mu(d\zeta)|\mathcal{F}G(s)(\zeta)|^2\right)^{\frac{p}{2}-1} \\ &\times \int_0^t ds \left(\sup_{x\in\mathbb{R}^d} E(|Z(s,x)|^p)\right) \int_{\mathbb{R}^d} \mu(d\zeta)|\mathcal{F}G(s)(\zeta)|^2, \end{aligned} \tag{4.3}$$

holds (see [10], [22]).

In this paper, we will consider the particular class of covariances Λ described in **(h1)** below.

(h1) Λ is an absolutely continuous measure, $\Lambda(dx) = f(x)dx$, $f \ge 0$. Its spectral measure $\mu = \mathcal{F}^{-1}\Lambda$ is such that, for all signed measures Φ and Ψ with finite total variation,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Phi(dx)\ \Psi(dy) f(x-y) = C \int_{\mathbb{R}^d} \mu(d\zeta)\mathcal{F}\Phi(\zeta)\overline{\mathcal{F}\Psi(\zeta)}, \tag{4.4}$$

for some positive and finite constant C.

Observe that (4.4) is a generalized version of Parseval's identity.

Remark 4.1. *Assume* $\int_{\mathbb{R}^d}\mu(d\zeta)|\mathcal{F}(|\Phi|)(\zeta)|^2<\infty$, $\int_{\mathbb{R}^d}\mu(d\zeta)|\mathcal{F}(|\Psi|)(\zeta)|^2<\infty$, *where here the notation* $|\cdot|$ *stands here for the total variation. Suppose also that* $f:\mathbb{R}^d\to[0,+\infty]$ *is lower semi-continuous. Then if* $\Phi=\Psi$, [18]*[Corollary 3.4] implies the validity of* (4.4) *(with* $C=(2\pi)^{-d}$*). By a polarity argument, the result can be extended to* $\Phi\neq\Psi$.

Assume **(h0)** and **(h1)**. Since for any $t>0$, $G(t,dx)$ is a non-negative finite measure with compact support, the identity (4.4) applied to $\Phi=G(t,dx)$ and $\Psi=G(s,dy)$, $s,t>0$ yields

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}G(t,dx)\ G(s,dy)f(x-y)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mu(d\zeta)\mathcal{F}G(t,\cdot)(\zeta)\overline{\mathcal{F}G(s,\cdot)(\zeta)}. \tag{4.5}$$

In particular,

$$J(t):=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}G(t,dy)G(t,dz)f(y-z)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mu(d\zeta)|\mathcal{F}G(t)(\zeta)|^2. \tag{4.6}$$

Using (2.6), we have

$$|\mathcal{F}G(t)(\zeta)|^2\le\frac{2}{1+|\zeta|^2}\ 1_{\{|\zeta|\ge1\}}+t^2\ 1_{\{|\zeta|<1\}}\le\frac{2(1+t^2)}{1+|\zeta|^2},\ t>0.$$

Therefore,

$$J(t)\le 2(1+t^2)C_\mu, \tag{4.7}$$

with

$$C_\mu:=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{\mu(d\zeta)}{1+|\zeta|^2}<\infty, \tag{4.8}$$

which implies, $\sup_{t\in[0,T]}J(t)<\infty$.

Assuming **(h0)** and **(h1)**, the stochastic integral $((GZ)\cdot W)(t,x)$ satisfies the sharper estimate (in comparison with (4.3)),

$$\begin{aligned}&\|((GZ)\cdot W)(t,x)\|_p^p\le(2\sqrt{p})^p\\ &\times E\left(\int_0^t ds\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\ G(t-s,x-dy)G(t-s,x-dz)f(y-z)Z(s,y)Z(s,z)\right)^{\frac{p}{2}},\end{aligned} \tag{4.9}$$

(see e.g. [10], [22]). This fact will be used repeatedly throughout the article.

We end this section with a technical lemma related with the identity (4.4). It will be applied at several points in the next proofs.

Lemma 4.2. *Let* $d\ge1$, $t>0$ *and* $G(t)$ *be the fundamental solution of the wave operator on* $\mathbb{R}^d$*. Let* φ, ψ *be bounded Borel measurable functions defined on* $\mathbb{R}^d$*. Let* Λ *be a symmetric measure satisfying* **(h1)**, *with corresponding spectral measure* $\mu=\mathcal{F}^{-1}\Lambda$ *satisfying* **(h0)**. *Then, for any* $s,t>0$ *and* $z\in\mathbb{R}^d$, *we have*

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi(x)G(t,dx)\psi(y)G(s,dy)f(x-y+z)\\ &\qquad=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mathcal{F}\left(\varphi G(t)\right)(\xi)\overline{\mathcal{F}\left(\psi G(s)\right)(\xi)}e^{-iz\cdot\xi}\ \mu(d\xi).\end{aligned} \tag{4.10}$$

Proof. By applying the translation $\tau_z x=x+z$, the left-hand side of (4.10) equals

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi(\tau_{-z}x)\tau_{-z}G(t,dx)\psi(y)G(s,dy)f(\tau_z x-y)=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}f(w-y)\ \Phi(dw)\Psi(dy),$$

where $\Phi(dw) = \varphi(\tau_{-z}w)\tau_{-z}G(t,dw)$ and $\Psi = \psi(y)G(s,dy)$. We recall that $\tau_{-z}G(t,dw)$ stands for the translation of the measure $G(t,dw)$ by $-z$ in the distribution sense (see e.g. [25][p. 55]).

Because of the assumptions on φ and ψ, the measures $\Phi(dw)$ and $\Psi(dy)$ are signed measures with finite total variation. We can therefore apply (4.4) to deduce

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(z-y)\ \Phi(dw)\Psi(dy) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathcal{F}f(\xi)\mathcal{F}(\Phi)(\xi)\overline{\mathcal{F}(\Psi)(\xi)}\ d\xi.$$

Using the identities

$$\begin{aligned}\mathcal{F}(\Phi)(\xi) &= \mathcal{F}\left(\tau_{-z}\varphi(\cdot)\tau_{-z}G(t,\cdot)\right)(\xi)\\ &= \int_{\mathbb{R}^d} e^{-i\xi\cdot v}\tau_{-z}\varphi(v)\tau_{-z}G(t,dv) = e^{-i\xi\cdot z}\mathcal{F}\left(\varphi G(t,\cdot)\right)(\xi),\end{aligned}$$

we obtain (4.10). □

Remark 4.3. *Lemma 6.5 in* [14] *gives a proof of* (4.10) *for $d=3$ by using the particular expression of $G(t)$ in this dimension.*

4.2. **Qualitative moment estimates.** We introduce a set of assumptions that ensure the existence and uniqueness of a random field solution to (1.3).

(he)

(i) The coefficients b and σ are Lipschitz continuous functions, therefore satisfying (2.12) with $g := b,\sigma$.

(ii) W is a spatially homogeneous noise as described in Section 4.1. Its covariance and spectral measures (Λ and μ, respectively) satisfy **(h0)** and **(h1)**.

(iii) The initial values u_0, v_0 are such that the function $(t,x)\mapsto I_0(t,x)$ defined in (2.8) is continuous and

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |I_0(t,x)| < \infty. \tag{4.11}$$

Theorem 4.4. *Assume that* **(he)** *is satisfied. Then there exists a random field solution $\left(u(t,x),(t,x)\in[0,T]\times\mathbb{R}^d\right)$ to* (1.3)*, and for any $p\in[1,\infty)$,*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \|u(t,x)\|_p < \infty. \tag{4.12}$$

This solution is unique in the class of jointly measurable, adapted processes u satisfying (4.12) *with $p=2$.*

In the case $u_0 = v_0 = 0$, this follows from Theorem 13 [6] applied to the wave operator. For non-null initial conditions, this follows from [7][Theorem 4.3].

Estimates of $\mathcal{N}_{\alpha,p}(u)$ for covariances Λ satisfying **(h0)** *and* **(h1)**

Proposition 4.5. *In addition to* **(he)**, *we assume that the initial values u_0, v_0, satisfy the following conditions:*

1. *for $d=2$, u_0 is a continuous, bounded, continuous differentiable function with bounded partial derivatives; v_0 is continuous and bounded;*
2. *for $d=3$, u_0 is a continuous, bounded, twice continuous differentiable function with bounded second order partial derivatives; v_0 is continuous and bounded.*

We also suppose that the covariance measure Λ satisfies **(h1)**, *and the Lipschitz constants $L(b)$, $L(\sigma)$ are such that $L(b) \ge \big(2^{12}\, 3^2\, C_\mu^2\, L(\sigma)^4\big) \vee \frac14$, where C_μ is given in (4.8). Then, for any $p \in \Big[2, \frac{\sqrt{L(b)}}{2^5\, 3\, C_\mu\, L(\sigma)^2}\Big]$ we have*

$$\mathcal{N}_{2\sqrt{L(b)},p}(u) \le C \left[\mathcal{T}_0 + \frac{c(b)}{L(b)} + \frac{c(\sigma)}{L(\sigma)}\right], \tag{4.13}$$

where C is a universal constant and

$$\mathcal{T}_0 = \begin{cases} \|u_0\|_\infty + \frac{1}{\sqrt{L(b)}}\big(\|\nabla u_0\|_\infty + \|v_0\|_\infty\big), & \text{if } \ d=2,\\ \|u_0\|_\infty + \|\Delta u_0\|_\infty + \frac{1}{\sqrt{L(b)}}\|v_0\|_\infty, & \text{if } \ d=3. \end{cases} \tag{4.14}$$

As a consequence, we deduce that for $t \in [0,T]$ and $p \in \Big[2, \frac{\sqrt{L(b)}}{2^5\, 3\, C_\mu\, L(\sigma)^2}\Big]$,

$$\sup_{x\in\mathbb{R}^d} E(|u(t,x)|^p) \le C^p\ e^{2pt\sqrt{L(b)}} \left[\mathcal{T}_0 + \frac{c(b)}{L(b)} + \frac{c(\sigma)}{L(\sigma)}\right]^p. \tag{4.15}$$

Proof. We will consider the contributions to $\mathcal{N}_{\alpha,p}$ of each of the terms $I_i(t,x)$ in (2.8) separately.

Estimates of $\mathcal{N}_{\alpha,p}(I_0)$

Let $d=2$; using [20][(1.11), (1.12)], we have for $t>0$ and $x \in \mathbb{R}^2$

$$\big|[G(t)*v_0](x)\big| \le t\,\|v_0\|_\infty, \quad \left|\frac{\partial}{\partial t}\big[G(t)*u_0\big](x)\right| \le C\left\{\|u_0\|_\infty + t\,\|\nabla u_0\|_\infty\right\}.$$

Hence, using (2.9) with $k=1$ we deduce that for any $\alpha>0$ and $p\in[2,\infty)$,

$$\mathcal{N}_{\alpha,p}(I_0) \le C\left[\|u_0\|_\infty + \frac{e^{-1}}{\alpha}\big(\|v_0\|_\infty + \|\nabla u_0\|_\infty\big)\right]. \tag{4.16}$$

Let $d=3$. Using (2.2) and (2.3), we obtain, for $t>0$ and $x\in\mathbb{R}^3$,

$$|[G(t)*v_0](x)| = \left|\int_{|y|=t} v_0(x-y)\ G(t,dy)\right| \le \|v_0\|_\infty \int_{|y|=t} G(t,dy) = t\|v_0\|_\infty.$$

By applying the formula

$$\frac{d}{dt}\big(G(t)*u_0\big) = \frac1t\big(G(t)*u_0\big) + \frac{1}{4\pi}\int_{\{|y|\le1\}} (\Delta u_0)(.+ty)dy$$

(see [26]), we have

$$\left|\frac{d}{dt}\big(G(t)*u_0\big)(x)\right| \le \|u_0\|_\infty + \frac13\|\Delta u_0\|_\infty.$$

Therefore, using (2.9) with $k=1$ we deduce,

$$\mathcal{N}_{\alpha,p}(I_0) \le \frac{e^{-1}}{\alpha}\|v_0\|_\infty + \|u_0\|_\infty + \frac13\,\|\Delta u_0\|_\infty. \tag{4.17}$$

Estimates of $\mathcal{N}_{\alpha,p}(I_1)$

Use the expresion of $I_1(t,x)$ given in (2.8) and then Minkovski's inequality along with (2.12) with $g := b$ and (2.3) to obtain

$$\begin{aligned}
\|I_1(t,x)\|_p &= \left\| \int_0^t ds \int_{\mathbb{R}^d} G(t-s,dy) b(u(s,x-y)) \right\|_p \\
&\le \int_0^t ds \int_{\mathbb{R}^d} G(t-s,dy) \left\| b(u(s,x-y)) \right\|_p \\
&\le \int_0^t ds \int_{\mathbb{R}^d} G(t-s,dy) \left[c(b) + L(b) \left\| u(s,x-y) \right\|_p \right] \\
&\le \frac{t^2}{2} c(b) + L(b) \int_0^t ds \sup_{x\in\mathbb{R}^d} \|u(s,x)\|_p \int_{\mathbb{R}^d} G(t-s,dy) \\
&= \frac{t^2}{2} c(b) + L(b) \int_0^t ds\ (t-s) \left(\sup_{x\in\mathbb{R}^d} \|u(s,x)\|_p \right).
\end{aligned}$$

From the above estimates, using an argument similar to that used to prove (3.8), we deduce,

$$\begin{aligned}
\mathcal{N}_{\alpha,p}(I_1) &= \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} e^{-\alpha t} \|I_1(t,x)\|_p \\
&\le c(b) \sup_{t\ge 0} \left(\frac{t^2}{2} e^{-\alpha t} \right) \\
&\quad + L(b) \sup_{t\in[0,T]} \int_0^t ds (t-s) e^{-\alpha(t-s)} \left(\sup_{(s,x)\in[0,T]\times\mathbb{R}^d} e^{-\alpha s} \|u(s,x)\|_p \right) \\
&\le \frac{2e^{-2}}{\alpha^2} c(b) + \frac{1}{\alpha^2} L(b) \mathcal{N}_{\alpha,p}(u), \qquad (4.18)
\end{aligned}$$

where we have used (2.9) with $k = 2$ and (2.10).

Estimates of $\mathcal{N}_{\alpha,p}(I_2)$

By applying (4.9) with $Z(s,y) := \sigma(u(s,y))$, then Minkowski's inequality and (2.12) with $g := \sigma$, we obtain

$$\begin{aligned}
\|I_2(t,x)\|_p^2 &\le 4p \Big\{ E\Big[\int_0^t ds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(t-s,x-dy) G(t-s,x-dz) f(y-z) \\
&\quad \times \sigma(u(s,y)) \sigma(u(s,z)) \Big]^{\frac{p}{2}} \Big\}^{\frac{2}{p}} \\
&\le 4p \int_0^t ds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(t-s,x-dy) G(t-s,x-dz) f(y-z) \|\sigma(u(s,y))\sigma(u(s,z))\|_{\frac{p}{2}} \\
&\le 4p \int_0^t ds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(t-s,x-dy) G(t-s,x-dz) f(y-z) \|\sigma(u(s,y))\|_p \|\sigma(u(s,z))\|_p \\
&\le 4p \int_0^t ds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(t-s,x-dy) G(t-s,x-dz) f(y-z) \\
&\quad \times \left[c(\sigma) + L(\sigma) \|u(s,y)\|_p \right] \left[c(\sigma) + L(\sigma) \|u(s,z)\|_p \right].
\end{aligned}$$

Thus, the inequality $2ab \leq a^2+b^2$, valid for any $a,b\in\mathbb{R}$ implies

$$\begin{aligned}\|I_2(t,x)\|_p^2 &\leq 4p\int_0^t ds \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(t-s,x-dy)G(t-s,x-dz)f(y-z)\\ &\qquad\times\big[c(\sigma)+L(\sigma)\|u(s,y)\|_p\big]^2\\ &\leq 8p\int_0^t ds \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(t-s,x-dy)G(t-s,x-dz)f(y-z)\\ &\qquad\times\big[c(\sigma)^2+L(\sigma)^2\|u(s,y)\|_p^2\big]. \end{aligned}\tag{4.19}$$

Using the notation introduced in (4.6), we can rewrite (4.19) as follows

$$\|I_2(t,x)\|_p^2 \leq 8p\left[c(\sigma)^2\int_0^t ds\; J(t-s) + L(\sigma)^2\int_0^t ds\; J(t-s)\sup_{y\in\mathbb{R}^d}\|u(s,y)\|_p^2\right].\tag{4.20}$$

From here, using the change of varables $s\mapsto t-s$, we have,

$$\begin{aligned}\mathcal{N}_{\alpha,p}(I_2) &= \sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^d} e^{-\alpha t}\|I_2(t,x)\|_p\\ &\leq \sqrt{8p}\,\nu_1(\alpha)\,c(\sigma)+\sqrt{8p}\,\nu_2(\alpha)\,L(\sigma)\mathcal{N}_{\alpha,p}(u),\end{aligned}\tag{4.21}$$

where

$$\begin{aligned}\nu_1(\alpha) &:= \sup_{t\in[0,T]}\left(e^{-2\alpha t}\int_0^t dsJ(s)\right)^{\frac12}<\infty,\\ \nu_2(\alpha) &:= \sup_{t\in[0,T]}\left(\int_0^t ds\; e^{-2\alpha s}J(s)\right)^{\frac12}<\infty.\end{aligned}\tag{4.22}$$

Thus, owing to (4.18), (4.21), we have

$$\begin{aligned}\mathcal{N}_{\alpha,p}(u) &\leq \mathcal{N}_{\alpha,p}(I_0)+\frac{2e^{-2}}{\alpha^2}c(b)+\sqrt{8p}\,c(\sigma)\,\nu_1(\alpha)\\ &\qquad+2\max\left[\frac{L(b)}{\alpha^2},\sqrt{8p}\,L(\sigma)\,\nu_2(\alpha)\right]\mathcal{N}_{\alpha,p}(u).\end{aligned}\tag{4.23}$$

Using (4.7), (2.9) with $k=1,3$ we obtain

$$\begin{aligned}\nu_1(\alpha) &\leq C_\mu^{\frac12}\sup_{t\in[0,T]}\Big(e^{-2\alpha t}\int_0^t 2\,(1+s^2)\,ds\Big)^{\frac12}\\ &\leq C_\mu^{\frac12}\,\Big(\frac{e^{-1}}{\alpha}+\frac{9}{4}\,\frac{e^{-3}}{\alpha^3}\Big)^{\frac12},\end{aligned}\tag{4.24}$$

where C_μ is defined in (4.8). Furthermore, the inequalities (4.7) and (2.11) imply

$$\begin{aligned}\nu_2(\alpha) &\leq C_\mu^{\frac12}\sup_{t\in[0,T]}\Big(\int_0^t 2(1+s^2)\,e^{-2\alpha s}ds\Big)^{\frac12}\\ &\leq C_\mu^{\frac12}\sup_{t\in[0,T]}\Big[\frac1\alpha-\frac1\alpha e^{-2\alpha t}+\frac{1}{2\alpha^3}-\frac{e^{-2\alpha t}}{2\alpha^3}\big(1+2\alpha t+2\alpha^2t^2\big)\Big]^{\frac12}\\ &\leq C_\mu^{\frac12}\,\Big(\frac1\alpha+\frac{1}{2\alpha^3}\Big)^{\frac12}.\end{aligned}\tag{4.25}$$

Thus, (4.23) – (4.25) yield

$$\mathcal{N}_{\alpha,p}(u) \le \mathcal{N}_{\alpha,p}(I_0) + \frac{2e^{-2}}{\alpha^2} c(b) + \sqrt{8p}\, c(\sigma)\, C_\mu^{\frac12} \left(\frac{e^{-1}}{\alpha} + \frac{9}{4}\, \frac{e^{-3}}{\alpha^3} \right)^{\frac12}$$
$$+ 2 \max\left[\frac{L(b)}{\alpha^2} \ ,\ \sqrt{8p}\, L(\sigma)\, C_\mu^{\frac12} \Big(\frac{1}{\alpha} + \frac{1}{2\alpha^3} \Big)^{\frac12} \right] \mathcal{N}_{\alpha,p}(u). \tag{4.26}$$

Choose $\alpha^2 = 4L(b)$. Since by assumption $L(b) \ge \frac14$, we have $\alpha \ge 1$, which yields

$$\frac{e^{-1}}{\alpha} + \frac{9}{4}\frac{e^{-3}}{\alpha^3} \le \frac{13}{8e} L(b)^{-\frac12},$$
$$\frac{1}{\alpha} + \frac{1}{2\alpha^3} \le \frac34 L(b)^{-\frac12}.$$

Moreover, using once more the assumption $L(b) \ge \left[2^{12}\, 3^2\, C_\mu^2\, L(\sigma)^4\right] \vee \frac14$, we see that for $\alpha^2 = 4L(b)$ and for any $p \in \left[2, \frac{\sqrt{L(b)}}{2^5\, 3\, C_\mu\, L(\sigma)^2}\right]$,

$$\max\left[\frac{L(b)}{\alpha^2} \ ,\ \sqrt{8p}\, L(\sigma)\, C_\mu^{\frac12} \left(\frac{1}{\alpha} + \frac{1}{2\alpha^3} \right)^{\frac12} \right] \le \max\left[\frac14, \sqrt{8p}\, L(\sigma)\, C_\mu^{\frac12} \left(\frac34 L(b)^{-\frac12} \right)^{\frac12} \right] = \frac14.$$

Hence, from (4.26), taking the upper bound $p \le \frac{\sqrt{L(b)}}{2^5\, 3\, C_\mu\, L(\sigma)^2}$, we deduce

$$\mathcal{N}_{2\sqrt{L(b)},p}(u) \le 2\mathcal{N}_{2\sqrt{L(b)},p}(I_0) + e^{-2}\frac{c(b)}{L(b)} + \left(\frac{13}{3e2^3} \right)^{\frac12} \frac{c(\sigma)}{L(\sigma)}$$
$$\le C_1 \mathcal{T}_0 + C_2 \left[\frac{c(b)}{L(b)} + \frac{c(\sigma)}{L(\sigma)} \right],$$

with $\mathcal{T}_0$ defined in (4.14), and we have used (4.16) and (4.17).

This completes the proof of (4.13). The inequality (4.15) follows from (4.13) using the definition of $\mathcal{N}_{\alpha,p}(u)$. □

4.3. **Uniform bounds on moments.** In this section, we address the problems of Section 3.2 in the setting of a noise W white in time and coloured in space, and dimensions $d = 2, 3$. The main task is to prove the analogue of Proposition 3.2 on moment estimates of increments in time and in space for the solution to equation (1.3) with globally Lipschitz coefficients. Since in comparison with the case $d = 1$ and space-time white noise, computations are much more intricate, for the sake of clarity, we divide the study of increments into several parts.

Increments of $I_0(t,x)$ in time and space.

Proposition 4.6. *Let $I_0(t,x)$, $(t,x) \in [0,T] \times \mathbb{R}^d$ be as in (2.8) and $R \ge 0$ be fixed.*

(1) *Let $d = 2$. Assume that u_0 is $\mathcal{C}^1$, ∇u_0 is Hölder continuous with exponent $\gamma_1 \in (0,1]$, and v_0 is Hölder continuous with exponent $\gamma_2 \in (0,1]$. Then, there exists a positive constant $C(T,R)$ such that, for any $t, \bar t \in [0,T]$, and any $x, \bar x \in B(0;R)$,*

$$|I_0(t,x) - I_0(\bar t, \bar x)| \le C(T,R) \left(\|v_0\|_{\infty,R+T} + \|v_0\|_{\gamma_2} + \|\nabla u_0\|_{\infty,R+T} + \|\nabla u_0\|_{\gamma_1} \right)$$
$$\times \left(|t - \bar t|^{\gamma_1 \wedge \gamma_2} + |x - \bar x|^{\gamma_1 \wedge \gamma_2} \right). \tag{4.27}$$

(2) *Let $d=3$. Assume that u_0 is $\mathcal{C}^2$, Δu_0 is Hölder continuous with exponent $\gamma_1 \in (0,1]$, and v_0 is Hölder continuous with exponent $\gamma_2 \in (0,1]$. Then, there exists a positive constant $C(T,R)$ such that, for any $t,\bar{t} \in [0,T]$, and any $x,\bar{x} \in B(0;R)$,*

$$|I_0(t,x)-I_0(\bar{t},\bar{x})| \leq C(T,R)\big[\|v_0\|_{\gamma_2}+\|\nabla u_0\|_{\infty,R+T}+\|\Delta u_0\|_{\gamma_1}\big]\times\big(|t-\bar{t}|^{\gamma_1\wedge\gamma_2}+|x-\bar{x}|^{\gamma_1\wedge\gamma_2}\big). \tag{4.28}$$

Proof. (2). Let $0 \leq t \leq \bar{t} \leq T$ and $x \in B(0;R)$ be fixed. Using (2.5), we have

$$\begin{aligned}
|[G(t)-G(\bar{t})]*v_0(x)| &= \left|\int_{\mathbb{R}^2} G(t,dy)\left(v_0(x-y)-v_0\left(x-\frac{\bar{t}}{t}y\right)\frac{\bar{t}}{t}\right)\right| \\
&\leq \frac{\bar{t}}{t}\int_{\mathbb{R}^2} G(t,dy)\ \left|v_0(x-y)-v_0\left(x-\frac{\bar{t}}{t}y\right)\right| \\
&\quad + \left|1-\frac{\bar{t}}{t}\right|\int_{\mathbb{R}^2} G(t,dy)\ |v_0(x-y)|.
\end{aligned}$$

Applying (2.3), we deduce

$$\begin{aligned}
\frac{\bar{t}}{t}\int_{\mathbb{R}^2} G(t,dy)\ \left|v_0(x-y)-v_0\left(x-\frac{\bar{t}}{t}y\right)\right| &\leq T\|v_0\|_{\gamma_2}|t-\bar{t}|^{\gamma_2}, \\
\left|1-\frac{\bar{t}}{t}\right|\int_{\mathbb{R}^2} G(t,dy)\ |v_0(x-y)| &\leq \|v_0\|_{\infty,R+T}|t-\bar{t}|.
\end{aligned}$$

Consequently,

$$\sup_{|x|\leq R}|[G(t)-G(\bar{t})]*v_0(x)| \leq C(T)\left(\|v_0\|_{\infty,R+T}+\|v_0\|_{\gamma_2}\right)||t-\bar{t}|^{\gamma_2}.$$

According to the computations in [20][p. 812-813], we have

$$\sup_{|x|\leq R}\left|\frac{\partial}{\partial t}\big[G(t)*u_0(x)-G(\bar{t})*u_0(x)\big]\right| \leq C(\|\nabla u_0\|_{\infty,R+T}|t-\bar{t}|+\|\nabla u_0\|_{\gamma_1})|t-\bar{t}|^{\gamma_1}.$$

Thus,

$$\sup_{|x|\leq R}|I_0(t,x)-I_0(\bar{t},x)| \leq C(T)\big(\|v_0\|_{\infty,R+T}+\|v_0\|_{\gamma_2}+\|\nabla u_0\|_{\infty,R+T}+\|\nabla u_0\|_{\gamma_1}\big)|t-\bar{t}|^{\gamma_1\wedge\gamma_2}. \tag{4.29}$$

Let now $0 \leq t \leq T$ and $x,\bar{x} \in B(0,R)$ be fixed; then

$$\begin{aligned}
|(G(t*v_0)(x)-(G(t*v_0)(\bar{x})| &\leq \int_{\mathbb{R}^2} G(t,y)|v_0(x-y)-v_0(\bar{x}-y)|\ dy \\
&\leq \|v_0\|_{\gamma_2}|x-\bar{x}|^{\gamma_2}\ \left(\int_{\mathbb{R}^2} G(t,y)\ dy\right) \\
&= t\|v_0\|_{\gamma_2}|x-\bar{x}|^{\gamma_2} \leq T\|v_0\|_{\gamma_2}|x-\bar{x}|^{\gamma_2},
\end{aligned} \tag{4.30}$$

where in the last inequality we have used (2.3).

According to the computations in [20][p. 815-816], we have

$$\left|\frac{\partial}{\partial t}\big[G(t)*u_0(x)-G(t)*u_0(\bar{x})\big]\right| \leq C\big(\|\nabla u_0\|_{\infty,T+R}|x-\bar{x}|+\|\nabla u_0\|_{\gamma_1}|x-\bar{x}|^{\gamma_1}\big), \quad t \in [0,T].$$

Therefore,

$$\sup_{0\leq t\leq T}|I_0(t,x)-I_0(t,\bar{x})| \leq C(T,R)\big(\|v_0\|_{\gamma_2}+\|\nabla u_0\|_{\infty,T+R}+\|\nabla u_0\|_{\gamma_1}\big)|x-\bar{x}|^{\gamma_1\wedge\gamma_2}. \tag{4.31}$$

From the estimates (4.29)–(4.31), we deduce (4.27).

(3). Let $0 \le t \le \bar{t} \le T$ and $x \in B(0,R)$ be fixed. According to [8][Lemma 4.9, p. 43], we have

$$\sup_{|x|\le R} \left\| \frac{\partial}{\partial t} \left(G(\cdot) * u_0\right)(x) \right\|_{\gamma_1} \le C\left(\|\nabla u_0\|_{\infty,R+T} + \|\Delta u_0\|_{\gamma_1}\right),$$
$$\sup_{|x|\le R} \|\left(G(\cdot) * v_0\right)(x)\|_{\gamma_2} \le C\|v_0\|_{\gamma_2},$$

where $C > 0$ is a universal constant. Consequently,

$$\sup_{|x|\le R} |I_0(t,x) - I_0(\bar{t},x)| \le C(T,R)\big(\|\nabla u_0\|_{\infty,R+T} + \|\Delta u_0\|_{\gamma_1} + \|v_0\|_{\gamma_2}\big)|t-\bar{t}|^{\gamma_1\wedge\gamma_2}. \quad (4.32)$$

Let $0 \le t \le T$ and $x, \bar{x} \in B(0,R)$ be fixed. Observe that the computations in (4.30) also hold in dimension $d = 3$, therefore yielding

$$\sup_{0\le t\le T} |(G(t * v_0)(x) - (G(t * v_0)(\bar{x})| \le T\|v_0\|_{\gamma_2}|x-\bar{x}|^{\gamma_2}.$$

Using the computations in [14][p. 362] (see also [8][Chapter 4]), we have

$$\sup_{0\le t\le T} \left| \frac{\partial}{\partial t}\big[G(t) * u_0(x) - G(t) * u_0(\bar{x})\big] \right| \le C\left(\|\nabla u_0\|_{\infty,R+T} + \|\Delta u_0\|_{\gamma_1}\right)|x-\bar{x}|^{\gamma_1}.$$

Hence,

$$\sup_{0\le t\le T} |I_0(t,x) - I_0(t,\bar{x})| \le C(T,R)\big(\|\nabla u_0\|_{\infty,R+T} + \|\Delta u_0\|_{\gamma_1} + \|v_0\|_{\gamma_2}\big)|x-\bar{x}|^{\gamma_1\wedge\gamma_2}. \quad (4.33)$$

The proof of (4.28) is a consequence of (4.32) and (4.33). □

Remark 4.7. *In comparison with the assumptions (1) and (2) in Proposition 4.5, in Proposition 4.6 we restrict the space variable to a bounded set, therefore having the boundedness hypotheses satisfied.*

Increments of $I_1(t,x)$ in time and space.

Proposition 4.8. *Let $I_1(t,x), (t,x) \in [0,T] \times \mathbb{R}^d$ be as in (2.8).*
1. Assume that the hypotheses **(he)** *are satisfied. Then there exists a positive constant $C(T)$ depending on T such that for any $(t,x), (\bar{t},\bar{x}) \in [0,T] \times \mathbb{R}^d$ and for any $p \in [2,\infty)$,*

$$\begin{aligned}&\|I_1(t,x) - I_1(\bar{t},\bar{x})\|_p \\ &\quad\le C(T)\Bigg\{L(b)\int_0^t ds\Bigg(\sup_{|z_1-z_2|=|x-\bar{x}|}\|u(s,z_1)-u(s,z_2)\|_p + \sup_{|z_1-z_2|\le|t-\bar{t}|}\|u(s,z_1)-u(s,z_2)\|_p\Bigg) \\ &\qquad\quad +|t-\bar{t}|\left[c(b) + L(b)\sup_{(t,x)\times\mathbb{R}^d}\|u(t,x)\|_p\right]\Bigg\}. \qquad (4.34)\end{aligned}$$

2. Assume the hypotheses of Proposition 4.5. Then there exists a positive constant $C(T)$ depending on T such that for any $p \in \left[2, \frac{\sqrt{L(b)}}{2^5 3 C_\mu L(\sigma)^2}\right]$ and any $(t,x), (\bar{t},\bar{x}) \in [0,T] \times \mathbb{R}^d$,

$$\begin{aligned}&\|I_1(t,x) - I_1(\bar{t},\bar{x})\|_p \\ &\quad\le C(T)\Bigg\{L(b)\int_0^t ds\Bigg(\sup_{|z_1-z_2|=|x-\bar{x}|}\|u(s,z_1)-u(s,z_2)\|_p + \sup_{|z_1-z_2|\le|t-\bar{t}|}\|u(s,z_1)-u(s,z_2)\|_p\Bigg)\end{aligned}$$

$$+|t-\bar t|\left[c(b)+L(b)e^{2T\sqrt{L(b)}}\mathcal{N}_{2\sqrt{L(b)},p}(u)\right]\Big\}, \tag{4.35}$$

with $\mathcal{N}_{2\sqrt{L(b)},p}(u)$ *satisfying* (4.13).

Proof. 1. Fix $t\in[0,T]$ and $x,\bar x\in\mathbb{R}^d$. The Minkoswki inequality, the Lipschitz continuity property of b and (2.3) yield

$$\begin{aligned}\|I_1(t,x)-I_1(t,\bar x)\|_p &= \left\|\int_0^t ds\int_{\mathbb{R}^d} G(t-s,dy)[b(u(s,x-y))-b(u(s,\bar x-y))]\right\|_p\\
&\le L(b)\int_0^t ds\int_{\mathbb{R}^d} G(t-s,dy)\,\|u(s,x-y)-u(s,\bar x-y)\|_p\\
&\le L(b)\int_0^t ds\ (t-s)\left[\sup_{|z_1-z_2|=|x-\bar x|}\|u(s,z_1)-u(s,z_2)\|_p\right]\\
&\le L(b)T\int_0^t ds \sup_{|z_1-z_2|=|x-\bar x|}\|u(s,z_1)-u(s,z_2)\|_p. \qquad (4.36)\end{aligned}$$

Let $0\le t\le \bar t\le T$. By the triangular inequality

$$\|I_1(\bar t,x)-I_1(t,x)\|_p\le T_1(p;t,\bar t,x)+T_2(p;t,\bar t,x),$$

with

$$T_1(p;t,\bar t,x)=\left\|\int_0^t ds\int_{\mathbb{R}^d}\big[G(\bar t-s,dy)-G(t-s,dy)\big]b(u(s,x-y))\right\|_p,$$

$$T_2(p;t,\bar t,x)=\left\|\int_t^{\bar t} ds\int_{\mathbb{R}^d}G(\bar t-s,dy)b(u(s,x-y))\right\|_p.$$

By the scaling property (2.4) of the fundamental solution $G(t)$,

$$\begin{aligned}&T_1(p;t,\bar t,x)\\
&\quad=\left\|\int_0^t ds\int_{\mathbb{R}^d}G(1,dz)\,[(t-s)b(u(s,x-(t-s)z))-(\bar t-s)b(u(s,x-(\bar t-s)z))]\right\|_p.\end{aligned}$$

Apply Minkowski's inequality to deduce

$$\begin{aligned}&T_1(p;t,\bar t,x)\\
&\quad\le\int_0^t ds\int_{\mathbb{R}^d}G(1,dz)\,\|(t-s)b(u(s,x-(t-s)z))-(\bar t-s)b(u(s,x-(\bar t-s)z))\|_p\\
&\quad\le L(b)\int_0^t ds\int_{\mathbb{R}^d}G(1,dz)(t-s)\,\|u(s,x-(t-s)z))-u(s,x-(\bar t-s)z))\|_p\\
&\qquad+|t-\bar t|\int_0^t ds\int_{\mathbb{R}^d}G(1,dz)\big[c(b)+L(b)\|u(s,x-(\bar t-s)z))\|_p\big], \qquad (4.37)\end{aligned}$$

where we have used the Lipschitz continuity property of b and (2.12) with $g=b$.

Since the support of $G(1,dz)$ is included in the closed ball $\overline{B(0,1)}$, we have

$$\begin{aligned}&\int_0^t ds\ (t-s)\int_{\mathbb{R}^d}G(1,dz)\,\|u(s,x-(t-s)z))-u(s,x-(\bar t-s)z))\|_p\\
&\qquad\le T\int_0^t ds\int_{\mathbb{R}^d}G(1,dz)\sup_{|z_1-z_2|\le|t-\bar t|}\|u(s,z_1)-u(s,z_2)\|_p\end{aligned}$$

$$= T\int_0^t ds \sup_{|z_1-z_2|\le|t-\bar t|} \|u(s,z_1)-u(s,z_2)\|_p \,.$$

As for the last term in (4.37), the identity (2.3) shows that it is bounded from above by

$$|t-\bar t|\Big\{Tc(b)+L(b)\int_0^t ds \sup_{x\in\mathbb{R}^d}\|u(s,x)\|_p\Big\}. \tag{4.38}$$

Thus,

$$\begin{aligned} T_1(p;t,\bar t,x) \le & T\Bigg(L(b)\int_0^t \sup_{|z_1-z_2|\le|t-\bar t|}\|u(s,z_1)-u(s,z_2)\|_p\, ds \\ & + |t-\bar t|\Big\{c(b)+L(b)\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\|u(t,x)\|_p\Big\}\Bigg). \end{aligned} \tag{4.39}$$

With the same kind of arguments as for $T_1(p;t,\bar t,x)$, we deduce the following upper bounds for $T_2(p;t,\bar t,x)$:

$$\begin{aligned} T_2(p;t,\bar t,x) &\le \int_t^{\bar t} ds \int_{\mathbb{R}^d} G(\bar t-s,dy)\ \|b(u(s,x-y))\|_p \\ &\le c(b)\int_t^{\bar t} ds\int_{\mathbb{R}^d} G(\bar t-s,dy) + L(b)\int_t^{\bar t} ds(\bar t-s)\sup_{x\in\mathbb{R}^d}\|u(s,x)\|_p \\ &\le c(b)\int_0^{\bar t-t} s\ ds + L(b)\int_0^{\bar t-t} ds\ s\sup_{x\in\mathbb{R}^d}\|u(\bar t-s,x)\|_p \\ &\le c(b)\frac{(\bar t-t)^2}{2} + L(b)\frac{(\bar t-t)^2}{2}\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\|u(t,x)\|_p \,. \end{aligned} \tag{4.40}$$

From (4.36), (4.39) and (4.40), we obtain (4.34).

2. Assume now the hypotheses of Proposition 4.5. The term defined in (4.38) is bounded by

$$\begin{aligned} &|t-\bar t|\Big\{Tc(b)+L(b)\int_0^t ds\sup_{x\in\mathbb{R}^d}\|u(s,x)\|_p\Big\} \\ &\quad\le C|t-\bar t|\Big\{Tc(b)+L(b)\int_0^t ds\ e^{2s\sqrt{L(b)}}\mathcal{N}_{2\sqrt{L(b)},p}(u)\Big\} \\ &\quad\le CT|t-\bar t|\ \Big\{c(b)+L(b)\ e^{2T\sqrt{L(b)}}\mathcal{N}_{2\sqrt{L(b)},p}(u)\Big\}. \end{aligned}$$

Therefore,

$$\begin{aligned} T_1(p;t,\bar t,x) \le T\Big(&L(b)\int_0^t \sup_{|z_1-z_2|\le|t-\bar t|}\|u(s,z_1)-u(s,z_2)\|_p \\ &+|t-\bar t|\ \Big\{c(b)+L(b)\ e^{2T\sqrt{L(b)}}\mathcal{N}_{2\sqrt{L(b)},p}(u)\Big\}\Big). \end{aligned} \tag{4.41}$$

As for $T_2(p;t,\bar t,x)$, concatenating with the last line in (4.40), we obtain,

$$T_2(p;t,\bar t,x) \le \frac{(\bar t-t)^2}{2}\Big[c(b)+L(b)\ e^{2T\sqrt{L(b)}}\mathcal{N}_{2\sqrt{L(b)},p}(u)\Big]. \tag{4.42}$$

From (4.36), (4.41) and (4.42), we obtain (4.35).

□

Space increments of $I_2(t,x)$

While keeping assumption **(h1)**, we consider a strengthening of **(h0)**, by adding condition (c') in [14][p. 367] relative to the spectral measure μ. More precisely, we introduce the following hypothesis:

(h2) There exists $\gamma \in (0,1)$ such that the Fourier transform of the tempered measure $|\zeta|^{2\gamma}\mu(d\zeta)$ is a non negative locally integrable function g_γ, and moreover,

$$\int_{\mathbb{R}^d} \frac{\mu(d\zeta)}{1+|\zeta|^{2-2\gamma}} < \infty.$$

Set

$$C_\mu^{(\gamma)} := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\mu(d\zeta)}{1+|\zeta|^{2-2\gamma}}. \tag{4.43}$$

Proposition 4.9. *Let $I_2(t,x)$, $(t,x) \in [0,T]\times\mathbb{R}^d$ be as in* (2.8).
1. Assume that the hypotheses **(he)**, **(h1)** *and* **(h2)** *are satisfied. Then, for any $p \in [2,\infty)$ and $t \in [0,T]$, there exists a positive constant C such that, for every $x, \bar{x} \in \mathbb{R}^d$, $t \in [0,T]$,*

$$\begin{aligned}\|I_2(t,x) - I_2(t,\bar{x})\|_p^2 \le\, & Cp\,(1+T^2) C_\mu L(\sigma)^2 \left(\int_0^t ds \sup_{|z_1-z_2|=|x-\bar{x}|} \|u(s,z_1)-u(s,z_2)\|_p^2\right) \\ & + Cp\,(T+T^3) C_\mu^{(\gamma)} |x-\bar{x}|^{2\gamma} \left[c(\sigma) + L(\sigma) \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \|u(s,y)\|_p\right]^2, \end{aligned} \tag{4.44}$$

where C_μ, $C_\mu^{(\gamma)}$ are defined in (4.8), (4.43), *respectively.*

2. Assume that the hypotheses of Proposition 4.5 hold. Then, for any $p \in \left[2, \frac{\sqrt{L(b)}}{2^5 3 C_\mu L(\sigma)^2}\right]$, there exists a positive constant C such that, for every $x, \bar{x} \in \mathbb{R}^d$, $t \in [0,T]$,

$$\begin{aligned}\|I_2(t,x) - I_2(t,\bar{x})\|_p^2 \le\, & Cp\,(1+T^2) C_\mu L(\sigma)^2 \left(\int_0^t ds \sup_{|z_1-z_2|=|x-\bar{x}|} \|u(s,z_1)-u(s,z_2)\|_p^2\right) \\ & + Cp\,(T+T^3) C_\mu^{(\gamma)} |x-\bar{x}|^{2\gamma} \left[c(\sigma) + L(\sigma) e^{2T\sqrt{L(b)}} \mathcal{N}_{2\sqrt{L(b)},p}(u)\right]^2, \end{aligned} \tag{4.45}$$

where C_μ, $C_\mu^{(\gamma)}$ are defined in (4.8), (4.43), *respectively, with $\mathcal{N}_{2\sqrt{L(b)},p}(u)$ satisfying* (4.13).

Proof. To simplify the presentation, we will use the notation of [14][Theorem 3.1] that we now recall. For $s \in [0,T]$ and $x, \bar{x}, y, z \in \mathbb{R}^d$, set $\xi = x - \bar{x}$ and

$$\begin{aligned} \Sigma_x(s,y) &= \sigma(u(s,x-y)), \\ \Sigma_{x,\bar{x}}(s,y) &= \sigma(u(s,x-y)) - \sigma(u(s,\bar{x}-y)), \\ h_1(s,y,z) &= f(y-z)\Sigma_{x,\bar{x}}(s,y)\Sigma_{x,\bar{x}}(s,z), \\ h_2(s,y,z) &= \big[f(y-z+\xi) - f(y-z)\big]\Sigma_x(s,z)\Sigma_{x,\bar{x}}(s,y), \\ h_3(s,y,z) &= \big[f(y-z-\xi) - f(y-z)\big]\Sigma_x(s,y)\Sigma_{x,\bar{x}}(s,z), \end{aligned}$$

$$h_4(s,y,z) = \big[2f(y-z) - f(y-z+\xi) - f(y-z-\xi)\big]\Sigma_x(s,y)\Sigma_x(s,z).$$

Fix $p \in [2,\infty)$ and apply the Burkholder-Davies-Gundy inequality to obtain

$$\|I_2(t,x) - I_2(t,\bar{x})\|_p^2 \le 4p\|Q(t;x,\bar{x})\|_{\frac{p}{2}}, \tag{4.46}$$

where

$$Q(t;x,\bar{x}) = \int_0^t ds \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \sigma(u(s,y))\big[G(t-s,x-dy) - G(t-s,\bar{x}-dy)\big] \\ \times f(y-z)\sigma(u(s,z))\big[G(t-s,x-dz) - G(t-s,\bar{x}-dz)\big].$$

Use the transfer of increments strategy introduced in [8] (used also in [14]), to deduce

$$Q(t;x,\bar{x}) = \sum_{i=1}^4 Q_i(t;x,\bar{x}),$$

where, for $i = 1,\dots,4$,

$$Q_i(t;x,\bar{x}) = \int_0^t ds \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(t-s,dy)G(t-s,dz)h_i(s,y,z).$$

Therefore,

$$\|I_2(t,x) - I_2(t,\bar{x})\|_p^2 \le 4p\sum_{i=1}^4 \|Q_i(t;x,\bar{x})\|_{\frac{p}{2}}.$$

In the proof of Theorem 3.2 in [14] (where $d = 3$), bounds from above for each term $\|Q_i(t;x,\bar{x})\|_{\frac{p}{2}}$ are established. We will here sketch the proofs of these bounds with special attention on the value of constants that are relevant in our context. We will also check that the arguments of the proofs hold for $d = 1,2$, thereby providing a unified approach to the analysis in dimensions $d = 1,2,3$.

Upper bound of $\|Q_1(t;x,\bar{x})\|_{\frac{p}{2}}$.

Using Minkowski's inequality, then the Cauchy-Schwarz inequality and the Lipschitz property of σ, we obtain

$$\begin{aligned}
\|Q_1(t;x,\bar{x})\|_{\frac{p}{2}} &\le \int_0^t ds \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(t-s,dy)G(t-s,dz)f(y-z)\|\Sigma_{x,\bar{x}}(s,y)\Sigma_{x,\bar{x}}(s,z)\|_{\frac{p}{2}} \\
&\le L(\sigma)^2 \int_0^t ds \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(t-s,dy)G(t-s,dz)f(y-z) \\
&\qquad \times \Big[\sup_{|z_1-z_2|=|x-\bar{x}|} \|u(s,z_1) - u(s,z_2)\|_p^2\Big] \\
&\le L(\sigma)^2 \int_0^t ds J(t-s)\Big[\sup_{|z_1-z_2|=|x-\bar{x}|} \|u(s,z_1) - u(s,z_2)\|_p^2\Big] \\
&\le 2L(\sigma)^2(1+T^2)C_\mu \int_0^t ds \sup_{|z_1-z_2|=|x-\bar{x}|} \|u(s,z_1) - u(s,z_2)\|_p^2,
\end{aligned} \tag{4.47}$$

where in the last inequality we have used (4.7).

For the study of the remaining terms $\|Q_i(t;x,\bar{x})\|_{\frac{p}{2}}$, $i = 2,3,4$, in order to be in the setting of Lemma 4.2, we use a truncation argument on the processes $\Sigma_x(s,y)$, $\Sigma_{x,\bar{x}}(s,y)$.

For $k\geq 1$, set $\Sigma_x^k(s,y)=\Sigma_x(s,y)1_{\{|\Sigma_x(s,y)|\leq k\}}$, $\Sigma_{x,\bar{x}}^k(s,y)=\Sigma_{x,\bar{x}}(s,y)1_{\{|\Sigma_{x,\bar{x}}(s,y)|\leq k\}}$, and

$$Q_i^k(t;x,\bar{x})=\int_0^t ds\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}G(t-s,dy)G(t-s,dz)h_i^k(s,y,z),\ i=2,3,4,$$

where each $h_i^k(s,y,z)$ is defined as $h_i(s,y,z)$ by replacing $\Sigma_x(s,y)$ and $\Sigma_{x,\bar{x}}(s,y)$ by $\Sigma_x^k(s,y)$ and $\Sigma_{x,\bar{x}}^k(s,y)$, respectively.

Upper bound for $\|Q_2^k(t;x,\bar{x})\|_{\frac{p}{2}}$.

Apply Lemma 4.2 to the bounded functions $\varphi(z)=\Sigma_x^k(s,z)$ and $\psi(y)=\Sigma_{x,\bar{x}}^k(s,y)$. Then, up to the factor $(2\pi)^{-d}$, $Q_2^k(t;x,\bar{x})$ is equal to

$$\int_0^t ds\int_{\mathbb{R}^d}\overline{\mathcal{F}\big(\Sigma_x^k(s,.)G(t-s,.)\big)(\zeta)}\ \mathcal{F}\big(\Sigma_{x,\bar{x}}^k(s,.)G(t-s,.)\big)(\zeta)\left[e^{-i\xi.\zeta}-1\right]\mu(d\zeta),$$

where $\xi=x-\bar{x}$. Since $|e^{-i\xi.\zeta}-1|\leq C|\xi.\zeta|^\gamma\leq C|\xi|^\gamma|\zeta|^\gamma$ is valid for any $\gamma\in(0,1]$, and $2\sqrt{ab}\leq(a+b)$ for $a,b\geq 0$, computations similar to that in [14][p. 368] imply

$$\|Q_2^k(t;x,\bar{x})\|_{\frac{p}{2}}\leq C\left(\|Q_2^{k,1}(t;x,\bar{x})\|_{\frac{p}{2}}+\|Q_2^{k,2}(t;x,\bar{x})\|_{\frac{p}{2}}\right),\tag{4.48}$$

where

$$Q_2^{k,1}(t;x,\bar{x}):=|\xi|^{2\gamma}\int_0^t ds\int_{\mathbb{R}^d}\left|\mathcal{F}\big(\Sigma_x^k(s,.)G(t-s,.)\big)(\zeta)\right|^2|\zeta|^{2\gamma}\mu(d\zeta),$$

$$Q_2^{k,2}(t;x,\bar{x}):=\int_0^t ds\int_{\mathbb{R}^d}\left|\mathcal{F}\big(\Sigma_{x,\bar{x}}^k(s,.)G(t-s,.)\big)(\zeta)\right|^2\mu(d\zeta).$$

Set

$$J^{(\gamma)}(t)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mu(d\zeta)|\zeta|^{2\gamma}|\mathcal{F}G(t)(\zeta)|^2.$$

A slight modification in the proof of (4.7) and assumption **(h2)** imply that for $C_\mu^{(\gamma)}$ defined in (4.43), $J^{(\gamma)}(t)$ can be upper estimated as follows

$$J^{(\gamma)}(t)\leq 2(1+t^2)C_\mu^{(\gamma)}<\infty.\tag{4.49}$$

Using the Plancherel identity, the Minkowski inequality with respect to the non negative measure $[G(t-s,.)*G(t-s,.)](y)g_\gamma(y)\,dy\,ds$, once more the Plancherel identity and the equality $\tilde{G}(s,.)=G(s,.)$, we deduce

$$\begin{aligned}
&\|Q_2^{k,1}(t;x,\bar{x})\|_{\frac{p}{2}}\\
&\quad\leq C|x-\bar{x}|^{2\gamma}\left\|\int_0^t ds\int_{\mathbb{R}^d}\left[\big(\Sigma_x^k(s,.)G(t-s,.)\big)*\big(\widetilde{\Sigma_x^k(s,.)G(t-s,.)}\big)\right](y)g_\gamma(y)dy\right\|_{\frac{p}{2}}\\
&\quad\leq C|x-\bar{x}|^{2\gamma}\int_0^t ds\int_{\mathbb{R}^d}[G(t-s,.)*G(t-s,.)](y)g_\gamma(y)\sup_{y,z\in\mathbb{R}^d}\|\Sigma_x^k(s,z)\Sigma_x^k(s,y+z)\|_{\frac{p}{2}}dy\\
&\quad\leq C|x-\bar{x}|^{2\gamma}\sup_{(s,y)\in[0,T]\times\mathbb{R}^d}\|\Sigma_x^k(s,y)\|_p^2\int_0^t ds\int_{\mathbb{R}^d}\mu(d\zeta)|\zeta|^{2\gamma}|\mathcal{F}G(t-s)(\zeta)|^2\\
&\quad\leq C|x-\bar{x}|^{2\gamma}T\left(1+T^2\right)C_\mu^{(\gamma)}\sup_{(s,y)\in[0,T]\times\mathbb{R}^d}\|\Sigma_x^k(s,y)\|_p^2.
\end{aligned}$$

The definition of $\Sigma_x(s,y)$ implies

$$\sup_{(s,y)\in[0,T]\times\mathbb{R}^d}\|\Sigma_x(s,y)\|_p\leq c(\sigma)+L(\sigma)\sup_{(s,y)\in[0,T]\times\mathbb{R}^d}\|u(s,y)\|_p.$$

Therefore,

$$\|Q_2^{k,1}(t;x,\bar{x})\|_{\frac{p}{2}} \le C|x-\bar{x}|^{2\gamma}(T+T^3)C_\mu^{(\gamma)}\left[c(\sigma)+L(\sigma)\sup_{(s,y)\in[0,T]\times\mathbb{R}^d}\|u(s,y)\|_p\right]^2 \quad (4.50)$$

for some universal positive constant C.

With similar arguments, we obtain

$$\begin{aligned}\|Q_2^{k,2}(t;x,\bar{x})\|_{\frac{p}{2}} \le& C\int_0^t ds\big[G(t-s,.)*G(t-s,.)\big](y)f(y)\\ &\times\sup_{y,z\in\mathbb{R}^d}\|\Sigma_{x,\bar{x}}^k(s,y)\Sigma_{x,\bar{x}}^k(y+z)\|_{\frac{p}{2}}dy\\ &\le C\int_0^t ds\sup_{y\in\mathbb{R}^d}\|\Sigma_{x,\bar{x}}^k(s,y)\|_p^2\int_{\mathbb{R}^d}|\mathcal{F}(G(t-s,.))(\zeta)|^2\,\mu(d\zeta)\\ &\le C\int_0^t\sup_{y\in\mathbb{R}^d}\|\Sigma_{x,\bar{x}}^k(s,y)\|_p^2 J(t-s)ds\\ &\le C\,(1+T^2)\,C_\mu\int_0^t\sup_{y\in\mathbb{R}^d}\|\Sigma_{x,\bar{x}}^k(s,y)\|_p^2 ds,\end{aligned}$$

where in the last inequality, we have used (4.7).

The definition of $\Sigma_{x,\bar{x}}(s,y)$ and the Lipschitz property of σ imply

$$\sup_{y\in[0,T]\times\mathbb{R}^d}\|\Sigma_{x,\bar{x}}(s,y)\|_p \le L(\sigma)\sup_{|z_1-z_2|=|x-\bar{x}|}\|u(s,z_1)-u(s,z_2)\|_p.$$

Hence, since $\|\Sigma_{x,\bar{x}}^k(s,y)\|_p\le\|\Sigma_{x,\bar{x}}(s,y)\|_p$,

$$\|Q_2^{k,2}(t;x,\bar{x})\|_{\frac{p}{2}} \le C(1+T^2)C_\mu L(\sigma)^2\int_0^t ds\sup_{|z_1-z_2|=|x-\bar{x}|}\|u(s,z_1)-u(s,z_2)\|_p^2, \quad (4.51)$$

for some universal constant $C>0$.

Summarising, (4.48), along with (4.50) and (4.51), we have

$$\begin{aligned}\|Q_2^k(t;x,\bar{x})\|_{\frac{p}{2}} \le& C(T+T^3)C_\mu^{(\gamma)}|x-\bar{x}|^{2\gamma}\Big[c(\sigma)+L(\sigma)\sup_{(s,y)\in[0,T]\times\mathbb{R}^d}\|u(s,y)\|_p\Big]^2\\ &+C(1+T^2)C_\mu L(\sigma)^2\int_0^t ds\sup_{|z_1-z_2|=|x-\bar{x}|}\|u(s,z_1)-u(s,z_2)\|_p^2, \quad (4.52)\end{aligned}$$

for some universal positive constant C.

Notice that, since $|e^{-i\xi.\zeta}-1|=|e^{i\xi.\zeta}-1|$, exchanging y and z we deduce that the upper bound estimate in (4.52) also holds for $\|Q_3^k(t;x,\bar{x})\|_{\frac{p}{2}}$.

Upper bound of $\|Q_4^k(t;x,\bar{x})\|_{\frac{p}{2}}$.

To upper bound $\|Q_4^k(t;x,\bar{x})\|_{\frac{p}{2}}$, we use Lemma 4.2 with $\varphi=\psi=\Sigma_x^k(s,.)$; this yields

$$|Q_4^k(t;x,\bar{x})| \le \frac{1}{(2\pi)^d}\int_0^t ds\int_{\mathbb{R}^d}dy\left|1-e^{-i\xi.\zeta}+1-e^{i\xi.\zeta}\right|\left|\mathcal{F}\big(\varphi G(t-s,.)\big)\right|^2\mu(d\zeta)$$

Since $|1-\cos(\xi.\zeta)|\le C(|\xi||\zeta|)^{2\gamma}$ holds for $\gamma\in(0,1]$, Plancherel's identity implies

$$|Q_4^k(t;x,\bar{x})| \le C|\xi|^{2\gamma}\int_0^t ds\int_{\mathbb{R}^d}dy\; g_\gamma(y)\big[\big(\Sigma_x^k(s,.)G(t-s,.)\big)*\big(\widetilde{\Sigma_x^k(s,.)G(t}-s,.)\big)\big](y).$$

Consider the non negative measure $g_\gamma(y)\big[G(t-s,.)*G(t-s,.)\big](y)ds\, dy$. The Minkowski inequality with respect to this measure, the Plancherel identity and (4.49) yield

$$\begin{aligned}\|Q_4^k(t;x,\bar x)\|_{\frac{p}{2}} &\le C|\xi|^{2\gamma}\int_0^t ds\int_{\mathbb{R}^d} dy\ g_\gamma(y)\big[G(t-s,.)*G(t-s,.)\big](y)\\ &\qquad\qquad \times \sup_{y,z\in\mathbb{R}^d}\|\Sigma_x^k(s,y)\Sigma_x^k(s,y+z)\|_{\frac{p}{2}}\\ &\le C|\xi|^{2\gamma}\int_0^t ds \sup_{y\in\mathbb{R}^d}\|\Sigma_x(s,y)\|_p^2\, 2(1+T^2)C_\mu^{(\gamma)}.\end{aligned}$$

Thus, an argument similar to that proving (4.50) implies

$$\|Q_4^k(t;x,\bar x)\|_{\frac{p}{2}} \le C(T+T^3)C_\mu^{(\gamma)}\,|x-\bar x|^{2\gamma}\Big[c(\sigma)+L(\sigma)\sup_{(s,y)\in[0,T]\times\mathbb{R}^d}\|u(s,y)\|_p\Big]^2. \quad (4.53)$$

The upper estimates (4.46), (4.47), (4.52) and (4.53) conclude the proof of (4.44).

The statement in part 2 is an immediate consequence of the definition of $\mathcal{N}_{2\sqrt{L(b)},p}(u)$ and Proposition 4.5. The proof of the proposition is complete. □

From Propositions 4.5–4.9, we derive estimates for space increments of the random field solution (1.3) with $d=2,3$. They will be used later on to deduce estimates for time increments of $I_2(t,x)$. To write the statement in a more compact form, we introduce some notation. Let

$$K_0(u_0,v_0)=\begin{cases}\|v_0\|_{\gamma_2}+\|\nabla u_0\|_{\infty,R+T}+\|\nabla u_0\|_{\gamma_1}, & d=2,\\ \|v_0\|_{\gamma_2}+\|\nabla u_0\|_{\infty,R+T}+\|\Delta u_0\|_{\gamma_1}, & d=3.\end{cases} \quad (4.54)$$

Proposition 4.10. *We are assuming the following.*

1. *The initial value functions u_0 and v_0 satisfy the conditions of Proposition 4.6 with some Hölder exponents $\gamma_1,\gamma_2\in(0,1]$.*
2. *The coefficients σ and b are globally Lipschitz continuous functions.*
3. *The covariance measure Λ of the noise W satisfies* **(h1)***, and the corresponding spectral measure μ satisfies* **(h2)***.*

(i) Fix $T,R>0$. Then, for any $p\in[2,\infty)$ and $\alpha>0$, there exist positive constants $c_1(T,R)$, $c_2(T)$ and $c_3(T)$ such that if

$$\begin{aligned} C_1 &:= c_1(T,R)\ K_0(u_0,v_0),\\ C_2 &:= c_2(T)\left(pC_\mu^{(\gamma)}\right)^{\frac12}\left[c(\sigma)+L(\sigma)\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\|u(t,x)\|_p\right]\\ &\le c_2(T)\left(pC_\mu^{(\gamma)}\right)^{\frac12}\left[c(\sigma)+L(\sigma)e^{\alpha T}\mathcal{N}_{\alpha,p}(u)\right],\\ C_3 &:= c_3(T)\left[L(b)^2+p\,C_\mu\,L(\sigma)^2\right], \end{aligned} \quad (4.55)$$

with C_μ, $C_\mu^{(\gamma)}$, defined in (4.8), (4.43), respectively, then for any $t\in[0,T]$, and $x,\bar x\in B(0;R)$,

$$\sup_{|z_1-z_2|\le|x-\bar x|}\|u(t,z_1)-u(t,z_2)\|_p^2 \le \exp(TC_3)\left(C_1^2|x-\bar x|^{2(\gamma_1\wedge\gamma_2)}+C_2^2|x-\bar x|^{2\gamma}\right). \quad (4.56)$$

Consequently,

$$\sup_{t\in[0,T]} \sup_{|z_1-z_2|\le|x-\bar{x}|} \|u(t,z_1)-u(t,z_2)\|_p \le \tilde{C}|x-\bar{x}|^{\nu_1}, \tag{4.57}$$

with $\nu_1=\min(\gamma,\gamma_1,\gamma_2)$ *and*

$$\tilde{C}=(C_1+C_2)\exp(TC_3/2). \tag{4.58}$$

(ii) Suppose furthermore that the Lipschitz constants $L(b)$, $L(\sigma)$ *are such that* $L(b)\ge \left(2^{12}\,3^2\,C_\mu^2\,L(\sigma)^4\right)\vee\frac{1}{4}$. *Then, for* $p\in\left[2,\frac{\sqrt{L(b)}}{2^5\,3\,C_\mu\,L(\sigma)^2}\right]$ *we have*

$$C_2\le c_2(T)\left(p\,C_\mu^{(\gamma)}\right)^{\frac{1}{2}}\left[c(\sigma)+L(\sigma)\,e^{2T\sqrt{L(b)}}\mathcal{N}_{2\sqrt{L(b)},p}(u)\right],$$

with $\mathcal{N}_{2\sqrt{L(b)},p}(u)$ *satisfying* (4.13).

Proof. (i) Recall the decomposition $u(t,z_1)-u(t,z_2)=\sum_{i=0}^2[I_i(t,z_1)-I_i(t,z_2)]$ (see (2.8)); we first prove

$$\begin{aligned}&\sup_{|z_1-z_2|\le|x-\bar{x}|}\|u(t,z_1)-u(t,z_2)\|_p^2\\ &\quad\le C_1^2|x-\bar{x}|^{2(\gamma_1\wedge\gamma_2)}+C_2^2|x-\bar{x}|^{2\gamma}+C_3\int_0^t ds\sup_{|z_1-z_2|\le|x-\bar{x}|}\|u(s,z_1)-u(s,z_2)\|_p^2.\end{aligned} \tag{4.59}$$

Indeed, the first term on the right-hand side comes from (4.31) and (4.33). The second one comes from the last term on the right-hand side of (4.44). Finally, the very last term is obtained by the sum of the upper bound (4.36) (in the proof of Proposition 4.8) and the first term on the right-hand side of (4.44).

Apply the Gronwall lemma to the real valued function $t\mapsto\sup\{\|u(t,z_1)-u(t,z_2)\|_p^2 : |z_1-z_2|\le|x-\bar{x}|\}$ to obtain (4.56), and then (4.57).

The claim *(ii)* follows form the definition of $\mathcal{N}_{2\sqrt{L(b)},p}(u)$ and Proposition 4.5. □

Time increments of $I_2(t,x)$

In order to deduce L^p estimates of increments in time of the stochastic integral term $I_2(t,x)$, additional assumptions on the covariance of the noise are needed.

(h3) The spectral measure μ is such that there exists $\nu>0$ and $C>0$ for which

$$\int_{\mathbb{R}^d}|\mathcal{F}G(t)(\zeta)|^2\;\mu(d\zeta)\le Ct^\nu, \tag{4.60}$$

for any $t\in[0,T]$.

(h4) The covariance density function f satisfies the following conditions (1) and (2).

(1) There exists $b>0$ and $C>0$ such that for any $h\in[0,T]$,

$$\begin{aligned}&\int_0^T ds\;s\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}G(1,dy)G(1,dz)\\ &\qquad\times\left|f(s(y+z)+h(y+z))-f(s(y+z)+hz)\right|\le Ch^b.\end{aligned} \tag{4.61}$$

(2) There exists $\bar{b} > 0$ and $C > 0$ such that for any $h \in [0, T]$,

$$\begin{aligned}&\int_0^T ds\ s^2 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(1, dy)G(1, dz)\\&\quad\times \left|f\big(s(y+z)+h(y+z)\big) - f\big(s(y+z)+hy\big) - f\big(s(y+z)+hz\big) + f\big(s(y+z)\big)\right|\\&\le Ch^{\bar{b}}.\end{aligned} \tag{4.62}$$

According to the discussion in Section 4.1 (see (4.6)–(4.8)), the supremum over t on a bounded interval of the left-hand side of (4.60) is finite. Assumption **(h3)** provides a qualitative estimate on the way this supremum depends on t.

Up to scalings, the assumption **(h4)** is on estimates of one and two-dimensional increments of the covariance density in a L^2 type norm. We shall see later that in the particular example of Riesz covariance densities, **(h4)** is a consequence from the semigroup property of Riesz kernels (see [8]).

Proposition 4.11. *Assume that the hypotheses (1)–(3) of Proposition 4.10 hold. Suppose also that the hypotheses* **(h3)** *and* **(h4)** *on the covariance of the noise are satisfied. Then there exists a constant $C(T,\nu)$ such that for any $p \in [2,\infty)$, $t, \bar{t} \in [0,T]$ and $x \in \mathbb{R}^d$,*

$$\begin{aligned}\|I_2(t,x) - I_2(\bar{t},x)\|_p^2 \le C(T,\nu)p\Bigg(& C_\mu L(\sigma)^2 \tilde{C}^2 |t-\bar{t}|^{2\nu_1}\\&+ \Big[c(\sigma) + L(\sigma) \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \|u(t,x)\|_p\Big]^2 \big\{|t-\bar{t}|^{1+\nu} + |t-\bar{t}|^{\min(b+1,\bar{b},\tilde{\alpha})}\big\}\\&+ L(\sigma)\tilde{C}\Big[c(\sigma) + L(\sigma) \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \|u(t,x)\|_p\Big] |t-\bar{t}|^{\nu_1+\min(b,1)}\Bigg),\end{aligned} \tag{4.63}$$

where $\nu_1 = \min(\gamma,\gamma_1,\gamma_2)$, $\tilde{C}$ is defined in (4.58), $\tilde{\alpha} = (1+\nu)\wedge 2$ if $\nu \neq 1$ and $\tilde{\alpha} < 2$ if $\nu = 1$.

If, as in Proposition 4.5, the Lipschitz constants $L(b)$, $L(\sigma)$ are such that $L(b) \ge \big(2^{12}\, 3^2\, C_\mu^2\, L(\sigma)^4\big) \vee \frac{1}{4}$, where C_μ is given in (4.8), then there exists a constant $\overline{C(\nu,T)}$ such that for $p \in \Big[2, \frac{\sqrt{L(b)}}{2^5\, 3\, C_\mu\, L(\sigma)^2}\Big]$, $t, \bar{t} \in [0,T]$ and $x \in \mathbb{R}^d$,

$$\begin{aligned}\|I_2(t,x) - I_2(\bar{t},x)\|_p^2 \le \overline{C(\nu,T)}p\Bigg(& C(T)C_\mu L(\sigma)^2 \tilde{C}^2 |t-\bar{t}|^{2\nu_1}\\&+ \Big[c(\sigma) + L(\sigma)e^{2T\sqrt{L(b)}}\mathcal{N}_{2\sqrt{L(b)},p}(u)\Big]^2 \big\{|t-\bar{t}|^{1+\nu} + |t-\bar{t}|^{\min(b+1,\bar{b},\tilde{\alpha})}\big\}\\&+ L(\sigma)\tilde{C}\Big[c(\sigma) + L(\sigma)e^{2T\sqrt{L(b)}}\mathcal{N}_{2\sqrt{L(b)},p}(u)\Big] |t-\bar{t}|^{\nu_1+\min(b,1)}\Bigg),\end{aligned} \tag{4.64}$$

with $\mathcal{N}_{2\sqrt{L(b)},p}(u)$ satisfying (4.13).

Proof. For $0 \le t \le \bar{t} \le T$ and $x \in \mathbb{R}^d$, set

$$I_{2,1}(t,\bar{t};x) = \int_t^{\bar{t}} \int_{\mathbb{R}^d} G(\bar{t}-s, x-dy)\sigma(u(s,y))\ W(ds,dy).$$

By applying Burkholder-Davis-Gundy's inequality, and then the Minkowski and Cauchy-Schwarz inequalities, we deduce

$$\begin{aligned}\|I_{2,1}(t,\bar t;x)\|_p^2 &\le 4p\Big\|\int_t^{\bar t} ds \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(\bar t-s,x-dy)G(\bar t-s,x-dz)f(y-z)\\ &\qquad \times \sigma(u(s,y))\sigma(u(s,z))\Big\|_{\frac{p}{2}}\\ &\le 4p\int_t^{\bar t} ds \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(\bar t-s,x-dy)G(\bar t-s,x-dz)f(y-z)\\ &\qquad \times \|\sigma(u(s,y))\sigma(u(s,z)\|_{\frac{p}{2}}\\ &\le 4p\int_t^{\bar t} ds J(\bar t-s)\sup_{y\in\mathbb{R}^d}\|\sigma(u(s,y))\|_p^2\\ &\le p\ C|t-\bar t|^{1+\nu}\left[c(\sigma)+L(\sigma)\sup_{(s,y)\in[0,T]\times\mathbb{R}^d}\|u(s,y)\|_p\right]^2,\end{aligned}$$

where J is defined in (4.6), and the last upper estimate is deduced from **(h3)** (see (4.60)).

Let

$$I_{2,2}(t,\bar t;x)=\int_0^t\int_{\mathbb{R}^d}[G(\bar t-s,x-dy)-G(t-s,x-dy)]\ \sigma(u(s,y))\ W(ds,dy). \tag{4.65}$$

We study the L_p norm of this term following the proof of [14][Theorem 4.1]. This uses the transfer of increments trick introduced in [8][Section 3.2]. Applying the Burkholder-Davies-Gundy inequality, we obtain

$$\|I_{2,2}(t,\bar t;x)\|_p^2\le 4p\sum_{i=1}^4\|R_i(t,\bar t;x)\|_{\frac{p}{2}},$$

where, letting $h:=\bar t-t$ and $\Theta_{t,x}(s,y)=\sigma(u(t-s,x-y))$, we set

$$\begin{aligned}R_1(t,\bar t;x)&=\int_0^t ds\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}G(1,dy)G(1,dz)(s+h)^2 f\big((s+h)y-(s+h)z\big)\\ &\quad\times\big[\Theta_{t,x}(s,(s+h)y)-\Theta_{t,x}(s,sy)\big]\big[\Theta_{t,x}(s,(s+h)z)-\Theta_{t,x}(s,sz)\big],\\ R_2(t,\bar t;x)&=\int_0^t ds\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}G(1,dy)G(1,dz)\\ &\quad\times\big[(s+h)^2 f\big((s+h)y-(s+h)z\big)-s(s+h)f\big(sy-(s+h)z\big)\big]\\ &\quad\times\big[\Theta_{t,x}(s,(s+h)z)-\Theta_{t,x}(s,sz)\big]\Theta_{t,x}(s,sy),\\ R_3(t,\bar t;x)&=\int_0^t ds\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}G(1,dy)G(1,dz)\\ &\quad\times\big[(s+h)^2 f\big((s+h)y-(s+h)z\big)-s(s+h)f\big((s+h)y-sz\big)\big]\\ &\quad\times\big[\Theta_{t,x}(s,(s+h)y)-\Theta_{t,x}(s,sy)\big]\Theta_{t,x}(s,sz),\\ R_4(t,\bar t;x)&=\int_0^t ds\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}G(1,dy)G(1,dz)\\ &\quad\times\big[(s+h)^2 f\big((s+h)y-(s+h)z\big)-s(s+h)f\big(sy-(s+h)z\big)\\ &\qquad -s(s+h)f\big((s+h)y-sz\big)+s^2 f\big(sy-sz\big)\big]\end{aligned}$$

$$\times\, \Theta_{t,x}(s,sy)\Theta_{t,x}(s,sz).$$

Notice that the linear growth and Lipschitz continuity assumptions on σ imply that for any $p\in[2,\infty)$, every $s,t\in[0,T]$ and $x,y,z\in\mathbb{R}^d$,

$$\begin{aligned}\|\Theta_{t,x}(s,y)\|_p &\le c(\sigma)+L(\sigma)\|u(t-s,x-y)\|_p,\\ \|\Theta_{t,x}(s,y)-\Theta_{t,x}(s,z)\|_p &\le L(\sigma)\|u(t-s,x-y)-u(t-s,x-z)\|_p.\end{aligned}\tag{4.66}$$

Therefore,

$$\sup_{0\le s\le t\le T;(x,y)\in\mathbb{R}^d}\|\Theta_{t,x}(s,y)\|_p\le c(\sigma)+L(\sigma)\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\|u(t,x)\|_p,\tag{4.67}$$

and by applying (4.57), we deduce that for every $r>0$,

$$\begin{aligned}\sup_{0\le s\le t\le T;(x,y,z)\in\mathbb{R}^d,|y-z|\le r}\|\Theta_{t,x}(s,y)-\Theta_{t,x}(s,z)\|_p &\le L(\sigma)\sup_{t\in[0,T],|y-z|\le r}\|u(t,y)-u(t,z)\|_p\\ &\le L(\sigma)\tilde{C}r^{\nu_1},\end{aligned}\tag{4.68}$$

where $\tilde{C}$ is defined in (4.58).

Upper bound of $\|R_1(t,\bar{t};x)\|_{\frac{p}{2}}$.

We apply the Minkowski and Cauchy-Schwarz inequalities. Then, since the support of the measure $G(1,dy)$ is included in the closed ball $\overline{B(0;1)}$, and because of (4.68), we obtain

$$\begin{aligned}\|R_1(t,\bar{t};x)\|_{\frac{p}{2}} &\le L(\sigma)^2\tilde{C}^2|t-\bar{t}|^{2\nu_1}\\ &\quad\times\int_0^t ds\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}G(1,dy)G(1,dz)(s+h)^2 f\big((s+h)y-(s+h)z\big).\end{aligned}\tag{4.69}$$

Consider the change of variables $((s+h)y,(s+h)z)\mapsto(y,z)$; using (2.4), (4.6) and (4.7), we deduce

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}G(1,dy)G(1,dz)(s+h)^2 f\big((s+h)y-(s+h)z\big)\\ &\qquad=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}G(s+h,dy)G(s+h,dz)f(y-z)\\ &\qquad=(2\pi)^{-d}\int_{\mathbb{R}^d}|\mathcal{F}G(s+h)(\zeta)|^2\,\mu(d\zeta)\le 2(1+(2T)^2))C_\mu.\end{aligned}\tag{4.70}$$

Hence, (4.69), (4.70) imply

$$\|R_1(t,\bar{t};x)\|_{\frac{p}{2}}\le 2\left[1+(2T)^2\right]C_\mu L(\sigma)^2\tilde{C}^2|t-\bar{t}|^{2\nu_1},\tag{4.71}$$

where $\tilde{C}$ is defined in (4.58).

Upper bound of $\|R_2(t,\bar{t};x)\|_{\frac{p}{2}}$ *and* $\|R_3(t,\bar{t};x)\|_{\frac{p}{2}}$.

We will only consider $\|R_2(t,\bar{t};x)\|_{\frac{p}{2}}$, since $\|R_3(t,\bar{t};x)\|_{\frac{p}{2}}$ is similar.

Set

$$\begin{aligned}R_{2,1}(t,\bar{t};x)=\int_0^t ds\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}&G(1,dy)G(1,dz)\\ &\times s(s+h)\big[f\big((s+h)y-(s+h)z\big)-f\big(sy-(s+h)z\big)\big]\\ &\times\big[\Theta_{t,x}(s,(s+h)z)-\Theta_{t,x}(s,sz)\big]\Theta_{t,x}(s,sy),\end{aligned}$$

Apply the change of variable $z \mapsto -z$ along with the Minkowski and Cauchy-Schwarz inequalities to obtain

$$\begin{aligned}
&\|R_{2,1}(t,\bar{t};x)\|_{\frac{p}{2}} \\
&\quad\le \sup_{0\le s\le t\le T;(x,z_1,z_2)\in\mathbb{R}^d,|z_1-z_2|\le h} \left(\|\Theta_{t,x}(s,z_1)-\Theta_{t,x}(s,z_2)\|_p\right) \sup_{0\le s\le t\le T;(x,y)\in\mathbb{R}^d} \left(\|\Theta_{t,x}(s,y)\|_p\right) \\
&\qquad\times \int_0^t ds \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(1,dy)G(1,dz)s(s+h)\big|f\big((s(y+z)+h(y+z)\big)-f\big(s(y+z)+hz\big)\big| \\
&\quad\le CTL(\sigma)\tilde{C}|t-\bar{t}|^{\nu_1+b}\Big[c(\sigma)+L(\sigma)\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\|u(t,x)\|_p\Big], \qquad (4.72)
\end{aligned}$$

where we have used (4.67), (4.68) and assumption **(h4)** (see (4.61)).

Define

$$\begin{aligned}
R_{2,2}(t,\bar{t};x) = \int_0^t ds \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} &G(1,dy)G(1,dz)h(s+h)f\big((s+h)y-(s+h)z\big) \\
&\times \big[\Theta_{t,x}(s,(s+h)z)-\Theta_{t,x}(s,sz)\big]\Theta_{t,x}(s,sy),
\end{aligned}$$

A computation similar to that used to upper estimate $\|R_{2,1}(t,\bar{t};x)\|_p$ implies

$$\begin{aligned}
&\|R_{2,2}(t,\bar{t};x)\|_{\frac{p}{2}} \\
&\quad\le \sup_{0\le s\le t\le T;(x,z_1,z_2)\in\mathbb{R}^d,|z_1-z_2|\le h} \left(\|\Theta_{t,x}(s,z_1)-\Theta_{t,x}(s,z_2)\|_p\right) \sup_{0\le s\le t\le T;(x,y)\in\mathbb{R}^d} \left(\|\Theta_{t,x}(s,y)\|_p\right) \\
&\qquad\qquad\times \int_0^t ds \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(1,dy)G(1,dz)h(s+h)\big|f\big((s(y+z)+h(y+z)\big)\big| \\
&\quad\le CL(\sigma)\tilde{C}|t-\bar{t}|^{\nu_1}\Big[c(\sigma)+L(\sigma)\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\|u(t,x)\|_p\Big] \\
&\qquad\qquad\times \int_0^t ds \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(1,dy)G(1,dz)h(s+h)\big|f\big((s+h)y-(s+h)z\big)\big|.
\end{aligned}$$

Using the change of variables $((s+h)y,(s+h)z)\mapsto(y,z)$, (2.4), (4.6) and **(h3)**, we obtain

$$\begin{aligned}
\int_0^t ds \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} &G(1,dy)G(1,dz)h(s+h)\left[f((s+h)y-(s+h)z)\right] \\
&= h\int_0^t \frac{ds}{s+h}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(s+h,dy)G(s+h,dz)f(y-z) \\
&= (2\pi)^{-d}h\int_0^t \frac{ds}{s+h}\int_{\mathbb{R}^d}\mu(d\zeta)|\mathcal{F}G(s+h)(\zeta)|^2 \\
&\le C\ h\int_0^t ds\ (s+h)^{\nu-1} \le C\,T^\nu\ h.
\end{aligned}$$

Thus,

$$\|R_{2,2}(t,\bar{t};x)\|_{\frac{p}{2}} \le CT^\nu L(\sigma)\tilde{C}|t-\bar{t}|^{\nu_1+1}\Big[c(\sigma)+L(\sigma)\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\|u(t,x)\|_p\Big]. \qquad (4.73)$$

Since $R_2(t,\bar{t};x)=R_{2,1}(t,\bar{t};x)+R_{2,2}(t,\bar{t};x)$, from (4.72) and (4.73), we deduce,

$$\|R_2(t,\bar{t};x)\|_{\frac{p}{2}} \le C(T+T^\nu)\,L(\sigma)\tilde{C}|t-\bar{t}|^{\nu_1+\min(b,1)}\Big[c(\sigma)+L(\sigma)\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\|u(t,x)\|_p\Big]. \qquad (4.74)$$

Upper bound of $\|R_4(t,\bar{t};x)\|_{\frac{p}{2}}$.

Applying Minkowski's inequality and using (4.67), we obtain

$$\|R_4(t,\bar{t};x)\|_{\frac{p}{2}} \le \Big[c(\sigma) + L(\sigma) \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \|u(t,x)\|_p\Big]^2 I(t,h),$$

where

$$\begin{aligned} I(t,h) = \int_0^t ds \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(1,dy)G(1,dz)\Big|&(s+h)^2 f\big((s+h)(y-z)\big) \\ &- s(s+h)f\big(sy-(s+h)z\big) - s(s+h)f\big((s+h)y-sz\big) + s^2 f\big(s(y-z)\big)\Big|. \end{aligned}$$

Use the change of variable $z \mapsto -z$ to see that $I(t,h) = \sum_{j=1}^4 \tilde{I}_j(t,h)$, with

$$\begin{aligned} \tilde{I}_1(t,h) &= \int_0^t ds \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(1,dy)G(1,dz)s^2 \\ &\quad \times \big|f\big((s+h)(y+z)\big) - f\big(sy+(s+h)z\big) - f\big((s+h)y+sz\big) + f\big(s(y+z)\big)\big|, \\ \tilde{I}_2(t,h) &= \int_0^t ds \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(1,dy)G(1,dz)\ sh \\ &\quad \times \big|f\big((s+h)y+(s+h)z)\big) - f\big(sy-(s+h)z\big)\big|, \\ \tilde{I}_3(t,h) &= \int_0^t ds \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(1,dy)G(1,dz)\ sh \\ &\quad \times \big|f\big((s+h)y+(s+h)z\big) - f\big((s+h)y+sz\big)\big|, \\ \tilde{I}_4(t,h) &= \int_0^t ds \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(1,dy)G(1,dz)\ h^2\big|f\big((s+h)y+(s+h)z\big)\big|. \end{aligned}$$

The hypothesis **(h4)** implies $\tilde{I}_1(t,h) \le C\, h^{\bar{b}}$ and $\tilde{I}_2(t,h)+\tilde{I}_3(t,h) \le C\, h^{b+1}$, (see (4.62) and (4.61), respectively). As for $\tilde{I}_4(t,h)$, we apply the change of variables $((s+h)y,(s+h)z) \mapsto (y,z)$, (2.4), (4.6) and **(h3)**; this yields

$$\begin{aligned} \tilde{I}_4(t,h) &= \int_0^t \frac{ds}{(s+h)^2} h^2 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(s+h,dy)G(s+h,dz)f(y-z) \\ &= (2\pi)^{-d} h^2 \int_0^t \frac{ds}{(s+h)^2} \int_{\mathbb{R}^d} \mu(d\zeta)|\mathcal{F}G(s+h)(\zeta)|^2 \\ &\le C h^2 \int_0^t (s+h)^{\nu-2}\ ds. \end{aligned}$$

This yields for $h \in (0,T]$

$$\tilde{I}_4(t,h) \le C \times \begin{cases} h^2\, T^{\nu-1} & \nu > 1, \\ h^{\nu+1}, & \nu < 1, \\ T^{\epsilon} h^{2-\varepsilon}, & \nu = 1, \end{cases}$$

where $\varepsilon > 0$ is arbitrarily small.

Summarising the estimates above, we obtain

$$\|I_2(t,x) - I_2(\bar{t},x)\|_p^2 \le 2\left(\|I_{2,1}(t,\bar{t};x)\|_p^2 + \|I_{2,2}(t,\bar{t},x)\|_p^2\right)$$

$$
\begin{aligned}
&\leq Cp\Bigg(\Big[c(\sigma)+L(\sigma)\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\|u(t,x)\|_p\Big]^2|t-\bar{t}|^{1+\nu}\\
&\quad+C_\mu L(\sigma)^2\tilde{C}^2|t-\bar{t}|^{2\nu_1}\\
&\quad+(T+T^\nu)L(\sigma)\tilde{C}\Big[c(\sigma)+L(\sigma)\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\|u(t,x)\|_p\Big]|t-\bar{t}|^{\nu_1+\min(b,1)}\\
&\quad+\tilde{C}(\nu,T)\Big[c(\sigma)+L(\sigma)\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\|u(t,x)\|_p\Big]^2|t-\bar{t}|^{\min(b+1,\bar{b},\tilde{\alpha})}\Bigg),
\end{aligned}
\tag{4.75}
$$

where $\tilde{\alpha}=(1+\nu)\wedge 2$ if $\nu\neq 1$, while $\tilde{\alpha}<2$ if $\nu=1$, and $\tilde{C}(\nu,T)$ is a positive constant. This completes the proof of (4.63).

From (4.63), using Proposition 4.5, we deduce (4.64). This concludes the proof of the proposition. □

From Propositions 4.6–4.11 we deduce Theorem 4.12 below, which is the main ingredient towards obtaining uniform bounds on moments.

In the next Theorem,

$$
\nu_1=\min(\gamma,\gamma_1,\gamma_2),\ \nu_2=\min\left(\nu_1,\frac{1}{2}[\nu_1+\min(b,1)],\frac{1+\nu}{2},\frac{b+1}{2},\frac{\bar{b}}{2},\frac{\tilde{\alpha}}{2}\right). \tag{4.76}
$$

We recall that γ_1, γ_2, are the Hölder exponents of the initial values (see Proposition 4.6), γ is the parameter in the assumption **(h2)**, ν is defined in **(h3)**, b and $\bar{b}$ in **(h4)**, and $\tilde{\alpha}$ in the last part of the proof of Proposition 4.11.

Let

$$
\bar{K}_0(u_0,v_0)=\begin{cases}\|v_0\|_{\gamma_2}+\|\nabla u_0\|_{\infty,R+T}+\|\nabla u_0\|_{\gamma_1}+\|v_0\|_{\infty,R+T}, & d=2,\\ \|v_0\|_{\gamma_2}+\|\nabla u_0\|_{\infty,R+T}+\|\Delta u_0\|_{\gamma_1}, & d=3.\end{cases} \tag{4.77}
$$

Comparing this definition with (4.54), we see that $K_0(u_0,v_0)\leq\bar{K}_0(u_0,v_0)$.

Theorem 4.12. *1. Suppose that the hypotheses (1)–(3) of Proposition 4.10 hold, and that the conditions* **(h3)** *and* **(h4)** *on the covariance of the noise are satisfied. Fix $T,R>0$. Then, for any $p\in[2,\infty)$, there exists a constant $C(p,T,R)$ such that, for any $t,\bar{t}\in[0,T]$, $x,\bar{x}\in B(0;R)$ and $\alpha>0$,*

$$
\frac{\|u(t,x)-u(\bar{t},\bar{x})\|_p}{|x-\bar{x}|^{\nu_1}+|t-\bar{t}|^{\nu_2}}\leq C(p,T,R)\Big[\mathcal{M}_1+\mathcal{M}_2+\mathcal{M}_3\ e^{T\alpha}\mathcal{N}_{\alpha,p}(u)\Big], \tag{4.78}
$$

where

$$
\begin{aligned}
\mathcal{M}_1&=\bar{K}_0(u_0,v_0)\left\{1+\left[L(b)+\sqrt{p}\left(1+\sqrt{C_\mu}\right)L(\sigma)\right]\exp\left(\frac{TC_3}{2}\right)\right\},\\
\mathcal{M}_2&=c(b)+\sqrt{p}c(\sigma)\left\{1+(C_\mu^{(\gamma)})^{1/2}\left[L(b)+\sqrt{p}\left(1+\sqrt{C_\mu}\right)L(\sigma)\right]\exp\left(\frac{TC_3}{2}\right)\right\},\\
\mathcal{M}_3&=\left[L(b)+\sqrt{p}\left(1+\sqrt{C_\mu}\right)L(\sigma)\right]\left\{1+(pC_\mu^{(\gamma)})^{1/2}L(\sigma)\exp\left(\frac{TC_3}{2}\right)\right\},
\end{aligned}
\tag{4.79}
$$

with $\bar{K}_0(u_0,v_0)$ and C_3 given in (4.77) and (4.55), respectively.

2. In addition to the assumptions of part 1., suppose that $L(b) \geq \left(2^{12}3^2C_\mu^2 L(\sigma)^4\right) \vee \frac{1}{4}$. *Then, for any* $p \in \left[2, \frac{\sqrt{L(b)}}{2^5 3 C_\mu L(\sigma)^2}\right]$,

$$\frac{\|u(t,x)-u(\bar{t},\bar{x})\|_p}{|x-\bar{x}|^{\nu_1}+|t-\bar{t}|^{\nu_2}} \leq C(p,T,R)\left[\mathcal{M}_1+\mathcal{M}_2+\mathcal{M}_3\ e^{2T\sqrt{L(b)}}\left(\mathcal{T}_0+\frac{c(b)}{L(b)}+\frac{c(\sigma)}{L(\sigma)}\right)\right], \tag{4.80}$$

with $\mathcal{T}_0$ *defined in* (4.14).

Proof. Fix $x \in B(0;R)$ and consider the time increment $\|u(t,x)-u(\bar{t},x)\|_p$, with $t,\bar{t} \in [0,T]$. Using the estimates (4.29), (4.32) for the increments of I_0 in dimension $d = 2,3$, respectively, then (4.41), (4.42) with α instead of $2\sqrt{L(b)}$ for the increments of I_1, and finally (4.64) and (4.58) for the increments of I_2, we obtain

$$\begin{aligned}\|u(t,x)-u(\bar{t},x)\|_p &\leq C(T,R)\Big\{\bar{K}_0(u_0,v_0)|t-\bar{t}|^{\min(\gamma_1,\gamma_2)}+\left[c(b)+L(b)e^{T\alpha}\mathcal{N}_{\alpha,p}(u)\right]|t-\bar{t}| \\ &\quad+\tilde{C}\left(\left[L(b)+\sqrt{p}\sqrt{C_\mu}L(\sigma)\right]|t-\bar{t}|^{\nu_1}+\sqrt{p}L(\sigma)|t-\bar{t}|^{\frac{1}{2}[\nu_1+\min(b,1)]}\right) \\ &\quad+\sqrt{p}\left[c(\sigma)+L(\sigma)e^{T\alpha}\mathcal{N}_{\alpha,p}(u)\right] \\ &\qquad\times\left[|t-\bar{t}|^{\frac{1}{2}[\nu_1+\min(b,1)]}+|t-\bar{t}|^{\frac{1+\nu}{2}}+|t-\bar{t}|^{\frac{1}{2}\min(b+1,\bar{b},\tilde{\alpha})}\right]\Big\},\end{aligned} \tag{4.81}$$

where we have applied the inequality $\sqrt{AB} \leq \frac{1}{2}(A+B)$ to the product of constants

$$A := \tilde{C}L(\sigma) \quad \text{and} \quad B := \left[c(\sigma)+L(\sigma)e^{T\alpha}\mathcal{N}_{\alpha,p}(u)\right]$$

appearing in the last line of (4.64) with α instead of $2\sqrt{L(b)}$.

Since by (4.57) we have $\sup_{t\in[0,T]}\|u(t,x)-u(t,\bar{x})\|_p \leq \tilde{C}|x-\bar{x}|^{\nu_1}$ for $x,\bar{x} \in B(0;R)$, we deduce that the L^p norm or space-time increment $\|u(t,x)-u(\bar{t},\bar{x})\|_p$ is bounded from above by the sum of the left-hand side of (4.81) and $\tilde{C}|x-\bar{x}|^{\nu_1}$. Using the definition of $\tilde{C}$ (see (4.58)) and grouping terms, we obtain the inequality (4.78).

The assertion of part 2. follows from Proposition 4.5 (see in particular (4.13)). This concludes the proof of the Proposition. □

From statement 2. of Proposition 4.12, in a similar way as in Section 3, we apply the version of the Kolmogorov continuity lemma given in [10] [Theorem A.3.1] to deduce uniform L^p moments estimates. Such estimates, stated in the following Proposition, are the key ingredient in the proof of existence and uniqueness of global random field solution to (1.3).

To simplify the notation, set

$$\mathcal{K}(c(b),c(\sigma),L(b),L(\sigma)) = \mathcal{M}_1+\mathcal{M}_2+\mathcal{M}_3 e^{2T\sqrt{L(b)}}\left(\mathcal{T}_0+\frac{c(b)}{L(b)}+\frac{c(\sigma)}{L(\sigma)}\right), \tag{4.82}$$

where $\mathcal{M}_j$, $j=1,2,3$ and $\mathcal{T}_0$ are defined in (4.79) and (4.14), respectively. Observe that, up to a constant factor depending on T, $\mathcal{K}(c(b),c(\sigma),L(b),L(\sigma))$ equals the right-hand side of (4.80).

Proposition 4.13. *Suppose that the hypotheses (1)–(3) of Proposition 4.10 hold. and also that the hypotheses* **(h3)** *and* **(h4)** *on the covariance of the noise are satisfied. Let*

ν_1 and ν_2 be the parameters defined in (4.76)*. Suppose that the Lipschitz coefficients $L(b)$ and $L(\sigma)$ satisfy $L(b) \ge \big(2^{12}3^2C_\mu^2 L(\sigma)^4\big) \vee \frac{1}{4}$ and*

$$\frac{\sqrt{L(b)}}{2^5 3 C_\mu L(\sigma)^2} > \frac{1}{\nu_1} + \frac{d}{\nu_2}, \quad d = 1, 2, 3. \tag{4.83}$$

Then, for any $p \in \Big(\frac{1}{\nu_1} + \frac{d}{\nu_2}, \frac{\sqrt{L(b)}}{2^5 3 C_\mu L(\sigma)^2}\Big]$, there exists positive constants C_1 and $C_2(p,T,R)$ such that

$$E\Big(\sup_{(t,x)\in[0,T]\times B(0;R)} |u(t,x)|^p\Big) \le 2^{p-1}C_1 + C_2(p,T,R)\mathcal{K}(c(b), c(\sigma), L(b), L(\sigma)), \tag{4.84}$$

with $\mathcal{K}(c(b), c(\sigma), L(b), L(\sigma))$ defined in (4.82)*.*

The proof is analogous to that of Proposition 3.3 and is omitted.

4.4. **Existence and uniqueness of a global solution.** In this section, we consider the equation (1.3) in spatial dimension $d = 2, 3$. We assume that the coefficients b and σ satisfy the hypothesis **(Cs)** of Section 3.3, thereby having superlinear growth. We also assume that b dominates σ, meaning condition **(Cd)** below.

(Cd) The parameters δ and a in (1.2) satisfy one of the properties:
(1) $\delta > 4a$,
(2) $\delta = 4a$ and θ_2 and σ_2 are such that $\theta_2 > 2^{12}3^2C_\mu^2\sigma_2^4\big(\frac{1}{\nu_1} + \frac{d}{\nu_2}\big)^2$, $d = 2, 3$,
where C_μ is defined in (4.8) and ν_1, ν_2 are given in (4.76).

The next theorem, which states existence and uniqueness of a global random field solution to (1.3) for $d = 2, 3$, is the main result of this section.

Theorem 4.14. *The hypothesis are as follows.*

(1) *The initial values u_0 and v_0 are functions with compact support included in $B(0;\rho)$ for some $\rho > 0$, and satisfy the hypotheses of Proposition 4.6 with some Hölder exponents $\gamma_1, \gamma_2 \in (0,1)$.*
(2) *The coefficients b and σ satisfy* **(Cs)** *and* **(Cd)** *with $\delta < \frac{1}{2}$.*
(3) *The covariance of the noise satisfies conditions* **(h1)**, **(h2)**, **(h3)** *and* **(h4)**.

Then, there exists a random field solution $\big(u(t,x), (t,x) \in [0,T]\times\mathbb{R}^d\big)$ to (1.3)*. This solution is unique and is such that*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |u((t,x)| < \infty, \ a.s.$$

Proof. We use the same approach as in the proof of Theorem 3.4. First, for $g = b, \sigma$, we consider the truncated globally Lipschitz functions b_N, σ_N, defined in (3.26). The assumption **(Cs)** imply that (3.27) holds. Moreover, by **(Cd)**, we see that the Lipschitz coefficients $L(b_N)$, $L(\sigma_N)$ satisfy the hypotheses of Proposition 4.13.

Let $u_N = \big(u_N(t,x), (t,x) \in [0,T]\times\mathbb{R}^d\big)$ be the unique global random field solution to (1.3) with coefficients b_N, σ_N. Under the standing hypotheses, we can apply Proposition 4.13 to the stochastic process u_N to deduce that, for any $p \in \Big(\frac{1}{\nu_1} + \frac{d}{\nu_2}, \frac{\sqrt{L(b_N)}}{2^5 3 C_\mu L(\sigma_N)^2}\Big]$ (and N large enough if necessary), there exist positive constants C_1 and $C_2(p,T,R)$, not depending on N, such that

$$E\Big(\sup_{(t,x)\in[0,T]\times B(0;R)} |u_N(t,x)|^p\Big) \le 2^{p-1}C_1 + C_2(p,T,R)\mathcal{K}(c(b_N), c(\sigma_N), L(b_N), L(\sigma_N)). \tag{4.85}$$

Here, $\mathcal{K}(c(b_N), c(\sigma_N), L(b_N), L(\sigma_N))$ is given by (4.82), with $c(b)$, $c(\sigma)$, $L(b)$, $L(\sigma)$ replaced by $c(b_N)$, $c(\sigma_N)$, $L(b_N)$, $L(\sigma_N)$. Recall that

$$c(b_N) = \theta_1, \quad c(\sigma_N) = \sigma_1, \quad L(b_N) = \theta_2(\ln(2N))^\delta, \quad L(\sigma_N) = \sigma_2(\ln(2N))^a,$$

(see (3.27)). Because of **(Cd)**, and since $\max(a, \delta) = \delta < \frac{1}{2}$, we have

$$\mathcal{K}(c(b_N), c(\sigma_N), L(b_N), L(\sigma_N)) = o(N^p). \tag{4.86}$$

Consider the sequence of increasing stopping times defined in (3.30). Using (4.86), we see that $\sup_N \tau_N = T$, a.s. By the standard localization argument (see the details of the proof of Theorem 3.4), we finish the proof. □

5. Examples of covariance densities

In this section, we give three examples of spatial covariances which satisfy the assumptions of section 4. For $d = 3$, the same covariances are studied in [14].

5.1. **Riesz kernels.** For $\beta \in (0, d)$, let $f_\beta : \mathbb{R}^d \to [0, +\infty]$ be defined by $f_\beta(x) = |x|^{-\beta}$ for $x \in \mathbb{R}^d \setminus \{0\}$, and $f_\beta(0) = +\infty$. The inverse Fourier transform, is

$$\left(\mathcal{F}^{-1} f\right)(\zeta) = c_{d,\beta} f_{d-\beta}(\zeta)\ d\zeta,\ c_{d,\beta} = 2^{-\beta+d/2} \frac{\Gamma\left(\frac{d-\beta}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)}, \tag{5.1}$$

where Γ denotes the Euler Gamma function (see [27][Chapter V]).

Let Λ be the non-negative definite tempered distribution given by $\Lambda(dx) = f_\beta(x)\ dx$. According to (5.1), its spectral measure is $\mu_\beta(d\zeta) = c_{d,\beta} f_{d-\beta}(\zeta)\ d\zeta$. Observe that the integral $\int_{\mathbb{R}^d} \frac{\mu_\beta(d\zeta)}{1+|\zeta|^2}$ converges if and only if $\beta \in (0, 2 \wedge d)$.

In the remaining of this section, we consider the dimensions $d = 2, 3$, and assume that $\beta \in (0, 2)$. From the previous discussion, we obtain that μ_β satisfies condition **(h0)**. Since f_β is a lower semi-continuous function, from Remark 4.1 we see that it satisfies **(h1)**.

Let $\gamma \in (0, 1)$. Using polar coordinates if $d = 2$ and spherical coordinates if $d = 3$, we have

$$\int_{\mathbb{R}^d} \frac{\mu_\beta(d\zeta)}{1 + |\zeta|^{2-2\gamma}} = C_{\beta,d} \int_0^\infty \frac{\rho^{\beta-1}}{1 + \rho^{2-2\gamma}} d\rho.$$

The integral on the right-hand side is finite if and only if $\gamma < (2-\beta)/2$. Since $|\zeta|^{2\gamma}\mu_\beta(d\zeta) = c_{d,\beta}|\zeta|^{-(d-\beta-2\gamma)}d\zeta$, and the Fourier transform of this measure is $g_\gamma(x) = \tilde{c}(\beta, d)|x|^{-(\beta+2\gamma)}$ (for some positive constant $\tilde{c}(\beta, d)$), if $\beta + 2\gamma < d$, the function $|\zeta|^{2\gamma}\mu_\beta(d\zeta)$ is locally integrable. Therefore, μ_β satisfies the condition **(h2)** for any $\gamma \in (0, (2 - \beta)/2)$.

Apply the change of variable $\eta = t\zeta$ to deduce

$$\int_{\mathbb{R}^d} |\mathcal{F}G(t)(\zeta)|^2\ \mu_\beta(d\zeta) = c_{d,\beta} \int_{\mathbb{R}^d} \frac{\sin^2(t|\zeta|)}{|\zeta|^2} |\zeta|^{-d+\beta} d\zeta = c_{d,\beta}\ t^{2-\beta} \int_{\mathbb{R}^d} \frac{\sin^2(|\eta|)}{|\eta|^{2+d-\beta}} d\eta.$$

Since the integral $I_{d,\beta} := \int_{\mathbb{R}^d} \frac{\sin^2(|\eta|)}{|\eta|^{2+d-\beta}} d\eta$ is finite, μ_β satisfies the condition **(h3)** with $\nu = 2 - \beta$ and $C := c_{d,\beta}\ I_{d,\beta}$.

The function f_β satisfies the condition **(h4)**(1) for any $b \in (0, \min(2 - \beta, 1))$. Indeed, the proof relies on [8][Lemma 2.6, p. 10] (which holds in any dimension $d \geq 1$) as follows.

Choose $b > 0$ satisfying $0 < \beta + b < d$. Letting $a := d - (\beta + b)$, we have $a + b \in (0, d)$. Then, by applying Lemma 2.6 (a) in [8], we have

$$f_\beta(s(y+z) + h(y+z)) - f_\beta(s(y+z) + hz) = |h|^b \int_{\mathbb{R}^d} dw |s(y+z) + hz - hw|^{-(\beta+b)} \times \left[|w+y|^{-(d-b)} - |w|^{-(d-b)} \right].$$

Consequenly, **(h4)**(1) will be established if we prove

$$\int_0^T ds\ s \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(1,dy)G(1,dz) \times \int_{\mathbb{R}^d} dw \big| s(y+z) + hz - hw \big|^{-(\beta+b)} \left| |w+y|^{-(d-b)} - |w|^{-(d-b)} \right| < \infty. \quad (5.2)$$

A small modification of the proof of Lemma 6.4 in [8] shows that (5.2) holds for $d = 2, 3$, and for any b such that $b \in (0, \min(2-\beta, 1))$. We refer also to [14][Proposition 5.3, p. 383-385] for a proof of (5.2) in dimension $d = 3$. Going through the details of the proof of this proposition, we see that it can be extended to $d = 2$, thanks to Lemma 4.2. This completes the proof of the validity of **(h4)**(1) for f_β, with $b \in (0, \min(2-\beta, 1))$.

Finally, we prove that f_β, satisfies the condition **(h4)**(2) for any $\bar{b} \in (0, 2-\beta)$. The proof relies on [8][Lemma 2.6, p. 10]. As in the proof of **(h4)**(1), we choose $\bar{b} > 0$ satisfying $0 < \beta + \bar{b} < d$. Letting $a := d - (\beta + \bar{b})$, we have $a + \bar{b} \in (0, d)$. Then, Lemma 2.6 (e) in [8] implies

$$\begin{aligned} &f_\beta\big(s(y+z) + h(y+z)\big) - f_\beta\big(s(y+z) + hy\big) - f_\beta\big(s(y+z) + hz\big) + f_\beta\big(s(y+z)\big) \\ &\quad \le |h|^{\bar{b}} \int_{\mathbb{R}^d} dw\ |y - hz|^{-(\beta+\bar{b})} \\ &\qquad \times \left[|w + hy + hz|^{-(d-\bar{b})} - |w + hy|^{-(d-\bar{b})} - |w + hz|^{-(d-\bar{b})} + |w|^{-(d-\bar{b})} \right]. \end{aligned}$$

Consequently, **(h4)**(2) will follow from

$$\int_0^T ds\ s^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(1,dy)G(1,dz) \int_{\mathbb{R}^d} dw |y - hz|^{-(\beta+\bar{b})} \times \left| |w + hy + hz|^{-(d-\bar{b})} - |w + hy|^{-(d-\bar{b})} - |w + hz|^{-(d-\bar{b})} + |w|^{-(d-\bar{b})} \right| < \infty. \quad (5.3)$$

With a slight modification (and simplification) of the proof of Lemma 6.5 in [8], we can check that (5.3) holds for $d = 2, 3$ and for any $\bar{b} \in (0, 2-\beta)$. Using ideas introduced in the proof of this lemma, [14][Proposition 5.3, p. 385-386] provides also a proof of (5.3) in dimension $d = 3$ with $\bar{b} \in (0, 2-\beta)$. Going through the details of the proof of this proposition, we see that it can be extended to $d = 2$, thanks once more to Lemma 4.2. Therefore, f_β satisfies **(h4)**(2) with $\bar{b} \in (0, 2-\beta)$.

Conclusion. Let $d = 2, 3$ and $\beta \in (0, 2)$. For spatially homogeneous Gaussian noises with covariance function given by (4.1) with $\Lambda(dx) = f_\beta(x)\ dx$, the parameters ν_1, ν_2 in (4.76) are $\nu_1 = \nu_2 = \min(\gamma, \gamma_1, \gamma_2)$, with $\gamma < \frac{2-\beta}{2}$.

Observe that, as a by-product, from (4.78) we deduce that, almost all sample paths of the solution to (1.3) are locally Hölder continuous, jointly in (t, x), with exponent $\theta \in]0, \min((2-\beta)/2, \gamma_1, \gamma_2[$. For $d = 3$, this is [8][Theorem 4.11, p. 48]. Moreover, the critical exponent $\min((2-\beta)/2, \gamma_1, \gamma_2)$ is sharp in both dimensions, $d = 2, 3$ (see [8], [9]).

5.2. **Bessel kernels.** For any $\kappa > 0$, the Bessel kernel is the function defined by $\tilde{f}_\kappa(x) = \int_0^\infty w^{\frac{\kappa-d-2}{2}} e^{-w} e^{-\frac{|x|^2}{4w}} dw$ for $x \in \mathbb{R}^d \setminus \{0\}$, $\tilde{f}_\kappa(0) = \infty$ if $0 < \kappa \le d$, and $\tilde{f}_\kappa(0) = c(d,\kappa)$ if $\kappa > d$, where $0 < c(d,\kappa) < \infty$ (see [27][Chapter V]) and [1]). The inverse Fourier transform is

$$\left(\mathcal{F}^{-1}\tilde{f}_\kappa\right)(\zeta) = C_{d,\kappa}\big(1+|\zeta|^2\big)^{-\frac{\kappa}{2}}. \tag{5.4}$$

Let Λ be the measure defined by $\Lambda(dx) = \tilde{f}_\kappa(x)\ dx$. From (5.4), we have that the corresponding spectral measure is $\tilde{\mu}_\kappa(d\zeta) = C_{d,\kappa}\big(1+|\zeta|^2\big)^{-\frac{\kappa}{2}}\ d\zeta$.

Throughout this section, we consider the case $d = 2,3$, and we assume $\kappa > d-2$. Our aim is to prove that hypotheses **(h0)**–**(h4)** are satisfied.

The function $\tilde{f}_\kappa$ is lower-semi continuous, and Remark 4.1 implies that condition **(h1)** holds.

Fix $\gamma \ge 0$. Using polar coordinates for $d = 2$ and spherical coordinates for $d = 3$, we obtain

$$\int_{\mathbb{R}^d} \frac{\tilde{\mu}_\kappa(d\zeta)}{1+|\zeta|^{2-2\gamma}} = C(d,\kappa)\int_0^\infty r^{d-1}\Big(\frac{1}{1+r^2}\Big)^{\frac{\kappa}{2}} \frac{1}{1+r^{2-2\gamma}}\ dr.$$

The integral on the right-hand side is finite if and only if $2\gamma < \kappa - d + 2$. Take $\gamma = 0$ to deduce that **(h0)** holds. Furthermore, for $\gamma \in \big(0, \min\big(\frac{\kappa-d+2}{2}, 1\big)\big)$, the constant $C^{(\gamma)}_{\tilde{\mu}_\kappa}$ defined in (4.43) is finite and therefore, **(h2)** holds.

We next check **(h3)**. For any $t > 0$ we have

$$\int_{\mathbb{R}^d} |\mathcal{F}G(t)(\zeta)|^2 \tilde{\mu}_\kappa(d\zeta) = C_{d,\kappa}\int_{\mathbb{R}^d} \frac{\sin^2(t|\zeta|)}{|\zeta|^2\,(1+|\zeta|^2)^{\frac{\kappa}{2}}}\ d\zeta = \tilde{C}_{d,\kappa}\int_0^\infty r^{d-1}\frac{\sin^2(tr)}{r^2}\big(1+r^2\big)^{-\frac{\kappa}{2}} dr.$$

Splitting the last integral, for any $t \in (0,T]$ we obtain

$$\begin{aligned}\int_0^\infty r^{d-1}\frac{\sin^2(tr)}{r^2}\big(1+r^2\big)^{-\frac{\kappa}{2}} dr &\le \int_0^1 t^2 r^{d-1} dr + \int_1^{\frac{T}{t}} t^2 r^{d-1-\kappa} dr + \int_{\frac{T}{t}}^\infty r^{d-1} r^{-2} r^{-\kappa} dr \\ &\le \frac{t^2}{d} + \int_1^{\frac{T}{t}} t^2 r^{d-1-\kappa} dr + \frac{1}{\kappa-d+2}\Big(\frac{t}{T}\Big)^{\kappa-d+2}.\end{aligned}$$

Furthermore, by setting $I(t) := \int_1^{\frac{T}{t}} t^2 r^{d-1-\kappa} dr$, we have

$$I(t) \le \begin{cases} \frac{t^2}{d-\kappa}\Big(\frac{T}{t}\Big)^{d-\kappa}, & \text{if} \quad d-2<\kappa<d, \\ t^2 \ln\Big(\frac{T}{t}\Big), & \text{if} \quad \kappa = d, \\ \frac{t^2}{\kappa-d}, & \text{if} \quad d<\kappa. \end{cases}$$

This implies

$$\int_{\mathbb{R}^d} |\mathcal{F}G(t)(\zeta)|^2 \tilde{\mu}_\kappa(d\zeta) \le C(d,\kappa,T)\ t^\nu,\ \ t \in [0,T],$$

where $C(d,\kappa,T)$ is some positive constant and $\nu < \min(2, \kappa-d+2)$. Therefore, **(h3)** holds with $\nu < \min(2,\kappa-d+2)$.

For $d = 3$, the validity of **(h4)** is proved in [14][Section 5.3]. Going through the arguments of this reference, we see that they also hold for $d = 2$. For the sake of completeness we give some details.

Let us first focus on **(h4)**(1). Set

$$I_1 = \int_0^T ds\, s \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(1,dy)G(1,dz)\big|\tilde f_\kappa\big((s+h)(y+z)\big) - \tilde f_\kappa\big((s+h)z+sy\big)\big|.$$

Fix $y,z\in\mathbb{R}^d$, $s,h\in[0,T]$ and $b\in(0,1)$. By writing the inequality in [14][p. 390, line 3] with $x:=(s+h)z+sy$ and $\xi:=y$, we see that

$$\begin{aligned}&\left|e^{-\frac{|(s+h)(y+z)|^2}{4w}} - e^{-\frac{|(s+h)z+sy|^2}{4w}}\right|\\ &\qquad \le C\left(\frac{h}{\sqrt{w}}\right)^b \int_0^1 d\lambda\ \left(e^{-\frac{|(s+h)z-(s+\lambda h)y|^2}{8w}} + e^{-\frac{|(s+h)(y+z)|^2}{4w}} + e^{-\frac{|(s+h)z+sy|^2}{4w}}\right).\end{aligned}$$

Hence, by the definition of $\tilde f_\kappa$,

$$\begin{aligned}\big|\tilde f_\kappa\big((s+h)(y+z)\big)-\tilde f_\kappa\big((s+h)z+sy\big)\big| &\le Ch^b\int_0^1 d\lambda \int_0^\infty dw\ w^{\frac{\kappa-b-d-2}{2}}\ e^{-w}\\ &\quad\times\left(e^{-\frac{|(s+h)z-(s+\lambda h)y|^2}{8w}} + e^{-\frac{|(s+h)(y+z)|^2}{4w}} + e^{-\frac{|(s+h)z+sy|^2}{4w}}\right),\end{aligned}$$

and therefore,

$$I_1 \le C\, h^b \int_0^\infty dw\ w^{\frac{\kappa-b-d-2}{2}}\ e^{-w}\sum_{i=1}^3 T_i(w),$$

where

$$\begin{aligned}T_1(w) &= \int_0^1 d\lambda \int_0^T s\,ds \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(1,dy)G(1,dz)e^{-\frac{|(s+h)z-(s+\lambda h)y|^2}{8w}},\\ T_2(w) &= \int_0^T s\,ds \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(1,dy)G(1,dz)e^{-\frac{|(s+h)(y+z)|^2}{4w}},\\ T_3(w) &= \int_0^T s\,ds \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(1,dy)G(1,dz)e^{-\frac{|(s+h)z+sy|^2}{4w}}.\end{aligned}$$

Apply the change of variables $x=(s+h)z$, $\bar x=(s+\lambda h)y$, and then Parseval's identity, to deduce

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(1,dy)G(1,dz)e^{-\frac{|(s+h)z-(s+\lambda h)y|^2}{8w}}\\ &\quad = \frac{1}{(s+h)(s+\lambda h)}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(s+h,dx)G(s+\lambda h,d\bar x)e^{-\frac{|x-\bar x|^2}{8w}}\\ &\quad = \left(\frac{2}{\pi}\right)^{\frac d2}\frac{1}{(s+h)(s+\lambda h)}\int_{\mathbb{R}^d} w^{\frac d2} e^{-2w|\zeta|^2}\left(\frac{\sin\big((s+h)|\zeta|\big)}{|\zeta|}\right)\left(\frac{\sin\big((s+\lambda h)|\zeta|\big)}{|\zeta|}\right)d\zeta.\end{aligned}\tag{5.5}$$

This yields

$$T_1(w)\ \le \left(\frac{2}{\pi}\right)^{\frac d2} w^{\frac d2}\int_0^1 d\lambda\int_0^T ds\ \frac{s}{(s+h)^{1-\epsilon}(s+\lambda h)^{1-\epsilon}}\int_{\mathbb{R}^d} e^{-2w|\zeta|^2}\ |\zeta|^{2\epsilon-2}\,d\zeta,\tag{5.6}$$

where in the last inequality, $\epsilon\in(0,1)$, and we have used the estimate $\frac{|\sin(a|x|)|}{a|x|}\le 1$, $a>0$, which implies $\frac{|\sin(a|x|)|}{a|x|}\le\big(\frac{\sin(a|x|)}{a|x|}\big)^{1-\epsilon}\le a^{\epsilon-1}|x|^{\epsilon-1}$.

Since $\epsilon\in(0,1)$, for every $\lambda\in(0,1)$ we have $\int_0^T\frac{s}{(s+h)^{1-\epsilon}(s+\lambda h)^{1-\epsilon}}ds\le\int_0^T s^{-1+2\epsilon}ds<\infty$.

Applying the change of variables $\eta = \sqrt{w}\zeta$, we obtain

$$
\begin{aligned}
T_1(w) &\leq C\, w^{\frac{d}{2}} \int_{\mathbb{R}^d} e^{-2|\eta|^2}\, |\eta|^{2\epsilon-2}\, w^{1-\epsilon}\, w^{-\frac{d}{2}} d\eta \\
&\leq C w^{1-\epsilon} \int_0^\infty e^{-2\rho^2}\, \rho^{2(\epsilon-1)}\, \rho^{d-1}\, d\rho \leq C w^{1-\epsilon}.
\end{aligned} \tag{5.7}
$$

Indeed, the last integral is finite if and only if $2\epsilon > 2-d$; since $d = 2, 3$ and $\epsilon > 0$ this constraint is satisfied. Similarly, for any $\epsilon \in (0,1)$,

$$
T_2(w) + T_3(w) \leq C w^{1-\epsilon}.
$$

Thus, for any $h \in [0,T]$ and $b \in (0,1)$, we have proved

$$
I_1 \leq C h^b \int_0^\infty w^{\frac{\kappa-b-d-2}{2}} e^{-w} w^{1-\epsilon}\, dw. \tag{5.8}
$$

By taking ε arbitrarily close to zero, we see that this integral is finite if $b < \kappa - d + 2$. Consequently, **(h4)**(1) is satisfied for $b < \min(\kappa - d + 2, 1)$.

Finally, we address the validity of **(h4)**(2), by using a similar approach as for **(h4)**(1). Set

$$
\begin{aligned}
I_2 = \int_0^T ds\ s^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} & G(1,dy) G(1,dz) \\
\times \big| \tilde{f}_\kappa\big((s+h)(y+z)\big) - \tilde{f}_\kappa\big((s+h)y + sz\big) & - \tilde{f}_\kappa\big(sy + (s+h)z\big) + \tilde{f}_\kappa\big(s(y+z)\big)\big|.
\end{aligned}
$$

Apply the first inequality in [14][p. 392] with $x := s(y+z)$, $\xi := y$ and $\eta := z$, to see that, for any $y, z \in \mathbb{R}^d$, $s, h \in [0,T]$ and $\bar{b} \in (0,2)$,

$$
\left| e^{-\frac{|(s+h)(y+z)|^2}{4w}} - e^{-\frac{|s(y+z)+hy|^2}{4w}} - e^{-\frac{|sy+(s+h)z|^2}{4w}} + e^{-\frac{|s(y+z)|^2}{4w}} \right| \leq C h^{\bar{b}}\, w^{-\frac{\bar{b}}{2}} q_{h,y,z}(s,w),
$$

where

$$
\begin{aligned}
q_{h,y,z}(s,w) = \int_0^1 d\lambda \int_0^1 d\mu \bigg(& e^{-\frac{|(s+\lambda h)y+(s+\mu h)z|^2}{8w}} + e^{-\frac{|(s+h)(y+z)|^2}{8w}} + e^{-\frac{|s(y+z)+hy|^2}{4w}} \\
& + e^{-\frac{|s(y+z)+hz|^2}{4w}} + e^{-\frac{|s(y+z)|^2}{4w}} \bigg).
\end{aligned} \tag{5.9}
$$

Therefore,

$$
\begin{aligned}
I_2 &\leq C\, h^{\bar{b}} \int_0^\infty dw\ w^{\frac{\kappa-\bar{b}-d-2}{2}}\, e^{-w} \int_0^T ds\ s^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(1,dy) G(1,dz) q_{h,y,z}(s,w) \\
&= C\, h^{\bar{b}} \int_0^\infty dw\ w^{\frac{\kappa-\bar{b}-d-2}{2}}\, e^{-w} \sum_{i=1}^5 S_i.
\end{aligned} \tag{5.10}
$$

In the last expression,

$$
S_1 = \int_0^1 d\lambda \int_0^1 d\mu \int_0^T ds\ s^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(1,dy) G(1,dz) e^{-\frac{|(s+\lambda h)y+(s+\mu h)z|^2}{8w}},
$$

and S_i, $i = 2, \ldots, 5$, are defined in a similar way, by taking each of the remaining exponential terms in (5.9).

As in (5.5),

$$
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(1,dy) G(1,dz) e^{-\frac{|(s+\lambda h)y+(s+\mu h)z|^2}{8w}}
$$

$$\leq \left(\frac{2}{\pi}\right)^{\frac{d}{2}} w^{\frac{d}{2}} \frac{1}{(s+\lambda h)^{1-\epsilon}(s+\mu h)^{1-\epsilon}} \int_{\mathbb{R}^d} e^{-2w|\zeta|^2}|\zeta|^{2\epsilon-2}\, d\zeta,$$

implying

$$S_1 \leq \left(\frac{2}{\pi}\right)^{\frac{d}{2}} w^{\frac{d}{2}} \int_0^1 d\lambda \int_0^1 d\mu \int_0^T ds\, \frac{s^2}{(s+\lambda h)^{1-\epsilon}(s+\mu h)^{1-\epsilon}} \int_{\mathbb{R}^d} e^{-2w|\zeta|^2}|\zeta|^{2\epsilon-2}\, d\zeta,$$

for any $\varepsilon \in (0,1)$.

Observe the analogy between this inequality and (5.6). Arguing as in the analysis of the right-hand side of (5.6), we obtain $S_1 \leq C(T)w^{1-\epsilon}$, and, with similar arguments, also $S_i \leq C(T)w^{1-\epsilon}$ for any $i = 2,\dots,5$. These estimates along with (5.10) yield

$$I_2 \leq C h^{\bar{b}} \int_0^\infty w^{\frac{\kappa-\bar{b}-d-2}{2}} e^{-w} w^{1-\epsilon}\, dw.$$

As in (5.8), by taking ε arbitrarily close to zero, we see that the last integral converges if $\bar{b} < \kappa - d + 2$, thereby proving that **(h4)**(2) holds with $\bar{b} < \min(\kappa - d + 2, 2)$.

Conclusion. Let $d = 2,3$, $\kappa > d-2$. For spatially homogeneous Gaussian noises with covariance function given by (4.1) with $\Lambda(dx) = \tilde{f}_\kappa(x)\, dx$, the parameters ν_1, ν_2 defined in (4.76) are $\nu_1 = \nu_2 = \min(\gamma,\gamma_1,\gamma_2)$, with $\gamma < \min\left(\frac{\kappa-d+2}{2}, 1\right)$.

Thus, we deduce that almost all sample paths of the solution to (1.3) are locally Hölder continuous, jointly in (t,x), with exponent $\theta \in \left]0, \min(\frac{\kappa-d+2}{2}, 1, \gamma_1, \gamma_2)\right[$. When $d = 3$, we recover the results in [14][p. 393]. Whether this Hölder exponent is sharp seems to be an open question.

5.3. **Fractional kernels.** Let $d = 2,3$ and $H = (H_i)_{1\leq i\leq d}$, with $\frac{1}{2} < H_i < 1$. Let $\bar{f}_H(x) = C_H \prod_{i=1}^d |x_i|^{2H_i-2}$, when $\prod_{i=1}^d x_i \neq 0$, where $C(H) = \prod_{i=1}^d H_i(2H_i-1)$, and $\bar{f}_H(x) = +\infty$ otherwise. The inverse Fourier transform of $\bar{f}_H$ is $\left(\mathcal{F}^{-1}\bar{f}_H\right)(\zeta) = C_H \prod_{i=1}^d |\zeta_i|^{1-2H_i}$, where C_H is some positive constant depending only on H.

Consider the non-negative definite tempered distribution $\Lambda(dx) = \bar{f}_H(x)dx$, whose spectral measure is $\bar{\mu}_H(\zeta) = C_H \prod_{i=1}^d |\zeta_i|^{1-2H_i}\, d\zeta$. In this section, we prove that $\bar{\mu}_H$ satisfies the conditions **(h0)**–**(h4)**.

Since the function $\bar{f}_H$ is lower semi-continuous, condition **(h1)** holds, by Remark 4.1.

We next check Condition **(h0)**. For any H as above, we have

$$I = \int_{\mathbb{R}^d} \frac{\bar{\mu}_H(d\zeta)}{1+|\zeta|^2} = C_H \int_{\mathbb{R}^d} \frac{\prod_{i=1}^d |\zeta_i|^{1-2H_i}}{1+|\zeta|^2} d\zeta = I(d)\tilde{I}(d),$$

where using polar (resp. spherical) coordinates when $d = 2$ (resp. $d = 3$), we have

$$I(d) = \int_0^\infty r^{d-1} r^{d-2\sum_{i=1}^d H_i}(1+r^2)^{-1} dr,$$

and

$$\tilde{I}(2) = 4\int_0^{\frac{\pi}{2}} |\cos(\theta)|^{1-2H_1} |\sin(\theta)|^{1-2H_2} d\theta, \tag{5.11}$$

$$\tilde{I}(3) = 2\,\tilde{I}(2) \int_0^{\frac{\pi}{2}} |\sin(\phi)||\sin(\phi)|^{2-2(H_1+H_2)} |\cos(\phi)|^{1-2H_3} d\phi. \tag{5.12}$$

The change of variable $u=\sin(\theta)$ on the interval $(0,\frac{\pi}{2})$ and the constraints on H_i imply

$$\tilde I(2) = C\int_0^1 u^{1-2H_2}(1-u^2)^{-H_1}du \le C\int_0^1 u^{1-2H_2}(1-u)^{-H_1}du<\infty.$$

Similarly, the change of variable $v=\sin(\phi)$ on the interval $(0,\frac{\pi}{2})$ yields

$$\int_0^{\frac{\pi}{2}} |\sin(\phi)|^{3-2(H_1+H_2)}|\cos(\phi)|^{1-2H_3}d\phi = 2\int_0^1 v^{3-2(H_1+H_2)}(1-v^2)^{-H_3}dv<\infty;$$

hence $\tilde I(3)<\infty$.

Finally, it is easy to see that $I(d)<\infty$ if and only if $2d-1-2\sum_{i=1}^d H_i > -1$ and $2d-3-2\sum_{i=1}^d H_i<-1$. Since $H_i<1$, the first constraint is satisfied. The second one is equivalent to $\sum_{i=1}^d H_i > d-1$. Hence, if $\sum_{i=1}^d H_i>d-1$, condition **(h0)** is satisfied.

Let us now check that condition **(h2)** is satisfied. Given $\gamma>0$, for $\tilde I(d)$ defined by (5.11) and (5.12) for $d=2,3$, we have

$$\int_{\mathbb{R}^d} \frac{\bar\mu(d\zeta)}{1+|\zeta|^{2-2\gamma}} = \tilde I(d)\int_0^\infty r^{d-1} r^{d-2\sum_{i=1}^d H_i}\big(1+r^{2-2\gamma}\big)^{-1}dr.$$

Set

$$\bar\kappa = \sum_{i=1}^d H_i - (d-1). \tag{5.13}$$

A computation similar to that used to check **(h0)** shows that this last integral is finite if and only if $2d-3-2\sum_{i=1}^d H_i + 2\gamma < -1$, that is $\gamma<\bar\kappa$, where $\bar\kappa$ is defined in (5.13).

To check condition **(h3)**, we use similar arguments to deduce that for $t\in[0,T]$, $\tilde I(d)$ defined as above

$$\begin{aligned}\int_{\mathbb{R}^d} \big|\mathcal{F}G(t)(\zeta)\big|^2\bar\mu_H(d\zeta) =& C_H\int_{\mathbb{R}^d}\Big|\frac{\sin(t|\zeta|)}{|\zeta|}\Big|^2\prod_{i=1}^d|\zeta_i|^{1-2H_i}d\zeta\\ &= C_H\tilde I(d)\int_0^\infty r^{d-1}\Big|\frac{\sin(tr)}{r}\Big|^2 r^{d-\sum_{i=1}^d H_i}dr\\ &= C\Big[\int_0^{t^{-1}} t^2 r^{2d-1-2\sum_{i=1}^d H_i}dr + \int_{t^{-1}}^\infty r^{d-1-2+d-2\sum_{i=1}^d H_i}dr\\ &= C\Big[t^{2-2(d-\sum_{i=1}^d H_i)} + t^{2\sum_{i=1}^d H_i - 2d+2}\Big].\end{aligned}$$

We deduce $\int_{\mathbb{R}^d}\big|\mathcal{F}G(t)(\zeta)\big|^2\bar\mu_H(d\zeta)\le Ct^{2\bar\kappa}$, that is (4.60) holds with $\nu=2\bar\kappa$, where $\bar\kappa$ is defined in (5.13).

The proof of **(h4)** is similar to that of a similar condition in [14]. We sketch it below for the sake of completeness. We start with **(h4)**(1). Apply the inequality in [14][p. 395, bottom] to see that if $d=3$,

$$\begin{aligned}|\bar f_H(x+hy)-\bar f_H(x)| \le& C_H\big||x_1+hy_1|^{2H_1-2}-|x_1|^{2H_1-2}\big||x_2+hy_2|^{2H_2-2}|x_3+hy_3|^{2H_3-2}\\ &+|x_1|^{2H_1-2}\big||x_2+hy_2|^{2H_2-2}-|x_1|^{2H_2-2}\big||x_3+hy_3|^{2H_3-2}\\ &+|x_1|^{2H_1-2}|x_2|^{2H_2-2}\big||x_3+hy_3|^{2H_3-2}-|x_1|^{2H_2-2}\big|,\end{aligned}$$

while for $d=2$

$$\begin{aligned}|\bar f_H(x+hy)-\bar f_H(x)| \le& C_H\big||x_1+hy_1|^{2H_1-2}-|x_1|^{2H_1-2}\big||x_2+hy_2|^{2H_2-2}\\ &+|x_1|^{2H_1-2}\big||x_2+hy_2|^{2H_2-2}-|x_1|^{2H_2-2}\big|.\end{aligned}$$

In both cases, we have $|\bar{f}_H(x+hy) - \bar{f}_H(x)| = \sum_{i=1}^d F_{d,i}(x,y,h)$. In the sequel, we only describe the computations in the case $d=3$; the case $d=2$ is easier and dealt with in a similar way.

Using the identity (2.11) in [8] for $d:=1$, $a = 2H_1 - 1 - \rho$, $b=\rho$ for some positive $\rho < (2\bar{\kappa}) \wedge \min(2H_i - 1; i=1,\cdots,d)$, we deduce since $a+b = 2H_1 - 1 \in (0,1)$,

$$\begin{aligned}
\big||x_1 - hy_1|^{2H_1-2} - |x_1|^{2H_1-2}\big| =& \Big|\int_{\mathbb{R}} dw|x_1 - w|^{2H_1-2-\rho}|w + hy_1|^{\rho-1} \\
&- \int_{\mathbb{R}} dw|x_1 - w|^{2H_1-2-\rho}|w|^{\rho-1}\Big| \\
=& \Big|\int_{\mathbb{R}} h\ dw|x_1 - hw|^{2H_1-2-\rho}h^{\rho-1}|w+y_1|^{\rho-1} \\
&- \int_{\mathbb{R}} h\ dw|x_1 - hw|^{2H_1-2-\rho}h^{\rho-1}|w|^{\rho-1}\Big|.
\end{aligned}$$

Hence, changing y_1 into $-y_1$, we have $\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(1,dy)G(1,dz)F_{3,1}((s+h)z + sy, y, h) \le h^\rho \sum_{j=1}^3 T_{1,j}(s,h)$, where

$$\begin{aligned}
T_{1,1}(s,h) =& \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(1,dy)G(1,dz)\int_{\{|w|\le 3\}} |(s+h)z_1 + sy_1 - hw|^{2H_1-2-\rho}|w - y_1|^{\rho-1} \\
&\times |(s+h)(z_2+y_2)|^{2H_2-2}|(s+h)(z_3+y_3)|^{2H_3-2}dw, \\
T_{1,2}(s,h) =& \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(1,dy)G(1,dz)\int_{\{|w|\le 3\}} |(s+h)z_1 + sy_1 - hw|^{2H_1-2-\rho}|w|^{\rho-1} \\
&\times |(s+h)(z_2+y_2)|^{2H_2-2}|(s+h)(z_3+y_3)|^{2H_3-2}dw, \\
T_{1,3}(s,h) =& \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(1,dy)G(1,dz)\int_{\{|w|> 3\}} |(s+h)z_1 + sy_1 - hw|^{2H_1-2-\rho} \\
&\times \big||w-y_1|^{\rho-1} - |w|^{\rho-1}\big|\ |(s+h)(z_2+y_2)|^{2H_2-2}|(s+h)(z_3+y_3)|^{2H_3-2}dw.
\end{aligned}$$

To prove an upper estimate of $T_{1,1}(s,h)$, let $\tilde{w} = w - y_1$; then using the scaling property (2.4) we obtain

$$\begin{aligned}
T_{1,1}(s,h) =& \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(1,dy)G(1,dz)\int_{\{|\tilde{w}|\le 4\}} |(s+h)(z_1+y_1) - h\tilde{w}|^{2H_1-2-\rho}|\tilde{w}|^{\rho-1} \\
&\times |(s+h)(z_2+y_2)|^{2H_2-2}|(s+h)(z_3+y_3)|^{2H_3-2}dw, \\
=& \frac{1}{(s+h)^2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(s+h,dy)G(s+h,dz)\int_{\{|\tilde{w}|\le 4\}} \bar{f}_{H_1-\frac{\rho}{2},H_2,H_3}\big(y+z-(h\tilde{w},0,0)\big)|\tilde{w}|^{\rho-1}d\tilde{w} \\
=& \frac{C}{(s+h)^2}\int_{\{|w|\le 4\}} dw \int_{\mathbb{R}^d} \Big|\frac{\sin\big((s+h)|\zeta|\big)}{|\zeta|}\Big|^2 |e^{ihw\zeta_1}||\zeta_1|^{1-2H_1+\rho}|\zeta_2|^{1-2H_2}|\zeta_3|^{1-2H_3}d\zeta.
\end{aligned}$$

Since $\frac{1}{2} < H_1 - \frac{\rho}{2} < 1$, the computations made to check **(h3)** imply

$$\int_0^T s\, T_{1,1}(s,h)ds \le C\int_0^T (s+h)^{2\bar{\kappa}-1}ds \le C(T) < \infty$$

for all $h \in [0,T]$.

We next upper estimate $T_{1,2}(s,h)$ using once more the scaling property; this yields

$$T_{1,2}(s,h) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(1,dy)G(1,dz)\int_{\{|w|\le 3\}} |(s+h)z_1 + sy_1 - hw|^{2H_1-2-\rho}\,|w|^{\rho-1}$$

$$\times |(s+h)(z_2+y_2)|^{2H_2-2}|(s+h)(z_3+y_3)|^{2H_3-2}dw,$$

$$= \frac{1}{(s+h)^2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(s+h,dy)G(s+h,dz)\int_{\{|w|\le 3\}}\Big|z_1+\frac{s}{s+h}y_1-hw\Big|^{2H_1-2-\rho}|w|^{\rho-1} \times |(s+h)(z_2+y_2)|^{2H_2-2}|(s+h)(z_3+y_3)|^{2H_3-2}dw.$$

Let $\psi(y)=\big(\frac{s}{s+h}y_1,y_2,y_3\big)$ and $G^\psi(s+h)$ denote the image of the measure $G(s+h,dy)$ by ψ. Then Fubini's theorem implies for $s>0$

$$\begin{aligned}
T_{1,2}(s,h) &= \frac{1}{(s+h)^2}\int_{\{|w|\le 3\}}|w|^{\rho-1}dw\int_{\mathbb{R}^d}\big[G^\psi(s+h)*G(s+h)\big](dx) \\
&\qquad \times |x_1|^{2H_1-\rho-2}|x_2|^{2H_2-2}|x_3|^{2H_3-2} \\
&\le C\frac{1}{(s+h)^2}\int_{\{|w|\le 3\}}|w|^{\rho-1}dw\int_{\mathbb{R}^d}\big|\mathcal{F}\big[G^\psi(s+h)*G(s+h)\big](\zeta)|e^{i\zeta_1 w}| \\
&\qquad \times |\zeta_1|^{1+\rho-2H_1}|\zeta_2|^{1-2H_2}|\zeta_3|^{1-2H_3}d\zeta \\
&\le C\frac{1}{(s+h)^2}\int_{\{|w|\le 3\}}|w|^{\rho-1}dw\int_{\mathbb{R}^d}\Big|\frac{\sin\big((s+h)|\zeta|\big)}{|\zeta|}\Big|\Big|\frac{\sin\big((s+h)\big|\big(\frac{s}{s+h}\zeta_1,\zeta_2,\zeta_3\big)\big|\big)}{\big|\big(\frac{s}{s+h}\zeta_1,\zeta_2,\zeta_3\big)\big|}\Big| \\
&\qquad \times |\zeta_1|^{1+\rho-2H_1}|\zeta_2|^{1-2H_2}|\zeta_3|^{1-2H_3}d\zeta \\
&\le C\int_{\{|w|\le 3\}}|w|^{\rho-1}dw\Big[\int_0^1 r^{d-1}r^{d+\rho-2\sum_{i=1}^d H_i}dr \\
&\qquad + \frac{1}{(s+h)^2}\,\frac{s+h}{s}\int_1^\infty r^{d-1}r^{d+\rho-2-2\sum_{i=1}^d H_i}dr\Big].
\end{aligned}$$

These integrals are convergent since $\sum_{i=1}^d H_i<d$ implies $2d+\rho-1-2\sum_{i=1}^d H_i>-1$, and since $\rho<2\bar\kappa$ implies $2d-3+\rho-2\sum_{i=1}^d H_i<-1$. Therefore, $\int_0^T sT_{1,2}(s,h)ds\le C(T)$ for all $h\in[0,T]$.

Finally,

$$\big||w+y_1|^{\rho-1}-|w|^{\rho-1}\big|\le\int_0^1\frac{\partial}{\partial\lambda}\big(|w+\lambda y_1|^{\rho-1}\big)d\lambda\le C|w|^{\rho-2}.$$

Notice that since $\rho<2H_1-1<1$, we have $\rho-2<-1$. Hence arguments similar to that used to upper estimate $T_{1,2}(s,h)$ imply $\int_0^T s\,T_{1,3}(s,h)ds\le C(T)<\infty$ for all $h\in[0,T]$.

A similar computation implies

$$\int_0^T ds\,s\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}G(1,dy)G(1,dz)\sum_{i=1}^d F_{3,i}((s+h)z+sy,y,h)\le Ch^\rho.$$

This shows that (4.61) holds with $b<(2\bar\kappa)\wedge\min(2H_i-1;i=1,\cdots,d)$.

To prove **(h4)**(2), we use the inequality

$$\begin{aligned}
&\big|\bar f_H\big((s+h)(y+z)\big)-\bar f_H\big((s+h)y+sz\big)-\bar f_H\big((s+h)z+sy\big)+\bar f_H\big(s(y+z)\big)\big| \\
&\quad\le\big|\bar f_H\big((s+h)(y+z)\big)-\bar f_H\big((s+h)z+sy\big)\big|+\big|\bar f_H\big((s+h)y+sz\big)+\bar f_H\big(s(y+z)\big)\big|.
\end{aligned}$$

The first difference is estimated by **(h4)**(1) and the second one is dealt with in a similar way. Therefore, (4.62) is satisfied with $\bar b<(2\bar\kappa)\wedge\min(2H_i-1;i=1,\cdots,d)$. This completes the proof.

Conclusion. Let $d = 2, 3$ and $H = (H_i)_{1\le i\le d}$, where $\frac{1}{2} < H_i < 1$. For spatially homogeneous Gaussian noises with covariance function given by (4.1) with $\Lambda(dx) = f_H(x)\,dx$, the parameters ν_1, ν_2 in (4.76) are

$$\nu_1 = \nu_2 = \min\left(\gamma_1, \gamma_2, \bar\kappa, \min\left(H_i - \frac{1}{2}; i = 1, \cdots, d\right)\right), \text{ with } \bar\kappa = \sum_{i=1}^d H_i - (d-1).$$

As a consequence, from (4.78) we deduce that almost all sample paths of the solution to (1.3) are locally Hölder continuous, jointly in (t, x), with exponent

$$\theta \in \left]0, \min\left(\gamma_1, \gamma_2, \bar\kappa, \min\left(H_i - \frac{1}{2}; i = 1, \cdots, d\right)\right)\right[.$$

For $d = 3$, this is [14][Theorem 6.1]. However, following [14][Theorem 6.2], the critical exponent must be $\min(\gamma_1, \gamma_2, \bar\kappa)$, and therefore the result is not optimal.

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