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CLEONICE F. BRACCIALI AND TERESA E. PÉREZ

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Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
Schwarzwaldstrasse 9-11
77709 Oberwolfach-Walke
Germany

Tel +49 7834 979 50
Fax +49 7834 979 55
Email admin@mfo.de
URL www.mfo.de

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# MULTIVARIATE HYBRID ORTHOGONAL FUNCTIONS

CLEONICE F. BRACCIALI AND TERESA E. PÉREZ

Abstract. We consider multivariate orthogonal functions satisfying hybrid orthogonality conditions with respect to a moment functional. This kind of orthogonality means that the product of functions of different parity order is computed by means of the moment functional, and the product of elements of the same parity order is computed by a modification of the original moment functional. Results about existence conditions, three term relations with matrix coefficients, a Favard type theorem for this kind of hybrid orthogonal functions are proved. In addition, a method to construct bivariate hybrid orthogonal functions from univariate orthogonal polynomials and univariate orthogonal functions is presented. Finally, we give a complete description of a sequence of hybrid orthogonal functions on the unit disk on $\mathbb{R}^2$, that includes, as particular case, the classical orthogonal polynomials on the disk.

## 1. Introduction

In [2] a class of orthogonal functions defined on the interval $[-1,1]$ has been studied. These functions can be defined as

$$w_n(x) = p_n(x) + \sqrt{1-x^2}\,q_{n-1}(x), \quad n \geqslant 0,$$

where $p_n(x)$ and $q_{n-1}(x)$ are real polynomials of respective degrees $n$ and $n-1$, and satisfy $p_n(-x) = (-1)^n p_n(x)$, $q_{n-1}(-x) = (-1)^{n-1} q_{n-1}(x)$.

Sequences of functions that have some orthogonality properties defined using a positive measure $\phi$ on the interval $[-1,1]$ was also considered in [2]. Namely, the sequence of functions $\{w_n\}_{n\geqslant 0}$ satisfies

$$\begin{aligned} \int_{-1}^{1} w_{2n+l}(x)\, w_{2m+l}(x)\, \sqrt{1-x^2}\, d\phi(x) = h_{2n+l}\, \delta_{n,m}, \quad l = 0,1, \\ \int_{-1}^{1} w_{2n+1}(x)\, w_{2m}(x)\, d\phi(x) = 0, \end{aligned} \tag{1.1}$$

for $n, m = 0, 1, 2, \ldots$, with $h_{2n+l} \neq 0$, $\delta_{n,m} = 0$, if $n \neq m$ and $\delta_{n,m} = 1$, if $n = m$. We refer to the functions $\{w_n\}_{n\geqslant 0}$ satisfying (1.1) as *hybrid orthogonal functions in one variable.*

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A function $w_n$ which satisfies the hybrid orthogonality properties (1.1), has exactly $n$ simple zeros on the interval $(-1,1)$, see [2, 5]. Hybrid orthogonal functions in one variable were introduced in [4], as a special example, and in [5], where an interesting connection with orthogonal polynomials on the unit circle was established.

In this paper we extend results obtained in both papers [2, 5] in two directions. On one hand, we consider hybrid orthogonality associated to moment functionals, and then, as occurs in one variable ([3, Chapter 1]) new questions about existence and a Favad type theorem arise in an natural way. On the other hand, we introduce the multivariate version of the hybrid orthogonality that has allowed us to give a common frame for this kind of hybrid orthogonality. In this way, a non trivial extension of univariate hybrid orthogonal functions associated with moment functionals to several variables is given. We work on $\mathbb{B}^d = \{\mathtt{x} \in \mathbb{R}^d : \|\mathtt{x}\| \leqslant 1\}$, the unit ball of $\mathbb{R}^d$, for $d \geqslant 1$, the natural extension of the interval $[-1,1]$.

The paper is structured as follows. Section 2 is devoted to recall and establish the basic facts about multivariate sets. Also, in this section, we define the functional systems that we deal with and the multivariate orthogonal functions systems satisfying hybrid orthogonality conditions with respect to a moment functional.

In Section 3 we give conditions for the existence of such type of hybrid orthogonal functions for a given moment functional. We prove three term relations with matrix coefficients, and a Favard type theorem that allows to recover the hybrid orthogonality from the three term relations.

A method to construct bivariate hybrid orthogonal functions system based in the well known Koornwinder's method ([6], [10]) is developed in Section 4. To construct systems of bivariate hybrid orthogonal functions we use both univariate hybrid orthogonal functions and univariate orthogonal polynomials. Also, the explicit expressions of the entries of the matrix coefficients of the three term relations are given.

In the last section we give two examples. The first example is a complete description of a sequence of hybrid orthogonal functions on the unit disk on $\mathbb{R}^2$ that extends a family studied in [2] to the bivariate case. This description includes, as particular case, the classical orthogonal ball polynomials ([6]). In the second example we connect univariate hybrid orthogonal functions with bivariate orthogonal polynomials on the bivariate semisphere $\mathbb{H}^1 = \{(x_1,x_2) \in \mathbb{R}^2 : x_1^2+x_2^2 = 1,\, x_2 \geqslant 0\}$.

## 2. Definitions and first properties

In this Section, we introduce the main tools that we need for the rest of the paper. We will work in any dimension $d \geqslant 1$, and then, the results given in [2] can be deduced as a particular case of our study. Sometimes, we particularize our results for the univariate case.

Let $d \geqslant 1$ be the number of variables. As usual, the Euclidean norm for $\mathtt{x} \in \mathbb{R}^d$ will be denoted by $\|\mathtt{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$, and the unit ball in $\mathbb{R}^d$ by

$$\mathbb{B}^d = \{\mathtt{x} \in \mathbb{R}^d : \|\mathtt{x}\|^2 \leqslant 1\}.$$

Along this paper, we work on the unit ball $\mathbb{B}^d$, that is, we suppose that $\|\mathtt{x}\|^2 \leqslant 1$, for every $\mathtt{x} = (x_1, x_2, \dots, x_d)$. When $d = 1$, we are working on the interval $[-1,1]$, as in [2].

We denote by $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}_0^d$ a multi-index, and we define $|\alpha| = \alpha_1+\alpha_2+\cdots+\alpha_d$. We order the multi-indexes by means of the *graded lexicographical order*, that is, $\alpha < \beta$ if and only if $|\alpha| < |\beta|$, and in the case $|\alpha| = |\beta|$, the first entry of $\beta - \alpha$ different from zero is positive.

Given $\mathtt{x} = (x_1, x_2, \ldots, x_d)$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}_0^d$, we say that

$$\mathtt{x}^\alpha = x_1^{\alpha_1}\, x_2^{\alpha_2}\, \cdots\, x_d^{\alpha_d}$$

is a monomial of total degree $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$. A real polynomial in $d$ variables of total degree $n$ is defined by a linear combination of monomials such as

$$p(\mathtt{x}) = \sum_{|\alpha|\leqslant n} a_\alpha\, \mathtt{x}^\alpha, \quad a_\alpha \in \mathbb{R}.$$

We denote by

$$\mathcal{P}_n^d = \operatorname{span}\{\mathtt{x}^\alpha : |\alpha| = n\},$$

the linear space of homogeneous polynomials of exact degree $n$ with real coefficients. We must observe that

$$r_n^d := \dim \mathcal{P}_n^d = \#\{\mathtt{x}^\alpha : |\alpha| = n\} = \binom{n+d-1}{d-1}, \quad n \geqslant 0.$$

That is, $r_n^d$ express the number of different monomials for a fixed degree $n \geqslant 0$. In addition, we define the linear spaces of multivariate real polynomials

$$\Pi_n^d = \bigcup_{m\leqslant n} \mathcal{P}_m^d, \quad \text{and} \quad \Pi^d = \bigcup_{n\geqslant 0} \Pi_n^d,$$

such that

$$s_n^d = \dim \Pi_n^d = \binom{n+d}{d} = \sum_{m=0}^{n} r_m^d.$$

Observe that, for $d = 1$ we get $r_n^1 = 1$, $s_n^1 = n + 1$, $\mathcal{P}_n^1 = \operatorname{span}\{x^n\}$, and $\Pi_n^1 = \operatorname{span}\{1, x, \ldots, x^n\}$. Therefore, we include the univariate case ([2]) as a part of our study.

A useful tool along this paper is the *canonical basis* of $\Pi^d$, formed as a sequence of column vectors of increasing size $r_n^d$, $\{\mathbb{X}_n\}_{n\geqslant 0}$, whose entries are all monomials of total degree $n$ ordered by using the reverse graded lexicographical order

$$\mathbb{X}_n = (\mathtt{x}^\alpha)_{|\alpha|=n}, \quad n \geqslant 0.$$

For instance, for $d = 3$, the first elements of this basis are given by

$$\mathbb{X}_0 = (1); \quad \mathbb{X}_1 = (x_1, x_2, x_3)^T; \quad \mathbb{X}_2 = (x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2)^T, \ldots$$

where the superscript $T$ means, as usual, the transpose.

We must observe that the set of entries of $\{\mathbb{X}_0, \mathbb{X}_1, \ldots, \mathbb{X}_n\}$ is a basis of $\Pi_n^d$, and by extension, we say that the set $\{\mathbb{X}_m\}_{m=0}^n$ form a basis of $\Pi_n^d$.

Following [6, p. 71], for $n \geqslant 0$, we introduce the matrices $L_{n,k}$ of size $r_n^d \times r_{n+1}^d$ defined by

$$x_k\, \mathbb{X}_n = L_{n,k}\, \mathbb{X}_{n+1}, \quad 1 \leqslant k \leqslant d, \tag{2.1}$$

that represent the raising operator given by the multiplication by $x_k$ expressed in the canonical basis. The matrices $L_{n,k}$, for $1 \leqslant k \leqslant d$ and $n \geqslant 0$, are matrices of full rank $r^d_n$. We compute

$$\|\mathtt{x}\|^2\,\mathbb{X}_n = \sum_{k=1}^{d} x_k^2\,\mathbb{X}_n = \sum_{k=1}^{d} L_{n,k}\,L_{n+1,k}\,\mathbb{X}_{n+2} = L_n^{(1)}\,\mathbb{X}_{n+2}, \tag{2.2}$$

defining the $r^d_n \times r^d_{n+2}$ matrix $L_n^{(1)} = \sum_{k=1}^{d} L_{n,k}\,L_{n+1,k}$. It is easy to verify that $L_n^{(1)}$ is a matrix of full rank $r^d_n$. For $d=1$, $L_n^{(1)} = L_{n,1} = 1$, for all $n \geqslant 0$.

Now, we define the linear space of functions that we study in this work. These new linear spaces are closely related to the linear spaces of polynomials defined before. Observe that the case $d=1$ is included as a particular case of this study.

For $m \geqslant 0$, let $\Omega^d_m$ be the linear space of functions defined on $\mathbb{B}^d$ generated by means of the basis

$$\Omega^d_{2n} = \operatorname{span}\{\mathbb{X}_{2n}, \sqrt{1-\|\mathtt{x}\|^2}\,\mathbb{X}_{2n-1}, \ldots, \sqrt{1-\|\mathtt{x}\|^2}\,\mathbb{X}_1, \mathbb{X}_0\},$$

in the even case, and, in the odd case, we get the basis

$$\Omega^d_{2n+1} = \operatorname{span}\{\mathbb{X}_{2n+1}, \sqrt{1-\|\mathtt{x}\|^2}\,\mathbb{X}_{2n}, \ldots, \mathbb{X}_1, \sqrt{1-\|\mathtt{x}\|^2}\,\mathbb{X}_0\}.$$

As before, the basis is the set of all entries of the vectors, and, for extension, we say that the set of vectors is a basis. Notice that the dimension of $\Omega^d_m$ is $s^d_m$, $m \geqslant 0$.

Observe that a function $R(\mathtt{x}) \in \Omega^d_m$ if it can be written in the form

$$R(\mathtt{x}) = P_m(\mathtt{x}) + \sqrt{1-\|\mathtt{x}\|^2}\,Q_{m-1}(\mathtt{x}), \tag{2.3}$$

where $P_m(\mathtt{x}) \in \Pi^d_m$, $Q_{m-1}(\mathtt{x}) \in \Pi^d_{m-1}$ are both symmetric polynomials, that is,

$$P_m(-\mathtt{x}) = (-1)^m\,P_m(\mathtt{x}), \quad \text{and} \quad Q_{m-1}(-\mathtt{x}) = (-1)^{m-1}\,Q_{m-1}(\mathtt{x}),$$

with $-\mathtt{x} = (-x_1, -x_2, \ldots, -x_d)$.

This means that, if $R \in \Omega^d_{2n}$ then $P_{2n}(\mathtt{x})$ is an even polynomial of degree at most $2n$, and $Q_{2n-1}(\mathtt{x})$ is an odd polynomial of degree at most $2n-1$. Likewise, if $R \in \Omega^d_{2n+1}$ then $P_{2n+1}(\mathtt{x})$ is an odd polynomial of degree at most $2n+1$, and $Q_{2n}(\mathtt{x})$ is an even polynomial of degree at most $2n$.

2.1. **Functional Systems.** For $n \geqslant 0$, we define the following vector of functions of increasing size $r^d_n$

$$\mathbb{W}_n = \mathbb{W}_n(\mathtt{x}) = \left(W_1^n(\mathtt{x}), W_2^n(\mathtt{x}), \ldots, W^n_{r^d_n}(\mathtt{x})\right)^T,$$

where $W^n_m(\mathtt{x}) \in \Omega^d_n$, for $1 \leqslant m \leqslant r^d_n$, and takes the form as in (2.3). Using the vector representation, it is clear that

$$\mathbb{W}_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} G^n_{n-2i}\,\mathbb{X}_{n-2i} + \sqrt{1-\|\mathtt{x}\|^2}\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} G^n_{n-(2i+1)}\,\mathbb{X}_{n-(2i+1)}, \tag{2.4}$$

where $G^n_{n-j}$ are $r^d_n \times r^d_{n-j}$ real matrices. The matrices $G^n_n$ and $G^n_{n-1}$ are respectively called the *first* and the *second leading coefficients* of the vector of functions $\mathbb{W}_n$.

Since all entries $W^n_m(\mathtt{x})$ are in $\Omega^d_n$, for $0 \leqslant m \leqslant r^d_n$, then by extension we write that $\mathbb{W}_n \in \Omega^d_n$, for $n \geqslant 0$.

We say that $\mathbb{W}_n$ is of *degree* $n$ if the matrix $G^n_n$ have full rank, that is, it is invertible. If $\mathbb{W}_n$ is of degree $n$, then, its entries $W^n_m(\mathtt{x})$, for $1 \leqslant m \leqslant r^d_n$, are independent functions, and the entries of two vectors $\mathbb{W}_n$ and $\mathbb{W}_m$ of respective degrees $n$ and $m$, $n \neq m$, are independent functions as well.

Observe that, if $G_n^n$ is an invertible matrix, then we can define a *monic* vector of functions from (2.4) in the form

$$\begin{aligned}\overline{\mathbb{W}}_n(\mathtt{x}) =&(G_n^n)^{-1}\mathbb{W}_n(\mathtt{x})\\ =&\mathbb{X}_n + \sum_{i=1}^{\lfloor \frac{n}{2}\rfloor} \overline{G}_{n-2i}^n\,\mathbb{X}_{n-2i} + \sqrt{1-\|\mathtt{x}\|^2}\sum_{i=0}^{\lfloor \frac{n-1}{2}\rfloor} \overline{G}_{n-(2i+1)}^n\,\mathbb{X}_{n-(2i+1)},\end{aligned}$$

where $\overline{G}_m^n = (G_n^n)^{-1}G_m^n$ are $r_n^d \times r_m^d$ real matrices. Obviously, if $\mathbb{W}_n(\mathtt{x}) \in \Omega_n^d$, then $\overline{\mathbb{W}}_n(\mathtt{x}) \in \Omega_n^d$.

Next Lemma collects some basic properties that we will use later.

**Lemma 2.1.** *Let* $\mathbb{W}_n$ *be a vector of functions defined as in* (2.4)*. Then,*

*(i)* $\sqrt{1-\|\mathtt{x}\|^2}\,\mathbb{W}_n(\mathtt{x}) \in \Omega_{n+1}^d$,

*(ii)* $x_k\,\mathbb{W}_n(\mathtt{x}) \in \Omega_{n+1}^d, \quad 1 \leqslant k \leqslant d$,

*(iii)* $x_k\sqrt{1-\|\mathtt{x}\|^2}\,\mathbb{W}_n(\mathtt{x}) \in \Omega_{n+2}^d, \quad 1 \leqslant k \leqslant d$,

*(iv)* $(1-\|\mathtt{x}\|^2)\,\mathbb{W}_n(\mathtt{x}) \in \Omega_{n+2}^d$.

*Proof.* (i) Computing directly in (2.4), using (2.2), and arranging it, we get

$$\begin{aligned}\sqrt{1-\|\mathtt{x}\|^2}\,\mathbb{W}_n(\mathtt{x}) =& \sum_{i=0}^{\lfloor \frac{n+1}{2}\rfloor} [G_{n-(2i-1)}^n - G_{n-(2i+1)}^n L_{n-(2i+1)}^{(1)}]\,\mathbb{X}_{n+1-2i}\\ &+ \sqrt{1-\|\mathtt{x}\|^2}\sum_{i=0}^{\lfloor \frac{n}{2}\rfloor} G_{n-2i}^n\,\mathbb{X}_{n+1-(2i+1)}.\end{aligned}$$

Now, we define

$$\begin{aligned}\widetilde{G}_{n+1-2i}^{n+1} &= G_{n-(2i-1)}^n - G_{n-(2i+1)}^n L_{n-(2i+1)}^{(1)}, \quad 0 \leqslant i \leqslant \lfloor \frac{n+1}{2}\rfloor,\\ \widetilde{G}_{n+1-(2i+1)}^{n+1} &= G_{n-2i}^n, \qquad\qquad 0 \leqslant i \leqslant \lfloor \frac{n}{2}\rfloor,\end{aligned} \tag{2.5}$$

such as $G_{-1}^n = G_{-2}^n = G_{n+1}^n$ are considered as zero matrices of appropriate sizes. Then, we can write

$$\begin{aligned}\sqrt{1-\|\mathtt{x}\|^2}\,\mathbb{W}_n(\mathtt{x}) =& \sum_{i=0}^{\lfloor \frac{n+1}{2}\rfloor} \widetilde{G}_{n+1-2i}^{n+1}\,\mathbb{X}_{n+1-2i}\\ &+ \sqrt{1-\|\mathtt{x}\|^2}\sum_{i=0}^{\lfloor \frac{n}{2}\rfloor} \widetilde{G}_{n+1-(2i+1)}^{n+1}\mathbb{X}_{n+1-(2i+1)},\end{aligned}$$

and therefore $\sqrt{1-\|\mathtt{x}\|^2}\,\mathbb{W}_n(\mathtt{x}) \in \Omega_{n+1}^d$. Observe that $\widetilde{G}_m^{n+1}$ are matrices of sizes $r_n^d \times r_m^d$, for $0 \leqslant m \leqslant n+1$.

(ii) Again, using (2.1) in (2.4), we deduce

$$x_k\mathbb{W}_n(\mathtt{x}) = \sum_{i=0}^{\lfloor \frac{n}{2}\rfloor} \widehat{G}_{n+1-2i}^{n+1,k}\mathbb{X}_{n+1-2i} + \sqrt{1-\|\mathtt{x}\|^2}\sum_{i=0}^{\lfloor \frac{n-1}{2}\rfloor} \widehat{G}_{n+1-(2i+1)}^{n+1,k}\mathbb{X}_{n+1-(2i+1)}, \tag{2.6}$$

where

$$\widehat{G}^{n+1,k}_{n+1-2i} = G^{n}_{n-2i}\, L_{n-2i,k}, \qquad 0 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$$

$$\widehat{G}^{n+1,k}_{n+1-(2i+1)} = G^{n}_{n-(2i+1)} L_{n-(2i+1),k}, \quad 0 \leqslant i \leqslant \lfloor \frac{n-1}{2} \rfloor,$$

and $\widehat{G}^{n+1,k}_{m}$ are matrices of respective sizes $r^d_n \times r^d_m$. Hence, we see that $x_k \mathbb{W}_n(\mathtt{x}) \in \Omega^d_{n+1}$, for $1 \leqslant k \leqslant d$.

Finally, (iii) and (iv) can be obtained by iterating (i) and (ii). □

Now, we can study a basis of the linear spaces $\Omega^d_n$.

**Lemma 2.2.** *Let $\{\mathbb{W}_n\}_{n\geqslant 0}$ be a sequence defined as in* (2.4)*. For $n = 0$, we define $\Gamma_0 = G^0_0$, and, for $n \geqslant 0$, we define the square $r^d_n + r^d_{n+1}$ matrices*

$$\Gamma_{n+1} = \begin{pmatrix} G^n_n & -G^n_{n-1} L^{(1)}_{n-1} \\ G^{n+1}_n & G^{n+1}_{n+1} \end{pmatrix}, \qquad n \geqslant 0.$$

*Denote $\rho_n = \det \Gamma_n$, for $n \geqslant 0$. Then,*

*(i) the set of vector functions*

$$\{\mathbb{W}_{2n}, \sqrt{1-\|\mathtt{x}\|^2}\mathbb{W}_{2n-1}, \mathbb{W}_{2n-2}, \sqrt{1-\|\mathtt{x}\|^2}\mathbb{W}_{2n-3}, \ldots, \sqrt{1-\|\mathtt{x}\|^2}\mathbb{W}_1, \mathbb{W}_0\},$$

*is a basis of $\Omega^d_{2n}$ if and only if $\prod_{i=0}^n \rho_{2i} \neq 0$.*

*(ii) The set*

$$\{\mathbb{W}_{2n+1}, \sqrt{1-\|\mathtt{x}\|^2}\mathbb{W}_{2n}, \mathbb{W}_{2n-1}, \sqrt{1-\|\mathtt{x}\|^2}\mathbb{W}_{2n-2}, \ldots, \mathbb{W}_1, \sqrt{1-\|\mathtt{x}\|^2}\mathbb{W}_0\},$$

*is a basis of $\Omega^d_{2n+1}$ if and only if $\prod_{i=0}^n \rho_{2i+1} \neq 0$.*

*Proof.* We consider only the even case since the odd case is similar.

We prove the result by studding the matrix of change of basis. For $n \geqslant 0$, we construct the column vector of the original basis of $\Omega^d_{2n}$

$$\mathcal{X}_{2n} = (\mathbb{X}^T_0, \sqrt{1-\|\mathtt{x}\|^2}\, \mathbb{X}^T_1, \cdots, \sqrt{1-\|\mathtt{x}\|^2}\, \mathbb{X}^T_{2n-1}, \mathbb{X}^T_{2n})^T,$$

of size $s^d_{2n} \times 1$. Let define the $s^d_{2n} \times 1$ vector containing the set of functions

$$\mathcal{W}_{2n} = (\mathbb{W}^T_0, \sqrt{1-\|\mathtt{x}\|^2}\, \mathbb{W}^T_1, \cdots, \sqrt{1-\|\mathtt{x}\|^2}\, \mathbb{W}^T_{2n-1}, \mathbb{W}^T_{2n})^T.$$

Then, there exists a square matrix $\mathcal{B}_{2n}$ of size $s^d_{2n}$ such that

$$\mathcal{B}_{2n}\, \mathcal{X}_{2n} = \mathcal{W}_{2n}, \quad n \geqslant 0.$$

We construct that matrix and study its non-singularity. For $n = 0$, since $\mathbb{W}_0 = G^0_0 \mathbb{X}_0 \neq 0$, it is clear that the $1 \times 1$ matrix $\mathcal{B}_0 = (G^0_0)$ is non-singular.

For $n = 1$, using (2.4) and Lemma 2.1, we get

$$\mathbb{W}_0 = G^0_0 \mathbb{X}_0$$

$$\sqrt{1-\|\mathtt{x}\|^2}\mathbb{W}_1 = \widetilde{G}^2_0 \mathbb{X}_0 + \widetilde{G}^2_1 \sqrt{1-\|\mathtt{x}\|^2}\mathbb{X}_1 + \widetilde{G}^2_2 \mathbb{X}_2$$

$$\mathbb{W}_2 = G^2_0 \mathbb{X}_0 + G^2_1 \sqrt{1-\|\mathtt{x}\|^2}\mathbb{X}_1 + G^2_2 \mathbb{X}_2,$$

where $G_0^0$ is a constant, $\widetilde{G}_i^2$ are matrices of size $r_1^d \times r_i^d$, and $G_i^2$ are matrices of size $r_2^d \times r_i^d$, for $i = 0, 1, 2$. Then, the $s_2^d$-size block matrix $\mathcal{B}_2$ reads as

$$\mathcal{B}_2 = \begin{pmatrix} G_0^0 & 0 & 0 \\ \widetilde{G}_0^2 & \widetilde{G}_1^2 & \widetilde{G}_2^2 \\ G_0^2 & G_1^2 & G_2^2 \end{pmatrix},$$

and the determinant of $\mathcal{B}_2$ is given by

$$\det \mathcal{B}_2 = G_0^0 \begin{vmatrix} \widetilde{G}_1^2 & \widetilde{G}_2^2 \\ G_1^2 & G_2^2 \end{vmatrix} = G_0^0 \begin{vmatrix} G_1^1 & -G_0^1 L_0^{(1)} \\ G_1^2 & G_2^2 \end{vmatrix} = \rho_0\, \rho_2,$$

by using the explicit expressions of $\widetilde{G}_i^2$ given in (2.5).

For $n \geqslant 1$, it can be checked that $\mathcal{B}_{2n}$ is a lower triangular block matrix such that its first block is the non-zero constant $G_0^0$, and the successive square diagonal blocks are

$$\begin{pmatrix} \widetilde{G}_{2m-1}^{2m} & \widetilde{G}_{2m}^{2m} \\ G_{2m-1}^{2m} & G_{2m}^{2m} \end{pmatrix} = \begin{pmatrix} G_{2m-1}^{2m-1} & -G_{2m-2}^{2m-1} L_{2m-2}^{(1)} \\ G_{2m-1}^{2m} & G_{2m}^{2m} \end{pmatrix} = \Gamma_{2m}, \quad 1 \leqslant m \leqslant n,$$

of size $r_{2m}^d + r_{2m+1}^d$. Then, $\det \mathcal{B}_{2n} = \rho_0\, \rho_2 \cdots \rho_{2n}$, and the result follows. □

An interesting consequence is the following result.

**Proposition 2.3.** *If* $\{\mathbb{W}_n, \sqrt{1-\|\mathtt{x}\|^2}\mathbb{W}_{n-1}, \mathbb{W}_{n-2}, \sqrt{1-\|\mathtt{x}\|^2}\mathbb{W}_{n-3}, \ldots\}$, *is a basis of* $\Omega_n^d$, *then, the following square matrices*

$$\gamma_n = G_n^n + G_{n-1}^n L_{n-1}^{(1)} (G_{n+1}^{n+1})^{-1} G_n^{n+1}, \tag{2.7}$$

$$\widetilde{\gamma}_{n+1} = G_{n+1}^{n+1} + G_n^{n+1} (G_n^n)^{-1} G_{n-1}^n L_{n-1}^{(1)}, \tag{2.8}$$

*are non-singular, and its respective inverses are given by*

$$\gamma_n^{-1} = (G_n^n)^{-1} - (G_n^n)^{-1} G_{n-1}^n L_{n-1}^{(1)} \widetilde{\gamma}_{n+1}^{-1} G_n^{n+1} (G_n^n)^{-1},$$

$$\widetilde{\gamma}_{n+1}^{-1} = (G_{n+1}^{n+1})^{-1} - (G_{n+1}^{n+1})^{-1} G_n^{n+1} \gamma_n^{-1} G_{n-1}^n L_{n-1}^{(1)} (G_{n+1}^{n+1})^{-1}.$$

*Proof.* Using Lemma 2.2, $\rho_n = \det \Gamma_n \neq 0$, for $n \geqslant 0$. Using the well know formulas for the determinant of a bock matrix, we deduce that

$$\begin{vmatrix} G_n^n & -G_{n-1}^n L_{n-1}^{(1)} \\ G_n^{n+1} & G_{n+1}^{n+1} \end{vmatrix} = \det[G_n^n]\, \det[G_{n+1}^{n+1} + G_n^{n+1} (G_n^n)^{-1} G_{n-1}^n L_{n-1}^{(1)}]$$

$$= \det[G_{n+1}^{n+1}]\, \det[G_n^n + G_{n-1}^n L_{n-1}^{(1)} (G_{n+1}^{n+1})^{-1} G_n^{n+1}].$$

Therefore, $\gamma_n$ and $\widetilde{\gamma}_{n+1}$ defined in (2.7) and (2.8), are non-singular. Using the Sherman-Morrison-Woodbury identity (see [7]), we get the expressions for the respective inverses. □

**Definition 2.4.** *A* functional system (FS) *is a sequence of vectors of functions* $\{\mathbb{W}_n\}_{n\geqslant 0}$ *defined as* (2.4), *such that the set*

$$\{\mathbb{W}_n, \sqrt{1-\|\mathtt{x}\|^2}\mathbb{W}_{n-1}, \mathbb{W}_{n-2}, \sqrt{1-\|\mathtt{x}\|^2}\mathbb{W}_{n-3}, \ldots\},$$

*are linearly independent.*

*Moreover, we say that it is a* monic functional system (MFS) *if the matrix leading coefficient* $G_n^n$ *is the identity matrix of size* $r_n^d$, *for* $n \geqslant 0$.

2.2. **Hybrid Orthogonality.** Given a sequence of real numbers $\{\mu_\alpha : |\alpha| = n, n \geqslant 0\}$, we define the functional

$$\begin{array}{rccc} \mathbf{u}: & \Pi^d & \longrightarrow & \mathbb{R} \\ & \mathtt{x}^\alpha & \longmapsto & \langle \mathbf{u}, \mathtt{x}^\alpha \rangle = \mu_\alpha \end{array}$$

extended by linearity to all polynomials as

$$\langle \mathbf{u}, p(\mathtt{x})\rangle = \sum_{|\alpha|\leqslant n} a_\alpha \langle \mathbf{u}, \mathtt{x}^\alpha\rangle = \sum_{|\alpha|\leqslant n} a_\alpha \mu_\alpha, \quad \text{for} \quad p(\mathtt{x}) = \sum_{|\alpha|\leqslant n} a_\alpha \mathtt{x}^\alpha.$$

In this case, $\mu_\alpha = \langle \mathbf{u}, \mathtt{x}^\alpha\rangle$, are called *moments*, and we say that $\mathbf{u}$ is a *moment functional.*

Let $A = \big(a_{i,j}(\mathtt{x})\big)_{i,j=1}^{n,m}$ be a matrix of multivariate polynomials. Then, the action of $\mathbf{u}$ over a matrix is given by

$$\langle \mathbf{u}, A\rangle = \big(\langle \mathbf{u}, a_{i,j}(\mathtt{x})\rangle\big)_{i,j=1}^{n,m},$$

and then, if $A = \big(a_{i,j}(\mathtt{x})\big)_{i,j=1}^{n,m}$ and $B = \big(b_{i,j}\big)_{i,j=1}^{m,h}$, is a matrix of constants, we get that

$$\langle \mathbf{u}, A\, B\rangle = \langle \mathbf{u}, A\rangle B,$$

is a $n \times h$ matrix with real entries, and, on the contrary, if $A = \big(a_{i,j}\big)_{i,j=1}^{n,m}$ is a matrix of constants, and $B = \big(b_{i,j}(\mathtt{x})\big)_{i,j=1}^{m,h}$ is a matrix of polynomials, then

$$\langle \mathbf{u}, A\, B\rangle = A\,\langle \mathbf{u}, B\rangle.$$

We introduce the moment functional $\mathbf{u}_{1/2}$ as the following perturbation of $\mathbf{u}$

$$\langle \mathbf{u}_{1/2}, p\, q\rangle = \langle \sqrt{1-\|\mathtt{x}\|^2}\,\mathbf{u}, p\, q\rangle = \langle \mathbf{u}, \sqrt{1-\|\mathtt{x}\|^2}\, p\, q\rangle. \tag{2.9}$$

Now, we define the *hybrid orthogonality* property for a *functional system.*

**Definition 2.5.** *A FS* $\{\mathbb{W}_n\}_{n\geqslant 0}$ *is a* hybrid orthogonal functional system (HOFS) *with respect to a moment functional* $\mathbf{u}$ *if satisfies the hybrid orthogonality conditions*

$$\begin{array}{llll} \langle \mathbf{u}, \mathbb{W}_{2m+1}\, \mathbb{W}_{2n}^T\rangle & = \mathtt{0}, & \forall n, m, & \\ \langle \mathbf{u}_{1/2}, \mathbb{W}_{2m+l}\, \mathbb{W}_{2n+l}^T\rangle & = \mathtt{0}, & n \neq m, & l = 0, 1, \\ \langle \mathbf{u}_{1/2}, \mathbb{W}_{2n+l}\, \mathbb{W}_{2n+l}^T\rangle & = H_{2n+l}, & n \geqslant 0, & l = 0, 1, \end{array} \tag{2.10}$$

*where* $H_{2n+l}$ *is a* $r^d_{2n+l} \times r^d_{2n+l}$ *symmetric and non-singular matrix, and* $\mathtt{0}$ *denotes the zero matrix of appropriate size.*

If $H_{2n+l}$ is a diagonal matrix, then the system is said to be a *mutually hybrid orthogonal functional system.*

A moment functional $\mathbf{u}$ is called *quasi-definite* if there exists a HOFS associated with $\mathbf{u}$. Moreover, $\mathbf{u}$ is *positive definite* if it is quasi-definite, and the non-singular matrices $H_{2n+l}$ are positive definite for $n \geqslant 0$, and $l = 0, 1$.

From (2.10), since $\mathbb{W}_0$ is a non-zero constant, we deduce that

$$\begin{aligned} &\langle \mathbf{u}, \mathbb{W}_{2m+1}\rangle = \mathtt{0}, \qquad m \geqslant 0, \\ &\langle \mathbf{u}_{1/2}, \mathbb{W}_{2m}\rangle = \langle \mathbf{u}, \sqrt{1-\|\mathtt{x}\|^2}\mathbb{W}_{2m}\rangle = \mathtt{0}, \quad m \geqslant 1. \end{aligned}$$

We prove that hybrid orthogonality for a sequence of functions as (2.4) implies the linear independence.

**Proposition 2.6.** *Let* $\mathbf{u}$ *be a moment functional, and let* $\{\mathbb{W}_n\}_{n\geqslant 0}$ *be a sequence of functions defined as in* (2.4) *satisfying the hybrid orthogonal conditions* (2.10)*. Then, the set*

$$\{\mathbb{W}_n, \sqrt{1-\|\mathtt{x}\|^2}\,\mathbb{W}_{n-1}, \mathbb{W}_{n-2}, \sqrt{1-\|\mathtt{x}\|^2}\,\mathbb{W}_{n-3}, \ldots\},$$

*is a basis of* $\Omega_n^d$.

*Proof.* We only prove the result for the even case because the odd case is analogous. We construct the following linear combination

$$a_0^T\,\mathbb{W}_0 + a_1^T\,\sqrt{1-\|\mathtt{x}\|^2}\,\mathbb{W}_1 + \cdots + a_{2n-1}^T\,\sqrt{1-\|\mathtt{x}\|^2}\,\mathbb{W}_{2n-1} + a_{2n}^T\,\mathbb{W}_{2n} = \mathtt{0}, \quad (2.11)$$

where $a_i$ are vectors of real constants of respective sizes $r_i^d \times 1$, for $0 \leqslant i \leqslant 2n$.

First, we multiply (2.11) by $\mathbb{W}_1^T$ and apply the moment functional $\mathbf{u}$ to obtain

$$a_0^T\langle \mathbf{u}, \mathbb{W}_0\mathbb{W}_1^T\rangle + a_1^T\langle \mathbf{u}, \sqrt{1-\|\mathtt{x}\|^2}\mathbb{W}_1\mathbb{W}_1^T\rangle + \cdots$$
$$+ a_{2n-1}^T\langle \mathbf{u}, \sqrt{1-\|\mathtt{x}\|^2}\mathbb{W}_{2n-1}\mathbb{W}_1^T\rangle + a_{2n}^T\langle \mathbf{u}, \mathbb{W}_{2n}\mathbb{W}_1^T\rangle = \mathtt{0}.$$

Then, by using (2.10), we deduce that, for $1 \leqslant j \leqslant n-1$,

$$\langle \mathbf{u}, \sqrt{1-\|\mathtt{x}\|^2}\mathbb{W}_{2j+1}\mathbb{W}_1^T\rangle = \langle \mathbf{u}_{1/2}, \mathbb{W}_{2j+1}\mathbb{W}_1^T\rangle = \mathtt{0},$$

and $\langle \mathbf{u}, \sqrt{1-\|\mathtt{x}\|^2}\mathbb{W}_1\mathbb{W}_1^T\rangle = \langle \mathbf{u}_{1/2}, \mathbb{W}_1\mathbb{W}_1^T\rangle = H_1$. Also, because of the different parity order, $\langle \mathbf{u}, \mathbb{W}_{2j}\mathbb{W}_1^T\rangle = \mathtt{0}$, for $0 \leqslant j \leqslant n$. In this way, we get

$$a_1^T H_1 = \mathtt{0},$$

since $H_1$ is an invertible matrix, we have $a_1 = \mathtt{0}$.

Similarly, multiplying (2.11) by $\mathbb{W}_{2j-1}^T$, for $2 \leqslant j \leqslant n$, and applying the moment functional $\mathbf{u}$, we get $a_{2j-1} = \mathtt{0}$, for $2 \leqslant j \leqslant n$. Then, equation (2.11) becomes

$$a_0^T\,\mathbb{W}_0 + a_2^T\,\mathbb{W}_2 + \cdots + a_{2n-2}^T\,\mathbb{W}_{2n-2} + a_{2n}^T\,\mathbb{W}_{2n} = \mathtt{0}.$$

Now, we multiply last equation by $\mathbb{W}_0^T$ and apply the moment functional $\mathbf{u}_{1/2}$ to get

$$a_0^T\langle \mathbf{u}_{1/2}, \mathbb{W}_0\mathbb{W}_0^T\rangle + a_2^T\langle \mathbf{u}_{1/2}, \mathbb{W}_2\mathbb{W}_0^T\rangle + \cdots$$
$$+ a_{2n-2}^T\langle \mathbf{u}_{1/2}\mathbb{W}_{2n-2}\mathbb{W}_0^T\rangle + a_{2n}^T\langle \mathbf{u}_{1/2}, \mathbb{W}_{2n}\mathbb{W}_0^T\rangle = \mathtt{0}.$$

Again, by (2.10), we know that $\langle \mathbf{u}_{1/2}, \mathbb{W}_{2m}\mathbb{W}_0^T\rangle = \mathtt{0}$, for $1 \leqslant m \leqslant n$. Then,

$$a_0^T\langle \mathbf{u}_{1/2}, \mathbb{W}_0\mathbb{W}_0^T\rangle = a_0^T H_0 = \mathtt{0},$$

therefore $a_0 = \mathtt{0}$. In a similar way, we show that $a_{2j} = \mathtt{0}$, for $1 \leqslant j \leqslant n$. We conclude that $a_i = \mathtt{0}$ in (2.11), for $0 \leqslant i \leqslant 2n$, which completes the proof. □

Next Lemma brings some consequences of the hybrid orthogonality.

**Lemma 2.7.** *Let* $\{\mathbb{W}_n\}_{n\geqslant 0}$ *be a HOFS associated with the moment functional* $\mathbf{u}$*. Then, for* $n \geqslant 1$*, the following statements hold*

*(i) For any function* $F(\mathtt{x}) \in \Omega_{n-(2i+1)}^d$, $0 \leqslant i \leqslant \lfloor \frac{n-1}{2} \rfloor$*, then*

$$\langle \mathbf{u}, F(\mathtt{x})\mathbb{W}_n^T\rangle = \mathtt{0}.$$

*(ii) For any function* $F(\mathtt{x}) \in \Omega_{n-2i}^d$, $1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$*, then*

$$\langle \mathbf{u}_{1/2}, F(\mathtt{x})\mathbb{W}_n^T\rangle = \mathtt{0}.$$

*(iii) For $1 \leqslant k \leqslant d$ and $1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$, we get*

$$\langle \mathbf{u}, x_k \mathbb{W}_{n-2i} \mathbb{W}_n^T \rangle = 0.$$

*(iv) For $1 \leqslant k \leqslant d$ and $1 \leqslant i \leqslant \lfloor \frac{n-1}{2} \rfloor$, we get*

$$\langle \mathbf{u}_{1/2}, x_k \mathbb{W}_{n-(2i+1)} \mathbb{W}_n^T \rangle = 0.$$

*(v) For $0 \leqslant i \leqslant \lfloor n/2 \rfloor$ and $0 \leqslant j \leqslant \lfloor (n+1)/2 \rfloor$, $i \neq j$, we get*

$$\langle \mathbf{u}_{1/2}, \sqrt{1-\|\mathtt{x}\|^2}\, \mathbb{W}_{n-2i} \mathbb{W}_{n+1-2j}^T \rangle = 0.$$

*Proof.* (i) Since $F(\mathtt{x}) \in \Omega_{n-(2i+1)}$, using the basis given in Lemma 2.2, there exist vectors of real constants of respective sizes $r_i^d \times 1$, such that $F(\mathtt{x})$ can be expressed in terms of that basis in the form

$$\begin{aligned} F(\mathtt{x}) = & c_{n-(2i+1)}^T \, \mathbb{W}_{n-(2i+1)} + c_{n-(2i+2)}^T \sqrt{1-\|\mathtt{x}\|^2}\, \mathbb{W}_{n-(2i+2)} \\ & + c_{n-(2i+3)}^T \, \mathbb{W}_{n-(2i+3)} + \cdots \end{aligned}$$

Therefore,

$$\begin{aligned} \langle \mathbf{u}, F(\mathtt{x}) \mathbb{W}_n^T \rangle = & c_{n-(2i+1)}^T \langle \mathbf{u}, \mathbb{W}_{n-(2i+1)} \mathbb{W}_n^T \rangle + + c_{n-(2i+2)}^T \langle \mathbf{u}_{1/2}, \mathbb{W}_{n-(2i+2)} \mathbb{W}_n^T \rangle \\ & + c_{n-(2i+3)}^T \langle \mathbf{u}, \mathbb{W}_{n-(2i+3)} \mathbb{W}_n^T \rangle + \cdots = 0, \end{aligned}$$

by using the hybrid orthogonality conditions (2.10).

(ii) As above, since $F(\mathtt{x}) \in \Omega_{n-2i}$, using the basis given in Lemma 2.2, there exist vectors of real constants of respective sizes $r_i^d \times 1$, such that $F(\mathtt{x})$ can be expressed in terms of that basis as

$$\begin{aligned} F(\mathtt{x}) = & c_{n-2i}^T \, \mathbb{W}_{n-2i} + c_{n-(2i+1)}^T \sqrt{1-\|\mathtt{x}\|^2}\, \mathbb{W}_{n-(2i+1)} \\ & + c_{n-(2i+2)}^T \, \mathbb{W}_{n-(2i+2)} + \cdots \end{aligned}$$

Therefore,

$$\begin{aligned} \langle \mathbf{u}_{1/2}, F(\mathtt{x}) \mathbb{W}_n^T \rangle = & c_{n-2i}^T \langle \mathbf{u}_{1/2}, \mathbb{W}_{n-2i} \mathbb{W}_n^T \rangle + c_{n-(2i+1)}^T \langle \mathbf{u}, (1-\|\mathtt{x}\|^2) \mathbb{W}_{n-(2i+1)} \mathbb{W}_n^T \rangle \\ & + c_{n-(2i+2)}^T \langle \mathbf{u}_{1/2}, \mathbb{W}_{n-(2i+2)} \mathbb{W}_n^T \rangle + \cdots = 0, \end{aligned}$$

by using the hybrid orthogonality conditions (2.10).

(iii) and (iv) can be deduced directly from (i) and (ii) because $x_k \mathbb{W}_{n-2i} \in \Omega_{n-2i+1}^d$ and $x_k \mathbb{W}_{n-(2i+1)} \in \Omega_{n-2i}^d$.

(v) We know that $(1-\|\mathtt{x}\|^2)\, \mathbb{W}_{n-2i} \in \Omega_{n-2i+2}^d$, $0 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$, thus we can express it in terms of a basis as

$$\begin{aligned} (1-\|\mathtt{x}\|^2)\, \mathbb{W}_{n-2i} = & E_{n-2i+2}^{n-2i+2} \mathbb{W}_{n-2i+2} + E_{n-2i+1}^{n-2i+2} \sqrt{1-\|\mathtt{x}\|^2} \mathbb{W}_{n-2i+1} \\ & + E_{n-2i}^{n-2i+2} \mathbb{W}_{n-2i} + \cdots \end{aligned}$$

where $E_m^{n-2i}$ are matrices of real constants of sizes $r_{n-2i}^d \times r_m^d$. Therefore,

$$\begin{aligned} \langle \mathbf{u}_{1/2}, & \sqrt{1-\|\mathtt{x}\|^2}\, \mathbb{W}_{n-2i} \mathbb{W}_{n+1-2j}^T \rangle = \langle \mathbf{u}, (1-\|\mathtt{x}\|^2)\, \mathbb{W}_{n-2i} \mathbb{W}_{n+1-2j}^T \rangle \\ & = E_{n-2i+2}^{n-2i+2} \langle \mathbf{u}, \mathbb{W}_{n-2i+2} \mathbb{W}_{n+1-2j}^T \rangle + E_{n-2i+1}^{n-2i+2} \langle \mathbf{u}_{1/2}, \mathbb{W}_{n-2i+1} \mathbb{W}_{n+1-2j}^T \rangle \\ & \quad + E_{n-2i}^{n-2i+2} \langle \mathbf{u}, \mathbb{W}_{n-2i} \mathbb{W}_{n+1-2j}^T \rangle + \cdots = 0, \end{aligned}$$

for $0 \leqslant j \leqslant \lfloor \frac{n+1}{2} \rfloor$, $i \neq j$, because of the hybrid orthogonality conditions (2.10). $\square$

The definition of hybrid orthogonality can be given in terms of the canonical basis $\{\mathbb{X}_n\}_{n \geqslant 0}$, as we show in the next result.

**Theorem 2.8.** *Let* $\mathbf{u}$ *be a moment functional, and let* $\{\mathbb{W}_n\}_{n\geqslant 0}$ *be a FS. Then,* $\{\mathbb{W}_n\}_{n\geqslant 0}$ *is a HOFS associated with* $\mathbf{u}$ *if, and only if*

$$\begin{aligned} \langle \mathbf{u}, \mathbb{X}_{n-(2i+1)}\, \mathbb{W}_n^T \rangle &= \mathtt{0}, \qquad 0 \leqslant i \leqslant \lfloor \tfrac{n-1}{2} \rfloor, \\ \langle \mathbf{u}_{1/2}, \mathbb{X}_{n-2i}\, \mathbb{W}_n^T \rangle &= \mathtt{0}, \qquad 1 \leqslant i \leqslant \lfloor \tfrac{n}{2} \rfloor, \\ \langle \mathbf{u}_{1/2}, \mathbb{X}_{n}\, \mathbb{W}_n^T \rangle &= S_n, \qquad n \geqslant 0, \end{aligned} \tag{2.12}$$

*where* $S_n$ *is a square non-singular real matrix of size* $r_n^d$.

*Proof.* The first two conditions are clear, since $\mathbb{X}_m \in \Omega_m^d$, for $m \geqslant 0$.

Reciprocally, suppose that hybrid orthogonality conditions (2.12) hold. Then, without loss of generality we consider $n > m$, use expression (2.4), and compute

$$\begin{aligned} \langle \mathbf{u}, \mathbb{W}_{2m+1}\mathbb{W}_{2n}^T \rangle = &\sum_{i=0}^{\lfloor \frac{2m+1}{2} \rfloor} G_{2m+1-2i}^{2m+1} \langle \mathbf{u}, \mathbb{X}_{2m+1-2i}\mathbb{W}_{2n}^T \rangle \\ &+ \sum_{i=0}^{\lfloor \frac{2m}{2} \rfloor} G_{2m-2i}^{2m+1} \langle \mathbf{u}_{1/2}, \mathbb{X}_{2m-2i}\mathbb{W}_{2n}^T \rangle = \mathtt{0}, \end{aligned}$$

that vanishes because $2m+1-2i < 2n$ and the change of parity in the first summation, and $2m-2i < 2n$ and the use of $\mathbf{u}_{1/2}$, in the second one.

Moreover, for $n > m$, and $l = 0, 1$,

$$\begin{aligned} \langle \mathbf{u}_{1/2}, \mathbb{W}_{2m+l}\mathbb{W}_{2n+l}^T \rangle = &\sum_{i=0}^{\lfloor \frac{2m+l}{2} \rfloor} G_{2m+l-2i}^{2m+l} \langle \mathbf{u}_{1/2}, \mathbb{X}_{2m+l-2i}\mathbb{W}_{2n+l}^T \rangle \\ + &\sum_{i=0}^{\lfloor \frac{2m+l-1}{2} \rfloor} G_{2m+l-(2i+1)}^{2m+l} \langle \mathbf{u}, (1-\|\mathtt{x}\|^2)\mathbb{X}_{2m+l-(2i+1)}\mathbb{W}_{2n+l}^T \rangle = \mathtt{0}, \end{aligned}$$

by using (2.2).

Now, we study $\langle \mathbf{u}_{1/2}, \mathbb{X}_n\, \mathbb{W}_n^T \rangle$ and $\langle \mathbf{u}_{1/2}, \mathbb{W}_n\, \mathbb{W}_n^T \rangle$. Since $\sqrt{1-\|\mathtt{x}\|^2}\mathbb{X}_n \in \Omega_{n+1}^d$, then there exist $r_n^d \times r_m^d$ matrices $\widetilde{F}_m^{n+1}$ such that

$$\sqrt{1-\|x\|^2}\mathbb{X}_n = \widetilde{F}_{n+1}^{n+1}\mathbb{W}_{n+1} + \widetilde{F}_n^{n+1}\sqrt{1-\|\mathtt{x}\|^2}\mathbb{W}_n + \widetilde{F}_{n-1}^{n+1}\mathbb{W}_{n-1} + \cdots$$

Then, using the hybrid orthogonality conditions (2.10) as well as the linearity, we directly compute

$$\begin{aligned} \langle \mathbf{u}_{1/2}, \mathbb{X}_n\mathbb{W}_n^T \rangle = &\langle \mathbf{u}, \sqrt{1-\|\mathtt{x}\|^2}\mathbb{X}_n\mathbb{W}_n^T \rangle \\ = &\widetilde{F}_{n+1}^{n+1}\langle \mathbf{u}, \mathbb{W}_{n+1}\mathbb{W}_n^T \rangle + \widetilde{F}_n^{n+1}\langle \mathbf{u}, \sqrt{1-\|\mathtt{x}\|^2}\mathbb{W}_n\mathbb{W}_n^T \rangle \\ &+ \widetilde{F}_{n-1}^{n+1}\langle \mathbf{u}, \mathbb{W}_{n-1}\mathbb{W}_n^T \rangle + \cdots \\ = &\widetilde{F}_n^{n+1} H_n. \end{aligned}$$

Now, we wish to compute the matrix $\widetilde{F}_n^{n+1}$. Using expression (2.4) and Lemma 2.1, we get

$$\begin{aligned}\sqrt{1-\|x\|^2}\mathbb{X}_n =& \widetilde{F}_{n+1}^{n+1}[G_{n+1}^{n+1}\mathbb{X}_{n+1} + G_n^{n+1}\sqrt{1-\|\mathtt{x}\|^2}\mathbb{X}_n + G_{n-1}^{n+1}\mathbb{X}_{n-1}+\cdots]\\ &+\widetilde{F}_n^{n+1}[\widetilde{G}_{n+1}^{n+1}\mathbb{X}_{n+1} + \widetilde{G}_n^{n+1}\sqrt{1-\|\mathtt{x}\|^2}\mathbb{X}_n+\cdots]+\cdots.\end{aligned}$$

Adjusting leading coefficients, and using again Lemma 2.1, we get the matrix linear system

$$\widetilde{F}_{n+1}^{n+1}G_{n+1}^{n+1} - \widetilde{F}_n^{n+1}G_{n-1}^{n}L_{n-1}^{(1)} = \mathtt{0},$$

$$\widetilde{F}_{n+1}^{n+1}G_n^{n+1} + \widetilde{F}_n^{n+1}G_n^{n} = I_{r_n^d},$$

with matrix unknowns $\widetilde{F}_{n+1}^{n+1}$ and $\widetilde{F}_n^{n+1}$. Observe that this linear system can be written as

$$(I_{r_n^d},\ \mathtt{0}) = (\widetilde{F}_n^{n+1},\ \widetilde{F}_{n+1}^{n+1})\begin{pmatrix} G_n^n & -G_{n-1}^n L_{n-1}^{(1)} \\ G_n^{n+1} & G_{n+1}^{n+1}\end{pmatrix},$$

where the coefficient matrix is the non singular square $r_n^d + r_{n+1}^d$ matrix $\Gamma_{n+1}$ defined in Lemma 2.2. Then, the system has unique solution. Since $G_{n+1}^{n+1}$ is non singular, then

$$\widetilde{F}_{n+1}^{n+1} = \widetilde{F}_n^{n+1}G_{n-1}^{n}L_{n-1}^{(1)}(G_{n+1}^{n+1})^{-1}.$$

Substituting this expression in the second equation, we get

$$\widetilde{F}_n^{n+1}[G_n^n + G_{n-1}^n L_{n-1}^{(1)}(G_{n+1}^{n+1})^{-1}G_n^{n+1}] = I_{r_n^d}.$$

Using Proposition 2.3 we know that $\gamma_n = G_n^n + G_{n-1}^n L_{n-1}^{(1)}(G_{n+1}^{n+1})^{-1}G_n^{n+1}$ is invertible, and then

$$\widetilde{F}_n^{n+1} = \gamma_n^{-1}.$$

In this way,

$$\langle \mathbf{u}_{1/2}, \mathbb{X}_n\mathbb{W}_n^T\rangle = \gamma_n^{-1}\langle \mathbf{u}_{1/2}, \mathbb{W}_n\mathbb{W}_n^T\rangle,$$

or equivalently,

$$S_n = \gamma_n^{-1}\, H_n.$$

Then, $S_n$ is non-singular if and only if $H_n$ is non-singular. That completes the proof. □

## 3. Existence, Three term relations and Favard type Theorem

First, we need to discuss the existence of a HOFS in terms of a given moment functional $\mathbf{u}$.

3.1. **Existence.** To simplify the notation we define $G_m^{n,T} := (G_m^n)^T$. We also consider the moment matrices

$$M_m^n = \langle \mathbf{u}, \mathbb{X}_n\,\mathbb{X}_m^T\rangle \quad \textit{and} \quad N_m^n = \langle \mathbf{u}_{1/2}, \mathbb{X}_n\,\mathbb{X}_m^T\rangle, \tag{3.1}$$

of respective sizes $r_n^d \times r_m^d$.

From (2.4) and $0 \leqslant j \leqslant \lfloor\frac{n-1}{2}\rfloor$, we can write

$$\langle \mathbf{u}, \mathbb{X}_{n-(2j+1)}\,\mathbb{W}_n^T\rangle = \sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\langle \mathbf{u}, \mathbb{X}_{n-(2j+1)}\,\mathbb{X}_{n-2i}^T\rangle G_{n-2i}^{n,T}$$

$$+\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\langle \mathbf{u}_{1/2}, \mathbb{X}_{n-(2j+1)}\,\mathbb{X}^T_{n-(2i+1)}\rangle G^{n,T}_{n-(2i+1)},$$

where $G^{n,T}_m$ are constant matrices of respective sizes $r^d_m \times r^d_n$, for $0 \leqslant m \leqslant n$ given in the expression (2.4).

Therefore, from Lemma 2.7 and using (3.1), we get

$$\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor} M^{n-(2j+1)}_{n-2i} G^{n,T}_{n-2i} + \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} N^{n-(2j+1)}_{n-(2i+1)} G^{n,T}_{n-(2i+1)} = \mathtt{0}, \tag{3.2}$$

for $0 \leqslant j \leqslant \lfloor (n-1)/2 \rfloor$. Also, using notation (3.1), we deduce

$$\begin{aligned}\langle \mathbf{u}_{1/2}, \mathbb{X}_{n-2j}\,\mathbb{W}^T_n\rangle &= \sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\langle \mathbf{u}_{1/2}, \mathbb{X}_{n-2j}\,\mathbb{X}^T_{n-2i}\rangle G^{n,T}_{n-2i}\\ &\quad+\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\langle \mathbf{u}, (1-\|\mathtt{x}\|^2)\,\mathbb{X}_{n-2j}\,\mathbb{X}^T_{n-(2i+1)}\rangle G^{n,T}_{n-(2i+1)}\\ &=\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor} N^{n-2j}_{n-2i} G^{n,T}_{n-2i} + \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} J^{n-2j}_{n-(2i+1)} G^{n,T}_{n-(2i+1)},\end{aligned}$$

for $0 \leqslant j \leqslant \lfloor n/2 \rfloor$, where $J^{n-2j}_{n-(2i+1)} = M^{n-2j}_{n-(2i+1)} - L^{(1)}_{n-2j} M^{n+2-2j}_{n-(2i+1)}$. Then, from Lemma 2.7, we get

$$\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor} N^{n-2j}_{n-2i} G^{n,T}_{n-2i} + \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} J^{n-2j}_{n-(2i+1)} G^{n,T}_{n-(2i+1)} = \mathtt{0}, \tag{3.3}$$

for $1 \leqslant j \leqslant \lfloor n/2 \rfloor$, and for $j = 0$,

$$\sum_{i=0}^{\lfloor\frac{n}{2}\rfloor} N^{n}_{n-2i} G^{n,T}_{n-2i} + \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} J^{n}_{n-(2i+1)} G^{n,T}_{n-(2i+1)} = S_n, \tag{3.4}$$

where $S_n = \langle \mathbf{u}_{1/2}, \mathbb{X}_n\,\mathbb{W}^T_n\rangle$.

Considering the matrix equations (3.2), (3.3) and (3.4), we get a linear system of $s^d_n = \sum_{i=0}^n r^d_i$ equations which solution is the column vector of the matrix coefficients of $\mathbb{W}_n$, in the form

$$\mathcal{M}_{s^d_n}\,\mathcal{G}_{s^d_n} = \mathcal{S}_{s^d_n},$$

where

$$\mathcal{G}_{s^d_n} = \begin{pmatrix} G^{n,T}_0\\ G^{n,T}_1\\ \vdots\\ G^{n,T}_{n-1}\\ G^{n,T}_n\end{pmatrix}, \qquad \mathcal{S}_{s^d_n} = \begin{pmatrix}\mathtt{0}\\ \mathtt{0}\\ \vdots\\ \mathtt{0}\\ S_n\end{pmatrix},$$

are column vectors of matrices of total size $s^d_n \times r^d_n$. The coefficient matrix is a square block matrix of size $s^d_n \times s^d_n$ whose structure depends on the parity of $n$ in the form

$$\mathcal{M}_{s_n^d} = \left(\begin{array}{ccccccc} M_0^{n-1} & N_1^{n-1} & M_2^{n-1} & N_3^{n-1} & \cdots & N_{n-1}^{n-1} & M_n^{n-1} \\ M_0^{n-3} & N_1^{n-3} & M_2^{n-3} & N_3^{n-3} & \cdots & N_{n-1}^{n-3} & M_n^{n-3} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ M_0^{1} & N_1^{1} & M_2^{1} & N_3^{1} & \cdots & N_{n-1}^{1} & M_n^{1} \\ \hline N_0^{n} & J_1^{n} & N_2^{n} & J_3^{n} & \cdots & J_{n-1}^{n} & N_n^{n} \\ N_0^{n-2} & J_1^{n-2} & N_2^{n-2} & J_3^{n-2} & \cdots & J_{n-1}^{n-2} & N_n^{n-2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ N_0^{0} & J_1^{0} & N_2^{0} & J_3^{0} & \cdots & J_{n-1}^{0} & N_n^{0} \end{array}\right),$$

for $n$ even, and for $n$ odd,

$$\mathcal{M}_{s_n^d} = \left(\begin{array}{ccccccc} N_0^{n-1} & M_1^{n-1} & N_2^{n-1} & \cdots & M_{n-2}^{n-1} & N_{n-1}^{n-1} & M_n^{n-1} \\ N_0^{n-3} & M_1^{n-3} & N_2^{n-3} & \cdots & M_{n-2}^{n-3} & N_{n-1}^{n-3} & M_n^{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ N_0^{0} & M_1^{0} & N_2^{0} & \cdots & M_{n-2}^{0} & N_{n-1}^{0} & M_n^{0} \\ \hline J_0^{n} & N_1^{n} & J_2^{n} & \cdots & N_{n-2}^{n} & J_{n-1}^{n} & N_n^{n} \\ J_0^{n-2} & N_1^{n-2} & J_2^{n-2} & \cdots & N_{n-2}^{n-2} & J_{n-1}^{n-2} & N_n^{n-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ J_0^{1} & N_1^{1} & J_2^{1} & \cdots & N_{n-2}^{1} & J_{n-1}^{1} & N_n^{1} \end{array}\right).$$

Then, we have the following result.

**Theorem 3.1.** *Let* $\mathbf{u}$ *be a moment functional. Then a hybrid orthogonal functional system associated with* $\mathbf{u}$ *exists if and only if* $\mathcal{M}_{s_n^d}$ *is a non singular matrix, for* $n \geqslant 0$.

3.2. **Three term relations.** In the following, we use the definition of *joint matrix* (see [6, p. 71]). Given $n \times m$ matrices $M_1, M_2, \ldots, M_d$, we define their *joint matrix* as

$$M = (M_1^T, M_2^T, \ldots, M_d^T)^T,$$

of size $dn \times m$.

In the next theorem we deduce matrix three term relations for a HOFS $\{\mathbb{W}_n\}_{n\geqslant 0}$.

**Theorem 3.2.** *Let* $\{\mathbb{W}_n\}_{n\geqslant 0}$ *be a HOFS associated with the moment functional* $\mathbf{u}$. *Then, for* $n \geqslant 0$ *and* $1 \leqslant k \leqslant d$, *there exist matrices* $A_{n,k}, B_{n,k}, C_{n,k}$ *of respective sizes* $r_n^d \times r_{n+1}^d$, $r_n^d \times r_n^d$ *and* $r_n^d \times r_{n-1}^d$ *such that*

$$x_k\, \mathbb{W}_n(\mathtt{x}) = A_{n,k}\, \mathbb{W}_{n+1}(\mathtt{x}) + B_{n,k}\, \sqrt{1-\|\mathtt{x}\|^2}\, \mathbb{W}_n(\mathtt{x}) + C_{n,k}\, \mathbb{W}_{n-1}(\mathtt{x}), \qquad (3.5)$$

*with* $\mathbb{W}_{-1}(\mathtt{x}) = 0$ *and* $\mathbb{W}_0(\mathtt{x}) = 1$.

*In addition,*

$$A_{n,k} = [\langle \mathbf{u}_{1/2}, x_k \mathbb{W}_n\, \mathbb{W}_{n+1}^T\rangle - B_{n,k}\langle \mathbf{u}_{1/2}, \sqrt{1-\|\mathtt{x}\|^2}\mathbb{W}_n\, \mathbb{W}_{n+1}^T\rangle]\; H_{n+1}^{-1},$$

$$B_{n,k} = \langle \mathbf{u}, x_k \mathbb{W}_n\, \mathbb{W}_n^T \rangle\ H_n^{-1},$$

$$C_{n,k} = [\langle \mathbf{u}_{1/2}, x_k \mathbb{W}_n\, \mathbb{W}_{n-1}^T \rangle - B_{n,k} \langle \mathbf{u}_{1/2}, \sqrt{1-\|\mathtt{x}\|^2} \mathbb{W}_n\, \mathbb{W}_{n-1}^T \rangle]\ H_{n-1}^{-1}.$$

*Proof.* For $1 \leqslant k \leqslant d$, and Lemma 2.1, we know that $x_k \mathbb{W}_n \in \Omega_{n+1}^d$, and it can be express in terms of the basis given in Lemma 2.2. Then,

$$x_k \mathbb{W}_n = \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} E_{n+1-2i}^{n+1,k}\, \mathbb{W}_{n+1-2i} + \sqrt{1-\|\mathtt{x}\|^2} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} E_{n+1-(2i+1)}^{n+1,k}\, \mathbb{W}_{n+1-(2i+1)}, \quad (3.6)$$

where the coefficients $E_m^{n+1,k}$ are real matrices of size $r_n^d \times r_m^d$.

From part (i) of Lemma 2.7, we know that

$$\langle \mathbf{u}, x_k \mathbb{W}_n \mathbb{W}_{n-2j}^T \rangle = \langle \mathbf{u}, \mathbb{W}_n (x_k \mathbb{W}_{n-2j})^T \rangle = \mathtt{0}, \quad 1 \leqslant j \leqslant \lfloor n/2 \rfloor,$$

since $x_k \mathbb{W}_{n-2j} \in \Omega_{n+1-2j}$. Then, we multiply (3.6) by $\mathbb{W}_{n-2j}^T$, for $1 \leqslant j \leqslant \lfloor n/2 \rfloor$, and apply the moment functional $\mathbf{u}$ to obtain

$$\mathtt{0} = \langle \mathbf{u}, x_k \mathbb{W}_n \mathbb{W}_{n-2j}^T \rangle = \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} E_{n+1-2i}^{n+1,k} \langle \mathbf{u}, \mathbb{W}_{n+1-2i}\, \mathbb{W}_{n-2j}^T \rangle$$
$$+ \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} E_{n+1-(2i+1)}^{n+1,k} \langle \mathbf{u}, \sqrt{1-\|\mathtt{x}\|^2}\, \mathbb{W}_{n+1-(2i+1)}\, \mathbb{W}_{n-2j}^T \rangle.$$

By using the hybrid orthogonality (2.10), the first summation vanishes, and all terms in the second summation vanishes except for $i = j$. Therefore,

$$\mathtt{0} = E_{n+1-(2j+1)}^{n+1,k} \langle \mathbf{u}_{1/2}, \mathbb{W}_{n-2j}\, \mathbb{W}_{n-2j}^T \rangle = E_{n-2j}^{n+1,k} H_{n-2j},$$

hence $E_{n-2j}^{n+1,k} = \mathtt{0}$, for $1 \leqslant j \leqslant \lfloor n/2 \rfloor$. Then, (3.6) can be written as

$$x_k \mathbb{W}_n(\mathtt{x}) = \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} E_{n+1-2i}^{n+1,k}\, \mathbb{W}_{n+1-2i}(\mathtt{x}) + E_n^{n+1,k} \sqrt{1-\|\mathtt{x}\|^2} \mathbb{W}_n(\mathtt{x}). \quad (3.7)$$

Now, $x_k \mathbb{W}_{n+1-2j} \in \Omega_{n+2-2j}$, and applying part (ii) of Lemma 2.7, we deduce that

$$\langle \mathbf{u}_{1/2}, x_k \mathbb{W}_n \mathbb{W}_{n+1-2j}^T \rangle = \langle \mathbf{u}_{1/2}, \mathbb{W}_n (x_k \mathbb{W}_{n+1-2j})^T \rangle = \mathtt{0}, \quad 2 \leqslant j \leqslant \lfloor (n+1)/2 \rfloor.$$

In this way, we multiply relation (3.7) by $\mathbb{W}_{n+1-2j}^T$, for $2 \leqslant j \leqslant \lfloor (n+1)/2 \rfloor$, apply the moment functional $\mathbf{u}_{1/2}$, and we get

$$\mathtt{0} = \langle \mathbf{u}_{1/2}, x_k \mathbb{W}_n\, \mathbb{W}_{n+1-2j}^T \rangle = \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} E_{n+1-2i}^{n+1,k} \langle \mathbf{u}_{1/2}, \mathbb{W}_{n+1-2i}\, \mathbb{W}_{n+1-2j}^T \rangle$$
$$+ E_n^{n+1,k} \langle \mathbf{u}_{1/2}, \sqrt{1-\|\mathtt{x}\|^2}\, \mathbb{W}_n\, \mathbb{W}_{n+1-2j}^T \rangle.$$

From the hybrid orthogonality conditions (2.10), only the term corresponding to $i = j$ in the first summation is different from zero, and using part (v) of Lemma 2.7, $\langle \mathbf{u}_{1/2}, \sqrt{1-\|\mathtt{x}\|^2}\, \mathbb{W}_n\, \mathbb{W}_{n+1-2j}^T \rangle = \mathtt{0}$. Hence,

$$\mathtt{0} = E_{n+1-2j}^{n+1,k} \langle \mathbf{u}_{1/2}, \mathbb{W}_{n+1-2j}\, \mathbb{W}_{n+1-2j}^T \rangle = E_{n+1-2j}^{n+1,k} H_{n+1-2j},$$

and $E_{n+1-2j}^{n+1,k} = \mathtt{0}$, for $2 \leqslant j \leqslant \lfloor (n+1)/2 \rfloor$. Then, (3.6) becomes the three term relation (3.5), defining $A_{n,k} = E_{n+1}^{n+1,k}$, $B_{n,k} = E_n^{n+1,k}$, and $C_{n,k} = E_{n-1}^{n+1,k}$.

Now, we compute the matrix coefficients. To get $B_{n,k}$, we multiply relation (3.5) by $\mathbb{W}_n^T$, apply the moment functional $\mathbf{u}$, deducing

$$\langle \mathbf{u}, x_k \mathbb{W}_n \mathbb{W}_n^T \rangle = A_{n,k} \langle \mathbf{u}, \mathbb{W}_{n+1} \mathbb{W}_n^T \rangle + B_{n,k} \langle \mathbf{u}_{1/2}, \mathbb{W}_n \mathbb{W}_n^T \rangle + C_{n,k} \langle \mathbf{u}, \mathbb{W}_{n-1} \mathbb{W}_n^T \rangle,$$

and, from the hybrid orthogonality (2.10), we obtain

$$B_{n,k} H_n = \langle \mathbf{u}, x_k \mathbb{W}_n \mathbb{W}_n^T \rangle.$$

Similarly, to obtain $A_{n,k}$, we multiply by $\mathbb{W}_{n+1}^T$, and apply the moment functional $\mathbf{u}_{1/2}$, obtaining

$$\begin{aligned}\langle \mathbf{u}_{1/2}, x_k \mathbb{W}_n \mathbb{W}_{n+1}^T \rangle = {} & A_{n,k} \langle \mathbf{u}_{1/2}, \mathbb{W}_{n+1} \mathbb{W}_{n+1}^T \rangle + B_{n,k} \langle \mathbf{u}_{1/2}, \sqrt{1 - \|\mathrm{x}\|^2} \mathbb{W}_n \mathbb{W}_{n+1}^T \rangle \\ & + C_{n,k} \langle \mathbf{u}_{1/2}, \mathbb{W}_{n-1} \mathbb{W}_{n+1}^T \rangle,\end{aligned}$$

and

$$A_{n,k} H_{n+1} = \langle \mathbf{u}_{1/2}, x_k \mathbb{W}_n \mathbb{W}_{n+1}^T \rangle - B_{n,k} \langle \mathbf{u}_{1/2}, \sqrt{1 - \|\mathrm{x}\|^2} \mathbb{W}_n \mathbb{W}_{n+1}^T \rangle.$$

Finally, to obtain $C_{n,k}$, we multiply (3.5) by $\mathbb{W}_{n-1}^T$ by means of the moment functional $\mathbf{u}_{1/2}$, and we get

$$C_{n,k} H_{n-1} = \langle \mathbf{u}_{1/2}, x_k \mathbb{W}_n \mathbb{W}_{n-1}^T \rangle - B_{n,k} \langle \mathbf{u}_{1/2}, \sqrt{1 - \|\mathrm{x}\|^2}\, \mathbb{W}_n \mathbb{W}_{n-1}^T \rangle.$$

Observe that

$$H_n C_{n+1,k}^T = \langle \mathbf{u}_{1/2}, x_k \mathbb{W}_n \mathbb{W}_{n+1}^T \rangle - \langle \mathbf{u}_{1/2}, \sqrt{1 - \|\mathrm{x}\|^2}\, \mathbb{W}_n \mathbb{W}_{n+1}^T \rangle B_{n+1,k}^T,$$

and then

$$A_{n,k} H_{n+1} + B_{n,k} \Lambda_n = H_n C_{n+1,k}^T + \Lambda_n B_{n+1,k}^T,$$

where $\Lambda_n = \langle \mathbf{u}_{1/2}, \sqrt{1 - \|\mathrm{x}\|^2}\, \mathbb{W}_n \mathbb{W}_{n-1}^T \rangle$. □

Now, we analyse the matrix coefficient of the three term relations (3.5).

**Proposition 3.3.** *Let* $\{\mathbb{W}_n\}_{n\geqslant 0}$ *be a HOFS associated with the moment functional* $\mathbf{u}$*, and let* $A_{n,k}, B_{n,k}$ *be the first coefficient matrices of the three term relations* (3.5)*. Then,*

$$A_{n,k} = [G_n^n L_{n,k} + G_{n-1}^n L_{n-1,k} (G_n^n)^{-1} G_{n-1}^n L_{n-1}^{(1)}] \widetilde{\gamma}_{n+1}^{-1}, \tag{3.8}$$

$$B_{n,k} = [G_{n-1}^n L_{n-1,k} - G_n^n L_{n,k} (G_{n+1}^{n+1})^{-1} G_n^{n+1}] \gamma_n^{-1}, \tag{3.9}$$

*where*

$$\begin{aligned}\gamma_n &= G_n^n + G_{n-1}^n L_{n-1}^{(1)} (G_{n+1}^{n+1})^{-1} G_n^{n+1}, \\ \widetilde{\gamma}_{n+1} &= G_{n+1}^{n+1} + G_n^{n+1} (G_n^n)^{-1} G_{n-1}^n L_{n-1}^{(1)},\end{aligned}$$

*were defined by* (2.7) *and* (2.8) *in Proposition 2.3.*

*Moreover,* $\operatorname{rank} A_{n,k} = r_n^d$*, and* $\operatorname{rank} A_n = r_{n+1}^d$*, where* $A_n$ *is the respective join matrix.*

*Proof.* Adjusting leading coefficients in (3.5) and using Lemma 2.1, we get the linear system

$$\begin{aligned}G_n^n L_{n,k} &= A_{n,k} G_{n+1}^{n+1} - B_{n,k} G_{n-1}^n L_{n-1}^{(1)}, \\ G_{n-1}^n L_{n-1,k} &= A_{n,k} G_n^{n+1} + B_{n,k} G_n^n,\end{aligned} \tag{3.10}$$

with matrix unknowns $A_{n,k}$, $B_{n,k}$. This linear system can be written as

$$(G^n_{n-1}L_{n-1,k},\ G^n_nL_{n,k}) = (B_{n,k},\ A_{n,k})\begin{pmatrix} G^n_n & -G^n_{n-1}L^{(1)}_{n-1} \\ G^{n+1}_n & G^{n+1}_{n+1}\end{pmatrix},$$

that is,

$$(G^n_{n-1}L_{n-1,k},\ G^n_nL_{n,k}) = (B_{n,k},\ A_{n,k})\Gamma_{n+1},$$

where $\Gamma_{n+1}$ is the non singular square $r^d_n + r^d_{n+1}$ matrix defined in Lemma 2.2. Then, the system has unique solution, and we will compute it.

Since $G^{n+1}_{n+1}$ is non singular, then first equation in (3.10) can be written up as

$$A_{n,k} = [G^n_nL_{n,k} + B_{n,k}G^n_{n-1}L^{(1)}_{n-1}](G^{n+1}_{n+1})^{-1}.$$

Substituting this expression in the second equation, we get

$$G^n_{n-1}L_{n-1,k} = [G^n_nL_{n,k} + B_{n,k}G^n_{n-1}L^{(1)}_{n-1}](G^{n+1}_{n+1})^{-1}G^{n+1}_n + B_{n,k}G^n_n.$$

Now, grouping terms in $B_{n,k}$, we deduce

$$B_{n,k}[G^n_n + G^n_{n-1}L^{(1)}_{n-1}(G^{n+1}_{n+1})^{-1}G^{n+1}_n] = G^n_{n-1}L_{n-1,k} - G^n_nL_{n,k}(G^{n+1}_{n+1})^{-1}G^{n+1}_n.$$

Using Proposition 2.3 we know that $\gamma_n = G^n_n + G^n_{n-1}L^{(1)}_{n-1}(G^{n+1}_{n+1})^{-1}G^{n+1}_n$ is invertible, and (3.9) follows.

Returning the system (3.10), we can do the same process starting in the second equation as

$$B_{n,k} = [G^n_{n-1}L_{n-1,k} - A_{n,k}G^{n+1}_n](G^n_n)^{-1},$$

substituting in the first one

$$G^n_nL_{n,k} = A_{n,k}G^{n+1}_{n+1} - [G^n_{n-1}L_{n-1,k} - A_{n,k}G^{n+1}_n](G^n_n)^{-1}G^n_{n-1}L^{(1)}_{n-1},$$

grouping terms in $A_{n,k}$,

$$A_{n,k}[G^{n+1}_{n+1} + G^{n+1}_n(G^n_n)^{-1}G^n_{n-1}L^{(1)}_{n-1}] = G^n_nL_{n,k} + G^n_{n-1}L_{n-1,k}(G^n_n)^{-1}G^n_{n-1}L^{(1)}_{n-1},$$

and using that the matrix $\widetilde{\gamma}_{n+1} = G^{n+1}_{n+1} + G^{n+1}_n(G^n_n)^{-1}G^n_{n-1}L^{(1)}_{n-1}$ is invertible by Proposition 2.3, we get (3.8).

Now, we study the rank of the matrix $A_{n,k}$. We write (3.10) as

$$G^n_nL_{n,k} + B_{n,k}G^n_{n-1}L^{(1)}_{n-1} = A_{n,k}G^{n+1}_{n+1},$$

$$G^n_{n-1}L_{n-1,k} - B_{n,k}G^n_n = A_{n,k}G^{n+1}_n.$$

Observe that we can express the matrix linear system in the form

$$(-B_{n,k},\ I_{r^d_n})\begin{pmatrix} G^n_n & -G^n_{n-1}L^{(1)}_{n-1} \\ G^n_{n-1}L_{n-1,k} & G^n_nL_{n,k}\end{pmatrix} = (A_{n,k}G^{n+1}_n,\ A_{n,k}G^{n+1}_{n+1}).$$

This matrix linear system has unique solution, and the block matrix of coefficients

$$\widetilde{\Gamma}_n = \begin{pmatrix} G^n_n & -G^n_{n-1}L^{(1)}_{n-1} \\ G^n_{n-1}L_{n-1,k} & G^n_nL_{n,k}\end{pmatrix}$$

has full rank $2r^d_n$. Therefore, by [12], we deduce that

$$2r^d_n = \operatorname{rank}\widetilde{\Gamma}_n = \operatorname{rank}G^n_n + \operatorname{rank}[G^n_nL_{n,k} + G^n_{n-1}L_{n-1,k}(G^n_n)^{-1}G^n_{n-1}L^{(1)}_{n-1}],$$

and then, since $\operatorname{rank}G^n_n = r^d_n$, the result follows.

Finally, in order to get the rank of the join matrix $A_n$, we extend the matrix linear system (3.10) to join matrices as follows,

$$\operatorname{diag}(G_n^n)L_n = A_n G_{n+1}^{n+1} - B_n G_{n-1}^n L_{n-1}^{(1)},$$

$$\operatorname{diag}(G_{n-1}^n)L_{n-1} = A_n G_n^{n+1} + B_n G_n^n,$$

where $L_n$, $A_n$ and $B_n$ are the respective join matrices. Working as above, we get $\operatorname{rank} A_n = r_{n+1}^d$. □

Finally, we study the expression of $C_{n,k}$. We need a technical Proposition.

**Proposition 3.4.** *Let $\{\mathbb{W}_n\}_{n\geqslant 0}$ be a HOFS associated with the moment functional $\mathbf{u}$. Then,*

*(i) Let $\kappa_{n,k}^{(1)} = G_{n-1}^{n-1}L_{n-1,k} + G_{n-2}^{n-1}L_{n-2}^{(1)}L_{n,k}(G_{n+1}^{n+1})^{-1}G_n^{n+1}$. Then*

$$\langle \mathbf{u}, x_k\sqrt{1-\|\mathbf{x}\|^2}\,\mathbb{W}_{n-1}\mathbb{W}_n^T\rangle = \kappa_{n,k}^{(1)}\gamma_n^{-1}H_n.$$

*(ii) Let $\kappa_n^{(2)} = -G_{n-2}^{n-1}L_{n-2}^{(1)} + G_{n-1}^{n-1}L_{n-1}^{(1)}(G_{n+1}^{n+1})^{-1}G_n^{n+1}$. Therefore,*

$$\langle \mathbf{u}, (1-\|\mathbf{x}\|^2)\mathbb{W}_{n-1}\mathbb{W}_n^T\rangle = \kappa_n^{(2)}\gamma_n^{-1}H_n.$$

*(iii) Let $\kappa_{n,k}^{(3)} = G_{n-1}^n L_{n-1,k} - G_n^n L_{n,k}(G_{n+1}^{n+1})^{-1}G_n^{n+1}$. Thus,*

$$\langle \mathbf{u}, x_k\mathbb{W}_n\mathbb{W}_n^T\rangle = \kappa_{n,k}^{(3)}\gamma_n^{-1}H_n.$$

*(iv) The explicit expression of $C_{n,k}$ is given by*

$$C_{n,k}^T = H_{n-1}^{-1}[\kappa_{n,k}^{(1)} - \kappa_n^{(2)}\gamma_n^{-1}\kappa_{n,k}^{(3)}]\gamma_n^{-1}H_n. \tag{3.11}$$

*Moreover, $\kappa_{n,k}^{(1)}$ and $[\kappa_{n,k}^{(1)} - \kappa_n^{(2)}\gamma_n^{-1}\kappa_{n,k}^{(3)}]$ are full rank matrices, as well as their respective join matrices.*

*Proof.* (i) Since $x_k\sqrt{1-\|\mathbf{x}\|^2}\,\mathbb{W}_{n-1} \in \Omega_{n+1}^d$, there exist $r_{n-1}^d \times r_m^d$ matrices of constants $\widetilde{E}_m^{n+1,k}$ such that

$$x_k\sqrt{1-\|\mathbf{x}\|^2}\,\mathbb{W}_{n-1} = \widetilde{E}_{n+1}^{n+1,k}\mathbb{W}_{n+1} + \widetilde{E}_n^{n+1,k}\sqrt{1-\|\mathbf{x}\|^2}\,\mathbb{W}_n + \widetilde{E}_{n-1}^{n+1,k}\mathbb{W}_{n-1} + \cdots. \tag{3.12}$$

Using the hybrid orthogonality, we compute

$$\begin{aligned}\langle \mathbf{u}, x_k\sqrt{1-\|\mathbf{x}\|^2}\,\mathbb{W}_{n-1}\mathbb{W}_n^T\rangle =& \widetilde{E}_{n+1}^{n+1,k}\langle \mathbf{u}, \mathbb{W}_{n+1}\mathbb{W}_n^T\rangle + \widetilde{E}_n^{n+1,k}\langle \mathbf{u}, \sqrt{1-\|\mathbf{x}\|^2}\,\mathbb{W}_n\mathbb{W}_n^T\rangle \\ &+ \widetilde{E}_{n-1}^{n+1,k}\langle \mathbf{u}, \mathbb{W}_{n-1}\mathbb{W}_n^T\rangle + \cdots \\ =& \widetilde{E}_n^{n+1,k}\langle \mathbf{u}_{1/2}, \mathbb{W}_n\mathbb{W}_n^T\rangle = \widetilde{E}_n^{n+1,k}H_n.\end{aligned}$$

Then, we need to compute $\widetilde{E}_n^{n+1,k}$. From part (iii) of Lemma 2.1, and comparing leading coefficients of both sides of (3.12), we obtain

$$-G_{n-2}^{n-1}L_{n-2}^{(1)}L_{n,k} = \widetilde{E}_{n+1}^{n+1,k}G_{n+1}^{n+1} - \widetilde{E}_n^{n+1,k}G_{n-1}^n L_{n-1}^{(1)},$$

$$G_{n-1}^{n-1}L_{n-1,k} = \widetilde{E}_{n+1}^{n+1,k}G_n^{n+1} + \widetilde{E}_n^{n+1,k}G_n^n,$$

that can be seen as a matrix linear system with matrix unknowns $\widetilde{E}_{n+1}^{n+1,k}$ and $\widetilde{E}_{n}^{n+1,k}$. We can express it by using again the matrix $\Gamma_{n+1}$ in the form

$$(G_{n-1}^{n} L_{n-1,k},\ -G_{n-2}^{n-1} L_{n-2}^{(1)} L_{n,k}) = (\widetilde{E}_{n}^{n+1,k}, \widetilde{E}_{n+1}^{n+1,k}) \begin{pmatrix} G_n^n & -G_{n-1}^{n} L_{n-1}^{(1)} \\ G_n^{n+1} & G_{n+1}^{n+1} \end{pmatrix},$$

and then, there exist unique solution. Computing as above, we substitute

$$\widetilde{E}_{n+1}^{n+1,k} = [\widetilde{E}_{n}^{n+1,k} G_{n-1}^{n} L_{n-1}^{(1)} - G_{n-2}^{n-1} L_{n-2}^{(1)} L_{n,k}](G_{n+1}^{n+1})^{-1}$$

in the second equation

$$G_{n-1}^{n-1} L_{n-1,k} = [\widetilde{E}_{n}^{n+1,k} G_{n-1}^{n} L_{n-1}^{(1)} - G_{n-2}^{n-1} L_{n-2}^{(1)} L_{n,k}](G_{n+1}^{n+1})^{-1} G_n^{n+1} + \widetilde{E}_{n}^{n+1,k} G_n^n,$$

and we obtain

$$\widetilde{E}_{n}^{n+1,k} = [G_{n-1}^{n-1} L_{n-1,k} + G_{n-2}^{n-1} L_{n-2}^{(1)} L_{n,k} (G_{n+1}^{n+1})^{-1} G_n^{n+1}] \gamma_n^{-1}.$$

The rank condition of the matrix $\kappa_{n,k}^{(1)}$ is deduced by means of a similar reasoning as in Proposition 3.3.

(ii) Again, $(1-\|\mathtt{x}\|^2)\mathbb{W}_{n-1} \in \Omega_{n+1}^d$, then we express

$$(1-\|\mathtt{x}\|^2)\mathbb{W}_{n-1} = \widehat{E}_{n+1}^{n+1} \mathbb{W}_{n+1} + \widehat{E}_{n}^{n+1} \sqrt{1-\|\mathtt{x}\|^2}\,\mathbb{W}_n + \widehat{E}_{n-1}^{n+1} \mathbb{W}_{n-1} + \cdots$$

Therefore,

$$\begin{aligned} \langle \mathbf{u}, (1-\|\mathtt{x}\|^2)\mathbb{W}_{n-1}\mathbb{W}_n^T \rangle = \widehat{E}_{n+1}^{n+1} \langle \mathbf{u}, \mathbb{W}_{n+1}\mathbb{W}_n^T \rangle + \widehat{E}_{n}^{n+1} \langle \mathbf{u}, \sqrt{1-\|\mathtt{x}\|^2}\,\mathbb{W}_n \mathbb{W}_n^T \rangle \\ + \widehat{E}_{n-1}^{n+1} \langle \mathbf{u}, \mathbb{W}_{n-1}\mathbb{W}_n^T \rangle + \cdots = \widehat{E}_{n}^{n+1} H_n. \end{aligned}$$

As above, we compute the matrix $\widehat{E}_{n}^{n+1}$ by comparing leading coefficients, obtaining

$$-G_{n-1}^{n-1} L_{n-1}^{(1)} = \widehat{E}_{n+1}^{n+1} G_{n+1}^{n+1} - \widehat{E}_{n}^{n+1} G_{n-1}^{n} L_{n-1}^{(1)},$$

$$-G_{n-2}^{n-1} L_{n-2}^{(1)} = \widehat{E}_{n+1}^{n+1} G_n^{n+1} + \widehat{E}_{n}^{n+1} G_n^n,$$

that has unique solution since $\Gamma_{n+1}$ is again the matrix coefficient. Working as before, we substitute

$$\widehat{E}_{n+1}^{n+1} = [\widehat{E}_{n}^{n+1} G_{n-1}^{n} L_{n-1}^{(1)} - G_{n-1}^{n-1} L_{n-1}^{(1)}](G_{n+1}^{n+1})^{-1}$$

in the second equation

$$-G_{n-2}^{n-1} L_{n-2}^{(1)} = [\widehat{E}_{n}^{n+1} G_{n-1}^{n} L_{n-1}^{(1)} - G_{n-1}^{n-1} L_{n-1}^{(1)}](G_{n+1}^{n+1})^{-1} G_n^{n+1} + \widehat{E}_{n}^{n+1} G_n^n,$$

we group the terms

$$\widehat{E}_{n}^{n+1} [G_n^n + G_{n-1}^{n} L_{n-1}^{(1)} (G_{n+1}^{n+1})^{-1} G_n^{n+1}] = G_{n-1}^{n-1} L_{n-1}^{(1)} (G_{n+1}^{n+1})^{-1} G_n^{n+1} - G_{n-2}^{n-1} L_{n-2}^{(1)},$$

and we obtain

$$\widehat{E}_{n}^{n+1} = [G_{n-1}^{n-1} L_{n-1}^{(1)} (G_{n+1}^{n+1})^{-1} G_n^{n+1} - G_{n-2}^{n-1} L_{n-2}^{(1)}] \gamma_n^{-1}.$$

(iii) The proof is analogous.

(iv) We only need to compute

$$\begin{aligned} H_{n-1} C_{n,k}^T &= \langle \mathbf{u}_{1/2}, x_k \mathbb{W}_{n-1}\mathbb{W}_n^T \rangle - \langle \mathbf{u}_{1/2}, \sqrt{1-\|\mathtt{x}\|^2}\,\mathbb{W}_{n-1}\mathbb{W}_n^T \rangle B_{n,k}^T \\ &= \kappa_{n,k}^{(1)} \gamma_n^{-1} H_n - \kappa_n^{(2)} \gamma_n^{-1} H_n H_n^{-1} \langle \mathbf{u}, x_k \mathbb{W}_n \mathbb{W}_n^T \rangle \\ &= \kappa_{n,k}^{(1)} \gamma_n^{-1} H_n - \kappa_n^{(2)} \gamma_n^{-1} \kappa_{n,k}^{(3)} \gamma_n^{-1} H_n \end{aligned}$$

$$= [\kappa_{n,k}^{(1)} - \kappa_n^{(2)}\gamma_n^{-1}\kappa_{n,k}^{(3)}]\gamma_n^{-1}H_n.$$

In order to study the rank of the matrix $[\kappa_{n,k}^{(1)} - \kappa_n^{(2)}\gamma_n^{-1}\kappa_{n,k}^{(3)}]$, we construct the block matrix

$$\begin{pmatrix} \gamma_n & \kappa_{n,k}^{(3)} \\ \kappa_{n,k}^{(2)} & \kappa_{n,k}^{(1)} \end{pmatrix},$$

we use that $\gamma_n$ and $\kappa_{n,k}^{(1)}$ are full rank matrix, and we work as Proposition 3.3. □

**Remark 3.5.** *For a fixed $1 \leqslant k \leqslant d$, we must observe that three term relation* (3.5) *can not be used to compute the monic HOFS. Although we can write*

$$A_{n,k}\mathbb{W}_{n+1}(\mathtt{x}) = [x_k I_{r_n^d} - B_{n,k}\sqrt{1-\|\mathtt{x}\|^2}]\mathbb{W}_n(\mathtt{x}) - C_{n,k}\mathbb{W}_{n-1}(\mathtt{x}), n \geqslant 0, \quad (3.13)$$

*the matrices $A_{n,k}$ are $r_n^d \times r_{n+1}^d$ matrices, except for the univariate case ($d = 1$) where $A_{n,1}$ are non-zero constant, for $n \geqslant 0$. We know that, since $A_{n,k}$ is a full rank matrix, there exists a (not unique) pseudo inverse only by the right side (*[8]*).*

*Following* [6, p. 72]*, in Proposition 3.3 we proved that the rank of the $dr_n^d \times r_{n+1}^d$ join matrix $A_n = (A_{n,1}^T, A_{n,2}^T, \ldots, A_{n,d}^T)^T$ is $r_{n+1}^d$, there exists a (not unique) $r_{n+1}^d \times dr_n^d$ block matrix $D_n = (D_{n,1}, D_{n,2}, \ldots, D_{n,d})$, with $D_{n,k}$ matrices of respective sizes $r_{n+1}^d \times r_n^d$, such that*

$$D_n A_n = (D_{n,1}, D_{n,2}, \ldots, D_{n,d}) \begin{pmatrix} A_{n,1} \\ A_{n,2} \\ \vdots \\ A_{n,d} \end{pmatrix} = \sum_{k=1}^{d} D_{n,k}A_{n,k} = I_{r_{n+1}^d}.$$

*Then, multiplying relations* (3.13) *by $D_{n,k}$, and summing, we get*

$$\mathbb{W}_{n+1}(\mathtt{x}) = \Big[\sum_{k=1}^{d} D_{n,k}x_k - \widehat{B}_n\sqrt{1-\|\mathtt{x}\|^2}\Big]\mathbb{W}_n(\mathtt{x}) - \widehat{C}_n\mathbb{W}_{n-1}(\mathtt{x}),$$

*for $n \geqslant 0$, where*

$$\widehat{B}_n = \sum_{k=1}^{d} D_{n,k}B_{n,k}, \qquad \widehat{C}_n = \sum_{k=1}^{d} D_{n,k}C_{n,k}.$$

*Then, we can compute the functions recursively.*

3.3. **Favard type Theorem.** Now we present a Favard type result for the hybrid orthogonal functions. See [3, p. 21] for Favard's Theorem for orthogonal polynomials on the real line, and [6, p. 73] for multivariate orthogonal polynomials.

**Theorem 3.6.** *For $n \geqslant 0$ and $1 \leqslant k \leqslant d$, let $A_{n,k}, B_{n,k}, C_{n,k}$ be matrices of respective sizes $r_n^d \times r_{n+1}^d$, $r_n^d \times r_n^d$ and $r_n^d \times r_{n-1}^d$ such that* $\operatorname{rank} A_{n,k} = r_n^d$, $\operatorname{rank} C_{n,k} = r_{n-1}^d$, *and* $\operatorname{rank} A_n = r_{n+1}^d$, $\operatorname{rank} C_n = r_n^d$, *where $A_n$ and $C_n$ are the respective join matrices.*

*Let $D_n = (D_{n,1}, D_{n,2}, \ldots, D_{n,d})$, with $D_{n,k}$ matrices of respective sizes $r_{n+1}^d \times r_n^d$, be a pseudo inverse of $A_n$.*

*Define the sequence of vector functions given by* $\mathbb{W}_{-1}(\mathtt{x}) = 0$, $\mathbb{W}_0(\mathtt{x}) = 1$, *and*

$$\mathbb{W}_{n+1}(\mathtt{x}) = \Big[\sum_{k=1}^{d} D_{n,k}x_k - \widehat{B}_n\sqrt{1-\|\mathtt{x}\|^2}\Big]\mathbb{W}_n(\mathtt{x}) - \widehat{C}_n\mathbb{W}_{n-1}(\mathtt{x}), \quad n \geqslant 0, \quad (3.14)$$

*where* $\widehat{B}_n = \sum_{k=1}^d D_{n,k}B_{n,k}$, *and* $\widehat{C}_n = \sum_{k=1}^d D_{n,k}C_{n,k}$, *and satisfying* (3.5) *for* $1 \leqslant k \leqslant d$. *Then,*

*(i)* $\{\mathbb{W}_n\}_{n\geqslant 0}$ *is a functional sequence (FS).*

*(ii) There exist a moment functional* $\mathbf{u}$ *such that* $\{\mathbb{W}_n\}_{n\geqslant 0}$ *is a HOFS associated with* $\mathbf{u}$.

*Proof.* (i) The matrices $D_{n,k}$ are full rank matrices, and then we can compute the functions recursively.

Now, we study the coefficient matrices. From Lemma 2.1, comparing the *first* and the *second leading coefficients* of expressions (2.6) and (3.5), we get

$$A_{n,k}G_{n+1}^{n+1} - B_{n,k}G_{n-1}^{n}L_{n-1}^{(1)} = G_n^n L_{n,k},$$

$$A_{n,k}G_n^{n+1} + B_{n,k}G_n^n = G_{n-1}^n L_{n-1,k},$$

for $1 \leqslant k \leqslant d$, that can be written as

$$(B_{n,k},\ A_{n,k})\begin{pmatrix} G_n^n & -G_{n-1}^n L_{n-1}^{(1)} \\ G_n^{n+1} & G_{n+1}^{n+1}\end{pmatrix} = (G_{n-1}^n L_{n-1,k},\ G_n^n L_{n,k}),$$

that is,

$$(B_{n,k},\ A_{n,k})\Gamma_{n+1} = (G_{n-1}^n L_{n-1,k},\ G_n^n L_{n,k}),$$

where $\Gamma_{n+1}$ is the square $r_n^d + r_{n+1}^d$ matrix defined in Lemma 2.2. In this way, we can see $\Gamma_{n+1}$ as the coefficient matrix of a linear system with unique solution, then $\Gamma_{n+1}$ is non-singular and the set

$$\{\mathbb{W}_n, \sqrt{1-\|\mathtt{x}\|^2}\mathbb{W}_{n-1}, \mathbb{W}_{n-2}, \sqrt{1-\|\mathtt{x}\|^2}\mathbb{W}_{n-3}, \ldots\},$$

is a basis of $\Omega_n^d$ for $n \geqslant 0$ by using Lemma 2.2.

(ii) We define a moment functional $\mathbf{u}$ by

$$\langle \mathbf{u}_{1/2}, \mathbb{W}_0\rangle = 1,$$

$$\langle \mathbf{u}_{1/2}, \mathbb{W}_{2n}\rangle = \langle \mathbf{u}, \sqrt{1-\|\mathtt{x}\|^2}\,\mathbb{W}_{2n}\rangle = \mathtt{0}, \quad n \geqslant 1,$$

$$\langle \mathbf{u}, \mathbb{W}_{2n+1}\rangle = \mathtt{0}, \quad n \geqslant 0.$$

We prove that the functional system $\{\mathbb{W}_n\}_{n\geqslant 0}$ defined by (3.14) satisfy the hybrid orthogonality conditions (2.10) by an inductive reasoning.

From the definition $\mathbb{W}_0 = 1$, and then

$$\langle \mathbf{u}_{1/2}, \mathbb{W}_0\mathbb{W}_0^T\rangle = 1,$$

$$\langle \mathbf{u}, \mathbb{W}_0\mathbb{W}_1^T\rangle = \langle \mathbf{u}, \mathbb{W}_1^T\rangle = \mathtt{0}.$$

Now, we follow the reasoning given in [6, p. 74] in order to prove the hybrid orthogonality (2.10). In fact, let $n \geqslant 0$ be an integer and suppose that

$$\langle \mathbf{u}, \mathbb{W}_n\mathbb{W}_{n+2i+1}^T\rangle = \mathtt{0},\ i \geqslant 0,$$

$$\langle \mathbf{u}_{1/2}, \mathbb{W}_n\mathbb{W}_{n+2i}^T\rangle = \mathtt{0},\ i \geqslant 1.$$

Now, we want to prove that result for $n+1$. In this way, for $i \geqslant 0$, we use (3.14) and induction hypothesis to compute

$$\begin{aligned}\langle \mathbf{u}, \mathbb{W}_{n+1}\mathbb{W}^T_{n+1+(2i+1)}\rangle &= \sum_{k=1}^{d} D_{n,k}\langle \mathbf{u}, x_k\mathbb{W}_n\mathbb{W}^T_{n+2i+2}\rangle \\ &\quad - \widehat{B}_n\langle \mathbf{u}_{1/2}, \mathbb{W}_n\mathbb{W}^T_{n+2i+2}\rangle - \widehat{C}_n\langle \mathbf{u}, \mathbb{W}_{n-1}\mathbb{W}^T_{n+2i+2}\rangle \\ &= \sum_{k=1}^{d} D_{n,k}\langle \mathbf{u}, \mathbb{W}_n(x_k\mathbb{W}_{n+2i+2})^T\rangle \\ &= \sum_{k=1}^{d} D_{n,k}\,\big[\langle \mathbf{u}, \mathbb{W}_n\mathbb{W}^T_{n+2i+3}\rangle A^T_{n+2i+2,k} \\ &\qquad + \langle \mathbf{u}_{1/2}, \mathbb{W}_n\mathbb{W}^T_{n+2i+2}\rangle B^T_{n+2i+2,k} \\ &\qquad + \langle \mathbf{u}, \mathbb{W}_n\mathbb{W}^T_{n+2i+1}\rangle C^T_{n+2i+2,k}\big] = \mathtt{0},\end{aligned}$$

and then, we have the first part of the induction. Next, we work with $\mathbf{u}_{1/2}$ and $i \geqslant 1$,

$$\begin{aligned}\langle \mathbf{u}_{1/2}, \mathbb{W}_{n+1}\mathbb{W}^T_{n+1+2i}\rangle &= \sum_{k=1}^{d} D_{n,k}\langle \mathbf{u}_{1/2}, x_k\mathbb{W}_n\mathbb{W}^T_{n+1+2i}\rangle \\ &\quad - \widehat{B}_n\langle \mathbf{u}_{1/2}, \sqrt{1-\|\mathtt{x}\|^2}\mathbb{W}_n\mathbb{W}^T_{n+1+2i}\rangle - \widehat{C}_n\langle \mathbf{u}_{1/2}, \mathbb{W}_{n-1}\mathbb{W}^T_{n+1+2i}\rangle \\ &= \sum_{k=1}^{d} D_{n,k}\langle \mathbf{u}_{1/2}, x_k\mathbb{W}_n\mathbb{W}^T_{n+1+2i}\rangle - \widehat{B}_n\langle \mathbf{u}, (1-\|\mathtt{x}\|^2)\mathbb{W}_n\mathbb{W}^T_{n+1+2i}\rangle,\end{aligned}$$

by using induction hypothesis. We know that $(1-\|\mathtt{x}\|^2)\mathbb{W}_n \in \Omega^d_{n+2}$, and then, there exists matrix coefficients of adequate size such that

$$(1-\|\mathtt{x}\|^2)\mathbb{W}_n = \sum_{i=0}^{\lfloor \frac{n+2}{2}\rfloor} E^{n+2}_{n+2-2i}\mathbb{W}_{n+2-2i} + \sqrt{1-\|\mathtt{x}\|^2}\sum_{i=0}^{\lfloor \frac{n+1}{2}\rfloor} E^{n+2}_{n+1-2i}\mathbb{W}_{n+1-2i}.$$

Therefore, $\langle \mathbf{u}, (1-\|\mathtt{x}\|^2)\mathbb{W}_n\mathbb{W}^T_{n+1+2i}\rangle = \mathtt{0}$, and then,

$$\langle \mathbf{u}_{1/2}, \mathbb{W}_{n+1}\mathbb{W}^T_{n+1+2i}\rangle = \sum_{k=1}^{d} D_{n,k}\langle \mathbf{u}_{1/2}, x_k\mathbb{W}_n\mathbb{W}^T_{n+1+2i}\rangle.$$

Substituting again the three term relation for $x_k\mathbb{W}_{n+1+2i}$, we get

$$\langle \mathbf{u}_{1/2}, \mathbb{W}_{n+1}\mathbb{W}^T_{n+1+2i}\rangle = \sum_{k=1}^{d} D_{n,k}\langle \mathbf{u}_{1/2}, \mathbb{W}_n(x_k\mathbb{W}_{n+2i+1})^T\rangle = \mathtt{0},$$

and the induction is complete.

Next, we prove that $H_n = \langle \mathbf{u}_{1/2}, \mathbb{W}_n\mathbb{W}^T_n\rangle$ is non-singular, for $n \geqslant 0$. First, $H_0 = \langle \mathbf{u}_{1/2}, \mathbb{W}_0\mathbb{W}^T_0\rangle = 1$, and then, it is invertible.

Now, we suppose that the square and symmetric matrix $H_n = \langle \mathbf{u}_{1/2}, \mathbb{W}_n\mathbb{W}^T_n\rangle$ is invertible. Using expression (3.11)

$$H_n\, C^T_{n+1,k} = M_{n+1,k}\gamma^{-1}_{n+1}H_{n+1}.$$

where $M_{n,k} = [\kappa_{n,k}^{(1)} - \kappa_n^{(2)}\gamma_n^{-1}\kappa_{n,k}^{(3)}]$, we get

$$\operatorname{diag}\{H_n, \dots, H_n\} C_{n+1}^T = M_{n+1}\gamma_{n+1}^{-1} H_{n+1}.$$

In this way, since $H_n$ is invertible, $\operatorname{diag}\{H_n, \dots, H_n\}$ is also invertible, and by hypothesis, $\operatorname{rank} C_{n+1} = r_{n+1}^d$. Therefore

$$\operatorname{rank}(M_{n+1}\gamma_{n+1}^{-1} H_{n+1}) = r_{n+1}^d.$$

Then,

$$\begin{aligned}\operatorname{rank} H_{n+1} &\geqslant \operatorname{rank}(M_{n+1}\gamma_{n+1}^{-1} H_{n+1}) \geqslant \operatorname{rank} M_{n+1} + \operatorname{rank}(\gamma_{n+1}^{-1} H_{n+1}) - r_{n+1}^d \\ &= \operatorname{rank} H_{n+1},\end{aligned}$$

and finally, $\operatorname{rank} H_{n+1} = r_{n+1}^d$. □

## 4. A method to construct bivariate hybrid orthogonal functions

To construct bivariate hybrid orthogonal functions, we develop a similar construction as the well known Koornwinder's method ([6], [10]), used to obtain orthogonal polynomials in $d = 2$ variables from univariate orthogonal polynomials.

More precisely, let $\omega_1(x)$ be an even weight function in one variable defined on the interval $(-1, 1)$. In this way, the moment functional is defined as

$$\langle \mathbf{v}_1, f \rangle = \int_{-1}^{1} f(x)\, \omega_1(x)\, dx,$$

for every univariate polynomial $f(x)$.

For $m \geqslant 0$, we denote by $\{p_n^{(m)}(x)\}_{n\geqslant 0}$ the family of polynomials orthogonal with respect to the even weight function $(1 - x^2)^{m+1}\, \omega_1(x)$ on $(-1, 1)$. Then, the polynomials are even functions, that is, $p_n^{(m)}(-x) = (-1)^n p_n^{(m)}(x)$, for $n, m \geqslant 0$ and $x \in (-1, 1)$.

Let $\{q_n(x)\}_{n\geqslant 0}$ be a sequence of univariate hybrid orthogonal functions in the sense of presented in [2], satisfying univariate hybrid orthogonal conditions as in (2.10), associated with a non symmetric weight function $\omega_2(x)$ on $(-1, 1)$, in the form

$$\langle \mathbf{v}_2, f \rangle = \int_{-1}^{1} f(x)\, \omega_2(x)\, dx.$$

We define the sequence of functions $\{\mathbb{W}_n\}_{n\geqslant 0}$, where

$$\mathbb{W}_n = (W_0^n(x_1, x_2), W_1^n(x_1, x_2), \dots, W_n^n(x_1, x_2))^T, \quad n \geqslant 0, \tag{4.1}$$

and

$$W_m^n(x_1, x_2) = p_{n-m}^{(m)}(x_1) \left(\sqrt{1 - x_1^2}\right)^m q_m\left(\frac{x_2}{\sqrt{1 - x_1^2}}\right), \quad 0 \leqslant m \leqslant n. \tag{4.2}$$

It is easy to check that the function $W_m^n(x_1, x_2)$ has degree $n$, for $0 \leqslant m \leqslant n$, and then $\{\mathbb{W}_n\}_{n\geqslant 0}$ defined as above is a FS.

Observe that there is an essential difference from the classical method by Koornwinder ([10]), apart from the fact that the second family is a hybrid orthogonal sequence, the polynomials $\{p_n^{(m)}\}_{n\geqslant 0}$ are orthogonal with respect to the weight function $\rho(x)^{2m+2}\omega_1(x)$, taking $\rho(x) = \sqrt{1 - x^2}$, and in the Koornwinder's classical construction they are orthogonal with respect to $\rho(x)^{2m+1}\omega_1(x)$.

We show the following result.

**Proposition 4.1.** *Let* $\{\mathbb{W}_n\}_{n\geqslant 0}$ *be a FS defined as in* (4.1)-(4.2). *Then,* $\{\mathbb{W}_n\}_{n\geqslant 0}$ *is a HOFS with respect to the weight function*

$$\omega(x_1,x_2) = \omega_1(x_1)\omega_2\left(\frac{x_2}{\sqrt{1-x_1^2}}\right),$$

*on the region* $R = \{(x_1,x_2) : -1 < x_1 < 1,\, -\sqrt{1-x_1^2} < x_2 < \sqrt{1-x_1^2}\}$.

*Proof.* We denote by $\mathbf{u}$ and $\mathbf{u}_{1/2}$ the moment functionals respectively defined by

$$\langle \mathbf{u}, p\rangle = \iint_R p(x_1,x_2)\,\omega(x_1,x_2)\,dx_1\,dx_2$$

and

$$\langle \mathbf{u}_{1/2}, p\rangle = \langle \mathbf{u}, \sqrt{1-\|\mathtt{x}\|^2}\,p\rangle = \iint_R p(x_1,x_2)\,\sqrt{1-x_1^2-x_2^2}\,\omega(x_1,x_2)\,dx_1\,dx_2.$$

For brevity, we denote $\rho(x) = \sqrt{1-x^2}$ and $t = x_2/\sqrt{1-x_1^2}$. Observe that

$$\rho(x_1)\,\rho(t) = \sqrt{1-x_1^2}\,\sqrt{1-t^2} = \sqrt{1-x_1^2-x_2^2} = \sqrt{1-\|\mathtt{x}\|^2}.$$

For $0 \leqslant m \leqslant n$ and $0 \leqslant k \leqslant h$, we compute the inner product of two functions as follows

$$\begin{aligned}
\langle \mathbf{u}, W_m^n\,W_k^h\rangle &= \iint_R W_m^n(x_1,x_2)W_k^h(x_1,x_2)\omega(x_1,x_2)dx_1dx_2\\
&= \iint_R p_{n-m}^{(m)}(x_1)\,p_{h-k}^{(k)}(x_1)\,\rho(x_1)^{m+k}\,\omega_1(x_1)\\
&\quad\times q_m\left(\frac{x_2}{\rho(x_1)}\right)\,q_k\left(\frac{x_2}{\rho(x_1)}\right)\,\omega_2\left(\frac{x_2}{\rho(x_1)}\right)dx_1dx_2 \qquad (4.3)\\
&= \int_{-1}^1 p_{n-m}^{(m)}(x_1)\,p_{h-k}^{(k)}(x_1)\,\rho(x_1)^{m+k+1}\,\omega_1(x_1)\,dx_1\\
&\quad\times\int_{-1}^1 q_m(t)q_k(t)\omega_2(t)dt,
\end{aligned}$$

and, in the same way,

$$\begin{aligned}
\langle \mathbf{u}_{1/2}, W_m^n\,W_k^h\rangle &= \iint_R W_m^n(x_1,x_2)\,W_k^h(x_1,x_2)\,\sqrt{1-\|\mathtt{x}\|^2}\omega(x_1,x_2)dx_1dx_2\\
&= \int_{-1}^1 p_{n-m}^{(m)}(x_1)\,p_{h-k}^{(k)}(x_1)\,\rho(x_1)^{m+k+2}\,\omega_1(x_1)\,dx_1 \qquad (4.4)\\
&\quad\times\int_{-1}^1 q_m(t)\,q_k(t)\,\rho(t)\,\omega_2(t)\,dt.
\end{aligned}$$

To prove the hybrid orthogonality we need to split it in the following four cases:

(i) $\langle \mathbf{u}, W_{2m+l}^{2n+1}\,W_{2k+l}^{2h}\rangle = 0$,

(ii) $\langle \mathbf{u}, W_{2m}^{2n+1}\,W_{2k+1}^{2h}\rangle = 0$,

(iii) $\langle \mathbf{u}_{1/2}, W_{2m}^{2n+l}\,W_{2k+1}^{2h+l}\rangle = 0$,

(iv) $\langle \mathbf{u}_{1/2}, W_{2m+i}^{2n+l}\,W_{2k+i}^{2h+l}\rangle = h_{n,m}^{(l,i)}\,\delta_{n,h}\,\delta_{m,k}, \quad h_{n,m}^{(l,i)} > 0$, for any $0 \leqslant m \leqslant n$, $0 \leqslant k \leqslant h$, and $l, i = 0, 1$.

Cases (i) and (iii) are deduced from the fact that the first integral in last term in (4.3) and (4.4) vanishes since in both cases $p_{n-m}^{(m)}(x)\, p_{h-k}^{(k)}(x)$ is an odd polynomial, and the function $\rho(x)^{m+k+1+l}\omega_1(x)$ is even, for any $n, m, h, k$ and $l = 0, 1$.

For case (ii), using the fact that $\{q_m(t)\}_{n\geqslant 0}$ are univariate hybrid orthogonal functions, the last integral in the last term in expressions (4.3) and (4.4) vanishes for any $m, k$, and then, for any $n, h$.

Case (iv) is deduced as case (ii), the last integral in the last term of (4.4) vanishes except for $m = k$. We denote

$$\int_{-1}^{1} q_m(t)^2 \sqrt{1-t^2}\, \omega_2(t) dt = h_m^{(q)} > 0,$$

and using the orthogonality of the polynomials $\{p_n^{(m)}\}_{n\geqslant 0}$, we get

$$\langle \mathbf{u}_{1/2}, W_{2m+i}^{2n+l}\, W_{2k+i}^{2h+l} \rangle = h_{2n+l-(2m+i)}^{(p)}\, h_{2m+i}^{(q)}\, \delta_{n,h}\, \delta_{m,k},$$

for $l = 0, 1$ and $i = 0, 1$, where

$$h_{n-m}^{(p)} = \int_{-1}^{1} p_{n-m}^{(m)}(x_1)^2\, \rho(x_1)^{2m+2}\omega_1(x_1) dx_1 > 0.$$

This completes the proof. □

For bivariate hybrid orthogonal functions constructed by this method, we give explicitly the matrices of the three term relations (3.5) for $k = 1, 2$,

$$x_k\, \mathbb{W}_n(\mathtt{x}) = A_{n,k}\, \mathbb{W}_{n+1}(\mathtt{x}) + B_{n,k} \sqrt{1-\|\mathtt{x}\|^2}\, \mathbb{W}_n(\mathtt{x}) + C_{n,k}\, \mathbb{W}_{n-1}(\mathtt{x}),$$

where $A_{n,k}, B_{n,k}, C_{n,k}$ are matrices of respective sizes $(n+1)\times(n+2)$, $(n+1)\times(n+1)$, and $(n+1)\times n$, such that $A_{n,k}$, $C_{n,k}$, $A_n$ and $C_n$ have full rank.

To this aim, we adapt the results given in [11].

As it is well known, the symmetric univariate orthogonal polynomial sequence $\{p_n^{(m)}(x)\}_{n\geqslant 0}$ satisfies a three term recurrence relation ([3], [13]). Then, there exist non zero constants $a_n^{(m)}, c_n^{(m)}$ such that

$$\begin{aligned} x\, p_n^{(m)}(x) &= a_n^{(m)}\, p_{n+1}^{(m)}(x) + c_n^{(m)}\, p_{n-1}^{(m)}(x), \quad n \geqslant 0, \\ p_{-1}^{(m)}(x) &= 0, \quad p_0^{(m)}(x) = 1, \quad m \geqslant 0. \end{aligned} \tag{4.5}$$

The univariate hybrid orthogonal sequence $\{q_m(t)\}_{m\geqslant 0}$ also satisfies a three term recurrence relation ([2])

$$\begin{aligned} t\, q_m(t) &= \tilde{a}_m\, q_{m+1}(t) + \tilde{b}_m \sqrt{1-t^2}\, q_m(t) + \tilde{c}_m\, q_{m-1}(t), \quad m \geqslant 0, \\ q_{-1}(t) &= 0, \quad q_0(t) = 1, \end{aligned} \tag{4.6}$$

with $\tilde{a}_m$ and $\tilde{c}_m$ non zero constants.

Moreover, we need relations between the symmetric adjacent families of orthogonal polynomials $\{p_n^{(m)}\}_{n\geqslant 0}$ and $\{p_n^{(m+1)}\}_{n\geqslant 0}$. In [11] it was proved the existence of the relations

$$p_n^{(m)}(x) = \delta_n^{(m)}\, p_n^{(m+1)}(x) + \zeta_n^{(m)}\, p_{n-2}^{(m+1)}(x), \tag{4.7}$$

$$\rho(x)^2\, p_n^{(m+1)}(x) = \eta_n^{(m)}\, p_{n+2}^{(m)}(x) + \vartheta_n^{(m)}\, p_n^{(m)}(x), \tag{4.8}$$

where $\delta_n^{(m)}, \zeta_n^{(m)}, \eta_n^{(m)}$ and $\vartheta_n^{(m)}$ are constants. Observe that the symmetry of the polynomials yields shorter relations than in the general case.

Therefore, we can give explicit expressions of the coefficient matrices for the three term relations for the bivariate hybrid orthogonal function sequence defined in (4.2). The matrices for the first three term relation are diagonal.

**Proposition 4.2.** *Let $\{\mathbb{W}_n\}_{n\geqslant 0}$ be a hybrid orthogonal FS constructed by means of (4.2). The matrix coefficients in the first three term relation (3.5) are given by*

$$\begin{aligned} A_{n,1} &= \operatorname{diag}\{a_{n-m}^{(m)} :\ 0 \leqslant m \leqslant n\}\, L_{n,1},\\ B_{n,1} &= \mathsf{0},\\ C_{n,1} &= L_{n-1,1}^T\ \operatorname{diag}\{c_{n-m}^{(m)} :\ 0 \leqslant m \leqslant n-1\}, \end{aligned}$$

*where $a_{n-m}^{(m)}$ and $c_{n-m}^{(m)}$, for $0 \leqslant m \leqslant n$, are the coefficients in (4.5), and the matrices $L_{n,1}, L_{n-1,1}$ were defined in (2.1).*

*Proof.* Multiplying (4.2) by $x_1$, and applying relation (4.5), we obtain

$$\begin{aligned} x_1\, W_m^n(x_1,x_2) &= x_1\, p_{n-m}^{(m)}(x_1)\, \rho(x_1)^m\, q_m\left(\frac{x_2}{\rho(x_1)}\right)\\ &= [a_{n-m}^{(m)} p_{n-m+1}^{(m)}(x_1) + c_{n-m}^{(m)} p_{n-m-1}^{(m)}(x_1)]\rho(x_1)^m\, q_m\left(\frac{x_2}{\rho(x_1)}\right)\\ &= a_{n-m}^{(m)}\, W_m^{n+1}(x_1,x_2) + c_{n-m}^{(m)}\, W_m^{n-1}(x_1,x_2). \end{aligned}$$

The result follows from the above relation for $m = 0, 1, 2, \ldots, n$, and the vector notation (4.1). ☐

In the next theorem, we prove that the matrix coefficients of the *second three term relation* for $\{\mathbb{W}_n\}_{n\geqslant 0}$ are tridiagonal.

**Proposition 4.3.** *The matrix coefficients of the second three term relation (3.5) for an orthogonal PS generated by (4.2) are given by the tridiagonal matrices*

$$A_{n,2} = \left(\begin{array}{cccc|c} 0 & \widehat{a}_n^{(0)} & & \bigcirc & 0\\ \widetilde{a}_{n-1}^{(1)} & 0 & \ddots & & \vdots\\ & \ddots & \ddots & \widehat{a}_1^{(n-1)} & 0\\ \bigcirc & & \widetilde{a}_0^{(n)} & 0 & \widehat{a}_0^{(n)} \end{array}\right),$$

*where*

$$\begin{aligned} \widehat{a}_{n-m}^{(m)} &= \tilde{a}_m\, \delta_{n-m}^{(m)}, & 0 \leqslant m \leqslant n,\\ \widetilde{a}_{n-m}^{(m)} &= \tilde{c}_m\, \eta_{n-m}^{(m-1)}, & 1 \leqslant m \leqslant n, \end{aligned}$$

$$C_{n,2} = \left(\begin{array}{cccc} 0 & \widehat{c}_n^{(0)} & & \bigcirc\\ \widetilde{c}_{n-1}^{(1)} & 0 & \ddots & \\ & \ddots & \ddots & \widehat{c}_2^{(n-2)}\\ \bigcirc & & \widetilde{c}_1^{(n-1)} & 0\\ \hline 0 & \ldots & 0 & \widehat{c}_0^{(n)} \end{array}\right),$$

*where*

$$\begin{aligned} \widehat{c}_{n-m}^{(m)} &= \tilde{a}_m\, \zeta_{n-m}^{(m)}, & 0 \leqslant m \leqslant n-2, \\ \widetilde{c}_{n-m}^{(m)} &= \tilde{c}_m\, \vartheta_{n-m}^{(m-1)}, & 1 \leqslant m \leqslant n-1, \end{aligned}$$

*and* $B_{n,2} = \operatorname{diag}\{\widetilde{b}_m :\ 0 \leqslant m \leqslant n\}$.

*Proof.* Multiplying (4.2) by $x_2$, denoting $t = x_2/\rho(x_1)$, and using (4.6), we get

$$\begin{aligned} x_2\, W_m^n(x_1,x_2) &= p_{n-m}^{(m)}(x_1)\, \rho(x_1)^{m+1}\, \frac{x_2}{\rho(x_1)}\, q_m\left(\frac{x_2}{\rho(x_1)}\right) \\ &= \tilde{a}_m\, p_{n-m}^{(m)}(x_1)\, \rho(x_1)^{m+1}\, q_{m+1}(t) \\ &\quad + \tilde{b}_m\, p_{n-m}^{(m)}(x_1)\, \rho(x_1)^{m+1} \sqrt{1-t^2}\, q_m(t) \\ &\quad + \tilde{c}_m\, \rho(x_1)^2\, p_{n-m}^{(m)}(x_1)\, \rho(x_1)^{m-1}\, q_{m-1}(t). \end{aligned} \tag{4.9}$$

The terms of the above sum will be studied separately. For the first term, using (4.7), we deduce

$$\begin{aligned} & p_{n-m}^{(m)}(x_1)\rho(x_1)^{m+1} q_{m+1}(t) \\ &\qquad = \left[\delta_{n-m}^{(m)}\, p_{n-m}^{(m+1)}(x_1) + \zeta_{n-m}^{(m)}\, p_{n-m-2}^{(m+1)}(x_1)\right] \rho(x_1)^{m+1}\, q_{m+1}(t) \\ &\qquad = \delta_{n-m}^{(m)}\, W_{m+1}^{n+1}(x_1,x_2) + \zeta_{n-m}^{(m)}\, W_{m+1}^{n-1}(x_1,x_2). \end{aligned}$$

Now, we consider the second term of (4.9). Since $\rho(x_1)\sqrt{1-t^2} = \sqrt{1-x_1^2-x_2^2}$, the second term yields

$$\begin{aligned} p_{n-m}^{(m)}(x_1)\, \rho(x_1)^{m+1} \sqrt{1-t^2}\, q_m(t) &= p_{n-m}^{(m)}(x_1)\, \rho(x_1)^m \sqrt{1-x_1^2-x_2^2}\, q_m(t) \\ &= \sqrt{1-x_1^2-x_2^2}\, W_m^n(x_1,x_2). \end{aligned}$$

For $m \geqslant 1$, last term in (4.9) is computed substituting (4.8) in the form

$$\begin{aligned} & \rho(x_1)^2\, p_{n-m}^{(m)}(x_1)\, \rho(x_1)^{m-1}\, q_{m-1}(t) \\ &\quad = \left[\eta_{n-m}^{(m-1)}\, p_{n-m+2}^{(m-1)}(x_1) + \vartheta_{n-m}^{(m-1)}\, p_{n-m}^{(m-1)}(x_1)\right] \rho(x_1)^{m-1}\, q_{m-1}(t) \\ &\quad = \eta_{n-m}^{(m-1)}\, W_{m-1}^{n+1}(x_1,x_2) + \vartheta_{n-m}^{(m-1)}\, W_{m-1}^{n-1}(x_1,x_2). \end{aligned}$$

Finally, replacing above expressions into (4.9), we get

$$\begin{aligned} x_2\, W_m^n(x_1,x_2) &= \tilde{c}_m\, \eta_{n-m}^{(m-1)}\, W_{m-1}^{n+1}(x_1,x_2) + \tilde{a}_m\, \delta_{n-m}^{(m)}\, W_{m+1}^{n+1}(x_1,x_2) \\ &\quad + \tilde{b}_m \sqrt{1-x_1^2-x_2^2}\, W_m^n(x_1,x_2) \\ &\quad + \tilde{c}_m\, \vartheta_{n-m}^{(m-1)}\, W_{m-1}^{n-1}(x_1,x_2) + \tilde{a}_m\, \zeta_{n-m}^{(m)}\, W_{m+1}^{n-1}(x_1,x_2). \end{aligned}$$

□

## 5. Examples

We present some special examples of bivariate HOFS generated by sequences of hybrid orthogonal functions of one variable satisfying univariate (2.10) and sequences of symmetric orthogonal polynomials.

For the first example we use the method described in Section 4.

### 5.1. **Bivariate hybrid functions on the disk.** 

In this example, we construct a family of bivariate hybrid functions on the unit disk in $\mathbb{R}^2$ as an extension of the classical disk polynomials ([6]) by using the Koornwinder-type construction described before. To this end, we use classical Gegenbauer polynomials as well as the univariate hybrid functions described in Example 2 in [2]. We study the bivariate hybrid orthogonal disk functions, and we deduce explicitly their matrix three term relations by using a similar technique as in [11].

In this way, we denote by $\{\bar{C}_n^{(\lambda)}\}_{n\geqslant 0}$ the sequence of monic classical Gegenbauer polynomials orthogonal on the interval $[-1,1]$ with respect to the weight function $\omega^{(\lambda)}(x) = (1-x^2)^{\lambda-1/2}$, for $\lambda > -1/2$. We include some equations for monic Gegenbauer polynomials, adapted to the monic case from relations in [1] and [13].

- Three term recurrence relation ([13, (4.7.17)])
$$x\,\bar{C}_n^{(\lambda)}(x) = \bar{C}_{n+1}^{(\lambda)}(x) + d_n^{(\lambda)}\,\bar{C}_{n-1}^{(\lambda)}(x),$$
where
$$d_n^{(\lambda)} = \frac{1}{4}\frac{n\,(n+2\lambda-1)}{(n+\lambda)(n+\lambda-1)}.$$
- Relation between adjacent families I ([13, (4.7.29)])
$$\bar{C}_n^{(\lambda)}(x) = \bar{C}_n^{(\lambda+1)}(x) + \zeta_n^{(\lambda)}\,\bar{C}_{n-2}^{(\lambda+1)}(x)$$
where
$$\zeta_n^{(\lambda)} = -\frac{1}{4}\frac{n\,(n-1)}{(n+\lambda)(n+\lambda-1)}.$$
- Relation between adjacent families II ([1, (22.7.21)])
$$(1-x^2)\bar{C}_n^{(\lambda+1)}(x) = -\bar{C}_{n+2}^{(\lambda)}(x) + \vartheta_n^{(\lambda)}\,\bar{C}_n^{(\lambda)}(x)$$
where
$$\vartheta_n^{(\lambda)} = \frac{1}{4}\frac{(n+2\lambda)\,(n+2\lambda+1)}{(n+\lambda)(n+\lambda+1)}.$$

In our case, we take $\omega_1(x) = (1-x^2)^{\lambda-1}$. For $m \geqslant 0$, we use the monic Gegenbauer polynomials orthogonal with respect to
$$(\sqrt{1-x^2})^{2m+2}\,(1-x^2)^{\lambda-1} = (1-x^2)^{\lambda+m} = \omega^{(\lambda+m+1/2)}(x).$$
Then, the family of univariate orthogonal polynomials will be taken as Gegenbauer polynomials of varying parameter, that is, $p_n^{(m)}(x) = \bar{C}_n^{(\lambda+m+1/2)}(x)$, for $n \geqslant 0$, where $\bar{C}_n^{(\lambda+m+1/2)}(x)$ denotes the $n$th monic Gegenbauer polynomial orthogonal with respect to the weight function $\omega^{(\lambda+m+1/2)}(x)$.

For the second family we use the univariate monic hybrid orthogonal functions given in the Example 2 in [2]. This family of functions, denoted here by $\{Q_m\}_{m\geqslant 0}$ is hybrid with respect to the weight function
$$e^{-2\eta\arccos(x)}\,(1-x^2)^{\lambda-1}, \quad \eta,\lambda\in\mathbb{R}, \quad \lambda > 1/2,$$

and satisfy the three term recurrence relation $Q_{-1}(x) = 0$, $Q_0(x) = 1$, and

$$x\, Q_m(x) = Q_{m+1}(x) + \tilde{b}_m \sqrt{1-x^2}\, Q_m(x) + \tilde{c}_m\, Q_{m-1}(x), \quad m \geqslant 0,$$

where

$$\tilde{b}_m = \frac{\eta}{m+\lambda-1}, \quad \tilde{c}_m = \frac{1}{4}\frac{m(m+2\lambda-1)}{(m+\lambda)(m+\lambda-1)}.$$

Observe that when $\eta = 0$, then $\tilde{b}_m = 0$, and the functions $Q_m(x)$ reduce to the Gegenbauer monic orthogonal polynomials $\bar{C}_m^{(\lambda)}(x)$ orthogonal with respect to the weight function $\omega^{(\lambda)}(x) = (1-x^2)^{\lambda-1/2}$ ([2]).

As it was showed in Proposition 4.1, the family of vector of functions $\{\mathbb{W}_n\}_{n\geqslant 0}$, where

$$\mathbb{W}_n = (W_0^n(x_1,x_2), W_1^n(x_1,x_2), \ldots, W_n^n(x_1,x_2))^T, \quad n \geqslant 0,$$

and

$$W_m^n(x_1,x_2) = \bar{C}_{n-m}^{(\lambda+m+1/2)}(x_1) \left(\sqrt{1-x_1^2}\right)^m Q_m\left(\frac{x_2}{\sqrt{1-x_1^2}}\right),$$

is a mutually HOFS associated with the weight function

$$\omega(x_1,x_2) = \omega_1(x_1)\omega_2\left(\frac{x_2}{\sqrt{1-x_1^2}}\right) = e^{-2\eta \arccos(x_2/\sqrt{1-x_1^2})}\,(1-x_1^2-x_2^2)^{\lambda-1}.$$

Now, we compute the three term relation for these bivariate hybrid orthogonal functions.

From Proposition 4.2, and the fact that we are using monic Gegenbauer orthogonal polynomials, we get

$$x_1\, \mathbb{W}_n(x_1,x_2) = L_{n,1}\, \mathbb{W}_{n+1}(x_1,x_2) + C_{n,1}\, \mathbb{W}_{n-1}(x_1,x_2),$$

where

$$C_{n,1} = L_{n-1,1}^T \operatorname{diag}\left\{\frac{1}{4}\frac{(n-m)(n+m+2\lambda)}{(n+\lambda+1/2)(n+\lambda-1/2)},\ 0 \leqslant m \leqslant n-1\right\}.$$

The matrix coefficients of the second three term relation

$$\begin{aligned} x_2\, \mathbb{W}_n(x_1,x_2) &= A_{n,2}\, \mathbb{W}_{n+1}(x_1,x_2) + B_{n,2}\sqrt{1-x_1^2-x_2^2}\, \mathbb{W}_n(x_1,x_2) \\ &\quad + C_{n,2}\, \mathbb{W}_{n-1}(x_1,x_2), \end{aligned}$$

are given in Proposition 4.3, and, in this case, we get

$$\begin{aligned} \widehat{a}_{n-m}^{(m)} &= 1, & 0 \leqslant m \leqslant n, \\ \widetilde{a}_{n-m}^{(m)} &= -\frac{1}{4}\frac{m(m+2\lambda-1)}{(m+\lambda)(m+\lambda-1)}, & 1 \leqslant m \leqslant n, \\ \widehat{c}_{n-m}^{(m)} &= -\frac{1}{4}\frac{(n-m)(n-m-1)}{(n+\lambda+1/2)(n+\lambda-1/2)}, & 0 \leqslant m \leqslant n-2, \\ \widetilde{c}_{n-m}^{(m)} &= \frac{1}{16}\frac{m(m+2\lambda-1)(n+m+2\lambda-1)\,(n+m+2\lambda)}{(m+\lambda)(m+\lambda-1)(n+\lambda-1/2)(n+\lambda+1/2)}, & 1 \leqslant m \leqslant n-1, \end{aligned}$$

and $B_{n,2} = \operatorname{diag}\{\widetilde{b}_m :\ 0 \leqslant m \leqslant n\}$.

Then we have done a complete description of a sequence of hybrid orthogonal functions on the unit ball on $\mathbb{R}^2$ that extends a family studied in [2] to the bivariate case. This description includes as particular case the classical orthogonal ball polynomials ([6]) in the case $\eta = 0$.

5.2. **Univariate hybrid orthogonal functions and bivariate polynomials.** Now, we relate univariate hybrid functions, introduced in [2] for the positive-definite case and extended as particular case of our results for $d = 1$, and bivariate orthogonal polynomials on the unit sphere.

For $d = 1$ and $-1 \leqslant x_1 \leqslant 1$, we consider functions that its explicit expression is given by (2.4)

$$w_n(x_1) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} g^n_{n-2i}\, x_1^{n-2i} + \sqrt{1-x_1^2} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} g^n_{n-(2i+1)}\, x_1^{n-(2i+1)}, \quad n \geqslant 0,$$

where $g^n_{n-j}$ are real numbers, $0 \leqslant j \leqslant n$, and suppose that they satisfy a hybrid orthogonality with respect to a moment functional $\mathbf{v}$ as was given in (2.10).

Now, we take $x_2 = \sqrt{1-x_1^2}$, and then, we work in the hemisphere $\mathbb{H}^1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1,\, x_2 \geqslant 0\}$.

Then, we study the bivariate polynomial

$$W_n(x_1, x_2) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} g^n_{n-2i}\, x_1^{n-2i} + \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} g^n_{n-(2i+1)}\, x_1^{n-(2i+1)}\, x_2, \quad n \geqslant 0.$$

That polynomial can be expressed in terms of the vector canonical basis $\{\mathbb{X}_n\}_{n\geqslant 0}$ as

$$W_n(x_1,x_2) = (g^n_n, g^n_{n-1}, 0, ..., 0) \begin{pmatrix} x_1^n \\ x_1^{n-1}x_2 \\ \vdots \\ x_1 x_2^{n-1} \\ x_2^n \end{pmatrix} + (g^n_{n-2}, g^n_{n-3}, 0, ..., 0) \begin{pmatrix} x_1^{n-2} \\ x_1^{n-3}x_2 \\ \vdots \\ x_1 x_2^{n-3} \\ x_2^{n-2} \end{pmatrix} + \cdots$$
$$= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \widetilde{G}^n_{n-2i}\, \mathbb{X}_{n-2i},$$

where $\widetilde{G}^n_{n-2i} = (g^n_{n-2i}, g^n_{n-(2i+1)}, 0, \ldots, 0)$ are real row vectors of size $n - 2i + 1$. Observe that for every bivariate polynomial $W_n(x_1, x_2)$ obtained as above we only use the first two entries of each element of the canonical basis $\{\mathbb{X}_n\}_{n\geqslant 0}$.

Moreover, the can study the orthogonality properties for this new family of polynomials. First, we can define a moment functional $\mathbf{u}$ over polynomials $W_n$ of different parity order as

$$\langle \mathbf{u}, W_{2n+1}\, W_{2m} \rangle = \langle \mathbf{v}, w_{2n+1}\, w_{2m} \rangle = 0.$$

The moment functional $\mathbf{u}_{1/2}$ defined as (2.9) belongs to the zero functional, since $\sqrt{1-\|\mathtt{x}\|^2} = 0$, for $\mathtt{x} \in \mathbb{H}^1$.

However, polynomials of the same parity order satisfy the following orthogonality property

$$\langle \widehat{\mathbf{u}}, W_{2n+l}W_{2m+l} \rangle = \langle \mathbf{u}, x_2 W_{2n+l}W_{2m+l} \rangle = \langle \mathbf{u}, \sqrt{1-x_1^2} W_{2n+l}W_{2m+l} \rangle$$
$$= \langle \mathbf{v}, \sqrt{1-x_1^2} w_{2n+l}w_{2m+l} \rangle = \langle \mathbf{v}_{1/2}, w_{2n+l}w_{2m+l} \rangle = h_{2n+l}\delta_{n,m},$$

for $l = 0, 1$, where $h_{2n+l} \neq 0$.

Therefore, the linear space of hybrid orthogonal functions $\Omega_n^1$ can be seen as a linear subspace of $\Pi_n^2(\mathbb{H}^1)$ with a special properties of orthogonality.

## References


[1] M. Abramowitz, I. A. Stegun, *Handbook of Mathematical Functions*, 9th printing. Dover, New York, 1972.

[2] C. F. Bracciali, J. H. McCabe, T. E. Pérez, A. Sri Ranga, *A class of orthogonal functions given by a three term recurrence formula*, Mathematics of Computation, **85** (2016), 1837-1859.

[3] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Mathematics and its Applications, vol. 13, Gordon and Breach, New York, 1978.

[4] D. K. Dimitrov, M. E. H. Ismail, A. Sri Ranga, *A class of hypergeometric polynomials with zeros on the unit circle: Extremal and orthogonal properties and quadrature formulas*, Applied Numerical Mathematics, **65** (2013), 41-52.

[5] D. K. Dimitrov, A. Sri Ranga, *Zeros of a family of hypergeometric para-orthogonal polynomials on the unit circle*, Mathematische Nachrichten, **286** (2013), 177-1791.

[6] C. F. Dunkl, Y. Xu, *Orthogonal Polynomials of Several Variables*, 2nd edition, Encyclopedia of Mathematics and its Applications, vol. **155**, Cambridge Univ. Press, Cambridge, 2014.

[7] G. H. Golub, C. F. Van Loan, *Matrix Computations*, 4th edition, Johns Hopkins Studies in the Mathematical Sciences, Johns Hopkins Univ. Press, Baltimore, MD, 2013.

[8] R. A. Horn, C. R. Johnson, *Matrix Analysis*, 2nd edition, Cambridge Univ. Press, Cambridge, 2013.

[9] M. E. H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable*, Cambridge Univ. Press, Cambridge, 2005.

[10] T. H. Koornwinder, *Two variable analogues of the classical orthogonal polynomials*. In R. Askey (ed.) Theory and Application of Special Functions, pp. 435-495. Academic Press, New York, 1975.

[11] M. Marriaga, T. E. Pérez, M. A. Piñar, *Three term relations for a class of bivariate orthogonal polynomials*, Mediterranean Journal of Mathematics, **14**(2) (2017), Art. 54, 26 pp.

[12] G. Marsaglia, G. P. H. Styan, *Equalities and inequalities for ranks of matrices*, Linear and Multilinear Algebra, **2** (1974), 269-292.

[13] G. Szegő, *Orthogonal Polynomials*, 4th ed., Amer. Math. Soc. Colloq. Publ. **23**, Amer Math Soc. Providence RI, 1978.



(C. F. Bracciali) Departamento de Matemática, IBILCE, UNESP - Universidade Estadual Paulista, 15054-000, São José do Rio Preto, SP, Brazil.

*E-mail address*: `cleonice.bracciali@unesp.br`

(T. E. Pérez) Instituto de Matemáticas IEMath - GR & Departamento de Matemática Aplicada, Facultad de Ciencias. Universidad de Granada, Spain.

*E-mail address*: `tperez@ugr.es`