Mathematisches
Forschungsinstitut
Oberwolfach

Member of the Leibniz Association

Oberwolfach Preprints

OWP 2020 - 08

DAVID BRANDER AND FARID TARI

Singularities and Bifurcations of Pseudospherical Surfaces

Mathematisches Forschungsinstitut Oberwolfach gGmbH
Oberwolfach Preprints (OWP) ISSN 1864-7596

Oberwolfach Preprints (OWP)

The MFO publishes a preprint series **Oberwolfach Preprints (OWP)**, ISSN 1864-7596, which mainly contains research results related to a longer stay in Oberwolfach, as a documentation of the research work done at the MFO. In particular, this concerns the Research in Pairs-Programme (RiP) and the Oberwolfach-Leibniz-Fellows (OWLF), but this can also include an Oberwolfach Lecture, for example.

A preprint can have a size from 1 - 200 pages, and the MFO will publish it on its website as well as by hard copy. Every RiP group or Oberwolfach-Leibniz-Fellow may receive on request 30 free hard copies (DIN A4, black and white copy) by surface mail.

In case of interest, please send a **pdf file** of your preprint by email to *rip@mfo.de* or *owlf@mfo.de*, respectively. The file should be sent to the MFO within 12 months after your stay as RiP or OWLF at the MFO.

There are no requirements for the format of the preprint, except that the introduction should contain a short appreciation and that the paper size (respectively format) should be DIN A4, "letter" or "article".

On the front page of the hard copies, which contains the logo of the MFO, title and authors, we shall add a running number (20XX – XX). Additionally, each preprint will get a Digital Object Identifier (DOI).

We cordially invite the researchers within the RiP or OWLF programme to make use of this offer and would like to thank you in advance for your cooperation.

Imprint:

Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
Schwarzwaldstrasse 9-11
77709 Oberwolfach-Walke
Germany

Tel +49 7834 979 50
Fax +49 7834 979 55
Email admin@mfo.de
URL www.mfo.de

The Oberwolfach Preprints (OWP, ISSN 1864-7596) are published by the MFO.

DOI 10.14760/OWP-2020-08

SINGULARITIES AND BIFURCATIONS OF PSEUDOSPHERICAL SURFACES

DAVID BRANDER AND FARID TARI

ABSTRACT. We study singularities and bifurcations of constant negative curvature surfaces in Euclidean 3-space via their association with Lorentzian harmonic maps. This preprint presents the basic results on this, the full proofs of which will appear in an article under preparation. We show that the generic bifurcations in 1-parameter families of such surfaces are the Cuspidal Butterfly, Cuspidal Lips, Cuspidal Beaks, 2/5 Cuspidal edge and Shcherbak bifurcations.

1. INTRODUCTION

This report summarizes research undertaken during a Research in Pairs visit by the authors at Mathematisches Forschungsinstitut Oberwolfach in May 2019. More details, as well as further results, will appear in a forthcoming publication (see [6]).

The subject is the study of the local singularities occuring on constant negative curvature (a.k.a. *pseudospherical*) surfaces in Euclidean space $\mathbb{R}^3$. If S is a simply connected Lorentz surface, a (Lorentzian)-harmonic map $N : S \to \mathbb{S}^2$ into the 2-sphere is a solution of the wave map equation

$$N \times N_{xy} = 0, \tag{1.1}$$

where (x, y) is any local null coordinate system. The wave map equation is equivalent to the integrability of the system

$$f_x = N \times N_x, \quad f_y = -N \times N_y, \tag{1.2}$$

and the solution $f : S \to \mathbb{R}^3$, unique up to a rigid motion, with the Riemannian metric induced from $\mathbb{R}^3$, has constant negative curvature $K = -1$ wherever it is immersed, and has N as its Gauss map. Conversely, the Gauss map of any pseudospherical surface is harmonic with respect to the Lorentz metric given by the second fundamental form. By Hilbert's theorem there are no complete pseudospherical immersions. However, at points where f (or equivalently N), is not immersed, the property of being Lorentzian-harmonic is well defined, even though the Gauss curvature of f is not. Thus, this formulation, which includes singularities, is a natural generalization of pseudospherical surfaces to global differential geometry. In this work we wish to understand the generic singularities and the bifurcations in generic 1-parameter families of solutions.

FIGURE 1. Example of a pseudospherical surface with singularities.

The analogous question for constant *positive* curvature surfaces (associated to *Riemannian* harmonic maps) was studied in [5]. In that case, the surface is always a parallel surface to a regular

2010 *Mathematics Subject Classification.* Primary 53A05; Secondary 53C43, 57R45.

Key words and phrases. Bifurcations, constant Gauss curvature, Cauchy problem, differential geometry, discriminants, frontals, integrable systems, loop groups, pseudospherical surfaces, singularities, wave fronts, wave maps.

surface, and hence the well developed theory of singularities of parallel surfaces (which are wave fronts) could be used. In contrast, a pseudospherical surface (as generalized here) may have points where it is not a wave front, and therefore a new approach needs to be deployed to deal with such points.

Local singularities of a pseudospherical surface f are completely determined by a certain k-jet of f at the singular point, and consequently of the k-jet of N at that point. Suppose that a given polynomial map P of degree k from a Lorentz surface into the 2-sphere satisfies the Lorentzian-harmonic condition up to order k: is there a germ of a Lorentzian-harmonic map into the 2-sphere with k-jet P? Theorem 2.1 below answers this question in the affirmative. This gives us a $4k$-dimensional parameterization of the space of k-jets that allows one to define the codimension of a singularity. This is the basis of our study of singularities and bifurcations below.

2. The space of germs of analytic Lorentzian-harmonic maps

Let $N : \Omega \subset \mathbb{R}^{1,1} \to \mathbb{S}^2$ be a real analytic map and (x,y) a null coordinate system in Ω. We may suppose that $O = (0,0) \in \Omega$ and that $N(0,0) = (0,0,1)$. Locally at O we can write $N(x,y) = \delta(x,y)(u(x,y),v(x,y),1)$, with u,v analytic functions on Ω vanishing at the origin, and $\delta = (1+u^2+v^2)^{-\frac{1}{2}}$. The wave map equation $N \times N_{xy} = 0$ is equivalent to the following system of semi-linear PDEs:

$$\begin{cases} u_{xy} - \dfrac{1}{1+u^2+v^2}\left(2uu_xu_y + v(u_xv_y+u_yv_x)\right) = 0, \\ v_{xy} - \dfrac{1}{1+u^2+v^2}\left(2vv_xv_y + u(u_xv_y+u_yv_x)\right) = 0. \end{cases} \tag{2.1}$$

Write $j^n u(x,y) = \sum_{k=1}^{n}\sum_{i=0}^{k} a_{ki}x^{k-i}y^i$ and $j^n v(x,y) = \sum_{k=1}^{n}\sum_{i=0}^{k} b_{ki}x^{k-i}y^i$ for the n-jets, at the origin, of u and v respectively. It follows from (2.1) that

$$\begin{aligned} j^n u(x,y) &= \sum_{k=1}^{n} a_{k0}x^k + \sum_{k=1}^{n} a_{kk}y^k + \sum_{k=3}^{n}\sum_{i=1}^{k-1} P_{ki}(a,b)x^{k-i}y^i, \\ j^n v(x,y) &= \sum_{k=1}^{n} b_{k0}x^k + \sum_{k=1}^{n} b_{kk}y^k + \sum_{k=3}^{n}\sum_{i=1}^{k-1} P_{ki}(b,a)x^{k-i}y^i, \end{aligned} \tag{2.2}$$

where $a_{ki} = P_{ki}(a,b), 1 \le i \le k-1$, are polynomial functions in $a_{l0}, a_{ll}, b_{l0}, b_{ll}$, with $1 \le l \le k-2$, and $b_{ki} = P_{ki}(b,a)$.

The important aspect of the expressions (2.2) is that all of the coefficients can be deduced from the knowledge of u, u_x, v and v_x along the non-characteristic curve $(x,y) = (t,t)$. Combined with standard existence and uniqueness theory for PDEs, this gives a parameterization of the space of k-jets for germs of Lorentzian-harmonic maps into $\mathbb{S}^2$:

Theorem 2.1. *For any $(\mathbf{a}_1,\mathbf{a}_2,\mathbf{b}_1,\mathbf{b}_2) \in \mathbb{R}^{4n}$, with $\mathbf{a}_1 = (a_{i0})_{1\le i\le n}$, $\mathbf{a}_2 = (a_{ii})_{1\le i\le n}$, $\mathbf{b}_1 = (b_{i0})_{1\le i\le n}$, and $\mathbf{b}_2 = (b_{ii})_{1\le i\le n}$, there is a local analytic Lorentzian-harmonic map into the 2-sphere determined by $(\mathbf{a}_1,\mathbf{a}_2,\mathbf{b}_1,\mathbf{b}_2)$ with n-jet as in* (2.2).

3. Frontals

A differentiable map $g : S \to \mathbb{R}^3$ from a surface into Euclidean space is called a *frontal map* and its image a *frontal* if there is a differentiable map $\nu : S \to \mathbb{S}^2 \subset \mathbb{R}^3$ such that dg is orthogonal to ν (see [10] for references). The map $L = (g,\nu) : S \to \mathbb{R}^3 \times \mathbb{S}^2$ is called a Legendrian lift, and g is called a *(wave) front* if L is regular (see [2]).

Wave fronts are well studied ([2]) and their singularities are special. They are the so-called Legendrian singularities and are determined by generating families of functions. Basically, they are discriminants of versal deformations of singularities of functions.

An example of a wave front is a parallel of a regular surface in the Euclidian 3-space. Singularities of parallels of a general surface $g : \Omega \to \mathbb{R}^3$ are studied by Bruce in [7] (see also [8]). Bruce considered the family of distance squared functions $F_{t_0} : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ given by $F_{t_0}((x,y),q) = |g(x,y) - q|^2 - t_0^2$. A parallel W_{t_0} of g is the discriminant of F_{t_0}, that is,

$$W_{t_0} = \left\{ q \in \mathbb{R}^3 : \exists (x,y) \in \Omega \text{ such that } F_{t_0}((x,y),q) = \frac{\partial F_{t_0}}{\partial x}((x,y),q) = \frac{\partial F_{t_0}}{\partial y}((x,y),q) = 0 \right\}.$$

For q_0 fixed, the function $F_{q_0,t_0}(x,y) = F_{t_0}(x,y,q_0)$ gives a germ of a function at a point on the surface. Varying q and t gives a 4-parameter family of functions F. Let $\mathscr{R}$ denote the group of germs of diffeomorphisms from the plane to the plane. Then, by a transversality theorem, for a generic surface, the possible singularities of F_{q_0,t_0} are those of $\mathscr{R}$-codimension 4, and these are as follows (with $\mathscr{R}$-models, up to a sign, in brackets): $A_1^{\pm}$ $(x^2 \pm y^2)$, A_2 $(x^2 + y^3)$, $A_3^{\pm}$ $(x^2 \pm y^4)$, A_4 $(x^2 + y^5)$ and $D_4^{\pm}$ $(y^3 \pm x^2 y)$.

If a deformation family F_{t_0} of the A_3-singularity is $\mathscr{R}$-versal then the parallel is a swallowtail. If it is not $\mathscr{R}$-versal then the singularities are called *non-transverse* $A_3^{\pm}$.

Bruce showed that F is always an $\mathscr{R}$-versal family of the $A_1^{\pm}$ and A_2 singularities. Consequently the parallels at such singularities are, respectively, regular surfaces or cuspidal edges. In particular, the A_2-transitions in wave fronts ([1]) do not occur on parallels of surfaces ([7]).

Singularities of frontals which are not wave fronts are recently being studied. However, there is so far no general theory that deals with bifurcations in families of frontals with non-isolated singularities (compare [9]), unlike the wave front case where one uses generating families of functions ([2]); the difficulty being that the map (f,N) from the surface to the unit cotangent bundle $T_1^*\mathbb{R}^3$ is not necessarily an immersion. See [10] for a survey article, new results and references on frontals.

As mentioned in the introduction, we deal with general pseudospherical frontals by using Theorem 2.1 to define a map to the k-jet space of Lorentzian-harmonic map-germs and use it to define the codimension of a singularity of the associated pseudospherical surface as well as the notion of generic families of such surfaces.

4. Pseudospherical frontals

The definition given by (1.2) implies that a pseudospherical surface f is a frontal with Legendrian lift (f,N), and it is easy to check that f is a wave front if and only if both N_x and N_y are non-vanishing, or, equivalently, both f_x and f_y are non-vanishing.

From the equations (1.2) for f, we have $|f_x| = |N_x|$, $|f_y| = |N_y|$ and $\langle f_x, f_y \rangle = -\langle N_x, N_y \rangle$, so $\mathrm{d}f$ and $\mathrm{d}N$ have the same rank at each point. Around a regular point we can write

$$|f_x| = |N_x| =: A \neq 0, \quad |f_y| = |N_y| =: B \neq 0, \quad \langle f_x, f_y \rangle = AB\cos\phi,$$

so the first fundamental form of f, with the metric induced from $\mathbb{R}^3$, is $I = A^2\mathrm{d}x^2 + 2AB\cos\phi\,\mathrm{d}x\mathrm{d}y + B^2\mathrm{d}y^2$. By the definition of f and by the harmonicity of N we have

$$\langle f_x, N_x \rangle = \langle f_y, N_y \rangle = 0, \quad f_{xy} = N_y \times N_x = f_x \times f_y = (AB\sin\phi)N,$$

so the second fundamental form of f is $II = 2AB\sin\phi\,\mathrm{d}x\mathrm{d}y$. Hence the null coordinates (x,y) are asymptotic coordinates for f, and the Gaussian curvature of f is constant $K = -1$.

Ishikawa and Machida [11] previously showed that swallowtails and cuspidal edges (Figure 2) are the only generic singularities for pseudospherical wave fronts. In order to study the bifurcations in generic 1-parameter families of such surfaces, we observe that a wave front pseudospherical

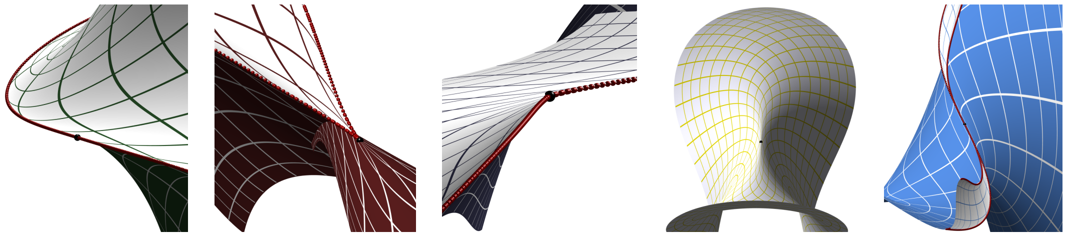

FIGURE 2. Pseudospherical surfaces with prescribed singularities. Cuspidal edge, swallowtail, cuspidal butterfly, cuspidal lips, cuspidal beaks.

surface f can be realised locally as a parallel of a regular surface g (in fact, the surface g is a linear Weingarten surface). We have the following result.

Theorem 4.1. *The evolution of parallels at a non-transverse A_3^+ (cuspidal lips), non-transverse A_3^- (cuspidal beaks) and A_4 (cuspidal butterfly) can be realized in generic 1-parameter families of pseudospherical surfaces; see Figure 3. These are the only bifurcations that can occur in generic 1-parameter family of pseudospherical wave fronts.*

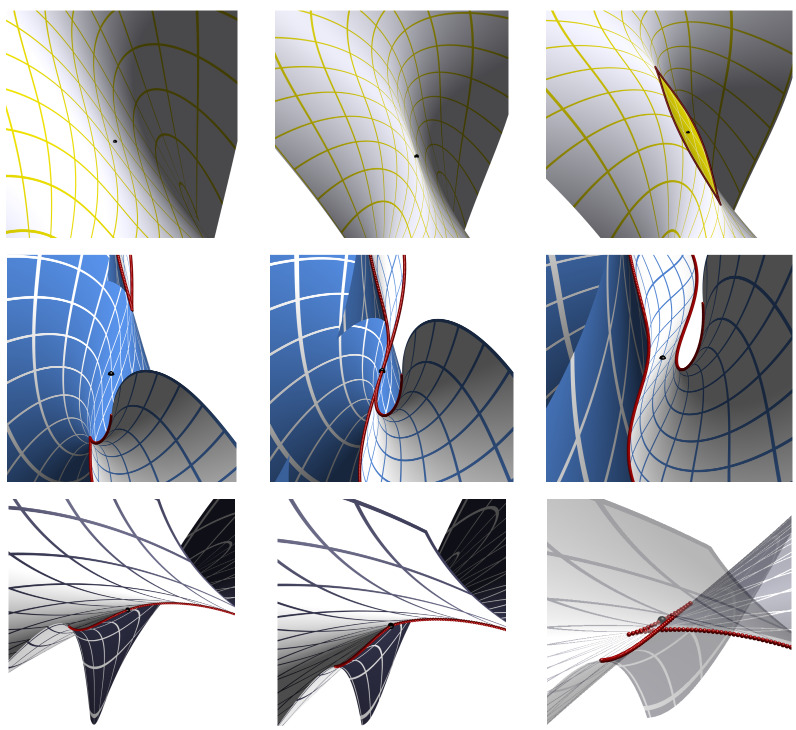

FIGURE 3. Cuspidal lips (top), cuspidal beaks (middle) and cuspidal butterfly (bottom) bifurcations.

The details of the proof of Theorem 4.1 are given in [6]. The idea is as follows: the recognition criteria in [12, 13, 14] can be used to find conditions on the k-jet of N for the various types of singularities to occur (recall that Theorem 2.1 asserts the existence of a Lorentizian harmonic map with a given k-jet). For the realization of the generic bifurcations we established in Theorem 4.1 and Theorem 4.2 in [5] general geometric criteria for determining such bifurcations. The criteria depend on certain k-jets of the family of surfaces. We use those criteria to construct generic 1-parameter families of Lorentzian-harmonic maps using Theorem 2.1. These give the 1-parameter families of pseudospherical surfaces which realize the generic bifurcations of parallels at non-transverse $A_3^\pm$ and A_4 singularities. For singularities more degenerate than these to occur requires more conditions on the k-jet, and a transversality argument shows that such singularities can be avoided on pseudospherical wave fronts.

A point where a pseudospherical frontal is not a wave front (i.e., where $f_x = 0$ or $f_y = 0$) is non-generic (the map from the surface to the k-jet space of germs of Lorentzian harmonic maps is not transverse to the relevant stratum). However, such singularities *do* occur in generic 1-parameter families of pseudospherical surfaces. We prove the following results in [6].

Theorem 4.2. *Suppose that N has rank 1, $N_y = 0$ or $N_x = 0$ and that its singular set is regular (so it is locally a null curve* [3]*). Then generically f is locally a $2/5$-cuspidal edge, i.e., it is locally diffeomorphic to the image of the map $(x,y) \mapsto (x,y^2,y^5)$. The bifurcations of f in a generic 1-parameter family f^s of pseudospherical surfaces with $f^0 = f$, are as shown in Figure 4, top row. The singular set of f^s is an ordinary 2/3 cuspidal edge for $s \neq 0$, and there is a birth of a double point curve on the surface f^s on one side of the bifurcation ($s < 0$ or $s > 0$). The singular curve in the source is timelike on one side of the transition and spacelike on the other side.*

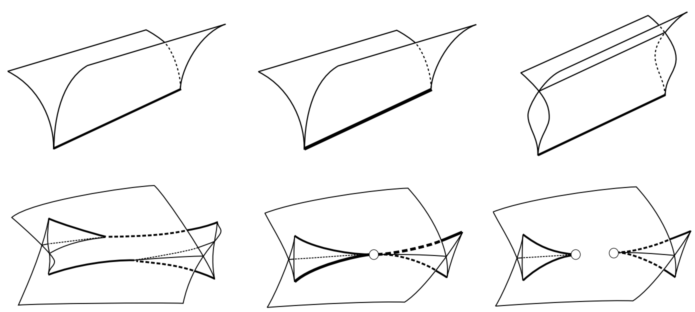

FIGURE 4. The 2/4-cuspidal edge bifurcation (top) and the Shcherbak bifurcation (bottom).

A surface $f : \mathbb{R}^2,0 \to \mathbb{R}^3,0$ is called a *Shcherbak surface* if f is equivalent, by smooth changes of coordinates in the source and target, to the map-germ $(x,y) \mapsto (x, xy^2 + y^3, xy^4 + \frac{6}{5}y^5)$; see Figure 4 (lower middle) and Figure 5.

Theorem 4.3. *Suppose that N has rank 1 and that $N_y = 0$ or $N_x = 0$. In generic 1-parameter families of pseudospherical surfaces f^s, the singular set of $f = f^0$ can have a Morse A_1^--singularity, where one of the branches is a null curve and the other is transverse to both null curves at the singular point. Then f^0 is a Shcherbak surface. The deformations in the family f^s as s varies near zero are as shown in Figure 4 and Figure 5. We have a birth of two swallowtail singularities on one side of the transition and none on the other side. On the side where there are no swallowtail singularities, we have a birth of two cusp singularities of the double point curve.*

For the proofs of Theorems 4.2 and 4.3 we use the induced map from the surface to the k-jet space of germs of Lorentzian harmonic maps. We show that the map is not transverse to the relevant strata, but that transversality holds in generic one parameter families of pseudospherical surface f^s. We then analyse the structure of the singular set and of the double point curve of the members of the family f^s.

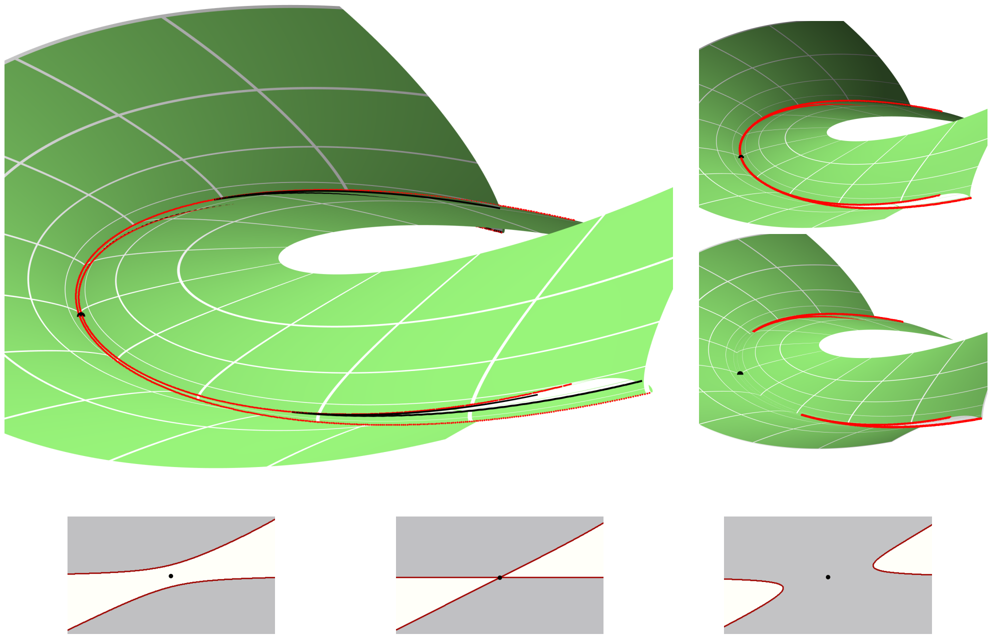

FIGURE 5. Shcherbak singularity bifurcation (see also Figure 4). The singular sets are the red curves. In the first image, the surface self intersection is shown by black curves. The singular sets in the coordinate domains are shown below.

5. NUMERICS

To compute examples of solutions, we modifed the solution of the Cauchy problem for pseudospherical surfaces given in [4] to find the loop group potentials that generate the unique solution to the Cauchy problem for a harmonic map N, with N and N_x prescribed along a non-characteristic curve $(x,y) = (t,t)$.

A *boundary potential* (see [4]) for the Cauchy problem along a curve parameterized by an interval J, is loop-algebra valued 1-form with Fourier-expansion in the loop parameter λ of the form $\psi = \sum_{i=-1}^{1} A_i(t)\lambda^i \mathrm{d}t$, where $A_i(t) \in \mathfrak{su}(2)$. The corresponding solution on the set $J \times J$ is obtained with the following steps:

(1) Solve the ODE $\mathrm{d}X = \psi$, with $X(t_0) = I$.
(2) At each $(x,y) \in J \times J$ perform a Birkhoff decomposition $X(x)^{-1}X(y) = H_-(x,y)H_+(x,y)$, where $H_\pm(x,y,\lambda)$ extends holomorphically in λ to $\mathbb{D}^\pm := \{\lambda \in \mathbb{C} \cup \{\infty\} \ : \ |\lambda^{\pm 1}| < 1\}$, normalized so that $H_-(x,y,\infty) = I$.
(3) The map $F(x,y,\lambda) = X(x,\lambda)H_-(x,y,\lambda)$ is an extended frame for the harmonic map $N(x,y) = F(x,y,1)\mathrm{diag}(i,-i)F(x,y,1)^{-1}$ which solves the given Cauchy problem for $N : I \times I \to \mathbb{S}^2 \subset \mathfrak{su}(2) \equiv \mathbb{R}^3$.
(4) The so-called Sym formula $f = \frac{\partial F}{\partial \lambda} F^{-1}\big|_{\lambda=1}$ also gives the pseudospherical surface f corresponding to N.

We implement the above procedure by approximating the Fourier expansion of a loop by a Laurent polynomial and then solving the ODE numerically. For polynomial loops the Birkhoff decomposition is equivalent to an LU-decomposition of matrices, so all of the above steps can be implemented.

Finding a formula for the boundary potential for the Cauchy problem with N and N_x prescribed along J, allows one to easily produce examples of all of the types of singularities discussed above, and the images in this report were generated this way. It is also possible to give simple criteria on the potentials that correspond to the different types of singularities. For example:

Theorem 5.1. *Any pseudospherical wave front can locally be represented by the boundary potential*

$$\psi(t) = \left(c(t)e_3 + e_1\lambda + (e_1 + a(t)e_2)\lambda^{-1}\right) \mathrm{d}t,$$

where e_i are an orthonormal basis for $\mathfrak{su}(2)$. The corresponding pseudospherical surface f has a singularity at $p = (t_0, t_0)$ if and only if $a(t_0) = 0$. At such a point, the germ of f at p is a:

(1) *Cuspidal edge (A_2) if and only if $a' \neq 0$.*
(2) *Swallowtail (A_3) if and only if $a' = 0$, $a'' \neq 0$ and $c \neq 0$.*
(3) *Cuspidal butterfly (A_4) if and only if $a'(t_0) = a'' = 0$, $a''' \neq 0$ and $c \neq 0$.*
(4) *Cuspidal lips (A_3^+) if and only if $a' = c = 0$, and $c'(a'' + c') < 0$.*
(5) *Cuspidal beaks (A_3^-) if and only if $a' = c = 0$, $a'' \neq 0$, and $c'(a'' + c') > 0$.*

See [6] for more details of this, as well as the construction for non-wave front singularities. In [6] we also treat the singularities of the Lorentzian harmonic map N, which we did not discuss in this report.

Acknowledgements: This research was supported through the programme "Research in Pairs" by the Mathematisches Forschungsinstitut Oberwolfach in 2019. The second author acknowledges financial support from São Paulo Research Foundation (FAPESP) grant 2019/07316-0 and CNPq grant 303772/2018-2.

REFERENCES

[1] VI Arnold. Wave front evolution and equivariant Morse lemma. *Commun. Pure and Appl. Math.*, 29:557–582, 1976.
[2] VI Arnold, SM Gusein-Zade, and AN Varchenko. *Singularities of differentiable maps I*, volume 82 of *Monographs in Mathematics*. Birkhäuser Boston, Inc., 1985.
[3] D Brander. Pseudospherical surfaces with singularities. *Ann. Mat. Pura Appl.*, 196:905–928, 2017.
[4] D Brander and M Svensson. The geometric Cauchy problem for surfaces with Lorentzian harmonic Gauss maps. *J. Differential Geom.*, 93:37–66, 2013.
[5] D Brander and F Tari. Families of spherical surfaces and harmonic maps. *Geometriae Dedicata*, 201:203–225, 2019.
[6] D Brander and F Tari. Wave maps and constant curvature surfaces: singularities and bifurcations. 2020. arXiv:1911.06856v2 [math.DG].
[7] JW Bruce. Wavefronts and parallels in Euclidean space. *Math. Proc. Cambridge Philos. Soc.*, 93:323–333, 1983.
[8] T Fukui and M Hasegawa. Singularities of parallel surfaces. *Tohoku Math. J.*, 64:387–408, 2012.
[9] G Ishikawa. Infinitesimal deformations and stability of singular legendre submanifolds. *Asian J. Math.*, 9:133–166, 2005.
[10] G Ishikawa. Singularities of frontals. *Adv. Stud. Pure Math.*, 78:55–106, 2018.
[11] G Ishikawa and Y Machida. Singularities of improper affine spheres and surfaces of constant Gaussian curvature. *Internat. J. Math.*, 17:269–293, 2006.
[12] S Izumiya and K Saji. The mandala of Legendrian dualities for pseudo-spheres of Lorentz-Minkowski space and "flat" spacelike surfaces. *J. Singul.*, 2:92–127, 2010.
[13] S Izumiya, K Saji, and M Takahashi. Horospherical flat surfaces in hyperbolic 3-space. *J. Math. Soc. Japan*, 62:789–849, 2010.
[14] M Kokubu, W Rossman, K Saji, M Umehara, and K Yamada. Singularities of flat fronts in hyperbolic space. *Pacific J. Math.*, 221:303–351, 2005.

DEPARTMENT OF APPLIED MATHEMATICS AND COMPUTER SCIENCE, RICHARD PETERSENS PLADS, BUILDING 324, TECHNICAL UNIVERSITY OF DENMARK, DK-2800 KGS. LYNGBY, DENMARK
Email address: dbra@dtu.dk

INSTITUTO DE CIÊNCIAS MATEMÁTICAS E DE COMPUTAÇÃO - USP, AVENIDA TRABALHADOR SÃO-CARLENSE, 400 - CENTRO, CEP: 13566-590 - SÃO CARLOS - SP, BRAZIL
Email address: faridtari@icmc.usp.br