Snapshots of modern mathematics
from Oberwolfach

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From Betti numbers to ℓ^2-Betti numbers

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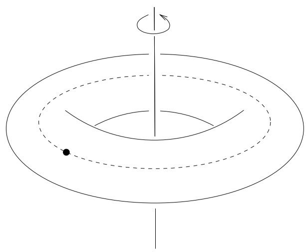
We provide a leisurely introduction to ℓ^2-Betti numbers, which are topological invariants, by relating them to their much older cousins, Betti numbers. In the end we present an open research problem about ℓ^2-Betti numbers.

1 A geometric problem

Large parts of mathematics are concerned with *symmetries* of various objects of interest. In topology, classical objects of interest are surfaces, like the one of a donut, which is also known as a *torus* and commonly denoted by $\mathbb{T}$. An apparent symmetry consists in the possibility of rotating $\mathbb{T}$ by arbitrary angles without changing its appearance as is indicated in the following figure.



In fact, if we follow the position of an arbitrary point on the torus while rotating it, we notice that the point travels along a circle. Any point not contained in this circle moves around yet another circle parallel to the first one. All these circles are called the *orbits* of the rotation *action*.

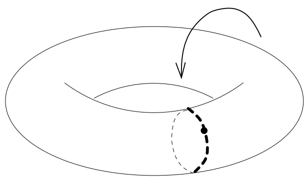
To say what this means in mathematical terms, let us first recall that the (standard) circle consists of all pairs of real numbers (x, y) such that $x^2 + y^2 = 1$. Any point (x, y) on the circle can equally be characterized by an angle $\alpha \in [0, 2\pi)$ in *radian*, meaning by the length of the arc between the points $(0, 1)$ and (x, y). Now, the key observation is that the circle forms a *group*, which means that we can "add" two points by adding corresponding angles, forfeiting full turns: for example $\frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}$, $\frac{3\pi}{2} + \pi = \frac{\pi}{2}$, $\pi + \pi = 0$, and so forth. [1]

The geometric situation described above can now be summarized mathematically by saying that the circle *acts* on $\mathbb{T}$: any angle α determines a transformation of $\mathbb{T}$, namely the rotation around the axis by the angle α. Under the action of this transformation, any point $z \in \mathbb{T}$ is mapped to another point called $r_\alpha(z) \in \mathbb{T}$, and we have the two important relations

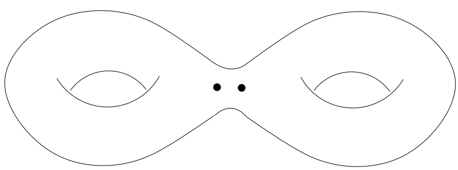
- $r_{\alpha+\beta}(z) = r_\alpha(r_\beta(z))$,
- $r_0(z) = z$,

which are valid for all angles α and β and for all points $z \in \mathbb{T}$. The action is moreover *continuous*: if points $z_1, z_2 \in \mathbb{T}$ are close to one another, then also the points $r_\alpha(z_1), r_\alpha(z_2) \in \mathbb{T}$ obtained by rotating about the central axis of an angle α are close to each other.

Rotating around the inscribed axis is not the only circle action one can find on $\mathbb{T}$. The following picture describes another one.



The orbits of this action are given by the dashed circle and all circles parallel to it. So one "wraps" the torus about an angle α towards the center of the hole. Nevertheless, depending on the surface under consideration, it can be tricky to find continuous circle actions. For example, consider the following surface:



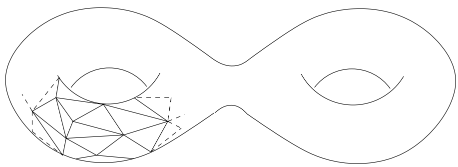
[1] For a more in-depth introduction to groups, see Snapshot 3/2018 *Computing with symmetries* by Colva M. Roney-Dougal.

We denote it by Σ_2, where the subscript two simply indicates that it has two handles. Or, in other words, that the *genus* of the bitorus is two – as opposed to the torus, whose genus is one.

Of course, one always has the trivial action defined by $r_\alpha(z) = z$ for all α, but that is not interesting. Other than that, we could try to wrap points toward the holes as in the last example. But then the point left from the middle would be transported leftwards while the point right from the middle would be transported rightwards so that they are not close anymore after the action: this violates the continuity condition. So it looks impossible to find a nontrivial, continuous circle action on this surface. But of course "it looks impossible" is not good enough in mathematics. We need *proof* that it *is* impossible.

2 Invariants and obstructions

A common method in mathematics to show the non-existence of a certain structure or property, like a circle action on a surface, is to find an *invariant* that serves as an *obstruction.* It turns out that the famous *Euler characteristic*, named after Leonhard Euler (1707–1783), is an invariant that obstructs circle actions. To explain this, let us first mention that the Euler characteristic is defined by means of a *triangulation.* A triangulation is obtained by "sprinkling" the surface with points and then "rebuilding" it by "gluing" triangles edge by edge, where the edges are formed by connecting the points as indicated in the following picture.



Then the Euler characteristic $\chi(\Sigma_2)$ of the surface is given by the formula

$$\chi(\Sigma_2) := V - E + F,$$

meaning that we count the number of vertices, subtract the number of edges, and add the number of faces. An important observation at this point is that no matter how the triangulation of Σ_2 was chosen, this number will always be -2. In this sense, the Euler characteristic is an *invariant* of the surface: it does not change if we modify the triangulation, and it does not change if we deform the surface, as if it was made from an elastic rubber material.

The reader is invited to convince herself that triangulating the torus $\mathbb{T}$ gives $\chi(\mathbb{T}) = 0$. A more conceptual way of showing the same thing is to start by noting that $\mathbb{T} = S^1 \times S^1$, that is, the torus is topologically equivalent to the "product" of two circles S^1. We then would like to apply the product formula $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$.

Here, triangulations and the Euler characteristic can be generalized to spaces of three dimensions and higher by using tetrahedra and their higher-dimensional counterparts, the *n-simplices* Δ^n. Here, the number n denotes the *dimension* of the simplex, such as $n = 1$ for an interval, $n = 2$ for a triangle, and $n = 3$ for a tetrahedron. Again, "triangulating" a space means to rebuild it by gluing n-simplices along their $(n-1)$-dimensional facets. The formula for the Euler characteristic of such a space X is then given by

$$\chi(X) = V - E + F - T \pm \cdots ,$$

now also taking the number T of tetrahedra and of higher-dimensional simplices into account. To verify the product formula, one notices that triangulations of X and Y yield a triangulation of $X \times Y$ by subdivision.

Since the torus is the product of two circles, $\mathbb{T} = S^1 \times S^1$, we thus have $\chi(\mathbb{T}) = \chi(S^1)^2$. But clearly $\chi(S^1) = 0$ because a triangulation of the circle is just a polygon – which always has as many edges as vertices. Therefore, $\chi(\mathbb{T}) = 0$. [2]

Now the point is that a nontrivial circle action on a surface of genus $g \geq 1$, or more generally on an *aspherical* space X (see Section 5), gives something similar to a product structure $X = S^1 \times Y$: the space can be glued from elementary spaces of type $S^1 \times \Delta^n$. All these elementary spaces have zero Euler characteristic by the product formula, from which one can conclude that the whole space must have vanishing Euler characteristic, too. Thus, the calculation $\chi(\Sigma_2) = -2$ shows that the genus two surface Σ_2 does not permit any non-trivial continuous circle action.

3 Homology and Betti numbers

As we just saw, the Euler characteristic is a powerful tool, capable of solving geometric problems. It should thus be worthwhile to refine this invariant in one way or another. Such a refinement is given by the *Betti numbers*, whose definition by Enrico Betti in 1871 can be seen as the moment when *algebraic topology* came into being. The precise technical definition is involved, but the idea is transparent: we start by considering the so called *cycles*, polygonal chains that form closed loops in a space, like the two cycles in the torus pictured below.

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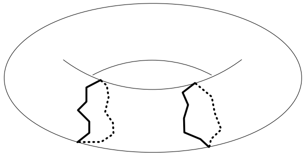
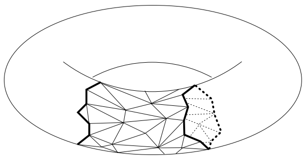
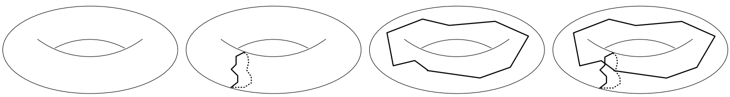
[2] For another snapshot related to triangulations and the Euler characteristic, see Snapshot 12/2016 *Footballs and donuts in four dimensions* by Steven Klee.



We do not want to distinguish two such cycles if their union forms the *boundary* of a *chain* of triangles. In this case, we say that the cycles agree up to homology, in other words, are *homologous*, as is the case for the two cycles we were considering, see the next figure.



It turns out that the torus $\mathbb{T}$ has only four cycles *up to homology* – the first of which is simply the trivial empty cycle:



In general, the surface of genus g will have 2^{2g} cycles up to homology. The exponent in this formula is called the *first Betti number* $b_1(\Sigma_g) = 2g$. To be precise, what we described is the first *mod 2* Betti number; to obtain the ordinary Betti numbers, one needs to take into account the "direction" of cycles. For "oriented" surfaces like the ones we are considering, Betti numbers and mod 2 Betti numbers agree.

Considering *cycles of triangles* up to *boundaries of chains of tetrahedra* in a space X gives the *second Betti number* $b_2(X)$. And in general, considering cycles of n-simplices up to boundaries of chains of $(n+1)$ simplices defines the *n-th Betti number* $b_n(X)$. The relation between Betti numbers and Euler characteristic is expressed by the *Euler–Poincaré formula*

$$\chi(X) = b_0(X) - b_1(X) + b_2(X) - b_3(X) \pm \cdots$$

where the sum ends after n steps if X can be triangulated using simplices of dimension at most n. Thus the Betti numbers determine the Euler characteristic, but not the other way around! The set of Betti numbers is a finer invariant than the Euler characteristic.

4 Covering spaces and ℓ^2-Betti numbers

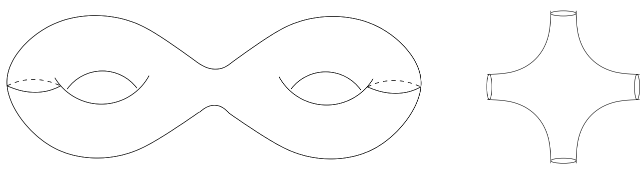
The virtue that Betti numbers carry more information than the Euler characteristic comes at a price. For the torus, the Euler–Poincaré formula reads

$$\chi(\mathbb{T}) = 1 - 2 + 1 = 0,$$

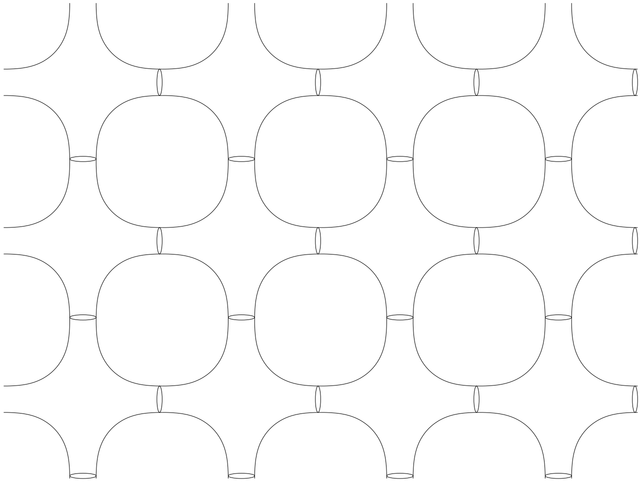
so individually, Betti numbers do not have the ability to obstruct circle actions on aspherical manifolds. However, there exists a variant of the Betti numbers, the so called *ℓ^2-Betti numbers* $b_n^{(2)}(X)$ which still satisfy the Euler–Poincaré formula

$$\chi(X) = b_0^{(2)}(X) - b_1^{(2)}(X) + b_2^{(2)}(X) - b_3^{(2)}(X) \pm \cdots .$$

But as opposed to ordinary Betti numbers, they also individually share many of the convenient features of the Euler characteristic, obstructing circle actions on aspherical manifolds only being one of them. ℓ^2-Betti numbers emerge by not only considering the space X itself, but instead the whole family of spaces that arises from X by "cutting and pasting" copies of X along closed loops. In our surface example, we could for instance cut along the two inscribed loops which, after some bending and deforming, results in the space on the right.

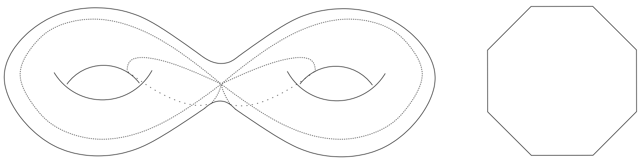


Infinitely many copies of this space can be glued together to form a "grid" of surfaces like the one below. This is an example of a *covering space*, a space that *locally*, within the immediate surroundings of any given point, looks the same as the original surface, but *globally*, as a whole, can be utterly different.

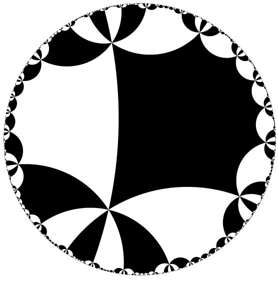


The pictured covering space is moreover *regular*: it has so many symmetries that any copy of the surface can be moved to any other by a translation of the whole space. The picture shows only one out of a myriad of different possibilities to produce a covering space by cutting and gluing.

But one particular covering stands out because it is a covering space of all the others: the *universal covering*. For the genus two surface, we obtain it as follows. First, cut along the indicated four loops.



It then takes quite some mental effort to see that with these cuts, the surface can be flattened out to an octagon. Now deform, squeeze, and glue infinitely many of these octagons until they tile a round disk as pictured below.



For better visibility, the octagons are alternately filled black and white. Now the idea is to define ℓ^2-Betti numbers as “asymptotic” Betti numbers when passing to larger and larger covering spaces of X that more and more look like the universal covering.

More precisely, we fix a sequence of regular covering spaces (X_k) of X such that each X_k is a covering space of the previous space X_{k-1} and such that each X_k is constructed from only finitely many copies of the original space X. The number d_k of copies of X used to construct X_k is also called the *number of sheets* of X_k. Since each X_k locally looks like X, we can consider points $x_k \in X_k$ such that the surroundings of each x_k in X_k look like the surroundings of a fixed point $x \in X$. Whenever we move the point x in X, we can move the corresponding points in X_k in the same direction. In this way, we assign to a path in X starting at x the so-called *lifted* path in X_k starting in x_k. A path in X is called a *noncontractible loop* if both its initial and end point is x and if it cannot be deformed continuously to the constant path at x. We say that (X_k) *converges to the universal covering* if for any noncontractible loop in X, there exists some k such that the lift of the loop to X_k has distinct initial and end point. We have finally collected all the preliminaries to define ℓ^2-Betti numbers.

Definition. *Fix a sequence of regular covering spaces (X_k) of X that converges to the universal covering. Then the n-th ℓ^2-*Betti number *of X is given by*

$$b_n^{(2)}(X) = \lim_{k\to\infty} \frac{b_n(X_k)}{d_k}.$$

This is not the usual way of defining ℓ^2-Betti numbers. It only applies to those spaces that possess such a sequence of regular covering spaces – which includes most spaces of interest. ℓ^2-Betti numbers were invented in 1976 by Michael Atiyah, and his definition [1] is quite different from the above. That ℓ^2-Betti numbers can be characterized by the above limit and that, in particular, the limit exists and is independent of the chosen sequence of coverings is the content of Wolfgang Lück’s *approximation theorem* [4] proven in 1994.

The circle S^1 has a such a sequence of regular self-coverings $S^1 \to S^1$ whose number of sheets goes to infinity. See `www.youtube.com/watch?v=vP7NAeeKjrw` for a movie presentation of these coverings. The above formula for ℓ^2-Betti numbers now implies that all ℓ^2-Betti numbers of S^1 are zero.

For our previous examples, it turns out that all ℓ^2-Betti numbers of the torus $\mathbb{T}$ are zero,

$$b_0^{(2)}(\mathbb{T}) = b_1^{(2)}(\mathbb{T}) = b_2^{(2)}(\mathbb{T}) = 0,$$

while for the bitorus Σ_2 we obtain

$$b_0^{(2)}(\Sigma_2) = b_2^{(2)}(\Sigma_2) = 0, \text{ and } b_1^{(2)}(\Sigma_2) = 2.$$

Taken together, they illustrate the statement that the ℓ^2-Betti numbers $b_i^{(2)}$ are an obstruction to circle actions: If a surface of genus $g \geq 1$ supports a circle action, then, in contrast to the Betti numbers b_i, all its ℓ^2-Betti numbers must be zero.

5 The Hopf–Singer conjecture

The Euler–Poincaré formula connecting Euler characteristic and ℓ^2-Betti numbers (see Section 4) is related to an open conjecture about aspherical manifolds, which motivates important research in this area.

Let us first explain the notion of an aspherical manifold. One calls a space *aspherical* if its universal covering is *contractible*, that is, can be continuously contracted to a single point. A space is an n-dimensional *manifold* if it "locally" looks like a ball in n-dimensional Euclidean space. We will also require that the manifolds under consideration are *closed*, which means that any real-valued continuous function on the manifold, which you should think of as a measurement of any kind taken at each point, has a finite maximal value. All surfaces of genus $g \geq 1$, such as the torus $\mathbb{T}$ and the bitorus Σ_2, are examples of closed aspherical 2-dimensional manifolds. The sphere is a closed 2-dimensional manifold that is not aspherical. Both the "grid" covering surface and the plane that we have already encountered are examples of aspherical 2-dimensional manifolds that are not closed. [3]

Higher-dimensional aspherical manifolds are an important object of research in topology. In case you hear about the Borel isomorphism, or the Farrell–Jones and Baum–Connes conjectures: they are all about aspherical manifolds or spaces! Here we want to present yet another conjecture about aspherical manifolds, which is attributed to Shiing-Shen Chern (1911–2004) and Heinz Hopf (1894–1971):

Conjecture. *Let M be a closed aspherical manifold of dimension $2n$. Then*

$$(-1)^n \chi(M) \geq 0.$$

It is almost impossible to "attack" this conjecture by investigating suitable triangulations of aspherical manifolds. Also, Betti numbers and their Euler–Poincaré formula do not help a lot: in fact, Betti numbers of aspherical manifolds are often non-zero, and it is hard to predict the sign of an alternating sum of non-zero numbers that you do not know exactly. Here, ℓ^2-Betti numbers and

[3] For a snapshot on manifolds from a geometric perspective, Snapshot 4/2019 *Positive Scalar Curvature and Applications* by Jonathan Rosenberg and David Wraith.

their Euler–Poincaré formula

$$\chi(X) = b_0^{(2)}(X) - b_1^{(2)}(X) + b_2^{(2)}(X) - b_3^{(2)}(X) \pm \cdots$$

are much more helpful, since ℓ^2-Betti numbers tend to vanish more often. The following conjecture is attributed to Isadore Singer and called *Hopf–Singer conjecture*. It clearly implies the Chern–Hopf conjecture. It is stated in writing for the first time in Józef Dodziuk's paper [2] (albeit in slightly less general form, namely for "non-positively curved" manifolds).

Conjecture. *Let M be a closed aspherical manifold of dimension $2n$. Then all its ℓ^2-Betti number vanish except possibly the one in degree n.*

Similarly, one conjectures the vanishing of all ℓ^2-Betti numbers for closed aspherical odd-dimensional manifolds.

The Hopf–Singer conjecture and ℓ^2-Betti numbers provide a strategy for proving the original conjecture by Chern and Hopf. This strategy was, for example, successfully implemented by Mikhail Gromov in the case of so-called hyperbolic Kähler manifolds [3]. The concentration of ℓ^2-Betti numbers in the middle dimension, thus the Hopf–Singer conjecture, remains open in general, though. They are still stimulating current research.

Image credits

The Octagonal tiling of Poincaré disk, Author: Anton Sherwood (own work), 2013. Public domain, accessed via `https://commons.wikimedia.org/wiki/File:H2chess_288b.png`, visited on October 29, 2019.

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