Mathematisches
Forschungsinstitut
Oberwolfach

Member of the Leibniz Association

Oberwolfach Preprints

OWP 2020 - 10

KARL HEINRICH HOFMANN AND LINUS KRAMER

On Weakly Complete Universal Enveloping Algebras of pro-Lie Algebras

Mathematisches Forschungsinstitut Oberwolfach gGmbH
Oberwolfach Preprints (OWP) ISSN 1864-7596

Oberwolfach Preprints (OWP)

The MFO publishes a preprint series **Oberwolfach Preprints (OWP)**, ISSN 1864-7596, which mainly contains research results related to a longer stay in Oberwolfach, as a documentation of the research work done at the MFO. In particular, this concerns the Research in Pairs-Programme (RiP) and the Oberwolfach-Leibniz-Fellows (OWLF), but this can also include an Oberwolfach Lecture, for example.

A preprint can have a size from 1 - 200 pages, and the MFO will publish it on its website as well as by hard copy. Every RiP group or Oberwolfach-Leibniz-Fellow may receive on request 30 free hard copies (DIN A4, black and white copy) by surface mail.

The full copyright is left to the authors. With the submission of a manuscript, the authors warrant that they are the creators of the work, including all graphics. The authors grant the MFO a perpetual, non-exclusive right to publish it on the MFO's institutional repository.

In case of interest, please send a **pdf file** of your preprint by email to *rip@mfo.de* or *owlf@mfo.de*, respectively. The file should be sent to the MFO within 12 months after your stay as RiP or OWLF at the MFO.

There are no requirements for the format of the preprint, except that the introduction should contain a short appreciation and that the paper size (respectively format) should be DIN A4, "letter" or "article".

On the front page of the hard copies, which contains the logo of the MFO, title and authors, we shall add a running number (20XX – XX). Additionally, each preprint will get a Digital Object Identifier (DOI).

We cordially invite the researchers within the RiP or OWLF programme to make use of this offer and would like to thank you in advance for your cooperation.

Imprint:

Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
Schwarzwaldstrasse 9-11
77709 Oberwolfach-Walke
Germany

Tel   +49 7834 979 50
Fax   +49 7834 979 55
Email admin@mfo.de
URL   www.mfo.de

The Oberwolfach Preprints (OWP, ISSN 1864-7596) are published by the MFO.

DOI   10.14760/OWP-2020-10

On Weakly Complete Universal Enveloping Algebras of pro-Lie algebras

Karl Heinrich Hofmann and Linus Kramer*

Abstract. Let $\mathbb{K}$ denote the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. A topological vector space over $\mathbb{K}$ is *weakly complete* if it is isomorphic to a power $\mathbb{K}^J$. The appropriate category is $\mathcal{W}$. For each topological Lie algebra $\mathfrak{g}$ there is a *weakly complete topological universal enveloping Hopf algebra* $\mathbf{U}(\mathfrak{g})$ over $\mathbb{K}$, which is defined by its universal property as follows:

If A is a weakly complete unital algebra, then A is known to be a projective limit of finite-dimensional quotient algebras (see e.g. K.H.Hofmann and S.A.Morris, The Structure of Compact Groups, 4th Ed., De Gruyter, 2020, Theorem A7.34). This implies that the weakly complete Lie algebra $A_{\rm Lie}$ obtained from A by considering the Lie bracket $[a,b]=ab-ba$ on A is a profinite-dimensional Lie algebra, that is, a projective limit of finite-dimensional quotient algebras. If $f\colon \mathfrak{g}\to A_{\rm Lie}$ is a morphism of topological Lie algebras, then there is a unique morphism of weakly complete unital algebras $f'\colon \mathbf{U}(\mathfrak{g})\to A$ such that for the natural morphism $\lambda_{\mathfrak{g}}\colon \mathfrak{g}\to\mathbf{U}(\mathfrak{g})_{\rm Lie}$ coming along with $\mathbf{U}(\mathfrak{g})$ we have $f=\mathbf{U}(f')\circ\lambda_{\mathfrak{g}}:\mathfrak{g}\to A_{\rm Lie}$. So the functor $\mathfrak{g}\mapsto\mathbf{U}(\mathfrak{g})$ from the category of pro-finite dimensional Lie algebras to the category of weakly complete unital algebras is left adjoint to the functor $A\mapsto A_{\rm Lie}$. The following facts are shown: (i) $\lambda_{\mathfrak{g}}$ is an embedding and so $\mathfrak{g}$ may be considered as a closed subalgebra of $\mathbf{U}(\mathfrak{g})_{\rm Lie}$. (ii) The subalgebra $\langle\mathfrak{g}\rangle$ generated by $\mathfrak{g}$ inside the abstract algebra $|\,\mathbf{U}(\mathfrak{g})|$ underlying $\mathbf{U}(\mathfrak{g})$ is naturally isomorphic to the classical enveloping algebra $U(|\mathfrak{g}|)$ of the abstract Lie algebra $|\mathfrak{g}|$ underlying the profinite-dimensional Lie algebra $\mathfrak{g}$. (iii) $\langle\mathfrak{g}\rangle$ is dense in $\mathbf{U}(\mathfrak{g})$. In short

$$\mathfrak{g}\subseteq U(|\mathfrak{g}|)=\langle\mathfrak{g}\rangle\subseteq\mathbf{U}(\mathfrak{g})=\overline{U(|\mathfrak{g}|)}.$$

If $\mathfrak{g}=\lim_{j\in J}\mathfrak{g}_j$, the projective limit representation of $\mathfrak{g}$ by its finite dimensional quotient algebras, then

$$\mathbf{U}(\lim_{j\in J}\mathfrak{g}_j)=\mathbf{U}(\mathfrak{g})\cong\lim_{j\in J}\mathbf{U}(\mathfrak{g}_j).$$

As in the case of the classical enveloping algebra $U(L)$, the weakly complete enveloping algebra $\mathbf{U}(\mathfrak{g})$ is in fact a symmetric Hopf algebra with a comultiplication $c\colon\mathbf{U}(\mathfrak{g})\to\mathbf{U}(\mathfrak{g})\otimes_{\mathcal{W}}\mathbf{U}(\mathfrak{g})$. Grouplike and primitive elements are definied in this Hopf algebra as usual. In the case of an abelian $\mathfrak{g}$ and its vector space dual $\mathfrak{g}'$ we establish a surjective morphism of symmetric Hopf-algebras $q_{\mathfrak{g}}\colon\mathbf{U}(\mathfrak{g})\to\mathbb{K}^{\mathfrak{g}'}$ whose kernel is the radical of $\mathbf{U}(\mathfrak{g})$. Here $\mathbb{K}^{\mathfrak{g}'}$ has componentwise addition and

*Both authors were supported by Mathematisches Forschungsinstitut Oberwolfach in the program RiP (Research in Pairs). Linus Kramer is funded by the Deutsche Forschungsgemeinschaft under Germany's Excellence Strategy EXC 2044-390685587, Mathematics Münster: Dynamics-Geometry-Structure.

multiplication and has a comultiplication $c\colon \mathbb{K}^{\mathfrak{g}'} \to \mathbb{K}^{\mathfrak{g}'\times\mathfrak{g}'}(\cong \mathbb{K}^{\mathfrak{g}'} \otimes_{\mathcal{W}} \mathbb{K}^{\mathfrak{g}'})$ such that $\varphi \in \mathbb{K}^{\mathfrak{g}'}$ and $\omega_1, \omega_2 \in \mathfrak{g}'$ implies

$$c(\varphi)(\omega_1, \omega_2) = \varphi(\omega_1 + \omega_2).$$

While $\lambda_{\mathfrak{g}}(\mathfrak{g})$ is always contained in $\mathbb{P}(\mathbf{U}(\mathfrak{g}))$ it is shown that $\mathbb{P}(\mathbb{K}^{\mathfrak{g}'})$ is larger by far than $q_{\mathfrak{g}}(\lambda_{\mathfrak{g}}(\mathfrak{g})) \cong \mathfrak{g}$. Various open questions are stated explicitly.
Mathematics Subject Classification 2010: 22E15, 22E65, 22E99.
Key Words and Phrases: Weakly complete vector space, weakly complete algebra, group algebra, Hopf algebra, pro-Lie algebra, universal enveloping algebra.

1. The Weakly Complete Enveloping Algebra of a Profinite-Dimensional Lie Algebra

In [5] we have initiated the theory of *weakly complete universal enveloping algebras over* $\mathbb{K}$ perhaps in some fashion that would resemble the universal enveloping algebra of a Lie algebra such as it is presented in the famous POINCARÉ-BIRKHOFF-WITT-Theorem (see e.g. [1], §2, n° 7, Théorème 1., p.30). This is not exactly the case, but exactly how close we come is discussed in this section.

So we let $\mathbb{K}$ again denote one of the topological fields $\mathbb{R}$ or $\mathbb{C}$. For a topological Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ we let $\mathcal{I}(\mathfrak{g})$ denote the filter basis of all closed ideals $I \subseteq \mathfrak{g}$ such that $\dim \mathfrak{g}/I < \infty$.

Definition 1.1. A topological Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ is called *profinite-dimensional* if $\mathfrak{g} = \lim_{I\in\mathcal{I}(\mathfrak{g})} \mathfrak{g}/I$. Let $\mathcal{WL}$ denote the category of profinite-dimensional Lie algebras (over $\mathbb{K}$) and continuous Lie algebra morphisms between them.

Notice that by its definition every profinite-dimensional Lie algebra is a pro-Lie algebra and is therefore weakly complete. A comment following Theorem 3.12 of [2] exhibits an example of a weakly complete $\mathbb{K}$-Lie algebra which is not a profinite-dimensional Lie algebra.

Let $\mathcal{WA}$ denote the category of weakly complete associative unital algebras over $\mathbb{K}$. However, instead of considering the full category of weakly complete Lie algebras over $\mathbb{K}$, in the following we consider $\mathcal{WL}$. The reason for this restriction is Theorem A7.34 of [7] stating that every weakly complete unital $\mathbb{K}$-algebra is the projective limit of its finite-dimensional quotient algebras. This implies at once the following

Lemma 1.2. *Let A be a weakly complete unital $\mathbb{K}$-algebra. Then the weakly complete Lie algebra A_{Lie} obtained by considering on the weakly complete vector space A the Lie algebra obtained with respect to the Lie bracket $[x, y] = xy - yx$ is profinite-dimensional.*

The functor which associates with a weakly complete associative algebra A the profinite-dimensional Lie algebra A_{Lie} is called the *underlying Lie algebra functor.*

Theorem 1.3. (The Existence Theorem of $\mathbf{U}$) *The underlying Lie algebra functor $A \mapsto A_{\mathrm{Lie}}$ from $\mathcal{WA}$ to $\mathcal{WL}$ has a left adjoint $\mathbf{U}\colon \mathcal{WL} \to \mathcal{WA}$.*

Proof. The category $\mathcal{WL}$ is complete. (Exercise. Cf. Theorem A3.48 of [7], p. 819.) The "Solution Set Condition" (of Definition A3.58 in [7], p. 824) holds. (Exercise: Cf. the proof Lemma 3.58 of [7].) Hence $\mathbf{U}$ exists by the Adjoint Functor Existence Theorem (i.e., Theorem A3.60 of [7], p. 825). ∎

In other words, for each weakly complete Lie algebra L there is a natural morphism $\lambda_{\mathfrak{g}}\colon \mathfrak{g} \to \mathbf{U}(\mathfrak{g})$ with the property that for each continuous Lie algebra morphism $f\colon \mathfrak{g} \to A_{\mathrm{Lie}}$ for some weakly complete associative unital algebra A there is a unique $\mathcal{WA}$-morphism $f'\colon \mathbf{U}(\mathfrak{g}) \to A$ such that $f = (f')_{\mathrm{Lie}} \circ \lambda_{\mathfrak{g}}$.

$$
\begin{array}{cc}
\mathcal{WL} & \mathcal{WA}\\ \hline
\begin{tikzcd} \mathfrak{g} \arrow[r, "\lambda_{\mathfrak{g}}"] \arrow[d, "\forall f"'] & \mathbf{U}(\mathfrak{g})_{\mathrm{Lie}} \arrow[d, "(f')_{\mathrm{Lie}}"] \\ A_{\mathrm{Lie}} \arrow[r, "\mathrm{id}"'] & A_{\mathrm{Lie}} \end{tikzcd} &
\begin{tikzcd} \mathbf{U}(\mathfrak{g}) \arrow[d, "\exists! f'"] \\ A. \end{tikzcd}
\end{array}
$$

If necessary we shall write $\mathbf{U}_{\mathbb{K}}$ instead of $\mathbf{U}$ whenever the ground field should be emphasized.

Definition 1.4. For each profinite-dimensional $\mathbb{K}$-Lie algebra, we shall call $\mathbf{U}_{\mathbb{K}}(\mathfrak{g})$ *the weakly complete enveloping algebra* of $\mathfrak{g}$ (over $\mathbb{K}$).

From Propositions A3.36 and A3.38 in [7] we derive immediately the following corollary, representing equivalent formulation for the fact that the universal enveloping functor $\mathbf{U}$ is left adjoint to $L \mapsto L_{\mathrm{Lie}}$:

Corollary 1.5. (i) *For each weakly complete unital $\mathbb{K}$-algebra A there is a natural $\mathcal{WA}$-morphism $\nu_A\colon\mathbf{U}(A_{\mathrm{Lie}})\to A$ such that for every $\mathcal{WA}$-morphism $\alpha\colon\mathbf{U}(\mathfrak{g}) \to A$ for some profinite-dimensional Lie algebra $\mathfrak{g}$ there is a unique $\mathcal{WL}$-morphism $\alpha'\colon \mathfrak{g} \to A_{\mathrm{Lie}}$ such that $\alpha = \nu_A \circ \mathbf{U}(\alpha')$.*

(ii) *For each $\mathfrak{g}$ in $\mathcal{WL}$, the category of profinite-dimensional Lie algebras, the composition*

$$\mathbf{U}(\mathfrak{g}) \xrightarrow{\mathbf{U}(\lambda_{\mathfrak{g}})} \mathbf{U}\left(\mathbf{U}(\mathfrak{g})_{\mathrm{Lie}}\right) \xrightarrow{\nu_{\mathbf{U}(\mathfrak{g})}} \mathbf{U}(\mathfrak{g})$$

is the identity of $\mathbf{U}(\mathfrak{g})$ and, likewise, for each A in $\mathcal{WA}$, the category of weakly complete algebras, the composition

$$A_{\mathrm{Lie}} \xrightarrow{\lambda_{A_{\mathrm{Lie}}}} \mathbf{U}(A_{\mathrm{Lie}})_{\mathrm{Lie}} \xrightarrow{(\nu_A)_{\mathrm{Lie}}} A_{\mathrm{Lie}}$$

is the identity of A_{Lie}.

The following Lemma will be useful.

Lemma 1.6. *Assume that* $\mathbf{V}\colon \mathcal{WL} \to \mathcal{WA}$ *is a functor endowed with a natural* $\mathcal{WL}$*-morphism* $\nu_{\mathfrak{g}}\colon \mathfrak{g} \to \mathbf{V}(\mathfrak{g})_{\mathrm{Lie}}$ *such that for each* $\mathcal{WL}$*-morphism* $f\colon \mathfrak{g} \to F$ *into a* finite-dimensional *unital associative algebra there is a unique continuous morphism* $f'\colon \mathbf{V}(\mathfrak{g}) \to F$ *such that* $f = (f')_{\mathrm{Lie}} \circ \nu_{\mathfrak{g}}$*. Then for each profinite-dimensional Lie algebra* $\mathfrak{g}$*, the morphism* $\nu_{\mathfrak{g}}$ *is an embedding into* $\mathbf{V}(\mathfrak{g})$ *with a closed image.*

Proof. Since each finite-dimensional quotient Lie algebra of $\mathfrak{g}$ has a faithful representation by the Theorem of Ado, and since the finite-dimensional quotients separate the points of $\mathfrak{g}$, the morphism $\nu_{\mathfrak{g}}$ is injective. However, injective morphisms of weakly complete vector spaces are open onto their images, and their images are closed. ∎

After this lemma it is no loss of generality if henceforth we assume that $\nu_{\mathfrak{g}}$ satisfying the hypotheses of the lemma is an inclusion map. Accordingly, in the circumstances of Lemma 1.6 we simply say that

the profinite-dimensional Lie algebra $\mathfrak{g}$ *is a closed Lie subalgebra of the weakly complete unital algebra* $\mathbf{V}(\mathfrak{g})_{\mathrm{Lie}}$ *such that every morphism of profinite-dimensional Lie algebras* $f\colon \mathfrak{g} \to A_{\mathrm{Lie}}$ *for a finite-dimensional unital algebra* A *"extends" to a morphism of weakly complete algebras* $f'\colon \mathbf{V}(\mathfrak{g}) \to A$*.*

In particular we have the following

Proposition 1.7. *For any profinite-dimensional Lie algebra* $\mathfrak{g}$ *the morphism* $\lambda_{\mathfrak{g}}\colon \mathfrak{g} \to \mathbf{U}(\mathfrak{g})_{\mathrm{Lie}}$ *is an embeddding of profinite-dimensional Lie algebras with a closed image, and we may in fact assume that* $\mathfrak{g}$ *is a closed Lie subalgebra of* $\mathbf{U}(\mathfrak{g})_{\mathrm{Lie}}$ *such that every morphism of profinite-dimensional Lie algebras* $f\colon \mathfrak{g} \to A_{\mathrm{Lie}}$ *for some weakly complete unital algebra* A *extends uniquely to a continuous algebra morphism* $f'\colon \mathbf{U}(\mathfrak{g}) \to A$*.*

Corollary 1.8. *For any profinite-dimensional Lie algebra* $\mathfrak{g}$*, the unital associative subalgebra* $\langle\mathfrak{g}\rangle$ *generated in* $\mathbf{U}(\mathfrak{g})$ *by* $\mathfrak{g}$ *is dense in* $\mathbf{U}(\mathfrak{g})$*.*

Proof. Let $f\colon \mathfrak{g} \to A_{\mathrm{Lie}}$ be a $\mathcal{WL}$-morphism for a weakly complete algebra A. By the universal property of $\mathbf{U}(\mathfrak{g})$ there is a unique $\mathcal{WA}$-morphism $f'\colon \mathbf{U}(\mathfrak{g}) \to A$ extending f. Then $f'|\overline{\langle\mathfrak{g}\rangle}\colon \overline{\langle\mathfrak{g}\rangle} \to A$ is a unique extension of f to a $\mathcal{WA}$-morphism. Hence $\langle\mathfrak{g}\rangle$ has the universal property of the weakly complete universal enveloping algebra $\mathbf{U}(\mathfrak{g})$ of $\mathfrak{g}$ and thus, by uniquenes, agrees with it. Therefore $\overline{\langle\mathfrak{g}\rangle} = \mathbf{U}(\mathfrak{g})$. ∎

Since every weakly complete unital algebra is a strict projective limit of all finite-dimensional quotient algebras, it will now turn out to be sufficient to test the universal property of the functor $\mathbf{U}$ for *finite-dimensional* unital associative algebras:

Proposition 1.9. *Assume that the profinite-dimensional Lie algebra* $\mathfrak{g}$ *is contained functorially in a weakly complete unital algebra* $\mathbf{V}(\mathfrak{g})$ *such that for each* finite-dimensional *unital algebra* A *and each morphism of profinite-dimensional Lie algebras* $f\colon \mathfrak{g} \to A_{\mathrm{Lie}}$ *there is a unique morphism of weakly complete unital algebras* $f'\colon \mathbf{V}(\mathfrak{g}) \to A$ *extending* f*. Then* $\mathbf{V}(\mathfrak{g}) \cong \mathbf{U}(\mathfrak{g})$ *naturally.*

Proof. We must test the universal property for arbitrary weakly complete unital algebras A in place of just finite-dimensional ones. So let A be one of the former and let $\mathcal{I}(A)$ be the filter basis of all closed ideals I such that $\dim A/I < \infty$. For $J \subseteq I$, $I, J \in \mathcal{I}(A)$ we have the natural quotient morphism $q_{IJ}\colon A/J \to A/I$ and so A and $\lim\{A/K, q_{IJ} : I, J, K \in \mathcal{I}(A), J \subseteq I\}$ are naturally isomorphic. For the remainder of the proof we identify A and $\lim_{I\in\mathcal{I}(A)} A/I$.

Now let $f\colon \mathfrak{g} \to A_{\mathrm{Lie}}$ be a morphism of profinite-dimensional Lie algebras. We must show that f extends to a morphism $f\colon \mathbf{V}(\mathfrak{g}) \to A$ of weakly complete algebras. For each $I \in \mathcal{I}(A)$, there is a morphism of profinite-dimensional algebras $f_I\colon \mathfrak{g} \to (A/I)_{\mathrm{Lie}}$, namely, $f_I = (q_I)_{\mathrm{Lie}} \circ f_{\mathrm{Lie}}$ with the quotient morphism $q_I\colon A \to A/I$ of weakly complete algebras. By hypothesis on $\mathbf{V}(\mathfrak{g})$ there is a unique extension $f'_I\colon \mathbf{V}(\mathfrak{g}) \to A/I$ of f_I to a morphism of weakly complete algebras. Now let $J \subseteq I$ in $\mathcal{I}(A)$. Then we have two morphisms of weakly complete Algebras

$$f'_I, \quad q_{IJ} \circ f'_J : \mathbf{V}(\mathfrak{g}) \to A/I. \tag{1}$$

By the definitions of f'_I and f'_J, their restrictions to $\mathfrak{g} \subseteq \mathbf{V}(\mathfrak{g})$ are f_I and $q_{IJ} \circ f_J$. Since $q_I = q_{IJ} \circ q_J$ and $f_I = q_I \circ f$ and $f_J = q_J \circ f$ as morphisms of complete vector spaces we conclude

$$(f_I)_{\mathrm{Lie}} = (q_{IJ} \circ f_J)_{\mathrm{Lie}}. \tag{2}$$

According to the uniqueness assertion in the property of $\mathbf{V}(\mathfrak{g})$ applied to (1) and (2), we obtain

$$(\forall J \subseteq I \text{ in } \mathcal{I}(A))\ f'_I = q_{IJ} \circ f'_J. \tag{3}$$

Then, by the universal property of the limit A, we obtain a unique morphism of weakly complete algebras $f'\colon \mathbf{V}(\mathfrak{g}) \to A$ such that

$$(\forall I \in \mathcal{I}(A))\ q_I \circ f' = f'_I\colon\ \mathbf{V}(\mathfrak{g}) \to A/I$$

with the quotient morphism $q_I\colon A \to A/I$. Furthermore for all $I \in \mathcal{I}(A)$ we have $(q_J)_{\mathrm{Lie}} \circ (f')_{\mathrm{Lie}}|\mathfrak{g} = (f'_I)_{\mathrm{Lie}}|\mathfrak{g} = f_I|\mathfrak{g} = (q_I)_{\mathrm{Lie}} \circ f$, and since the q_I separate the points of A, we get $f'_{\mathrm{Lie}}|\mathfrak{g} = f$. ∎

As a left-adjoint functor, $\mathbf{U}$ preserves colimits. That does not exclude the possibility that it preserves certain limits. The following is a relevant example. Let $\mathfrak{g}$ be a profinite-dimensional Lie algebra. Let again denote $\mathcal{I}(\mathfrak{g})$ the filter basis of closed ideals I of $\mathfrak{g}$ such that $\mathfrak{g}/I$ is a finite-dimensional Lie algebra. Accordingly, the natural morphism $\mathfrak{g} \to \lim_{I\in\mathcal{I}(\mathfrak{g})} \mathfrak{g}/I$ is an isomorphism of profinite-dimensional Lie algebras. Let $p_I\colon \mathfrak{g} \to \mathfrak{g}/I$ and $p_{IJ}\colon \mathfrak{g}/J \to \mathfrak{g}/I$ for $J \subseteq I$ in $\mathcal{I}(\mathfrak{g})$ be the natural quotient morphisms in the complete category of weakly complete unital algebras $\mathcal{WA}$. Then $\{\mathbf{U}(\mathfrak{g}/K), \mathbf{U}(p_{I,J}) : I, J, K \in \mathcal{I}(\mathfrak{g}), I \subseteq J\}$ is an inverse system in $\mathcal{WA}$. Define $\mathbf{V}(\mathfrak{g}) = \lim_{J\in\mathcal{I}(\mathfrak{g})} \mathbf{U}(\mathfrak{g}/J)$ and let $\omega\colon \mathbf{U}(\mathfrak{g}) \to \mathbf{V}(\mathfrak{g})$ be the natural morphism attached to the limit in such a fashion that

$$\begin{array}{ccc}
\mathbf{U}(\mathfrak{g}) & \xrightarrow{\ \omega\ } & \mathbf{V}(\mathfrak{g}) \\
{\scriptstyle \mathbf{U}(p_J)}\big\downarrow & & \big\downarrow{\scriptstyle \pi_J} \\
\mathbf{U}(\mathfrak{g}/J) & \xrightarrow[\mathrm{id}]{} & \mathbf{U}(\mathfrak{g}/J)
\end{array}$$

is commutative for all $J\in\mathcal{I}(\mathfrak{g})$, where the $\pi_J\colon \mathbf{V}(\mathfrak{g})\to\mathbf{U}(\mathfrak{g}/J)$ are the limit maps satisfying $\pi_I=\mathbf{U}(p_{IJ})\circ\pi_J$ for $J\subseteq I$ in $\mathcal{I}(\mathfrak{g})$.

Theorem 1.10. ($\mathbf{U}$ preserves some projective limits) *The natural $\mathcal{WA}$-morphism $\omega_{\mathfrak{g}}\colon \mathbf{U}(\mathfrak{g})\to\mathbf{V}(\mathfrak{g})$ is an isomorphism. In short:*

For a profinite-dimensional Lie algebra $\mathfrak{g}$ with its filter basis $\mathcal{I}(\mathfrak{g})$ of cofinite-dimensional ideals I we have

$$\mathfrak{g}\cong\lim_{I\in\mathcal{I}(\mathfrak{g})}\mathfrak{g}/I \text{ in } \mathcal{WL} \text{ and } \mathbf{U}(\mathfrak{g})\cong\lim_{I\in\mathcal{I}(\mathfrak{g})}\mathbf{U}(\mathfrak{g}/I) \text{ in } \mathcal{WA}.$$

Proof. By Corollary 1.9 it suffices when we show that for each morphism $f\colon\mathfrak{g}\to F_{\mathrm{Lie}}$ of profinite-dimensional Lie groups for any finite-dimensional algebra F there is a unique morphism of weakly complete algebras $f'\colon\mathbf{V}(\mathfrak{g})\to F$ such that $f=(f')_{\mathrm{Lie}}\circ\omega_{\mathfrak{g}}\colon\mathfrak{g}\to F_{\mathrm{Lie}}$.

Now, since $\dim F<\infty$, there is an $I_0\in\mathcal{I}(\mathfrak{g})$ such that $I_0\subseteq\ker f$. Then the filterbasis $\mathcal{I}_0:=\{I\in\mathcal{I}(\mathfrak{g}) : I\subseteq I_0\}$ is cofinal in $\mathcal{I}$, so that the natural morphism

$$\lim_{I\in\mathcal{I}_0}\mathbf{U}(\mathfrak{g}/I)\xrightarrow{\alpha}\lim_{I\in\mathcal{I}(\mathfrak{g})}\mathbf{U}(\mathfrak{g}/I)=\mathbf{V}(\mathfrak{g})$$

is an isomorphism. We henceforth identify $\mathbf{V}(\mathfrak{g})$ and $\lim_{I\in\mathcal{I}_0}\mathbf{U}(\mathfrak{g}/I)$. Then for each $J\in\mathcal{I}_0$ let $\pi_J\colon\mathbf{V}(\mathfrak{g})=\lim_{I\in\mathcal{I}_0}\mathbf{U}(\mathfrak{g}/I)\to\mathbf{U}(\mathfrak{g}/J)$ denote the limit morphism in $\mathcal{WA}$. The $\mathcal{WL}$ morphism $f\colon\mathfrak{g}\to F_{\mathrm{Lie}}$ factors uniquely through the quotient $\mathfrak{g}/J$ as

$$f=(\mathfrak{g}\xrightarrow{\mathrm{quot}_J}\mathfrak{g}/J\xrightarrow{f_J}F_{\mathrm{Lie}})$$

for suitable $\mathcal{WL}$-morphisms $f_J\colon\mathfrak{g}/J\to F_{\mathrm{Lie}}$ By the universal property of $\mathbf{U}$ we find $\mathcal{WA}$-morphisms $f'_J\colon\mathbf{U}(\mathfrak{g}/J)\to F$ extending f_J. If $J_2\subseteq J_1$ inside $\mathcal{I}_0$ we have an $\mathcal{WL}$ morphism $p_{J_1J_2}\colon\mathfrak{g}/J_2\to\mathfrak{g}/J_1$ such that

$$p_{J_1}=p_{J_1J_2}\circ p_{J_2},$$

and since $\mathbf{U}$ is a functor, it responds with

$$\mathbf{U}(p_{J_1})=\mathbf{U}(p_{J_1J_2})\circ\mathbf{U}(p_{J_2})$$

in $\mathcal{WA}$. Now, by the universal property of the limit, there is a *unique* $\mathcal{WA}$-morphism $\lambda^*\colon\mathfrak{g}\to(\mathbf{V}(\mathfrak{g}))_{\mathrm{Lie}}=(\lim_{I\in\mathcal{I}_0}\mathbf{U}(\mathfrak{g}/J))_{\mathrm{Lie}}$ such that the following diagrams commute for all $J\in\mathcal{I}_0$:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\lambda^*"] \arrow[d, "p_J"'] & \mathbf{V}(\mathfrak{g})_{\mathrm{Lie}}, \arrow[d, "(\pi_J)_{\mathrm{Lie}}"] \\ \mathfrak{g}/J \arrow[r, "\lambda_{\mathfrak{g}/J}(=\mathrm{incl})"] & \mathbf{U}(\mathfrak{g}/J)_{\mathrm{Lie}}, \end{tikzcd} \qquad \begin{tikzcd} \mathbf{V}(\mathfrak{g}) \arrow[d, "\pi_J"] \\ \mathbf{U}(\mathfrak{g}/J). \end{tikzcd}$$

For $J\subseteq I$ in $\mathcal{I}$, we consider the following commutative diagram

$$\begin{tikzcd} \mathbf{V}(\mathfrak{g}) \arrow[r, "="] \arrow[d, "\pi_J"'] & \mathbf{V}(\mathfrak{g}) \arrow[d, "\pi_I"] \\ \mathbf{U}(\mathfrak{g}/J) \arrow[r, "\mathbf{U}(p_{IJ})"] \arrow[d, "f'_J"'] & \mathbf{U}(\mathfrak{g}/I) \arrow[d, "f'_I"] \\ F \arrow[r, "="'] & F \end{tikzcd}$$

and conclude that

$$(\forall J \subseteq I \text{ in } \mathcal{I})\, f'_J \circ \pi_J = f'_I \circ \pi_I.$$

Therefore we have a unique $\mathcal{WA}$-morphism $f'\colon \mathbf{V}(\mathfrak{g}) \to F$ such that

$$(\forall I \in \mathcal{I}_0)\, f' = (\mathbf{V}(\mathfrak{g}) \xrightarrow{\mathbf{U}(\pi_I)} \mathbf{U}(\mathfrak{g}/I) \xrightarrow{f'_J} F).$$

The commutative diagram

$$\begin{array}{ccccc}
\mathfrak{g} & \xrightarrow{\lambda^*} & (\mathbf{V}(\mathfrak{g}))_{\rm Lie} & & \mathbf{V}(\mathfrak{g}) \\
{\scriptstyle p_{J_0}}\downarrow & & \downarrow{\scriptstyle (\pi_{J_0})_{\rm Lie}} & & \downarrow{\scriptstyle \pi_{J_0}} \\
\mathfrak{g}/J_0 & \xrightarrow{(\lambda_{J_0})_{\rm Lie}} & (\mathbf{U}(\mathfrak{g}/J_0))_{\rm Lie} & & \mathbf{U}(\mathfrak{g}/J_0) \\
{\scriptstyle f_{J_0}}\downarrow & & \downarrow{\scriptstyle (f'_{J_0})_{\rm Lie}} & & \downarrow{\scriptstyle f'_{J_0}} \\
F_{\rm Lie} & \xrightarrow{=} & F_{\rm Lie} & & F
\end{array}$$

commutes where $f = f_{J_0} \circ p_{J_0}$ and $f' = f'_{J_0} \circ \mathbf{U}(\pi_{J_0})$ shows that $f = (f')_{\rm Lie} \circ \lambda^*$, that is f' extends f. ∎

2. The Abstract Enveloping Algebra $U(L)$ of a Lie Algebra L

We briefly recall that the functor which assigns to a unital $\mathbb{K}$-algebra X the underlying Lie algebra $X_{\rm Lie}$ (with the underlying $\mathbb{K}$ vector space of X as vector space structure endowed with the bracket operation $(x, y) \mapsto [x, y] := xy - yx$ as Lie bracket) has a left adjoint functor U which assigns to a Lie algebra L a unital associative algebra $U(L)$ and a natural Lie algebra morphism $\rho_L\colon L \to U(L)_{\rm Lie}$ such that for each Lie algebra morphim $f\colon L \to X_{\rm Lie}$ for a unital algebra X there is a unique morphism of unital algebras $f'\colon U(L) \to X$ such that $f = f'_{\rm Lie} \circ \rho_L$. The algebra $U(L)$ is called the *universal enveloping algebra of* L. A large body of text book literature is available on it. A prominent result is the Poincaré-Birkhoff-Witt Theorem on the structure of $U(L)$ which implies in particular that $\rho_L\colon L \to U(L)_{\rm Lie}$ is injective.

From the Theorem of Poincaré, Birkhoff and Witt it is known that ρ is injective. One may therefore assume that $L \subseteq U(L)$ such that ρ is the inclusion function. (See [1] or [3].) In this parlance the universal property reads as follows: *For each unital algebra A, each Lie algebra morphism $f\colon L \to A_{\rm Lie}$ extends uniquely to an algebra morphism $f'\colon U(L) \to A$.*

Also from the Theorem of Poincaré, Birkhoff and Witt we know that

$U(L)$ is the unital algebra generated by L, i.e., $U(L) = \langle L \rangle$.

The main result of the present section will be a complete clarification of the relation of the weakly complete enveloping algebra $\mathbf{U}(\mathfrak{g})$ of a profinite-dimensional Lie algebra $\mathfrak{g}$ and the universal enveloping algebra $U(|\mathfrak{g}|)$ of the (abstract) Lie algebra $|\mathfrak{g}|$ underlying $\mathfrak{g}$.

Lemma 2.1. *For a profinite-dimensional Lie algebra $\mathfrak{g}$ there is a natural morphism $\varepsilon_{\mathfrak{g}}\colon U(|\mathfrak{g}|) \to |\mathbf{U}(\mathfrak{g})|$ of unital algebras such that*

(i) *the following diagram is commutative:*

$$\begin{array}{ccc}
|\mathfrak{g}| & \xrightarrow{\text{incl}} & U(|\mathfrak{g}|) \\
{\scriptstyle \mathrm{id}_{|\mathfrak{g}|}}\downarrow & & \downarrow{\scriptstyle \varepsilon_{\mathfrak{g}}} \\
|\mathfrak{g}| & \xrightarrow[|\,\text{incl}\,|]{} & |\,\mathbf{U}(\mathfrak{g})|.
\end{array}$$

(ii) *The image of* $\varepsilon_{\mathfrak{g}}$ *is dense in* $\mathbf{U}(\mathfrak{g})$.

(iii) *The morphism* $\varepsilon_{\mathfrak{g}}$ *is injective if* $\mathfrak{g}$ *is finite-dimensional.*

Proof. (i) The claim is a direct consequence of the universal property of the functor U.

(ii) We have $\mathrm{im}(\varepsilon_{\mathfrak{g}}) = \varepsilon_{\mathfrak{g}}(U(|\mathfrak{g}|)) = \varepsilon_{\mathfrak{g}}(\langle|\mathfrak{g}|\rangle = \langle\varepsilon_{\mathfrak{g}}(\mathfrak{g})\rangle = \langle|\mathfrak{g}|\rangle$ in $|\,\mathbf{U}(\mathfrak{g})|$. From Corollary 1.8 we know that $\langle|\mathfrak{g}|\rangle$ is dense in $\mathbf{U}(\mathfrak{g})$.

(iii) If $\mathfrak{g}$ is finite-dimensional, then every finite-dimensional Lie algebra representation $\rho\colon \mathfrak{g} \to A_{\mathrm{Lie}} = \mathrm{End}(V)_{\mathrm{Lie}}$ for a finite-dimensional vector space V extends to an associative representation $\rho'\colon\ \mathbf{U}(\mathfrak{g}) \to \mathrm{End}(V)$. Then $\rho\circ\varepsilon_{\mathfrak{g}}\colon U(\mathfrak{g}) \to \mathrm{End}(V)$ is an extension to an associative representation of $U(\mathfrak{g})$ which is unique. By Harish-Chandra's Lemma (see Dixmier [3], 2.5.7), the extensions to representations to associative representations of $U(\mathfrak{g})$ of finite-dimensional Lie algebra representations of $\mathfrak{g}$ separate the points of $U(\mathfrak{g})$ and so the claim follows. ∎

The remainder of this section now is devoted to remove the restriction to finite-dimensionality in Lemma 2.2(iii). That is, we want to show

Lemma 2.2. *For a profinite-dimensional Lie algebra* $\mathfrak{g}$, *the algebra morphism* $\varepsilon_{\mathfrak{g}}\colon U(|\mathfrak{g}|) \to |\,\mathbf{U}(\mathfrak{g})|$ *is injective.*

The proof will occupy the remainder of this section. We shall resort to the existing literature on $U(L)$ such as [1] or [3]. We are given the profinite-dimensional Lie algebra $\mathfrak{g}$ and we write $L := |\mathfrak{g}|$ for the underlying Lie algebra. So $L \subseteq U(L)_{\mathrm{Lie}}$. For any basis B of the vector space L and any total order on B, the set of finite products $b_1 \cdots b_n$ for finite increasing sequences $b_1 \le \cdots \le b_n$ form a basis B^* of $U(L)$ by [1], Corollary 3 of Section 7 of §2.

Accordingly, if $u \in U(L)$ is an arbitrary nonzero element, there is a a finite subset $F \subseteq B$ such that there is are finite set $F_u \subseteq B^*$ for which each $b \in F_u$ is of the form $b_1 \cdots b_n$ with $b_j \in F$ for $j = 1, \ldots, n$, and that the following lemma applies:

Lemma 2.3. *There is a finite-dimensional vector space* H *of* L *and there is a closed Lie ideal* J *of* $\mathfrak{g}$ *such that*

(i) $u \in \operatorname{span} F_u$.

(ii) $F \subseteq H$.

(iii) $L = H \oplus J$

Proof. The vector space $V = \operatorname{span} F$ is finite-dimensional. Let C be the boundary of a compact 0-neighborhood in V. Then $V = \mathbb{K}{\cdot}C$. Since $\mathfrak{g}$ is profinite-dimensional, there is a filterbasis $\mathcal{I}$ of closed ideals I of $\mathfrak{g}$ such that $\dim \mathfrak{g}/I < \infty$ for $I \in \mathcal{I}$ such that $\mathfrak{g} \cong \lim_{I\in\mathcal{I}} \mathfrak{g}/I$. In particular, $\bigcap \mathcal{I} = \{0\}$. Therefore $\bigcap_{I\in\mathcal{I}}(C\cap I) = C\cap\{0\} = \emptyset$. Since C is compact, the filter basis $\{C\cap I : I \in \mathcal{I}\}$ with empty intersection consists of compact sets and therefore must contain the empty set. Thus there is a $J \in \mathcal{I}$ such that $C \cap J = \emptyset$. Suppose there were a nonzero $t \in \mathbb{K}$ and a $c \in C$ such that $t{\cdot}c \in J$, then $c = t^{-1}{\cdot}(t{\cdot}c) \in t^{-1}{\cdot}J = J$ which is impossible. Thus $V \cap J = \mathbb{K}{\cdot}C \cap J = \{0\}$. Since L/J is finite-dimensional there is some finite-dimensional vector subspace H of L such that $V \subseteq H$, $H + J = L$ and $H \cap J = \{0\}$. ∎

Now we choose a new totally ordered basis B_u of L as follows: From the algebraic theory of enveloping algebras we firstly derive the following background informaton: If J is a Lie ideal in L, then the kernel of the natural morphism $U(L) \to U(L/J)$ is the linear span of $U(L){\cdot}J$, that is, the $U(L)$-submodule generated in the $U(L)$-module $U(L)$ by the vector subspace J of $U(L)$. This ideal of $U(L)$ is commonly written $U(L)J$. (See [1], Proposition 3, Section 3 of §2, and [3], Proposition 2.2.14.)

Then we find a finite set F' that complements F to a totally ordered basis $F\dot\cup F'$ of H such that F precedes F'. Then choose a totally ordered basis F'' of J and define $B_u = F\dot\cup F'\dot\cup F''$ with the total order such that F' precedes F''.

For any increasing finite sequence S of elements $a_1 \le \cdots \le a_m$ from $F \cup F'$ we write $a_S = a_1 \cdots a_m$, and for the empty sequence S write $a_S = 1$ in $U(L)$. Similarly we define b_T for finite increasing sequences $b_1 \le \cdots \le b_n$ from F'' with $b_\emptyset = 1$. Then the set B_u^* of all $a_S b_T$, as S ranges through the set of finite increasing sequences of elements from $F \cup F'$ and T through the set of all increasing sequences of elements from X'', is a basis of $U(L)$.

Lemma 2.4. *Let $x = a_S b_T = x_1 \cdots x_p$ with $x_k \in B$ be a basis element in B_u. Then the following statements are equivalent:*

(i) $T \ne \emptyset$.

(ii) $x_p \in J$.

(iii) $x \in U(L)J$.

Proof. The equivalence of (i) and (ii) and the implication of (iii) from (ii) follow immediately from the definitions. So we have to show that (iii) implies (ii). By (iii), x is a linear combination of products yb with $y \in U(L)$ and $b \in J$. So y is a linear combination of basis elements $a_S b_T \in \mathbb{B}$. Thus x is a linear combination of elements $a_S b_T b$ with $b \in J$. We claim that $b_T b$ is a linear combination of basis elements $b_{T"}$. But any element of the form $a_S b_{T"}$ then is a basis element and so must necessarily agree with $a_S b_T = x$.

For a proof of our claim we now assume that $R = \{b_1, \ldots, b_n\}$ is a sequence of elements from B. For $m < n$ we call R to be (n,m)-*adjusted* if $(b_1 \le \cdots \le b_m \le b_{m+2} \le \cdots \le b_n$. It is to be understood that "$(n,0)$ adjusted" means"increasing" and that $(n-1,n)$-adjusted means that $\{b_1,\dots,b_{n-1}\}$ is increasing. Let R be (n,m) adjusted. If R is increasing, we are done. Otherwise $b_{m+1} < b_m$. Then we observe that $\{b_1,\dots,b_{m+1},b_m,\dots,b_n\}$ is increasing and that $b_R = b_1\cdots b_n = b_1\cdots(b_m b_{m+1})\cdots b_n = b_1\cdots(b_{m+1}b_m)\cdots b_n + b_1\cdots[b_m,b_{m+1}]\cdots b_n$. By hypothesis, J is a Lie subalgebra of $U(L)_{\rm Lie}$, and thus there are elements $b'_1,\cdots,b'_q\in B$ and scalars t_k, $k=1,\dots,q$ such that $[b_m,b_{m+1}] = t_1{\cdot}b'_1+\cdots+t_q{\cdot}b'_q$. The elements $b_1\cdots b_{m-1}b'_jb_m+2\cdots b_n$ are $(m-1,n-1)$ adjusted. Thus

(a) *for each (n,m)-adjusted R there exists, firstly, an $(n,m-1)$ adjusted sequence R' and there exist, secondly, $(n-1,m-1)$-adjusted sequences R_j, $j=1,\dots,q$ such that*

$$b_R = b_{R'} + \text{ a linear combination of the elements } b_{R_j}.$$

Let us order $\{(n,m) : n\in\mathbb{N}, m<n\}$ lexicographically. Then (a) implies

(b) *For each (n,m)-adjusted R with $m>0$ the element b_R is a linear combination of elements $b_{R'}$ with (n',m')-adjusted sequences R' with $(n',m')<(n,m)$.*

Now for an increasing T the element b_Tb with $b\in B$ is $(n,n-1)$-adjusted. Applying (b) recursively we find that b_Tb is a linear combination of elements $b_{T''}$ where the T'' is $(n',0)$ adjusted with $n'\le n$. Any such element $b_{T''}$ is basis element in B. This is what we had to show. ■

Lemma 2.5. $u\notin U(L)J$

Proof. By Lemma 2.3, the set $B_u^{(2)}$ of basis elements $a_Sb_T\in B_u$ which span $U(L)J$ is exactly those for which $b_T\ne 1$. Set $B_u^{(1)} = B_U\setminus B_u^{(2)}$ and $W=\operatorname{span} B_u^{(1)}$. Then $U(L)=W\oplus U(L)J$. But $u\in\operatorname{span}(F_u)\subseteq W$. The assertion follows. ■

Now let $|p_J|\colon L\to L/J\subseteq U(L/J)$ with $p_J\colon \mathfrak{g}\to\mathfrak{g}/J$ being the quotient map extending uniquely to an algebra morphism $U(|p_J|)\colon U(L)\to U(L/J)$ with kernel $U(L)J$. Then by Lemma 2.5 we know that $U(|p_J|)(u)$ is nonzero in $U(L/J)$ and from Lemma 2.1 we infer that $\varepsilon_{\mathfrak{g}/J}$ is injective. The commutative diagram

$$\begin{array}{ccccc}
|\mathfrak{g}| & \xrightarrow{\text{incl}} & U(|\mathfrak{g}|) & \xrightarrow{\varepsilon_{\mathfrak{g}}} & |\,\mathbf{U}(\mathfrak{g})| \\
{\scriptstyle |p_J|}\big\downarrow & & {\scriptstyle U(|p_J|)}\big\downarrow & & \big\downarrow{\scriptstyle |\,\mathbf{U}(p_J)|} \\
|\mathfrak{g}/J| & \xrightarrow{\text{incl}} & U(|\mathfrak{g}/J|) & \xrightarrow{\varepsilon_{\mathfrak{g}/J}} & |\,\mathbf{U}(\mathfrak{g}/J)|
\end{array}$$

then shows that $\varepsilon_{\mathfrak{g}}(u)\ne 0$. Therefore, since $u\in U(L)\setminus\{0\}$ was arbitrary, $\varepsilon_{\mathfrak{g}}$ is injective, leaving the elements of $L=|\mathfrak{g}|$ fixed. This completes the proof of Lemma 2.2. Thus $U(|\mathfrak{g}|)$ may be considered as a subalgebra of $|\,\mathbf{U}(\mathfrak{g})|$, containing $|\mathfrak{g}|\subseteq|\,\mathbf{U}(\mathfrak{g})|$.

This may be rephrased in the following Theorem which summarizes our efforts to elucidate the close relation between $U(|\mathfrak{g}|)$ and $\mathbf{U}(\mathfrak{g})$:

Theorem 2.6. (The Relation of $U(-)$ and $\mathbf{U}(-)$) *For any profinite-dimensional real or complex Lie algebra $\mathfrak{g}$ considered as a closed Lie subalgebra of $\mathbf{U}(\mathfrak{g})_{\rm Lie}$, the associative unital subalgebra $\langle\mathfrak{g}\rangle$ generated algebraically by $\mathfrak{g}$ in $\mathbf{U}(\mathfrak{g})$ is naturally isomorphic to $U(|\mathfrak{g}|)$ (under an isomorphism fixing the elements of $\mathfrak{g}$) and is dense in $\mathbf{U}(\mathbb{G})$.*

In a slightly careless sense we may memorize this as saying:
For a profinite-dimensional Lie algebra $\mathfrak{g}$ *we have*

$$(*)\qquad \mathfrak{g} \subseteq \langle \mathfrak{g} \rangle = U(\mathfrak{g}) \subseteq \overline{U(\mathfrak{g})} = \mathbf{U}(\mathfrak{g}).$$

3. The Weakly Complete Universal Enveloping Algebra as a Hopf Algebra

For some of the proofs we refer to [5].

Proposition 3.1. *The universal enveloping functor* $\mathbf{U}$ *is multiplicative, that is, there is a natural isomorphism* $\alpha_{\mathfrak{g}_1\mathfrak{g}_2}\colon \mathbf{U}(\mathfrak{g}_1 \times \mathfrak{g}_2) \to \mathbf{U}(\mathfrak{g}_1) \otimes_{\mathcal{W}} \mathbf{U}(\mathfrak{g}_2)$.

For a proof see [5], Proposition 6.3.

Lemma 3.2. *For any weakly complete unital algebra* A*, the vector space morphism* $\Delta_A\colon A \to A \otimes_{\mathcal{W}} A$*,* $\Delta_A(a) = a \otimes 1 + 1 \otimes a$ *is a morphism of weakly complete Lie algebras* $A_{\mathrm{Lie}} \to (A \otimes_W A)_{\mathrm{Lie}}$.

Cf. [5], Lemma 64.

Recall the natural morphism $\lambda_{\mathfrak{g}}\colon \mathfrak{g} \to \mathbf{U}(\mathfrak{g})_{\mathrm{Lie}}$ which we consider as an inclusion morphism. By Lemma 3.2,

$$p_{\mathfrak{g}} = \Delta_{U(\mathfrak{g})} \circ \lambda_{\mathfrak{g}}\colon \mathfrak{g} \to (\mathbf{U}(\mathfrak{g}) \otimes \mathbf{U}(\mathfrak{g}))_{\mathrm{Lie}}$$

is a morphism of weakly complete Lie algebras. The universal property of $\mathbf{U}$, $p_{\mathfrak{g}}$ yields a unique natural morphism of weakly complete associative unital algebras $\gamma_{\mathfrak{g}}\colon \mathbf{U}(\mathfrak{g}) \to \mathbf{U}(\mathfrak{g}) \otimes_{\mathcal{W}} \mathbf{U}(\mathfrak{g})$ such that $p_{\mathfrak{g}} = (\gamma_{\mathfrak{g}})_{\mathrm{Lie}} \circ \lambda_{\mathfrak{g}}$. Let $k_{\mathfrak{g}}\colon \mathfrak{g} \to \{0\}$ denote the constant morphism. Together with the identity $e_{\mathfrak{g}} : \{0\} to \mathfrak{g}$ we get the constant morphism $e_{\mathfrak{g}} \circ k_g\colon \mathfrak{g} \to \mathfrak{g}$. Now we have

Proposition 3.3. ($\mathbf{U}(\mathfrak{g})$ as a Hopf algebra) *Each weakly complete enveloping algebra* $\mathbf{U}(\mathfrak{g})$ *is a weakly complete symmetric Hopf algebra with the comultiplication* $\gamma_{\mathfrak{g}}$ *and the coidentity* $\mathbf{U}(k_{\mathfrak{g}})\colon \mathbf{U}(\mathfrak{g}) \to \mathbb{K}$.

See [5], Corollary 6.5.

The significance of the aspect expressed in this proposition is the fact that an essential portion of the noteworthy theory of weakly complete symmetric Hopf algebras has meanwhile entered the textbook literature as is exemplified by [7]. (See [7], Appendix A3, Appendix A7, Chapter 3–Part 3.)

Definition 3.4. Let A be a weakly complete symmetric Hopf algebra, i.e. a group object in the monoidal category $(\mathcal{W}, \otimes_W)$ of weakly complete vector spaces (see [7], Appendix 7 and Definition A3.62), with comultiplication $c\colon A \to A \otimes A$ and coidentity $k\colon A \to \mathbb{K}$.

An element $a \in A$ is called *grouplike* if $c(a) = a \otimes a$ and $k(a) = 1$. The subgroup of grouplike elements in the group of units A^{-1} will be denoted $\mathbb{G}(A)$.

An element $a \in A$ is called *primitive*, if $c(a) = a \otimes 1 + 1 \otimes a$. The Lie algebra of primitive elements of A_{Lie} will be denoted $\mathbb{P}(A)$.

Proposition 3.5. *Any weakly complete unital algebra A has an everywhere defined exponential function $\exp\colon A \to A^{-1}$ into the pro-Lie group A^{-1} of invertible elements defined as $\exp x = 1+x+\frac{1}{2!}\cdot x^2+\frac{1}{3!}\cdot x^3+\cdots$. As a function $\exp\colon A_{\mathrm{Lie}} \to A^{-1}$ it is the exponential function of the pro-Lie group A^{-1} in the sense of pro-Lie groups.*

If A is in fact a weakly complete symmeric Hopf algebra then the set $\mathbb{G}(A)$ of grouplike elements is a closed pro-lie subgroup of the pro-Lie group A^{-1}, and the set $\mathbb{P}(A)$ of primitive elements is a closed Lie subalgebra of the pro-finite dimensional Lie algebra A_{Lie} and $\exp(\mathbb{P}(A)) \subseteq \mathbb{G}(A)$ in such a fashion that the restriction and corestriction of exp *is the exponential function* $\exp_{\mathbb{G}(A)} : \mathbb{P}(A) \to \mathbb{G}(A)$ *of the pro-Lie group $\mathbb{G}(A)$.*

(See e.g. [2], [7], [8].)

Theorem 3.6. (The Weakly Complete Enveloping Hopf Algebra) *Let $\mathfrak{g}$ be a weakly complete Lie algebra and $\mathbf{U}(\mathfrak{g})$ its weakly complete enveloping algebra comntaining $\mathfrak{g}$ according to* Theorem 2.6. *Then the following statements hold:*

(i) *The weakly complete algebra $\mathbf{U}(\mathfrak{g})$ is a strict projective limit of finite-dimensional associative unital algebras and the group of units $\mathbf{U}(\mathfrak{g})^{-1}$ is dense in $\mathbf{U}(\mathfrak{g})$. It is an almost connected pro-Lie group, connected in the case of $\mathbb{K} = \mathbb{C}$. The algebra $\mathbf{U}(\mathfrak{g})$ has an exponential function* $\exp\colon \mathbf{U}(\mathfrak{g})_{\mathrm{Lie}} \to \mathbf{U}(\mathfrak{g})^{-1}$, *The Lie algebra $\mathfrak{L}(\mathbf{U}(\mathfrak{g}))$ of $\mathbf{U}(\mathfrak{g})^{-1}$ is $\mathbf{U}(\mathfrak{g}))_{\mathrm{Lie}}$.*

(ii) *The pro-Lie algebra $\mathbb{P}(\mathbf{U}(\mathfrak{g}))$ is the Lie algebra of the pro-Lie group $\mathbb{G}(\mathbf{U}(\mathfrak{g}))$ of grouplike elements.*

(iii) *The pro-Lie algebra $\mathbb{P}(\mathbf{U}(\mathfrak{g}))$ of primitive elements of $\mathbf{U}(\mathfrak{g})$ contains $\mathfrak{g}$.*

(See [5] Theorem 3.4. Cf. also [7], Theorem A3.102 and its proof for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.)

Corollary 3.7. *The weakly complete enveloping algebra $\mathbf{U}(\mathfrak{g})$ of a nondegenerate profinite-dimensional weakly complete Lie algebra $\mathfrak{g}$ has nontrivial grouplike elements.*

The universal enveloping Hopf algebra $U(L)$ of a Lie algebra L contains no nontrivial grouplike elements.
There remains the question under which circumstances we then have in fact $\mathfrak{g} = \mathbb{P}(\mathbf{U}(\mathfrak{g}))$. In the classical setting of the discrete enveloping Hopf algebra in characteristic 0 this is the case: see e.g. [9], Theorem 5.4 on p. LA 3.10. In the next section we shall see that this is not the case in general.

4. The Abelian Case

In [5] the weakly complete group algebras were lucidly illustrated in the abelian case due to Pontryagin duality. We propose that a similar procedure helps illustrating the construction of a quotient algebra $\mathbf{U}(\mathfrak{g})$ which is defined by a weaker universal property if the Lie algebra $\mathfrak{g}$ is abelian. In that case, in reality, $\mathfrak{g}$ is just a weakly complete vector space.

Definition 4.1. We shall call a complex weakly complete abelian unital $\mathbb{K}$-algebra *semisimple* if the set $\mathrm{Spec}(A) = \mathrm{Alg}_{\mathbb{K}}(A, \mathbb{C})$ of morphisms $A \to \mathbb{C}$ of weakly complete unital $\mathbb{K}$-algebras separates the points.

Note that the $\mathbb{R}$-algebra $A = \mathbb{C}$ fails to be semisimple while the $\mathbb{C}$-algebra $A = \mathbb{C}$ is semisimple. Therefore we shall use due caution in the use of this concept of semisimplicity used here.

Typically, any power $\mathbb{K}^J$ for any set J is a weakly complete semisimple K-algebra. Now we let $\mathfrak{g}$ be a weakly complete $\mathbb{K}$-vector space and discuss what will appear to be the weakly complete symmetric $\mathbb{K}$-Hopf algebra $\mathbb{K}^{\mathfrak{g}'}$, where $\mathfrak{g}' = \mathcal{W}(\mathfrak{g}, \mathbb{K})$ is the $\mathbb{K}$-dual of $\mathfrak{g}$. We shall write ω for the elements of $\mathfrak{g}'$. The pairing between $\mathfrak{g}'$ and $\mathfrak{g}$ we write $\langle \omega, x\rangle = \omega(x)$. By the duality of weakly complete vector spaces (see [7],Theorem A7.9) we may identify canonically $\mathfrak{g}$ and the algebraic dual $\mathfrak{g}'^*$ of the $\mathbb{K}$-vector space $\mathfrak{g}'$:

$$(*) \qquad \mathfrak{g} \cong \mathfrak{g}'^* = \mathcal{V}_{\mathbb{K}}(\mathfrak{g}', \mathbb{K}).$$

For describing a comultiplication $\gamma_{\mathfrak{g}}\colon \mathbb{K}^{\mathfrak{g}'} \to \mathbb{K}^{\mathfrak{g}'} \otimes_{\mathcal{W}} \mathbb{K}^{\mathfrak{g}'}$ we identify $\mathbb{K}^{\mathfrak{g}'} \otimes_{\mathcal{W}} \mathbb{K}^{\mathfrak{g}'}$ with $\mathbb{K}^{\mathfrak{g}' \times \mathfrak{g}'}$ according to Proposition 3.1. Then $\gamma_{\mathfrak{g}}\colon \mathbb{K}^{\mathfrak{g}'} \to \mathbb{K}^{\mathfrak{g}' \times \mathfrak{g}'}$ is given as follows

$$(\forall \varphi \in \mathbb{K}^{\mathfrak{g}'},\, \omega_1, \omega_2 \in \mathfrak{g}')\, \gamma_{\mathfrak{g}}(\varphi)(\omega_1, \omega_2) = \varphi(\omega_1 + \omega_2).$$

The function $k_{\mathfrak{g}}\colon \mathbb{K}^{\mathfrak{g}'} \to \mathbb{K}$, $k_{\mathfrak{g}}\varphi = \varphi(0)$ is a coidentity, and the function $\sigma_{\mathfrak{g}}\colon \mathbb{K}^{\mathfrak{g}'} \to \mathbb{K}^{\mathfrak{g}'}$, $\sigma_{\mathfrak{g}}(\varphi)(\omega) = -\omega$ a symmetry. The function const: $\mathbb{K}^{\mathfrak{g}'} \to \mathbb{K}^{\mathfrak{g}'}$ is the constant morphism given by $\mathrm{const}\,\varphi(\omega) = 0$ for all $\omega \in \mathfrak{g}'$.

Lemma 4.2. *The semisimple abelian weakly complete unital algebra $\mathbf{V}(\mathfrak{g}) := \mathbb{K}^{\mathfrak{g}'}$ is a symmetric $\mathbb{K}$-Hopf algebra in $\mathcal{W}$ with respect to the comultiplication $\gamma_{\mathfrak{g}}$, the coidentity $k_{\mathfrak{g}}$ and the symmetry $\sigma_{\mathfrak{g}}$.*

Proof. The assignment $\mathfrak{g} \mapsto \mathbf{V}(\mathfrak{g}) : \mathcal{W} \to \mathcal{WA}$ is clearly a functor $\mathbf{V}$ from the category of weakly complete vector spaces to the category of weakly complete commutative unital algebras. The function $\gamma_{\mathfrak{g}}\colon \mathbb{K}^{\mathfrak{g}'} \to \mathbb{K}(\mathfrak{g}') \otimes_{\mathcal{W}} \mathbb{K}(\mathfrak{g}')$ is, up to natural isomorphism, the result of applying $\mathcal{V}$ to the diagonal morphism $x \mapsto (x, x) : \mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$ and is, therefore, a morphism of $\mathcal{WA}$. The coassociativity of the diagonal morphism implies the coassociativity of $\gamma_{\mathfrak{g}}$. Applying $\mathbf{V}$ to the $\mathfrak{g} \to \{0\}$ yields the coidentity $k_{\mathfrak{g}}\colon \mathbf{V}(\mathfrak{g}) \to \mathbb{K}$. Finally, applying $\mathbf{V}$ to the inversion $x \mapsto -x$ of $\mathcal{W}$ yields the symmetry $\sigma_{\mathfrak{g}}\colon \mathbf{V}(g) \to \mathbf{V}(g)$. All in all we obtain the desired commutative diagram

$$\begin{array}{ccc}
\mathbf{V}(\mathfrak{g}) \otimes_{\mathcal{W}} \mathbf{V}(\mathfrak{g}) & \xrightarrow[\sigma_{\mathfrak{g}} \otimes_{\mathcal{W}} \mathrm{id}]{} & \mathbf{V}(\mathfrak{g}) \otimes_{\mathcal{W}} \mathbf{V}(\mathfrak{g}) \\
\gamma_{\mathfrak{g}} \big\uparrow & & \big\downarrow \mathrm{mult} \\
\mathbf{V}(\mathfrak{g}) & \xrightarrow[\mathrm{const}]{} & \mathbf{V}(\mathfrak{g}).
\end{array}$$

■

Lemma 4.3. *There is a natural $\mathcal{W}$-embedding $\nu_{\mathfrak{g}} \colon \mathfrak{g} \to \mathbf{V}(\mathfrak{g})$ such that for any $\mathcal{W}$-morphism $f \colon \mathfrak{g} \to A$ into the underlying vector space of a semisimple abelian weakly complete $\mathbb{K}$-algebra A there is a unique morphism $f^{\#} \colon \mathbf{V}(\mathfrak{g}) \to A$ such that $f = f^{\#} \circ \nu_{\mathfrak{g}}$.*

Proof. For all $x \in \mathfrak{g}$ and all $\omega \in \mathfrak{g}'$ we define $\nu_{\mathfrak{g}}(x)(\omega) = \langle \omega, x \rangle$. Because of $\mathfrak{g}'^* \cong \mathfrak{g}$ we know that $\nu_{\mathfrak{g}}$ is an isomorphism onto its image.

Now let $f \colon \mathfrak{g} \to A$ be a $\mathcal{W}$-morphism into a weakly complete semisimple abelian algebra. We must show that there is a $\mathcal{WA}$-morphism $f^{\#} \colon \mathbf{V}(\mathfrak{g}) \to A$ such that $f = f^{\#} \circ \nu_{\mathfrak{g}}$. We now apply the method of the proof of Proposition 1.9 to conclude that without loss of generality we may assume that A is finite dimensional. But then, since A is semisimple, it is a product of simple subalgebras each one of which is isomorphic to $\mathbb{K}$. Then, by the universal property of a product, we may assume that $A = \mathbb{K}$. Then $f \in \mathfrak{g}'$. Then we have $f^{\#} = \mathrm{pr}_f \colon \mathbb{K}^{g'} \to \mathbb{K}$: Indeed, for $x \in \mathfrak{g}'$ we have $(f^{\#} \circ \nu_{\mathfrak{g}})(x) = \mathrm{pr}_f(\nu_g(x)) = \nu_{\mathfrak{g}}(x)(f) = f(x)$, and the uniqueness of $f^{\#} = \mathrm{pr}_f$ is clear from this calculation. ■

Proposition 4.4. (Semisimple Weakly Complete *Abelian* Universal Enveloping Algebras) *The semisimple weakly complete symmetric $\mathbb{K}$-Hopf-algebra $\mathbf{V}(\mathfrak{g}) = \mathbb{K}^{\mathfrak{g}'}$ over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ has the following properties*

(i) *The addition and multiplication in the $\mathbb{K}$-algebra $\mathbf{V}(\mathfrak{g})$ is clculated pointwise. The exponential function of $\mathbf{V}(\mathfrak{g}) = \mathbb{K}^{\mathfrak{g}'}$ is computed componentwise.*

(ii) *An element $\varphi \in \mathbf{V}(\mathfrak{g})$ is primitive, that is, satisfies $\gamma_{\mathfrak{g}}(\varphi) = \varphi \otimes 1 + 1 \otimes \varphi$ iff*

$$(\forall \omega_1, \omega_2 \in \mathfrak{g})\, \varphi(\omega_1 + \omega_2) = \varphi(\omega_1) + \varphi(\omega_2).$$

(iii) *The weakly complete vector space $\mathbb{P}(\mathbf{V}(\mathfrak{g})) = \mathcal{AB}(\mathfrak{g}', (\mathbb{K}, +))$ is a closed $\mathbb{K}$-vector subspace of $\mathbb{K}^{\mathfrak{g}'}$ where $\mathcal{AB}$ is the category of abelian groups.*

(iv) *For $\varphi \in \mathbb{K}^{\mathfrak{g}'}$ the element $\varphi \otimes \varphi \in \mathbb{K}^{\mathfrak{g}' \times \mathfrak{g}'}$ is given by $(\varphi \otimes \varphi)(\omega_1, \omega_2) = \varphi(\omega_1)\varphi(\omega_2)$.*

Accordingly, a $\varphi \in \mathbf{V}(\mathfrak{g})$ is grouplike, that is, satisfies $\gamma_{\mathfrak{g}}(\varphi) = \varphi \otimes \varphi$ iff

$$(\forall \omega_1, \omega_2 \in \mathfrak{g}),\ \varphi(\omega_1 + \omega_2) = \varphi(\omega_1)\varphi(\omega_2).$$

(v) *$\mathbb{G}(\mathbf{V}(\mathfrak{g})) = \mathcal{AB}(\mathfrak{g}', \mathbb{K}^{\times})$ with $\mathbb{K}^{\times} = (\mathbb{K} \setminus \{0\}, \cdot)$, and this pro-Lie group is a closed subgroup of $(\mathbb{K}^{\mathfrak{g}'})^{-1} = (\mathbb{K}^{\times})^{\mathfrak{g}'}$. Its Lie algebra is $\mathbb{P}(\mathbf{V}(\mathfrak{g}))$. If we abbreviate $\mathbb{G}(\mathbf{V}(\mathfrak{g}))$ by G, then the exponential function of G is $\exp_G \colon \mathfrak{L}(G) = \mathcal{AB}(\mathfrak{g}', (\mathbb{K}, +)) \to \mathcal{AB}(\mathfrak{g}', \mathbb{K}^{\times}) = G$ and is calculated componentwise.*

Proof. (i) We know that the bidual $\mathfrak{g}'^*$ is naturally isomorphic to $\mathfrak{g}$, and this bidual $\mathfrak{g}'^* = \mathcal{V}_{\mathbb{K}}(\mathfrak{g}', \mathbb{K})$ is obviously a vector subspace of $\mathbb{K}^{\mathfrak{g}'}$. By Lemma 4.3 the morphism $\nu_{\mathfrak{g}}$ has the isomorphism $\mathfrak{g} \to \mathfrak{g}'^*$ as corestriction to the image. Thus $\nu_{\mathfrak{g}}(x)(\omega) = \langle \omega, x\rangle$ for $x \in \mathfrak{g}$ and $\omega \in \mathfrak{g}'$. If view of the definition, the assertions in (i) are clear.

(ii) First we note, observing the isomorphism $\mathbb{K}^{\mathfrak{g}'} \otimes_{\mathcal{W}} \mathbb{K}^{\mathfrak{g}'} \cong \mathbb{K}^{\mathfrak{g}' \times \mathfrak{g}'}$, that for $\varphi \in \mathbb{K}^{\mathfrak{g}'}$ and $\omega_1, \omega_2 \in \mathfrak{g}'$ we have the following properties:

(a) $(\varphi \otimes 1)(\omega_1, \omega_2) = \varphi(\omega_1)$,

(b) $(1 \otimes \varphi)(\omega_1, \omega_2) = \varphi(\omega_2)$

(c) $(\varphi \otimes 1 + 1 \otimes \varphi)(\omega_1, \omega_2) = \varphi(\omega_1) + \varphi(\omega_2)$.

(d) $(\varphi \otimes \varphi)(\omega_1, \omega_2) = \varphi(\omega_1)\varphi(\omega_2)$.

(e) $(\forall x \in \mathfrak{g})\, (\nu_{\mathfrak{g}}(x) \otimes 1)(\omega_1, \omega_2) = \langle \omega_1, x\rangle$,

(f) $(\forall x \in \mathfrak{g})\, (1 \otimes \nu_{\mathfrak{g}}(x))(\omega_1, \omega_2) = \langle \omega_2, x\rangle$,

(g) $(\forall x \in \mathfrak{g})\, (1 \otimes \nu_{\mathfrak{g}}(x) + \nu_{\mathfrak{g}}(x) \otimes 1)(\omega_1, \omega_2) = \langle \omega_1 + \omega_2, x\rangle$.

Now, the comultiplication $\gamma_{\mathfrak{g}} \colon \mathbb{K}^{\mathfrak{g}'} \to \mathbb{K}^{\mathfrak{g}' \times \mathfrak{g}'}$ is the unique algebra morphism such that $\gamma_{\mathfrak{g}} \circ \nu_{\mathfrak{g}} \colon \mathfrak{g} \to \mathbb{K}^{\mathfrak{g}' \times \mathfrak{g}'}$ is precisely $x \mapsto \nu_{\mathfrak{g}}(x) \otimes 1 + 1 \otimes \nu_g(x)$, i.e., for $\omega_1, \omega_2 \in \mathfrak{g}'$ we have

$$(\nu_{\mathfrak{g}}(x) \otimes 1 + 1 \otimes \nu_{\mathfrak{g}}(x))(\omega_1, \omega_2) = \nu_{\mathfrak{g}}(x)(\omega_1 + \omega_2) = \langle \omega_1 + \omega_2, x\rangle.$$

This uniquely defines the algebra morphism $\gamma_{\mathfrak{g}} \colon \mathbb{K}^{\mathfrak{g}'} \to \mathbb{K}^{\mathfrak{g}' \times \mathfrak{g}'}$ via $\gamma_{\mathfrak{g}}(\varphi)(\omega_1, \omega_2) = \varphi(\omega_1 + \omega_2)$.

(iii) follows from (ii)

(iv) In view of the definition of $\mathbb{P}(\mathbb{K}^{\mathfrak{g}'})$, statement (iv) is statement (iii) reformulated.

(v) In (d) above we observed $(\varphi \otimes \varphi)(\omega_1, \omega_2) = \varphi(\omega_1)\varphi(\omega_2)$. Then (v) follows from (ii).

(vi) The first assertion is a reformulation of (v). For the second one we refer to Proposition 3.5 and our item (i) where we observed that the exponential function is calculated pointwise in $\mathbb{K}^{\mathfrak{g}'}$. ■

Corollary 4.5. *If $\mathfrak{g}$ is a nonzero weakly complete vector space then we have a commutative diagram*

$$\begin{array}{ccc}
\mathbb{P}(\mathbf{V}(\mathfrak{g})) & \xrightarrow{\cong} & \mathcal{AB}(\mathfrak{g}', (\mathbb{K}, +)) \\
{\scriptstyle\text{inc}}\uparrow & & \uparrow{\scriptstyle\text{inc}} \\
\mathfrak{g} & \xrightarrow{\cong} & \mathcal{V}(\mathfrak{g}', \mathbb{K}).
\end{array}$$

The inclusion $\mathfrak{g} \xrightarrow{\text{inc}} \mathbb{P}(\mathbf{V}(\mathfrak{g}))$ is proper.

Proof. The commuting diagram is a consequence of Proposition 4.4. The inclusion of $\mathcal{V}(\mathfrak{g}', \mathbb{K})$, the vector space of $\mathbb{K}$-linear functions $\mathfrak{g}' \to \mathbb{K}$ is strictly smaller than the abelian group of abelian group homomorphisms $\mathfrak{g} \to (\mathbb{K}, +)$. ∎

Exercise 4.6. The torsion free ranks of the abelian groups underlying $\mathfrak{g}$ and $\mathcal{AB}(\mathfrak{g}', (\mathbb{K}, +))$ agree.

The class of weakly complete abelian $\mathbb{K}$-Hopf algebras $\mathbf{V}(\mathfrak{g})$ with generating vector subspace $\mathfrak{g}$ introduced in this section supplies us with readily accessible examples for which the natural inclusion $\mathfrak{g} \mapsto \mathbb{P}(\mathbf{V}(\mathfrak{g}))$ is proper. This fact has consequences for the group $\mathbb{G}(\mathbf{V}(\mathfrak{g}))$ of group like elements. It is now important to illustrate that the Hopf algebras $\mathbf{V}(\mathfrak{g})$ are not completely satisfactory in their universal properties in as much as they are short off the *universal* Hopf algebras $\mathbf{U}(\mathfrak{g})$.

Example 4.7. We let $A_2 = \mathbb{K}^2$ with componentwise addition and scalar multiplication and a multiplication defined by

$$(\forall x_1, x_2, y_1, y_2 \in \mathbb{K})\, (x_1, y_1)(x_2, y_2) = (y_1 x_2 + y_2 x_1, y_1 y_2). \tag{1}$$

Then A_2 is a 2-dimensional abelian algebra with identity $\mathbf{1} = (0, 1)$ and the following properties:

(i) There is a faithful representation

$$(x, y) \mapsto \begin{pmatrix} y & x \\ 0 & y \end{pmatrix} : A_2 \to M_2(\mathbb{K}) \tag{2}$$

by 2×2 matrices.

(ii) The group of units A_2^{-1} is $(\mathbb{K}^\times)^2$ with $\mathbb{K}^\times = \mathbb{K} \setminus \{0\}$.

(iii) An element $c = (x, y)$ satisfies $c^2 = 0$ iff $y = 0$. The radical $R(A_2)$ of A_2 is $\mathbb{K} \times \{0\}$.

(iv) For each morphism $F\colon \mathbf{V}(\mathfrak{g}) = \mathbb{K}^{\mathfrak{g}'} \to A_2$ of weakly complete unital algebras, $F^{-1}(R(A)) \subseteq \ker F$.

Proof. Statements (i), (ii) and (iii) are straightforward. Proof of (iv): Let $\varphi\colon \mathfrak{g}' \to \mathbb{K}$ be an element of $\mathbb{K}^{\mathfrak{g}'}$ such that $F(\varphi) \in R(A)$. Then $F(\varphi^2) = F(\varphi)^2 = 0$ by (iii). Thus $\varphi^2 \in \ker F$. Any ideal of $\mathbb{K}^{\mathfrak{g}'}$ is a partial product $\cong \mathbb{K}^X$ with $X \subseteq \mathfrak{g}'$. But then $\varphi^2 \in \mathbb{K}^X$ implies $\varphi \in \mathbb{K}^X$, and so $\varphi \in \ker F$. ∎

This example serves the purpose of showing that $\mathfrak{g} \mapsto \mathbf{V}(\mathfrak{g})$ is not suitable to serve as universal enveloping algebra functor in the abelian case. Indeed, let $\mathfrak{g} = R(A)$, a 1-dimensional $\mathbb{K}$-vector space and $\mathfrak{g} \to \mathfrak{g}'^* \subseteq \mathbb{K}^{\mathfrak{g}'}$ giving the embedding $\nu_{\mathfrak{g}}\colon \mathfrak{g} \to \mathbb{K}^{\mathfrak{g}'}$. Suppose that $F\colon K^{\mathfrak{g}'} \to A_2$ is a morphism of weakly complete unital algebras such that $F \circ \nu_{\mathfrak{g}} : \mathfrak{g} \to A$ is the inclusion. Then $F(\mathfrak{g}'^*) = R(A)$. By Example 4.7 (iv) this implies $F(\mathfrak{g}'^*) = \{0\}$. Thus $R(A) = \{0\}$, which is a contradiction that shows that F cannot exist as supposed.

4.1. The relation of $\mathbf{U}(\mathfrak{g})$ and $\mathbf{V}(\mathfrak{g})$ However, if we identify $\nu_{\mathfrak{g}}(\mathfrak{g}) \subseteq \mathbf{V}(\mathfrak{g})$ with $\mathfrak{g}$ we may assume $\mathfrak{g} \subseteq \mathbf{V}(\mathfrak{g})$ just as we assumed that $\mathfrak{g} \subseteq \mathbf{U}(\mathfrak{g})$. So for each weakly complete vector space $\mathfrak{g}$, the universal property of $\mathbf{U}(\mathfrak{g})$ according to Theorem 1.3 yields a unique $\mathcal{WA}$-morphism $q_{\mathfrak{g}}\colon \mathbf{U}(\mathfrak{g}) \to \mathbf{V}(\mathfrak{g})$ inducing the identity map on $\mathfrak{g}$ as contained in both $\mathbf{U}(\mathfrak{g})$ and $\mathbf{V}(\mathfrak{g})$.

Proposition 4.8. *The function*

$$(*) \qquad q_{\mathfrak{g}}\colon \mathbf{U}(\mathfrak{g}) \to \mathbf{V}(\mathfrak{g})$$

is a surjective morphism of weakly complete symmetric Hopf-algebras over $\mathbb{K}$ whose kernel is the radical $R(\mathbf{U}(\mathfrak{g}))$ of $\mathbf{U}(\mathfrak{g})$.

Proof. The Corollary 1.8 shows that the image of $q_{\mathfrak{g}}$ is dense in $\mathbf{V}(\mathfrak{g})$. Since the images of morphisms of weakly complete vector spaces are closed (see [7], Theorem A7.12(b)), the morphism $q_{\mathfrak{g}}$ is surjective.

Since $\mathcal{V}(\mathfrak{g}) = \mathbb{K}^{\mathfrak{g}'}$ is semisimple, the radical $R := R(\mathbf{U}(\mathfrak{g}))$ is contained in $\ker q_{\mathfrak{g}}$. Conversely, every $\mathcal{W}$-morphism $f\colon \mathfrak{g} \to A$ into a semisimple commutative $\mathcal{WA}$-algebra extends uniquely to a $\mathcal{WA}$-morphism $f'\colon \mathbf{U}(\mathfrak{g}) \to A$ by Proposition 1.7, which, by semisimplicity of A, vanishes on the radical R and thus factors through a $\mathcal{WA}$-morphism $f^{\#}\colon \mathbf{U}(\mathfrak{g})/R \to A$. Thus the quotient $\mathbf{U}(\mathfrak{g})/R$ containing the isomorphic copy $(\mathfrak{g}+R)/R$ of $\mathfrak{g}$ shares the universal property of $\mathbf{V}(\mathfrak{g})$ of Lemma 4.3. This implies that $\ker q_{\mathfrak{g}} = R$. ∎

While it is clear that $q_{\mathfrak{g}}(\mathbb{P}(\mathbf{U}(\mathfrak{g}))) \subseteq \mathbb{P}(\mathbf{V}(\mathfrak{g}))$, it is an open question, whether equality holds.

Problem 1. Does the quotient morphism $q_{\mathfrak{g}}\colon \mathbf{U}(\mathfrak{g}) \to \mathbf{V}(\mathfrak{g})$ induce a quotient morphism $\mathbb{P}(\mathbf{U}(\mathfrak{g})) \to \mathbb{P}(\mathbf{V}(\mathfrak{g}))$ of profinite dimensional Lie algebras for abelian Lie algebras $\mathfrak{g}$?

A positive answer to this Problem would yield a negative answer to the question whether $\mathfrak{g} = \mathbb{P}(\mathbf{U}(\mathfrak{g}))$ for all profinite-dimensional Lie algebras $\mathfrak{g}$.

Note in passing that the Example 4.7 above is just the simplest occurrence of the construction of a weakly complete unital $\mathbb{K}$-algebra $\mathfrak{g} = \mathfrak{v} \times \mathbb{K}$ for an arbitrary weakly complete vector space $\mathfrak{v}$, componentwise addition and scalar multiplication, and the multiplication $(x,s)(y,t) = (t{\cdot}x + s{\cdot}y, st)$, yielding the radical $\mathfrak{v} \times \{0\}$ with 0-multipliction.

5. Enveloping Algebras Versus Group Algebras

Enveloping algbras and group algebras share the common festure that both are symmetric Hopf algebras. We point out a concrete relationship. Recall that for a compact group we naturally identify G with the group of grouplike elements of $\mathbb{R}[G]$ (cf. [2], Theorems 8.7, 8.9 and 8.12), and that $\mathfrak{L}(G)$ may be identified with the pro-Lie algebra $\mathbb{P}(\mathbb{R}[G])$ of primitive elements. (Cf. also Theorem 3.5 above.) We may also assume that $\mathfrak{L}(G)$ is contained the set $\mathbb{P}(\mathbf{U}(\mathfrak{L}(G)))$ of primitive elements of $\mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G))$.

Theorem 5.1. (i) *Let G be a compact group. Then there is a natural morphism of weakly complete algebras $\omega_G\colon \mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G)) \to \mathbb{R}[G]$ fixing the elements of $\mathfrak{L}(G)$ elementwise.*

(ii) *The image of ω_G is the closed subalgebra $\mathbb{R}[G_0]$ of $\mathbb{R}[G]$.*

(iii) *The pro-Lie group $\mathbb{G}(\mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G)))$ is mapped onto $G_0 = \mathbb{G}(\mathbb{R}[G_0]) \subseteq \mathbb{R}[G]$. The connected pro-Lie group $\mathbb{G}(\mathbf{U}_R(\mathfrak{L}(G)))_0$ maps surjectively onto G_0 and $\mathbb{P}(\mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G)))$ onto $\mathbb{P}(\mathbb{R}[G])$.*

Proof. (i) follows at once from the universal property of $\mathbf{U}$.

(ii) As a morphism of weakly complete Hopf algebras, ω_G has a closed image which is generated as a weakly complete subalgebra by $\mathfrak{L}(G)$ which is $\mathbb{R}[G_0]$ by Corollary 3.3 (ii) of [5].

(iii) The morphism ω_G of weakly complete Hopf algebras maps grouplike elements to grouplike elements, whence we have the commutative diagram

$$
\begin{array}{ccc}
\mathfrak{L}(G) \subseteq \mathbb{P}(\mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G))) & \xrightarrow{\mathbb{P}(\omega_G)} & \mathbb{P}(\mathbb{R}(G)) = \mathfrak{L}(G)\\
{\scriptstyle \exp_{\mathbb{G}(\mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G)))}}\Big\downarrow & & \Big\downarrow{\scriptstyle \exp_G}\\
\mathbb{G}(\mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G))) & \xrightarrow[\mathbb{G}(\omega_G)]{} & \mathbb{G}(\mathbb{R}[G]) = G.
\end{array}
$$

Since $\mathbb{P}(\omega)$ is a retraction and the image of $\exp_G$ topologically generates G_0, the image of $\mathbb{G}(\omega_G) \circ \exp_{\mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G))}$ topologically generates G_0. Since the image of the exponential function of the pro-Lie group $\mathbb{G}(\mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G)))$ generates topologically its identity component, $\mathbb{G}(\omega_G)$ maps this identity component onto G_0.

Since $\mathfrak{L}(G) \subseteq \mathbb{P}(\mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G))$, and since also any morphism of Hopf algebras maps a primitive element onto a primitive element we know $\omega_G(\mathbb{P}(\mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G)))) = \mathbb{P}(\mathbb{R}[G])$. ∎

Problem 2. Is $\mathbb{G}(\mathbf{U}_{\mathbb{R}}(\mathfrak{L}(G)))$ connected?

An overview of the situation is helpful.
Let us write $\mathfrak{g} = \mathfrak{L}(G)$:

$$
\begin{array}{lccccccc}
 & & & & & & & \mathbb{R}[G]\\
 & & & & & & & \Big|\\
 & \mathbf{U}_{\mathbb{R}}(\mathfrak{g}) & \xrightarrow{\omega_G,\text{onto}} & \mathbb{R}[G_0] & & & & \Big|\\
 & \Big| & & \Big| & & & & \Big|\\
(D) & \mathbb{G}(\mathbf{U}_{\mathbb{R}}(G)) & \longrightarrow & \Big| & & \longrightarrow & & \mathbb{G}(\mathbb{R}[G]) = G\\
 & \Big| & & \Big| & & & & \Big|\\
 & \mathbb{G}(\mathbf{U}_{\mathbb{R}}(\mathfrak{g}))_0 & \xrightarrow{\text{onto}} & \mathbb{G}(\mathbb{R}[G_0]) = G_0 & & = & & G_0\\
 & {\scriptstyle \exp_{\mathbf{U}_{\mathbb{R}}(\mathfrak{g})}}\Big\uparrow & & \Big\uparrow{\scriptstyle \exp_{\mathbb{R}[G]}} & & = & & \Big\uparrow{\scriptstyle \exp_G}\\
 & \mathfrak{g} \subseteq \mathbb{P}(\mathbf{U}_{\mathbb{R}}(\mathfrak{g})) & \xrightarrow[\text{retract}]{} & \mathbb{P}(\mathbb{R}[G_0]) = \mathbb{P}(\mathbb{R}[G]) & & = & & \mathfrak{g}.
\end{array}
$$

5.1. Lie's Third Fundamental Theorem for profinite-dimensional Lie algebras In the light of these relations of the weakly complete enveloping algebra $\mathbf{U}(\mathfrak{g})$ and the weakly complete group algebra $\mathbb{K}[G]$ as weakly complete $\mathbb{K}$-Hopf algebras it may be illuminating to review the basic relationship between pro-Lie groups G

and their Lie algebras $\mathfrak{L}(G)$. Let us therefore recall in compact form what we know about the relationship providing a profinite dimensional Lie algebra $\mathfrak{g}$ to any pro-Lie group G and its reversal. The reversal $\mathfrak{g} \to G$ is historically known as Sophus Lie's Third Fundamental Theorem. As soon as one leaves the classical domain of finite dimensional real or complex Lie groups, there emerge several aspects of this theorem in the Literature:

Theorem 5.2. (Sophus Lie's Third Principal Theorem) For every profinite-dimensional real Lie algebra $\mathfrak{g}$ there is a simply connected pro-Lie group $\Gamma(\mathfrak{g})$ whose Lie algebra $\mathfrak{L}(\Gamma(\mathfrak{g}))$ is (isomorphic to) $\mathfrak{g}$.

For a systematic proof see [6], or e.g. [8]. Theorem 5.2 is also cited in [7], Theorem A7.29. For the definition of simple connectivity see [7], Definition A2.6. Let us recall here that for an abelian $\mathfrak{g}$ (that is, a weakly complete real vector space), the underlying vector space of $\mathfrak{g} \cong \mathfrak{L}(\Gamma(\mathfrak{g}))$ is isomorphic to $\Gamma(\mathfrak{g})$ via $\exp_{\mathfrak{L}(\Gamma(\mathfrak{g}))} : \mathfrak{L}(\Gamma(\mathfrak{g})) \to \Gamma(\mathfrak{g})$.

Recall that a real Lie algebra $\mathfrak{g}$ is called "compact" if it is isomorphic to the Lie algebra of a compact group (apologetically defined in [7] Definition 6.1 in that fashion). We know a real Lie algebra to be *compact* if and only if there exists a set X and a family $\mathcal{S}$ of compact simple simply connected Lie groups S such that $\mathfrak{g} = \mathbb{R}^X \times \prod \mathcal{S}$, where we wrote $\prod \mathcal{S}$ for $\prod_{S \in \mathcal{S}} S$. Now from [7], Theorem 9.76 we obtain the following statement:

Theorem 5.3. (Sophus Lie's Third Principal Theorem for Compact Lie Algebras) For every compact real Lie algebra $\mathfrak{g}$ there is a projective connected compact group $\mathbb{P}(\mathfrak{g})$ whose Lie algebra $\mathfrak{L}(\mathbb{P}(\mathfrak{g}))$ is (isomorphic to) $\mathfrak{g}$.

Every compact connected group G with $\mathfrak{L}(G) \cong \mathfrak{g}$ is a quotient of $\mathbb{P}(\mathfrak{g})$. modulo some central 0-dimensional subgroup. For details see [7], discussion following Lemma 9.72, notably Theorem 9.76 and Theorem 9.76bis. For the abelian case see [7], Theorem 8.78ff. Notice that for a compact real Lie algebra $\mathfrak{g}$ the projective compact connected group $\mathbb{P}(\mathfrak{g})$ is simply connected if and only if $\mathfrak{g}$ is semisimple. By contrast, if $\mathfrak{g} = \mathbb{R}^X$ for some set then $\mathbb{P}(\mathfrak{g}) = (\widehat{\mathbb{Q}})^X$ (see [7], Proposition 8.81), a compact connected abelian group that fails to be simply connected while $\pi_1(\mathbb{P}(\mathfrak{g})) = \{0\}$ (see [7], Theorem 8.62).

Let us contemplate how the present concept of weakly complete enveloping algebras belongs to the circle of ideas of Lie's Third Fundamental Theorem.

Let $\mathfrak{g}$ be a profinite-dimensional Lie algebra over $\mathbb{R}$ and write $\mathbb{G}$ for the group $\mathbb{G}(\mathbf{U}(\mathfrak{g}))$ of grouplike elements and $\mathbb{P}$ for the Lie algebra $\mathbb{P}(\mathbf{U}(\mathfrak{g}))$ of primitive elements of $\mathbf{U}(\mathfrak{g})$. By Theorem 3.6 above, $\mathbb{G}$ is a pro-Lie group and $\mathbb{P}$ its profinite-dimensional Lie algebra $\mathfrak{L}(\mathbb{G})$ containing $\mathfrak{g}$ and the exponential function $\exp : \mathbf{U}(\mathfrak{g})_{\mathrm{Lie}} \to \mathbf{U}(\mathfrak{g})^{-1}$ of $\mathbf{U}(\mathfrak{g})$ induces the exponential function $\exp_{\mathbb{G}} : \mathfrak{L}(\mathbb{G}) = \mathbb{P} \to \mathbb{G}$. Now $\mathfrak{g}$ is a closed Lie subalgebra of $\mathbf{U}(\mathfrak{g})_{\mathrm{Lie}}$ by Proposition 1.7, and is contained in $\mathbb{P}$ by Theorem 3.6(iii). Inside $\mathbb{G}$ we can form the closed connected pro-Lie subgroup $G^* = \overline{\langle \exp_{\mathbb{G}}(\mathfrak{g}) \rangle}$. Clearly $\mathfrak{g} \subseteq \mathfrak{L}(G^*)$.

Problem 3. Is $\mathfrak{g} = \mathfrak{L}(G^*)$?

Example 5.4. Let $\mathfrak{g}$ be a compact semisimple Lie algebra. Then $\mathfrak{g} = \mathfrak{L}(G)$ for the compact projective group $G = \mathbb{P}(\mathfrak{g})$. In this case, $G = \Gamma(\mathfrak{g})$, and we have a commutative diagram

$$(D_1)\qquad \begin{tikzcd}
\mathbf{U}(\mathfrak{g}) \arrow[r, "{\omega_G, onto}"] \arrow[d, dash] & \mathbb{R}[G] \arrow[d, dash] \\
\mathbb{G}_0 \arrow[r, "\text{retract}"] & G = \mathbb{G}(\mathbb{R}[G]) \\
\mathfrak{g} \subseteq \mathbb{P} \arrow[r, "\text{retract}"] \arrow[u, "\exp_{\mathbb{G}}"] & \mathfrak{g}. \arrow[u, "\exp_G"']
\end{tikzcd}$$

We do not know what $\mathbb{G} = \mathbb{G}(\mathbf{U}(\mathfrak{g}))$ and $\mathbb{P} = \mathbb{P}(\mathbf{U}(\mathfrak{g}))$ are even if $\mathfrak{g} = \mathrm{so}(3)$ in which case $G = SU(2)$. Still, in this case $\exp_{\Gamma(\mathfrak{g})} \colon \mathfrak{g} \to \Gamma(\mathfrak{g})$ is surjective (cf. [7], Theorems 6.30, 9.19(ii) and Theorem 9.32(ii)). Therefore, $G^* = \overline{\langle \exp_{\mathbb{G}}(\mathfrak{g}) \rangle} = \exp_{\mathbb{G}}(\mathfrak{g}) \cong G$ and therefore the answer to Problem 1 is positive. The group $\mathbb{G}$ of grouplike elements of $\mathbf{U}(\mathfrak{g})$ is a semidirect product of some unknown closed normal subgroup N by G. From the content of Diagram (D_1) we do not know anything about N.

Problem 4. Is $N = \{1\}$?

The following example is the opposite to the preceding one:

Example 5.5. Let $\mathfrak{g} = \mathbb{R}^X$ for some set X. Then $G = \mathbb{P}(\mathfrak{g}) = (\widehat{\mathbb{Q}})^X$ and $\Gamma(\mathfrak{g}) = \mathbb{R}^X$.

Here we invoke the information we collected in Section 4 and incorporate it into the following diagram:

$$(D_2)\qquad \begin{tikzcd}
\mathbf{U}(\mathfrak{g}) \arrow[r, "q_{\mathfrak{g}}"] \arrow[d, dash] & \mathbf{V}(\mathfrak{g}) = \mathbb{R}^{\mathbb{R}^{(X)}} \arrow[r] \arrow[d, dash] & \mathbb{R}[G] \arrow[d, dash] \\
\mathbb{G}_0 \arrow[r] & \mathbb{G}(\mathbf{V}(\mathfrak{g})) \cong \mathcal{AB}(\mathbb{R}^{(X)}, \mathbb{R}) \arrow[r] & G = (\widehat{\mathbb{Q}})^X \\
\mathfrak{g} \subseteq \mathbb{P} \arrow[r] \arrow[u, "\exp_{\mathbb{G}}"] & \mathbb{P}(\mathbf{V}(\mathfrak{g})) \cong \mathcal{AB}(\mathbb{R}^{(X)}, \mathbb{R}) \arrow[r] \arrow[u, "\exp_{\mathbb{P}(\mathbf{V}(\mathfrak{g}))}"'] & \mathfrak{g}. \arrow[u, "\exp_G"']
\end{tikzcd}$$

We recall that $q_{\mathfrak{g}}$ is the quotient morphism of $\mathbf{U}(\mathfrak{g})$ modulo its radical.

Problem 5. Does $q_{\mathfrak{g}}$ induce surjective morphisms between the groups of grouplike elements and the Lie subalgebra of primitive elements?

Recall that in the isomorphism in the last line of diagram (D_2) the subspace of $\mathcal{AB}(\mathbb{R}^{(X)}, \mathbb{R})$ corresponding to $\mathfrak{g}$ is $\mathcal{W}_{\mathbb{R}}(\mathbb{R}^{(X)}, \mathbb{R}) \cong \mathbb{R}^X$.

5.2. The relation of enveloping algebras and group algebras in the abelian case. In [5] we gave an explicit description of the weakly complete complex group algebra $\mathbb{C}[G]$ of a compact abelian group G which resembles the description we gave in Proposition 4.4 for $\mathbf{V}(\mathfrak{g}) \cong \mathbf{U}(\mathfrak{g})/R(\mathbf{U}(\mathfrak{g}))$. Indeed we showed that for a compact abelian group G we have a natural isomorphism

$$\mathbb{C}[G] \cong \mathbb{C}^{\widehat{G}}.$$

In the case of the real field we pointed out that $\mathbb{R}[G]$ is the fixed point set of an involution of $\mathbb{C}[G]$, namely, the involution σ given by $\sigma(p)(\chi) = \overline{p}(-\chi)$ for $p \in \mathbb{C}^{\widehat{G}}$ and $\chi \in \widehat{G}$.

In [7] Theorem A7.10 we showed that real weakly complete vector topological spaces have Pontryagin duality. This was shown via the fact that for any weakly complete real vector space $\mathfrak{g}$ the natural morphism $\mathfrak{g}' = \mathrm{Hom}(\mathfrak{g}, \mathbb{R}) \to \mathrm{Hom}(\mathfrak{g}, \mathbb{T}) = \widehat{\mathfrak{g}}$ is an isomorphism, where $\widehat{\mathfrak{g}}$ is the Pontryagin dual of the topological abelian group $\mathfrak{g}$. Hence we have the natural isomorphisms

$$\mathbb{K}^{\mathfrak{g}'} \cong \mathbb{K}^{\widehat{\mathfrak{g}}}.$$

However, for $\mathbb{K} = \mathbb{C}$, the last group was recognized in [5], Theorem 5.1 as the Group Hopf algebra $\mathbb{C}[\mathfrak{g}]$.

Problem 6. For a weakly complete $\mathbb{K}$-vector space $\mathfrak{g}$, considered as a weakly complete abelian $\mathbb{K}$-Lie algebra, what is the precise relation between $\mathbf{V}_{\mathbb{K}}(\mathfrak{g}) = \mathbb{K}^{\mathfrak{g}'}$ and $\mathbb{K}[\mathfrak{g}]$, $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$?

Acknowledgments. An essential part of this text was written while the authors were partners in the program Research in Pairs at the Mathematisches Forschungsinstitut Oberwolfach MFO in the Black Forest from February 2 through 22, 2020. The authors are grateful for the environment and infrastructure of MFO which made this research possible.

References

[1] Bourbaki, N., ≪Groupes et algèbres de Lie≫, Hermann, Paris, 1971.

[2] Dahmen, R., and K. H. Hofmann, *The Pro-Lie Group Aspect of Weakly Complete Algebras and Weakly Complete Group Hopf Algebras*, J. of Lie Theory **29** (2019), 413–455.

[3] Dixmier, J., "Enveloping Algebras", North-Holland, 1977, xvi+375pp.

[4] Hamilton, A., *A Poincaré-Birkhoff-Witt Theorem for Profinite Pronilpotent Lie Algebras*, J. of Lie Theory **29** (2019), 611–618.

[5] Hofmann, K.H., and L. Kramer, *On weakly complete Group Algebras of Compact Groups*, J. of Lie Theory **30** (2020), 21–39.

[6] Hofmann, K. H., and S. A. Morris, *Sophus Lie's Third Fundamental Theorem and the Adjoint Functor Theorem,* J. of Group Theory **8** (2005), 115–133.

[7] Hofmann, K. H., and S. A. Morris, "The Structure Theory of Compact Groups–A Primer for the Student–a Handbook for the Expert", 4th Edition, De Gruyter Studies in Mathematics **25**, Berlin/Boston, 2020, xxvi+1006pp.

[8] Hofmann, K. H., and S. A. Morris, "The Lie Theory of Connected Pro-Lie Groups,–A Structure Theory for Pro-Lie Algebras, Pro-Lie Groups, and Connected Locally Compact Groups", European Mathematical Society Publishing House, Zürich, 2006, xii+663pp.

[9] Serre, J.-P., "Lie Algebras and Lie Groups", W.A. Benjamin, Inc., New York-Amsterdam, 1965.

Karl Heinrich Hofmann
Fachbereich Mathematik
Technische Universität Darmstadt
Schlossgartenstraße 7
64289 Darmstadt, Germany
hofmann@mathematik.tu-darmstadt.de

Linus Kramer
Mathematisches Institut
Universität Münster
Einsteinstraße 62
48149 Münster, Germany
linus.kramer@uni-muenster.de