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Representation Theory of Quivers and Finite Dimensional Algebras

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Abstract. Methods and results from the representation theory of quivers and finite dimensional algebras have led to many interactions with other areas of mathematics. Such areas include the theory of Lie algebras and quantum groups, commutative algebra, algebraic geometry and topology, and in particular the theory of cluster algebras. The aim of this workshop was to further develop such interactions and to stimulate progress in the representation theory of algebras.

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Introduction by the Organizers

The representation theory of quivers is probably one of the most fruitful parts of modern representation theory because of its various links to other mathematical subjects. This has been the reason for devoting a substantial part of this Oberwolfach meeting to problems that can be formulated and solved involving quivers and their representations. The interaction with neighbouring mathematical subjects like geometry, topology, and combinatorics is one of the traditions of such Oberwolfach meetings; it continues to be a source of inspiration. There were 28 lectures given at the meeting, and what follows is a quick survey of their main themes.

Gentle algebras and geometry. The link between derived categories of gentle algebras and the geometry of graded surfaces has been established in the last 5 years by Haiden-Katzarkov-Kontsevich and Opper-Plamondon-Schroll. This led to many applications and generalizations in representation theory. In his talk, Plamondon explained this geometric model and its use to get a complete geometric derived invariant for gentle algebras. While Schroll explained how to generalize this geometric model to get a model for the derived category of skew-gentle algebras, Brüstle used group actions to generalize the geometric derived invariant to skew-gentle algebras. Zvonareva gave also a complete derived invariant for Brauer graph algebras, using A_∞-structures associated to surfaces, following Haiden-Katzarkov and Kontsevich ideas. The class of gentle algebras has also been generalized to quasi-gentle algebras by Burban, who explained why this bigger class yields tame algebras. Torsions classes and pairs for derived categories of gentle algebras have been also discussed in talks by Laking and Chan. Finally, the link between partially wrapped Fukaya categories and derived categories of non commutative algebras generalizes beyond the 2-dimensional case. Higher Auslander algebras seem to be the higher analogue when the dimension of the symplectic variety is greater than 2. Jasso reported about the A_n case which has been recently treated in joint work.

Cluster categorification. The Grassmannian variety of a vector space was one of the first motivating example of a cluster variety. Their categorifications have been studied by Geiss Leclerc and Schröer using preprojective algebras, and by Jensen, King and Su using the singularity category of a quotient of the preprojective algebra. The Grassamnnian variety comes with an action of the affine braid group discovered by Fraser. Keller explained their joint work on this subject: it is possible to lift this action as a categorical action. The Jensen-King-Su categorification has also been generalized by the authors to categorify the quantum Grassmannian, and this new approach has been reported by King. Concerning acyclic categorification, Hubery discussed the non-algebraically closed situation, which surprisingly was still open. Finally, Jacobian algebras which naturally appear everywhere in cluster categorification, were discussed by Davison, who provided a rich source of finite dimensional Jacobian algebras.

Persistent modules. Topological data analysis using persistent homology naturally produces linear algebra data, which may be in the form of quiver representations, or more general persistence modules of some type. The analysis of such data, for example using barcodes, involves the decomposition into indecomposable representations and fits therefore into representation theory. Two experts from this field reported about recent developments and interesting connections to representation theory. Botnan gave a survey about the decomposition of persistent modules, their barcodes, and the interleaving distance between persistent modules. One of the challenges in the subject is the generalisation from dimension one (so representations of the real line $\mathbb{R}^1$) to higher dimensions. The talk by Oudot was devoted to some specific results in this direction for the case $\mathbb{R}^2$ and based on appropriate generalizations of interval modules.

Module varieties. Recent results on various schemes associated with quiver representations were reported during the workshop. Thomas considered the Jordan type $J(X)$ of generic nilpotent endomorphisms of a representation X of a Dynkin quiver Q. For a cominuscule vertex i of Q, if all direct summands of X are supported at i, one can recover X from $J(X)$. Moreover, this correspondence gives a generalization of the Robinson–Schensted–Knuth correspondence for type A_n. For a gentle algebra associated with a triangulation of a marked surface, Schröer considered certain irreducible components of the module varieties, called generically τ-reduced. These components with decorations correspond bijectively with laminations on the marked surface, and the generic Caldero–Chapoton functions coincides with the bangle functions. For a representation M of a quiver Q, Cerulli-Irelli discussed geometric properties of the quiver Grassmaniann X of M. He explained that X has a cellular decomposition in many cases, e.g. if Q is Dynkin. If M is rigid, then X satisfies a certain weaker condition (S) about the Borel–Moore homology. He posed some conjectures.

Homologically defined algebras and representations. A number of recent developments on homological aspects of representation theory were presented during the workshop. For an algebraic triangulated category $\mathcal{D}$ and the heart $\mathcal{H}$ of a t-structure of $\mathcal{D}$, there is a triangle functor G, called the realization functor, from the bounded derived category of $\mathcal{H}$ to $\mathcal{D}$. Chen gave a necessary and sufficient condition for G to be an equivalence when the t-structure is given by a HRS tilt. Qiu introduced a notion of global dimension of a triangulated category by using Bridgeland's stability conditions. He explained some results on the explicit values of the global dimension of the bounded derived category of Dynkin quivers, smooth projective curves and $\mathbb{P}^2$. Gnedin discussed silting theory of orders Λ over commutative Noetherian complete local rings R. For a regular sequence $\mathbf{x}$ of R and the factor algebra $\overline{\Lambda} = \Lambda/\mathbf{x}\Lambda$, there is a bijection between the silting complexes of Λ and those of $\overline{\Lambda}$. He also gave a result about tilting complexes, and an application to ribbon graph orders and twisted Brauer graph algebras.

Minamoto discussed a certain central extension of the preprojective algebra of an acyclic quiver Q, called the quiver Heisenberg algebra. It contains various information on the category of representations of Q, and its $\mathbb{Z}/2\mathbb{Z}$-covering is the 3-preprojective algebra of a certain explicit 2-hereditary algebra. Herschend talked about a $\mathbb{Z}$-graded hypersurface singularity R of dimension 1. The stable category of the $\mathbb{Z}$-graded Cohen-Macaulay R-modules has a tilting object, whose endomorphism algebra is 2-hereditary and given by a quiver with potential. Its cluster category is equivalent to the stable category of $\mathbb{Z}/2\mathbb{Z}$-graded Cohen-Macaulay R-modules.

Modules with double centralizer property are called faithfully balanced. Rognerud gave a combinatorial description of the basic faithfully balanced modules over a path algebra of the linearly oriented A_n quiver, and calculate the number of them. Marczinzik discussed two generalizations of Gorenstein algebras, namely the class of weakly Gorenstein algebras, and the class of Cohen-Macaulay artin algebras due to Auslander-Reiten. He explained some properties and examples, and also posed questions. Skowronski introduced weighted surface algebras, which are symmetric of tame representation type and periodic of period 4. He conjectured that an algebra of generalized quaternion type is either a weighted surface algebras or belongs to nine ‘exotic’ familes of algebras. Then he explained a partial result.

Further topics. Tensor categories or monoidal categories arise frequently in representation theory. One can define algebras and modules in such tensor categories and this leads to a Morita theory. The talk of Kinser was devoted to a description of Morita equivalence classes and enhanced by a series of interesting examples. Tensor categories also provide a possible setting for defining support varieties, but interesting results often depend on an altervative approach via chomological varieties. Koenig reported about finiteness conditions for such support varieties using Hochschild cohomology. Monomorphism categories form another convenient setting in representation theory. For instance, they arise as categories of Gorenstein projective objects when one generalizes quiver representations. The talk of Külshammer was devoted to the Auslander-Reiten theory of such monomorphism categories, involving the general theory of relative Nakayama functors.

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Workshop: Representation Theory of Quivers and Finite Dimensional Algebras

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Abstracts

Grassmannian braiding categorified

Bernhard Keller
(joint work with Chris Fraser)

1. Grassmannian braiding

Let $n \geq 3$ be an integer. The *extended affine braid group on n strands*, denoted by $\widetilde{\mathrm{Br}}_n$, is the fundamental group of the space of configurations of n distinct points in an annulus. It has generators σ_i, $1 \leq i \leq n$, and ρ as depicted in Figure 1. We refer to [5] for more information on this group.

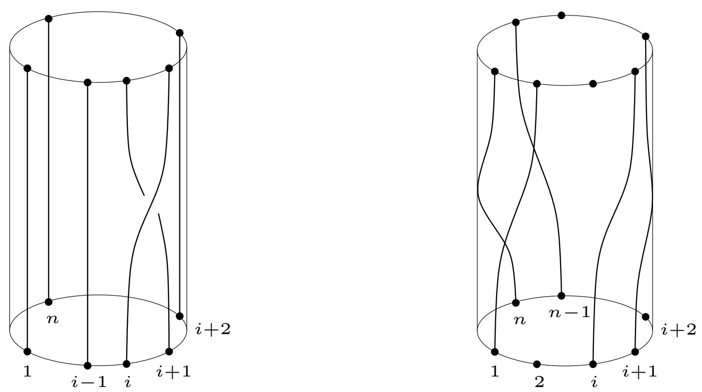

Figure 1. The affine braid generators σ_i and ρ

Let $1 \leq k \leq n-1$ be an integer. Let $\mathrm{Gr}(k,n)$ be the Grassmannian of k–dimensional subspaces in $\mathbb{C}^n$. Via the Plücker embedding

$$\mathrm{Gr}(k,n) \longrightarrow \mathbb{P}^N(\mathbb{C})$$

taking a subspace U to $\Lambda^k U$ it embeds into projective space, where $N = \binom{n}{k} - 1$. The cone $\widetilde{\mathrm{Gr}}(k,n)$ over the Grassmannian embeds into $\mathbb{C}^N$ and the pullbacks of the coordinate functions are the Plücker coordinates, which generate the coordinate ring $\mathcal{A} = \mathbb{C}[\widetilde{Gr}(k,n)]$. Explicitly, if we represent a k-tuple of vectors in $\mathbb{C}^n$ by a $k \times n$-matrix, the Plücker coordinate P_I associated with an increasing sequence $i_1 < \cdots < i_k$ of elements of $\{1, \ldots, n\}$ is the minor on the columns indexed by $i_1, \ldots, i_k$. The Plücker coordinate P_I is called *frozen* if the classes modulo n of the indices $i_1, \ldots, i_k$ form an interval. Let $\underline{\mathcal{A}}$ denote the quotient of $\mathcal{A}$ by the ideal generated by all $P_I - 1$, where P_I is frozen.

Theorem 1.1 (Scott [13])**.** *The algebra* $\underline{\mathcal{A}}$ *is the cluster algebra associated with the triangle product quiver* $A_{k-1} \boxtimes A_{n-k-1}$ *(cf. section 3.3 of* [10] *for the notation).*

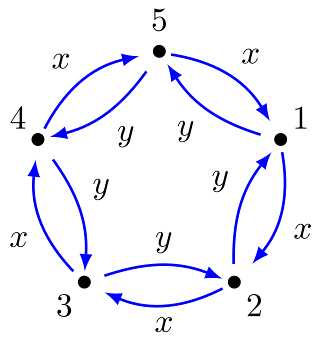

FIGURE 2. The quiver of the preprojective algebra Π for $n = 5$

Theorem 1.2 (Fraser [4]). *Let d be the greatest common divisor of k and n. The group $\widetilde{\mathrm{Br}}_d$ naturally acts on $\underline{\mathcal{A}}$ by cluster automorphisms (i.e. automorphisms taking cluster variables to cluster variables and clusters to clusters).*

Notice that the action of the Theorem is not faithful as ρ^n acts by the identity but ρ is of infinite order in $\widetilde{\mathrm{Br}}_n$. Now let Δ and Δ' be simply laced Dynkin diagrams with Coxeter numbers k and $n-k$. By the periodicity theorem for pairs of Dynkin diagrams [10], if we take ρ to the Zamolodchikov transformation of the cluster algebra $\mathcal{A}(\Delta \boxtimes \Delta')$ associated with $\Delta \boxtimes \Delta'$, we also obtain an action of the cyclic group $\langle \rho \rangle \subset \widetilde{\mathrm{Br}}_n$ taking ρ^n to the identity.

Conjecture 1.3. *The action of $\langle \rho \rangle$ extends to an action of $\widetilde{\mathrm{Br}}_d$ on $\mathcal{A}(\Delta \boxtimes \Delta')$.*

Using the quiver mutation applet [9] we have checked the conjecture for $A_r \boxtimes D_s$ for the following pairs (r, s): $(2, 7)$, $(2, 10)$, $(3, 5)$, $(3, 9)$, $(4, 6)$.

2. Categorification

Let Π denote the completed preprojective algebra of type $\widetilde{A}_{n-1}$ over $\mathbb{C}$. It is presented by the quiver of Figure 2 subject to the n relations $xy - yx$. Let B be the quotient of Π by all relations $x^k - y^{n-k}$. It was introduced by Jensen–King–Su in [8]. The algebras Π and B are noetherian, the preprojective algebra Π is of global dimension 2 and bimodule 2-Calabi-Yau [2, 7] and the algebra B is Iwanaga–Gorenstein of infinite global dimension, cf. [8]. Let $\mathrm{mod}B$ denote the category of finitely generated right B-modules, $\mathcal{D}^b(\mathrm{mod}B)$ its bounded derived category and $\mathrm{per}(B)$ its perfect derived category, i.e. the full subcategory whose objects are quasi-isomorphic to bounded complexes of finitely generated projective B-modules. Let $\mathrm{sg}(B)$ denote the singularity category of B, i.e. the quotient $\mathcal{D}^b(\mathrm{mod}B)/\mathrm{per}(B)$, cf. [1, 11]. Equivalently, it may be described as the stable category of Gorenstein projective modules over B. The triangulated category $\mathrm{sg}(B)$ is Hom-finite and 2-Calabi-Yau. A triangle equivalent category was first constructed in a different manner by Geiss–Leclerc–Schröer [6].

Theorem 2.1 (Geiss–Leclerc–Schröer [6]). *The cluster algebra $\underline{\mathcal{A}}$ is categorified by $\mathrm{sg}(B)$.*

Since the algebra Π is bimodule 2-Calabi-Yau, the simple modules S_i over Π are 2-spherical in the bounded derived category $\mathcal{D}^b(\mathrm{mod}\Pi)$, i.e. $\mathrm{Ext}^*_\Pi(S_i, S_i)$ is the cohomology algebra of a 2-sphere (with complex coefficients). Therefore, from Seidel–Thomas' work [14], we get a braid group action:

Theorem 2.2 (Seidel–Thomas [14]). *The group $\widetilde{\mathrm{Br}}_n$ acts on $\mathcal{D}^b(\mathrm{mod}\Pi)$ taking ρ to the automorphism induced by counterclockwise rotation of the quiver and σ_i to the spherical twist tw_{S_i} such that for an object X, we have a triangle*

$$\mathrm{RHom}(S_i, X) \otimes S_i \longrightarrow X \longrightarrow \mathrm{tw}_{S_i}(X) \longrightarrow \Sigma\mathrm{RHom}(S_i, X) \otimes S_i\ .$$

Now notice that we have an embedding $\widetilde{\mathrm{Br}}_d \subset \widetilde{\mathrm{Br}}_n$ taking ρ to ρ^d and σ_i to the product of the (commuting) σ_j such that $j \equiv i$ modulo d. Whence an action of $\widetilde{\mathrm{Br}}_d$ on $\mathcal{D}^b(\mathrm{mod}\Pi)$.

Theorem 2.3. *a) There is an action of $\widetilde{\mathrm{Br}}_d$ on $\mathcal{D}^b(\mathrm{mod}B)$ such that the functor $?\overset{L}{\otimes}_\Pi B : \mathcal{D}^b(\mathrm{mod}\Pi) \to \mathcal{D}^b(\mathrm{mod}B)$ becomes equivariant.*

b) The action of a) induces an action on $\mathrm{sg}(B) = \mathcal{D}^b(\mathrm{mod}B)/\mathrm{per}(B)$ such that the canonical cluster character [12]

$$\underline{CC} : \mathrm{sg}(B) \to \underline{\mathcal{A}}$$

becomes equivariant for the action of Theorem 1.2 on $\underline{\mathcal{A}}$.

Let $\mathcal{A}_{loc}$ denote the localization of $\mathcal{A}$ with respect to all frozen Plücker coordinates. The more precise version of Theorem 1.2 proved in [4] yields an action of $\widetilde{\mathrm{Br}}_d$ on $\mathcal{A}_{loc}$ by *pseudo-isomorphisms*. These are certain algebra morphisms introduced in [3] (where they were called quasi-isomorphisms) which act non trivially on the frozen variables and take cluster variables to products of cluster variables with Laurent monomials in the frozen variables. The following theorem shows that the lift of the $\widetilde{\mathrm{Br}}_d$-action from $\underline{\mathcal{A}}$ to $\mathcal{A}_{loc}$ corresponds to the lift of the action on the singularity category $\mathrm{sg}(B)$ to the derived category $\mathcal{D}^b(\mathrm{mod}B)$.

Theorem 2.4. *There is a 'refined cluster character'*

$$CC_{loc} : \mathcal{D}^b(\mathrm{mod}B) \to \mathcal{A}_{loc}$$

which is equivariant for the $\widetilde{\mathrm{Br}}_d$-actions introduced above and compatible with $\underline{CC}$.

Acknowledgements

Thanks are due to the authors of [5] for Figure 1 and those of [8] for Figure 2. I am grateful to Chris Fraser, Alastair King, Matthew Pressland and Yilin Wu for helpful remarks on a draft of this report.

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Poset Representations in Data Analysis

Magnus Bakke Botnan

Topological data analysis (TDA) is a novel approach to data science in which the "shape" of the data is inferred through topological invariants. One of the most notable tools in TDA, *persistent homology*, assigns a collection of intervals to an input data set. These intervals are to be interpreted as topological features in the data. It is customary to treat short intervals as "noise" and long bars as "significant", with the precise notion of what constitutes a long interval being application-dependent.

The idea is as follows: given a filtered topological space

$$X_0 \subseteq X_1 \subseteq X_2 \ldots \subseteq X_n$$

(typically a filtered simplicial complex), one obtains a sequence of homology vector spaces and linear maps by applying the p-th (singular) homology functor with coefficients in a field

$$H_p(X_0) \to H_p(X_1) \to \ldots \to H_p(X_n).$$

The resulting object is nothing more than a representation of the linear quiver and thus decomposes into a direct sum of constant modules. The support of these modules constitute the *barcode* and correspond to the intervals mentioned above.

1. Single parameter

It is convenient to work in the following more general setting. Consider $\mathbb{R}$ as a category with respect to its total order in the obvious way. We define a *persistence module* to be a functor $M\colon \mathbb{R} \to \mathrm{vec}_k$ where vec_k is the category of finite-dimensional k-vector spaces.

Theorem 1.1. [5, 3] *Any persistence module M decomposes as a direct sum of interval modules, i.e. constant modules k_I where $I \subseteq \mathbb{R}$ is an interval.*

The collection of intervals (the barcode) in the decomposition of M is denoted by $B(M)$. One way of obtaining such a functor is as follows: let $f\colon Y \to \mathbb{R}$ be any function and define $M_t := f^{-1}(-\infty, t]$. Under the assumption that the sublevel sets have finitely generated homology, it follows that M defines a persistence module. A natural question arises: let g be a real-valued function defined on Y satisfying $||f - g||_\infty \leq \epsilon$ and let $N_t := g^{-1}(-\infty, t]$. Will the barcodes of M and N also be at most ϵ apart? The answer turns out to be yes, and it is one of the fundamental results in the field of TDA. One way to prove this is through the use of *interleavings*. Let $M(\epsilon)$ denote the persistence module $M(\epsilon)_t = M_{t+\epsilon}$. We say that $M, N\colon \mathbb{R} \to \mathrm{vec}$ are *ϵ-interleaved* if there exist morphisms $f : M \to N(\epsilon)$ and $g\colon N \to M(\epsilon)$ such that $g_{t+\epsilon} \circ f_t = M(t \leq t + 2\epsilon)$ and $f_{t+\epsilon} \circ g_t = N(t \leq t + 2\epsilon)$.

Definition 1.2. The *interleaving distance* between M and N is

$$d_I(M, N) = \inf_{\epsilon}\{\epsilon : \text{there exists an } \epsilon\text{-interleaving between } M \text{ and } N\}.$$

It is a small exercise to show that the persistence modules M and N associated to the functions f and g are ϵ-interleaved, and therefore $d_I(M, N) \leq \epsilon$. What is important, is that the interleaving distance has a combinatorial counterpart which is defined in terms of collections of intervals. This distance is denoted by $d_B(B(M), B(N))$ and called the *bottleneck distance between M and N*. Through this equivalence, called the *isometry theorem*[4], we obtain the stability result:

$$d_B(B(M), B(N)) = d_I(M, N) \leq \epsilon.$$

2. Multiple parameters

Motivated by applications - such as to clustering[1] - it is natural to consider persistence modules indexed by other types of posets, such as $\mathbb{R}^2$ or any finite subgrid thereof. The representation theory in this setting is however extremely complicated and the pipeline from the linear setting does not directly generalize. Therefore new, potentially *stable*, invariants are needed to push the field forward. Likewise, the interleaving distance mentioned above *does* generalize to higher dimensions but there is no corresponding bottleneck distance. Furthermore, whereas computing the interleaving distance in the linear case, through its equivalence with the bottleneck distance, comes down to a bipartite matching problem, it has recently been shown that computing its multi-parameter analogue is NP-hard[2].

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The Robinson–Schensted–Knuth correspondence via quiver representations

Hugh Thomas

(joint work with Alexander Garver, Rebecca Patrias)

This talk was based on the preprint [2]. A shorter version of the preprint, aimed more at combinatorialists, is also available [3].

1. Generic nilpotent endomorphisms

Let Q be a Dynkin quiver. Given a representation X (typically decomposable), we consider a nilpotent endomorphism ϕ of X. At each vertex, it induces ϕ_j, a nilpotent endomorphism of X_j. We can ask about the sizes of the Jordan blocks of ϕ_j. We have the following result (which I state more generally).

Proposition 1. For X an A-module, A a finite-dimensional algebra, and ϕ a nilpotent endomorphism of X, there is a well-defined generic behaviour of the Jordan block sizes of ϕ_j.

Denote this $J(X)$, with $J(X)_j$ the block sizes of ϕ_j. We think of $J(X)_j$ as a partition of $\dim(X_j)$. If need be, we order the parts in weakly decreasing order, and refer to $J(X)_j^1 \geq J(X)_j^2 \geq \dots$.

We are interested in the question of when an object X in a subcategory of $\operatorname{rep} Q$ can be recovered from $J(X)$.

For example, consider the A_3 quiver $1 \to 2 \leftarrow 3$. Consider $M = P_2^a \oplus P_1^b \oplus P_3^c \oplus I_2^d \oplus I_1^e \oplus I_3^f$. Then

$$J(X)_1 = (b + d + e), J(X)_2 = (a + \max(b, c) + d, \min(b, c)), J(X)_3 = (c + d + f).$$

We observe that from these four integers it is obviously impossible to recover (a, b, c, d, e, f), but if we assume that $e = f = 0$, it is indeed possible to recover (a, b, c, d). This is a small example of the phenomenon which we will be interested in.

Let Q be a Dynkin quiver, with $Q_0 = \{1, \dots, n\}$. Let i be a (co)minuscule vertex. That says that every indecomposable representation has dimension at most 1 at vertex i.

These are the "very good" nodes. All nodes in type A_n are (co)minuscule, as well as the most type-A-ish in other types. These are the leaves in D_n, the leaves furthest from the branch point in E_6, and the leaf furthest from the branch point in E_7. There are none in E_8.

Let $\mathcal{C}_i$ be the subcategory of representations of Q consisting of sums of copies of representations which are supported at i.

Theorem 1. $X \in \mathcal{C}_i$ can be recovered up to isomorphism from $J(X)$.

We say that $\mathcal{C}_i$ is "Jordan recoverable."

Note that our theorem wouldn't hold if we chose a non-minuscule vertex, such as the middle vertex of D_4. We can write the highest root of D_4 as a sum of roots all of which include the central node in five different ways, so there are five different isoclasses of representations of that dimension in the analogous additive category, but there are only two different possibilities for the Jordan data, so obviously we can't reconstruct the representation from the Jordan data.

Question 1. What additive subcategories $\mathcal{C}$ of $\operatorname{rep} Q$ are Jordan recoverable?

There is a specific way in which we might hope to recover X from $J(X)$. Given a collection of Jordan data J (i.e., a partition J_i at each vertex), for $i \in Q_0$, let V_i be a vector space of the right dimension. Choose a nilpotent linear transformation N_i with Jordan form given by J_i.

Now, consider representations of Q on the vector spaces V_i, such that the collection (N_i) defines an endomorphism. Such representations form an irreducible variety cut out by linear equations, and have a dense open set where the dimension vectors of their indecomposable summands are well-defined. (This generalizes Kac's canonical decomposition of dimension vectors [5]; to recover Kac's result, set all the $N_i = 0$.) If this allows us to reconstruct X for any $X \in \mathcal{C}$, we say that $\mathcal{C}$ is canonically Jordan recoverable.

We have the following strengthening of Theorem 1:

Theorem 2. $\mathcal{C}_i$ is canonically Jordan recoverable.

We will now consider two examples where we start with a representation, calculate the Jordan form of a generic nilpotent endomorphism, and then reconstruct the representation as described above.

Consider $110 \oplus 011$. Its only nilpotent endomorphism is 0, whose Jordan form is ((1),(1,1),(1)). The nilpotent linear transformations are zero. So we are just asking about the canonical decomposition in the sense of Kac of the dimension vector (1,2,1), the generic representation with that dimension vector. Nothing forces the images of the vector spaces over 1 and 3 to line up, so they don't, and we recover $110 \oplus 011$.

Now consider $111 \oplus 010$. The generic Jordan form of a nilpotent endomorphism is $(1), (2), (1)$. The nilpotent linear transformation N_i is non-zero at vertex 2, but zero elsewhere. The images of the generator of the vector space at 1 and 3 must lie in the kernel of N_2 (for N to be an endomorphism). This forces the images to line up, so we get $111 \oplus 010$.

Looking more closely at the Jordan block sizes we calculated in our original example, we notice that, assuming $X \in \mathcal{C}_2$, the sizes of the different blocks at different vertices satisfy some inequalities. Specifically, we see that the block size at 1 is between the two block sizes at 2, and the same thing at 3. In order to make a general statement, we need a definition.

2. Structure of Jordan block size data for $\mathcal{C}_i$

Taking the AR-quiver of $\operatorname{rep} Q$ and restricting to $\mathcal{C}_i$, we get a poset with minimal element P_i and maximal element I_i. We call this poset $\mathcal{M}_i$.

In type A_n, for example, with vertex i as the chosen node, the resulting poset is the product of an i-element chain and an $n+1-i$-element chain.

Because applying reflection functors at vertices other that i does not change the vector space at i of a representation, and we can get from any orientation of Q to any other without reflecting at i, this poset does not depend on the orientation of Q. This is the "minuscule poset" associated to Q and i. The minuscule poset has other interpretations, in terms of Schubert calculus, and in terms of Weyl group combinatorics [6].

One feature of the minuscule poset is that there is a natural map π from $\mathcal{M}_i$ to Q_0, where $\pi(X) = j$ iff X and P_j are in the same τ-orbit.

In general, we have the following description of the Jordan form data:

Theorem 3. Let $X \in \mathcal{C}_i$. Define a function from $\mathcal{M}_i$ to $\mathbb{Z}_{\geq 0}$ by setting $f(x)$ for $x \in \pi^{-1}(j)$ to be the elements of $J(X)_j$, in decreasing order as we go up the poset, adding zeros if necessary so that f is defined everywhere. The result is an order-reversing map from $\mathcal{M}_i$ to $\mathbb{Z}_{\geq 0}$. Further, the map from isomorphism classes of objects in $\mathcal{C}_i$ to order-reversing maps from $\mathcal{M}_i$ to $\mathbb{Z}_{\geq 0}$ is a bijection.

In particular, this theorem asserts a bound on the number of Jordan blocks at vertex j, namely $|\pi^{-1}(j)|$.

It might not be too surprising that in type A_n, it is possible to work out this combinatorics explicitly. And in fact essentially this was done, without the language of quivers, for a particular choice of orientation, by Gansner [1], who showed that it realizes a rather general version of the Robinson–Schensted–Knuth map. The bijection between arbitrary functions from $\mathcal{M}_i$ to $\mathbb{Z}_{\geq 0}$ (thought of as encoding multiplicities of indecomposables) and order-reversing maps from $\mathcal{M}_i$ to $\mathbb{Z}_{\geq 0}$ (Jordan form data) can be considered a generalization of this map. For an excellent introduction to the Robinson–Schensted–Knuth map in a form compatible with our treatment, see [4].

3. Affine generalization

There is a generalization of the categories $\mathcal{C}_i$ to affine type. There is a natural poset, and a map from isomorphism classes of representations to order-reversing maps from the poset to the non-negative integers, but the map is now an injection rather than a bijection. We are studying this in ongoing work.

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Quiver Heisenberg Algebras

Hiroyuki Minamoto

(joint work with Martin Herschend)

1. Quiver Heisenberg Algebras

Let K be a field of characteristic 0.

We denote by $\overline{Q}$ the double quiver of Q. We recall that the mesh relation ρ is defined as

$$\rho := \sum_{\alpha\in Q_1} \alpha\alpha^* - \sum_{\alpha\in Q_1} \alpha^*\alpha.$$

For an arrow $a \in \overline{Q}_1$, we define the *quiver Heisenberg relation* $\eta_a \in K\overline{Q}$ to be the commutator of a and ρ, namely,

$$\eta_a := [a, \rho] = a\rho - \rho a.$$

We define the quiver Heisenberg algebra $\Lambda(Q)$ to be

$$\Lambda(Q) := K\overline{Q}/(\eta_a| \ \ a \in \overline{Q}_1).$$

1.0.1. *Previous results.* As is explained in Section A.1, the algebra $\Lambda(Q)$ is a special case of classes algebras introduced before. Specializing previous results, we obtain the following results.

For an ADE Dynkin quiver Q, the following assertions hold:

(1) (Etingof-Latour-Rains [4]) $\Lambda(Q)$is a symmetric algebra.
(2) (Eu-Schedler [6]) $\Lambda(Q)$ is stably 3-Calabi-Yau.
(3) (Etingof-Rains [3])

$$\dim \Lambda(Q) = \sum_{M:\in \mathrm{ind} KQ} (\dim M)^2 = \frac{rh^2(h+1)}{12}$$

where h is the Coxeter number of Q.

1.0.2. *Our results.* We equip $\overline{Q}$ with the grading $\deg Q_0 := 0, \deg \alpha := 0, \deg \alpha^* = 1$ for $\alpha \in Q_1$. The algebra $\Lambda = \Lambda(Q)$ inherits the grading. We denote Λ_n the degree n-part of n. We note $\Lambda_0 = KQ$.

Since $\Pi = \Lambda/(\rho)$, we have an exact sequence $\Lambda(-1) \xrightarrow{\rho} \Lambda \to \Pi \to 0$. Looking at the degree 1-part of this sequence, we obtain an exact sequence $\theta : KQ \xrightarrow{\rho} \Lambda_1 \to \Pi_1 \to 0$.

For Q we define an algebra $A(Q)$ to be

$$A(Q) := \begin{pmatrix} KQ & \Lambda_1 \\ 0 & KQ \end{pmatrix}.$$

The following theorem together with the above results obtained before shows that $\Lambda(Q)$ and $A(Q)$ can be looked as one-dimensional higher version of the preprojective algebras $\Pi(Q)$ and the path algebras KQ.

The following assertions hold.

(1) If M is an indecomposable non-injective KQ-module, then the map $\rho \otimes M : M \to \Lambda_1 \otimes_{KQ} M$ is injective and the exact sequence $\theta \otimes M$ is an Auslander-Reiten sequence

$$\theta \otimes M : 0 \to M \xrightarrow{\rho \otimes M} \Lambda_1 \otimes_{KQ} M \to \tau^{-1} M \to 0.$$

(2) Let $\mu : \Lambda_1 \otimes_{KQ} \Lambda_1 \to \Lambda_2$ be the multiplication map. Then there exists a map $\Pi_1 \xrightarrow{\eta_*} \Lambda_1 \otimes_{KQ} \Lambda_1$ such that $\mathrm{Ker}\ \mu = \mathrm{Im}\ \eta_*$.

(3) We have an algebra isomorphism

$$\Lambda = \mathsf{T}_{KQ}(\Lambda_1)/(\mathrm{Im}\ \eta_*)$$

where T denotes the tensor algebra.

(4) As KQ-modules we have

$${}_{KQ}\Lambda(Q) \cong \bigoplus_{M \in \mathrm{ind}\mathcal{P}(KQ)} M^{\oplus \dim M}$$

where $\mathcal{P}(KQ)$ denotes the full subcategory of the preprojective KQ-modules.

(5) $\Lambda(Q)$ is finite dimensional if and only if Q is an ADE-quiver if and only if $A(Q)$ is 2-representation finite algebra.

(6) $\Lambda(Q)$ is infinite dimensional if and only if Q is not an ADE-quiver $A(Q)$ is 2-representation infinite algebra.

Assume this is the case. Then Λ is graded coherent and 3-Calabi-Yau.

(7) In any case, the 2-quasi-Veronese algebra of $\Lambda(Q)$ is isomorphic to the 3-preprojective algebra of $A(Q)$. 2-APR-tilting operations on $A(Q)$ are compatible with reflections of a quiver Q.

2. Algebraic McKay correspondence

Let Q be an ADE-Dynkin quiver, $\widehat{Q}$ the extended quiver, $0 \in \widehat{Q}_0$ the extended vertex and $G < \mathrm{SL}(2)$ the corresponding finite subgroup. Let H denote the Heisenberg algebra in two variables

$$H := K\langle x, y\rangle/([x,[x,y]],[y,[x,y]]).$$

Then G naturally acts on H and the slew group algebra $H*G$ is Morita equivalent to $\Lambda(\widehat{Q})$. We note that by [7, Corollary 1.7] the fixed subalgebra H^G is AS-Gorenstein. Using a slight generalization of a result by Amiot-Iyama-Reiten [1], we obtain a Heisenberg analogue of algebraic McKay correspondence.

Namely, under the above situation, we have a triangle equivalence $\mathsf{D}^{\mathrm{b}}(A(Q) \mod) \simeq \underline{\mathrm{CM}}\ H^G$ which descents to give a triangle equivalence $\mathcal{C}_2(A(Q)) \simeq \underline{\mathrm{CM}}^{\mathbf{Z}/2\mathbf{Z}} H^G$.

Appendix A. Related algebras

We point out the following isomorphism of algebras

$$\Lambda(Q) \cong (K[z]\overline{Q})/(\rho - z),$$

from which we see that $\Lambda(Q)$ is a special case of algebras introduced before.

(1) The central extension of the preprojective algebras by Etingof-Rains [3]

$$\Pi(Q)_{\lambda,\mu} := (K[z]\overline{Q})/(\rho_i - (\lambda_i z + \mu_i)e_i \mid i \in Q_1).$$

where $\lambda_i, \mu_i \in K$ for each $i \in Q_0$.

This algebra is a special case of the following algebra.

(2) The $N = 1$-quiver algebra by Cachazo-Katz-Vafa [2]

$$\Pi(Q)_P := (K[z]\overline{Q})/(\rho_i - P_i(z)e_i \mid i \in Q_1).$$

where $P_i(z) \in K[z]$ for each $i \in Q_0$.

This algebra is obtained as a pull back of the following family of algebras.

(3) The deformation family of the preprojective algebras Crawley-Boevey-Holland [5]

$$\Pi(Q)_\bullet := (K[x_1, \cdots, x_r]\overline{Q})/(\rho_i - x_i \mid i \in Q_0)$$

where $r = \#Q_0$.

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Comparing support varieties over different algebras

Steffen Koenig
(joint work with Yiping Chen)

Support varieties of modules or complexes over an algebra A are defined when A satisfies a finite generation condition (Fg), which forces A to be Gorenstein. When A is derived equivalent to another algebra B, then validity of (Fg) and support variety theory can be transfered, as shown by Külshammer, Psaroudakis and Skartsæterhagen [6]. We propose methods to transfer such information from A to B in the absence of a derived equivalence.

(Fg) condition and support varieties can be redefined in terms of small stable Hochschild cohomology (defined in [2]). Building on the long exact sequences involving Hochschild cohomology established in [5], it is shown that small stable Hochschild cohomology behaves better under recollements of derived module categories involving for instance one term having finite global dimension. In such a situation, the support varieties of objects in the middle term of the recollement are seen to coincide with their images in the outer term that is not assumed to have finite global dimension. Similarly, validity of the (Fg) condition of the middle term is determined on that outer term.

These results allow to develop a tool box (using for instance glueing in the sense of [1] and the split pairs in [4]) for removing or adding arrows or vertices or glueing algebras without changing (Fg) or support varieties.

This implies that gentle algebras, or more generally Gorenstein quadratic monomial algebras, satisfy (Fg) and their indecomposable objects have the same support varieties as objects over Nakayama algebras. The tools also work for other algebras such as cluster tilted algebras of Dynkin type.

The approach using recollements of triangulated categories complements the approach via abelian recollements developed in published and forthcoming work of Psaroudakis, Solberg and others, using for instance eventually homological isomorphisms as in [7]. Validity of the (Fg) condition for Gorenstein monomial algebras in general has been shown in parallel and independent work by Dotsenko, Gélinas and Tamaroff [3], using a very detailed study of A_∞-structures on Yoneda extension algebras.

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Dissections of surfaces and derived invariants for gentle algebras

Pierre-Guy Plamondon
(joint work with C. Amiot, S. Opper, S. Schroll)

Our aim in this talk is to see how a surface model for the derived category of a gentle algebra yields a complete derived invariant for this class of algebras. This is very much related to recent work on partially wrapped Fukaya categories of surfaces with stops (see [8, 9]), although we will focus on the algebraic and combinatorial aspects. This is mainly a report on [11, 1]. We fix a field $\mathbf{k}$.

1. Surfaces and dissections

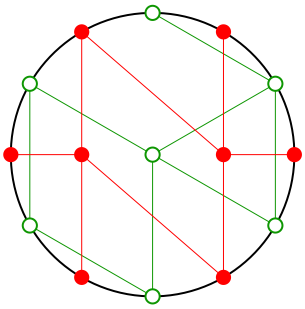
By *surface* we will always mean *surface with boundary*, that is, smooth compact oriented surface with a finite number of open discs removed. Let S be a surface and let $M = M_\circ \sqcup M_\bullet$ be a finite set of *marked points* on the boundary of S such that each boundary component of S contains at least one marked point, and that points $\circ$ in $M_\circ$ and $\bullet$ in $M_\bullet$ alternate on each boundary component. Let $P = P_\circ \sqcup P_\bullet$ be a finite set of *punctures* in the interior of S.

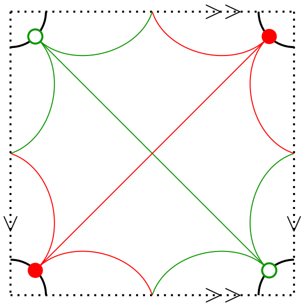
A $\circ$-*arc* (or $\bullet$-*arc*) is a smooth map from $[0, 1]$ to the surface S whose endpoints are in $M_\circ \cup P_\circ$ (or $M_\bullet \cup P_\bullet$, respectively) and whose interior is in the interior of S. We consider arcs up to (smooth) homotopy.

From now on, let (S, M, P) be a surface with marked points and punctures.

Definition 1.1. A finite collection Δ of pairwise distinct $\circ$-arcs (or $\bullet$-arcs) on the surface (S, M, P) is *admissible* if the arcs of Δ are pairwise non-intersecting in their interior, and the arcs of Δ do not enclose a subsurface containing no $\bullet$ (or no $\circ$, respectively). An *admissible* $\circ$-*dissection* (or *admissible* $\bullet$-*dissection*) is a maximal admissible collection of $\circ$-arcs (or $\bullet$-arcs, respectively).

Example 1.2. The following pictures show admissible $\circ$-dissections and their dual $\bullet$-dissections.

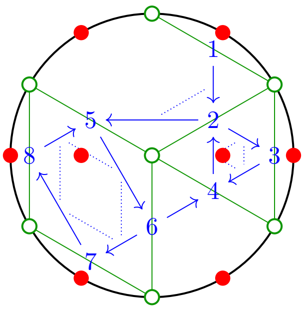




2. The gentle algebra of a dissection

Definition 2.1. Let Δ be an admissible $\circ$-dissection of (S, M, P). We define the algebra $A(\Delta) = \mathbf{k}Q(\Delta)/I(\Delta)$ as follows. The quiver $Q(\Delta)$ has vertices in bijection with the arcs of Δ, and there is an arrow from i to j for each $\circ$ shared by i and j and such that j appears immediately after i around that $\circ$ in the counter-clockwise order. The ideal $I(\Delta)$ of the path algebra $\mathbf{k}Q(\Delta)$ is generated by the path of length 2 defined as follows: whenever i, j and k are adjacent arcs in a polygon of Δ in the counter-clockwise order, and if $\alpha : i \to j$ and $\beta : j \to k$ are the corresponding arrows, then $\beta\alpha \in I(\Delta)$.

Example 2.2. Below is the quiver with relations associated to the first $\circ$-dissection of Example 1.2. Dotted lines represent relations.



Algebras obtained in this way are (possibly infinite-dimensional) gentle algebras, as introduced in [3]. In fact, this construction defines a bijection between gentle algebras (up to isomorphism) and dissected surfaces (up to oriented homeomorphism) [5, 11, 12].

3. Derived invariants

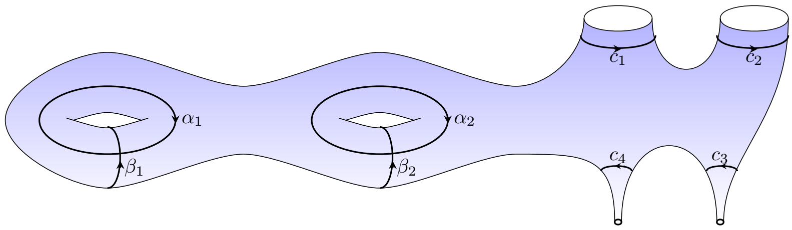
From now on, assume that (S, M, P, Δ) is a dissected surface with $P_\circ = \emptyset$. This is equivalent to assuming that the algebra $A(\Delta)$ is finite-dimensional. We are interested in the bounded derived category of $A(\Delta)$, and more specifically in the homotopy category $K^b(\operatorname{proj} A(\Delta))$. Our first result is that we can interpret objects and morphisms in $K^b(\operatorname{proj} A(\Delta))$ using curves on S and their intersections. We recall that in [6, 7], the indecomposable objects of $K^b(\operatorname{proj}(A(\Delta))$ were classified into string and band objects.

Theorem 3.1 ([11])**.** *Graded* $\circ$*-arcs on* (S, M, P) *correspond to string objects in the category* $K^b(\operatorname{proj} A(\Delta))$*; graded closed curves of winding number zero (with respect to* Δ^**), together with an indecomposable* $\mathbf{k}[t, t^{-1}]$*-module, correspond to band objects. Intersections between them correspond to the bases of the homomorphism spaces described in* [2]*.*

This result allows us to describe tilting and silting objects in the derived category of a gentle algebra.

Theorem 3.2 ([1, 10])**.** *Let* X *be a basic silting object of* $K^b(\operatorname{proj} A(\Delta))$*. Then the indecomposable summands of* X *correspond to graded* $\circ$*-arcs in* (S, M, P) *that form an admissible* $\circ$*-dissection of* (S, M, P)*.*

Finally, let $(c_1, \ldots, c_r)$ be closed curves on S, each freely homotopic to either one of the boundary components of S or to one of the punctures in $P_\bullet$. Number the boundary components of S from 1 to b. Let $\mathcal{G} = \{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ be a geometric symplectic basis, as illustrated below, and denote by $w^{\Delta^*}(\gamma)$ the winding number of a curve γ with respect to Δ^*.



Theorem 3.3 ([1])**.** *The derived equivalence class of the algebra* $A(\Delta)$ *is determined by the following data: the genus* g *of* S*; the number* p *of punctures in* $P_\bullet$*; the multiset* $\{n_1, \ldots, n_b\}$*, where each* n_i *is the number of marked points on the* i*-th boundary component of* S*; if* $g = 1$*, the number* $\gcd(w^{\Delta^*}(\gamma), w^{\Delta^*}(c_i) + 2 \mid \gamma \in \mathcal{G}, c_i \in \mathcal{B})$*; if* $g \geq 2$*, the existence of* $\gamma \in \mathcal{G} \cup \mathcal{B}$ *such that* $w^{\Delta^*}(\gamma)$ *is odd, or (if no such* γ *exists) the existence of* $c_i \in \mathcal{B}$ *such that* $w^{\Delta^*}(c_i) \equiv 0 \mod 4$*, or (if no such* c_i *exists) the number* $\sum_{i=1}^{g}(\frac{1}{2}w^{\Delta^*}(\alpha_i) + 1)(\frac{1}{2}w^{\Delta^*}(\beta_i) + 1) \mod 2$*.*

Problem 3.4. *Given only the quiver with relations* $(Q(\Delta), I(\Delta))$ *defining* $A(\Delta)$*, find an algorithm to compute all the data appearing in Theorem 3.3.*

The algorithm found in [4] computes the numbers $p, n_1, \ldots, n_b$. The genus g is easy to deduce using the fact that $Q(\Delta)$ is a retract of the surface. The other data of Theorem 3.3 remain to be computed algorithmically.

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Skew-gentle algebras and orbifolds

Sibylle Schroll
(joint work with Daniel Labardini-Fragoso, Yadira Valdivieso-Díaz)

Originating in cluster theory, geometric surface models have been instrumental in connecting representation theory with other areas of mathematics such as, for example, homological mirror mirror symmetry, see for example [9, 10] and [7]. In this extended abstract based on [6], we give an orbifold model for the bounded derived category $D^b(A)$ of a skew-gentle algebra A which encodes the indecomposable objects in $D^b(A)$ in terms of graded curves in the orbifold.

Skew-gentle algebras were introduced in [8] in the context of the study of the Auslander Reiten theory of clannish algebras. They are closely linked to the well-studied class of gentle algebras.

A quiver is a quadruple $Q = (Q_0, Q_1, s, t)$ consisting of a pair of finite sets, the vertex set Q_0 and the arrow set Q_1, and two maps $s, t : Q_1 \to Q_0$. We think of Q as a directed graph and of elements in Q_1 as the arrows, that is, directed edges $s(a) \to t(a)$, for $a \in Q_1$. Throughout let K be an algebraically closed field.

Definition 1. A K-algebra A is skew-gentle if A is Morita equivalent to an algebra KQ/I where $Q_1 = Q_1' \cup S$ with $S \subset \{a \in Q_1 \mid s(a) = t(a)\}$, $I = I' \cup \{\varepsilon^2 - \varepsilon \mid \varepsilon \in S\}$, and where KQ'/I' is a locally gentle algebra with $Q' = (Q_0, Q_1')$. Furthermore, if $\varepsilon \in S$ then $s(\varepsilon)$, as a vertex in Q', is a single arrow source, a single arrow sink or there exist exactly one arrow $a \in Q_1$ and one arrow $b \in Q_1$ with $s(b) = t(a)$ and $ab \in I'$. We call the elements of S special loops.

We note that in the above definition even if the skew-gentle algebra KQ/I is finite dimensional, the ideal I is not necessarily admissible. We refer to [8] for an admissible presentation KQ^{sg}/I^{sg} isomorphic to KQ/I.

Finite dimensional skew-gentle algebras are tame [5] and derived tame [2, 3]. In this paper we will use the description of the indecomposable objects in $D^b(A)$, for A finite dimensional skew-gentle, in terms of generalised homotopy strings and bands as given in [2].

Given a skew-gentle algebra KQ/I with underlying gentle algebra KQ'/I', we will now construct an orbifold dissection into generalised polygons. For this, recall from [11] that gentle algebras (up to isomorphism) are in bijection with surface dissections into a special set of polygons (up to homeomorphism). Let $(\Sigma', G', G'^*, M', P^*)$ be the surface dissection corresponding to KQ'/I' as defined in [11, §1], where Σ' is an oriented surface with boundary and marked points, G' is a dissection of Σ' into polygons, G'^* is a graph dual to G', M' are those vertices of G' which are in the boundary of Σ' and P^* are those vertices of G'^* which are in the interior of Σ'.

We recall that the edges in G' correspond to the vertices in $Q'_0 = Q_0$. Suppose an edge $g' \in G'$ corresponds to a vertex v in Q'_0 which is the start of a special loop. Then g' cuts out a digon in (Σ', G'). We now describe a local replacement operation resembling in each step two consecutive Whitehead moves: namely, we contract the special edge g' and expand it in the orthogonal direction producing in the process an edge g connecting a new marked point on the boundary and an orbifold point ω of order 2 as illustrated locally in the following example:

FIGURE 1. Local replacement in Σ' of the special edge g' in G' and the resulting local picture in O with new edge g connecting the new marked point $X = Y$ with a new orbifold point ω.

We do this for every edge corresponding to the start of a special loop and we obtain in this way a dissection of an orbifold which we will denote by $(O, G, G^*, M, P^*, \Omega)$, where O is an orbifold with Ω its set of orbifold points (which are all of order two), G is a dissection of O into polygons (including the edges to orbifold points), G^* is the graph dual to G, M is the set of boundary vertices of G, and P^* are the vertices of G^* in the interior of O. By construction we have a bijection between Ω and the set of special loops S and we call a polygon in G containing at least one orbifold point a *generalised polygon*.

We then show the following result, which has also been shown in [1].

Proposition 2. With the notation above, there is a bijection between skew-gentle algebras up to isomorphism and homeomorphism classes of dissections of orbifolds into polygons and generalised polygons such that each (generalised) polygon has either exactly one boundary segment with a marked point in M or is a (generalised) polygon with only internal edges with exactly one marked point in P^* in its interior. The bijection is given by the map which associates to a skew-gentle algebra KQ/I the orbifold dissection $(O, G, G^*, M, P^*, \Omega)$.

We show that a skew-gentle algebra A is Koszul and that its Koszul dual is skew-gentle with associated orbifold dissection induced by the dual graph of the orbifold dissection of A. More precisely, we show the following:

Proposition 3. Let A be a finite dimensional skew-gentle algebra. Then A is Koszul and the Koszul dual $A^!$ is skew-gentle. Furthermore, if A has orbifold dissection $(O, G, G^*, M, P^*, \Omega)$ then $A^!$ has orbifold dissection $(\tilde{O}, G^*, G, M^*, P, \Omega)$ where $\tilde{O}$ is the orbifold associated to $A!$, M^* are the boundary vertices of G^* and P corresponds to the set of non-boundary vertices of G.

We note that similarly to the case of gentle algebras, a skew-gentle algebra A is finite dimensional if and only if all the vertices of G are on the boundary of O, that is $P = \emptyset$. Furthermore, A is of finite global dimension if and only if $P^* = \emptyset$.

Let O be an orbifold as above. We recall the notion of O-free homotopy from [4].

Definition 4. Two oriented closed curves γ and γ' in an orbifold O with orbifold points of order 2 are O-homotopic if they are related by a finite number of moves given by either a homotopy in the complement of the orbifold points or a skein relation as in Figure 2 taking place in a disk D containing exactly one orbifold point ω. A segment of a curve with no self-intersection in D and passing through ω is O-homotopic relative to its endpoints to a segment spiralling around ω in either direction exactly once.

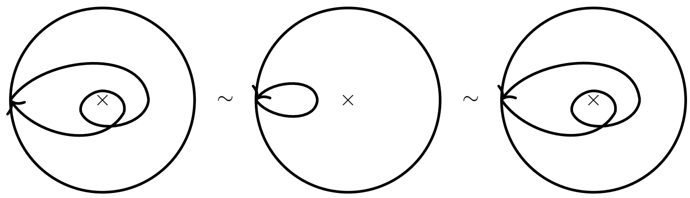

FIGURE 2. Skein relations.

Let $\mathbb{H}$ denote the upper half plane and let $\Gamma < PSL(2, \mathbb{R})$ be a Fuchsian group such that O is a suborbifold of $\mathbb{H}/\Gamma$ with geodesic boundary. It is shown in [4] that there is a natural bijection between the set of conjugacy classes of Γ and the set of O-free homotopy classes of closed oriented curves in O.

We are now in a position to state the main result of [6].

Theorem 5. Let A be a finite-dimensional skew-gentle algebra with orbifold dissection $(O, G, G^*, M, P^*, \Omega)$. Then the homotopy strings and bands of A, giving rise to the indecomposable objects in $D^b(A)$, are in bijection with graded curves (γ, f) where

(1) γ is an O-homotopy class of curves in O connecting marked points in $M \cup P^*$ or γ is an O-homotopy class of certain closed curves in O.

(2) Given a curve γ as in (1), set $f : \gamma \cap G^* \to \mathbb{Z}$ to be the map such that, denoting the segment connecting two consecutive (in the direction of γ) intersection points x_i, x_{i+1} of γ with G^* by γ_i, we have

$$f(x_{i+1}) = \begin{cases} f(x_i) + 1 & \text{if } \exists m \in M \text{ to the left of } \gamma_i \text{ in } R_i \\ f(x_i) - 1 & \text{if } \exists m \in M \text{ to the right of } \gamma_i \text{ in } R_i \end{cases}$$

where R_i is the (generalised) polygon of G^* containing γ_i.

(3) The open curves correspond to homotopy strings and the closed curves corresponding to homotopy bands are exactly those that have combinatorial winding number induced by f equal to zero.

Remark 6. (1) We note that since A is finite-dimensional the segment γ_i will lie in exactly one (generalised) polygon of G^* which by construction has exactly one boundary segment with a marked point in M. Thus the map f is well-defined.

(2) In ongoing work we are showing the connection between 'well-graded' intersections of two graded curves and maps between the corresponding objects in $D^b(A)$.

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Derived equivalences between skew-gentle algebras using orbifolds

Thomas Brüstle
(joint work with Claire Amiot)

Gentle algebras provide an example of a class of algebras whose derived category can be described explicitly. The class of gentle algebras contains all finite dimensional path algebras of type $\mathbb{A}$ and $\widetilde{\mathbb{A}}$, and has been shown to be stable under derived equivalences. More recently, gentle algebras have been found to be deeply connected to the combinatorics and geometry of marked surfaces: The Jacobian algebra of a triangulation of an unpunctured surface $(\mathcal{S}, M)$ is a gentle algebra. Thus certain gentle algebras appear as endomorphism ring of cluster-tilting objects in the cluster category $C(\mathcal{S}, M)$ associated to the cluster algebra of a marked surface $(\mathcal{S}, M)$ without punctures. Building on this, a geometric model for the objects in the cluster category $C(\mathcal{S}, M)$ can be given associating strings and bands with curves and closed curves.

Obviously, triangulations of surfaces yield only certain gentle algebras. This shortcoming has been overcome in [BCS] and [OPS] by relating every gentle algebra to a dissection of a marked surface, cutting $(\mathcal{S}, M)$ into polygons. Using this correspondence one can obtain a geometric description of the module category of a gentle algebra, or of its derived category.

Independently, Haiden, Katzarkov and Kontsevich establish in [HKK] a description of the (partially wrapped) Fukaya category of a surface $\mathcal{S}$ with stops using the derived category of a (graded) gentle algebra associated to these data, also given by a dissection of $\mathcal{S}$. Combining results in [OPS] and [LP], a geometric interpretation of the derived equivalence relation for gentle algebras is given in [APS] and [O].

We explain in this talk how to extend these results to orbifolds $\bar{\mathcal{S}}$ admitting a two-fold cover. The two-fold cover $\mathcal{S}$ corresponds to a gentle algebra which comes equipped with a $\mathbb{Z}_2$-action. The corresponding skew-group algebra is studied in [GePe], called skew-gentle algebra. This class of algebras contains in particular all path algebras of type $\mathbb{D}$ and $\widetilde{\mathbb{D}}$. We employ this point of view, where a description of the derived category of a skew-gentle algebra can be obtained using the $\mathbb{Z}_2$-action, and the known results for gentle algebras.

Looking back to the cluster algebra of a triangulated surface, the orbifold points correspond to punctures, and the fact that the Jacobian algebra admits a $\mathbb{Z}_2$-action corresponds to having all orbifold points lying in a self-folded triangle. The description of the cluster category for punctured surfaces with skew-gentle algebras has been given in [AP] using a $\mathbb{Z}_2$-action on the category and on the surface. We follow a similar approach, generalizing it to study the derived category in the case of an orbifold allowing a dissection such that all orbifold points are uniquely connected by an arc to the boundary (this is the polygonal equivalent of the self-folded triangle in the cluster situation).

Of course, the class of skew-gentle algebras is not stable under derived equivalences, not even the simplest case of type $\mathbb{D}$ satisfies this. It is however natural to ask what is the geometric interpretation of the derived equivalence relation for skew-gentle algebras.

To answer this, we study the $\mathbb{Z}_2$-action, both on the algebraic side of the skew-gentle algebras, and on their geometric realizations. To any dissected surface which is invariant under the action of an order-2 diffeomorphism (with finitely many fixed points), we associate a gentle algebra Λ together with a $\mathbb{Z}_2$ action and an orbifold together with an orbifold dissection.

We give a geometric interpretation of the derived equivalence relation for skew-gentle algebras when the equivalence is given by a $\mathbb{Z}_2$-invariant tilting object. The $\mathbb{Z}_2$-invariant line field η of the double cover induces a line field $\bar{\eta}$ on the orbifold, and we have the following characterization:

Theorem 1. Two skew-gentle algebras $\bar{\Lambda}$ and $\bar{\Lambda}'$ are derived equivalent via a $\mathbb{Z}_2$-invariant tilting object if and only if there exists a diffeomorphism between their corresponding orbifolds sending $\bar{\eta}$ to $\bar{\eta}'$ up to homotopy.

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Derived equivalence classification of Brauer graph algebras via A_∞-algebras

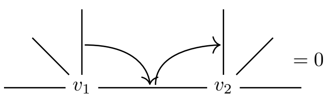
Alexandra Zvonareva

(joint work with Sebastian Opper)

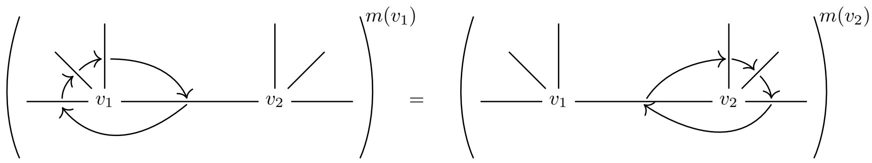
Brauer graph algebras, or equivalently symmetric special biserial algebras, first appeared in modular representation theory in form of blocks with cyclic [5] or dihedral defect group [6]. Roughly speaking, a Brauer graph algebra can be constructed from the data of a graph Γ with a cyclic ordering of edges around each vertex and a multiplicity function on the set of its vertices $m : V(\Gamma) \to \mathbb{N}_{>0}$. Note that such a graph can be embedded into a unique closed oriented surface $\mathcal{S}_\Gamma$ in such a way that its complement is a union of open discs and the ordering of edges comes from the orientation of the surface (when drawn on the plane, the ordering of the edges will always be assumed to go clockwise). In this case the graph Γ cuts the surface $\mathcal{S}_\Gamma$ into polygons called the faces of Γ, if a certain face of Γ is an n-gon, then n will be called the perimeter of that face.

To the data (Γ, m) one can associate a path algebra of a quiver with relations $A \simeq kQ_\Gamma/I_\Gamma$ (we assume that k is an algebraically closed field), the algebras of this form are called Brauer graph algebras. The quiver Q_Γ can be constructed as follows: the vertices of Q_Γ correspond to the edges of Γ, the arrows of Q_Γ are induced from the ordering of edges (see figures).

The arrows of Q_Γ are naturally divided into cycles, corresponding to vertices of Γ. The ideal of relations I_Γ is generated by two types of relations, graphically they can be given as follows:



Composing not in the cyclic ordering is 0.



For an edge (v_1, v_2), going $m(v_1)$ times around the cycle corresponding to v_1 is the same as going $m(v_2)$ times around the cycle corresponding to v_2.

The derived equivalence classification of Brauer graph algebras was considered in the literature over the past years. The classification for the case of Brauer graph algebras of finite representation type was provided by Rickard [9]. Antipov classified Brauer graph algebras up to derived equivalence in the case when the surface $\mathcal{S}_\Gamma$ is a sphere [1]. He also produced a list of combinatorial invariants of the derived category of any Brauer graph algebra (see Theorem) [1, 3]. Furthermore, the class of Brauer graph algebras is closed under derived equivalence [2, 3].

It turns out that Brauer graph algebras can be consider as a part of a bigger class of A_∞-algebras, which can be constructed from a full collection of arcs on a surface, equipped with a line field. Working in this much larger class of algebras it becomes easier to produce equivalences of derived categories. Using ideas of Bocklandt [4] and Haiden-Katzarkov-Kontsevich [7], as well as results of Lekili-Polishchuk [8] we can prove the following theorem:

Theorem 1. Let B and B' be Brauer graph algebras with Brauer graphs Γ and Γ' (assume that neither Γ nor Γ' is a loop with multiplicity 1 or an edge with multiplicity 2 at both ends). Then, B and B' are derived equivalent if and only if the following conditions are satisfied.

(1) Γ and Γ' have the same number of vertices, edges and faces, i.e. the surfaces of B and B' are homeomorphic;

(2) the multisets of perimeters of faces and the multisets of the multiplicities at vertices of Γ and Γ' coincide;

(3) Either both or none of Γ and Γ' are bipartite.

The exception for the case of a loop with multiplicity 1 or an edge with multiplicity 2 at both ends has to be made, since in this case the Brauer graph is not invariant under isomorphism of algebras [3].

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Torsion pairs for cluster-tilted algebras of type $\tilde{A}$

Rosanna Laking
(joint work with Karin Baur)

We presented an account of the classification of cosilting modules over cluster-tilted algebras of type $\tilde{A}$ contained in [3]. In particular, we emphasised the strong links between cosilting modules and the torsion pairs in the category of finite-dimensional modules.

Let A be a finite-dimensional algebra over a field k. We will denote the category of left A-modules by $\mathrm{Mod}(A)$ and the full subcategory of finite-dimensional left A-modules by $\mathrm{mod}(A)$. In an abelian category $\mathcal{A}$ (e.g. $\mathrm{mod}(A)$ or $\mathrm{Mod}(A)$) we define a *torsion pair* to be a pair of full subcategories $(\mathcal{T}, \mathcal{F})$ such that $\mathrm{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0$ and such that every object $X \in \mathcal{A}$ has a subobject $t(X) \in \mathcal{T}$ such that $X/t(X) \in \mathcal{F}$. We may consider the torsion pairs in $\mathrm{mod}(A)$ and $\mathrm{Mod}(A)$ and ask how they relate to each other.

Since $\mathrm{mod}(A)$ is closed under subobjects, we have that every torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathrm{Mod}(A)$ restricts to a torsion pair $(\mathcal{T} \cap \mathrm{mod}(A), \mathcal{F} \cap \mathrm{mod}(A))$ in $\mathrm{mod}(A)$. Note that there may be many torsion pairs in $\mathrm{Mod}(A)$ that restrict to the same torsion pair in mod(A). Crawley-Boevey [5, Lem. 4.4] showed that, if (t, f) is a torsion pair in mod(A), then $(\varinjlim t, \varinjlim f)$ is a torsion pair in Mod(A). Moreover, he showed that $\varinjlim f$ consists of the modules F such that $\mathrm{Hom}(t, F) = 0$ and so we have that $(\varinjlim t, \varinjlim f)$ restricts to (t, f). Torsion pairs in Mod(A) of this form are therefore in bijection with torsion pairs in mod(A) and it turns out that they can be characterised as those with $\mathcal{F} = \varinjlim \mathcal{F}$. Such torsion pairs are known as torsion pairs of *finite type*.

Another characterisation of torsion pairs of finite type in Mod(A) has recently emerged from silting theory: a torsion pair is of finite type if and only if it is a cosilting torsion pair (see, for example, [1]). In particular, for every torsion pair $(\mathcal{T}, \mathcal{F})$ of finite type in Mod(A), there exists a cosilting module C such that $\mathcal{T}$ consists of the modules T such that $\mathrm{Hom}_A(T, C) = 0$ and $\mathcal{F}$ is exactly the class of modules cogenerated by C.

Cosilting modules C are not necessarily finite-dimensional, however they are pure-injective [4, 8], which, over a finite-dimensional algebra, means that they arise as direct summands of set-indexed products of finite-dimensional modules. The pure-injective modules are a particularly well-behaved class of modules and in some cases it is even possible to classify the indecomposable pure-injective modules over a given algebra. For example, Prest and Puninski [6] classify the indecomposable pure-injective modules over domestic string algebras (when k is algebraically closed).

Since every algebra A that is cluster-tilted algebras of type $\tilde{A}$ is a domestic string algebra, their classification yields the following complete list of indecomposable pure-injective A-modules :

- string modules indexed by finite and infinite strings (in the sense of [7]);
- band modules indexed by the set $\mathrm{k}^* \cup \{\infty, -\infty\}$;
- a generic module.

It was shown in [2] that the class of cluster-tilted algebras of type $\tilde{A}$ coincides with the class of surface algebras coming from triangulations Γ of the annulus with finitely many marked points in its boundary (where *triangulation* means a maximal collection of noncrossing arcs whose endpoints are marked points). Moreover, the finite-dimensional string modules are in one-to-one correspondence with arcs on the surface (that are not contained in Γ) up to end-point fixing homotopy. We show that the string modules indexed by infinite strings are in bijection with so-called asymptotic arcs.

Let A be a cluster-tilted algebra of type $\tilde{A}$ corresponding to an annulus S with marked points M. Using the geometric description of the strings modules, we may classify the cosilting modules and hence all torsion pairs in mod(A). First we consider the finite-dimensional cosilting modules. For every triangulation of (S, M), the corresponding direct sum of string modules is a finite-dimensional cosilting A-module. Moreover, every finite-dimensional cosilting module is equivalent to such a module.

The infinite-dimensional cosilting A-modules are similar but they may contain direct summands that are Prüfer, adic or generic modules. Every maximal collection T of arcs containing at least one asymptotic arc together with a partition $P_1 \sqcup P_2 = \mathrm{k}^*$ yields a cosilting A-module. The direct summands of this module are given by the collection of string modules corresponding to the arcs in T, the Prüfer modules parametrised by P_1, the adic modules parametrised by P_2 and the generic module. Moreover, such collections parametrise the equivalence classes of infinite-dimensional cosilting modules.

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Torsion classes of gentle algebras

Aaron Chan

(joint work with Laurent Demonet)

For an abelian category $\mathcal{A}$ consisting only of finite length objects, recall that a full subcategory is called torsion class if it is closed under taking quotients and extensions. Torsion classes play a central role in the homological behaviour of a finite-length abelian category, c.f. (aisles of) t-structures in triangulated categories.

The combinatorics arising from the collection $\mathsf{tors}(\mathcal{A})$ of torsion classes is just as interesting. In fact, it is a complete lattice, meaning that any set of torsion classes has a greatest common lower bound given by their intersection, and a least common upper bound given by the smallest torsion class containing their union. For various lattice theoretic results, particularly in the case when $\mathsf{tors}(\mathcal{A})$ is finite, see [3].

In general, classifying torsion classes is an extremely difficult task - especially if one does not want to appeal to big module theory (see [1]). Nevertheless, in the case where $\mathcal{A}$ is the category of finitely generated modules over a so-called special biserial algebra, there are various expositions in the literature that give us some hopes in completing this task.

Firstly, for special biserial algebras, the classification of finite-dimensional indecomposable modules as well as their homomorphisms are completely known. These can be described easily using the so-called string combinatorics. Indeed, there already exist descriptions, in terms of strings, of the Ext-projective generator (or the associated two-term silting complex) of a functorially finite torsion class. However, the combinatorics in these cases can be cumbersome to work with. Taking various ideas and techniques from [3], it occurs to us that the better setting is to take $\mathcal{A}$ to be the category $\mathsf{fl}.\Lambda$ of finite length modules over a (possibly infinite-dimensional) gentle algebra Λ.

Secondly, it is now well-known that gentle algebras are in correspondence with marked surfaces [5]. The finite length indecomposable modules can be interpreted as closed curves or arcs (meaning curves that connect marked points) on the associated surface. This connection between gentle algebras with topology gives us an alternative, and often useful, way to think about various homological behaviour. Indeed, the functorially finite torsion classes also admit nice interpretation in terms of surface combinatorics. For example, in the case of Jacobian algebras arising from surface triangulations, the Ext-projective generator of a functorially finite torsion class can be combinatorially described by a maximal set of pairwise non-crossing arcs.

One of the ideas in Fomin and Thurston article [4] is that, in order for the combinatorics to be better-behaved (from the perspective of computing g-vectors), one can look at laminations of the surface instead of maximal non-crossing sets. Algebraically, this means that one will work with two-term (silting) complexes instead of their zeroth homology. Combinatorially, all we need to do is to perturb the endpoints of an arc. Namely, if an endpoint is in the interior (resp. boundary), then its neighbourhood in the curve is modified to wind indefinitely towards (resp. hit a boundary slightly away from) this point, in an appropriately chosen direction. Such a modification will change a maximal non-crossing set of arcs into a maximal set of pairwise non-crossing curves which is called a lamination. In particular, the g-vector of (the two-term silting complex associated to) the corresponding Ext-projective generator is just the shear coordinate of the lamination.

Even though a general torsion class in $\mathsf{fl}.\Lambda$ does not necessarily admit a Ext-projective generator, inspired by Fomin-Thurston's idea, we speculate that there is still a combinatorial replacement for the Ext-projective generator, namely, we simply relax the restriction of a lamination so that it can contain curves that go off indefinitely wherever they want.

Finally, by "orienting crossings" between two curves, one can give the collection of non-crossing sets of arcs the structure of a poset. It can be observed from existing results in the literature that this poset structure is compatible with the poset structure on the collection of functorially finite torsion classes. Now, it is natural to expect this combinatorics of oriented crossings can be generalized to the setting of generalized laminations.

Combining these ideas, we obtain an isomorphism (of complete lattices) between $\mathsf{tors}(\mathsf{fl}.\Lambda)$ and the complete lattice of generalized laminations of the associated surface. This talk aims to explain the necessary definitions that goes into this result so that one can now compute torsion classes from generalized laminations.

In view of the recent article by Baur and Laking [2], we expect that our result essentially gives a description of all cosilting modules over gentle algebras, even though the classification of the pure-injective (big) modules over a lot of gentle algebras (precisely, those of non-domestic type) seems to be unknown. To finish, let us remark also that there are still various aspects of the theory that remains mysterious - for example, how one could compute the Bongartz (co)completion explicitly using the combinatorics of generalized laminations. Unfortunately for us, who want to stay inside the finite length setting as much as possible, it seems that a better understanding of the Hom-spaces between the cosilting modules over the gentle algebras seems to be necessary.

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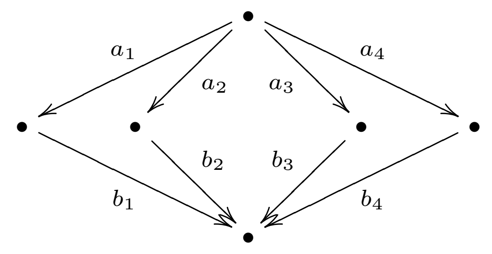
Derived tameness of quasi-gentle algebras

Igor Burban

(joint work with Yuriy Drozd)

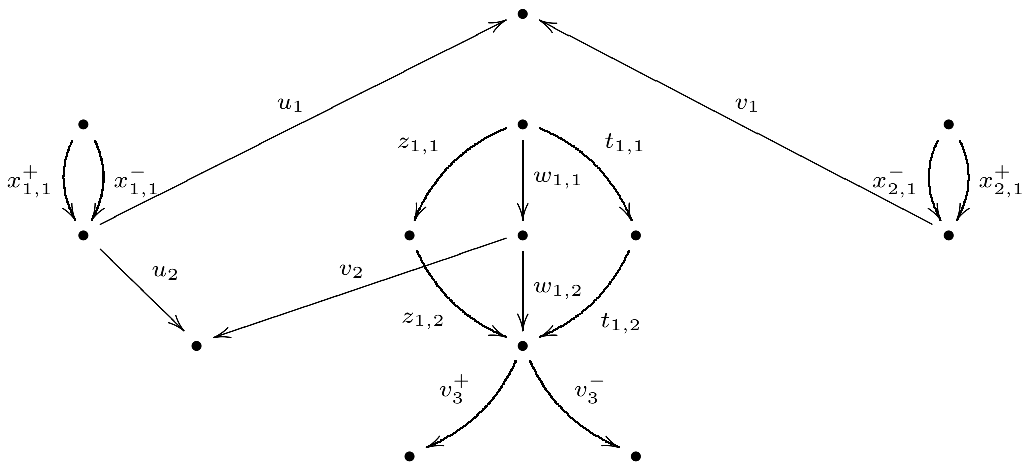
Quasi-gentle algebras were introduced in [4]. As special cases, these class of algebras includes the gentle and skew-gentle algebras.

Example (degenerate tubular algebra). The path algebra of the quiver



subject to the relations $b_2a_2 = b_3a_3$ and $b_1a_1 + b_2a_2 + b_4a_4 = 0$, is a quasi-gentle algebra.

Example. The path algebra of the quiver



modulo the relations:

$$\begin{cases} z_{1,2}z_{1,1} + w_{1,2}w_{1,1} + t_{1,2}t_{1,1} = 0 \\ v_3^{\pm} w_{1,2} = 0 \\ u_1 x^-_{1,1} = 0 = v_1 x^+_{2,1} \\ u_2 x^+_{1,1} = 0 = v_2 w_{1,1} \end{cases}$$

is a quasi-gentle algebra.

Theorem (Burban & Drozd). Let Λ be a quasi-gentle algebra over an algebraically closed field $\mathbb{k}$ and $\mathsf{Hot}^*\big(\mathsf{pro}(\Lambda)\big)$ be the homotopy category of complexes of finitely generated projective Λ-modules, where $* \in \{+,-,b,\emptyset\}$. Then $\mathsf{Hot}^*\big(\mathsf{pro}(H)\big)$ is representation-tame.

Following the scheme, proposed for the first time in [2], a proof of this result is obtained by constructing a representation embedding

$$\mathsf{Hot}^*\big(\mathsf{pro}(\Lambda)\big) \xrightarrow{\Phi} \mathsf{Rep}^*\big(\mathfrak{X}_\Lambda\big)$$

(i.e. Φ is a $\mathbb{k}$-linear functor, which reflects indecomposability and isomorphism classes of objects), where $\mathsf{Rep}^*\big(\mathfrak{X}_\Lambda\big)$ is the category of representations of a *bunch of semi-chains* $\mathfrak{X}_\Lambda$, known to be representation tame by a work of Bondarenko [1].

A description of $\mathfrak{X}_\Lambda$ is essentially based on a detailed study of the Auslander–Reiten quiver of the category of vector bundles on a weighted projective line of type $(2,2,n)$, made by Kussin, Lenzing and Meltzer [5].

As a consequence of the derived tameness of quasi-gentle algebras, we complete the classification of derived-tame non-commutative nodal curves [4].

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Two decomposition results for bipersistence modules

Steve Oudot
(joint work with Magnus Botnan, Jérémy Cochoy, Vadim Lebovici)

Recent work by Botnan and Crawley-Boevey [3] shows that pointwise finite-dimensional (pfd) representations of posets over an arbitrary field $\mathbf{k}$ decompose as direct sums of indecomposables with local endomorphism ring. Here we are interested in the poset $\mathbb{R}^d$ equipped with the product order. This choice is motivated by applications in topological data analysis (TDA), where representations of this poset arise naturally. While the poset is of wild representation type (for $d > 1$), we are only interested in a subclass of its indecomposables, called *interval modules*, which by definition are indicator representations $\mathbf{k}_I$ of connected and convex subsets (called *intervals*) I of $\mathbb{R}^d$. Here, connectivity and convexity are understood in the product order, see Figure 1 for an example where $d = 2$.

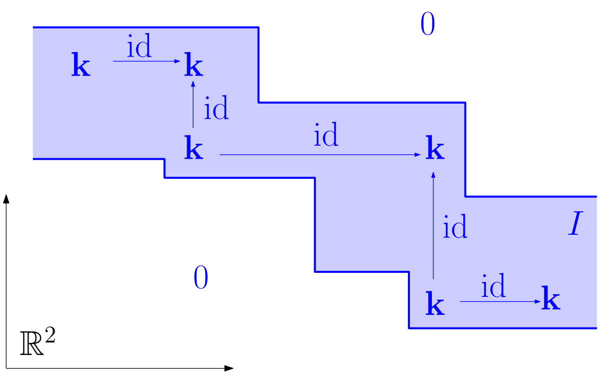

Figure 1. An interval $I \subset \mathbb{R}^2$ and its associated interval module $\mathbf{k}_I$.

These indecomposables play a key role in TDA. Indeed, given a pfd representation M, the collection of the supports of the interval summands appearing in its direct-sum decomposition can be used as a descriptor of M—called its *barcode*—and thereby also as a descriptor of the data from which the representation originates. This descriptor is purely geometric by nature, therefore easy to interpret for practitioners, and efficient to encode and manipulate on a computer. Furthermore, its stability properties, proven in the TDA literature [7, 8], make it a relevant choice to derive consistent estimators in statistical analysis.

From the computational point of view, an important question is to be able to determine quickly whether a given representation M of $\mathbb{R}^d$ has interval summands or not, and if so, to extract these summands. An obvious solution for this consists in decomposing M then checking its summands one by one, however this is far too costly in practice. Rather, one seeks to find *local*, easily checkable conditions, that guarantee the existence of such summands, or at least, that test whether M entirely decomposes into interval summands. Such conditions can be useful not only for algorithms, but also for the mathematical analysis of certain classes of problems in data analysis, for which they may help assert that such problems do (or do not) yield interval summands when TDA is applied to them.

The work presented here tackles this question in the special case where $d = 2$. Note that the case where $d = 1$ is already known, and the question in that case is actually trivial because every pfd representation decomposes into interval summands without any further conditions [6]. In the case $d = 2$ however, we can show that no local condition for full interval-decomposability exists, for any reasonable notion of locality. For instance, restricting the focus to the fully commutative square grid $G_n = [\![1, n]\!]^2$, we can exhibit non-thin indecomposable representations of G_n whose restrictions to any strict subgrid of G_n fully decompose into interval summands. One such representation is shown below, where $\iota_j : \mathbf{k} \to \mathbf{k}^{n-1}$ denotes the canonical inclusion of the j-th copy of $\mathbf{k}$ into the direct sum $\mathbf{k}^{n-1}$.

$$
\begin{array}{ccccccccc}
\mathbf{k} & \xrightarrow{\iota_1} & \mathbf{k}^{n-1} & \xrightarrow{\mathrm{id}} & \mathbf{k}^{n-1} & \overset{\mathrm{id}}{\dashrightarrow} & \mathbf{k}^{n-1} & \xrightarrow{\mathrm{id}} & \mathbf{k}^{n-1} \\
\uparrow & & \uparrow{\scriptstyle \iota_2} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \mathrm{id}} \\
0 & \longrightarrow & \mathbf{k} & \xrightarrow{\iota_2} & \mathbf{k}^{n-1} & \overset{\mathrm{id}}{\dashrightarrow} & \mathbf{k}^{n-1} & \xrightarrow{\mathrm{id}} & \mathbf{k}^{n-1} \\
\uparrow & & \uparrow & & & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \mathrm{id}} \\
 & & & & \ddots & & \mathbf{k}^{n-1} & \xrightarrow{\mathrm{id}} & \mathbf{k}^{n-1} \\
 & & & & & & \uparrow{\scriptstyle \iota_{n-1}} & & \uparrow{\scriptstyle \mathrm{id}} \\
0 & \longrightarrow & 0 & \longrightarrow & & \longrightarrow & \mathbf{k} & \xrightarrow{\iota_{n-1}} & \mathbf{k}^{n-1} \\
\uparrow & & \uparrow & & & & \uparrow & & \uparrow{\scriptstyle \begin{bmatrix} 1 \\ | \\ 1 \end{bmatrix}} \\
0 & \longrightarrow & 0 & \longrightarrow & & \longrightarrow & 0 & \longrightarrow & \mathbf{k}
\end{array}
\tag{1}
$$

This negative result settles the question in 2-d. From here, one might ask whether there are reasonable subclasses of the interval modules that enjoy a local characterization. This new question can be answered positively, and the classes of summands of interest are the ones of so-called *rectangle modules* and *block modules*. A rectangle is a product of two 1-d intervals, and a block is a rectangle that extends to infinity in at least 2 directions, more precisely it is either an upper-right quadrant, or a lower-left quadrant, or a vertical band, or a horizontal band (other types of rectangles extending to infinity in 2 directions or more are forbidden). The corresponding interval modules can be characterized locally as follows: an interval module over $\mathbb{R}^2$ is a rectangle (resp. block) module if and only if all its restrictions to squares $\{x, x'\} \times \{y, y'\} \subset \mathbb{R}^2$ are. More generally, we can establish the following decomposition theorem:

Theorem 1 ([2, 3, 5]). A pfd representation of $\mathbb{R}^2$ decomposes as a direct sum of rectangle (resp. block) modules if and only if all its restrictions to squares $\{x, x'\} \times \{y, y'\} \subset \mathbb{R}^2$ do.

This result (in fact two results, distinguishing between the rectangle case and block case) has important implications in TDA, for instance to the stability theory of the class of representations of $\mathbb{R}^2$ called *level-sets persistence modules* [1, 4], which find applications e.g. in the study of Reeb graphs and their approximations from data. From there, a number of open questions arise, most notably:

- Working out local conditions that assert the presence of interval (resp. rectangle or block) summands in the decomposition, not just the full interval (resp. rectangle or block) decomposability. Such conditions would greatly benefit to the algorithms computing decompositions in general.
- Extending the analysis to larger classes of indecomposables beyond the interval modules, and to larger classes of posets beyond $\mathbb{R}^2$. In particular, understanding what in the poset structure allows us to derive such local conditions as above.

These questions will be the subject of further investigations by the TDA community in the near future. Our hope is that the representation theory community will be interested as well, and take part in this effort.

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Derived equivalences via HRS-tilting

Xiao-Wu Chen
(joint work with Zhe Han, Yu Zhou)

Let $\mathcal{A}$ be an abelian category. A *torsion pair* $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$ consists of two full subcategories subject to the following conditions.

(1) $\mathrm{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0$, that is, $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$ for any $T \in \mathcal{T}$ and $F \in \mathcal{F}$;
(2) For any object X in $\mathcal{A}$, there exists a short exact sequence

$$0 \longrightarrow T \longrightarrow X \longrightarrow F \longrightarrow 0$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$, called the *decomposition sequence* of X.

The decomposition sequence is unique up to isomorphism.

Following [7], a torsion pair $(\mathcal{T}, \mathcal{F})$ is said to be *tilting* provided that any object in $\mathcal{A}$ is isomorphic to a sub object of some object in $\mathcal{T}$; dually, the torsion pair is *cotilting* provided that any object is isomorphic to a factor object of some object in $\mathcal{F}$. As these terminologies suggest, torsion pairs arise naturally in the classical tilting theory [6, 5].

Denote by $\mathbf{D}^b(\mathcal{A})$ the bounded derived category of $\mathcal{A}$. We identify objects in $\mathcal{A}$ with stalk complexes concentrated in degree zero. The key observation is made in [7]: associated to any torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$, the following full subcategory of $\mathbf{D}^b(\mathcal{A})$

$$\mathcal{B} = \{X \in \mathbf{D}^b(\mathcal{A}) \mid H^{-1}(X) \in \mathcal{F}, H^0(X) \in \mathcal{T}, H^i(X) = 0 \text{ for } i \neq -1, 0\}$$

is abelian, called the *(forward) HRS-tilt* of $\mathcal{A}$ with respect to $(\mathcal{T}, \mathcal{F})$. Indeed, we have a bounded t-structure $(\mathcal{U}^{\leq 0}, \mathcal{U}^{\geq 0})$ on $\mathbf{D}^b(\mathcal{A})$, where

$$\mathcal{U}^{\leq 0} = \{U \in \mathbf{D}^b(\mathcal{A}) \mid H^0(U) \in \mathcal{T}, H^i(U) = 0 \text{ for } i > 0\}, \text{ and}$$
$$\mathcal{U}^{\geq 0} = \{V \in \mathbf{D}^b(\mathcal{A}) \mid H^{-1}(V) \in \mathcal{F}, H^i(V) = 0 \text{ for } i < -1\}).$$

As $\mathcal{B}$ is the *heart* of this t-structure, it is naturally an abelian category.

Let us recall some general facts on bounded t-structures. Let $\mathcal{D}$ be a triangulated category, and let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a bounded t-structure. Then we have the heart $\mathcal{H} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ and its bounded derived category $\mathbf{D}^b(\mathcal{H})$. By a *realization functor* of the bounded t-structure, we mean a triangle functor

$$G \colon \mathcal{D}^b(\mathcal{H}) \longrightarrow \mathcal{D}$$

whose restriction on $\mathcal{H}$ is isomorphic to the inclusion $\mathcal{H} \hookrightarrow \mathcal{D}$. If the triangulated category $\mathcal{D}$ is *algebraic*, that is, triangle equivalent to the stable category of a Frobenius category, such a realization functor always exists [9]. We mention that the original construction of a realization functor via filtered triangulated categories is given by [1]; compare [2].

We observe that a realization functor is unique on the level of objects. However, it is a very subtle issue whether a realization functor is unique. Despite the lack of uniqueness, we still often say *the* realization functor.

In general, a realization functor is not an equivalence. It is a standard fact that a fully-faithful realization functor is dense, and thus an equivalence [2]. The converse is somehow surprising to us, although the proof is standard; see [4].

Theorem A. *Let $G\colon \mathcal{D}^b(\mathcal{H}) \to \mathcal{D}$ be the realization functor as above. Assume that G is dense. Then G is fully-faithful, and thus a triangle equivalence.*

The following classical result unifies the corresponding derived equivalences induced by classical tilting modules over artin algebras [6] and tilting sheaves on weighted projective lines [5].

Theorem. (Happel-Reiten-Smalø) *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{A}$, and let $\mathcal{B}$ be the forward HRS-tilt. Assume that $(\mathcal{T}, \mathcal{F})$ is tilting or cotilting. Then the corresponding realization functor*

$$G\colon \mathbf{D}^b(\mathcal{B}) \longrightarrow \mathbf{D}^b(\mathcal{A})$$

is a triangle equivalence.

We mention that there are torsion pairs, neither tilting nor cotilting, whose corresponding realization functor is a derived equivalence. Indeed, the examples arise from two-term tilting complexes [8, 3]. We point out that the well-known HW-reflection is induced from a term-term tilting complex.

As the HRS-tilt plays an essential role in both quasi-tilted algebras and stability conditions for certain geometric objects, it might be of great interest to know when precisely the realization functor in an HRS-tilt is a derived equivalence. The following main result answers this question in full generality; see [4].

Theorem B. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{A}$, and let $\mathcal{B}$ be the forward HRS-tilt. Denote by $G\colon \mathbf{D}^b(\mathcal{B}) \to \mathbf{D}^b(\mathcal{A})$ the corresponding realization functor. Then the following statements are equivalent.*

1. *(1) The realization functor G is an equivalence.*
2. *(2) The subcategory $\mathcal{A}$ lies in the essential image of G.*
3. *(3) For each object $X \in \mathcal{A}$, there is an exact sequence in $\mathcal{A}$*

$$\eta_X\colon 0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow X \longrightarrow T^0 \longrightarrow T^1 \longrightarrow 0$$

with $F^i \in \mathcal{F}$ and $T^i \in \mathcal{T}$, such that the corresponding class $[\eta_X]$ in the Yoneda extension group $\mathrm{Yext}^3_{\mathcal{A}}(T^1, F^0)$ vanishes.

The proof of Theorem B uses the *backward HRS-tilt* of $\mathcal{B}$ with respect to the induced torsion pair. One of the key ingredients is a categorical version of [3, Proposition 4.1 and Theorem 4.4].

We mention that the condition (3) is intrinsic. In view of it, the classical result of Happel-Reiten-Smalø follows immediately. Unlike the decomposition sequence, the exact sequence η_X is not unique in general. There is an example in [4] to show that the vanishing condition on $[\eta_X]$ is necessary.

In view of the condition (2), the following question is natural.

Question. *Let $\mathcal{A}$ be an abelian category, and let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a bounded t-structure. Denote by $\mathcal{H} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ the heart, and by $G\colon \mathbf{D}^b(\mathcal{H}) \to \mathbf{D}^b(\mathcal{A})$*

the corresponding realization functor. Assume that $\mathcal{A}$ is contained in the essential image of G. Is G a derived equivalence?

The answer to this question is affirmative, provided that the abelian category $\mathcal{A}$ is hereditary. Indeed, in this situation, the realization functor G is dense. Then the assertion follows from Theorem A.

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Silting theory of orders modulo a regular sequence

Wassilij Gnedin

Let R be a commutative complete local Noetherian ring, and let $\mathbf{x}$ be some R-regular sequence of elements in the maximal ideal $\mathfrak{m}$ of R. In [2], Eisenbud studied the question how the homological algebra of the ring R differs from that of its lower-dimensional quotient $\overline{\mathrm{R}} = \mathrm{R}/\mathbf{x}\mathrm{R}$. We shall be concerned with a non-commutative analogue of this question in the framework of derived categories.

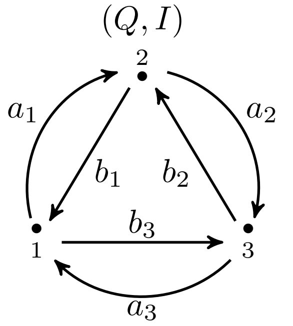
To simplify the exposition, we assume that the base ring R is regular. Let Λ be an R-*order*, by which we mean an R-algebra Λ such that Λ is finitely generated and free as an R-module. In particular, the R-algebra Λ is $\mathbf{x}$-regular.

We would like to compare the derived representation theory of the ring Λ to that of its quotient $\overline{\Lambda} = \Lambda/\mathbf{x}\Lambda$. Both rings have the same finite number n of isomorphism classes of of simple modules. However, the natural *push down functor*

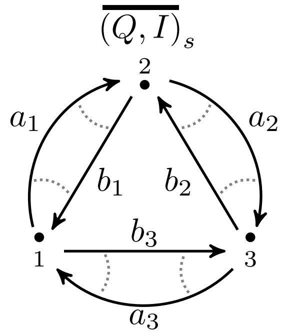
$$\mathbb{P}\colon \mathcal{D} = \mathrm{D}^-(\operatorname{mod}\Lambda) \longrightarrow \overline{\mathcal{D}} = \mathrm{D}^-(\operatorname{mod}\overline{\Lambda}), \qquad L^\bullet \longmapsto \overline{L^\bullet} = L^\bullet \overset{\mathbb{L}}{\otimes}_\Lambda \overline{\Lambda}$$

is usually not dense and does not reflect isomorphism classes of objects.

Example. Let Λ be the arrow ideal completion of the preprojective algebra of affine type $\widetilde{\mathbb{A}}_2$. Its quiver (Q, I) is shown on the left:



$$b_1a_1 = a_3b_3 \quad b_2a_2 = a_1b_1 \quad b_3a_3 = a_2b_2$$



$$b_ia_i = 0 \quad a_ib_i = 0 \quad i\in\{1,2,3\}$$

It can be shown that the ring Λ is an R-order with respect to the structure morphism

$$\mathrm{R} = \Bbbk[\![s,t]\!] \longrightarrow \Lambda, \quad s \longmapsto \sum_{i=1}^{3} b_ia_i, \quad t \longmapsto \text{sum of all 3-cycles in } (Q, I).$$

Any element $p \in \mathfrak{m} = (s, t)$ is R-regular and $\overline{\Lambda}_p = \Lambda/p\Lambda$ is a one-dimensional order. For the special choice $p = s$, the $\Bbbk[\![t]\!]$-order $\overline{\Lambda}_s$ is isomorphic to the completed path algebra of the gentle quiver $\overline{(Q, I)}_s$ on the right. We note that the perfect complex $P_1 \xrightarrow{a_1\cdot} P_2 \xrightarrow{b_1\cdot} P_1$ of $\overline{\Lambda}_s$ cannot be lifted to a perfect complex of Λ.

For any number $n \in \mathbb{N}^+$ the sequence $\mathbf{x} = (s, t^n)$ is R-regular as well. The finite-dimensional $\Bbbk$-algebra $\overline{\Lambda}_{s,t^n} = \Lambda/\mathbf{x}\Lambda$ is isomorphic to the path algebra of the quiver $\overline{(Q, I)}_s$ with the additional relations:

$$(a_3a_2a_1)^n = (b_1b_2b_3)^n \quad (a_1a_3a_2)^n = (b_2b_3b_1)^n \quad (a_2a_1a_3)^n = (b_3b_1b_2)^n$$

We will return to the R-order Λ and its quotients $\overline{\Lambda}_p$, $\overline{\Lambda}_s$ and $\overline{\Lambda}_{s,t^n}$ later.

Silting theory and three bijections

For any perfect complexes $L^\bullet, M^\bullet$ of Λ we set $L^\bullet \geq M^\bullet$ if $\mathrm{Hom}_{\mathcal{D}}(L^\bullet, M^\bullet[p]) = 0$ for any positive number $p \in \mathbb{N}^+$. Let $\mathsf{presilt}_n\Lambda$ denote the set of isomorphism classes of basic perfect complexes $L^\bullet$ with n indecomposable summands satisfying $L^\bullet \geq L^\bullet$. The set $\mathsf{silt}\,\Lambda$ consists all elements in $\mathsf{presilt}_n\Lambda$ which generate the perfect derived category of Λ. Finally, its subset $\mathsf{tilt}\,\Lambda$ is given by all complexes $T^\bullet \in \mathsf{silt}\,\Lambda$ such that $\mathrm{Hom}_{\mathcal{D}}(T^\bullet, T^\bullet[i]) = 0$ for any integer $i \in \mathbb{Z}\backslash\{0\}$.

Our study of silting complexes is motivated by work of Aihara and Iyama [1]. They have shown that $(\mathsf{silt}\,\Lambda, \leq)$ is a partially ordered set, which is a derived invariant of the ring Λ. Given a silting complex $L^\bullet$, it is possible to compute new silting complexes from $L^\bullet$ by an explicit process called *mutation.* In contrast to silting complexes, the class of tilting complexes is not closed under mutation.
At last, silting objects are closely related to certain t-structures.
The same notions and results apply to the quotient ring $\overline{\Lambda}$.

It turns out that the push down is well-behaved on silting complexes:

Theorem 1. In the setup above, the functor $\mathbb{P}$ induces three well-defined bijections

$$\begin{array}{ccccc}
\mathsf{presilt}_n \Lambda & \hookleftarrow\!\!\!\longrightarrow & \mathsf{silt}\,\Lambda & \hookleftarrow\!\!\!\longrightarrow & \mathsf{tilt}^*\Lambda \\
\psi_{\mathsf{p}} \downarrow 1{:}1 & & \psi_{\mathsf{s}} \downarrow 1{:}1 & & \psi_{\mathsf{t}} \downarrow 1{:}1 \\
\mathsf{presilt}_n \overline{\Lambda} & \hookleftarrow\!\!\!\longrightarrow & \mathsf{silt}\,\overline{\Lambda} & \hookleftarrow\!\!\!\longrightarrow & \mathsf{tilt}\,\overline{\Lambda}
\end{array}$$

$$\begin{aligned}
\text{where } \mathsf{tilt}^*\Lambda &:= \{\, T^\bullet \in \mathsf{tilt}\,\Lambda \mid \text{the R-algebra } \operatorname{End}_{\mathcal{D}}(T^\bullet) \text{ is } \mathbf{x}\text{-regular} \,\} \\
&= \{\, T^\bullet \in \mathsf{tilt}\,\Lambda \mid \operatorname{Hom}_{\overline{\mathcal{D}}}(\overline{T^\bullet}, \overline{T^\bullet}[-1]) = 0 \,\}.
\end{aligned}$$

Moreover, for any two complexes $L^\bullet, M^\bullet \in \mathsf{presilt}\,\Lambda$ the following statements hold:

(a) It holds that $L^\bullet \geq M^\bullet$ in $\mathcal{D}$ if and only if $\overline{L^\bullet} \geq \overline{M^\bullet}$ in $\overline{\mathcal{D}}$.
(b) There is an isomorphism of $\overline{\mathrm{R}}$-algebras

$$\overline{\mathrm{R}} \otimes_{\mathrm{R}} \operatorname{End}_{\mathcal{D}}(L^\bullet) \cong \operatorname{End}_{\overline{\mathcal{D}}}(\overline{L^\bullet})$$

In particular, there is an isomorphism $(\mathsf{silt}\,\Lambda, \leq) \cong (\mathsf{silt}\,\overline{\Lambda}, \leq)$ of posets.

Let us comment on the three bijections above:

Assuming that the map ψ_{p} is well-defined, its bijectivity follows from the following "lifting folklore" for any complex $P^\bullet \in \overline{\mathcal{D}}$:

(a) If $\operatorname{Hom}_{\overline{\mathcal{D}}}(P^\bullet, P^\bullet[2]) = 0$, then $P^\bullet$ has some lift $L^\bullet \in \mathcal{D}$ with respect to $\mathbb{P}$.
(b) If $\operatorname{Hom}_{\overline{\mathcal{D}}}(P^\bullet, P^\bullet[1]) = 0$, then any two lifts of $P^\bullet$ are isomorphic in $\mathcal{D}$.

These results were established by Rickard in a setup including group theory [4], and by Yoshino in commutative algebra [5]. Theorem 1 can be generalized to a framework unifying both setups.

The surjectivity of the map ψ_{s} follows by dg-categorical arguments due to Keller.

It was shown originally by Rickard that the map ψ_{t} is well-defined and bijective in the case that $\overline{\Lambda} = \Lambda/\mathfrak{m}\Lambda$. In this setup, $\mathsf{tilt}^*\Lambda$ is given by all complexes $T^\bullet \in \mathsf{tilt}\,\Lambda$ such that $\operatorname{End}_{\mathcal{D}}(T^\bullet)$ is an R-order.

Theorem 1 is inspired by Rickard's work [4]. It has the following consequences:

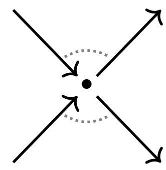
Corollary. In the setup above, the following statements are true:

(a) It holds that $\mathsf{presilt}_n\Lambda = \mathsf{silt}\,\Lambda$ if and only if $\mathsf{presilt}_n\overline{\Lambda} = \mathsf{silt}\,\overline{\Lambda}$.
(b) Let Γ be some $\mathbf{x}$-regular R-algebra derived equivalent to the R-order Λ. Then the quotients $\overline{\Gamma} = \Gamma/\mathbf{x}\Gamma$ and $\overline{\Lambda}$ are derived equivalent, too.
(c) Let $\mathcal{C}$ be some class of R-orders which is closed under derived equivalences. Then $\overline{\mathcal{C}} = \{\overline{\Lambda} \mid \Lambda \in \mathcal{C}\}$ is closed under derived equivalences as well.

It is an open problem whether any presilting complex with n summands is already silting. For the R-order Λ, this problem can be reduced to the corresponding problem for the finite-dimensional $\Bbbk$-algebra $\Lambda/\mathfrak{m}\Lambda$.

An application to ribbon graph orders

A finite quiver (Q, I) is 2-*regular gentle* if at any vertex of Q we have the situation on the right. Any such quiver comes from a ribbon graph. The arrow ideal completion Λ of its infinite-dimensional path algebra $\Bbbk Q/I$ will be called a *ribbon graph order*. The ring Λ becomes a $\Bbbk[\![t]\!]$-order by identifying t with the sum of all repetition-free cyclic paths in the quiver (Q, I).



It can be shown that ribbon graph orders have unique homological features which allow to deduce the first part of the following statement:

Theorem 2. Both classes $\mathcal{R}$ of ribbon graph orders and $\overline{\mathcal{R}}$ of *twisted Brauer graph algebras* are closed under derived equivalences.

A ring $\overline{\Lambda} \in \overline{\mathcal{R}}$ is isomorphic to some symmetric Brauer graph algebra if and only if the field $\Bbbk$ has characteristic two or the underlying graph is bipartite [3].

Example (continued). Because the R-order Λ is symmetric, it can be shown that $\mathsf{tilt}^*\Lambda = \mathsf{silt}\,\Lambda$. For any $p \in \mathfrak{m}$ and any $n \in \mathbb{N}^+$ Theorem 1 yields the bijections:

$$\mathsf{tilt}\,\overline{\Lambda}_p \xleftarrow{1:1} \mathsf{tilt}\,\Lambda \xrightarrow{1:1} \mathsf{tilt}\,\overline{\Lambda}_s \xrightarrow{1:1} \mathsf{tilt}\,\overline{\Lambda}_{s,t^n}$$

It holds that $\overline{\Lambda}_s \in \mathcal{R}$ and $\overline{\Lambda}_{s,t^n} \in \overline{\mathcal{R}}$. By a result of Burban and Drozd, the ribbon graph order $\overline{\Lambda}_s$ is derived-tame. On the other hand, the preprojective algebra Λ, most of its quotient rings $\overline{\Lambda}_p$ and all Brauer graph algebras $\overline{\Lambda}_{s,t^n}$ are derived-wild.

Thus, the tilting theory of a family of rings is reduced to a single feasible case.

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Schemes of modules over gentle algebras and laminations of surfaces

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(joint work with Daniel Labardini-Fragoso, Christof Geiß)

1. Introduction

We study some geometric aspects of the representation theory of gentle algebras. This class of finite-dimensional algebras was defined by Assem and Skowroński [3], who were classifying iterated tilted algebra of path algebras of extended Dynkin type $\widetilde{A}$. Gentle algebras are special biserial, which implies that their module categories can be described combinatorially, see [18] and also [5].

The irreducible components of the affine schemes of modules over gentle algebras are easy to classify. As a first main result, we describe all smooth points of these schemes, and we show that most components are generically reduced.

A special class of gentle algebras are Jacobian algebras arising from triangulations of unpunctured marked surfaces $(\mathbb{S}, \mathbb{M})$. For these we obtain a bijection between the set of generically τ-reduced decorated irreducible components and the set of laminations of the surface. This bijection is compatible with the parametrization of these two sets via g-vectors and shear coordinates, and it has some application to cluster algebras, a class of combinatorially defined commutative algebras discovered by Fomin and Zelevinsky [10]. Initially meant as a tool to describe parts of Lusztig's dual canonical basis of quantum groups in a combinatorial way, cluster algebras turned out to appear at numerous different places of mathematics and mathematical physics. The generically τ-reduced decorated components parametrize the generic Caldero-Chapoton functions, which belong to the upper cluster algebra $\mathcal{U}_{(\mathbb{S},\mathbb{M})}$ associated with $(\mathbb{S}, \mathbb{M})$. In many cases, these generic Caldero-Chapoton functions are known to form a basis, called the *generic basis*, of $\mathcal{U}_{(\mathbb{S},\mathbb{M})}$, see for example [11] and [17]. We use the bijection mentioned above to show that the generic basis coincides with the Musiker-Schiffler-Williams bangle basis (see [15]) of the cluster algebra $\mathcal{A}_{(\mathbb{S},\mathbb{M})}$ associated with $(\mathbb{S}, \mathbb{M})$. (In most of these cases, we have $\mathcal{A}_{(\mathbb{S},\mathbb{M})} = \mathcal{U}_{(\mathbb{S},\mathbb{M})}$.)

In the following, we describe our results in more detail.

2. Main results

2.1. **Gentle algebras.** Let $Q = (Q_0, Q_1, s, t)$ be a quiver. Thus by definition, Q_0 and Q_1 are finite sets, where the elements of Q_0 and Q_1 are the *vertices* and *arrows* of Q, respectively. Furthermore, s and t are maps $s, t\colon Q_1 \to Q_0$, where $s(a)$ and $t(a)$ are the *starting vertex* and *terminal vertex* of an arrow $a \in Q_1$, respectively. A *loop* in Q is an arrow $a \in Q_1$ with $s(a) = t(a)$.

A basic algebra $A = KQ/I$ is a *gentle algebra* provided the following hold:

(i) For each $i \in Q_0$ we have $|\{a \in Q_1 \mid s(a) = i\}| \leq 2$ and $|\{a \in Q_1 \mid t(a) = i\}| \leq 2$.

(ii) The ideal I is generated by a set ρ of paths of length 2.

(iii) Let $a, b, c \in Q_1$ such that $a \neq b$ and $t(a) = t(b) = s(c)$. Then exactly one of the paths ca and cb is in I.
(iv) Let $a, b, c \in Q_1$ such that $a \neq b$ and $s(a) = s(b) = t(c)$. Then exactly one of the paths ac and bc is in I.

A gentle algebra $A = KQ/I$ is a Jacobian algebra in the sense of [6] if and only if the following hold:

(v) Q is connected.
(vi) Q does not have any loops.
(vii) Let $a, b \in Q_1$ such that $s(a) = t(b)$ and $ab \in I$. Then there exists an arrow $c \in Q_1$ with $s(c) = t(a)$ and $t(c) = s(b)$ such that $bc, ca \in I$.

The gentle Jacobian algebras are exactly the Jacobian algebras associated to triangulations of unpunctured marked surfaces. This follows from [2, Section 2].

2.2. **Generically τ-reduced irreducible components of module schemes.** Let Q be a quiver with $Q_0 = \{1, \ldots, n\}$, and let $A = KQ/I$ be a basic algebra. For $\mathbf{d} \in \mathbb{N}^n$ let $\mathrm{Irr}(A, \mathbf{d})$ be the set of irreducible components of the affine scheme $\mathrm{mod}(A, \mathbf{d})$ of A-modules with dimension vector $\mathbf{d}$. Let

$$\mathrm{Irr}(A) := \bigcup_{\mathbf{d} \in \mathbb{N}^n} \mathrm{Irr}(A, \mathbf{d}).$$

The group

$$\mathrm{GL}_{\mathbf{d}} := \prod_{i=1}^{n} \mathrm{GL}_{d_i}(K)$$

acts on $\mathrm{mod}(A, \mathbf{d})$ by conjugation, where $\mathbf{d} = (d_1, \ldots, d_n)$. The orbit of $M \in \mathrm{mod}(A, \mathbf{d})$ is denoted by $\mathcal{O}_M$. The orbits in $\mathrm{mod}(A, \mathbf{d})$ correspond bijectively to the isomorphism classes of A-modules with dimension vector $\mathbf{d}$.

For $M \in \mathrm{mod}(A, \mathbf{d})$ let

$$\begin{aligned} c_A(M) &:= \max\{\dim(Z) \mid Z \in \mathrm{Irr}(A, \mathbf{d}),\ M \in Z\} - \dim \mathcal{O}_M, \\ e_A(M) &:= \dim \mathrm{Ext}^1_A(M, M), \\ h_A(M) &:= \dim \mathrm{Hom}_A(M, \tau_A(M)). \end{aligned}$$

Here τ_A denotes the Auslander-Reiten translation of A.

For each $Z \in \mathrm{Irr}(A)$ there is a dense open subset $U \subseteq Z$ such that the maps c_A, e_A and h_A are constant on U. These generic values are denoted by $c_A(Z)$, $e_A(Z)$ and $h_A(Z)$. Voigt's Lemma and the Auslander-Reiten formulas imply that

$$c_A(Z) \leq e_A(Z) \leq h_A(Z).$$

Clearly, an irreducible component Z is generically reduced if and only if $c_A(Z) = e_A(Z)$. We say that Z is *generically τ-reduced* provided

$$c_A(Z) = e_A(Z) = h_A(Z).$$

Such irreducible components were first defined and studied in [11], where they ran under the name *strongly reduced components.*

Let $\mathrm{Irr}^\tau(A)$ be the subset of $\mathrm{Irr}(A)$ consisting of the generically τ-reduced components.

Recall that an A-module M is *rigid* (resp. *τ-rigid*) if $\mathrm{Ext}^1_A(M,M) = 0$ (resp. $\mathrm{Hom}_A(M, \tau_A(M)) = 0$). By the Auslander-Reiten formulas, any τ-rigid module is rigid, wheras the converse is wrong in general. Each rigid A-module M yields a generically reduced component $Z = \overline{\mathcal{O}_M}$. If M is τ-rigid, then this Z is generically τ-reduced.

2.3. **Laminations of marked surfaces and generically τ-reduced components.** A *lamination* of an unpunctured marked surface $(\mathbb{S}, \mathbb{M})$ is (roughly speaking) a set of homotopy classes of curves and loops in $(\mathbb{S}, \mathbb{M})$, which do not intersect each other, together with a positive integer attached to each class. Let $\mathrm{Lam}(\mathbb{S}, \mathbb{M})$ be the set of such laminations.

Let T be a triangulation of $(\mathbb{S}, \mathbb{M})$, and let A_T be the associated gentle Jacobian algebra. Instead of studying irreducible components of the schemes $\mathrm{mod}(A_T, \mathbf{d})$, one can equip the components with a *decoration*, which is just an extra integer datum. Similarly as before, one defines generically τ-reduced decorated irreducible components. Let $\mathrm{decIrr}^\tau(A_T)$ be the set of such components.

Theorem 2.1. *Let $(\mathbb{S}, \mathbb{M})$ be an unpunctured marked surface, and let T be a triangulation of $(\mathbb{S}, \mathbb{M})$. Let $A = A_T$ be the associated Jacobian algebra. Then there is a natural bijection*

$$\eta\colon \mathrm{Lam}(\mathbb{S}, \mathbb{M}) \to \mathrm{decIrr}^\tau(A).$$

In their ground breaking work, Fomin, Shapiro and Thurston [8] proved that the laminations of $(\mathbb{S}, \mathbb{M})$ consisting of curves are in bijection with the cluster monomials of a cluster algebra $\mathcal{A}_{(\mathbb{S},\mathbb{M})}$ associated with $(\mathbb{S}, \mathbb{M})$. (Cluster algebras were introduced by Fomin and Zelevinsky [10].) Musiker, Schiffler and Williams [15] extended this by defining a set

$$\mathcal{B}_T := \{\psi_L \mid L \in \mathrm{Lam}(\mathbb{S}, \mathbb{M})\} \subset \mathcal{U}_{(\mathbb{S},\mathbb{M})}$$

of *bangle functions*, whose elements are parametrized by $\mathrm{Lam}(\mathbb{S}, \mathbb{M})$, and which (by results in [14]) contains all cluster monomials. Here $\mathcal{U}_{(\mathbb{S},\mathbb{M})}$ denotes the upper cluster algebra associated with $(\mathbb{S}, \mathbb{M})$. A result by W. Thurston (see [9, Theorem 12.3]) says that there is a bijection

$$\mathrm{sh}_T\colon \mathrm{Lam}(\mathbb{S}, \mathbb{M}) \to \mathbb{Z}^n$$

sending a lamination to its shear coordinate. Combining a theorem by Brüstle and Zhang [4, Theorem 1.6] with a result by Adachi, Iyama and Reiten [1, Theorem 4.1], one gets a bijection between the laminations in $\mathrm{Lam}(\mathbb{S}, \mathbb{M})$, which consists only of curves, and the set of generically τ-reduced components in $\mathrm{Irr}^\tau(A_T)$, which have a dense orbit. Plamondon [16] proved that there is a bijection

$$g_T\colon \mathrm{decIrr}^\tau(A_T) \to \mathbb{Z}^n$$

sending a component to its g-vector. Theorem 2.1 extends the bijection mentioned above to a bijection $\eta\colon \mathrm{Lam}(\mathbb{S}, \mathbb{M}) \to \mathrm{decIrr}^\tau(A_T)$ such that $g_T \circ \eta = \mathrm{sh}_T$.

Let

$$\mathcal{G}_T := \{\phi_Z \mid Z \in \operatorname{decIrr}^{\tau}(A_T)\} \subset \mathcal{U}_{(\mathbb{S},\mathbb{M})}$$

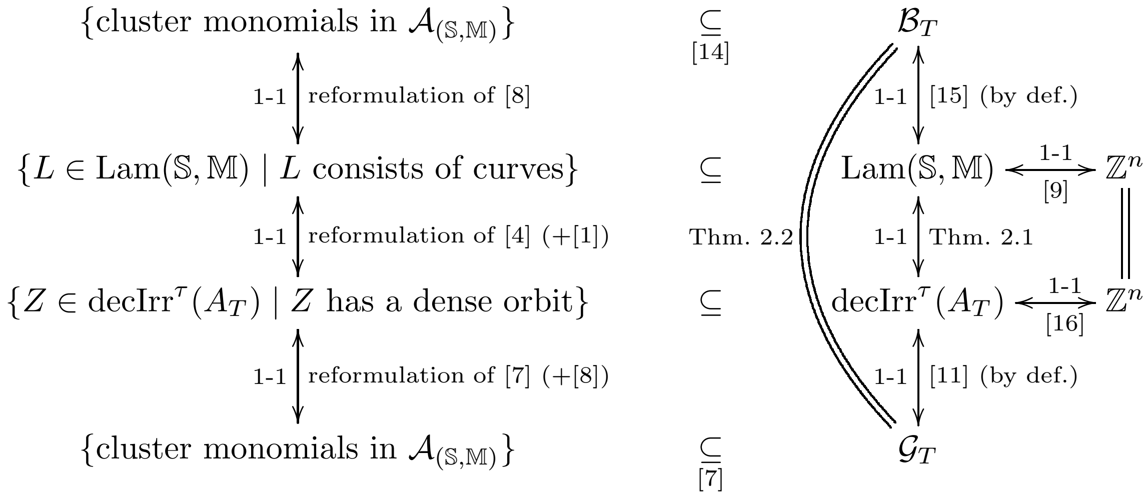
be the set of *generic Caldero-Chapoton functions* as defined in [11]. By [7], the set $\mathcal{G}_T$ contains all cluster monomials. Furthermore, by [16, Theorem 1.3], the set $\mathcal{G}_T$ is (in a certain sense) independent of the choice of the triangulation T of $(\mathbb{S},\mathbb{M})$.

The proof of the next theorem is based on the bijection from Theorem 2.1.

Theorem 2.2. $\mathcal{B}_T = \mathcal{G}_T$.

It is known in most cases (for example, if $|\mathbb{M}| \geq 2$) that $\mathcal{A}_{(\mathbb{S},\mathbb{M})} = \mathcal{U}_{(\mathbb{S},\mathbb{M})}$ (see [12, 13]) and that $\mathcal{B}_T$ is a basis of $\mathcal{U}_{(\mathbb{S},\mathbb{M})}$ (see [15]).

The following diagram summarizes the situation:



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The symplectic geometry of higher Auslander algebras, an overview

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(joint work with Tobias Dyckerhoff, Yankı Lekili)

Let $\mathbf{k}$ be a commutative ring. Let $\mathbb{D}$ be the 2-dimensional unit disk and $\Lambda_n \subset \partial\mathbb{D}$ the set of $(n+1)$-th roots of unity, where $n \geq 0$. To these data one can associate [Aur10b, Aur10a] a *partially wrapped Fukaya category* $\mathcal{W}(\mathbb{D}, \Lambda_n)$, which is an idempotent-complete triangulated A_∞-category. After choosing appropriate generators, the aforementioned Fukaya category can be described combinatorially as the perfect derived category of $\mathbf{k}$-linear representations of the linearly oriented quiver

$$A_n := (1 \to 2 \to \cdots \to n)$$

of Dynkin type $\mathbb{A}_n$. As originally observed by Waldhausen [Wal85] (in a slightly different language), the derived functors induced by the morphisms between the various quivers A_n, $n \geq 0$ are part of a *co*-simplicial object $\mathsf{perf}\, A_\bullet$. Consequently, for each A_∞-category $\mathcal{A}$ there is an associated *simplicial* object

$$\mathsf{Fun}_{\mathbf{k}}(\mathcal{W}(\mathbb{D}, \Lambda_\bullet), \mathcal{A}) \overset{(a)}{\simeq} \mathsf{Fun}_{\mathbf{k}}(\mathsf{perf}\, A_\bullet, \mathcal{A}) \overset{(b)}{\simeq} \mathsf{S}(\mathcal{A})_\bullet$$

whose triangulated A_∞-category of n-cells is given by the A_∞-category of A_∞-functors $\mathcal{W}(\mathbb{D}, \Lambda_n) \to \mathcal{A}$. The simplicial object $\mathsf{S}(\mathcal{A})_\bullet$, called the *Waldhausen* S*-construction of* $\mathcal{A}$, is the main ingredient in the construction of the *Waldhausen* K*-theory space* $K(\mathcal{A})$ of $\mathcal{A}$, for we have the formula

$$K(A) := \Omega|\mathsf{S}(\mathcal{A})_\bullet^{\simeq}|.$$

In summary, the quasi-equivalent simplicial objects above provide an explicit connection between

- the (partially) wrapped Floer theory of the 2-dimensional unit disk,
- the derived representation theory of Dynkin quivers of type $\mathbb{A}$ and
- the Waldhausen K-theory of A_∞-categories.

Let $d \geq 1$ be a natural number. In previous work with Dyckerhoff and Walde [DJW19] we have described a higher-dimensional generalisation of the quasi-equivalence (b) above, which now takes the form

$$\mathsf{Fun}_{\mathbf{k}}(\mathsf{perf}\, A_{\bullet,d}, \mathcal{A}) \simeq \mathsf{S}^{\langle d \rangle}(\mathcal{A})_\bullet \tag{1}$$

and relates the *d-dimensional Waldhausen* S*-construction* $\mathsf{S}^{\langle d\rangle}(\mathcal{A})_\bullet$ of $\mathcal{A}$ (introduced by Hesselholt and Madsen [HM15] in the case $d = 2$ and by Dyckerhoff [Dyc17] and Poguntke [Pog17] in general) to the derived representation theory of *Iyama's d-dimensional Auslander algebras of type* $\mathbb{A}$ [Iya11]. The relevance of the simplicial object $\mathsf{S}^{\langle d\rangle}(\mathcal{A})_\bullet$ in K-theory stems from the homotopy equivalence

$$K(A) \simeq \Omega^d |\mathsf{S}^{\langle d\rangle}(\mathcal{A})_\bullet^{\simeq}|,$$

which, by letting d vary, exhibits $K(\mathcal{A})$ as a so-called connective spectrum.

In recent work with Dyckerhoff and Lekili [DJL19] we extend the above discussion by providing a d-dimensional analogue

$$\mathsf{Fun}_{\mathbf{k}}(\mathcal{W}(\mathsf{Sym}^d\mathbb{D}, \Lambda_\bullet^{(d)}), \mathcal{A}) \simeq \mathsf{Fun}_{\mathbf{k}}(\mathsf{perf}\, A_{\bullet,d}, \mathcal{A})$$

of the quasi-equivalence (a) above, induced by quasi-equivalences

$$\mathcal{W}(\mathsf{Sym}^d\mathbb{D}, \Lambda_n^{(d)}) \simeq \mathsf{perf}\, A_{n,d} \tag{2}$$

of triangulated A_∞-categories. In (2), the left-hand side denotes the partially wrapped Fukaya category associated to the d-fold symmetric product

$$\mathsf{Sym}^d\mathbb{D} := \underbrace{\mathbb{D} \times \cdots \times \mathbb{D}}_{d \text{ times}} / \mathfrak{S}_d$$

equipped with the stops

$$\Lambda_n^{(d)} := \bigcup_{p\in\Lambda_n} \{p\} \times \mathsf{Sym}^{d-1}\mathbb{D},$$

we refer the reader to [Aur10b, Aur10a] for the details of this construction. The existence of a quasi-equivalence in (2) is established by leveraging general generation results of Auroux [Aur10b, Aur10a] together with the explicit computation of the quasi-isomorphism type of the derived endomorphism algebra of an explicit set of generators of $\mathcal{W}(\mathsf{Sym}^d\mathbb{D}, \Lambda_n^{(d)})$ following and idea of Lipshitz, Ozsváth and Thurston [LOT15]. In representation-theoretic terms, we construct an explicit tilting object in $\mathcal{W}(\mathsf{Sym}^d\mathbb{D}, \Lambda_n^{(d)})$ whose endomorphism $\mathbf{k}$-algebra is isomorphic to $A_{n,d}$.

As an application of our results, and as a consequence of Koszul duality for augmented A_∞-categories, in [DJL19] we also establish the existence of quasi-equivalences

$$\mathcal{W}(\mathsf{Sym}^d\mathbb{D}, \Lambda_n^{(d)}) \simeq \mathcal{W}(\mathsf{Sym}^{n-d}\mathbb{D}, \Lambda_n^{(n-d)}), \tag{3}$$

$n \geq d \geq 1$, thereby providing a symplectic proof of a result of Beckert [Bec18] concerning the derived equivalence between the $\mathbf{k}$-algebras $A_{n,d}$ and $A_{n,n-d}$ obtained by a delicate calculus of homotopy Kan extensions in stable derivators.

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Geometric properties of (certain) quiver Grassmannians

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(joint work with Francesco Esposito, Hans Franzen, Markus Reineke)

Let Q be a quiver with set of vertices Q_0 and set of arrows Q_1, and let M be a finite dimensional complex representation of Q. Let us denote by $\mathbf{d} = (d_i) \in \mathbb{Z}_{\geq 0}^{Q_0}$ the dimension vector of M. We identify M with a point of the vector space $R_{\mathbf{d}}(Q) = \bigoplus_{\alpha:i\to j\in Q_1} \mathrm{Hom}(\mathbf{C}^{\mathbf{d}_i}, \mathbf{C}^{\mathbf{d}_j})$. Given another dimension vector $\mathbf{e} \in \mathbb{Z}_{\geq 0}^{Q_0}$, following Schofield [8], we define the incidence variety

$$\mathrm{Gr}_{\mathbf{e}}^{Q}(\mathbf{d}) = \{((N_i)_{i\in Q_0}, (M_\alpha)_{\alpha\in Q_1}) \in \mathrm{Gr}_{\mathbf{e}}(\mathbf{d}) \times R_{\mathbf{d}}(Q) |\, M_\alpha(N_i) \subseteq N_j,\, \forall \alpha : i \to j\}.$$

where $\mathrm{Gr}_{\mathbf{e}}(\mathbf{d}) := \prod_{i\in Q_0} \mathrm{Gr}_{\mathrm{e}_i}(\mathbf{C}^{\mathrm{d_i}})$. A point of $\mathrm{Gr}_{\mathbf{e}}^{Q}(\mathbf{d})$ is hence a pair consisting of a collection of subspaces N together with a Q-representation M such that N is a Q-subrepresentation of M. It is endowed with the the two maps

$$\begin{array}{ccc} & \mathrm{Gr}_{\mathbf{e}}^{Q}(\mathbf{d}) & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ \mathrm{Gr}_{\mathbf{e}}(\mathbf{d}) & & R_{\mathbf{d}}(Q) \end{array}$$

induced by the two projections. The map p_1 is a vector bundle and the map $p_2 : \mathrm{Gr}_{\mathbf{e}}^{Q}(\mathbf{d}) \to R_{\mathbf{d}}(Q)$ is proper. The image of p_2 is the closed subvariety of $R_{\mathbf{d}}(Q)$ consisting of Q-representations of dimension vector $\mathbf{d}$ which admit a subrepresentation of dimension vector $\mathbf{e}$. The group $\mathrm{G}_{\mathbf{d}} = \prod_{i\in Q_0} \mathrm{GL}_{\mathrm{d}_i}(\mathbf{C})$ acts naturally on $\mathrm{Gr}_{\mathbf{e}}^{Q}(\mathbf{d})$ and on $R_{\mathbf{d}}(Q)$ and p_2 is $\mathrm{G}_{\mathbf{d}}$-equivariant. The fiber of a point $p_2^{-1}(M) =: \mathrm{Gr}_{\mathbf{e}}(M)$ is called a *quiver Grassmannian*.

Quiver Grassmannians are interesting projective varieties naturally arising in different parts of mathematics. What we show in our paper [3] is that some of their geometric properties can be studied using Auslander-Reiten theory.

In order to state the resut we need to recall a few definitions from [5]. An *α–partition* of a complex algebraic variety X is a finite partition (X_i) of X whose parts can be ordered so that

$$X_1 \amalg \cdots \amalg X_i \text{ is closed in } X \text{ for every } i. \tag{1}$$

Clearly, every piece of an α–partition is locally closed. Property (1) can be reformulated by

$$\overline{X}_i \subseteq X_1 \amalg \cdots \amalg X_{i-1} \tag{2}$$

for every i. A *cellular decomposition* of X is an α–partition whose parts X_i are (complex) affine spaces. For example, Grassmannian manifolds admit a cellular decomposition. The variety $X = \{[x : y : z] \in \mathbb{P}^2 | xyz = 0\}$ does not admit a cellular decomposition. The existence of a cellular decomposition for X is rare but when happens it implies wonderful homological properties: we denote by $H_i(X)$ the i–th space of the Borel–Moore homology of X. Following [5, Sec. 1.7] we say that an algebraic variety X has *property (**S**)* if $H_{odd}(X)$ is zero, $H_{even}(X)$ has no torsion and the cycle map $\varphi_i : A_i(X) \to H_{2i}(X)$ is an isomorphism for all i. (Here $A_k(X)$ denotes the Chow group generated by k–dimensional irreducible subvarieties modulo rational equivalences (see [10, Sec. 1.3])). Cellular decomposition implies property (S) but the opposite is not true (a counterexample is the famous example of Barlow of an irrational surface with trivial H^1 [1]).

Coming back to quiver Grassmannians, we say that a Q-representation M has property (C) (resp. (S)) if every quiver Grassmannian $\mathrm{Gr}_{\mathbf{e}}(M)$ attached to M admits a cellular decomposition (has property (S), respectively).

Recall that a Q-representation M is rigid if $\mathrm{Ext}^1(M, M) = 0$. Our main result is the following.

Theorem 0.1. [3] Let Q be a connected quiver and let M be a Q-representation.
(1) Q Dynkin $\Longrightarrow$ M has property (C).
(2) Q extended Dynkin, M indecomposable $\Longrightarrow$ M has property (C).
(3) M rigid $\Longrightarrow$ M has property (S).

The key technical result to prove part (1) and (2) is what we call the reduction theorem: it states that if there exists a short exact sequence $\xi : 0 \to X \to Y \to S \to 0$ such that $\mathrm{Ext}^1(S, X) \simeq \mathbf{C}\xi$ then one can obtain property (C) or (S) of the middle term Y as a consequence of the same property for the external terms X and S together with two extra representations $X_S \subset X$ and S/S^X which are trivial if $\xi = 0$. We call such sequences ξ generating. This result implies that if $Y = X \oplus S$ and $\mathrm{Ext}^1(S, X) = 0$ then if both X and S have property (C) the same holds for Y. In particular, for preprojective or preinjective representations, the problem reduces to the study of the indecomposables. Now, for Dynkin quivers we found a way to deal with indecomposable inductively, using the fact that their dimension vector is minuscule (a part for type E_8 which is treated separately). In the affine case the situation is much more complicated and the result is obtained by using Auslander-Reiten theory. By the reduction theorem, part (2) of theorem 0.1 implies that every representation of an extended Dynkin quiver whose regular part is rigid has property (C). Part (2) of theorem 0.1 has been obtained before in [12] for affine type D and in [4] for the Kronecker quiver, with different techniques.

Part (3) of theorem 0.1 is obtained as a consequence of a stronger property: we show that quiver Grassmannians associated to rigid quiver representations admit the *decomposition of the diagonal*. This property implies property (S). It is worth noticing that in a recent preprint Hans Franzen has shown that quiver Grassmannians associated to rigid quiver representations are rational [9].

1. Conjectures and open problems

The following is a list of conjectures and open problems.

(1) We conjecture that every rigid Q-representation has property (C).
(2) I conjecture that every representation of an extended Dynkin quiver has property (C). What remains to do here is to show that the regular representations with more than two summands have property (C).
(3) It would be interesting to study the behaviour of property (C) or property (S) under Fomin-Zelevinky mutation i.e. if M has such property is it true that also the mutation $\mu_k(M)$ has?
(4) It would be interesting to study other geometric properties of quiver Grassmannians attached to rigid quiver representations: e.g. are they Fano?
(5) Maksimau [13] has recently adapted the proof of part (3) of theorem 0.1 to the case of flags of subrepresentations of a rigid quiver representation. This has applications in the theory of quiver Hecke algebras. He also shows property (C) in type A and D. Type E is open.
(6) It is interesting to study the behaviour of cellular decomposition under degeneration of quiver representations. Together with F. Esposito, G. Fourier, X. Fang [2] we study it in the case of the equioriented quiver of type A, generalizing a result of Lanini and Strickland [11].
(7) Since the family p_2 is proper, by the decomposition theorem the direct image of the constant sheaf on $\mathrm{Gr}_{\mathbf{e}}(\mathbf{d})$ decomposes as a finite direct sum of simple perverse sheaves which are the intersection cohomology of the orbit closures. The support of the decomposition has been studied by Fang and Reineke [6] in the case when the family gives irreducible linear degenerations of the complete flag variety. It is an open problem to determine the coefficients of the decomposition, i.e. the multiplicities of the simples.

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Combinatorics of faithfully balanced modules for Nakayama algebras

Baptiste Rognerud
(joint work with William Crawley-Boevey, Biao Ma, Julia Sauter)

We consider the category $\operatorname{mod}\Lambda$ of finitely generated left Λ-modules, where Λ is a finite-dimensional algebra over a field K. Recall that a module M is said to be *balanced*, or to have the *double centralizer property* if the natural map from Λ to $\operatorname{End}_{\operatorname{End}_{\Lambda(M)}}(M)$ is surjective, and it is said to be *faithfully balanced* if the natural map is bijective. Balanced and faithfully balanced modules appear in various places in the literature on ring theory, such as Schur-Weyl duality, Thrall's notion of a QF-1 algebra and Morita theory.

In general the behaviour of faithfully balanced modules is rather mysterious. We will illustrate this by studying these modules for the path algebra Λ_n of a linearly oriented A_n quiver. More generally our methods can be applied to Nakayama algebras. Our main tool is the reformulation of the double centralizer property in terms of certain subcategories of $\operatorname{mod}\Lambda$. For a Λ-module M we denote by $\operatorname{cogen}(M)$ the full subcategory of $\operatorname{mod}\Lambda$ consisting of the submodules of direct sums of M and we denote by $\operatorname{cogen}^1(M)$ the full subcategory of $\operatorname{mod}\Lambda$ consisting of the modules $X \in \operatorname{cogen}(M)$ such that the minimal left $\operatorname{add}(M)$-approximation of X has cokernel in $\operatorname{cogen}(M)$. Dually, we define $\operatorname{gen}(M)$ and $\operatorname{gen}_1(M)$, then we have the following useful characterization of faithfully balanced modules.

Proposition 1. *Let M be a Λ-module. Then the following are equivalent*

(1) M is faithfully balanced.
(2) $\operatorname{cogen}^1(M)$ *contains the finitely generated projective modules.*
(3) $\operatorname{gen}_1(M)$ *contains the finitely generated injective modules.*

As a corollary we have easy proofs of the following well-known results.

Proposition 2. *Let M be a Λ-module. If M is a generator, a cogenerator, a tilting module or a cotilting module, then M is faithfully balanced.*

Proof. Since Proposition 1 is self-dual, it is enough to prove the result for cogenerators and tilting modules. If M is a cogenerator, then $\operatorname{cogen}^1(M) = \operatorname{mod}\Lambda$.

If M is a tilting module, there is an exact sequence

$$0 \to \Lambda \overset{\alpha_0}{\to} M_1 \to \cdots \overset{\alpha_{n-1}}{\to} M_n \to 0,$$

where the $M_i's$ are in $\operatorname{add}(M)$. If K_{n-1} denotes the kernel of α_{n-1}, we easily see that $0 \to K_{n-1} \to M_{n-1}$ is a left $\operatorname{add}(M)$-approximation of K_{n-1} and that $\operatorname{Ext}^i(K_{n-1}, T) = 0$ for $i \geq 1$. Then, by induction we show that $A \to M_1$ is a left $\operatorname{add}(M)$-approximation with cokernel in $\operatorname{cogen}(M)$. □

The indecomposable modules for Λ_n are in bijection with the set of intervals $[i, j]$ for $1 \leq i \leq j \leq n$. We display them as the blocks of a Young diagram of staircase shape. The box with coordinates (i, j) corresponds to the module M_{ij} with top and socle the simple modules $S[i]$ and $S[j]$. The left hand column is the indecomposable projective modules, the top row is the indecomposable injective modules and the modules M_{ii} are the simple modules $S[i]$. By a *leaf* we mean an element of the set $L = \{(1, 0), (2, 1), \ldots, (n + 1, n)\}$.

We define *cohooks* for $(i, j) \in I_n$ and *virtual cohooks* for $(i, j) \in L$ by the formula

$$\operatorname{cohook}(i, j) = \{M_{kj} : 1 \leq k < i\} \cup \{M_{i\ell} : n \geq \ell > j\}$$

In the Young diagram the cohook of (i, j) consists of all the boxes on the left and above (i, j). Using this combinatorial gadget we can characterize the faithfully balanced modules for the algebra Λ_n, along with its generalization to Nakayama algebras and a version for balanced modules.

Theorem 3. *A Λ_n-module M is faithfully balanced if and only if it satisfies the following conditions:*

- *(FB0) M_{1n} is a summand of M;*
- *(FB1) if M_{ij} is a summand of M, $(i, j) \neq (1, n)$, then* $\operatorname{cohook}(i, j)$ *contains a summand of M; and*
- *(FB2) every virtual cohook contains a summand of M.*

With this characterization, we can count faithfully balanced modules for Λ_n and we obtain the following theorem.

Theorem 4. *In the expansion of the polynomial*

$$h_n(x_1, \ldots, x_n) = \prod_{r=1}^{n} \left(\prod_{s=1}^{r} (1 + x_s) - 1 \right),$$

the coefficient of the monomial $x_1^{t_1} \ldots x_n^{t_n}$ is the number of basic faithfully balanced Λ_n-modules M with t_i indecomposable summands having top $S[i]$ (or equivalently in row i of the Young diagram), for all i.

It follows that the number of basic faithfully balanced modules for Λ_n is the 2-factorial number

$$[n]_2! := \prod_{i=1}^{n}(2^i - 1).$$

Also, any basic faithfully balanced module for Λ_n has at least n summands, and the number with exactly n summands is $n!$. For comparison, note that the number of basic tilting modules for Λ_n is the nth Catalan number. This also shows that the frequency of isomorphism classes of basic faithfully balanced modules in the set of isomorphism classes of all basic modules is almost constant and approximatively equal to 0.29.

Let $\mathrm{fb}(n)$ be the set of faithfully balanced modules with exactly n indecomposable summands. Since $|\,\mathrm{fb}(n)| = n!$, we would like to find an *explicit* bijection with a family of objects classically counted by $n!$. Moreover, we would like this bijection to preserve the very rich combinatorics of tilting modules. For that, we introduce the notion of *interleaved trees*. They can be though as binary trees together with a shuffle of their leaves. They naturally generalizes binary trees, which can be seen as interleaved trees with a trivial shuffle of the leaves.

Theorem 5. *Given n, there are explicit bijections between the following types of objects:*

- *(i) faithfully balanced modules for Λ_n with exactly n indecomposable summands;*
- *(ii) interleaved trees with n vertices;*
- *(iii) increasing binary trees with n vertices;*
- *(iv) functions $f : \{1, \dots, n\} \to \{1, \dots, n\}$ such that $f(i) \leq i$ for all i.*

These restrict to bijections between basic tilting modules; binary trees; well-ordered increasing binary trees and non-decreasing self-bounded functions.

Finally to give more structure on the set $\mathrm{fb}(n)$, we show that the relation given by $N \trianglelefteq M$ if and only if $\mathrm{cogen}(N) \subseteq \mathrm{cogen}(M)$ and $\mathrm{gen}(N) \supseteq \mathrm{gen}(M)$ is a partial order on $\mathrm{fb}(n)$. If we restrict this partial order to the set of basic tilting modules, we recover the usual partial ordering of tilting modules by inclusion of the corresponding torsion-free classes (or by reverse inclusion of torsion classes). It is well-known that this poset is actually a lattice isomorphic to the Tamari lattice. For faithfully balanced modules, we prove the following result.

Theorem 6. *(1) The poset $(\mathrm{fb}(n), \trianglelefteq)$ is lattice.*
(2) The Tamari lattice is a sub-lattice of $(\mathrm{fb}(n), \trianglelefteq)$.
(3) The cover relations in $(\mathrm{fb}(n), \trianglelefteq)$ are given by exchanging exactly one indecomposable summand.

In other words, we have obtained a lattice structure on a set with $n!$ elements which extends naturally the Tamari lattice and which is *not* isomorphic to the weak order on the symmetric group as it can be seen below.

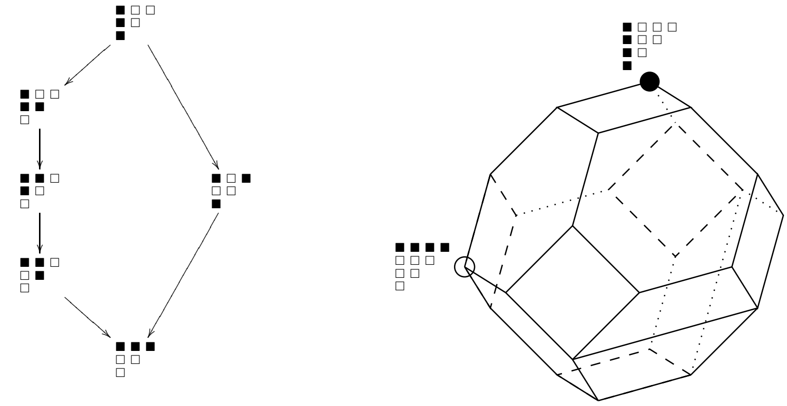

FIGURE 1. The Hasse diagram of $(\mathrm{fb}(3), \trianglelefteq)$ and the graph of the Hasse diagram of $(\mathrm{fb}(4), \trianglelefteq)$.

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Monomorphism categories for Eilenberg–Moore categories

JULIAN KÜLSHAMMER

(joint work with Nan Gao, Sondre Kvamme, Chrysostomos Psaroudakis)

Let $\Bbbk$ be a field and let Q be a finite (or more generally locally bounded) quiver. Then the path algebra $\Bbbk Q$ is of global dimension one, and therefore in particular 1-Iwanaga-Gorenstein. A $\Bbbk Q$-module M is Gorenstein projective if and only if it is in the image of the inverse Nakayama functor $\mathrm{Hom}_{\Bbbk Q}(D(\Bbbk Q), -)$ if and only if it is projective. This happens if and only if the map

$$M_{\mathtt{i},\mathrm{in}}\colon \bigoplus_{\alpha\colon s(\alpha)\to \mathtt{i}} M_{s(\alpha)} \xrightarrow{(M_\alpha)_\alpha} M_{\mathtt{i}}$$

is a monomorphism for all $\mathtt{i} \in Q_0$.

More generally, for any finite dimensional algebra Λ, the path algebra $\Lambda Q \cong \Lambda \otimes_{\Bbbk} \Bbbk Q$ is no longer of global dimension one, but in some sense of still of relative global dimension one, since for each ΛQ-module M there is the standard resolution:

$$0 \to \Lambda Q \otimes_{\Lambda Q_0} \Lambda Q_1 \otimes_{\Lambda Q_0} M \to \Lambda Q \otimes_{\Lambda Q_0} M \to M \to 0,$$

where modules of the form $\Lambda Q \otimes_{\Lambda Q_0} N$, for a ΛQ_0-module N, can be regarded as relative projective, similarly to a familiar setup in group representation theory (see for example [1, Section 3.6]). This in particular implies that if Λ is selfinjective, then the algebra ΛQ is 1-Iwanaga-Gorenstein, i.e. of finite injective dimension considered as a left or right module over itself.

In this setup a module M is Gorenstein projective if and only if the analogous map $M_{\mathtt{i},\mathrm{in}}$ is a monomorphism for all $\mathtt{i} \in Q_0$ and for all $\mathtt{i} \in Q_0$ the Λ-module

$\operatorname{coker}(M_{\mathtt{i}}, \mathrm{in})$ is Gorenstein projective. The monomorphism category, that is the category of all modules for which only the former condition is satisfied, also contains the relative projective modules. It is a functorially finite resolving subcategory of $\operatorname{mod} \Lambda Q$, in particular it has almost split sequences. In the case that Q is the quiver of Dynkin type A_2, that is that $\Lambda Q \cong \left(\begin{smallmatrix} \Lambda & \Lambda \\ 0 & \Lambda \end{smallmatrix}\right)$, this subcategory was studied extensively by Ringel and Schmidmeier. It is easy to see that it is precisely the image of the kernel functor sending a Λ-representation (M_1, M_2, M_α) to $(\ker(M_\alpha), M_1, \mathrm{incl})$. This functor behaves like a relative inverse Nakayama functor in that it sends relative injective modules to relative projective modules. For the case of a general quiver, one can prove that there is also a relative inverse Nakayama functor, namely $\operatorname{Hom}_{\Lambda Q}(\operatorname{Hom}_{\Lambda Q_0}(\Lambda Q, \Lambda Q_0), -)$, which sends relative injective modules to relative projective modules. It can be computed explicitly using the dual of the standard resolution above. The essential image of the relative inverse Nakayama functor is precisely the monomorphism category of ΛQ, which can therefore be regarded as the category of relative Gorenstein projective modules.

More recently, motivated by applications to the theory of cluster algebras, interest has grown in certain generalisations of ΛQ called generalised species (or modulated quivers), see [6, 4]. The idea is to vary the algebra Λ associated to each vertex and relating them via bimodules on the arrows. A generalised species is a quiver Q together with a finite dimensional algebra $\Lambda_{\mathtt{i}}$ for each vertex $\mathtt{i} \in Q_0$ and $\Lambda_{t(\alpha)}$-$\Lambda_{s(\alpha)}$-bimodule Λ_α associated to each arrow, such that Λ_α is finitely generated projective from either side, and together with an isomorphism of bimodules

$$\operatorname{Hom}_{\Lambda_{s(\alpha)}^{\mathrm{op}}}(\Lambda_\alpha, \Lambda_{s(\alpha)}) \cong \operatorname{Hom}_{\Lambda_{t(\alpha)}}(\Lambda_\alpha, \Lambda_{t(\alpha)}) =: \Lambda_{\alpha^*}.$$

A representation of such a species is then given by a $\Lambda_{\mathtt{i}}$-module $M_{\mathtt{i}}$ for each vertex $\mathtt{i} \in Q_0$ together with a $\Lambda_{t(\alpha)}$-linear maps $M_\alpha \colon \Lambda_\alpha \otimes_{\Lambda_{s(\alpha)}} M_{s(\alpha)} \to M_{t(\alpha)}$. The category of representations of a species is equivalent to the category of modules over the tensor algebra $T_{\prod_{\mathtt{i}} \Lambda_{\mathtt{i}}}(\bigoplus_{\alpha \in Q_1} \Lambda_\alpha)$. Analogous statements regarding the description of Gorenstein projectives, relative projectives, and monomorphism category can be proven. In particular, the monomorphism category is a functorially finite resolving subcategory, whence has almost split sequences. Dually, one can define the epimorphism category, which is a functorially finite coresolving subcategory.

The main goal of our work in progress (the first part of which can be found in [3]) is to study the relationship between the Auslander–Reiten theory of $\operatorname{mod} \prod_{\mathtt{i}} \Lambda_{\mathtt{i}}$, $\operatorname{mod} T_{\prod_{\mathtt{i}} \Lambda_{\mathtt{i}}}(\bigoplus_{\alpha \in Q_1} \Lambda_\alpha)$, and of the monomorphism category. We obtain the following results generalising the work of Ringel and Schmidmeier:

Theorem. *Let $(\Lambda_{\mathtt{i}}, \Lambda_\alpha)$ be a generalised species over a locally bounded quiver. Denote by ν^- the relative inverse Nakayama functor.*

(1) Let

$$0 \to L \to M \to N \to 0$$

be an almost split sequence in $\operatorname{mod} T_{\prod_{\mathtt{i}} \Lambda_{\mathtt{i}}}(\bigoplus_\alpha \Lambda_\alpha)$ *with* N *being in the epimorphism category, but not projective in there. Then*

$$0 \to \nu^-(L) \to \nu^-(M) \to \nu^-(N) \to 0$$

is the sum of an almost split sequence and a split sequence in the monomorphism category.

(2) The Auslander-Reiten translation τ *in the monomorphism category can be computed as*

$$\tau = \mathrm{Mimo} \circ \left(\prod_{\mathtt{i}} \tau_{\Lambda_{\mathtt{i}}} \right) \circ \nu$$

where ν *denotes the relative Nakayama functor,* $\prod_{\mathtt{i}} \tau_{\Lambda_{\mathtt{i}}}$ *denotes the operation of applying the Auslander-Reiten translation of* $\operatorname{mod} \Lambda_{\mathtt{i}}$ *'pointwise' to the representation and* Mimo *denotes the minimal right approximation to the monomorphism category.*

We also provide an explicit description of the minimal right approximation Mimo. Our results are deduced from the general theory of relative Nakayama functors developed in [5] in a slightly more general setup, which also reveals potential connections to the theory of categorification as in [2]. More precisely we work with an abelian 'base' category $\mathcal{C}$, here given by $\operatorname{mod} \prod_{\mathtt{i}} \Lambda_{\mathtt{i}}$ as well as with endofunctors X and Y on $\mathcal{C}$. In the special case given by the functors $X = \bigoplus_\alpha \Lambda_\alpha \otimes_{\prod_{\mathtt{i}} \Lambda_{\mathtt{i}}} -$ and $Y = \bigoplus_\alpha \Lambda_{\alpha^*} \otimes_{\prod_{\mathtt{i}} \Lambda_{\mathtt{i}}} -$. The following properties are crucial to develop the theory:

(P1) X and Y form an ambidextrous adjunction $X \dashv Y \dashv X$, i.e. they are Frobenius functors;

(P2) The coproducts and products of powers of X exist and the natural map $\coprod_{i \geq 0} X^i \to \prod_{i \geq 0} X^i$ is an isomorphism. Similary for Y;

(P3) There is no epimorphism $XM \to M$ as well as no monomorphism $M \to YM$ for any $M \neq 0$ in $\mathcal{C}$.

The third property is an analogue of Nakayama's lemma in this more general setup and follows from (P1) and (P2) under the assumption of enough projectives and injectives.

The first part of the theorem above follows from a general statement on preservation of almost split morphisms under adjoint functors applied to the setup of the adjunction $\nu \dashv \nu^-$ of the relative Nakayama functor and the relative inverse Nakayama functor.

Theorem. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be abelian categories and let* $L\colon \mathcal{A} \to \mathcal{B}$ *be left adjoint to* $R\colon \mathcal{B} \to \mathcal{A}$. *Assume that the counit of the adjunction is a (pointwise) monomorphism. If* $g\colon M \to M'$ *is a right almost split morphism in* $\mathcal{B}$ *then* $R(g)\colon R(M) \to R(M')$ *is a right almost split morphism in* $\mathrm{Im}(R)$.

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Refined invariants of finite-dimensional Jacobi algebras

Ben Davison

Finite-dimensional Jacobi algebras. Jacobi algebras play a central role in the theory of 3–Calabi–Yau algebras, the 3-dimensional minimal model programme and Donaldson–Thomas theory; see [1] [2], [13] for a small selection of the background material approaching the study of these algebras from a primarily algebraic point of view. We will approach these algebras via a seemingly innocent question, especially appropriate for this conference: When are they finite-dimensional?

First we define these algebras. Let Q be a quiver, that is, a pair of finite sets Q_1 and Q_0, to be thought of as arrows and vertices, respectively, along with maps $s, t\colon Q_1 \to Q_0$. To this data we associate the free path algebra $\mathbb{C}Q$ in the usual way: a $\mathbb{C}$-basis is provided by paths in the quiver, multiplication is defined by concatenation of paths. We complete with respect to the ideal generated by the arrows, to obtain the algebra $\widehat{\mathbb{C}Q}$. Given a single cyclic path (always assumed to have length at least three) $a_1 \ldots a_n$ we define

$$\partial a_1 \ldots a_n / \partial a = \sum_{a_i = a} a_{i+1} \ldots a_n a_1 \ldots a_{i-1}.$$

We extend the above definition to formal linear combinations of cyclic paths in Q via linearity and continuity. We define I_W to be the closure of the two-sided ideal containing all $\partial W/\partial a$ for $a \in Q_1$, and set $\widehat{\mathrm{Jac}}(Q, W) = \widehat{\mathbb{C}Q}/I_W$.

The main question for the talk is: when is $\widehat{\mathrm{Jac}}(Q, W)$ finite-dimensional? For the sake of brevity we restrict our attention to quivers with one vertex, though the conjectures and theorems we present generalise to a wide class of quivers. We denote by $Q^{(l)}$ the quiver with one vertex and l loops.

(0) If $l = 0$ there is not much to say: $W = 0$, and $\widehat{\mathrm{Jac}}(Q^{(0)}, W) \cong \mathbb{C}$ is obviously a finite-dimensional $\mathbb{C}$-algebra.
(1) If $l = 1$, and $W \neq 0$ then up to formal automorphism we have $W = x^{d+1}$ for some $d \geq 2$, and $\widehat{\mathrm{Jac}}(Q^{(1)}, W) \cong \mathbb{C}[x]/(x^d)$ is finite-dimensional. If $W = 0$, then $\widehat{\mathrm{Jac}}(Q, W) \cong \mathbb{C}[\![x]\!]$ is infinite-dimensional. Thus *any* finite-dimensional quotient of the one loop quiver algebra is a Jacobi algebra.
(2) If $l = 2$, the situation is more complicated. Now there are nonzero potentials that give rise to infinite-dimensional algebras, for instance labelling the loops x and y, $\widehat{\mathrm{Jac}}(Q^{(2)}, y^3) \cong \mathbb{C}[\![x, y]\!]/(y^2)$. On the other hand, if

$W = x^2y + y^4$ then [3] $\widehat{\mathrm{Jac}}(Q^{(2)}, W)$ is a 9-dimensional algebra, arising from considering noncommutative deformations of the Laufer flopping curve [9].

(3) There is a famous Jacobi algebra with $l = 3$, given by taking $W = xyz - xzy$. Then $\widehat{\mathrm{Jac}}(Q, W) \cong \mathbb{C}[\![x, y, z]\!]$. This is certainly not finite-dimensional though! We will return to the $l = 3$ case shortly.

Jacobi algebras from geometry. The Laufer example above points towards a rich source of finite-dimensional Jacobi algebras, which we recall. Say we are given a resolution of singularities $f: X \to Y$ with $Y = \mathrm{Spec}(R)$ an isolated Gorenstein singularity, with exceptional fibre $\mathbb{P}^1 \subset X$ which we denote C. Consider the A_∞-Yoneda algebra of $\mathscr{O}_E$, which we denote $A = \mathrm{Ext}_\infty(\mathscr{O}_E, \mathscr{O}_E)$. The noncommutative deformation theory of $\mathscr{O}_E$ is represented by $\mathrm{H}^0(A^!)$, where $A^!$ is the A_∞-Koszul dual of A in the sense of e.g. [10]. This algebra can be defined as follows: it is the free tensor algebra on $\mathrm{Ext}^1(\mathscr{O}_C, \mathscr{O}_C)$, with relations given by the noncommutative derivatives of $W(x) = \sum_{i\geq 2}\langle b_i(x, \ldots, x), x\rangle$ where $b_i: (A^1)^{\otimes i} \to A^2$ are the (higher) composition maps in A, and $\langle \bullet, \bullet\rangle$ is the pairing from Serre duality. In particular, $\mathrm{H}^0(A^!)$ is a Jacobi algebra, with underlying quiver $Q^{(e)}$ where $e = \mathrm{ext}^1(\mathscr{O}_C, \mathscr{O}_C)$.

The interest for us comes from a result of Donovan and Wemyss [4], stating that the algebra $\Lambda = \mathrm{H}^0(A^!)$, which they call the *contraction algebra*, is finite-dimensional, since we have just seen that it is a Jacobi algebra. Furthermore, in [4] it is conjectured that Λ is a complete invariant of (a formal neighbourhood of) the flopping curve; a weakened form of this conjecture is proved in [6]. We'll be focusing on a different conjecture, due to Brown and Wemyss, stating that *all*[1] finite-dimensional Jacobi algebras arise this way.

Attack on the conjecture: We will try (and fail!) to disprove this conjecture using features of contraction algebras that "remember" the geometric origin of Λ. As a warmup application of this strategy, note that by [11] there are only three possibilities for the normal bundle of C: writing $\mathcal{N}_{C,X} \cong \mathscr{O}_C(a) \oplus \mathscr{O}_C(b)$ with $a \geq b$, we must have either

(0) $(a, b) = (-1, -1)$, the "Atiyah flop". This case is unique up to local isomorphism

(1) $(a, b) = (0, -2)$, Reid's "pagoda" [12]. Up to local isomorphism there are $\mathbb{N}$ of these, all explicitly written down.

(2) $(a, b) = (1, -3)$: the zoo. Here the classification of these curves becomes challenging...

In each case we have $\mathrm{ext}^1(\mathscr{O}_C, \mathscr{O}_C) = a + 1$, and this is the number of loops in the underlying quiver of Λ. So if we find a Jacobi algebra with underlying quiver $Q^{(l)}$ for $l \geq 3$, we disprove the conjecture. Unfortunately for this approach, by a result of Iyudu and Smoktunowicz [8] there is no such Jacobi algebra! Furthermore, in the cases (0) and (1) the classification of flopping curves matches our by-hand

[1]At least for one-vertex quivers: for quivers with many vertices the statement is more involved.

classification of Jacobi algebras for one-vertex quivers. So to try to break the conjecture we focus on the two loop quiver and $(1,-3)$-curves.

Fortunately, there is a *lot* of algebraic geometry studying this "zoo", since flopping curves are one of the fundamental objects of the minimal model programme in 3-dimensional birational geometry. In particular, associated to our curve are Gopakumar–Vafa invariants, **positive** numbers $n_1, n_2, \ldots$, which count rational curves in deformations of X. Now define

$$\mathrm{CC}_\nu(\Lambda, t) = \sum_{i=0}^{\dim(\Lambda)} \chi(\mathrm{Gr}_i(\Lambda), \nu)(-t)^i$$

a modification of the usual Caldero-Chapoton character of Λ, considered as a right module over itself, where we take the Euler characteristic weighted by Behrend's microlocal function ν, essentially to take account of the non-reduced/singular structure of the above Grassmannians. Then by [7] there is an equality

$$\mathrm{CC}(\Lambda) = \prod_i (1-t^i)^{in_i}. \tag{1}$$

Note that the left hand side is defined for *any* algebra, and so the numbers n_i are also. In general, these are known as the DT invariants of Λ, and they can be negative, for instance setting $\Lambda = \mathbb{C}[x]$ we find $n_1 = -1$ and $n_i = 0$ for $i \geq 2$. So we have a more advanced trap for the Brown–Wemyss conjecture: we just need to find a finite-dimensional Jacobi algebra with at least one negative DT invariant. But alas, instead we have the following theorem, which should thus be interpreted as strong evidence for their conjecture (which remains open):

Theorem 0.1. [5] *Let Λ be a finite-dimensional Jacobi algebra. Then all of the DT invariants, defined in the same way as n_i in* (1), *are non-negative integers.*

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2-hereditary algebras from hypersurfaces

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(joint work with Osamu Iyama)

Let k be a field and $R = k[x,y]/(f)$, where $f = f_1 f_2 \cdots f_n$ for some linear forms $f_i(x,y) = \alpha_i x + \beta_i y \in k[x,y]$ satisfying $(f_i) \neq (f_j)$ for any $i \neq j$. We equip R with the standard $\mathbb{Z}$-grading: $\deg x = \deg y = 1$. Then R is a 1-dimensional hypersurface and an isolated singularity. Since R is Gorenstein the category $\mathsf{CM}^{\mathbb{Z}} R$ of $\mathbb{Z}$-graded Cohen-Macaulay modules is a Frobenius category and so its stable category $\underline{\mathsf{CM}}^{\mathbb{Z}} R$ is triangulated. Our aim is to apply tilting techniques to obtain 2-hereditary algebras from $\underline{\mathsf{CM}}^{\mathbb{Z}} R$.

Let $d \geq 1$. We recall that d-hereditary algebras are a class of finite dimensional algebras of global dimension d, which were introduced in [HIO] to play the role of hereditary algebras in higher dimensional Auslander-Reiten theory. In particular, d-hereditary algebras come in two types: finite and infinite, which are called d-representation finite and d-representation infinite respectively. If A is a d-representation finite algebra, then its $(d+1)$-preprojective algebra $\Pi_{d+1}(A)$ is finite dimensional, selfinjective and stably $(d+1)$-Calabi-Yau. If, on the other hand, A is d-representation infinite, then $\Pi_{d+1}(A)$ is infinite dimensional, has global dimension $d+1$ and is a $(d+1)$-Calabi-Yau algebra. For more details we refer to [HIO]. From now on we focus on the case $d = 2$.

We define graded R-modules

$$T_i := S/(f_1 \cdots f_i), \quad T := \bigoplus_{i=1}^{n} T_i, \quad U := T \oplus T(1).$$

The completion of T was shown in [BIKR, Theorem 4.1] to be a 2-cluster tilting object in the category of Cohen-Macaulay modules over the completion of R. Our first result says that U gives a tilting object in the graded setting.

Theorem 1. *The following statements hold.*

(1) U is a tilting object in $\underline{\mathsf{CM}}^{\mathbb{Z}}R$.
(2) $\underline{\Lambda} := \underline{\mathrm{End}}^{\mathbb{Z}}_R(U)$ is a 2-representation finite algebra.
(3) $\underline{\mathsf{CM}}^{\mathbb{Z}}R$ is triangle equivalent to $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\underline{\Lambda})$.

Since $\underline{\Lambda}$ is 2-representation finite, there is a canonical 2-cluster tilting subcategory in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\underline{\Lambda})$ (see [HIO]). Using the equivalence in Theorem 1(3) we obtain a corresponding 2-cluster tilting subcategory in $\underline{\mathsf{CM}}^{\mathbb{Z}}R$, which can be lifted to $\mathsf{CM}^{\mathbb{Z}}R$. More precisely, we show the following.

Theorem 2. *The following statements hold.*

(1) $\mathsf{CM}^{\mathbb{Z}}R$ has a 2-cluster tilting subcategory

$$\mathsf{add}\{U(2\ell) \mid \ell \in \mathbb{Z}\} = \mathsf{add}\{T(\ell) \mid \ell \in \mathbb{Z}\}.$$

(2) $\mathsf{CM}^{\mathbb{Z}/2\mathbb{Z}}R$ has a 2-cluster tilting object U.
(3) $\mathsf{CM}\,R$ has a 2-cluster tilting object T.

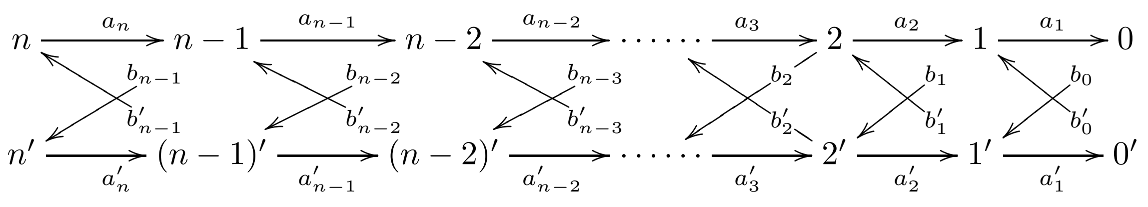
Next we use a graded version of Knörrer periodicity (see [Y]) to obtain similar results for a hypersurface of dimension 3. Set $R^\sharp := k[x,y,u,v]/(f(x,y)-uv)$, with $\mathbb{Z}$-grading given by $\deg x = \deg y = 1$, $\deg u = a$ and $\deg v = n-a$ for some $1 \le a \le n-1$. Knörrer periodicity gives a triangle equivalence

$$\mathcal{K}^G : \underline{\mathsf{CM}}^G R \to \underline{\mathsf{CM}}^G R^\sharp.$$

for each $G \in \{0, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}\}$, which sends T_i to $T_i^\sharp := (u, f_1 \cdots f_i)$ for $1 \le i \le n-1$ and T_n to $T_n^\sharp := R^\sharp$. Thus we obtain 2-cluster tilting objects

$$T^\sharp := \bigoplus_{i=1}^{n} T_i^\sharp \quad \text{and} \quad U^\sharp := T^\sharp \oplus T^\sharp(1)$$

in $\mathsf{CM}\,R^\sharp$ and $\mathsf{CM}^{\mathbb{Z}/2\mathbb{Z}}R^\sharp$ respectively. We show that $\mathrm{End}^{\mathbb{Z}/2\mathbb{Z}}_{R^\sharp}(U^\sharp)$ is a 3-Calabi-Yau algebra and can be realized as the Jacobian algebra of a certain quiver with potential (see [DWZ] for definitions of Jacobian algebras and [G] for the relationship to 3-Calabi-Yau algebras). More precisely, let Q be the following quiver with $2n$ vertices



where we identify $(n,n') = (0,0')$ if a is even and $(n,n') = (0',0)$ if a is odd. Set

$$\gamma_{ij} := \det \begin{pmatrix} \alpha_i & \alpha_j \\ \beta_i & \beta_j \end{pmatrix}$$

and define the potential

$$W := \sum_{i=1}^{n} \frac{1}{\gamma_{i-1,i}} \left(b'_{i-1} b_{i-2} a_{i-1} a_i - \frac{\gamma_{i-1,i+1}}{\gamma_{i,i+1}} b'_{i-1} a'_i b_{i-1} a_i + b_{i-1} b'_{i-2} a'_{i-1} a'_i \right)$$

where by convention $\gamma_{0,i} = \gamma_{n,i}$ and

$$(a_0, a'_0, b_{-1}, b'_{-1}) = \begin{cases} (a_n, a'_n, b_{n-1}, b'_{n-1}) & \text{if } a \text{ is even,} \\ (a'_n, a_n, b'_{n-1}, b_{n-1}) & \text{if } a \text{ is odd.} \end{cases}$$

Moreover, denote by $(\underline{Q}, \underline{W})$, the quiver with potential obtained by removing the vertices n and n' from (Q, W). We denote the Jacobian algebras of (Q, W) and $(\underline{Q}, \underline{W})$ by $P(Q, W)$ and $P(\underline{Q}, \underline{W})$ respectively.

Theorem 3. *The following statements hold.*

(1) $\underline{\mathrm{End}}^{\mathbb{Z}/2\mathbb{Z}}_{R^\sharp}(U^\sharp) \simeq P(\underline{Q}, \underline{W})$ *is selfinjective and stably* 3*-Calabi-Yau.*

(2) $\mathrm{End}^{\mathbb{Z}/2\mathbb{Z}}_{R^\sharp}(U^\sharp) \simeq P(Q, W)$ *is a* 3*-Calabi-Yau algebra.*

It turns out that we can realize $P(Q, W)$ and $P(\underline{Q}, \underline{W})$ as the 3-preprojective algebras of certain 2-hereditary algebras, by introducing suitable gradings. For this purpose let C be a set of arrows in Q such that each cycle in the potential W contains exactly one arrow from C. Moreover, assume that $C \cap \{a_i, a'_i \mid 1 \leq i \leq n\}$ contains exactly a elements. Define a grading on Q by setting

$$\deg(a) = \begin{cases} 1 & \text{if } a \in C \\ 0 & \text{if } a \notin C \end{cases} \quad \text{for } a \in Q_1.$$

This induces a grading on $\underline{Q}$ and the Jacobian algebras $P(Q, W)$ and $P(\underline{Q}, \underline{W})$. We denote the corresponding graded Jacobian algebras by $P(Q, W, C)$ and $P(\underline{Q}, \underline{W}, C)$ and their degree zero parts by $P(Q, W)_C$ and $P(\underline{Q}, \underline{W})_C$ respectively.

By applying certain degree shifts to the summands of $U^\sharp$ we obtain $U^\sharp_C \in \mathsf{CM}^{\mathbb{Z}} R^\sharp$, which is isomorphic to $U^\sharp$ in $\mathsf{CM}^{\mathbb{Z}/2\mathbb{Z}} R^\sharp$ and satisfies the following.

Theorem 4. *In the notation above we have the following results.*

(1) $\underline{\Lambda}^\sharp_C := \underline{\mathrm{End}}^{\mathbb{Z}}_{R^\sharp}(U^\sharp_C) \simeq P(\underline{Q}, \underline{W})_C$ *is a* 2*-representation finite algebra.*

(2) $\Pi_3(\underline{\Lambda}^\sharp_C) \simeq \underline{\mathrm{End}}^{\mathbb{Z}/2\mathbb{Z}}_{R^\sharp}(U^\sharp_C) \simeq P(\underline{Q}, \underline{W}, C)$ *as* $\mathbb{Z}$*-graded algebras.*

(3) $\Lambda^\sharp_C := \mathrm{End}^{\mathbb{Z}}_{R^\sharp}(U^\sharp_C) \simeq P(Q, W)_C$ *is a* 2*-representation infinite algebra.*

(4) $\Pi_3(\Lambda^\sharp_C) \simeq \mathrm{End}^{\mathbb{Z}/2\mathbb{Z}}_{R^\sharp}(U^\sharp_C) \simeq P(Q, W, C)$ *as* $\mathbb{Z}$*-graded algebras.*

Finally, we describe the cluster categories $\mathsf{C}(\underline{Q}, \underline{W})$, $\mathsf{C}(Q, W)$ of the quivers with potential $(\underline{Q}, \underline{W})$, (Q, W) and the cluster categories $\mathsf{C}(\underline{\Lambda}^\sharp_C)$, $\mathsf{C}(\Lambda^\sharp_C)$ of the algebras $\underline{\Lambda}^\sharp_C$ and $\Lambda^\sharp_C$. See [A] for relevant definitions.

For $G \in \{\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}\}$ consider the Serre quotients $\mathsf{qmod}^G R^\sharp := \mathsf{mod}^G R^\sharp / \mathsf{fd}^G R^\sharp$, where $\mathsf{fd}^G R^\sharp$ denotes the full subcategory of $\mathsf{mod}^G R^\sharp$ consisting of all finite dimensional modules.

Theorem 5. *In the notation above we have the following triangle equivalences.*

(1) $\underline{\mathsf{CM}}^{\mathbb{Z}} R^\sharp \simeq \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\underline{\Lambda}^\sharp_C)$ *and* $\underline{\mathsf{CM}}^{\mathbb{Z}/2\mathbb{Z}} R^\sharp \simeq \mathsf{C}(\underline{Q}, \underline{W}) \simeq \mathsf{C}(\underline{\Lambda}^\sharp_C)$.

(2) $\mathsf{D}^{\mathrm{b}}(\mathsf{qmod}^{\mathbb{Z}} R^\sharp) \simeq \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\Lambda^\sharp_C)$ *and* $\mathsf{D}^{\mathrm{b}}(\mathsf{qmod}^{\mathbb{Z}/2\mathbb{Z}} R^\sharp) \simeq \mathsf{C}(Q, W) \simeq \mathsf{C}(\Lambda^\sharp_C)$.

Note that similar results were obtained in [AIR].

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Global dimension for triangulated categories via stability conditions

Yu Qiu

Classical, the global dimension is defined for an algebra A. There are various definitions and we will take the following one:

$$\operatorname{gldim} A\colon = \sup\{k \in \mathbb{Z} \mid \operatorname{Hom}_{\mathcal{D}^b(A)}(\mathcal{H}_A, \mathcal{H}_A[k]) \neq 0\}$$

where $\mathcal{H}_A = \mod A$ is a *heart* in $\mathcal{D}^b(A)$. A classical question is to compare $\operatorname{gldim} B$ for any algebra B that is derived equivalent to A. The minimal value among all such $\operatorname{gldim} B$ may be regarded as the global dimension of $\mathcal{D}^b(A)$.

We can generalize such a definition to any heart $\mathcal{H}$ in a fixed triangulated category $\mathcal{D}$ as

$$\operatorname{gldim} \mathcal{H}\colon = \sup\{k \in \mathbb{Z} \mid \operatorname{Hom}_{\mathcal{D}}(\mathcal{H}, \mathcal{H}[k]) \neq 0\}$$

Note that here the Hom is not taken in the derived category $\mathcal{D}^b(\mathcal{H})$ of $\mathcal{H}$. A heart $\mathcal{H}$ is equivalent to a (bounded) *t-structure* $(\mathcal{P}^{\perp}, \mathcal{P})$ in $\mathcal{D}$, which is a torsion pair such that the torsion part $\mathcal{P}$ is closed under shift [1] (and a technical condition boundedness). Towards a $\mathbb{R}$-generalization of global dimension, it is natural to consider the $\mathbb{R}$-generalization of t-structure, the slicing.

Definition 1. A *slicing* $\mathcal{P}$ is a $\mathbb{R}$-collection $\{\mathcal{P}\phi \mid \phi \in \mathbb{R}\}$ of full additive (in fact abelian) subcategories in $\mathcal{D}$ such that $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$ and, for any $\phi \in \mathbb{R}$, there are t-structures $\Big(\mathcal{P}(-\infty, \phi), \mathcal{P}[\phi, +\infty)\Big)$ and $\Big(\mathcal{P}(-\infty, \phi], \mathcal{P}(\phi, +\infty)\Big)$.

Here $\mathcal{P}(I)$ is the full extension-closed subcategory of $\mathcal{D}$ generated by zero object and objects in $\mathcal{P}(\varphi), \varphi \in I$. Then we can define the global dimension for a slicing $\mathcal{P}$ (in fixed $\mathcal{D}$) as

$$\operatorname{gldim} \mathcal{P} = \sup\{\phi_2 - \phi_1 \mid \operatorname{Hom}_{\mathcal{D}}(\mathcal{P}(\phi_1), \mathcal{P}(\phi_2)) \neq 0\}$$

Note that when $\mathcal{D} = \mathcal{D}^b(A)$ and consider the slicing $\mathcal{P}_A$ with $\mathcal{P}_A(0) = \mathcal{H}_A$ and $\mathcal{P}_A(\phi) = 0$ for $\phi \notin \mathbb{Z}$, we have indeed

$$\operatorname{gldim} \mathcal{P}_A = \operatorname{gldim} A.$$

Therefore, it is natural to regard gldim as a function on the space of slicing and its infimum can be regard as the global dimension of $\mathcal{D}$. However, such a space is too big and does not have good properties. More rigid structure is better to be added, which leads to Bridgeland stability conditions.

Definition 2. A *stability condition* $\sigma = (Z, \mathcal{P})$ on a triangulated category consists of a central charge $Z \in \operatorname{Hom}_{\mathbb{Z}} K(\mathcal{D}), \mathbb{C})$, where $K(\mathcal{D})$ is the Grothendieck group of $\mathcal{D}$, and a slicing $\mathcal{P}$ such that they are compatible in the sense that

- for any $M \in \mathcal{P}(\phi)$, one has $Z(M) = m(M) \cdot e^{\mathbf{i}\pi\phi}$ for some $m(M) \in \mathbb{R}_{\geq 0}$.

The following is a crucial result, due to Bridgeland, that equips complex structure on the space of stability conditions.

Theorem 3. [B] *The space* $\operatorname{Stab}\mathcal{D}$ *of stability conditions* σ *satisfying support property on* $\mathcal{D}$ *form a complex manifold with dimension* $\operatorname{rank} K(\mathcal{D})$ *and local coordinate* Z*.*

There is a natural $\mathbb{C}$-action $\operatorname{Stab}(\mathcal{D})$ by $s \cdot (Z, \mathcal{P}) = (Z \cdot e^{-\mathbf{i}\pi s}, \mathcal{P}_{\Re(s)})$, where $\mathcal{P}_x(\phi) = \mathcal{P}(\phi + x)$. There is also a natural action induced by $\operatorname{Aut}(\mathcal{D})$: $\Phi(Z, \mathcal{P}) = \big(Z \circ \Phi^{-1}, \Phi(\mathcal{P})\big)$.

So it is natural to define $\operatorname{gldim}\sigma \colon = \operatorname{gldim}\mathcal{P}$ for $\sigma = (Z, \mathcal{P})$ and we have the following.

Lemma 4. [IQ1] gldim *is a continuous function on* $\operatorname{Aut}\mathcal{D} \backslash \operatorname{Stab}\mathcal{D}/\mathbb{C}$*. with values in* $\mathbb{R}_{\geq 0} \cup \{+\infty\}$*.*

Definition 5. The infimum of gldim on $\operatorname{Stab}\mathcal{D}$, denoted by $\operatorname{gd}\mathcal{D}$, is the *global dimension* of a triangulated category $\mathcal{D}$.

We have the following calculation of $\operatorname{gd}\mathcal{D}$ in some examples.

Theorem 6. [Q1] *Let* $\mathcal{D}(Q) = \mathcal{D}^b(\mathbf{k}Q)$ *be the bounded derived category of the path algebra of an acyclic quiver* Q *(and* $\mathbf{k}$ *is a field). Then we have*

- *If* Q *is a Dynkin quiver, then* $\operatorname{gd}\mathcal{D}(Q) = 1 - 2/h$*, where* h *is the Coxeter number of* Q*.*
- *Otherwise,* $\operatorname{gd}\mathcal{D}(Q) = 1$*.*

Moreover, in the Dynkin case, there is a unique stability condition σ_G*, up to* $\mathbb{C}$*-action, with* $\operatorname{gldim}\sigma_G = 1 - 2/h$*. In fact,* σ_G *is also the unique solution, up to* $\mathbb{C}$*-action, of the Gepner equation (cf.* [KST]*)*

$$\tau(\sigma) = (-2/h) \cdot \sigma.$$

Theorem 7. [KOT] *Let* $\mathcal{D}(X) = \mathcal{D}^b(\operatorname{coh} X)$ *be the bounded derived category of coherent sheaves a smooth projective curve* X *with genus* g*. Then* $\operatorname{gd}\mathcal{D}(X) = 1$*. The infimum* 1 *is reachable by some stability conditions if and only if* $g \leq 1$*.*

We conjecture the following.

Conjecture 8. *Let* $\operatorname{gd}\mathcal{D}$ *be the global dimension of a triangulated category* $\mathcal{D}$*.*

- *If the subspace* $\operatorname{gldim}^{-1}(\operatorname{gd}\mathcal{D})$ *is non-empty, then it is contractible.*

- $\operatorname{gldim}^{-1}(\mathrm{gd}, x)$ *contracts to* $\operatorname{gldim}^{-1}(\mathrm{gd}\,\mathcal{D}, y)$ *for any real number* $\mathrm{gd}\,\mathcal{D} < y < x$.

Here, we may take the preimage of gldim *in a preferable connected component of* $\mathcal{D}$.

Note that such a conjecture implies the general contractibility conjecture of $\operatorname{Stab}\mathcal{D}$. The conjecture holds for $\mathcal{D}^b(\operatorname{coh}\mathbb{P}^1)$ and $\mathcal{D}^b(\mathbf{k}A_2)$ by direct calculation, see [Q1]. We have the following.

Theorem 9. [FLLQ] *Let* $\mathcal{D}(\mathbb{P}^2) = \mathcal{D}^b(\operatorname{coh}\mathbb{P}^2)$ *be the bounded derived category of coherent sheaves on* $\mathbb{P}^2$. *Then* $\mathrm{gd}\,\mathcal{D}(\mathbb{P}^2) = 2$ *and Conjecture 8 holds (for the principal component of* $\operatorname{Stab}\mathcal{D}(\mathbb{P}^2)$*).*

Finally, here is a version of Gabriel's theorem, which is due to [KOT] but we state it in a slightly more general form in [Q2].

Theorem 10. [KOT, Q2] *Let* $\mathcal{D}$ *be a connected triangulated category. Then* $\operatorname{gldim}\mathcal{D} < 1$ *if and only if* $\mathcal{D} = \mathcal{D}(Q)/\iota$ *for some Dynkin quiver* Q *and* $\iota \in \operatorname{Aut}\mathcal{D}(Q)$ *being induced from some graph automorphism of* Q.

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On generalisations of Iwanaga-Gorenstein algebras

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(joint work with Aaron Chan on Cohen-Macaulay Artin algebras)

Assume all algebras are connected, non-semisimple Artin algebras and all modules are finitely generated right modules unless stated otherwise. Recall that an Artin algebra A is called Iwanaga-Gorenstein in case the left and right injective dimension of the regular module A are finite, in which case they coincide. In this case the selfinjective dimension of A is defined as the injective dimension of A. $P^{<\infty}$ denotes the full subcategory of modules having finite projective dimension and $I^{<\infty}$ denotes the full subcategory of modules having finite injective dimension. It is well known that A is Iwanaga-Gorenstein if and only if $P^{<\infty} = I^{<\infty}$. We will discuss two generalisations of Iwanaga-Gorenstein algebras here. The first one is due to Auslander and Reiten from [AR] nearly 30 years ago and the second is a recent generalisation due to Ringel and Zhang from [RZ]. At the end we briefly discuss an open problem on the injective dimension of the Jacobson radical that is known to be true for Iwanaga-Gorenstein algebras.

1. Cohen-Macaulay Artin algebras

In the last section of [AR], Auslander and Reiten introduced Cohen-Macaulay Artin algebras as a generalisation of Iwanaga-Gorenstein algebras. By definition, A is Cohen-Macaulay in case there exists an A-bimodule W, called the dualizing module, such that there is an equivalence of categories $Hom_A(W, -) : I^{<\infty} \to P^{<\infty}$. In this case W is a cotilting module as a left and right A-module with several special properties, we refer to [AR2] for more information and equivalent characterisations of Cohen-Macaulay Artin algebras. The CM dimension of a Cohen-Macaulay Artin algebra is defined as the injective dimension of W, which coincides with the finitistic dimension of the algebra by proposition 1.6. of [AR2]. It is easy to see that a Cohen-Macaulay Artin algebra A is Iwanaga-Gorenstein if and only if $W = A$. The full subcategory of Cohen-Macaulay A-modules for an Cohen-Macaulay Artin algebra A is defined as $CM(A) := \{X | Ext_A^i(X, W) = 0$ for all $i \geq 1\}$. Following Iyama and Solberg [IS], an Iwanaga-Gorenstein algebra A is called minimal Auslander-Gorenstein in case $id(A) \leq d \leq domdim(A)$ for some $d \geq 2$. This class of algebras generalised the higher Auslander algebras in [I] that are in a bijective correspondence with cluster tilting modules and are by definition just the minimal Auslander-Gorenstein algebras of finite global dimension. We define minimal Auslander-Cohen-Macaulay algebras as Cohen-Macaulay Artin algebras that satisfy the condition $id(W) \leq d \leq domdim(A)$ for some $d \geq 2$. This generalises minimal Auslander-Gorenstein algebras, which are exactly those minimal Auslander-Cohen-Macaulay algebras that are Iwanaga-Gorenstein. One can show that for minimal Auslander-Cohen-Macaulay Artin algebras A, one has $domdim(W) = domdim(A)$ and one can use this to get a generalisation of several results on minimal Auslander-Gorenstein algebras such as the correspondence to precluster-tilting modules that is theorem 4.5. in [IS]. In [AR2], after proposition 3.1., Auslander and Reiten posed the question whether for a Cohen-Macaulay Artin algebra A with CM dimension $d > 0$, $CM(A) = \Omega^d(mod - A)$ implies that A is Iwanaga-Gorenstein. We show that in case A is minimal Auslander-Cohen-Macaulay with CM dimension $d \geq 2$ one always has $CM(A) = \Omega^d(mod - A)$. Thus in order to give a negative answer to the question of Auslander and Reiten it would be enough to find a minimal Auslander-Cohen-Macaulay algebra with CM dimension $d \geq 2$ that is not Iwanaga-Gorenstein. With the help of the GAP-package QPA, see [QPA], we were able to find such an algebra and a general plan for a construction of such algebras. Namely let $A = kQ/I$ be the quiver algebra where Q is the quiver with 3 vertices 1,2 and 3 and arrows a_1 from 1 to 3, a_2 from 2 to 3, a_3 from 3 to 1 and a_4 from 3 to 2 and set $I =< a_1a_3, a_3a_1 - a_4a_2, a_2a_3a_1a_4 >$. Then

A is a minimal Auslander-Cohen-Macaulay Artin algebra of CM dimension 2 that is not Iwanaga-Gorenstein. It has the property that $CM(A) = \Omega^2(mod - A)$, thus giving a negative answer to the question of Auslander and Reiten. In fact we were able to construct an infinite family of minimal Auslander-Cohen-Macaulay Artin algebras that are not Iwanaga-Gorenstein algebras and it is work in progress to be able to obtain a general construction of Cohen-Macaualay Artin algebras from noetherian (non-artin) rings coming from commutative algebra.

2. Weakly Gorenstein algebras

In [AB], Auslander and Bridger introduced the concept of Gorenstein projective modules for a general Artin algebra A. An A-module M is Gorenstein projective in case $Ext^i_A(M, A) = 0 = Ext^i(D(A), \tau(M))$ for all $i > 0$. In case A is Iwanaga-Gorenstein, this is equivalent to the single condition $Ext^i_A(M, A) = 0$. For nearly 40 years it was an open question whether $Ext^i_A(M, A) = 0$ already implies that M is Gorenstein projective for a general Artin algebra A until in [JS] a first counterexample was given. In [RZ], Ringel and Zhang defined semi-Gorenstein projective modules as modules M with $Ext^i_A(M, A) = 0$ for all $i > 0$ and defined an algebra A to be weakly Gorenstein in case every semi-Gorenstein projective module is already Gorenstein projective. Iwanaga-Gorenstein algebras are weakly Gorenstein, but also many other classes of algebras such as representation-finite algebras. Ringel and Zhang gave a first systematic study of semi-Gorenstein projective modules and weakly Gorenstein algebras and gave another example of an algebra that is not weakly Gorenstein. It seems that no other example of non-weakly Gorenstein algebras have appeared in the literature and both examples were local algebras. Weakly Gorenstein algebras satisfy the Nakayama conjecture and from the point of view of finding possible counterexamples to this conjecture it is important to find examples of non-weakly Gorenstein algebras. We give a first systematic construction of non-weakly Gorenstein algebras by using symmetric algebras. Namely we prove that in case A is a symmetric algebra with A-modules X and M such that $Ext^l_A(X, M) \neq 0$ for some $l \geq 1$ but $Ext^i_A(X, M) = 0$ for all $i \geq l + 1$, then the algebra $B := End_A(A \oplus X)$ is not weakly Gorenstein. We can construct like this many non-weakly Gorenstein algebras using for example quantum exterior algebras, we refer to [M2] for more details and examples of explicit constructions. We remark that all known examples of non-weakly Gorenstein algebras have relations containing field elements that are not roots of unity. In particular, non-weakly Gorenstein finite dimensional algebras are not known over finite fields. We are also able to show that a very large class of algebras are weakly Gorenstein. Namely for an Artin algebra A, define $sGp(A)$ as the full subcategory of semi-Gorenstein projective A-modules and $\phi_n(A) := sGp(A) \cap add(\Omega^n(mod - A))$. We call A ϕ_n-finite in case $\phi_n(A)$ contains only finitely many indecomposable modules. In [RZ], it was shown by Ringel and Zhang that in case A is ϕ_1-finite, then A is weakly Gorenstein. A corollary of this result is that any torionless-finite algebra is weakly Gorenstein. We have shown in [M3] that the result holds for any n, namely every ϕ_n-finite algebra A is weakly Gorestein. As a corollary we can prove that any monomial algebra or any algebra of the form $End_B(M)$ is weakly Gorenstein in case B is representation-finite with an arbitrary B-module M.

3. The injective dimension of the Jacobson radical

At the end we briefly discuss a question on the Jacobson radical J of a general Artin algebra. By work of Auslander it is known that the projective dimension of the Jacobson radical of A is equal to the global dimension of A minus one. It seems however, that the injective dimension of the Jacobson radical of A has not been studied yet. We conjecture that the injective dimension of J is always equal to the global dimension of A. This is true in case A is Iwanaga-Gorenstein and in some other cases such as when A is a Nakayama algebra or a radical square zero algebra, see [M].

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Categorifying acyclic cluster algebras via hereditary algebras

Andrew Hubery

Cluster algebras. Cluster algebras were introduced by Fomin and Zelevinsky [4] as a tool to study total positivity and Lusztig's theory of canonical bases in Lie theory, but have since found applications in many areas of mathematics, including Poisson geometry, integral systems and Teichmüller theory.

Briefly, a cluster algebra is an integral subring $\mathcal{A}$ of $\mathcal{F} = \mathbb{Q}(x_1, \dots, x_n)$, determined by an initial seed and mutation rule. More precisely, a seed consists of an ordered set of generators $(y_1, \dots, y_n)$ of $\mathcal{F}$ together with an exchange matrix B which encodes how one can mutate in any of the n positions to obtain a new seed. Starting from an initial seed, one thus obtains via repeated mutation a distinguished set of elements of $\mathcal{F}$, called cluster variables, and the cluster algebra is defined to the the integral subring of $\mathcal{F}$ generated by all these cluster variables.

Cartan lattices. A generalised Cartan lattice Γ consists of a lattice $\mathbb{Z}^n$ equipped with an integral-valued bilinear form $\langle -, - \rangle$ such that, setting $a_{ij} := \langle e_i, e_j \rangle$ for the standard basis vectors e_i, we have $a_{ij} \leq 0$ for $i < j$ and $a_{ij} = 0$ for $i > j$, whereas a_{ii} is positive and divides both a_{ij}, a_{ji} for all j. It follows that $\mathbb{Z}^n$ together with the associated symmetric bilinear form $(x, y) := \langle x, y \rangle + \langle y, x \rangle$ is a Cartan datum in the sense of Lusztig, and the matrix with entries $a_{ii}^{-1}(a_{ij} + a_{ji})$ is a symmetrisable generalised Cartan matrix.

Attached to every generalised Cartan lattice Γ is a cluster algebra $\mathcal{A}_\Gamma$, where the exchange matrix of the initial seed has entries $a_{ii}^{-1}(a_{ij} - a_{ji})$. The cluster algebras which arise in this way are called acyclic, reflecting the fact that the matrix (a_{ij}) is upper triangular.

Hereditary algebras. Starting from a finite dimensional hereditary k-algebra Λ, we naturally obtain a generalised Cartan lattice by taking its Grothendieck group $K_0(\operatorname{mod}\Lambda)$, with standard basis given by the simple modules, and Euler form $\langle [M], [N] \rangle := \dim \operatorname{Hom}_\Lambda(M, N) - \dim \operatorname{Ext}^1_\Lambda(M, N)$. Recall that Λ is hereditary provided $\operatorname{Ext}^2_\Lambda(M, N) = 0$ for all modules M, N, in which case the Euler form on the module category does indeed descend to a bilinear form on the Grothendieck group.

Observe that if $\Lambda = kQ$ is the path algebra of an (acyclic) quiver, then the bilinear form satisfies $a_{ii} = 1$ for all i. On the other hand, if k is a finite field, then every generalised Cartan lattice arises as the Grothendieck group of some finite dimensional hereditary k-algebra.

In a sequence of papers, including [1, 2, 3], an explicit connection was made between the representation theory of a path algebra $\mathbb{C}Q$ and the corresponding cluster algebra $\mathcal{A}_Q$. More precisely, there is an explicit map from $\mathbb{C}Q$-modules to $\mathcal{F}$ which restricts to a bijection between the exceptional modules and the non-initial cluster variables, in such a way that the clusters (or seeds) correspond bijectively to the support-tilting modules, and the mutation of clusters corresponds to the usual mutation of support-tilting modules.

The natural question was therefore whether this beautiful relationship can be extended to cover all finite dimensional hereditary algebras and all acyclic cluster cluster algebras. There are three ingredients needed to obtain such a result.

(A) Since clusters are by definition obtained via mutation from the initial cluster, it is necessary that the same result holds for all support-tilting modules for Λ.

(B) One needs an analogue of the Caldero-Chapoton map [2], from (rigid) Λ-modules to $\mathcal{F}$. One would also like to obtain from such a map that the image of a rigid module is a Laurent polynomial with non-negative coefficients and denominator given by the dimension vector of the module.

(C) Finally one has to check that this map respects the two notions of mutation; in other words, one needs to prove the relevant cluster multiplication formula, generalising Caldero-Keller [3].

Main results. Let Λ be a finite dimensional hereditary k-algebra, and write Γ for the associated generalised Cartan lattice. A (right) Λ-module M is said to be rigid provided $\mathrm{Ext}^1(M,M) = 0$, and exceptional provided it is both rigid and indecomposable. A rigid module M is (basic) tilting if its indecomposable summands form a basis of the Grothendieck group $K_0(\mathrm{mod}\,\Lambda)$. More generally, M is support-tilting if there is an idempotent $e \in \Lambda$ such that $Me = 0$ and M is a tilting module for $\Lambda/\Lambda e\Lambda$.

We call $d \in \mathbb{Z}^n$ a dimension vector provided it has only non-negative coefficients. These are precisely the elements of the form $[M]$ for some Λ-module M. The dimension vectors of the exceptional modules are called positive real Schur roots, and are known to depend only on Γ. One can characterise them in terms of the poset of non-crossing partitions attached to Γ. More generally, the dimension vectors of the rigid modules depend only on Γ, and for each such rigid dimension vector d and hereditary algebra Λ of type Γ, there exists a unique (up to isomorphism) rigid module M of dimension vector d.

(A) In [5] I showed that the support-tilting modules for Λ form a simplicial polytope, where the vertices of the polytope correspond to the exceptional modules together with a complete set of primitive orthogonal idempotents in Λ, and the facets of the polytope correspond to the support-tilting modules. In particular, any two support-tilting modules are mutation-equivalent.

(B) The original Caldero-Chapoton map, for a module M over the path algebra $\mathbb{C}Q$, has coefficients which are (sums of) Euler characteristics of quiver Grassmannians attached to M. For a general hereditary algebra Λ it is unclear what the replacement of the Euler characteristic should be. Using the theory of Ringel-Hall algebras and quantum groups, however, I showed in [6] that for each pair of dimension vectors (d,e) in Γ such that d is rigid, there is a polynomial $g_{(d,e)}(t) \in \mathbb{Z}[t]$ with the following property:

For each finite field k and each finite dimensional hereditary k-algebra Λ of type Γ, if M is a rigid module of dimension vector d, then $g_{(d,e)}(|k|)$ equals the number of submodules of M of dimension vector e.

One can then construct an analogue of the Caldero-Chapoton map using the numbers $g_{(d,e)}(1)$ instead of Euler characteristics of quiver Grassmannians. Moreover, in the quiver case, it is known that if the quiver Grassmannian of a rigid module is non-empty, then it is smooth, projective, and defined over the integers. Thus one may use the Weil conjectures to deduce that its Euler characteristic is positive and equals the number $g_{(d,e)}(1)$. In fact, in this case the polynomial $g_{(d,e)}(t)$ always has non-negative coefficients. For general Γ one can use results of Deligne and Lusztig to show that the number $g_{(d,e)}(1)$ is again positive, and conjecturally it will also have non-negative coefficients.

(C) The main announcement of my talk was that, using this analogue of the Caldero-Chapoton map, one can now prove the cluster-multiplication theorem for general Γ, [7]. This therefore completes the programme stated above connecting the combinatorics of acyclic cluster algebras with the support-tilting modules for finite dimensional hereditary algebras.

Furthermore, one sees that the image under the Caldero-Chapoton map of a rigid module of dimension vector d is a Laurent polynomial with non-negative coefficients and denominator x^d.

One can now relate exceptional modules to non-initial cluster variables in such a way that the coefficients appearing in the cluster variable express knowledge about the corresponding submodule Grassmannians. This connection holds for all finite dimensional hereditary algebras over either an algebraically-closed field (so one has the path algebra of a quiver), or over a finite field. The question remains, however, about other fields, in particular in characteristic zero. A simple example would be an algebra of the form $\begin{pmatrix} \mathbb{R} & \mathbb{H} \\ 0 & \mathbb{H} \end{pmatrix}$ involving the real numbers and the quaternions. Is there an appropriate analogue of the Euler characteristic of the submodule Grassmannian which covers all such cases?

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Tensor algebras in finite tensor categories

Ryan Kinser

(joint work with Pavel Etingof and Chelsea Walton)

1. Preliminaries

We assume throughout that $\Bbbk$ is an algebraically closed field, and all categories, functors, etc. are $\Bbbk$-linear. A standard reference for our terminology and conventions is [EGNO15]. Here, a *finite tensor category* is given by the following data, subject to some natural axioms:

- $\mathcal{C}$ an abelian category which is equivalent to the category of finite dimensional representations of some finite-dimensional associative $\Bbbk$-algebra,
- $-\otimes -\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ a biexact functor which is bilinear on morphisms,

- a *unit object* $\mathbf{1} \in \mathcal{C}$ such that $\mathsf{End}_{\mathcal{C}}(\mathbf{1}) = \Bbbk$ and both $\mathbf{1} \otimes -$ and $- \otimes \mathbf{1}$ are autoequivalences of $\mathcal{C}$,
- an arbitrary, fixed isomorphism $\mathbf{1} \otimes \mathbf{1} \xrightarrow{\sim} \mathbf{1}$,
- a functorial *associativity isomorphism* $a_{X,Y,Z} \colon (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$.

In particular, we assume all objects are *rigid* (possess both right and left duals). Examples include the category Vec of finite-dimensional vector spaces, $\mathsf{Rep}(G)$ and Vec_G where G is a finite group, $\mathsf{Rep}(H)$ where H is a finite-dimensional Hopf algebra, and $\mathsf{Bimod}(A)$ where A is a finite-dimensional semisimple algebra. But in general we do not assume that objects have underlying vector spaces (e.g. the Yang-Lee tensor category).

An *associative algebra in* $\mathcal{C}$ is an object $A \in \mathcal{C}$ with morphisms $m \colon A \otimes A \to A$ and $u \colon \mathbf{1} \to A$ satisfying the diagrams of the usual definition of an algebra. In a general tensor category $\mathcal{C}$, we always have the algebras $\mathbf{1}$ and $X \otimes X^*$ for any $X \in \mathcal{C}$. Examples arising in nature include $\Bbbk$-algebras with additional structure, such as actions of or gradings by a finite group, and more generally, algebras with an action of a finite-dimensional Hopf algebra.

A *right* A*-module* is an object $M \in \mathcal{C}$ along with an action $M \otimes A \to M$ satisfying the diagrams of the usual definition of a module. Right A-modules form an abelian category $\mathsf{Mod}_{\mathcal{C}}(A)$, which carries a left action of $\mathcal{C}$ by taking $X \in \mathcal{C}$, $M \in \mathsf{Mod}_{\mathcal{C}}(A)$, and forming $X \otimes M \in \mathsf{Mod}_{\mathcal{C}}(A)$ in the natural way. Algebras A and B in $\mathcal{C}$ are *Morita equivalent* if there is an equivalence of categories $\mathsf{Mod}_{\mathcal{C}}(A) \simeq \mathsf{Mod}_{\mathcal{C}}(B)$ which is compatible with the $\mathcal{C}$-action.

Now let S be an algebra in $\mathcal{C}$, and E an S-bimodule. We form the *tensor algebra*

$$T_S(E) := S \oplus E \oplus (E \otimes_S E) \oplus (E \otimes_S E \otimes_S E) \oplus \cdots$$

using tensor product over the algebra S, which can be naturally defined for S-bimodules in $\mathcal{C}$. (This may actually lie in $\mathsf{Ind}(\mathcal{C})$ instead of $\mathcal{C}$).

Etingof and Ostrik introduced the following notion as a good generalization of semisimplicity of an algebra [EO04]: the algebra S is *exact* if $P \in \mathcal{C}$ projective and $M \in \mathsf{Mod}_{\mathcal{C}}(S)$ arbitrary imply $P \otimes M$ is projective in $\mathsf{Mod}_{\mathcal{C}}(S)$. If $\mathcal{C}$ itself is semisimple, then S is exact if and only if $\mathsf{Mod}_{\mathcal{C}}(S)$ is semisimple. When $\mathcal{C}$ is not semisimple (e.g. $\mathcal{C} = \mathsf{Rep}(G)$ and $\operatorname{char} \Bbbk$ divides $|G|$), exact algebras give more flexibility but they still have nice properties. For example, $\mathbf{1}$ is always an exact algebra in any finite tensor category. We always assume below that S is exact.

We say $T_S(E)$ is *equivalent to* $T_{S'}(E')$ if there exists a Morita equivalence $S \sim S'$ in $\mathcal{C}$ which identifies E with E' via the induced equivalence on bimodule categories.

Theorem 1. [EKW, Theorem 3.11] Let $\mathcal{C}$ be a finite tensor category. Then equivalence classes of tensor algebras are in bijection with pairs (S, E) where we take one S from each Morita equivalence class, and for each choice of S, one E from each conjugacy class (conjugacy by invertible S-bimodules).

In the case $\mathcal{C} = \mathsf{Vec}$, the theorem above translates to the fact that each tensor algebra $T_S(E)$ (with S semisimple and E a finite-dimensional bimodule) is Morita equivalent to the path algebra of a quiver. Future work will develop a similar combinatorial model for tensor algebras in more general $\mathcal{C}$. When $\mathcal{C}$ is a *fusion category* (i.e. semisimple as well), Ocneanu rigidity can be applied in the above situation to conclude that there are only finitely many "building blocks" for constructing all tensor algebras in $\mathcal{C}$ (see [EKW, Cor. 3.16]). Future work will study quotients of tensor algebras in tensor categories and other generalizations of classical theory of finite-dimensional algebras to tensor categories.

2. Categorical Morita equivalence and sketch of an example

From now on we assume the characteristic of $\Bbbk$ is zero. The Kac-Paljutkin Hopf algebra H_8 is generated as a $\Bbbk$-algebra by x, y, z subject to the relations

$$x^2 = y^2 = 1,\ z^2 = \frac{1}{2}(1 + x + y - xy),\ xy = yx,\ xz = zy,\ yz = zx.$$

Comultiplication is given by x, y being group-like and

$$\Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z).$$

This 8-dimensional algebra is semisimple, but not isomorphic as a Hopf algebra to any group algebra or dual group algebra. To illustrate our main result, we set the goal of classifying tensor algebras in $\mathsf{Rep}(H_8)$ (this includes our original motivation, classifying Hopf actions of H_8 on path algebras, as a special case).

Tensor categories $\mathcal{C}, \mathcal{D}$ are *categorically Morita equivalent* if there exists an exact algebra A in $\mathcal{C}$ such that $\mathcal{D}$ is tensor equivalent to $\mathsf{Bimod}_{\mathcal{C}}(A)$. This is an equivalence relation. In this situation, there is an induced bijection on algebras, module categories, bimodule categories, etc., and thus we obtain a bijection between tensor algebras in $\mathcal{C}$ and $\mathcal{D}$ (up to equivalence). For example, if G is a finite group, then $\mathsf{Rep}(G)$ and Vec_G are categorically Morita equivalent (take the group algebra $A = \Bbbk G$ in Vec_G).

Let G be a finite group and consider a 3-cocycle (with trivial G-action on $\Bbbk^\times$)

$$\omega \colon G \times G \times G \to \Bbbk^\times.$$

We obtain the tensor category Vec_G^ω whose objects are G-graded vector spaces, with simple objects $\delta_g = \Bbbk$ concentrated in degree $g \in G$, tensor product determined by $\delta_g \otimes \delta_h = \delta_{gh}$, and associativity isomorphism determined by

$$a_{g,h,k} \colon (\delta_g \otimes \delta_h) \otimes \delta_k \xrightarrow{\sim} \delta_g \otimes (\delta_h \otimes \delta_k)$$

being multiplication by $\omega(g, h, k)$. Fixing G, tensor-equivalence classes of such categories are in bijection with $H^3(G, \Bbbk^\times)$.

Let D_8 be the dihedral group of order 8, presented by the same generators and relations as H_8 above, but with $z^2 = 1$. Take ω to be any cohomologically non-trivial 3-cocycle on D_8, and consider the algebra $A = \Bbbk\langle z \rangle$ in $\mathcal{C} = \mathsf{Vec}_{D_8}^\omega$. Then $\mathsf{Rep}(H_8)$ is tensor equivalent to $\mathsf{Bimod}_{\mathcal{C}}(A)$, so categorical Morita equivalence induces a bijection between tensor algebras in $\mathsf{Rep}(H_8)$ and $\mathcal{C}$ (up to equivalence of tensor algebras).

Work of Ostrik [Ost03b] and Natale [Nat17] gives a classification of semisimple algebras in Vec_G^ω up to Morita equivalence: they are parametrized by pairs (L, ψ)

where $L \leq G$ and $\psi\colon L \times L \to \Bbbk^\times$ is a 2-cochain such that $d\psi = \omega|_L$. Equivalence is rather complicated, but explicitly computable.

From the above, one can compute that there are 6 equivalence classes of indecomposable semisimple algebras in $\mathsf{Vec}^\omega_{D_8}$, thus in $\mathsf{Rep}(H_8)$ as well. Explicit representatives can be found in [EKW, §5.3]. This means that there will be 6 flavors of "vertices" in a combinatorial, quiver-type model for equivalence classes of tensor algebras in $\mathsf{Rep}(H_8)$. Turning to the analogue of "arrows" in such a model, for each pair of indecomposable semisimple algebras (S_1, S_2) in $\mathsf{Rep}(H_8)$, one needs to classify the indecomposable (S_1, S_2)-bimodules in $\mathsf{Rep}(H_8)$. Again using the categorical Morita equivalence, this can be done explicitly by using [Ost03b, Nat17]. We leave this for future consideration along with other classes of examples.

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Symmetric periodic algebras

Andrzej Skowroński
(joint work with Karin Erdmann)

This is report on the joint project with Karin Erdmann concerning the Morita equivalence classification of symmetric periodic tame algebras over an algebraically closed field K.

By general theory (see [1, 2, 9]), every basic, indecomposable, symmetric, periodic, tame algebra of polynomial growth is socle equivalent to an orbit algebra $\mathrm{T}(B)/G$ of the trivial extension algebra $\mathrm{T}(B)$ of a tilted algebra B of Dynkin type or a tubular algebra B, with respect to free action of a finite cyclic group G. The following problem was raised in [4].

Problem. Let A be an indecomposable symmetric tame algebra of non-polynomial growth for which all simple modules are periodic. Is it true that A is a periodic algebra of period 4?

Following [5], an algebra A is said to be of *generalized quaternion type* if A is symmetric, indecomposable, tame of infinite type, and all simple modules are periodic of period 4. We note that every indecomposable, symmetric, representation-infinite periodic algebra of period 4 is an algebra of generalized quaternion type. Moreover, the class of algebras of quaternion type described in [3] is the class of algebras of generalized quaternion type with nonsingular Cartan matrix.

During the talk we introduced weighted surface algebras $\Lambda = \Lambda(S, \vec{T}, m_\bullet, c_\bullet, b_\bullet)$ in the general form as follows.

A *triangulation quiver* is a pair (Q, f) where $Q = (Q_0, Q_1, s, t)$ is a finite connected 2-regular quiver and $f : Q_1 \to Q_1$ is a permutation of the set Q_1 of arrows such that $f^3 = \mathrm{id}$ and $s(f(\alpha)) = t(\alpha)$ for any arrow $\alpha \in Q_1$. Every triangulation quiver is the triangulation quiver $(Q(S, \vec{T}), f)$ of a directed triangulated surface $(S, \vec{T})$ [4].

Let (Q, f) be a triangulation quiver, $\bar{}: Q_1 \to Q_1$ the involution which assigns to an arrow $\alpha \in Q_1$ the arrow $\bar{\alpha} \neq \alpha$ with $s(\alpha) = s(\bar{\alpha})$, $g = \bar{f} : Q_1 \to Q_1$ the associated permutation, and $\mathcal{O}(g)$ the set of all g-orbits in Q_1. We consider three functions: a *weight function* $m_\bullet : \mathcal{O}(g) \to \mathbb{N} \setminus \{0\}$, a *parameter function* $c_\bullet : \mathcal{O}(g) \to K \setminus \{0\}$, and a *border function* $b_\bullet : Q_1 \to K$ such that $b_\alpha = 0$ if $\alpha \neq f(\alpha)$. For each arrow $\alpha \in Q_1$, we denote by $\mathcal{O}(\alpha)$ the g-orbit of α in Q_1, and set $n_\alpha = |\mathcal{O}(\alpha)|$, $m_\alpha = m_{\mathcal{O}(\alpha)}$, $c_\alpha = c_{\mathcal{O}(\alpha)}$, and define two paths $A_\alpha = \alpha g(\alpha) \dots g^{m_\alpha n_\alpha - 2}(\alpha)$ and $B_\alpha = \alpha g(\alpha) \dots g^{m_\alpha n_\alpha - 1}(\alpha)$.

We assume that $m_\alpha n_\alpha \geq 2$ for all arrows $\alpha \in Q_1$ (and further restrictions (see [7, Sections 2 and 3]), and call $\alpha \in Q_1$ *virtual* if $m_\alpha n_\alpha = 2$. Then the associated *weighted triangulation algebra* $\Lambda(Q, f, m_\bullet, c_\bullet, b_\bullet)$ is the quotient algebra KQ/I of the path algebra KQ of Q over K by the ideal $I = I(Q, f, m_\bullet, c_\bullet, b_\bullet)$ generated by:

(1) $\alpha f(\alpha) - c_{\bar{\alpha}} A_{\bar{\alpha}} - b_\alpha B_{\bar{\alpha}}$, for all arrows α of Q,
(2) $\alpha f(\alpha) g(f(\alpha))$ for all arrows α of Q such that $f^2(\alpha)$ is not virtual,
(3) $\alpha g(\alpha) f(g(\alpha))$ for all arrows $\alpha \in Q$ such that $f(\alpha)$ is not virtual.

If $(Q, f) = (Q(S, \vec{T}), f)$ for a directed triangulated surface $(S, \vec{T})$, then $\Lambda(S, \vec{T}, m_\bullet, c_\bullet, b_\bullet) = \Lambda(Q(S, \vec{T}), f, m_\bullet, c_\bullet, b_\bullet)$ is called a *weighted surface algebra*.

The following theorem describes basic properties of weighted surface algebras.

Theorem 1. Let $\Lambda = \Lambda(S, \vec{T}, m_\bullet, c_\bullet, b_\bullet)$ be a weighted surface algebra. Then the following statements hold:

(i) Λ is a symmetric algebra with $\dim_K \Lambda = \sum_{\mathcal{O} \in \mathcal{O}(g)} m_{\mathcal{O}} n_{\mathcal{O}}^2$.
(ii) Λ is a periodic algebra of period 4.
(iii) Λ is a tame algebra, and (with few exceptions) of non-polynomial growth.

There are known nine (exotic) families of algebras of generalized quaternion type which are not weighted surface algebras, six of them were discovered and studied recently (see [6, 8, 10] and forthcoming joint papers with T. Holm).

We stated the following conjecture.

Galactic Conjecture. Every algebra of generalized quaternion type is Morita equivalent to a weighted surface algebra or one of the nine exotic algebras.

It was confirmed for algebras whose Gabriel quiver is 2-regular and has at least three vertices [5, Main Theorem].

Theorem 2. Let A be a basic, indecomposable algebra whose Gabriel quiver is 2-regular and has at least three vertices. The following statements are equivalent:

(i) A is of generalized quaternion type.
(ii) A is a symmetric tame periodic algebra of period 4.
(iii) A is isomorphic to a weighted surface algebra or a higher tetrahedral algebra.

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Categorification and the Quantum Grassmannian

Alastair King
(joint work with Bernt Tore Jensen and Xiuping Su)

One aim of this talk is to illuminate the slogan "quantization is categorification" by showing how the quasi-commutation indices for quantization of the Grassmannian cluster algebra are read off from the dimensions of certain vector spaces that can be found within the categorification of this cluster algebra from [5].

1. The quantum Grassmannian.
The homogeneous coordinate ring $\mathbb{C}\big[\mathrm{Gr}_{k,n}\big]$ of the Grassmannian is the subring of the coordinate ring $\mathbb{C}\big[\mathrm{M}_{k,n}\big]$ of all $k \times n$ matrices generated by the $k \times k$ minors Δ_J for each k-subset $J \subseteq \{1, \dots, n\}$. We refer to such J as 'labels'.

From work of Fomin-Zelevinsky [4] and Scott [10], we know that $\mathbb{C}\big[\mathrm{Gr}_{k,n}\big]$ is a cluster algebra with several clusters of minors, corresponding to triangulations of an n-gon, when $k = 2$, and to Postnikov diagrams in general. See Figure 1 for an example with $n = 5$ and $k = 2$.

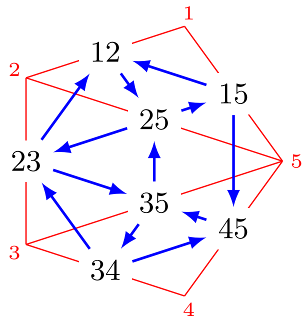

FIGURE 1. A cluster of minors from a triangulation.

To quantize, you first deform the coordinate ring of matrices to a non-commutative ring $\mathbb{C}_q\big[\mathrm{M}_{k,n}\big]$ and then modify the definition of the minors Δ_J. This yields a quantum cluster algebra in the sense of Berenstein-Zelevinsky [2] with essentially the same clusters, but, for example, with minors replaced by quantum minors. The seed data now consists of the exchange matrix (or quiver) and an additional matrix $L = (\ell_{ij})$ encoding the quasi-commutation rules: $X_i X_j = q^{\ell_{ij}} X_j X_i$.

The quasi-commutation rules for quantum minors were computed by Leclerc-Zelevinsky [7] as follows:

Theorem 1. *Two quantum minors Δ_I and Δ_J quasi-commute if and only if **either** $J \setminus I = J' \cup J''$ with $J' < (I \setminus J) < J''$, in which case*

$$q^{|J'|}\Delta_I\Delta_J = q^{|J''|}\Delta_J\Delta_I, \tag{1}$$

***or** vice versa, i.e. swapping I and J. Then I, J are said to be 'non-crossing'.*

Indeed, every cluster of (quantum) minors corresponds to a maximal non-crossing set $\mathcal{C}$ of labels. A first observation is that (1) can be rewritten as

$$q^{\kappa(I,J)}\Delta_I\Delta_J = q^{\kappa(J,I)}\Delta_J\Delta_I, \tag{2}$$

where $\kappa(I, J) = \mathrm{MaxDiag}(\lambda_I \setminus \lambda_J)$ is defined for any pair I, J as the largest diagonal distance (or height in Russian orientation) in the set difference of the partitions λ_I, λ_J whose 'profiles' are the labels I, J. See [9] for more detail and Figure 2 for an example.

2. Categorification.

In [5], an additive categorification of $\mathbb{C}\big[\mathrm{Gr}_{k,n}\big]$ is given by the category $\mathrm{CM}(B)$ of (finitely-generated) Cohen-Macaulay modules for the twisted group ring $B = R{*}G$, where

$$R = \mathbb{C}[[x, y]]/(x^k - y^{n-k}) \quad \text{and} \quad G = \{\zeta \in \mathbb{C} : \zeta^n = 1\},$$

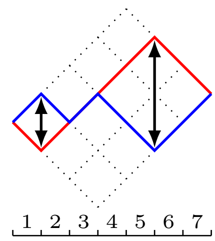

FIGURE 2. Showing that $\kappa(245, 167) = 1$ and $\kappa(167, 245) = 2$.

acting with weights 1 on x and -1 on y. We can also describe B as the (complete) path algebra of a quiver Q with $Q_0 = G^\vee = \mathbb{Z}_n$, with arrows $x_j\colon (j-1) \to j$ and $y_j\colon j \to (j-1)$ and with relations $xy = yx$ and $x^k = y^{n-k}$. See Figure 3 in the case $n = 7$ (so vertices 0 and 7 are identified).

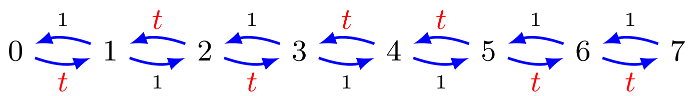

FIGURE 3. Quiver of B with the rank 1 module M_{245}.

Note that the centre of B is the subring $Z = R^G = \mathbb{C}[[t]]$, where $t = xy$, and that Cohen-Macaulay B-modules are precisely those that are free over Z. As representations of the quiver Q, a Cohen-Macaulay module has the same rank at each vertex. In particular, for each label J, we can define a rank 1 module M_J with Z at each vertex, $y_j = t$ for $j \in J$ and $x_j = t$ for $j \notin J$, while the other arrow of each pair is 1. See Figure 3 for an example.

There is a cluster character $\Phi\colon \operatorname{CM}(B) \to \mathbb{C}\big[\operatorname{Gr}_{k,n}\big]$, so that $\Phi(M_J) = \Delta_J$ and another key feature of this categorification is the following [5].

Proposition 2. *Labels I, J are non-crossing if and only if* $\operatorname{Ext}^1(M_I, M_J) = 0$.

Thus any maximal non-crossing set $\mathcal{C}$ of labels also determines a maximal rigid (hence cluster tilting) object $T_{\mathcal{C}} = \bigoplus_{J\in\mathcal{C}} M_J$ in $\operatorname{CM}(B)$. The main result of [1] is that $A = \operatorname{End}_B(T_{\mathcal{C}})^{\mathrm{op}}$ is the 'dimer algebra' associated to the Postnikov diagram $D(\mathcal{C})$ which can be defined for any such $\mathcal{C}$ by [8]. In particular, the quiver of A has vertex set $\mathcal{C}$.

A further feature of the categorification $\operatorname{CM}(B)$ is recently proved in [6].

Theorem 3. *For any labels I, J, we have $\kappa(I, J) = \dim K(M_I, M_J)$, where*

$$K(M, N) = \operatorname{Hom}_Z(M_0, N_0)/\operatorname{Hom}_B(M, N). \tag{3}$$

Note that $M_0 = e_0 M$ is the 'fibre' of M at the vertex 0. The forgetful functor $\operatorname{CM}(B) \to \operatorname{CM}(Z)\colon M \mapsto M_0$ has a right adjoint $\mathcal{J}\colon \operatorname{CM}(Z) \to \operatorname{CM}(B)$, so that $\operatorname{Hom}_Z(M_0, N_0) \cong \operatorname{Hom}_B(M, \mathcal{J}N_0)$ and further the canonical map $N \to \mathcal{J}N_0$ is injective, so that we effectively have an inclusion $\operatorname{Hom}_B(M, N) \subseteq \operatorname{Hom}_Z(M_0, N_0)$. This explains how (3) is to be understood.

In addition, this means that, for the cluster tilting object $T_{\mathcal{C}}$ as above, $K(T_{\mathcal{C}}, N)$ is a finite dimensional A-module. We can now define $\kappa(M,N) = \dim K(M,N)$ for any M, N in $\mathrm{CM}(B)$, and, as a compact notation, write the dimension vector of $K(T_{\mathcal{C}}, N)$ as $\kappa(T_{\mathcal{C}}, N) \in \mathbb{N}^{\mathcal{C}}$.

To see why Theorem 3 is true, we can consider the labels I, J as 'profiles' of the modules M_I, M_J, as in [5], drawn like the profiles of the partitions λ_I, λ_J as in Figure 2. Then the codimension of $\mathrm{Hom}_B(M_I, M_J)$ in $\mathrm{Hom}_Z(e_0 M_I, e_0 M_J)$ is computed by moving the profile of M_I to just below the profile of M_J and finding the height difference at vertex 0, as in Figure 4.

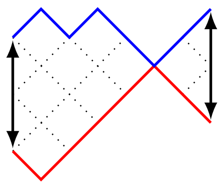

FIGURE 4. Showing that $\dim K(M_{167}, M_{245}) = 2$.

Now, since we can recover the quasi-commutation rules for quantum minors from the dimensions $\kappa(M,N)$ for rank 1 modules, it is natural to ask what we get for higher rank modules.

Indeed, a key point of categorification is that there is a process of mutation of cluster tilting objects parallel to mutation of cluster seeds and so, for every cluster, there is a cluster tilting object $T = \bigoplus_i T_i$ in $\mathrm{CM}(B)$ for which the quiver of $\mathrm{End}_B(T)^{\mathrm{op}}$ is the quiver of the corresponding seed. From this we can read off the exchange matrix B as the anti-symmetrized adjacency matrix. To extend this to quantum seeds, we define a matrix $L = (\ell_{ij})$ by

$$\ell_{ij} = \kappa(T_j, T_i) - \kappa(T_i, T_j)$$

and show in [6] that

Theorem 4. *The matrices B and L associated to any cluster tilting object T in $\mathrm{CM}(B)$ are compatible, in the sense of [2]. Furthermore, under mutation of cluster tilting objects, they undergo mutation of quantum seed data.*

As a consequence, the dimensions $\kappa(M,N)$ for higher rank summands of cluster tilting objects also determine the quasi-commmutation rules for higher degree quantum cluster variables.

3. Newton-Okounkov bodies

Associated to any maximal non-crossing collection $\mathcal{C}$, and thus Postnikov diagram $D(\mathcal{C})$, there is a 'network chart' (cf. [9]), which is a coordinate system on an affine open subset of $\mathrm{Gr}_{k,n}$ in which the minors are given by partition functions for dimer configurations on the associated bipartite graph, as follows:

$$\Delta_J = \sum_{\mu:\partial\mu = J} y^{\mathrm{wt}(\mu)}, \tag{4}$$

where μ ranges over perfect matchings and $\mathrm{wt}(\mu) \in \mathbb{N}^{\mathcal{C}}$, so that y^w is a formal variable. In [9], the right-hand side of (4) is called the 'flow polynomial' and is (equivalently) expressed as a sum over flows rather than perfect matchings.

In [9], Rietsch-Williams show that this flow polynomial has a common factor $y^{\mathrm{wt}(\mu_0)}$, for a certain minimal matching (or flow) μ_0, and that $\mathrm{wt}(\mu_0)_I = \kappa(I, J)$. As a consequence of this, they deduce, in effect, that the dimension vectors $\kappa(T_{\mathcal{C}}, M_J)$ are the integral points in a Newton-Okounkov body for $\mathrm{Gr}_{k,n}$

In recent work [3], it is shown that, when $T = T_{\mathcal{C}}$ and $M = M_J$, there is a one-to-one correspondence

$$\{\mu : \partial\mu = J\} \quad \longleftrightarrow \quad \{X \subseteq \mathrm{Hom}_B(T, M) : eX = M\} \tag{5}$$

where e is the boundary idempotent of A, for which $eAe = B$.

We can now further show that, under this correspondence, $\mathrm{wt}(\mu) = \dim \mathrm{Wt}(X)$, where $\mathrm{Wt}(X) = \mathrm{Hom}_Z(T_0, M_0)/X$. From this it immediately follows that $\kappa(T, M)$ is a minimal weight, because $K(T, M)$ is a quotient of $\mathrm{Wt}(X)$ for each X.

Since the right-hand side of (4) is defined for any M it is natural to conjecture a general formula for any cluster character in the network chart

$$\Phi(M) = \sum_{X : eX = M} y^{\mathrm{wt}(X)}, \tag{6}$$

where the sum is over $X \subseteq \mathrm{Hom}_B(T, M)$ and $\mathrm{wt}(X) = \dim \mathrm{Wt}(X)$. Since this sum may be infinite, it should be really be formulated as sum over dimension vectors weighted by the Euler characteristics of suitable quiver Grassmannians, along the lines of a CC formula. We can also extract the common factor and obtain

$$\Phi(M) = y^{\kappa(T,M)} \sum_{d \in \mathbb{N}^{\mathcal{C}}} \chi(\mathrm{Gr}^d(\underline{\mathrm{Hom}}_B(T, M)) y^d, \tag{7}$$

where Gr^d denotes the quiver Grassmannian of quotients of dimension vector d. It is then also natural to conjecture that the dimension vectors $\kappa(T, M)$ for M in $\mathrm{CM}(B)$ of rank r are all the integral points in the Newton-Okounkov body scaled up by r.

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