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Algebraic K-theory

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Abstract. Algebraic K-theory has seen a fruitful development during the last three years. Part of this recent progress was driven by the use of ∞-categories and related techniques originally developed in algebraic topology. On the other hand we have seen continuing progress based on motivic homotopy theory which has been an important theme in relation to algebraic K-theory for twenty years.

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Introduction by the Organizers

The workshop *Algebraic K-theory* was well attended with over fifty participants from various backgrounds. It covered a wide range of topics in algebraic K-theory and its applications. Two major lines of investigation in algebraic K- theory were emphasized during the workshop. On the one hand using Lurie's theory of ∞-*categories*, the true nature of algebraic K-theory is now much better understood, and the results are now beginning to show in earnest. Through this approach to K-theory there has been a fruitful interaction to number theory and algebraic topology, in particular to recent progress in p-adic Hodge theory. On the other hand we have seen continuing progress related to techniques from *motivic homotopy theory.*

Results related to ∞-category techniques

At the workshop, Clausen, Mathew, Nikolaus, Tamme, and Wang reported on results depending in an essential way on ∞-categories, while Gerhardt and Strunk presented results based on a more classic point of view in topological Hochschild homology.

Clausen explained his work with Mathew on Selmer K-theory, which is defined in the same broad generality as algebraic K-theory and agrees with étale K-theory (in degrees ≥ -1) for all (qcqs) schemes. This has a numerous important consequences. To wit, combined with recent work of Bhatt–Morrow–Scholze, it facilitates the definition of étale motivic cohomology for all (qcqs) schemes. Tamme reported on work with Land, which gives a conceptual treatment of the excision problem in K-theory and topological cyclic homology. This both makes for stronger results and eliminates the heavy calculational input that was necessary in earlier approaches.

In a similar vein, Mathew explained work with Antieau, Morrow, and Nikolaus, which makes use of the recent p-adic extension of Gabber rigidity by Clausen–Mathew–Morrow to give a conceptual proof of a result of Beilinson that identifies, up to a shift, the K-theory with $\mathbb{Q}_p$-coefficients of a p-complete ring R relative to the ideal pR with the cyclic homology with $\mathbb{Q}_p$-coefficients of R. In the process, some finiteness assumptions that were required for the original proof were eliminated. Nikolaus reported on work with Krause that applies the ∞-categorical setup to give much simplified proofs of a number of calculations in topological Hochschild homology, including Bökstedt's fundamental periodicity theorem, and determines the topological Hochschild homology in several new cases. Wang gave a report on ongoing work to understand the topological cyclic homology of all complete intersections over $\mathbb{Z}_p$. The new method that he presented is made possible by the much simplified setup for topological cyclic homology by Nikolaus–Scholze.

Gerhardt presented a variant of topological Hochschild homology for rings equipped with an action by a finite cyclic group. Strunk gave a proof of an essential case of Vorst's conjecture on regularity in positive characteristic refining earlier work of Geisser–Hesselholt.

Results related to number theory

Another recurring theme was the application of algebraic K-theory in number theory. Bräunling explained that for a regular $\mathbb{Z}$-order $\mathfrak{A}$ in a semi-simple $\mathbb{Q}$-algebra A, the equivariant Tamagawa number of Burns–Flach appears naturally as an element in Clausen's K-homology group $K_1(\mathrm{lc}_{\mathfrak{A}})$. Lichtenbaum reported that his conjectures on special values of L-functions in terms of Weil-étale cohomology are compatible with the functional equation. A striking result was the proof by Suzuki, using his rational étale site, of a conjecture of Grothendieck on a perfect duality between the groups of components of the special fibers of the Néron models of an abelian variety over a henselian discrete valuation field and its dual. Szamuely reported on work with Rössler that extends Beilinson's ℓ-adic height pairing to the case of homologically trivial cycles on a scheme smooth and projective over any finitely generated field of characteristic $p > 0$. In Beilinson's work, the field was required to be of transcendence at most degree one over a finite field.

Results related to motivic homotopy and homology

Motivic cohomology, which bears the same relationship to algebraic K-theory as singular cohomology does to topological K-theory, was again a major theme at the workshop, as was the motivic homotopy theory introduced by Morel–Voevodsky to parallel ordinary homotopy theory. In particular, Röndigs reported on his work with Østvær and Spitzweck that completely identifies the stable motivic homotopy groups of spheres in real dimension one, $\pi_{n+1,n}(\mathbb{S})$, in terms of the Milnor-Witt K-theory of the base field, which is assumed to be of characteristic different from two. Wickelgren reported work in unstable motivic homotopy theory that upgrades a classical degree formula to an identity in the Grothendieck-Witt group of the base field and of applications to curve counting. Rülling explained work with Saito that assigns a motivic conductor to a reciprocity sheaf. The motivic conductor unifies and extends various earlier conductors in abelian ramification theory. Iwasa reported work with Kai that defines a motivic filtration on the K-theory spectrum $K(X, D)$ of a regular and separated noetherian scheme X relative to an effective Cartier divisor D and identifies, up to small torsion, the graded pieces of the filtration on $K_0(X, D)$ in terms of algebraic cycles with modulus. Kelly spoke about work with Eberhardt that aims to apply methods of motivic homotopy theory in geometric representation theory.

Levine and Yakerson both reported on K-theory of quadratic forms. Levine explained that, based on an observation of Morel the motivic stable homotopy category decomposes in two parts $\mathrm{SH}(k)[1/2]^+$ and $\mathrm{SH}(k)[1/2]^-$, and that oriented theories such as motivic cohomology, algebraic K-theory, and motivic bordism do not see the latter part. However, this "dark matter" part can be detected by SL-oriented theories introduced by Ananevskiy, such as hermitian K-theory and Milnor–Witt motivic cohomology. Yakerson reported on work with Bachmann that gives a description of Milnor–Witt motivic cohomology in terms of algebraic cycles similar to the description of motivic cohomology given by Bloch's higher Chow groups.

Finally, going back to the original construction of algebraic K-theory by Quillen, Zakharevich explained that by focusing on a notion of exact squares rather than exact sequences, Quillen's Q-construction and Waldhausen's S-construction can be applied to the category of varieties over a field to yield a K-theory of varieties spectrum $K_0(\mathrm{Var}/k)$, the group of components of which is the Grothendieck group of varieties $K_0(\mathrm{Var}/k)$.

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Workshop: Algebraic K-theory

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Abstracts

On Selmer *K*-theory

Dustin Clausen
(joint work with Akhil Mathew)

The aim of this talk was to present some of the results of [1]. The general context is the following. We consider arbitrary qcqs algebraic spaces X and their algebraic K-theory spectra $K(X)$, defined in terms of perfect complexes as in [2]. Here is the main motivating problem. While $K(-)$ is known to satisfy Nisnevich descent by [2], it is also known *not* to satisfy the even more useful etale descent. This lack of etale descent is an obstruction to giving the tightest possible connection between algebraic K-theory and standard cohomology theories such as etale cohomology and de Rham cohomology.

Selmer K-theory is a new theory $K^{Sel}(-)$, first mentioned in [3], which repairs this "defect" of algebraic K-theory. The imprecise summary of our main results is as follows:

(1) K^{Sel} satisfies etale descent;
(2) K^{Sel} possesses the same formal properties/structure as K itself;
(3) There is a natural comparison map $K \to K^{Sel}$ which is quite close to being an isomorphism.

Now let us give the precise versions. To avoid distraction, however, we will save the actual definition for later. For now it's enough to know that K^{Sel} is a contravariant functor from qcqs algebraic spaces to spectra, admitting a natural transformation $K \to K^{Sel}$. We state the precise versions of (1),(2),(3) above, then give further results (4), (5).

(1) The presheaf of spectra K^{Sel} on qcqs algebraic spaces is an etale sheaf in the sense of [4]: for every qcqs algbraic space X and etale covering sieve S in qcqs algebraic spaces over X, the comparison map

$$K^{Sel}(X) \xrightarrow{\sim} \varprojlim_{(Y\to X)\in S} K^{Sel}(Y)$$

is an equivalence.

(2) The functor K^{Sel}, as well as the natural transformation $K \to K^{Sel}$, factor through the functor Perf which sends a qcqs algebraic space X contravariantly to its stable ∞-category of perfect complexes. Thus K^{Sel} can be defined on the level of stable ∞-categories. Moreover it is a *localizing invariant* in the sense of [5].

(3) Consider the sequence of natural transformations

$$K \to K^{et} \to K^{Sel},$$

where K^{et} denotes the etale sheafification of the presheaf K, again in the sense of [4]. Suppose X is a qcqs algebraic space. Then $K^{et}(X) \to K^{Sel}(X)$ is an isomorphism on homotopy groups in degrees ≥ -1.

(4) If X is a qcqs algebraic space of finite Krull dimension and with a global bound on the virtual Galois cohomological dimensions of its residue fields, then K^{Sel} on qcqs algebraic spaces etale over X is even an *etale Postnikov sheaf*: it identifies with the inverse limit of the etale sheafification of its Postnikov tower.

(5) The functor $R \mapsto K^{Sel}(\mathrm{Spec}(R))$ from commutative rings to spectra commutes with all filtered colimits. So does $R \mapsto K^{et}(\mathrm{Spec}(R))$.

We will say some words about the proofs of these results and their relations to some previous work towards the end; for now we just make some remarks.

First, the most crucial property is (2). It implies for example that K^{Sel} has the same proper pushforward functoriality as K, and that K^{Sel} has a projective bundle formula just like K. Both of these properties fail for the more naive fix K^{et}. In fact (3) implies that K^{et} does satisfy these properties/structure in a good range of homotopy, but that is not at all obvious from the definition of K^{et} as a sheafification; the only way we know how to see it is by comparison with K^{Sel}.

A related remark is that properties (2) and (3) uniquely characterize K^{Sel} as a presheaf on qcqs algebraic spaces in terms of K. Indeed, they imply that Bass's delooping procedure can be applied to the presheaf $(K^{et})_{\geq d}$ for any $d \geq -1$, and that the resulting non-connective delooping of etale K-theory identifies with K^{Sel}. This is delooping is indeed quite a non-trivial procedure, as for example $K^{Sel}(\mathbb{Z})$ has interesting homotopy groups in arbitrarily negative degrees.

Property (4) addresses a subtlety in the theory of sheaves of spectra, which is that it's possible to have an etale sheaf of spectra, even on such a simple finite-dimensional scheme as $\mathrm{Spec}(\mathbb{F}_p)$ or $\mathrm{Spec}(\mathbb{C}[T])$, all of whose stalks vanish but which has a large space of global sections. This is an unfamiliar phenomenon which does not arise in linear settings such as the derived category of etale sheaves of abelian groups. It can appear for sheaves of spectra essentially because the sphere spectrum has homotopy going all the way up to (homological) infinity. What (4) guarantees is that this problem nonetheless does *not* arise in the setting of Selmer K-theory. Indeed (4) gives a conditionally convergent descent spectral sequence

$$H^p(X_{et}; \pi_q^{et} K^{Sel}) \Rightarrow \pi_{q-p} K^{Sel}(X)$$

showing that K^{Sel} is controlled by its homotopy group sheaves to a large extent. In this context we remark that the homotopy group sheaves of K^{Sel} with finite coefficients can be more-or-less understood in terms of standard cohomology theories (etale cohomology and prismatic cohomology), but with rational coefficients we have $K^{Sel} \otimes \mathbb{Q} = K \otimes \mathbb{Q}$, and rational algebraic K-theory is as mysterious as ever. This property (4) is very subtle, and we have to use the norm residue isomoprhism theorem to verify it.

Finally, we remark that the last listed property, that K^{et} commutes with filtered colimits, is by no means a formal consequence of the fact that K commutes with filtered colimits. In fact, this is somehow the deepest property of all the properties mentioned above, in this its proof has the essentially all of the previous results as logical dependencies. It is also quite useful in practice.

Now let us give the definition of K^{Sel}. It is as a (homotopy) pullback:

$$K^{Sel} := L_1 K \times_{L_1 \mathrm{TC}} \mathrm{TC}\,.$$

Here L_1 is a certain functor from spectra to spectra, specifically it is the Bousfield localization at the non-connective complex K-theory spectrum KU, and TC is the functor of *topological cyclic homology.* With this definition, it is tautological (from the trace map $K \to \mathrm{TC}$) that there is a natural transformation $K \to K^{Sel}$, and that (2) is satisfied. The hard part is seeing what $K^{Sel}(Perf(X))$ is for X a qcqs algebraic space.

It turns out that for certain specific X and after completion at a prime p, this pullback construction reduces to constructions considered previously. Namely, if X lives over $\mathbb{Z}[1/p]$, then $K^{Sel}(X)_{\widehat{p}}$ identifies with $L_{K(1)}K(X)$, the construction considered by Thomason in his work [6] relating K-theory and etale cohomology. On the other hand, if X is proper over a p-complete commutative ring R, then $K^{Sel}(X)_{\widehat{p}} = \mathrm{TC}(X)_{\widehat{p}}$, as considered by Geisser-Hesselholt in their work [7] relating K-theory to de Rham-Witt theory. Several of our results (1)-(5) were already proved by these originators in their settings, perhaps under more restrictive hypotheses. Thus our work is most properly viewed as a generalization and synthesis of these previous works. The key tool that we use to obtain this synthesis, besides the fundamental ingredients going back to [6], is the recent rigidity result of [8]. We also contribute and make use of a fairly thorough analysis of the distinction between etale descent and etale *hyperdescent* for presheaves of spectra. This a technical but important distinction which we already mentioned in discussing (4) above.

We also remark that forthcoming joint work of the two of us with Bhargav Bhatt implies that all of the terms in the seemingly ad hoc pullback square defining $K^{Sel}(X)$ have a natural geometric interpretation, after p-completion. The $\mathrm{TC}(X)$ part corresponds to the formal completion $X_{\widehat{p}}$, the $L_1K(X)$ part corresponds to the generic fiber $X[1/p]$, and the remaining term $L_1\,\mathrm{TC}(X)$ corresponds to the natural locus on which these can be compared, namely the rigid analytic generic fiber $X_{\widehat{p}}[1/p]$.

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On the K-theory of pullbacks

Georg Tamme
(joint work with Markus Land)

In this talk I explained a proof of the following theorem and some of its consequences. See [9] for more details and references.

Theorem. *Every pullback square of $\mathbb{E}_1$-ring spectra*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array} \tag{1}$$

naturally refines to a commutative diagram of $\mathbb{E}_1$-ring spectra

$$\begin{array}{ccccc} A & \longrightarrow & B & & \\ \downarrow & & \downarrow & \searrow & \\ A' & \longrightarrow & A' \otimes_A^{B'} B & & \\ & \searrow & & \searrow & \\ & & & & B' \end{array} \tag{2}$$

such that any localizing invariant sends the inner square in (2) *to a pullback square. Moreover, the underlying spectrum of the $\mathbb{E}_1$-ring $A' \otimes_A^{B'} B$ is equivalent to $A' \otimes_A B$, and the underlying maps of spectra in* (2) *are the canonical ones.*

Examples of squares of discrete rings where this theorem applies are affine elementary Nisnevich squares, analytic isomorphism squares, and Milnor squares. As consequences we deduce a generalisation of results of Suslin and Wodzicki [12, 11] on excision in algebraic K-theory, and a generalisation of results of Geisser–Hesselholt on torsion in birelative K-groups associated with a Milnor square [7].

Let k be a connective $\mathbb{E}_\infty$-ring.

Definition. A localizing invariant E of k-linear categories is called *truncating (on k-algebras)* if the canonical map $E(A) \to E(\pi_0(A))$ is an equivalence for every connective $\mathbb{E}_1$-k-algebra A.

Here, for any $\mathbb{E}_1$-ring spectrum A we write $E(A)$ for $E(\mathrm{Perf}(A))$ where $\mathrm{Perf}(A)$ is the ∞-category of perfect A-modules in spectra. Examples of truncating invariants are the following: periodic cyclic homology $\mathrm{HP}(-/\mathbb{Q})$ on $\mathbb{Q}$-algebras (by a theorem of Goodwillie [8]), the fibre of the cyclotomic trace (by Dundas–Goodwillie–McCarthy [3]), Weibel's homotopy K-theory for $\mathbb{Z}$-algebras, rational K-theory on $\mathbb{Z}/N$-algebras ($N \geq 1$) by a result of Weibel [13], $K(1)$-local K-theory on $\mathbb{Z}/N$-algebras. The latter assertion is proven in joint work in preparation with Markus Land and Lennart Meier and also by a different method in forthcoming work of Bhatt–Clausen–Mathew.

As a direct consequence of the theorem we obtain:

Corollary. *Any truncating invariant satisfies excision, i.e. it sends Milnor squares to pullback squares, and nilinvariance.*

This in particular implies excision results of Cuntz–Quillen [2] for $\mathrm{HP}(-/\mathbb{Q})$ and Cortiñas [1], Geisser–Hesselholt [6], Dundas–Kittang [4, 5] for the fibre of the cyclotomic trace. As another consequence we get that $L_{K(1)}K(\mathbb{Z}/p^n) \simeq L_{K(1)}K(\mathbb{F}_p)$, which vanishes by Quillen's computation of the algebraic K-theory of finite fields [10]. Here p is the prime which is implicit in the Morava K-theory $K(1)$ of height 1.

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Duality for cohomology of local fields and curves with coefficients in abelian varieties

Takashi Suzuki

1. Grothendieck's conjecture: local duality

Let K be a complete discrete valuation field with perfect residue field k of characteristic $p > 0$. Let A be an abelian variety over K with Néron model $\mathcal{A}$ and special fiber $\mathcal{A}_x$ (where $x = \operatorname{Spec} k$). The group of connected components $\pi_0(\mathcal{A}_x)$

is a finite étale group over k. From the dual abelian variety B, we have corresponding objects $\mathcal{B}$, $\mathcal{B}_x$ and $\pi_0(\mathcal{B}_x)$. Grothendieck, in SGA 7 I [Gro72, Exposé IX], constructed a canonical pairing

$$\pi_0(\mathcal{A}_x) \times \pi_0(\mathcal{B}_x) \to \mathbb{Q}/\mathbb{Z}$$

as the obstruction to extending the Poincaré bundle on $A \times B$ to the Néron models $\mathcal{A} \times \mathcal{B}$, and conjectured that it is a perfect pairing. Based on the duality formalism [Suz13] in a certain Grothendieck site called the *rational étale site* $\operatorname{Spec} k^{\mathrm{rat}}_{\mathrm{et}}$, his conjecture is proved in [Suz14]:

Theorem 1. Grothendieck's pairing is perfect.

Some previously known cases include: semistable A (Grothendieck, Werner); finite k (McCallum); mixed characteristic K (Bégueri); prime-to-p part (Grothendieck, Bertapelle). See [Suz14, §1.1] for more cases with detailed references. The site $\operatorname{Spec} k^{\mathrm{rat}}_{\mathrm{et}}$ is the category of finite products of perfections of finitely generated fields over k endowed with the étale topology. The étale cohomology complex $R\Gamma(K, A)$ can be equipped with a canonical structure of an object of the derived category of sheaves of abelian groups on $\operatorname{Spec} k^{\mathrm{rat}}_{\mathrm{et}}$. Denote the resulting object by $R\mathbf{\Gamma}(K, A)$. Theorem 1 follows from the following Tate-duality type statement in [Suz14]:

Theorem 2. There exists a canonical isomorphism

$$R\mathbf{\Gamma}(K, A)^{\mathrm{SDSD}} \xrightarrow{\sim} R\mathbf{\Gamma}(K, B)^{\mathrm{SD}}[1],$$

where SD denotes the derived sheaf-Hom $R\,\mathbf{Hom}_k(\,\cdot\,, \mathbb{Z})$ for (a pro-category version of) $\operatorname{Spec} k^{\mathrm{rat}}_{\mathrm{et}}$.

The point is that this statement admits Galois descent. Hence its proof reduces to the semistable case. The semistable case is proved using the result of Grothendieck and Werner mentioned above.

2. Global duality

There is a global function field version of the above local duality ([Suz18]). Let K now be the function field of a proper smooth geometrically connected curve X over k. The Néron model $\mathcal{A}$ of an abelian variety A/K is now considered over X. The étale cohomology complex $R\Gamma(X, \mathcal{A})$ has a canonical structure of an object of the derived category of $\operatorname{Spec} k^{\mathrm{rat}}_{\mathrm{et}}$, denoted by $R\mathbf{\Gamma}(X, \mathcal{A})$. Let $\mathcal{B}^0$ be the part of $\mathcal{B}$ with connected fibers.

Theorem 3. There exists a canonical morphism

$$R\mathbf{\Gamma}(X, \mathcal{A})^{\mathrm{SDSD}} \to R\mathbf{\Gamma}(X, \mathcal{B}^0)^{\mathrm{SD}},$$

whose mapping cone is the rational profinite Tate module $T(\mathbf{H}^1(X, \mathcal{A})_{\mathrm{div}}) \otimes \mathbb{Q}$ of the maximal divisible subsheaf of $\mathbf{H}^1(X, \mathcal{A}) = H^1 R\mathbf{\Gamma}(X, \mathcal{A})$ placed in degree zero.

The sheaf $\mathbf{H}^1(X, \mathcal{A})$ is represented by the perfection of a smooth group scheme over k with unipotent identity component. Its group of $\overline{k}$-points is canonically isomorphic to the Tate-Shafarevich group of $A \times_k \overline{k}$. This theorem generalizes the perfectness of the Cassels-Tate pairing in the finite base field case and includes the non-degeneracy of the height pairing. In the joint work with Geisser [GS18], when k is finite, we applied this theorem to formulate and prove a Weil-étale version of the BSD formula for A assuming the finiteness of the Tate-Shafarevich group of A and using the result of Kato-Trihan [KT03].

3. Speculations and picture in the motivic direction

The above theories are under the following very speculative and highly conjectural picture. The abelian variety A above should be generalized to a mixed étale motive $\in \mathrm{DM}_{\mathrm{et}}(K)$. Its cohomology should be viewed as an object of $\mathrm{DM}_{\mathrm{et}}(k)$. The problem is that these categories of motives are $\mathbb{Z}[1/p]$-linear, hence ignore p-torsion and unipotent groups. There should instead be the category of mixed étale motives "with p-torsion": "$\widetilde{\mathrm{DM}}_{\mathrm{et}}(k)$", "$\widetilde{\mathrm{DM}}_{\mathrm{et}}(K)$". There is no such definition yet, but Milne-Ramachandran's $\mathbb{Z}$-integral motives [MR13] and Kahn-Saito-Yamazaki's motives with modulus [KSY15] should be relevant here. The category $\widetilde{\mathrm{DM}}_{\mathrm{et}}(k)$ should admit some integral p-adic realization functor (which, in Milne-Ramachandran's setting, values in the category of coherent complexes of graded modules over the Raynaud ring R). Composing this functor with the mapping fiber of (divided) Frobenius minus one in the integral p-adic category, we should get unipotent groups (cf. [MR15, Proposition 4.4]), which should recover the object $R\mathbf{\Gamma}(K, A)$ in §1.

Now we hope that the Néron model $\mathcal{A}$ over the curve X should be viewed as an object of "$\widetilde{\mathrm{DM}}_{\mathrm{et}}(X)$", the hypothetical category of étale motives with p-torsion over X. Cisinski-Déglise [CD16] establishes a six operations formalism for the category of étale motives without p-torsion $\mathrm{DM}_{\mathrm{et}}(X)$. As a motivic refinement of Theorem 3, the structure morphism $\pi \colon X \to \operatorname{Spec} k$ should produce objects $R\pi_* \mathcal{A}$ and $R\pi_* \mathcal{B}^0$ of $\widetilde{\mathrm{DM}}_{\mathrm{et}}(k)$ and they should be dual to each other.

Furthermore, for any morphism of any varieties $X \to Y$ over k, there should be a six operations formalism (including Verdier duality) between $\widetilde{\mathrm{DM}}_{\mathrm{et}}(X)$ and $\widetilde{\mathrm{DM}}_{\mathrm{et}}(Y)$, which would be viewed as an ultimate generalization of Grothendieck's conjecture, Cassels-Tate pairings and height pairings.

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"The category of varieties is exact" and other true nonsensical statements

Inna Zakharevich
(joint work with Jonathan Campbell)

The Grothendieck ring of varieties over k — denoted $K_0(\mathrm{Var}_k)$ — is defined to be the free abelian group generated by varieties over k, modulo the relation that for every closed immersion $Y \hookrightarrow X$ there is a relation $[X] = [Y] + [X\backslash Y]$. The ring structure is induced by the Cartesian product of varieties. In recent years two separate constructions [1, 4] of a "K-theory of varieties" producing a spectrum $K(\mathrm{Var}_k)$ whose π_0 is the Grothendieck ring have been published. In the course of working with these constructions, it quickly became clear that in many ways the category of varieties "behaves like" an exact category from the perspective of algebraic K-theory: there are "dévissage" and "localization" theorems (analogous to Quillen's [3]) that can be proved, and there is a "Q-construction" that ought to work in the context of varieties. This observation is reasonable from the perspective of motives: if an abelian category of mixed motives exists then there is an "abelian approximation" to the category of varieties, and thus it stands to reason that the category of varieties should behave similarly to an exact category from the point of view of K-theory. On the other hand, this does not appear reasonable at all: the category of varieties does not satisfy any of the axioms for an exact category, including the most basic one of having a zero object, and thus attempting to treat it as exact seems to be doomed to failure.

This talk described a new approach to exact categories, called *CGW-categories*, which bridges this gap: both exact categories and geometric categories (such as the category of varieties and the category of polytopes) produce examples of CGW-categories. A CGW-category has an associated K-theory space (which is a spectrum if the category satisfies some extra assumptions). In addition, there is a class of CGW-categories called "ACGW-categories" which behave similarly to abelian categories, in the sense that there are "dévissage" and "localization" theorems analogous to Quillen's for ACGW-categories.

A key feature of the definition of a CGW-category is that their behavior very closely approximates that of exact (and abelian) categories. CGW-categories don't just satisfy the same properties as abelian categories; many of the proofs for abelian categories (even those that rely on the additive structure) can often be translated to CGW-categories. In this talk we illustrated this phenomenon for Quillen's proof of dévissage by taking the original proof and showing which points needed to be modified to work for general ACGW-categories. A similar modification can be made for the proof of localization, and we expect that many other contexts in which abelian categories appear can be generalized to ACGW-categories. In particular, we expect that many homological techniques can be generalized to ACGW-categories, producing a theory of homology (and possibly cohomology) that will work in a much greater variety of contexts.

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Reciprocity sheaves and abelian ramification theory

Kay Rülling

(joint work with Shuji Saito)

We report on [8] in which it is shown that the theory of reciprocity sheaves gives a unified picture of various classical abelian ramification phenomena.

1. Reciprocity sheaves, following Kahn-Saito-Yamazaki, see [2], [9]. Let k be a fixed perfect base field. In the following, a pair (X, D) consists of a separated finite type k-scheme X and an effective (possibly empty) Cartier divisor D on X, such that $X \setminus |D|$ is smooth. A *compactification* of (X, D) is a pair $(\overline{X}, \overline{D} + B)$, where $\overline{X}$ is a proper k-scheme and B and $\overline{D}$ are effective Cartier divisors such that $X = \overline{X} \setminus |B|$ and $D = \overline{D}_{|X}$. Given two pairs $\mathcal{X} = (X, D)$ and $\mathcal{Y} = (Y, E)$ we denote by $\underline{\mathbf{M}}\mathbf{Cor}(\mathcal{X}, \mathcal{Y})$ the free abelian group with generators the integral closed subschemes $V \subset X \setminus |D| \times Y \setminus |E|$ which are finite and surjective over a component of $X \setminus |D|$ satisfying the property that the normalization of the closure $\tilde{V} \to X \times Y$ is proper over X and the inequality $D_{|\tilde{V}} \geq E_{|\tilde{V}}$ holds. We obtain a category $\underline{\mathbf{M}}\mathbf{Cor}$ with objects the pairs (X, D) and morphisms as defined above; the composition is induced by the usual composition of finite correspondences.

Let $\mathcal{X} = (\overline{X}, D)$ be a pair with $U = \overline{X} \setminus |D|$ and assume $\overline{X}$ is proper. For $S \in \mathbf{Sm}_k$ we define

$$h_0(\mathcal{X})(S) := \mathrm{Coker}(\underline{\mathbf{M}}\mathbf{Cor}((\mathbb{P}^1_S, \{\infty\}_S), \mathcal{X}) \xrightarrow{i_0^* - i_1^*} \mathbf{Cor}(S, U)).$$

This defines a presheaf with transfers $h_0(\mathcal{X})$ on $\mathbf{Sm}_k$. Let F be a presheaf with transfers on $\mathbf{Sm}_k$ and let $\mathcal{X} = (X, D)$ be a pair with $U = X \setminus |D|$. Set

$$\widetilde{F}(\mathcal{X}) := \left\{ a \in F(U) \,\middle|\, \begin{array}{l} \text{the Yoneda map } \mathbf{Cor}(U,-) \to F \text{ defined by } a \text{ factors via} \\ h_0(\overline{\mathcal{X}}), \text{ for some compactification } \overline{\mathcal{X}} \text{ of } \mathcal{X} \end{array} \right\}.$$

One can think of this as sections on U with poles on X controlled by D and some finite poles at infinity. If C is a proper smooth curve over a function field K, then $h_0(C, D)(K) = \mathrm{CH}_0(C, D)$ is the Chow group with modulus as defined by Serre; in this case we obtain a pairing

$$\widetilde{F}(C, D) \otimes_{\mathbb{Z}} \mathrm{CH}_0(C, D) \to F(K). \tag{1}$$

The assignment $\mathcal{X} \to \widetilde{F}(\mathcal{X})$ defines a presheaf on $\underline{\mathbf{M}}\mathbf{Cor}$. A *reciprocity presheaf* is a presheaf with transfers F on $\mathbf{Sm}_k$ such that for all $X \in \mathbf{Sm}_k$ we have

$$F(X) = \bigcup_{\overline{\mathcal{X}}} \widetilde{F}(\overline{\mathcal{X}}),$$

where the union is over all compactifications of $(X, \emptyset)$. We say F is a *reciprocity sheaf* if it is a sheaf in the Nisnevich topology on $\mathbf{Sm}_k$.

2. Let F be a reciprocity sheaf. Denote by Φ the set of henselian discrete valuation rings of geometric type over k and by $\Phi_{\leq n}$ the subset of those $L \in \Phi$ with $\mathrm{trdeg}(L/k) \leq n$. For $L \in \Phi$ denote by $\mathcal{O}_L$ and $\mathfrak{m}_L$ the ring of integers and the maximal ideal, respectively. Set

$$\widetilde{F}(\mathcal{O}_L, \mathfrak{m}_L^{-n}) := \widetilde{F}(\operatorname{Spec} \mathcal{O}_L, n \cdot \text{closed point}).$$

We define the *motivic conductor* $c^F = \{c_L^F : F(L) \to \mathbb{N}_0\}_{L \in \Phi}$ by

$$c_L^F(a) := \min\{n \geq 0 \mid a \in \widetilde{F}(\mathcal{O}_L, \mathfrak{m}_L^{-n})\}.$$

Definition 3 ([8, §4])**.** We say F has *level* $n \in [1, \infty]$ if for all $X \in \mathbf{Sm}_k$ and all $a \in F(\mathbb{A}^1_X)$ the condition $c^F_{k(x)(t)_\infty}(a_x) \leq 1$, for all at most $(n-1)$-dimensional points $x \in X$, implies $a \in F(X)$. Here $k(x)(t)_\infty = \mathrm{Frac}(\mathcal{O}^h_{\mathbb{P}^1_{k(x)}, \infty})$ and $a_x \in \widetilde{F}(k(x)(t)_\infty)$ denotes the pullback of a.

Theorem 4 ([8, Thm 4.15, Thm 4.29])**.** (1) *Let $X \in \mathbf{Sm}_k$ be connected, $a \in F(\mathbb{A}^1_X)$, and set $K := k(X)$.*

$$c^F_{K(t)_\infty}(a) \leq 1 \Longrightarrow a \in F(X).$$

(2) *Assume F has level $n \leq \infty$. For $a \in F(X \setminus |D|)$ we have*

$$a \in \widetilde{F}(X, D) \Longleftrightarrow \begin{array}{l} \textit{there exists a compactification } (\overline{X}, \overline{D} + B) \textit{ of } (X, D) \textit{ such that} \\ c_L^F(\rho^* a) \leq v_L(\rho^*(\overline{D} + B)), \textit{ for all } \rho \in X(L) \textit{ and all } L \in \Phi_{\leq n}. \end{array}$$

If F is a homotopy invariant sheaf and $a \in F(L)$, then $c_L^F(a) = 0$, if $a \in F(\mathcal{O}_L)$, and $c_L^F(a) = 1$, else. This implies:

Corollary 5. *Denote by $h^0_{\mathbb{A}^1}(F)$ the maximal $\mathbb{A}^1$-invariant subsheaf of F. Then $h^0_{\mathbb{A}^1}(F) = F^{c^F \leq 1}$.*

We have the following general procedure to compute the motivic conductor: on any presheaf with transfers we define a general notion of conductor; the motivic conductor is the minimal conductor; one gets lower bounds for the motivic conductor by local symbol computations. Using this we show:

Theorem 6 ([8, Thm 5.2]). *Let G be a smooth commutative k-group. Then G is a reciprocity sheaf of level* 1 *and the motivic conductor is determined by the Rosenlicht-Serre modulus on curves* [10, III].

Theorem 7 ([8, Thm 6.4, Cor's 6.7, 6.8]). *Assume* $\mathrm{char}(k) = 0$ *and* $q \geq 0$. *The q-th differentials relative to k, Ω^q, is a reciprocity sheaf of level $q+1$ and for $L \in \Phi$ with local parameter t we have* $\widetilde{\Omega^q}(\mathcal{O}_L, \mathfrak{m}^{-n}) = \frac{1}{t^{n-1}}\Omega^q_{\mathcal{O}_L}(\log)$; $h^0_{\mathbb{A}^1}(\Omega^q)(X) = H^0(\overline{X}, \Omega^q_{\overline{X}}(\log D))$, *where* $(\overline{X}, D)$ *is an SNCD compactification of X; the closed forms* $Z\Omega^q$ *have level q.*

Corollary 8. *Let Y be a normal affine Cohen-Macaulay k-scheme,* $\dim Y = d$. *Then Y has rational singularities if and only if there exists an effective Cartier divisor D on Y whose support contains Y_{sing} such that the sheaf* $Y_{\mathrm{Zar}} \ni U \mapsto \widetilde{\Omega^d}(U, D_{|U})$ *is* $(S2)$.

Theorem 9 ([8, Thm 6.11, Cor 6.12]). *Assume* $\mathrm{char}(k) = 0$ *(as above). Denote by* $\mathrm{MIC}_1(X)$ *the group of isomorphism classes of integrable rank 1 connections on X. Then* $X \mapsto \mathrm{MIC}_1(X)$ *is a reciprocity sheaf of level 1; the motivic conductor of a rank 1 connection on $L \in \Phi$ is equal to its irregularity as defined in* [4] *(up to a shift by* $+1$*);* $h^0_{\mathbb{A}^1}(\mathrm{MIC}_1)(X)$ *are the regular singular rank 1 connections on X in the sense of Deligne.*

The pairing (1) for $F = \mathrm{MIC}_1$, was constructed before in [1, §4].

Theorem 10 ([8, Thm 8.8, Cor 8.10]). *Assume* $\mathrm{char}(k) = p > 0$ *and ℓ is a prime different from p. Let* $\mathrm{Lisse}^1(X)$ *be the group of isomorphism classes of* $\overline{\mathbb{Q}}_\ell$*-lisse rank 1 sheaves on X. Then* $X \mapsto \mathrm{Lisse}^1(X)$ *is a reciprocity sheaf of level 1; the motivic conductor is equal to the Artin conductor (defined via the Brylinski-Kato-Matsuda filtration cf.* [3], [7]*);* $h^0_{\mathbb{A}^1}(\mathrm{Lisse}^1)(X)$ *are the tamely ramified 1-dimensional* $\overline{\mathbb{Q}}_\ell$*-representations of* $\pi_1^{\mathrm{ab}}(X)$ *(defined using curve-tameness, see* [5]*).*

If we restrict to finite monodromy we obtain the pairing (1) for $F = H^1_{\text{ét}}(-, \mathbb{Q}/\mathbb{Z})$; this is the pairing from geometric class field theory in case K is a finite field.

It seems the following motivic conductor was not considered before.

Theorem 11 ([8, Thm 9.12]). *Assume* $\mathrm{char}(k) = p > 0$. *Let G be a commutative finite k-group. Denote by* $H^1(G)(X) := H^1_{\mathrm{fppf}}(X, G)$ *the group of isomorphism classes of G-torsors on X. Then* $X \mapsto H^1(G)(X)$ *is a reciprocity sheaf of level 2; it has level 1 if G has no infinitesimal unipotent part; the motivic conductor for $G = \alpha_p$ or for G without infinitesimal unipotent part is computed explicitly; if we write $G = G' \times G_u$ with G_u unipotent and G' without unipotent part, then* $h^0_{\mathbb{A}^1}(H^1(G))(X) = H^1(G')(X) \oplus H^1(G_u)(\overline{X})$, *where $\overline{X}$ is a smooth compactification of X.*

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The Beilinson fiber square and syntomic cohomology

Akhil Mathew

(joint work with Benjamin Antieau, Matthew Morrow and Thomas Nikolaus)

For a ring R, we let $K(R)$ denote its algebraic K-theory and $\mathrm{HC}^-(R)$ denote its negative cyclic homology; one has a natural Chern character map

$$K(R) \to \mathrm{HC}^-(R),$$

and the following classical result.

Theorem (Goodwillie [10])**.** *Let R be a $\mathbb{Q}$-algebra and $I \subset R$ a nilpotent ideal. Then the induced square*

$$\begin{array}{ccc} K(R) & \longrightarrow & \mathrm{HC}^-(R) \\ \downarrow & & \downarrow \\ K(R/I) & \longrightarrow & \mathrm{HC}^-(R/I) \end{array}$$

is homotopy cartesian. Moreover, one has an identification $K(R, I) \simeq \mathrm{HC}(R, I)[1]$, for HC *denoting cyclic homology.*

The conclusion of Goodwillie's theorem fails for algebras which do not contain $\mathbb{Q}$. One can refine the construction $\mathrm{HC}^-(R)$ to the topological cyclic homology $\mathrm{TC}(R)$ of Bökstedt-Hsiang-Madsen [3], and the cyclotomic trace $K(R) \to \mathrm{TC}(R)$. The Dundas-Goodwillie-McCarthy theorem [6] then gives the analogous fiber square integrally, with HC^- replaced by TC.

Recently, Beilinson [1] constructed a version of Goodwillie's original result in a p-adic setting; we state a (slightly simplified) version of the result below. More precisely, Beilinson considers the "continuous $\mathbb{Q}_p$-K-theory" of a ring R (of bounded p^∞-torsion),[1]

$$K^{\mathrm{cts}}(R;\mathbb{Q}_p) = \left(\varprojlim K(R/p^n)_{\hat{p}}\right)[1/p].$$

One also analogously defines the "$\mathbb{Q}_p$-cyclic homology," which simplifies to

$$\mathrm{HC}(R;\mathbb{Q}_p) = \mathrm{HC}(R)_{\hat{p}}[1/p].$$

Theorem (Beilinson [1]). *Let R be a p-adically complete ring with bounded p^∞-torsion and such that R/p has finite stable range. Then there is a natural equivalence*

$$K^{\mathrm{cts}}(R,(p);\mathbb{Q}_p) \simeq \mathrm{HC}(R;\mathbb{Q}_p)[1]. \tag{1}$$

Remark.

(1) Beilinson's proof relies on some p-adic Lie theory, in particular the Lazard isomorphism between continuous group and Lie algebra cohomology.

(2) Both sides of (1) are naturally obtained by obtained by inverting p on the p-complete spectra $K^{\mathrm{cts}}(R;\mathbb{Z}_p) = \varprojlim_n K(R/p^n)_{\hat{p}}$ and $\mathrm{HC}(R;\mathbb{Z}_p)$. In fact, [1] gives an equivalence of these spectra up to "quasi-isogeny;" the denominators involved are bounded in any range of homological degrees.

We give a simple proof of Beilinson's result using topological cyclic homology, and refine it to a fiber square. More precisely, we show the following:

Theorem. *Let R be any ring. Then there is a natural "crystalline Chern character" $K(R/p;\mathbb{Q}_p) \to \mathrm{HP}(R;\mathbb{Q}_p)$ and a homotopy cartesian square*

$$\begin{array}{ccc} K^{\mathrm{cts}}(R;\mathbb{Q}_p) & \longrightarrow & K(R/p;\mathbb{Q}_p) \\ \downarrow & & \downarrow \\ \mathrm{HC}^-(R;\mathbb{Q}_p) & \longrightarrow & \mathrm{HP}(R;\mathbb{Q}_p) \end{array} \tag{2}$$

On the level of relative terms, we can take homotopy fibers of the horizontal maps to obtain an identification analogous to Beilinson's (i.e., between the same objects, but a priori a different isomorphism). To prove the above result, we use the Dundas-Goodwillie-McCarthy theorem together with the new description of topological cyclic homology given by Nikolaus-Scholze [11]. In particular, the fiber square (2) (which is constructed with TC replacing K) follows from basic properties of $\mathrm{THH}(\mathbb{F}_p)$ as a cyclotomic spectrum.

As an application of these results, we study the sheaves $\mathbb{Z}_p(i)$ defined by Bhatt-Morrow-Scholze [4]. The $\mathbb{Z}_p(i)$, considered a p-adic analog of the usual étale Tate twists (for $\mathbb{Z}[1/p]$-algebras), are defined on a wide class of p-complete rings (the

[1] For commutative rings, it suffices to take $K(R)$, p-complete, and then invert p, by the results of [5].

quasisyntomic rings, including all local complete intersection rings and all perfectoid rings). The $\mathbb{Z}_p(i)$ have the following features:

(1) In low degrees, they are easy to understand: $\mathbb{Z}_p(0)(R) = R\Gamma(\mathrm{Spec}(R), \mathbb{Z}_p)$ is (pro)-étale cohomology with $\mathbb{Z}_p$-coefficients, and $\mathbb{Z}_p(1)(R) = R\Gamma(\mathrm{Spec}(R), \mathbb{G}_m)_{\hat{p}}[-1]$.
(2) For smooth $\mathbb{F}_p$-algebras, $\mathbb{Z}_p(i)$ is the cohomology (shifted by $-i$) of the logarithmic de Rham-Witt forms $W\Omega^i_{\log}$.
(3) For formally smooth algebras over $\mathcal{O}_C$ (for C a complete algebraically closed nonarchimedean field of mixed characteristic), the $\mathbb{Z}_p(i)$ are identified with the truncations $\tau^{\leq i}$ of the p-adic nearby cycles.
(4) In general, the $\mathbb{Z}_p(i)$ admit a purely algebraic (and site-theoretic) description in terms of the prismatic cohomology of Bhatt-Scholze [BS19]; the construction in particular is completely different from cycle complexes.
(5) For any quasisyntomic ring, there is a "motivic" filtration on TC with associated graded given by the $\mathbb{Z}_p(i)[2i], i \geq 0$.

Since it is known that TC and p-adic étale K-theory agree in nonnegative degrees (cf. [8] and [5]), one expects the $\mathbb{Z}_p(i)$ to be a very general form of étale motivic cohomology, which can be defined in highly non-noetherian situations which are very far from regular (so more traditional definitions via higher Chow groups cannot be used).

Our main result, which is an analog of a result of Geisser [9] for étale motivic cohomology, gives a description the $\mathbb{Z}_p(i)$ for $i \leq p-2$ solely in terms of de Rham (rather than prismatic) theory.

Definition (Divided Frobenius operators)**.** Let R be a quasisyntomic ring. We consider the p-adic derived de Rham cohomology of R, denoted $L\Omega_R$ (cf. [2]) and its derived Hodge filtration $L\Omega^{\geq i}_R$. Note that $L\Omega_R$ is also equipped with a (crystalline) Frobenius φ, arising from the Frobenius on R/p. For $i < p$, one can show that there is a unique natural map

$$\varphi_i : L\Omega^{\geq i}_R \to L\Omega_R$$

such that $p^i\varphi_i = \varphi|_{L\Omega^{\geq i}}$.

The following result is proved using the techniques of [4], i.e., via a direct quasisyntomic descent, and taking sheafified homotopy groups in the square (2).

Theorem. *Let R be a p-torsion-free quasisyntomic ring. Then there is a natural pullback square*

$$\begin{array}{ccc} \mathbb{Q}_p(i)(R) & \longrightarrow & \mathbb{Q}_p(i)(R/p) \\ \downarrow & & \downarrow \\ (L\Omega^{\geq i}_R)_{\mathbb{Q}_p} & \longrightarrow & (L\Omega_R)_{\mathbb{Q}_p} \end{array}$$

For $i \leq p-2$, one obtains an analog of the above fiber square, but with $\mathbb{Z}_p$-coefficients everywhere.

Definition (Syntomic cohomology, after Fontaine-Messing [7])**.** For $i \leq p-2$, we define

$$\mathbb{Z}_p(i)^{FM} = \mathrm{fib}(\varphi_i - 1 : L\Omega_R^{\geq i} \to L\Omega_R)$$

and for all i, we define

$$\mathbb{Q}_p(i)^{FM} = \mathrm{fib}(\varphi - p^i : L\Omega_R^{\geq i} \to L\Omega_R).$$

We prove the following result, cf. [9] for the analog in motivic cohomology.

Theorem. *For $i \leq p-2$, we have equivalences of quasisyntomic sheaves $\mathbb{Z}_p(i)^{FM} \simeq \mathbb{Z}_p(i)$. For all i, we have equivalences $\mathbb{Q}_p(i)^{FM} \simeq \mathbb{Q}_p(i)$.*

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Bökstedt periodicity and quotients of DVRs

Thomas Nikolaus

(joint work with Achim Krause)

The main goal in this talk, which is based on the paper [4], is to explain how to compute the topological Hochschild homology groups for quotients of DVRs. While the result in this generality is new, we recover BrunÕs result [1] on $\mathrm{THH}_*(\mathbb{Z}/p^n)$ quite elegantly.

Along the way we give a short argument for Bökstedt periodicity (making use of a universal description of the dual Steenrod algebra as an $\mathbb{E}_2$-algebra) and generalizations over various bases. Our strategy will also give a very efficient way to redo the computations for THH for complete DVRs which are originally due to Lindenstrauss–Madsen. The basic idea of proof is to work relative to the ring spectrum $\mathbb{S}[z]$ and then use a descent style spectral sequence to get the absolute THH.

Recall that Bökstedt's seminal result states that one has an isomorphism of rings $\mathrm{THH}_*(\mathbb{F}_p) = \mathbb{F}_p[x]$ with x in degree 2. We show that this result can be easily deduced from the fact that the dual Steenrod algebra $\mathbb{F}_p \otimes_{\mathbb{S}} \mathbb{F}_p$[1] is the free $\mathbb{E}_2$-algebra over $\mathbb{F}_p$ on a single generator in degree 1. This structural result for the dual Steenrod algebra follows by combining work of Araki-Kudo [6], Dyer-Lashof [5], Milnor [7] and Steinberger [8].

Now we let A be a discrete valuation ring of mixed characteristic $(0, p)$ with perfect residue field. Choose a uniformizer $\pi \in A$. We consider A as an algebra over the commutative ring spectrum $\mathbb{S}[z] = \Sigma^\infty_+ \mathbb{N}$ by sending z to π. Then one has the following variant of Bökstedt's result, which is due to Lurie and Bhatt-Morrow-Scholze [9]:

Theorem. *We have an isomorphism of rings*

$$\mathrm{THH}_*(A/\mathbb{S}[z]; \mathbb{Z}_p) \cong A[x].$$

where $\mathrm{THH}_*(A/\mathbb{S}[z], \mathbb{Z}_p)$ *denotes the p-completion of the topological Hochschild homology relative to* $\mathbb{S}[z]$.

Similarly we show that for a quotient $A' = A/\pi^k$ we have $\mathrm{THH}_*(A'/\mathbb{S}[z], \mathbb{Z}_p) \cong A'[x]\langle y\rangle$. Here y is a free divided power variable in degree 2.

Now we are interested in the absolute topological Hochschild homology groups $\mathrm{THH}_*(A; \mathbb{Z}_p)$ and $\mathrm{THH}_*(A'; \mathbb{Z}_p)$ rather than the relative versions $\mathrm{THH}_*(A/\mathbb{S}[z], \mathbb{Z}_p)$ and $\mathrm{THH}_*(A'/\mathbb{S}[z], \mathbb{Z}_p)$ computed above. The key is the following spectral sequence:

Theorem. *There is a multiplicative, convergent spectral sequence*

$$\mathrm{THH}_*(A/\mathbb{S}[z]; \mathbb{Z}_p) \otimes \Lambda(dz) \Rightarrow \mathrm{THH}_*(A; \mathbb{Z}_p)$$

and similar for A' *(or more generally for any* $\mathbb{Z}[z]$*-algebra in place of* A*).*

Using this spectral sequence and the computation of the relative THH one can obtain the absolute THH in the cases of interested relatively easy, since there are very few differentials to compute. We get

$$\mathrm{THH}_*(A; \mathbb{Z}_p) = \begin{cases} A & \text{for } * = 0 \\ A/nE'(\pi) & \text{for } * = 2n-1 \\ 0 & \text{otherwise} \end{cases}$$

[1]Here we consider rings implicitly as ring spectra and tensor them together over the sphere spectrum $\mathbb{S}$. The result is accordingly a spectrum itself.

where $E(z)$ is the minimal polynomial of π over the Witt vectors $W(A/\pi) \subseteq A$. This result is (with a different proof) due to Lindenstrauss-Madsen. We also get the following new result:

Theorem. *For $A' = A/\pi^k$ as above we get that* $\mathrm{THH}_*(A')$ *is given by the homology of the DGA*

$$A'[x]\langle y\rangle \otimes \Lambda(dz)$$

with differential ∂ determined by $\partial x = E'(z)dz$ and $\partial y^{[i]} = k\pi^{k-1}y^{[i-1]}dz$

Finally we explain how this result can be evaluated for different A and k and explain how it gives (a multiplicative) version of Brun's computation [1] of $\mathrm{THH}_*(\mathbb{Z}/p^n)$.

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Motivic filtration on relative K-theory via algebraic cycles with modulus

Ryomei Iwasa

(joint work with Wataru Kai)

Let X be a regular separated noetherian scheme and D an effective Cartier divisor on X. This talk presents a construction of a homotopy-coniveau type filtration on the relative K-theory spectrum $K(X, D)$, which is defined to be the the homotopy fiber of the canonical morphism $K(X) \to K(D)$ between the K-theory spectra of X and D.

It is easy to describe the induced filtration on $K_0(X, D) = \pi_0 K(X, D)$, which has been studied in [3, 4] (and in [2] for the case $(X, D) = (\Delta^n_A, \partial\Delta^n_A)$ for a regular ring A). For each $p \geq 0$, we define $F^p K_0(X, D)$ to be the image of the canonical morphism $\mathrm{colim}_Z\, K^Z_0(X) \to K_0(X, D)$, where Z runs over all closed subsets of codimension $\geq p$ in X which do not meet D. The main theorem of [4] says that, under the assumption that D admits an affine open neighborhood in X, the graded piece $\mathrm{Gr}^p_F K_0(X,D)$ is isomorphic to Binda-Saito's Chow group with modulus $\mathrm{CH}^p(X|D)$ up to $(p-1)!$-power torsion. The key for this theorem is to establish an exact sequence ([4, Theorem 2.2])

$$\operatorname*{colim}_{W\colon \text{modulus } D} K_0^W(X\times\Delta^1) \longrightarrow \operatorname*{colim}_{Z\cap D=\varnothing} K_0^Z(X) \longrightarrow K_0(X,D) \longrightarrow 0,$$

where W runs over all closed subsets in $X\times\Delta^1$ satisfying the modulus condition along D (in the sense of Binda-Saito [1]) and Y runs over all closed subsets in X not meeting D. Since we have an analogous exact sequence for $\mathrm{CH}^*(X|D)$ essentially by definition, we obtain a comparison between $K_0(X,D)$ and $\mathrm{CH}^*(X|D)$.

The above exact sequence suggests to consider the following: For each $n\geq 0$, we define

$$K^{(0)}(X,D;n) := \operatorname*{colim}_{Z\colon \text{modulus } D} K^Z(X\times\Delta^n),$$

where Z runs over all closed subsets of $X\times\Delta^n$ satisfying the modulus condition along D. Then the assignment $n\mapsto K^{(0)}(X,D;n)$ forms a simplicial spectrum in a natural way. In [5], we have proved that there exists a morphism of spectra

$$\Upsilon\colon |K^{(0)}(X,D;\bullet)| \to K(X,D)$$

which is functorial in (X,D) and extends the canonical morphism $K^{(0)}(X,D;0) = \mathrm{colim}_Z K^Z(X) \to K(X,D)$. The above exact sequence is equivalent to say that $\pi_0\Upsilon$ is an isomorphism. We will discuss the construction of Υ in more detail, where we will see some relation between the K-theory and the modulus condition.

The morphism Υ allows us to construct a filtration $\{K^{(p)}(X,D;\bullet)\}_{p\geq 0}$ on $K(X,D)$ as in [7]. We hope that the graded piece $K^{(p/p+1)}(X,D;\bullet)$ is Zariski-locally equivalent to the cycle complex with modulus $z^p(X|D,\bullet)$. This is still not achieved, but I will explain some ideas to prove it and show some partial results obtained by now. The last part is also based on a joint work with Amalendu Krishna [6].

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On Vorst's conjecture

Florian Strunk
(joint work with Moritz Kerz, Georg Tamme)

In this talk, we report on joint work with Moritz Kerz and Georg Tamme on Vorst's conjecture. We assume that all rings appearing in this abstract are unital and commutative. Recall that a ring A is called *K_n-regular*, if the canonical map $K_n(A) \cong K_n(A[X_1, \ldots, X_m])$ is an isomorphism for all positive integers m. Quillen proved that a noetherian regular ring is K_n-regular for all n. In [4], Vorst conjectured a (possibly stronger) converse to this result for algebras over a field:

Conjecture (Vorst)**.** Let A be an algebra essentially of finite type over a field k and of Krull-dimension d. If A is K_{d+1}-regular, then A is regular.

It is easy to see that a K_1-regular ring is reduced. This implies Vorst's conjecture for 0-dimensional rings. In [4] Vorst showed his conjecture for 1-dimensional k-algebras. In [1] Cortiñas, Haesemeyer and Weibel showed the conjecture in the case $\mathrm{char}(k) = 0$. For a perfect field k of positive characteristic, Geisser and Hesselholt showed (a slightly weaker version of) the conjecture in [2] assuming a strong form of resolution of singularities. In the preprint [3], we remove this assumption:

Theorem. *Let A be an excellent $\mathbb{F}_p$-algebra such that $[k(x) : k(x)^p] < \infty$ for all points $x \in \mathrm{Spec}(A)$. If A is K_{e+1}-regular, then A is regular, where e denotes the* p-dimension

$$p\text{-}\dim(A) := \sup\{\dim_{k(x)} \Omega_{k(x)} + \mathrm{height(x)}\,|x \in \mathrm{Spec}(A)\}$$

of A.

This theorem implies the conjecture for A being of finite type over a field of positive characteristic. Moreover, it questions the necessity of the assumption to be essentially of finite type over a field in Vorst's conjecture. In fact, this was questioned already by Vorst himself in [4]. Using the above-mentioned theorem of Cortiñas, Haesemeyer and Weibel and Artin–Hironaka algebraization of isolated singularities, we can remove this assumptions in the characteristic zero case:

Theorem. *Let A be an excellent noetherian ring of characteristic zero and of Krull-dimension d. If A is K_{d+1}-regular, then A is regular.*

Finally, we can also prove a mixed-characteristic result for curves:

Theorem. *Let A be an excellent noetherian ring with* $\dim(A) \leq 1$ *such that the residue field $k(\mathfrak{m})$ for every maximal ideal $\mathfrak{m} \subset A$ is perfect and of characteristic different from two. If A is K_2-regular, then A is regular.*

One might therefore ask for an analogue of Vorst's conjecture for every excellent noetherian ring.

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K-theory and local compactness

Oliver Braunling

In the talk we discussed a reformulation of the non-commutative "Equivariant Tamagawa Number Conjecture" (ETNC) of Burns and Flach [4]. This conjecture generalizes most of the famous conjectures on special L-values of motives. For example, the Birch–Swinnerton-Dyer conjecture is a tiny unproven special case, while for example Dirichlet's Analytic Class Number Formula is a tiny proven special case. Our new viewpoint has mostly to do with the *equivariant* enrichment though, where we offer a different perspective based on a perhaps somewhat new philosophy.

This is mostly material of our preprints [2] and [3]. Our reformulation is based on replacing ordinary K-theory by its variant taking the locally compact topologies of the p-adics and reals into account, which naturally occur when trivializing the so-called *fundamental line*. This in turn is based on a similar idea which Clausen had previously proposed in the context of class field theory [6]. He had considered the reciprocity maps of classical class field theory, namely

$$\begin{aligned} \mathbb{Z} &\longrightarrow \mathrm{Gal}(F^{\mathrm{ab}}/F) && \text{if } F \text{ is a finite field,} \\ F^{\times} &\longrightarrow \mathrm{Gal}(F^{\mathrm{ab}}/F) && \text{if } F \text{ is a local field,} \\ \mathbb{A}^{\times}/F^{\times} &\longrightarrow \mathrm{Gal}(F^{\mathrm{ab}}/F) && \text{if } F \text{ is a number field,} \end{aligned} \tag{1}$$

where $\mathbb{A}$ denotes the adèles. Among the many other wonderful ideas of [6], he proposed that the first K-group $K_1(\mathsf{LCA}_F)$, where LCA_F is the category of locally compact topological F-vector spaces, gives exactly the group on the left in the above diagram, irrespective of which of the three kinds of fields we use for F. This means that there is a *uniform* way to look at the left side of these reciprocity maps (one could call this the *automorphic* side). In joint work with Peter Arndt, we then had given a complete computation of the spectrum $K(\mathsf{LCA}_F)$ in the three cases occurring in Equation 1, besides just the K_1-group. In particular, this confirms various predictions of Clausen. This is written up in [1].

In the process of working on this, a relation to the Equivariant Tamagawa Number Conjecture (ETNC) emerged. This came as quite a surprise. A **Tamagawa number** is classically a volume and as such a positive real number. It is a certain normalization of a Haar measure on the adelic points of a linear group G. This concept has undergone wide generalizations. First, by work of Bloch, linear groups have been generalized to no longer necessarily affine group varieties, so that abelian varieties could also be considered, revealing that the Birch–Swinnerton-Dyer conjecture could be considered a type of Tamagawa number formula. Next, Bloch and Kato [5] replaced these by general pure motives, and skipping ahead a few further steps, in the context of the Burns–Flach formulation, instead of being a volume, a Tamagawa number became a value in the relative K-group $K_0(\mathfrak{A}, \mathbb{R})$. Here $\mathfrak{A}$ is an order in a finite-dimensional semisimple $\mathbb{Q}$-algebra A (so, in the case of equivariance by a finite group H one would use $A := \mathbb{Q}[H]$, the group algebra). In the classical case, $\mathfrak{A} = \mathbb{Z}$, one has a canonical isomorphism

$$K_0(\mathbb{Z}, \mathbb{R}) \cong \mathbb{R}^{\times}_{>0},$$

and by this identification the interpretation as a volume is still visible. A relative K-group, like the group $K_0(\mathfrak{A}, \mathbb{R})$ here, is defined as the relevant term to complete a long exact sequence; in the case at hand this sequence is

$$\cdots \longrightarrow K_1(\mathfrak{A}) \longrightarrow K_1(A_{\mathbb{R}}) \longrightarrow K_0(\mathfrak{A}, \mathbb{R}) \longrightarrow K_0(\mathfrak{A}) \longrightarrow \cdots \tag{2}$$

Besides fitting in the right spot, no further interpretation comes with relative K-groups. We have a new viewpoint on this. Based on Clausen's computations [6], we have the left column in

$$\begin{array}{ccc} \text{universal det-functor on } \mathsf{LCA}_{\mathbb{Z}} & \longrightarrow & ? \\ \| & & \| \\ \text{Haar measure} & & \text{equivariant Haar measure} \end{array}$$

which interprets the Haar measure torsor as the universal det-functor of the category $\mathsf{LCA}_{\mathbb{Z}}$ of locally compact abelian groups. If we now were to completely ignore the work of Burns and Flach (no worries – just temporarily for exposition reasons!), anything which is called an "equivariant Tamagawa number" should probably be a certain normalization of an "equivariant Haar measure". But what is an *equivariant* Haar measure? What does this even mean?

The above diagram suggests to study

$$\mathsf{LCA}_{\mathfrak{A}},$$

the category of locally compact abelian groups with a right action by $\mathfrak{A}$. Then it would be plausible that its universal det-functor should match the torsor of (hypothetical) $\mathfrak{A}$-equivariant Haar measures, whatever their definition might be. Namely, by work of Deligne, the universal det-functor of an exact category can (under identifying Picard groupoids with stable 1 homotopy types) be viewed as the truncation functor of the K-theory spectrum to its stable 1 homotopy type.

This turns out not to be an overly false viewpoint. It fits philosophically, but it also turns out that the K-theory of this category naturally spits out *exactly* the relative K-group which had also appeared in Burns–Flach formulation, although they did not have this picture/yoga of equivariant Haar measures available at the time. We prove:

Theorem ([2]). *Suppose $\mathfrak{A}$ is a regular order in a finite-dimensional semisimple $\mathbb{Q}$-algebra A. Then there is a canonical long exact sequence*

$$\cdots \longrightarrow K_n(\mathfrak{A}) \longrightarrow K_n(A_{\mathbb{R}}) \longrightarrow K_n(\mathsf{LCA}_{\mathfrak{A}}) \xrightarrow{a} K_{n-1}(\mathfrak{A}) \longrightarrow \cdots, \tag{3}$$

and for all n, there are canonical isomorphisms

$$K_n(\mathsf{LCA}_{\mathfrak{A}}) \cong K_{n-1}(\mathfrak{A}, \mathbb{R}).$$

This shows that the equivariant Tamagawa number $T\Omega$ of Burns and Flach, living naturally in $K_0(\mathfrak{A}, \mathbb{R})$ in their construction in [4] can also be interpreted as an element in $K_1(\mathsf{LCA}_{\mathfrak{A}})$. A full construction of $T\Omega$ along these lines is then given in [3]. It is shown that the real and p-adic trivialization of the fundamental line simply give a closed loop in the K-theory space $K(\mathsf{LCA}_{\mathfrak{A}})$, and thus an element of π_1. This is our viewpoint on what the equivariant Tamagawa number should be. We prove that this idea is compatible with the formulation of Burns–Flach.

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A generalization of Beilinson's geometric height pairing

Tamás Szamuely
(joint work with Damian Rössler)

Let $\mathbf{F}$ be a finite field, B a smooth proper integral $\mathbf{F}$-scheme of finite type with function field $K = \mathbf{F}(B)$, and X a smooth projective K-variety of dimension d. Fox a prime ℓ invertible in $\mathbf{F}$.

In the first section of his seminal paper [1], Beilinson worked in the case when B is a curve and constructed (unconditionally) an ℓ-adic height pairing

$$CH^p_{\text{hom}}(X)_{\mathbf{Q}} \otimes CH^q_{\text{hom}}(X)_{\mathbf{Q}} \to \mathbf{Q}_\ell \tag{1}$$

for rational Chow groups of homologically trivial cycles and $p + q = d + 1$. This served as a motivation for his (conditional) construction of the height pairing in the number field case.

Together with Damian Rössler we have extended Beilinson's work to a base B of arbitrary dimension. More precisely, we have defined a pairing

$$CH^p_{\text{hom}}(X)_{\mathbf{Q}} \otimes CH^q_{\text{hom}}(X)_{\mathbf{Q}} \to H^2_{\text{ét}}(B, \mathbf{Q}_\ell(1)), \tag{2}$$

still for $p + q = d + 1$.

For $\dim B = 1$ one may pass to the algebraic closure and compose with the trace isomorphism $H^2_{\text{ét}}(\overline{B}, \mathbf{Q}_\ell(1)) \xrightarrow{\sim} \mathbf{Q}_\ell$ of Poincaré duality to obtain a $\mathbf{Q}_\ell$-valued pairing. This pairing is the same as (1).

We conjecture that our pairing is of motivic origin. More precisely:

Conjecture 1. There exists a pairing

$$CH^p_{\text{hom}}(X)_{\mathbf{Q}} \otimes CH^q_{\text{hom}}(X)_{\mathbf{Q}} \to \text{Pic}(B)_{\mathbf{Q}}$$

that, after composition with the cycle map $\text{Pic}(B)_{\mathbf{Q}} \to H^2_{\text{ét}}(B, \mathbf{Q}_\ell(1))$ gives back (2) for all ℓ invertible in $\mathbf{F}$.

Such a pairing should also exist for $\mathbf{F}$ replaced by $\mathbf{Q}$. In that case, there should also be a variant for Arakelov Chow groups.

There is evidence for the conjecture in the case when X is an abelian variety: the construction of Moret-Bailly ([2], §III.3) gives such a pairing, in fact over an arbitrary field in place of $\mathbf{F}$ and only assuming B to be normal. In general, the Tate conjecture would imply that the pairing (2) comes from a pairing with values in $\text{Pic}(X)_{\mathbf{Q}_\ell}$.

The idea of the construction is as follows. Consider the ℓ-adic Abel-Jacobi maps

$$CH^p_{\text{hom}}(X)_{\mathbf{Q}} \to H^1_{\text{ét}}(K, H^{2p-1}(X_{\overline{K}}, \mathbf{Q}_\ell(p)),$$
$$CH^q_{\text{hom}}(X)_{\mathbf{Q}} \to H^1_{\text{ét}}(K, H^{2q-1}(X_{\overline{K}}, \mathbf{Q}_\ell(q)).$$

Given cycle classes on the left, we may represent their images in the Galois cohomology groups by classes in $H^1_{\text{ét}}(U, \mathbf{R}^{2p-1}\pi_*\mathbf{Q}_\ell(p))$ and $H^1_{\text{ét}}(U, \mathbf{R}^{2q-1}\pi_*\mathbf{Q}_\ell(q))$, respectively, for a suitable open subscheme $U \subset B$.

On the other hand, setting $b := \dim B$ there are natural maps

$$H^{1-b}_{\text{ét}}(B, j_{!*}\mathbf{R}^{2p-1}\pi_*\mathbf{Q}_\ell(p)[b]) \to H^1_{\text{ét}}(U, \mathbf{R}^{2p-1}\pi_*\mathbf{Q}_\ell(p)),$$
$$H^{1-b}_{\text{ét}}(B, j_{!*}\mathbf{R}^{2q-1}\pi_*\mathbf{Q}_\ell(q)[b]) \to H^1_{\text{ét}}(U, \mathbf{R}^{2q-1}\pi_*\mathbf{Q}_\ell(q))$$

which can be shown to be injective. Using a weight argument, we verify that the images of cycle classes in $H^1_{\text{ét}}(U, \mathbf{R}^{2p-1}\pi_*\mathbf{Q}_\ell(p))$ and $H^1_{\text{ét}}(U, \mathbf{R}^{2q-1}\pi_*\mathbf{Q}_\ell(q))$ land in the subgroups $H^{1-b}_{\text{ét}}(B, j_{!*}\mathbf{R}^{2p-1}\pi_*\mathbf{Q}_\ell(p)[b])$ and $H^{1-b}_{\text{ét}}(B, j_{!*}\mathbf{R}^{2q-1}\pi_*\mathbf{Q}_\ell(q)[b])$, respectively. Finally, for these "perverse" subgroups we construct a cohomological pairing

$$H^{1-b}_{\text{ét}}(B, j_{!*}\mathbf{R}^{2p-1}\pi_*\mathbf{Q}_\ell(p)[b]) \times H^{1-b}_{\text{ét}}(B, j_{!*}\mathbf{R}^{2q-1}\pi_*\mathbf{Q}_\ell(q)[b]) \to H^2_{\text{ét}}(B, \mathbf{Q}_\ell(1))$$

based on fibrewise Poincaré duality.

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Characteristic classes of vector bundles in Witt cohomology

Marc Levine

We give an overview of the theory of Euler classes and Pontryagin classes in motivic cohomology theories that have Thom classes for bundles with trivialized determinants and for which the algebraic Hopf map η acts invertibly, as developed by Panin-Walter and Ananevskiy. We specialize to the theory of cohomology in the Witt sheaves, where we use "splitting to the normalizer of the torus" give a formula for the characteristic classes of symmetric powers and tensor products. This gives a complete calculus for characteristic classes in this theory. As a simple application we give an arithmetic count of the lines on a hypersurface of degree $2n-1$ in $\mathbb{P}^{n+1}$, generalizing the classical count, the case of the real lines for real hypersurfaces handled by Okonek–Teleman and the arithmetic count for lines on a cubic surface by Kass–Wickelgren.

$\mathbb{A}^1$-degree for counting rational curves

Kirsten Wickelgren
(joint work with Jesse Leo Kass, Marc Levine, Jake Solomon)

We report on joint work in progress giving an arithmetic enrichment of the degree of a map between smooth, proper schemes over k. When the target is appropriately connected and the map is relatively oriented, potentially after removing divisors mapping to codimension at least 2, this degree is valued in the Grothendieck–Witt ring $\mathrm{GW}(k)$ of bilinear forms over k.

The motivating examples are evaluation maps from Kontsevich moduli spaces, following work of Jake Solomon [2]. These examples give $\mathrm{GW}(k)$-valued enrichments of curve counting invariants such as an arithmetic count of the degree d rational plane curves through $3d-1$ points, which can be related to Marc Levine's enriched Welschinger invariants [1].

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Computations of prismatic cohomology and topological cyclic homology

Guozhen Wang
(joint work with Ruochuan Liu)

We introduce a spectral sequence computing periodic topological cyclic homology of complete intersection rings. Its E_2-term can be described as the Ext groups over a certain Hopf algebroid, similar to the construction of Adams spectral sequence. It is believed that this E_2-term computes the prismatic cohomology defined in [2]. and the spectral sequence is isomorphic to the BMS spectral sequence introduced in [1]. Krause and Nikolaus [3] has used a similar method base on the resolution of the base of THH to study DVR's and their quotiens.

1. Topological Hochschild Homology

Let A be an E_∞-ring spectrum.

$$THH(A) = A^{\otimes \mathbb{T}}$$

is the free $\mathbb{T}$-E_∞-ring spectrum generated by A.

There is a cyclotomic structure defined on $THH(A)$, i.e. an E_∞-homomorphism

$$\phi : THH(A) \to THH(A)^{tC_p}$$

equivariant with respect to the group isomorphism $\mathbb{T} \cong \mathbb{T}/C_p$.

2. Topological Cyclic Homology

Topological periodic homology:

$$TP(A) = (THH(A)^{tC_p})^{h\mathbb{T}}$$

Topological negative cyclic homology:

$$TC^-(A) = THH(A)^{h\mathbb{T}}$$

Topological cyclic homology $TC(A)$ is the equalizer of the canonical map

$$can : TC^-(A) \to TP(A)$$

and the Frobenius

$$\phi : TC^-(A) \to TP(A)$$

3. Relative THH

Let

$$S_n = \mathbb{S}[z_0, \dots, z_n]$$

be the E_∞-ring spectrum $\mathbb{S} \wedge \mathbb{N}_+^{n+1}$. Then the ∞-category of S_n-modules is symmetric monoidal.

For any E_∞-S_n-algebra A, define

$$THH(A/S_n) = A^{\otimes_{S_n} \mathbb{T}}$$

as the free $\mathbb{T}$-E_∞-S_n-algebra generated by A.

There is a cyclotomic structure on S_n with trivial $\mathbb{T}$-action such that the Frobenius

$$\phi : S_n \to S_n^{t\mathbb{T}}$$

is defined by sending z_i to z_i^p.

Theorem 1. *For any E_∞-S_n-algebra A, $THH(A/S_n)$ has a structure as a cyclotomic E_∞-spectrum over S_n.*

Relative TP:

$$TP(A/S_n) = THH(A/S_n)^{t\mathbb{T}}$$

Relative TC^-:

$$TC^-(A/S_n) = THH(A/S_n)^{h\mathbb{T}}$$

4. Complete Intersections

Let

$$P = \mathbb{Z}_p[x_1, \dots, x_n]$$

Suppose $f_1(x), \dots, f_k(x) \in P$ are polynomials. We assume the following:

The sequence

$$p, f_1(x), \dots, f_k(x)$$

forms a regular sequence in P.

Set

$$R = P/(f_1(x), \dots, f_k(x))$$

Then R and R/p are complete intersections.

P and R are S_n-algebras via the maps

$$S_n \to P \to R$$

sending z_0 to p and z_i to x_i.

5. Relative TP for P

Convention: We will implicitly complete everything with respect to p and the filtration in the Tate spectral sequence (the Nygaard filtration).

Theorem 2.

$$THH(P/S_n) \cong P[u]$$

with $|u| = 2$ *being the Bökstedt element.*

Theorem 3.

$$TP_0(P/S_n) = \mathbb{Z}_p[x_0, \ldots, x_n]^\wedge$$

with Nygaard filtration defined by powers of $x_0 - p$.

$$TP_*(P/S_n) = TP_0(P/S)[\sigma^\pm]$$

$$\phi(x_i) = x_i^p$$

6. Relative THH for R

Theorem 4. *Structure of* $THH(R/S_n)$*:*

- $THH(R/S_n)$ *is concentrated in even degrees.*
- *The Tate and homotopy fixed point spectral sequences for* $THH(R/S_n)$ *collapses.*
- *There is a filtration on* $THH(R/S_n)$ *such that the graded pieces is isomorphic to*

  $$R[u] \otimes \Gamma(\delta_{f_1(x)}, \ldots, \delta_{f_k(x)})$$

  with $|\delta_{f_i(x)}| = 2$.

7. Relative TP for R

Lemma 1. $x_0^p - p$ is not a zero divisor in $TP_0(R/S_n)$.

Lemma 2. If $\alpha \in TP_0(R/S_n)$ has Nygaard filtration i, then $\phi(\alpha)$ is divisible by $\phi(x_0 - p)^i$.

Theorem 5. $TP_0(R/S_n)$ *is the completion under the Nygaard filtration of the* δ*-ring over* $\mathbb{Z}_p[x_0, \ldots, x_n]$ *generated by* $e_{f_1(x)}, \ldots, e_{f_n(x)}$ *modulo the relations*

$$e_{f_i(x)}(x_0^p - p) = \phi(f_i(x))$$

Proof. Define

$$f_i^{[1]} = \frac{f_i(x)^p - e_{f_i(x)}(x_0 - p)^p}{p}$$

Inductively, we define $f_i^{[k]}$ and $e_{f_i(x)}^{[k]}$ by:

- Define

  $$f_i^{[k]} = \frac{(f_i^{[k-1]})^p - e_{f_i(x)}^{[k-1]}(x_0 - p)^{p^k}}{p}$$

- $f_i^{[k]}$ lies in $\mathbb{Z}_p[x_0, \ldots, x_n][e_{f_i(x)}, \ldots, e_{f_i(x)}^{[k-1]}]$.

- $f_i^{[k]}$ lies in Nygaard filtration p^k.
- $\phi(f_i^{[k]})$ is divisible by $(x_0^p - p)^{p^k}$.
- Define $e^{[k]}_{f_i(x)}$ by the equation

$$e^{[k]}_{f_i(x)}(x_0^p - p)^{p^k} = \phi(f_i^{[k]})$$

- By construction

$$\phi(f_i^{[k]}) = \frac{\phi(f_i^{[k-1]})^p - \phi(e^{[k-1]}_{f_i(x)})(x_0^p - p)^{p^k}}{p}$$

- Since $x_0^p - p$ is not a zero divisor,

$$e^{[k]}_{f_i(x)} = \frac{(e^{[k-1]}_{f_i(x)})^p - \phi(e^{[k-1]}_{f_i(x)})}{p}$$

- $(f_i^{[k]})^p - \phi(f_i^{[k]})$ is divisible by p.

□

8. Resolution of the base

We have an Adams resolution for $\mathbb{S}$:

$$\mathbb{S} \to S_n \to S_n \otimes_{\mathbb{S}} S_n \to S_n^{\otimes 3} \to \dots$$

R is a $S_n^{\otimes m}$-algebra via the map

$$S_n^{\otimes m} \to S_n \to R$$

Theorem 6. *We have the following convergence:*

- $THH(R) \xrightarrow{\cong} Tot(THH(R/S_n^{\otimes \bullet}))$.
- $TP(R) \xrightarrow{\cong} Tot(TP(R/S_n^{\otimes \bullet}))$.
- $TC^-(R) \xrightarrow{\cong} Tot(TC^-(R/S_n^{\otimes \bullet}))$.

Proof. We have

$$THH(R/S_n^{\otimes 2}) \cong THH(R/S_n) \otimes_{THH(R)} THH(R/S_n)$$

so $THH(R) \to Tot(THH(R/S_n^{\otimes \bullet}))$ is the Adams resolution for $THH(R)$ with respect to $THH(R/S)$. □

9. The Adams type spectral sequence

Theorem 7.

- $TP_0(R/S_n^{\otimes 2})$ *is flat over* $TP_0(R/S_n)$.
- $(TP_0(R/S_n), TP_0(R/S_n^{\otimes 2}))$ *forms a Hopf algebroid.*
- $TP_*(R/S_n)$ *is a* $TP_0(R/S_n^{\otimes 2})$*-comodule.*
- $TP_*(R/S_n^{\otimes \bullet})$ *is isomorphic to the cobar complex:*

$$C^\bullet(TP_*(R/S_n), TP_0(R/S_n^{\otimes 2}), TP_0(R/S_n))$$

- *We have a spectral sequence*

$$Ext^j_{TP_0(R/S_n^{\otimes 2})}(TP_0(R/S_n), TP_i(R/S_n)) \Rightarrow TP_{i-j}(R) \tag{3}$$

Conjecture 1. Comparing with works of Bhatt, Morrow and Scholze,

(1) The Ext groups

$$Ext^j_{TP_0(R/S_n^{\otimes 2})}(TP_0(R/S_n), TP_i(R/S_n))$$

is isomorphic to the prismatic cohomology of R.

(2) The spectral sequence (3) is isomorphic to the BMS spectral sequence.

10. Structure of $TP_0(R/S_n^{\otimes 2})$

$$R = \mathbb{Z}_p[x_1, \ldots, x_n, x_0', x_1', \ldots, x_n']/(f_1(x), \ldots, f_k(x), x_0' - p, x_1' - x_1, \ldots, x_n' - x_n)$$

$TP_0(R/S_n^{\otimes 2})$ is the completion under the Nygaard filtration of the δ-ring generated over $TP_0(R/S_n)[x_0', \ldots, x_n']$ by $e_{x_0'-p}, e_{x_1'-x_1}, \ldots, e_{x_n'-x_n}$, modulo the relations

$$e_{x_0'-p}(x_0^p - p) = x_0'^p - p$$

$$e_{x_i'-x_i}(x_0^p - p) = x_i'^p - x_i^p$$

Lemma 4. $e_{x_0'-p}$ is a unit in $TP_0(R/S_n^{\otimes 2})$.

11. Hopf Algebroid Structures

Units:

$$\eta_L(x_i) = x_i$$

$$\eta_R(x_i) = x_i'$$

Comultiplication:

$$\psi(x_i) = x_i \otimes 1$$

$$\psi(x_i') = 1 \otimes x_i'$$

Comodule structure on $TP_*(R/S_n)$:

$$TP_*(R/S_n) = TP_0(R/S_n)[\sigma^{\pm}]$$

$$\psi(\sigma) = \epsilon^{-1}\sigma$$

$$\epsilon^{-1}\phi(\epsilon) = e_{x_0'-p}$$

12. Naturality

Let

$$\tilde{R} = \mathbb{Z}_p[y_1, \ldots, y_l]/(h_1(y), \ldots, h_m(y))$$

with $p, h_1(y), \ldots, h_m(y)$ a regular sequence.

Let $g_1(x), \ldots, g_l(x) \in \mathbb{Z}_p[x_1, \ldots, x_n]$ be polynomials such that

$$h_i(y) \in (f_1(x), \ldots, f_k(x), y_1 - g_1(x), \ldots, y_l - g_l(x))$$

Then $g_i(x)$ defines a ring homomorphism $g : \tilde{R} \to R$

$$\begin{array}{ccccc} S_l & \longrightarrow & S_{l+n} & \longleftarrow & S_n \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z}_p[y_i] & \longrightarrow & \mathbb{Z}_p[x_j, y_i] & \longleftarrow & \mathbb{Z}_p[x_j] \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{R} & \longrightarrow & R & \longleftarrow & R \end{array}$$

Theorem 8. *We have the following:*

- *There is a morphism of Hopf algebroids*

$$(TP_0(\tilde{R}/S_l), TP_0(\tilde{R}/S_l^{\otimes 2})) \to (TP_0(R/S_{l+n}), TP_0(R/S_{l+n}^{\otimes 2}))$$

- *There is a Morita equivalence*

$$(TP_0(R/S_n), TP_0(R/S_n^{\otimes 2}) \to (TP_0(R/S_{l+n}), TP_0(R/S_{l+n}^{\otimes 2}))$$

- *We have a morphism of spectral sequences*

$$\begin{array}{ccc} Ext_{TP_0(\tilde{R}/S_l^{\otimes 2})}(TP_*(\tilde{R}/S_l)) & \longrightarrow & Ext_{TP_0(R/S_n^{\otimes 2})}(TP_*(R/S_n)) \\ \Downarrow & & \Downarrow \\ TP_*(\tilde{R}) & \longrightarrow & TP_*(R) \end{array}$$

13. Localization

Let $h(x) \in \mathbb{Z}[x_1, \ldots, x_n]$ be a polynomials. We have:

$$R[h^{-1}] = \mathbb{Z}[x_1, \ldots, x_n, y]/(f_1(x), \ldots, f_k(x), yh(x) - 1)$$

Theorem 9. *The Hopf algebroid*

$$(TP_0(R[h^{-1}]/S_{n+1}), TP_0(R[h^{-1}]/S_{n+1}^{\otimes 2}))$$

is Morita equivalent to

$$(TP_0(R/S_n)[h(x)^{-1}], TP_0(R/S_n^{\otimes 2})[h(x)^{-1}])$$

It follows that our construction gives a Zariski sheaf of Hopf algebroids.

14. Locally complete intersection varieties

Let $\mathbb{P}^n$ be the projective variety over $\mathbb{Z}_p$, and Y a closed subscheme of $\mathbb{P}^n$ such that $Y_{\mathbb{F}_p}$ is a locally complete intersection

The is a Frobenius homomorphism

$$\phi : \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}$$

Y can be also be regarded as a subscheme of

$$\mathbb{P}^n \times_{\mathbb{Z}_p} \mathbb{A}^1$$

via

$$Y \to \mathbb{P}^n \xrightarrow{x_0 \mapsto p} \mathbb{P}^n \times_{\mathbb{Z}_p} \mathbb{A}^1$$

15. The sheaf of relative TP

Let I be the sheaf of ideal in $\mathcal{O}_{\mathbb{P}^n \times_{\mathbb{Z}_p} \mathbb{A}^1}$ defining Y. Let $T_n = \mathbb{P}^n_{\mathbb{S}} \times_{\mathbb{S}} \mathbb{A}^1_{\mathbb{S}}$.

Theorem 10. *We have a Zariski sheaf of Hopf algebroids*

$$(TP_0(\mathcal{O}_Y/\mathcal{O}_{T_n}), TP_0(\mathcal{O}_Y/\mathcal{O}_{T_n^2}))$$

Theorem 11. *$TP_0(\mathcal{O}_Y/\mathcal{O}_{T_n})$ is the completion under the Nygaard filtration of the sheaf of δ-rings generated over $\mathcal{O}_{\mathbb{P}^n \times_{\mathbb{Z}_p} \mathbb{A}^1}$ by adding the local sections $\frac{\phi(f)}{\phi(x_0 - p)}$ for f a local section of I.*

$$TP_0(\mathcal{O}_Y/\mathcal{O}_{T_n^2})$$

can be described similarly. Note that the support of these sheaves are on Y.

Definition 12. We call a sheaf of $TP_0(\mathcal{O}_Y/\mathcal{O}_{T_n^2})$-comodules to be a $\mathcal{P}$-module.

$\mathcal{P}$-modules can be viewed as $TP_0(\mathcal{O}_Y/\mathcal{O}_{T_n})$-modules with a certain type of connection.

16. Breuil-Kisin Twists

Let BK be the free $TP_0(\mathcal{O}_Y/\mathcal{O}_{T_n})$-module of rank 1 generated by σ, such that

$$\psi(\sigma) = \epsilon^{-1}\sigma$$

$$\epsilon^{-1}\phi(\epsilon) = \frac{x_0'^p - p}{x_0^p - p}$$

For any $\mathcal{P}$-module A, we define its Breuil-Kisin twist by

$$A\{i\} = A \otimes BK^{\otimes i}$$

17. TP for locally complete intersections

Theorem 13. *The category of $\mathcal{P}$-modules is an abelian category.*

Theorem 14. *There is a spectral sequence*

$$Ext^j_{\mathcal{P}\text{-}Mod}(TP_0(\mathcal{O}_Y/\mathcal{O}_{T_n}), TP_0(\mathcal{O}_Y/\mathcal{O}_{T_n})\{i\}) \Rightarrow TP_{2i-j}(Y) \tag{5}$$

Conjecture 2. $Ext^j_{\mathcal{P}\text{-}Mod}(TP_0(\mathcal{O}_Y/\mathcal{O}_{T_n}), TP_0(\mathcal{O}_Y/\mathcal{O}_{T_n})\{i\})$ is isomorphic to the prismatic cohomology of Y, and (5) is isomorphic to the BMS spectral sequence.

18. Hopf algebroid for $\mathbb{Z}_p$

$$TP_0(\mathbb{Z}_p/\mathbb{S}[z]) = \mathbb{Z}_p[x]$$

$TP_0(\mathbb{Z}_p/\mathbb{S}[z, z'])$ is the completion of the δ-ring over $\mathbb{Z}_p[x, y]$ generated by e, modulo the relation

$$e(x^p - p) = y^p - p$$

The structure maps are

$$\eta_L(x) = x$$
$$\eta_R(x) = y$$

19. The Algebraic Tate Spectral Sequence

Using the Nygaard filtration, we have the algebraic Tate spectral sequence:

$$Ext_{Gr_* TP_0(\mathbb{Z}_p/\mathbb{S}[z,z'])}(Gr_* TP_*(\mathbb{Z}_p/\mathbb{S}[z])) \Rightarrow Ext_{TP_0(\mathbb{Z}_p/\mathbb{S}[z,z'])}(TP_*(\mathbb{Z}_p/\mathbb{S}[z]))$$

and its mod p analogue.

Theorem 15.
- *The E_2-term of the mod p algebraic Tate spectral sequence is*
$$\mathbb{F}_p[f, g]/g^2$$
with f represented by x^p and g the Bockstein image of f.
- *The algebraic Tate spectral sequence is isomorphic to the Tate spectral sequence.*

20. The Tate differentials

Theorem 16. *The element*

$$log(e) - \frac{1}{p}log(\phi(e)) = \frac{x^p - y^p}{p} + \frac{x^{2p} - y^{2p}}{2p^2} + \dots$$

lies in $TP_0(\mathbb{Z}_p/\mathbb{S}[z, z'])$.

So we have

$$x^p - y^p \doteq \frac{x^{2p} - y^{2p}}{p} + \dots \quad \text{mod } p$$

Theorem 17.

$$d_p(f) \doteq fg$$

Applying Frobenius, we get

$$x^{p^2} - y^{p^2} \doteq \frac{x^{2p^2} - y^{2p^2}}{p} + \dots \mod p$$

Expanding

$$p^{2p}(log(e) - \frac{1}{p}log(\phi(e)))$$

we get

$$\frac{x^{2p^2} - y^{2p^2}}{p} \doteq \frac{x^{2p^2+p} - y^{2p^2+p}}{p} + \dots \mod p$$

These imply:

Theorem 18.

$$d_{p^2+p}(f^p) = f^{2p}g$$

Let

$$\psi(k) = \frac{p^k - 1}{p - 1}$$

By induction, we have:

Theorem 19.

$$d_{p\psi(k)}(f^{p^{k-1}}) \doteq g f^{p^{k-1}+\psi(k)-1}$$

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Higher Chow-Witt groups

Maria Yakerson

(joint work with Tom Bachmann)

One of the fascinating features of algebraic K-theory of smooth schemes is its close connection with algebraic cycles. For example, Chern character induces a rational isomorphism between K_0 and the Chow ring. Higher K-theory $K_n(X)$ is rationally isomorphic to the sum of higher Chow groups $\oplus_{q\in\mathbb{Z}} CH^q(X, n)$, which are defined as cohomology groups of Bloch's cycle complex of weight q, built out of algebraic cycles of codimension q on $X \times \Delta^\bullet$ that are "in good position". Integrally, there is a spectral sequence, whose terms are given by higher Chow groups, which converges to the K-theory groups. This is the motivic spectral sequence, more often written in terms of motivic cohomology. Motivic cohomology groups are defined as Zariski hypercohomology groups of the corresponding Suslin complex $\mathbb{Z}(q)$, which is an analogue of the singular cochain complex in algebro-geometric settings. A hard theorem due to Friedlander-Suslin and Voevodsky (see e.g. [MVW06]) provides a comparison between higher Chow groups and motivic cohomology, which is important at least for three reasons. This result gives a geometric model for motivic cohomology, it automatically implies vanishing in certain bidegrees of motivic cohomology and provides us with a more explicit construction for motivic cohomology since Bloch's cycle complex satisfies Zariski descent by the localization theorem of Levine [Lev08].

For hermitian K-theory there is a generalized motivic spectral sequence [Bac17], which also degenerates rationally, whose terms are given by certain motivic cohomology groups and by Milnor-Witt motivic cohomology of Calmès-Fasel [CF17]. The latter are constructed via generalized Suslin complexes $\tilde{\mathbb{Z}}(q)$, which in turn are defined by replacing cycles in the definition of Voevodsky's finite correspondences with cycles with additional data of unramified quadratic forms on them. One can ask an analogous question: is there an algebro-geometric description of Milnor-Witt motivic cohomology, in the flavour of higher Chow groups? We give a positive answer to this question. To do so, we construct a generalization of Bloch's cycle complex of weight q which we call Bloch-Levine-Rost-Schmid complex and denote by $C^*(-, K_q^{MW}, q)$, where K_q^{MW} stands for the unramified sheaf of Milnor-Witt K-theory. See [BY18, Definition 4.1] and [BY18, Corollary 1.4] for details.

Theorem 1. Let k be a perfect field of characteristic $\neq 2$, X a smooth k-scheme, $q \geqslant 0$, $n \in \mathbb{Z}$. Then there is a canonical isomorphism between the Milnor-Witt motivic cohomology and the higher Chow-Witt groups:

$$\mathbb{H}^{2q-n}(X, \tilde{\mathbb{Z}}(q)) \simeq \widetilde{CH}^q(X, n),$$

where the right-hand side is defined as the $(q-n)$-th Zariski hypercohomology of the BLRS complex $C^*(X, K_q^{MW}, q)$.

As indicated by the name, the BLRS complex is constructed by splicing together two complexes: the levelwise truncation of Levine's homotopy coniveau tower [Lev08] and (a part of) the Rost-Schmid complex of Morel [Mor12]. The complex $C^*(X, M, q)$ is defined for any strictly homotopy invariant sheaf M on the category of smooth k-schemes. For example, in case of the sheaf of Milnor K-theory K_q^M the Rost-Schmid part of the complex $C^*(X, K_q^M, q)$ vanishes and we get the (shifted) Bloch's cycle complex. In the case $M = K_q^{MW}$ the Bloch-Levine part of the BLRS complex is built out of cycles "in good position" with unramified quadratic forms on them. However, we don't know if the analogue of Levine's localization theorem holds more generally.

Question 1. Does the BLRS complex $C^*(-, K_q^{MW}, q)$ satisfy Zariski descent?

To prove Theorem 1, it is enough to compute the homotopy coniveau tower for sheaves K_q^{MW} (considered as discrete S^1-spectra), since Milnor-Witt motivic cohomology is represented in the motivic stable homotopy category $SH(k)$ by the effective cover of the homotopy module of Milnor-Witt K-theory [BF18], and the slice tower is known to be implemented by the homotopy coniveau tower

[Lev08]. Hence Theorem 1 follows from the following more general result (see [BY18, Theorem 1.4]).

Theorem 2. Let k be a perfect field, X a smooth affine k-scheme with trivial canonical sheaf, and M a strictly homotopy invariant sheaf that has an infinite $\mathbb{G}_m$-delooping (i.e. extends to a homotopy module). Then for any $q \geqslant 0$ there is a canonical equivalence of spectra:

$$M^{(q)}(X) \simeq C^*(X, M, q),$$

where left-hand side is the q-th layer of the homotopy coniveau tower of M at X.

The main part in the proof is a moving lemma for the Rost-Schmid complex that allows to move cycles into "good position", which we prove by adapting the techniques from Levine's moving lemma for the homotopy coniveau tower [Lev06]. The method of "hard moving" uses transfers, which in the unoriented case (e.g. $M = K_q^{MW}$) require a trivialization of the canonical sheaf of X.

Remark 1. The proof of Theorem 2 only requires that $\Omega^q_{\mathbb{G}_m} M$ has an infinite $\mathbb{G}_m$-delooping, i.e. that the q-th contraction M_{-q} extends to a homotopy module.

Having Remark 1 in mind, we may wonder how restrictive is the condition that M_{-q} extends to a homotopy module. In general, proving that a motivic S^1-spectrum has an infinite $\mathbb{G}_m$-delooping amounts to constructing coherent framed tranfers for it, according to the motivic recognition principle [EHK+18]. However, we show that in our situation of discrete sheaves M which are strictly homotopy invariant hence unramified, providing coherent framed transfers reduces to providing for each finitely generated field extension K/k an action of the Grothendieck-Witt group $GW(K)$ on $M(K)$ and transfer maps $M(K(x)) \to M(K)$ for monogenic field extensions $K(x)/K$, with some compatibility properties. This simplification allows us to prove the following theorem (see [BY18, Theorem 1.6]).

Theorem 3. Let k be a perfect field, M a strictly homotopy invariant sheaf, $q \geqslant 2$ ($q \geqslant 3$ if $char\, k > 0$). Then M_{-q} is a homotopy module.

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A motivic formalism in representation theory

Shane Kelly

(joint work with Jens Niklas Eberhardt)

Categories of mixed ℓ-adic sheaves and mixed Hodge modules are indispensable tools in geometric representation theory. They are used in the proof of the Kazhdan-Lusztig conjecture, uncover hidden gradings in categories of representations or categorify objects such as Hecke algebras, representations of quantum groups and link invariants, to name a few. But they are—by their nature—limited to characteristic zero coefficients. In this talk, I will discuss a formalism of mixed sheaves with coefficients in characteristic p following ideas of Soergel, Virk, and Wendt which allows us to make use of motivic sheaves as developed by Ayoub, Cisinski, Déglise, Morel, Voevodsky, etc.

In this talk, G is always a connected reductive linear algebraic group, equipped with a choice of Borel subgroup B and a split maximal torus T. We always work over an algebraically closed field $k = \overline{k}$, but we assume that G, B, T are defined over the integers $\mathbb{Z}$.

Recall that to every character $\lambda : T \to GL_1$ is associated a *standard* representation $\nabla(\lambda)$: one extends $T \to GL_1$ to $B \to GL_1$, and then takes global sections of line bundle $\mathcal{O}(\lambda)$ of λ-invariant functions. Then

$$\nabla(\lambda) := \Gamma(G/B, \mathcal{O}(\lambda))$$

has a canonical G-action induced by the action of G on G/B. Every standard representation has a unique nonzero irreducible subrepresentation,

$$L(\lambda) \subseteq \nabla(\lambda),$$

and every irreducible representation of G is of this form. A major goal of representation theory is to compute

$$[\nabla(\nu) : L(\mu)], \tag{1}$$

the number of times the irreducible representation $L(\mu)$ appears as a subquotient of the standard representation $\nabla(\nu)$ for any two given characters μ, ν.

It turns out that for "nice" λ, ν, this can be computed in a modified category of representations introduced by Soergel, the *modular category* $\mathcal{O}$. The modular category $\mathcal{O}$ is a subquotient $\mathcal{A}/\mathcal{N}$ of the category Rep_G of representations. The idea for the definition of $\mathcal{O}$ is the following. There is a canonical *shifted p-dilated* action $\cdot_p : W \times X \to X$ (which we do not describe here) of the Weyl group $W = N_G(T)/T$ on the lattice X of characters. Moreover, there is a canonical orbit $W \cdot_p \xi = \{x \cdot_p \xi : x \in W\}$ for this action ($\xi \in X$ is some fixed canonical element).

We *want* to define $\mathcal{A}$ as the full subcategory of representations whose irreducible subquotients lie in $W \cdot_p \xi$. However this is not a Serre abelian category, so we define

> $\mathcal{A} :=$ the smallest full Serre abelian subcategory of Rep_G containing the irreducible representations $\{L(x \cdot_p \xi) : x \in W\}$.

The problem with $\mathcal{A}$ is that in addition to the irreducible representations $L(x \cdot_p \xi)$, it also contains other irreducible representations. This is where $\mathcal{N}$ comes in.

> $\mathcal{N} :=$ the smallest full Serre abelian subcategory of $\mathcal{A}$ containing the irreducible representations *not* in $\{L(x \cdot_p \xi) : x \in W\}$.

The canonical functor $\mathcal{A} \to \mathcal{A}/\mathcal{N} = \mathcal{O}$ is denoted $M \mapsto \overline{M}$. The modular category $\mathcal{O}$ has many advantages over Rep_G. In particular, the simple objects are indexed by the Weyl group. Moreover, for every character λ such that $L(\lambda) \in \mathcal{A}$, there is a "smallest" epimorphism from a projection object

$$P(\lambda) \to \overline{L(\lambda)}.$$

These projective objects admit a composition series whose subquotients are images of standard representations, and the numbers (1) are in fact equal to the number of times $\overline{\nabla(\nu)}$ appears in the composition series of $P(\mu)$

$$[\nabla(\nu) : L(\mu)] = [P(\mu) : \overline{\nabla(\nu)}]$$

whenever $L(\mu), \nabla(\nu) \in \mathcal{A}$.

Now we insert some geometry. The hom spaces between projectives of $\mathcal{O}$ can be calculated using certain perverse sheaves called *parity sheaves* on the $\mathbb{C}$-points of the variety G/B of cosets

$$\bigoplus_i \hom_{D^b((G/B)(\mathbb{C}),\overline{\mathbb{F}}_p)}(E_x, E_y[i]) \cong \hom_{\mathcal{O}}(P(\mu_x), P(\mu_y))$$

Here, $x, y \in W = N_G T/T$ and we are using the canonical indexing described above. A natural question is: can this calculation be upgraded to an equivalence of categories between the modular category $\mathcal{O}$ and some category of sheaves. The answer is given by motivic sheaves.

In ℓ-adic étale cohomology, for every finite type morphism $f : X \to Y$ of noetherian schemes, we have adjunctions

$$f^* : D_{et}(Y) \rightleftarrows D_{et}(X) : f_*, \qquad f_! : D_{et}(X) \rightleftarrows D_{et}(Y) : f^!$$

Satisfying lots of nice properties ($\mathbb{A}^1$-invariance, smooth base change, proper base change, purity, Verdier duality, ...

$$H^i_{et}(X, \mathbb{Q}_\ell(n)) \cong \hom_{D_{et}(X)}(\mathbb{Q}_\ell, \mathbb{Q}_\ell(n)[i]).$$

Theorem (Eberhardt-K. Using work of Ayoub, Cisinski-Deglise, Geisser-Levine, Spitzweck, ...). *We can define a $\overline{\mathbb{F}}_p$-linear tensor triangulated category $H(X, \overline{\mathbb{F}}_p)$ for every finite type $\overline{\mathbb{F}}_p$-scheme X, with the same three adjunctions and compatibilities as above, and such that when X is smooth we have*

$$CH^n(X, 2n-i, \overline{\mathbb{F}}_p) \cong \hom_{H(X,\overline{\mathbb{F}}_p)}(\overline{\mathbb{F}}_p, \overline{\mathbb{F}}_p(n)[i]).$$

The important point for our representation theoretic motivations is that this category has no higher Ext's on affine spaces.

$$\hom_{H(\mathbb{A}^n,\overline{\mathbb{F}}_p)}(\overline{\mathbb{F}}_p(n)[i],\overline{\mathbb{F}}_p(m)[j]) \cong \begin{cases} \overline{\mathbb{F}}_p & n=m, i=j \\ 0 & \text{otherwise} \end{cases}$$

Now we consider motivic analogues of perverse sheave which are locally constant along a stratification: stratified mixed Tate motives.

An *affinely stratified variety* is a variety X with a finite partition $\mathcal{S}$ into locally closed subvarieties (called the strata of X)

$$X = \bigcup_{s\in\mathcal{S}} X_s$$

such that each stratum X_s is isomorphic to $\mathbb{A}^n$ for some n, and the closure $\overline{X}_s$ is a union of strata. The embeddings are denoted by $j_s : X_s \hookrightarrow X$. The motivating example we have in mind is G/B with the stratification $G/B = \cup_{x\in W} BxB/B$. The category of *mixed Tate motives* on X, is the smallest full triangulated subcategory of $H(X)$ containing the motives $\overline{\mathbb{F}}_p(i)[j]$.

$$\mathrm{MTDer}(X) = \langle \overline{\mathbb{F}}_p(n)[j]\rangle \subset H(X,\overline{\mathbb{F}}_p).$$

and the category of *stratified mixed Tate motives* on $(X,\mathcal{S})$ is

$$\mathrm{MTDer}_{\mathcal{S}}(X) = \{M \in H(X,\overline{\mathbb{F}}_p) : j_s^* M \in \mathrm{MTDer}(X_s)\ \forall\ s\in\mathcal{S}\}.$$

In other words, the category of motives which are mixed Tate along each X_s.

By the Ext vanishing mentioned above, there is a canonical equivalence between $\mathrm{MTDer}_{\mathcal{S}}(\mathbb{A}^n)$ and the bounded derived category of graded $\overline{\mathbb{F}}_p$-vector spaces,

$$\mathrm{MTDer}_{\mathcal{S}}(\mathbb{A}^n) \cong D^b(\overline{\mathbb{F}}_p\text{-vec.sp.}^{\mathbb{Z}}).$$

This equivalence sends $\overline{\mathbb{F}}_p(i)$ to the one dimensional vector space of weight i. In particular, it makes sense to talk about the *dimension* of objects of $\mathrm{MTDer}_{\mathcal{S}}(\mathbb{A}^n)$.

The subcategory of stratified mixed Tate motives is of great interest in representation theory. Using Soergel's results one can prove the following.

Theorem (Eberhardt-K.)**.** *Let G be a semisimple simply connected split algebraic group over $\overline{\mathbb{F}}_p$ and $G^\vee$ the Langlands dual group. Then there is an equivalence of categories*

$$\mathrm{MTDer}_{(B^\vee)}(G^\vee/B^\vee) \xrightarrow{\sim} \mathrm{Der}^{\mathrm{b}}(\mathcal{O}^{2\mathbb{Z}}(G))$$

between the category of stratified mixed Tate motives on $G^\vee/B^\vee$ and the derived evenly graded modular category $\mathcal{O}^{2\mathbb{Z}}(G)$.

In the above theorem, we have to assume that both the torsion index of G is invertible in $\mathbb{F}_p$ and p is bigger than the Coxeter number of G.

Now, using the six operations, given $x, y \in W = N_G T/T$ one can construct a special object $\mathcal{P}_x \in \mathrm{MTDer}_{\mathcal{B}}(SL_n/B)$ such that

$$\dim j_y^* \mathcal{P}_x = [\Delta(\nu) : L(\mu)]$$

where $j_y : \mathbb{A}^n \to SL_n/B$ is the inclusion of the stratum corresponding to y, and μ, ν are the characters corresponding to $x, y \in W$.

(Topological) Hochschild homology with equivariant input

Teena Gerhardt
(joint work with Andrew Blumberg, Michael Hill, Tyler Lawson)

In the trace method approach, algebraic K-theory is approximated by invariants which are more computable. For a ring A there is a map from its algebraic K-theory to its Hochschild homology, called the Dennis trace map: $K_i(A) \to \mathrm{HH}_i(A)$. Hochschild homology can be defined via the cyclic bar construction, $B^{cy}_{\cdot}(A)$. This is a simplicial abelian group with q simplices

$$B^{cy}_q(A) = A^{\otimes q+1}.$$

The face and degeneracy maps are given as follows.

$$d_i(a_0 \otimes a_1 \otimes \dots a_q) = \begin{cases} a_0 \otimes a_1 \otimes \dots a_i a_{i+1} \otimes \dots a_q & \text{for } 0 \le i < q \\ a_q a_0 \otimes a_1 \otimes \dots a_{q-1} & \text{for } i = q \end{cases}$$

$$s_i(a_0 \otimes a_1 \otimes \dots a_q) = a_0 \otimes a_1 \otimes \dots a_i \otimes 1 \otimes a_{i+1} \otimes \dots a_q. \qquad 0 \le i \le q$$

Let $C_{\cdot}(A)$ denote the chain complex $(B^{cy}_{\cdot}(A), \Sigma(-1)^i d_i)$. Then the Hochschild homology of A is the homology of this chain complex: $\mathrm{HH}_*(A) = H_*(C_{\cdot}(A))$. Equivalently, Hochschild homology can be defined as the homotopy groups of the geometric realization of the cyclic bar construction, $\mathrm{HH}_*(A) = \pi_*(|B^{cy}(A)|)$.

Hochschild homology of a ring has a topological analogue for ring spectra, topological Hochschild homology (THH), defined roughly by replacing the rings in the cyclic bar construction with ring spectra, and the tensor products with smash products. Topological Hochschild homology plays an essential role in the trace method approach to algebraic K-theory. Indeed, for a ring A, the Dennis trace map lifts through THH:

$$K(A) \to \mathrm{THH}(A) \to \mathrm{HH}(A).$$

Here $\mathrm{THH}(A)$ denotes the topological Hochschild homology of the Eilenberg-MacLane spectrum HA. Understanding the S^1-spectrum $\mathrm{THH}(A)$ is a crucial step towards computing algebraic K-theory via trace methods. The map from topological Hochschild homology to Hochschild homology above is called linearization, and it induces an isomorphism in degree 0: $\pi_0\mathrm{THH}(A) \cong \mathrm{HH}_0(A)$.

The construction of topological Hochschild homology via the cyclic bar construction allows for generalizations. In recent work, Angeltveit, Blumberg, Gerhardt, Hill, Lawson, and Mandell [1] define a C_n-relative version of topological Hochschild homology, $\mathrm{THH}_{C_n}(R)$, which takes as input a C_n-equivariant ring spectrum R. This is defined using a C_n-twisted variant of the cyclic bar construction, where one of the face maps is twisted by the action of a generator of C_n. This leads to the question: What is the algebraic analogue of C_n-relative THH? Answering this question requires a theory of Hochschild homology for Green functors.

In equivariant homotopy theory, Mackey functors play the role of abelian groups. The category of G-Mackey functors has a symmetric monoidal structure via box product. A Green functor for G is a monoid object in the symmetric monoidal category of G-Mackey functors.

Let $G \subset S^1$ be a finite subgroup, and $\underline{R}$ a Green functor for G. We can define the Hochschild homology of $\underline{R}$ as $H_*(B^{cy,G}_{\cdot}(\underline{R}))$, where $B^{cy,G}$ denotes a "G-twisted" cyclic bar construction in Green functors. However, to define an algebraic analog of C_n-relative THH one needs a relative version of this construction, i.e. a theory of G-twisted Hochschild homology for H-Green functors, where $H \subset G \subset S^1$.

To define such a relative theory, one needs a notion of norms for Mackey functors. Constructions in Mackey functors are often closely related to analogous constructions in orthogonal G-spectra, and that is indeed the case for norms. An H-Mackey functor $\underline{M}$ has an associated Eilenberg-MacLane spectrum $H\underline{M}$, which is an H-equivariant spectrum. The norm of $\underline{M}$ can then be defined using the Hill-Hopkins-Ravenel norm in spectra: $N_H^G \underline{M} := \underline{\pi}_0(N_H^G H\underline{M})$ (see [4]). We can now define the G-twisted Hochschild homology of an H-Green functor $\underline{R}$ (see [2]):

$$\underline{\mathrm{HH}}_H^G(\underline{R})_* := H_*(B^{cy,G}_{\cdot} N_H^G(\underline{R})).$$

This theory is indeed an algebraic analogue of relative topological Hochschild homology.

Theorem 1. Let $H \subset G \subset S^1$ be finite subgroups, and R a (-1)-connected H-equivariant commutative ring spectrum, there is a linearization map

$$\underline{\pi}_k^G \mathrm{THH}_H(R) \to \underline{\mathrm{HH}}_H^G(\underline{\pi}_0^H R)_k.$$

As in the classical case, this linearization map induces an isomorphism in degree 0:

$$\underline{\pi}_0^G\big(\mathrm{THH}_H(R)\big) \cong \underline{\mathrm{HH}}_H^G(\underline{\pi}_0^H R)_0.$$

Further, there are spectral sequences relating this algebraic theory and the topological theory, as in the classical case. Thus, Hochschild homology for Green functors can be used as a computational tool to understand H-relative THH.

The isomorphism in degree 0 in the theorem above motivates a definition of Witt vectors for Green functors, based on Hesselholt and Madsen's work showing that the fixed points of topological Hochschild homology are closely related to Witt vectors [3]. In particular, they show that for a commutative ring R

$$\pi_0(\mathrm{THH}(R)^{C_n}) \cong \mathbb{W}_{\langle n\rangle}(R),$$

where $\langle n\rangle$ denotes the truncation set of natural numbers that divide n. In [2] the Witt vectors for Green functors are defined as follows.

Definition 1. Let $H \subset G \subset S^1$ be finite subgroups, and let $\underline{R}$ be a Green functor for H. The G-Witt vectors for $\underline{R}$ are defined by

$$\underline{\mathbb{W}}_G(\underline{R}) := \underline{\mathrm{HH}}_H^G(\underline{R})_0.$$

Given this definition, the following analog of the Hesselholt-Madsen result holds.

Proposition 1. For $H \subset G \subset S^1$ finite subgroups, and R a (-1)-connected commutative ring orthogonal H-spectrum,

$$\underline{\pi}_0^G \mathrm{THH}_H(R) \cong \underline{\mathbb{W}}_G(\underline{\pi}_0^H R).$$

Ordinary rings can be viewed as Green functors for the trivial group, and in this case the linearization map of Theorem 1 yields new trace maps from algebraic K-theory, lifting the Dennis trace. One can also define an algebraic analogue of TR-theory using a type of cyclotomic structure on the Hochschild homology of Green functors.

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Stable homotopy groups of motivic spheres

Oliver Röndigs

This talk constitutes a progress report on the project of computing homotopy groups of the sphere spectrum $\mathbf{1}_F$ over a field F of characteristic not two. Most of the results have been obtained in joint work with Markus Spitzweck and Paul Arne Østvær.

Let $\mathrm{SH}(F)$ denote the Morel-Voevodsky $\mathbf{P}^1$-stable homotopy category of F. Given $\mathsf{E} \in \mathrm{SH}(F)$ and integers s, w, let $\pi_{s,w}\mathsf{E}$ denote the abelian group $[\Sigma^{s,w}\mathbf{1}_F, \mathsf{E}]$ of maps in $\mathrm{SH}(F)$. To clarify the grading, $\Sigma^{0,1}\mathbf{1}$ is represented by (the $\mathbf{P}^1$-suspension spectrum of) $\mathbf{A}^1 \smallsetminus \{0\}$ pointed by 1, and $\Sigma^{1,0}\mathbf{1}$ is represented by (the $\mathbf{P}^1$-suspension spectrum of) the simplicial circle S^1, considered as a constant pointed simplicial presheaf. Let

$$\pi_s\mathsf{E} := \pi_{s,\star}\mathsf{E} := \bigoplus_{w \in \mathbb{Z}} \pi_{s,w}\mathsf{E}$$

denote the direct sum, considered as a $\mathbb{Z}$-graded module over the $\mathbb{Z}$-graded ring $\pi_{0,\star}\mathbf{1}$. The latter has the following form by [1].

Theorem 1 (Morel)**.** For every field F, $\pi_0\mathbf{1}$ is the Milnor-Witt K-theory of F.

The Milnor-Witt K-theory of F is denoted $K^{\mathrm{MW}}(F)$, or simply K^{MW}, following the convention that the base field may be ignored in the notation. Its generators are denoted $\eta \in K^{\mathrm{MW}}_{-1} = \pi_{0,1}\mathbf{1}$ and $[u] \in K^{\mathrm{MW}}_1(F) = \pi_{0,-1}\mathbf{1}_F$ for every unit $u \in F^\times$. They correspond to the "obvious" elements given by the first Hopf map $\mathbf{A}^2 \smallsetminus \{0\} \to \mathbf{P}^1$ and the points $[u] \colon S^0 \to \mathbf{A}^1 \smallsetminus \{0\}$.

The "obvious" elements in $\pi_1\mathbf{1}$ are the second Hopf map $\nu \in \pi_{1,2}\mathbf{1}$ given by the Hopf construction on $\mathbf{SL}_2 \simeq \mathbf{A}^2 \smallsetminus \{0\}$ and the first topological Hopf map $\eta_{\mathrm{top}} \in \pi_{1,0}\mathbf{1}$. In his Arolla talk 2012, Dan Dugger conjectured that these two generate $\pi_1\mathbf{1}$ as a K^{MW}-module. Voevodsky's slice filtration is used for the following computation of $\pi_1\mathbf{1}$, which is expressed via the unit map $\mathbf{1} \to \mathbf{kq}$ to the very effective cover of the $\mathbf{P}^1$-spectrum representing hermitian K-theory.[1]

Theorem 2 (R.-Spitzweck-Østvær)**.** Let F be a field of exponential characteristic $e \neq 2$. The unit map $\mathbf{1} \to \mathbf{kq}$ induces an isomorphism $\pi_0\mathbf{1} \xrightarrow{\cong} \pi_0\mathbf{kq}$, and a surjection $\pi_{1,\star}\mathbf{1} \to \pi_{1,\star}\mathbf{kq}$ whose kernel coincides with $K^{\mathrm{Mil}}_{2-\star}/24$ after inverting e.

The proof of Theorem 2 implies that also the unit map for Voevodsky's Eilenberg-MacLane spectrum (which by theorems of Voevodsky and Levine coincides with the zero slice $\mathsf{s}_0\mathbf{1}$) induces a surjection $\pi_{1,\star}\mathbf{1} \to \pi_{1,\star}\mathrm{M}\mathbb{Z} = H^{-1-\star}(F;\mathbb{Z}(-\star))$ to an often nontrivial group. Since both η_{top} and ν are sent to zero under this map, Dugger's conjecture is false.[2] However, it is precisely the zero slice which prevents the validity of Dugger's conjecture, since it holds for the first effective cover $\mathsf{f}_1\mathbf{1} \to \mathbf{1}$.

Theorem 3 (R.)**.** The K^{MW}-module map

$$K^{\mathrm{MW}}_{2+\star} \oplus K^{\mathrm{MW}}_{\star} \to \pi_{1-\star}\mathsf{f}_1\mathbf{1}$$

sending (a,b) to $a \cdot \nu + b \cdot \eta_{\mathrm{top}}$ induces an isomorphism

$$K^{\mathrm{MW}}_{2+\star}\{\nu\} \oplus K^{\mathrm{MW}}_{\star}\{\eta_{\mathrm{top}}\}/(\eta\nu, 2\eta_{\mathrm{top}}, \eta^2\eta_{\mathrm{top}} - 12\nu) \cong \pi_{1,-\star}\mathsf{f}_1\mathbf{1}$$

after inverting the exponential characteristic.

All "obvious" elements in $\pi_2\mathbf{1}$, namely $\nu^2 \in \pi_{2,4}\mathbf{1}$, $\eta_{\mathrm{top}}\nu \in \pi_{2,2}\mathbf{1}$, and $\eta^2_{\mathrm{top}} \in \pi_{2,0}\mathbf{1}$, have order two. However, $\pi_2\mathbf{1}$ and even $\pi_2\mathsf{f}_1\mathbf{1}$ may contain elements of higher order.

Theorem 4 (R.-Spitzweck-Østvær)**.** Let F be a field of exponential characteristic $e \neq 2$. If e is odd, or if the virtual cohomological dimension of F at 2 is at most two, or if $\sqrt{-1} \in F$, or if $\sqrt{2} \in F$, the kernel of $\pi_{2,\star}\mathbf{1} \to \pi_{2,\star}\mathbf{kq}$ coincides with $K^{\mathrm{Mil}}_{4-\star}/2 \oplus Q_{1-\star}$ after inverting e. Here Q_{1-n} is the quotient of $H^{1-n}(F;\mathbb{Z}(2-n))/24$ modulo the subgroup given by those elements whose image (under the canonical map) in $H^{1-n}(F;\mathbb{Z}/24(2-n))$ is hit by the composition $H^{-2-n}(F;\mathbb{Z}/2(1-n)) \xrightarrow{\mathrm{Sq}^2\mathrm{Sq}^1} H^{1-n}(F;\mathbb{Z}/2(2-n)) \xrightarrow{\mathrm{inc}} H^{1-n}(F;\mathbb{Z}/24(2-n))$.

The direct summand $K^{\mathrm{Mil}}_{4-\star}/2$ is generated by ν^2. For any field of characteristic not 2, the map $\pi_{2,2}\mathbf{1} \to \pi_{2,2}\mathbf{kq}$ is the zero map from $K^{\mathrm{Mil}}_2/2$ to the zeroth symplectic K-group (which is the group of even integers), and the image of the map $\pi_{2,1}\mathbf{1} \to \pi_{2,1}\mathbf{kq}$ is the group μ_{24} of 24-th roots of unity.

[1]The published reference [2, Theorem 5.5] uses the effective cover instead.

[2]In the case $F = \mathbb{Q}$, $\pi_{1,-2}\mathbf{1}_{\mathbb{Q}}$ surjects onto $H^1(\mathbb{Q},\mathbb{Z}(2)) \cong \mathbb{Z}/24$, and this surjection does not split. The group $H^1(\mathbb{Q}(\sqrt{-1}),\mathbb{Z}(2))$ even contains a free summand.

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Compatibility of special-value conjectures with the functional equation

Stephen Lichtenbaum

Let X be a regular scheme, projective and flat over the integers. We give a conjecture expressing the special values of the zeta-function of X in terms of Weil-étale motivic cohomology, singular cohomology, and de Rham cohomology. This conjecture is compatible with Serre's functional equation. The key ingredient in the proof of compatibility is an Euler characteristic formula due to Bloch, Kato, and T. Saito.

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