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OWP 2020 - 14

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Hopf Algebras in Combinatorics

Volume 1

Mathematisches Forschungsinstitut Oberwolfach gGmbH
Oberwolfach Preprints (OWP) ISSN 1864-7596

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Imprint:

Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
Schwarzwaldstrasse 9-11
77709 Oberwolfach-Walke
Germany

Tel +49 7834 979 50
Fax +49 7834 979 55
Email admin@mfo.de
URL www.mfo.de

The Oberwolfach Preprints (OWP, ISSN 1864-7596) are published by the MFO.

DOI 10.14760/OWP-2020-14

Hopf Algebras in Combinatorics

Volume 1 of 2

Version of July 27, 2020

Darij Grinberg

Drexel University

Victor Reiner

University of Minnesota, Twin Cities

$$\begin{tikzcd}
 & A \otimes A \arrow[rr, "S \otimes \mathrm{id}_A"] & & A \otimes A \arrow[dr, "m"] & \\
A \arrow[ur, "\Delta"] \arrow[rr, "\epsilon"] \arrow[dr, "\Delta"'] & & \mathbf{k} \arrow[rr, "u"] & & A \\
 & A \otimes A \arrow[rr, "\mathrm{id}_A \otimes S"'] & & A \otimes A \arrow[ur, "m"'] &
\end{tikzcd}$$

Oberwolfach Preprints, 2020

Hopf Algebras in Combinatorics, volume **1** of 2.

Version of 27 July 2020,
equivalent to arXiv:1409.8356v7 up to editorial changes.

Darij Grinberg:
Drexel University
Korman Center, Room 263
15 S 33rd Street
Philadelphia PA, 19104
USA
(temporary:)
Mathematisches Forschungsinstitut Oberwolfach,
Schwarzwaldstrasse 9–11
77709 Oberwolfach
Germany
darijgrinberg@gmail.com
https://www.cip.ifi.lmu.de/~grinberg/

Victor Reiner:
School of Mathematics
University of Minnesota
Minneapolis, MN 55455
USA
reiner@math.umn.edu
https://www-users.math.umn.edu/~reiner/

Keywords: Hopf algebra, combinatorics, symmetric functions, quasisymmetric functions.

Typeset with LaTeX.

Future versions of this text will be available from the first author's website:
https://www.cip.ifi.lmu.de/~grinberg/algebra/HopfComb.pdf
https://www.cip.ifi.lmu.de/~grinberg/algebra/HopfComb-sols.pdf
(version with solutions).

Note to the Reader

This is the first volume of the July 2020 edition of "Hopf Algebras in Combinatorics", an introduction to combinatorial Hopf algebras with particular focus on symmetric and quasisymmetric functions.

This text surveys some of the most fundamental Hopf algebras appearing in combinatorics. After introducing coalgebras, bialgebras and Hopf algebras in general, we study the Hopf algebra of *symmetric functions*; we prove Zelevinsky's axiomatic characterization of it as a "PSH" (*positive self-adjoint Hopf algebra*) and its application to the *representation theory* of symmetric and (briefly) finite general linear groups. We then continue with the *quasisymmetric* and the *noncommutative symmetric functions*, some Hopf algebras formed from graphs, posets and matroids, and the *Malvenuto–Reutenauer Hopf algebra of permutations.* Among other results, we survey the Littlewood–Richardson rule and other symmetric function identities, Zelevinsky's structure theorem for PSHs, the antipode formula for P-partition enumerators, the Aguiar–Bergeron–Sottile universal property of QSym, the theory of Lyndon words, the Gessel–Reutenauer bijection, and Hazewinkel's polynomial freeness of QSym.

The text is written with a graduate student reader in mind (and originates from a one-semester graduate class held by the second author at the University of Minnesota). It assumes a good familiarity with multilinear algebra and – for the representation-theoretical applications – basic group representation theory; otherwise it is meant to be rather self-contained.

The text has been edited over 9 years; it is still likely to be rough at some edges, but has proven useful at least to its authors. It may still grow (note the strategic gap in the numbering between Chapters 8 and 11) and improve. The authors will appreciate any comments and corrections sent to `darijgrinberg@gmail.com` and `reiner@math.umn.edu`.

This version of the text is essentially the version posted on the arXiv as `arXiv:1409.8356v7`; it differs only in some minor editorial changes (spacing, display of formulas, and the occasional trivial rewording of a sentence) and in the page numbering. The numbering of results and equations is identical between this and the arXiv version.

For printing reasons, this version of the text is split into two volumes. Volume 1 (this one) covers general Hopf algebra theory and the symmetric functions (including their PSH-characterization and representation-theoretical applications), while Volume 2 covers the "larger" combinatorial Hopf algebras (and includes the bibliography, the index and a short section containing hints to the exercises from Chapter 1.

Parts of this text have been written during stays at the Mathematisches Forschungsinstitut Oberwolfach (2019 and 2020)[1] and at the Institut

[1]This research was supported through the programme "Oberwolfach Leibniz Fellows" by the Mathematisches Forschungsinstitut Oberwolfach in 2019 and 2020.

Mittag–Leffler Djursholm (Spring 2020, supported by the Swedish Research Council under grant no. 2016-06596); DG thanks both for their hospitality.

Darij Grinberg
July 27, 2020

Victor Reiner
July 27, 2020

HOPF ALGEBRAS IN COMBINATORICS

DARIJ GRINBERG AND VICTOR REINER

Contents

Date: July 27, 2020 (with minor corrections July 28, 2020).

Key words and phrases. Hopf algebra, combinatorics, symmetric functions, quasisymmetric functions.

Introduction

The concept of a Hopf algebra crystallized out of algebraic topology and the study of algebraic groups in the 1940s and 1950s (see [8] and [35] for its history). Being a fairly elementary algebraic notion itself, it subsequently found applications in other mathematical disciplines, and is now particularly commonplace in representation theory[1].

These notes concern themselves (after a brief introduction into the algebraic foundations of Hopf algebra theory in Chapter 1) with the Hopf algebras that appear in combinatorics. These Hopf algebras tend to have bases naturally parametrized by combinatorial objects (partitions, compositions, permutations, tableaux, graphs, trees, posets, polytopes, etc.), and their Hopf-algebraic operations often encode basic operations on these objects[2]. Combinatorial results can then be seen as particular cases of general algebraic properties of Hopf algebras (e.g., the multiplicativity of the Möbius function can be recovered from the fact that the antipode of a Hopf algebra is an algebra anti-endomorphism), and many interesting invariants of combinatorial objects turn out to be evaluations of Hopf morphisms. In some cases (particularly that of symmetric functions), the rigidity in the structure of a Hopf algebra can lead to enlightening proofs.

One of the most elementary interesting examples of a combinatorial Hopf algebra is that of the symmetric functions. We will devote all of Chapter 2 to studying it, deviating from the usual treatments (such as in Stanley [206, Ch. 7], Sagan [186] and Macdonald [142]) by introducing the Hopf-algebraic structure early on and using it to obtain combinatorial results. Chapter 3 will underpin the importance of this algebra by proving Zelevinsky's main theorem of PSH theory, which (roughly) claims that a Hopf algebra over $\mathbb{Z}$ satisfying a certain set of axioms must be a tensor product of copies of the Hopf algebra of symmetric functions. These axioms are fairly restrictive, so this result is far from curtailing the diversity of combinatorial Hopf algebras; but they are natural enough that, as we will see in Chapter 4, they are satisfied for a Hopf algebra of representations of symmetric groups. As a consequence, this Hopf algebra will be revealed isomorphic to the symmetric functions – this is the famous Frobenius correspondence between symmetric functions and characters of symmetric groups, usually obtained through other ways ([73, §7.3], [186, §4.7]). We will further elaborate on the representation theories of wreath products and general linear groups over finite fields; while Zelevinsky's PSH theory does not fully explain the latter, it illuminates it significantly.

In the next chapters, we will study further examples of combinatorial Hopf algebras: the quasisymmetric functions and the noncommutative symmetric functions in Chapter 5, various other algebras (of graphs, posets, matroids, etc.) in Chapter 7, and the Malvenuto-Reutenauer Hopf algebra of permutations in Chapter 8.

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[1] where it provides explanations for similarities between group representations and Lie algebra representations

[2] such as concatenating two compositions, or taking the disjoint union of two graphs – but, more often, operations which return a multiset of results, such as cutting a composition into two pieces at all possible places, or partitioning a poset into two subposets in every way that satisfies a certain axiom

The main prerequisite for reading these notes is a good understanding of graduate algebra[3], in particular multilinear algebra (tensor products, symmetric powers and exterior powers)[4] and basic categorical language[5]. In Chapter 4, familiarity with representation theory of finite groups (over $\mathbb{C}$) is assumed, along with the theory of finite fields and (at some places) the rational canonical form of a matrix. Only basic knowledge of combinatorics is required (except for a few spots in Chapter 7), and familiarity with geometry and topology is needed only to understand some tangential remarks. The concepts of Hopf algebras and coalgebras and the basics of symmetric function theory will be introduced as needed. We will work over a commutative base ring most of the time, but no commutative algebra (besides, occasionally, properties of modules over a PID) will be used.

These notes began as an accompanying text for Fall 2012 Math 8680 Topics in Combinatorics, a graduate class taught by the second author at the University of Minnesota. The first author has since added many exercises (and solutions[6]), as well as Chapter 6 on Lyndon words and the polynomiality of QSym. The notes might still grow, and any comments, corrections and complaints are welcome!

The course was an attempt to focus on examples that we find interesting, but which are hard to find fully explained currently in books or in one paper. Much of the subject of combinatorial Hopf algebras is fairly recent (1990s onwards) and still spread over research papers, although sets of lecture notes do exist, such as Foissy's [70]. A reference which we discovered late, having a great deal of overlap with these notes is Hazewinkel, Gubareni, and Kirichenko [93]. References for the purely algebraic theory of Hopf algebras are much more frequent (see the beginning of Chapter 1 for a list). Another recent text that has a significant amount of material in common with ours (but focuses on representation theory and probability applications) is Méliot's [153].

Be warned that our notes are highly idiosyncratic in choice of topics, and they steal heavily from the sources in the bibliography.

Warnings: Unless otherwise specified ...

- $\mathbf{k}$ here usually denotes a commutative ring[7].
- all maps between $\mathbf{k}$-modules are $\mathbf{k}$-linear.
- every ring or $\mathbf{k}$-algebra is associative and has a 1, and every ring morphism or $\mathbf{k}$-algebra morphism preserves the 1's.

[3]William Schmitt's expositions [193] are tailored to a reader interested in combinatorial Hopf algebras; his notes on modules and algebras cover a significant part of what we need from abstract algebra, whereas those on categories cover all category theory we will use and much more.

[4]Keith Conrad's expository notes [40] are useful, even if not comprehensive, sources for the latter.

[5]We also will use a few nonstandard notions from linear algebra that are explained in the Appendix (Chapter 11).

[6]The version of the notes you are reading does not contain said solutions. The version that does can be downloaded from `http://www.cip.ifi.lmu.de/~grinberg/algebra/HopfComb-sols.pdf` or compiled from the sourcecode.

[7]As explained below, "ring" means "associative ring with 1". The most important cases are when $\mathbf{k}$ is a field or when $\mathbf{k} = \mathbb{Z}$.

- all $\mathbf{k}$-algebras A have the property that $(\lambda 1_A)\, a = a\, (\lambda 1_A) = \lambda a$ for all $\lambda \in \mathbf{k}$ and $a \in A$.
- all tensor products are over $\mathbf{k}$ (unless a subscript specifies a different base ring).
- 1 will denote the multiplicative identity in some ring like $\mathbf{k}$ or in some $\mathbf{k}$-algebra (sometimes also the identity of a group written multiplicatively).
- for any set S, we denote by id_S (or by id) the identity map on S.
- The symbols $\subset$ (for "subset") and $<$ (for "subgroup") don't imply properness (so $\mathbb{Z} \subset \mathbb{Z}$ and $\mathbb{Z} < \mathbb{Z}$).
- the n-th symmetric group (i.e., the group of all permutations of $\{1, 2, \ldots, n\}$) is denoted $\mathfrak{S}_n$.
- A permutation $\sigma \in \mathfrak{S}_n$ will often be identified with the n-tuple $(\sigma(1), \sigma(2), \ldots, \sigma(n))$, which will occasionally be written without commas and parentheses (i.e., as follows: $\sigma(1)\sigma(2)\cdots\sigma(n)$). This is called the *one-line notation* for permutations.
- The product of permutations $a \in \mathfrak{S}_n$ and $b \in \mathfrak{S}_n$ is defined by $(ab)(i) = a(b(i))$ for all i.
- *Words* over (or in) an *alphabet* I simply mean finite tuples of elements of a set I. It is customary to write such a word $(a_1, a_2, \ldots, a_k)$ as $a_1 a_2 \ldots a_k$ when this is not likely to be confused for multiplication.
- $\mathbb{N} := \{0, 1, 2, \ldots\}$.
- if i and j are any two objects, then $\delta_{i,j}$ denotes the *Kronecker delta* of i and j; this is the integer 1 if $i = j$ and 0 otherwise.
- a *family* of objects indexed by a set I means a choice of an object f_i for each element $i \in I$; this family will be denoted either by $(f_i)_{i \in I}$ or by $\{f_i\}_{i \in I}$ (and sometimes the "$i \in I$" will be omitted when the context makes it obvious – so we just write $\{f_i\}$).
- several objects $s_1, s_2, \ldots, s_k$ are said to be *distinct* if every $i \neq j$ satisfy $s_i \neq s_j$.
- similarly, several sets $S_1, S_2, \ldots, S_k$ are said to be *disjoint* if every $i \neq j$ satisfy $S_i \cap S_j = \varnothing$.
- the symbol $\sqcup$ (and the corresponding quantifier $\bigsqcup$) denotes a disjoint union of sets or posets. For example, if $S_1, S_2, \ldots, S_k$ are k sets, then $\bigsqcup_{i=1}^k S_i$ is their disjoint union. This disjoint union can mean either of the following two things:
  - It can mean the union $\bigcup_{i=1}^k S_i$ in the case when the sets $S_1, S_2, \ldots, S_k$ are disjoint. This is called an "internal disjoint union", and is simply a way to refer to the union of sets while simultaneously claiming that these sets are disjoint. Thus, of course, it is only well-defined if the sets are disjoint.
  - It can also mean the union $\bigcup_{i=1}^k \{i\} \times S_i$. This is called an "external disjoint union", and is well-defined whether or not the sets $S_1, S_2, \ldots, S_k$ are disjoint; it is a way to assemble the sets $S_1, S_2, \ldots, S_k$ into a larger set which contains a copy of each of their elements that "remembers" which set this element comes from.

The two meanings are different, but in the case when $S_1, S_2, \ldots, S_k$ are disjoint, they are isomorphic. We hope the reader will not have a hard time telling which of them we are trying to evoke.

Similarly, the notion of a direct sum of $\mathbf{k}$-modules has two meanings ("internal direct sum" and "external direct sum").

- A sequence $(w_1, w_2, \ldots, w_k)$ of numbers (or, more generally, of elements of a poset) is said to be *strictly increasing* (or, for short, *increasing*) if it satisfies $w_1 < w_2 < \cdots < w_k$. A sequence $(w_1, w_2, \ldots, w_k)$ of numbers (or, more generally, of elements of a poset) is said to be *weakly increasing* (or *nondecreasing*) if it satisfies $w_1 \leq w_2 \leq \cdots \leq w_k$. Reversing the inequalities, we obtain the definitions of a *strictly decreasing* (a.k.a. *decreasing*) and of a *weakly decreasing* (a.k.a. *nonincreasing*) sequence. All these definitions extend in an obvious way to infinite sequences. Note that "nondecreasing" is not the same as "not decreasing"; for example, any sequence having at most one entry is both decreasing and nondecreasing, whereas the sequence $(1, 3, 1)$ is neither.

Hopefully context will resolve some of the ambiguities.

1. What is a Hopf algebra?

The standard references for Hopf algebras are Abe [1] and Sweedler [213], and some other good ones are [33, 36, 47, 93, 107, 118, 157, 176, 196, 225]. See also Foissy [70] and Manchon [149] for introductions to Hopf algebras tailored to combinatorial applications. Most texts only study Hopf algebras over fields (with exceptions such as [36, 33, 225]). We will work over arbitrary commutative rings[8], which requires some more care at certain points (but we will not go deep enough into the algebraic theory to witness the situation over commutative rings diverge seriously from that over fields).

Let's build up the definition of Hopf algebra structure bit-by-bit, starting with the more familiar definition of algebras.

1.1. Algebras.

Recall that an *associative* $\mathbf{k}$*-algebra* is defined to be a $\mathbf{k}$-module A equipped with an associative $\mathbf{k}$-bilinear map $\operatorname{mult} : A \times A \to A$ (the *multiplication map* of A) and an element $1 \in A$ (the *(multiplicative) unity* or *identity* of A) that is neutral for this map mult (that is, it satisfies $\operatorname{mult}(a, 1) = \operatorname{mult}(1, a) = a$ for all $a \in A$). If we recall that

- $\mathbf{k}$-bilinear maps $A \times A \to A$ are in 1-to-1 correspondence with $\mathbf{k}$-linear maps $A \otimes A \to A$ (by the universal property of the tensor product), and
- elements of A are in 1-to-1 correspondence with $\mathbf{k}$-linear maps $\mathbf{k} \to A$,

then we can restate this classical definition of associative $\mathbf{k}$-algebras as follows in terms of $\mathbf{k}$-linear maps[9]:

Definition 1.1.1. An *associative* $\mathbf{k}$*-algebra* is a $\mathbf{k}$-module A equipped with a $\mathbf{k}$-linear *associative operation* $A \otimes A \xrightarrow{m} A$, and a $\mathbf{k}$-linear *unit* $\mathbf{k} \xrightarrow{u} A$, for which the following two diagrams are commutative:

(1.1.1)
$$\begin{tikzcd}
 & A \otimes A \otimes A \arrow[dl, "m \otimes \mathrm{id}"'] \arrow[dr, "\mathrm{id} \otimes m"] & \\
A \otimes A \arrow[dr, "m"] & & A \otimes A \arrow[dl, "m"'] \\
 & A &
\end{tikzcd}$$

(1.1.2)
$$\begin{tikzcd}
A \otimes \mathbf{k} \arrow[d, "\mathrm{id} \otimes u"'] & A \arrow[l] \arrow[r] \arrow[d, "\mathrm{id}"'] & \mathbf{k} \otimes A \arrow[d, "u \otimes \mathrm{id}"] \\
A \otimes A \arrow[r, "m"] & A & A \otimes A \arrow[l, "m"']
\end{tikzcd}$$

where the maps $A \to A \otimes \mathbf{k}$ and $A \to \mathbf{k} \otimes A$ are the isomorphisms sending $a \mapsto a \otimes 1$ and $a \mapsto 1 \otimes a$.

We abbreviate "associative $\mathbf{k}$-algebra" as "$\mathbf{k}$-algebra" (associativity is assumed unless otherwise specified) or as "algebra" (when $\mathbf{k}$ is clear from

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[8]and we will profit from this generality in Chapters 3 and 4, where we will be applying the theory of Hopf algebras to $\mathbf{k} = \mathbb{Z}$ in a way that would not be possible over $\mathbf{k} = \mathbb{Q}$

[9]Explicitly speaking, we are replacing the $\mathbf{k}$-bilinear multiplication map $\operatorname{mult} : A \times A \to A$ by the $\mathbf{k}$-linear map $m : A \otimes A \to A,\ a \otimes b \mapsto \operatorname{mult}(a, b)$, and we are replacing the element $1 \in A$ by the $\mathbf{k}$-linear map $u : \mathbf{k} \to A,\ 1_{\mathbf{k}} \mapsto 1$.

the context). We sometimes refer to m as the "multiplication map" of A as well.

As we said, the multiplication map $m : A \otimes A \to A$ sends each $a \otimes b$ to the product ab, and the unit map $u : \mathbf{k} \to A$ sends the identity $1_{\mathbf{k}}$ of $\mathbf{k}$ to the identity 1_A of A.

Well-known examples of $\mathbf{k}$-algebras are *tensor* and *symmetric algebras*, which we can think of as algebras of *words* and *multisets*, respectively.

Example 1.1.2. If V is a $\mathbf{k}$-module and $n \in \mathbb{N}$, then the *n-fold tensor power* $V^{\otimes n}$ of V is the $\mathbf{k}$-module $\underbrace{V \otimes V \otimes \cdots \otimes V}_{n \text{ times}}$. (For $n = 0$, this is the $\mathbf{k}$-module $\mathbf{k}$, spanned by the "empty tensor" $1_{\mathbf{k}}$.)

The *tensor algebra* $T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$ on a $\mathbf{k}$-module V is an associative $\mathbf{k}$-algebra spanned (as $\mathbf{k}$-module) by decomposable tensors $v_1 v_2 \cdots v_k := v_1 \otimes v_2 \otimes \cdots \otimes v_k$ with $k \in \mathbb{N}$ and $v_1, v_2, \ldots, v_k \in V$. Its multiplication is defined $\mathbf{k}$-linearly by

$$m\left(v_1 v_2 \cdots v_k \otimes w_1 w_2 \cdots w_\ell\right) := v_1 v_2 \cdots v_k w_1 w_2 \cdots w_\ell$$

[10] for all $k, \ell \in \mathbb{N}$ and $v_1, v_2, \ldots, v_k, w_1, w_2, \ldots, w_\ell$ in V. The unit map $u : \mathbf{k} \to T(V)$ sends $1_{\mathbf{k}}$ to the empty tensor $1_{T(V)} = 1_{\mathbf{k}} \in \mathbf{k} = V^{\otimes 0}$.

If V is a free $\mathbf{k}$-module, say with $\mathbf{k}$-basis $\{x_i\}_{i \in I}$, then $T(V)$ has a $\mathbf{k}$-basis of decomposable tensors $x_{i_1} \cdots x_{i_k} := x_{i_1} \otimes \cdots \otimes x_{i_k}$ indexed by *words* $(i_1, \ldots, i_k)$ in the alphabet I, and the multiplication on this basis is given by concatenation of words:

$$m(x_{i_1} \cdots x_{i_k} \otimes x_{j_1} \cdots x_{j_\ell}) = x_{i_1} \cdots x_{i_k} x_{j_1} \cdots x_{j_\ell}.$$

Recall that a *two-sided ideal* of a $\mathbf{k}$-algebra A is defined to be a $\mathbf{k}$-submodule J of A such that all $j \in J$ and $a \in A$ satisfy $ja \in J$ and $aj \in J$. Using tensors, we can restate this as follows: A *two-sided ideal* of a $\mathbf{k}$-algebra A means a $\mathbf{k}$-submodule J of A satisfying $m(J \otimes A) \subset J$ and $m(A \otimes J) \subset J$. Often, the word "two-sided" is omitted and one just speaks of an ideal.

It is well-known that if J is a two-sided ideal of a $\mathbf{k}$-algebra A, then one can form a *quotient algebra* A/J.

Example 1.1.3. Let V be a $\mathbf{k}$-module. The *symmetric algebra* $\mathrm{Sym}(V) = \bigoplus_{n \geq 0} \mathrm{Sym}^n(V)$ is the quotient of $T(V)$ by the two-sided ideal generated by all elements $xy - yx$ with x, y in V. When V is a free $\mathbf{k}$-module with basis $\{x_i\}_{i \in I}$, this symmetric algebra $S(V)$ can be identified with a (commutative) polynomial algebra $\mathbf{k}[x_i]_{i \in I}$, having a $\mathbf{k}$-basis of (commutative)

[10] Some remarks about our notation (which we are using here and throughout these notes) are in order.

Since we are working with tensor products of $\mathbf{k}$-modules like $T(V)$ – which themselves are made of tensors – here, we must specify what the $\otimes$ sign means in expressions like $a \otimes b$ where a and b are elements of $T(V)$. Our convention is the following: When a and b are elements of a tensor algebra $T(V)$, we always understand $a \otimes b$ to mean the pure tensor $a \otimes b \in T(V) \otimes T(V)$ rather than the product of a and b inside the tensor algebra $T(V)$. The latter product will plainly be written ab.

The operator precedence between $\otimes$ and multiplication in $T(V)$ is such that multiplication in $T(V)$ binds more tightly than the $\otimes$ sign; e.g., the term $ab \otimes cd$ means $(ab) \otimes (cd)$. The same convention applies to any algebra instead of $T(V)$.

monomials $x_{i_1} \cdots x_{i_k}$ as $\{i_1, \ldots, i_k\}_{\text{multiset}}$ runs through all finite multisubsets[11] of I, and with multiplication defined $\mathbf{k}$-linearly via multiset union[12].

Note that the $\mathbf{k}$-module $\mathbf{k}$ itself canonically becomes a $\mathbf{k}$-algebra. Its associative operation $m : \mathbf{k} \otimes \mathbf{k} \to \mathbf{k}$ is the canonical isomorphism $\mathbf{k} \otimes \mathbf{k} \to \mathbf{k}$, and its unit $u : \mathbf{k} \to \mathbf{k}$ is the identity map.

Topology and group theory give more examples.

Example 1.1.4. The *cohomology algebra* $H^*(X; \mathbf{k}) = \bigoplus_{i \geq 0} H^i(X; \mathbf{k})$ with coefficients in $\mathbf{k}$ for a topological space X has an associative *cup product.* Its unit $\mathbf{k} = H^*(\mathbf{pt}; \mathbf{k}) \xrightarrow{u} H^*(X; \mathbf{k})$ is induced from the unique (continuous) map $X \to \mathbf{pt}$, where $\mathbf{pt}$ is a one-point space.

Example 1.1.5. For a group G, the *group algebra* $\mathbf{k}G$ has $\mathbf{k}$-basis $\{t_g\}_{g \in G}$ and multiplication defined $\mathbf{k}$-linearly by $t_g t_h = t_{gh}$, and unit defined by $u(1) = t_e$, where e is the identity element of G.

1.2. **Coalgebras.** In Definition 1.1.1, we have defined the notion of an algebra entirely in terms of linear maps; thus, by reversing all arrows, we can define a dual notion, which is called a *coalgebra.* If we are to think of the multiplication $A \otimes A \to A$ in an algebra as *putting together* two basis elements of A to get a sum of basis elements of A, then coalgebra structure should be thought of as *taking basis elements apart.*

Definition 1.2.1. A *co-associative* $\mathbf{k}$*-coalgebra* is a $\mathbf{k}$-module C equipped with a *comultiplication*, that is, a $\mathbf{k}$-linear map $C \xrightarrow{\Delta} C \otimes C$, and a $\mathbf{k}$-linear *counit* $C \xrightarrow{\epsilon} \mathbf{k}$ for which the following diagrams (which are exactly the diagrams in (1.1.1) and (1.1.2) but with *all arrows reversed*) are commutative:

(1.2.1)
$$\begin{tikzcd}
 & C \otimes C \otimes C & \\
C \otimes C \arrow[ur, "\Delta \otimes \mathrm{id}"] & & C \otimes C \arrow[ul, "\mathrm{id} \otimes \Delta"'] \\
 & C \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

(1.2.2)
$$\begin{tikzcd}
C \otimes \mathbf{k} \arrow[r] & C & \mathbf{k} \otimes C \arrow[l] \\
C \otimes C \arrow[u, "\mathrm{id} \otimes \epsilon"] & C \arrow[l, "\Delta"] \arrow[r, "\Delta"'] \arrow[u, "\mathrm{id}"] & C \otimes C \arrow[u, "\epsilon \otimes \mathrm{id}"']
\end{tikzcd}$$

Here the maps $C \otimes \mathbf{k} \to C$ and $\mathbf{k} \otimes C \to C$ are the isomorphisms sending $c \otimes 1 \mapsto c$ and $1 \otimes c \mapsto c$.

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[11]By a *multisubset* of a set S, we mean a multiset each of whose elements belongs to S (but can appear arbitrarily often).

[12]The *multiset union* of two finite multisets A and B is defined to be the multiset C with the property that every x satisfies

$$(\text{multiplicity of } x \text{ in } C) = (\text{multiplicity of } x \text{ in } A) + (\text{multiplicity of } x \text{ in } B).$$

Equivalently, the multiset union of $\{a_1, a_2, \ldots, a_k\}_{\text{multiset}}$ and $\{b_1, b_2, \ldots, b_\ell\}_{\text{multiset}}$ is $\{a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_\ell\}_{\text{multiset}}$. The multiset union is also known as the *disjoint union* of multisets.

We abbreviate "co-associative **k**-coalgebra" as "**k**-coalgebra" (co-associativity, i.e., the commutativity of the diagram (1.2.1), is assumed unless otherwise specified) or as "coalgebra" (when **k** is clear from the context).

Sometimes, the word "coproduct" is used as a synonym for "comultiplication"[13].

One often uses the *Sweedler notation*

$$\Delta(c) = \sum_{(c)} c_1 \otimes c_2 = \sum c_1 \otimes c_2 \tag{1.2.3}$$

to abbreviate formulas involving Δ. This means that an expression of the form $\sum_{(c)} f(c_1, c_2)$ (where $f : C \times C \to M$ is some $\mathbf{k}$-bilinear map from $C \times C$ to some $\mathbf{k}$-module M) has to be understood to mean $\sum_{k=1}^{m} f(d_k, e_k)$, where $k \in \mathbb{N}$ and $d_1, d_2, \ldots, d_k \in C$ and $e_1, e_2, \ldots, e_k \in C$ are chosen such that $\Delta(c) = \sum_{k=1}^{m} d_k \otimes e_k$. (There are many ways to choose such k, d_i and e_i, but they all produce the same result $\sum_{k=1}^{m} f(d_k, e_k)$. Indeed, the result they produce is $F(\Delta(c))$, where $F : C \otimes C \to M$ is the $\mathbf{k}$-linear map induced by the bilinear map f.) For example, commutativity of the left square in (1.2.2) asserts that $\sum_{(c)} c_1 \epsilon(c_2) = c$ for each $c \in C$. Likewise, commutativity of the right square in (1.2.2) asserts that $\sum_{(c)} \epsilon(c_1) c_2 = c$ for each $c \in C$. The commutativity of (1.2.1) can be written as $\sum_{(c)} \Delta(c_1) \otimes c_2 = \sum_{(c)} c_1 \otimes \Delta(c_2)$, or (using nested Sweeedler notation to unravel the two remaining Δ's) as

$$\sum_{(c)} \sum_{(c_1)} (c_1)_1 \otimes (c_1)_2 \otimes c_2 = \sum_{(c)} \sum_{(c_2)} c_1 \otimes (c_2)_1 \otimes (c_2)_2.$$

The **k**-module **k** itself canonically becomes a **k**-coalgebra, with its comultiplication $\Delta : \mathbf{k} \to \mathbf{k} \otimes \mathbf{k}$ being the canonical isomorphism $\mathbf{k} \to \mathbf{k} \otimes \mathbf{k}$, and its counit $\epsilon : \mathbf{k} \to \mathbf{k}$ being the identity map.

Example 1.2.2. Let **k** be a field. The *homology* $H_*(X; \mathbf{k}) = \bigoplus_{i \geq 0} H_i(X; \mathbf{k})$ for a topological space X is naturally a coalgebra: the (continuous) *diagonal embedding* $X \to X \times X$ sending $x \mapsto (x, x)$ induces a coassociative map

$$H_*(X; \mathbf{k}) \to H_*(X \times X; \mathbf{k}) \cong H_*(X; \mathbf{k}) \otimes H_*(X; \mathbf{k})$$

in which the last isomorphism comes from the *Künneth theorem* with field coefficients **k**. As before, the unique (continuous) map $X \to \mathbf{pt}$ induces the counit $H_*(X; \mathbf{k}) \xrightarrow{\epsilon} H_*(\mathbf{pt}; \mathbf{k}) \cong \mathbf{k}$.

Exercise 1.2.3. Let C be a **k**-module, and let $\Delta : C \to C \otimes C$ be a **k**-linear map. Prove that there exists *at most one* **k**-linear map $\epsilon : C \to \mathbf{k}$ such that the diagram (1.2.2) commutes.

For us, the notion of a coalgebra serves mostly as a stepping stone towards that of a Hopf algebra, which will be the focus of these notes. However, coalgebras have interesting properties of their own (see, e.g., [150]).

[13] although the word "coproduct" already has a different meaning in algebra

1.3. **Morphisms, tensor products, and bialgebras.** Just as we rewrote the definition of an algebra in terms of linear maps (in Definition 1.1.1), we can likewise rephrase the standard definition of a morphism of algebras:

Definition 1.3.1. A *morphism of algebras* is a $\mathbf{k}$-linear map $A \xrightarrow{\varphi} B$ between two $\mathbf{k}$-algebras A and B that makes the following two diagrams commute:

(1.3.1)
$$\begin{tikzcd} A \arrow[r, "\varphi"] & B \\ A \otimes A \arrow[u, "m_A"] \arrow[r, "\varphi\otimes\varphi"] & B \otimes B \arrow[u, "m_B"'] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "\varphi"] & & B \\ & \mathbf{k} \arrow[ul, "u_A"] \arrow[ur, "u_B"'] & \end{tikzcd}$$

Here the subscripts on m_A, m_B, u_A, u_B indicate for which algebra they are part of the structure (e.g., the map u_A is the map u of the algebra A); we will occasionally use such conventions from now on.

Similarly, a *morphism of coalgebras* is a $\mathbf{k}$-linear map $C \xrightarrow{\varphi} D$ between two $\mathbf{k}$-coalgebras C and D that makes the reverse diagrams commute:

(1.3.2)
$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d, "\Delta_C"'] & D \arrow[d, "\Delta_D"] \\ C \otimes C \arrow[r, "\varphi\otimes\varphi"] & D \otimes D \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "\varphi"] \arrow[dr, "\epsilon_C"'] & & D \arrow[dl, "\epsilon_D"] \\ & \mathbf{k} & \end{tikzcd}$$

As usual, we shall use the word "*homomorphism*" as a synonym for "morphism", and we will say "$\mathbf{k}$-coalgebra homomorphism" for "homomorphism of coalgebras" (and similarly for algebras and other structures).

As usual, the word "*isomorphism*" (of algebras, of coalgebras, or of other structures that we will define further below) means "invertible morphism whose inverse is a morphism as well". Two algebras (or coalgebras, or other structures) are said to be *isomorphic* if there exists an isomorphism between them.

Example 1.3.2. Let $\mathbf{k}$ be a field. Continuous maps $X \xrightarrow{f} Y$ of topological spaces induce algebra morphisms $H^*(Y;\mathbf{k}) \to H^*(X;\mathbf{k})$, and coalgebra morphisms $H_*(X;\mathbf{k}) \to H_*(Y;\mathbf{k})$.

Coalgebra morphisms behave similarly to algebra morphisms in many regards: For example, the inverse of an invertible coalgebra morphism is again a coalgebra morphism[14]. Thus, the invertible coalgebra morphisms are precisely the coalgebra isomorphisms.

Definition 1.3.3. Given two $\mathbf{k}$-algebras A, B, their tensor product $A \otimes B$ also becomes a $\mathbf{k}$-algebra defining the multiplication bilinearly via

$$m((a \otimes b) \otimes (a' \otimes b')) := aa' \otimes bb',$$

or, in other words, $m_{A\otimes B}$ is the composite map

$$A \otimes B \otimes A \otimes B \xrightarrow{\mathrm{id} \otimes T \otimes \mathrm{id}} A \otimes A \otimes B \otimes B \xrightarrow{m_A \otimes m_B} A \otimes B$$

where T is the *twist map* $B \otimes A \to A \otimes B$ that sends $b \otimes a \mapsto a \otimes b$. (See Exercise 1.3.4(a) below for a proof that this $\mathbf{k}$-algebra $A \otimes B$ is well-defined.)

[14] The easy proof of this fact is left to the reader.

Here we are omitting the topologist's sign in the twist map which should be present for graded algebras and coalgebras that come from cohomology and homology: For homogeneous elements a and b of two graded modules A and B, the topologist's twist map $T : B \otimes A \to A \otimes B$ sends

$$b \otimes a \longmapsto (-1)^{\deg(b)\deg(a)} a \otimes b \tag{1.3.3}$$

instead of $b \otimes a \mapsto a \otimes b$. This means that, if one is using the topologists' convention, most of our examples which we later call *graded* should actually be considered to live in only *even* degrees (which can be achieved, e.g., by artificially doubling all degrees in their grading). We will, however, keep to our own definitions (so that our twist map T will always send $b \otimes a \mapsto a \otimes b$) unless otherwise noted. Only in parts of Exercise 1.6.5 will we use the topologist's sign. Readers interested in the wide world of algebras defined using the topologist's sign convention (which is also known as the *Koszul sign rule*) can consult [65, Appendix A2]; see also [87] for applications to algebraic combinatorics[15].

The unit element of $A \otimes B$ is $1_A \otimes 1_B$, meaning that the unit map $\mathbf{k} \overset{u_{A \otimes B}}{\to} A \otimes B$ is the composite

$$\mathbf{k} \longrightarrow \mathbf{k} \otimes \mathbf{k} \xrightarrow{u_A \otimes u_B} A \otimes B\ .$$

Similarly, given two coalgebras C, D, one can make $C \otimes D$ a coalgebra in which the comultiplication and counit maps are the composites of

$$C \otimes D \xrightarrow{\Delta_C \otimes \Delta_D} C \otimes C \otimes D \otimes D \xrightarrow{\mathrm{id} \otimes T \otimes \mathrm{id}} C \otimes D \otimes C \otimes D$$

and

$$C \otimes D \xrightarrow{\epsilon_C \otimes \epsilon_D} \mathbf{k} \otimes \mathbf{k} \longrightarrow \mathbf{k}\ .$$

(See Exercise 1.3.4(b) below for a proof that this $\mathbf{k}$-coalgebra $C \otimes D$ is well-defined.)

Exercise 1.3.4. (a) Let A and B be two $\mathbf{k}$-algebras. Show that the $\mathbf{k}$-algebra $A \otimes B$ introduced in Definition 1.3.3 is actually well-defined (i.e., its multiplication and unit satisfy the axioms of a $\mathbf{k}$-algebra).

(b) Let C and D be two $\mathbf{k}$-coalgebras. Show that the $\mathbf{k}$-coalgebra $C \otimes D$ introduced in Definition 1.3.3 is actually well-defined (i.e., its comultiplication and counit satisfy the axioms of a $\mathbf{k}$-coalgebra).

It is straightforward to show that the concept of tensor products of algebras and of coalgebras satisfy the properties one would expect:

- For any three $\mathbf{k}$-coalgebras C, D and E, the $\mathbf{k}$-linear map

  $$(C \otimes D) \otimes E \to C \otimes (D \otimes E)\,, \qquad (c \otimes d) \otimes e \mapsto c \otimes (d \otimes e)$$

  is a coalgebra isomorphism. This allows us to speak of the $\mathbf{k}$-coalgebra $C \otimes D \otimes E$ without worrying about the parenthesization.
- For any two $\mathbf{k}$-coalgebras C and D, the $\mathbf{k}$-linear map

  $$T : C \otimes D \to D \otimes C, \qquad c \otimes d \mapsto d \otimes c$$

  is a coalgebra isomorphism.

[15] To be precise, [87] works with the related concept of *superalgebras*, which are graded by elements of $\mathbb{Z}/2\mathbb{Z}$ rather than $\mathbb{N}$ but use the same sign convention as the topologists have for algebras.

- For any $\mathbf{k}$-coalgebra C, the $\mathbf{k}$-linear maps
$$\begin{array}{lll} C \to \mathbf{k} \otimes C, & c \mapsto 1 \otimes c & \text{and} \\ C \to C \otimes \mathbf{k}, & c \mapsto c \otimes 1 & \end{array}$$
are coalgebra isomorphisms.
- Similar properties hold for algebras instead of coalgebras.

One of the first signs that these definitions interact nicely is the following straightforward proposition.

Proposition 1.3.5. *When A is both a $\mathbf{k}$-algebra and a $\mathbf{k}$-coalgebra, the following are equivalent:*
- *The maps Δ and ϵ are morphisms for the algebra structure (A, m, u).*
- *The maps m and u are morphisms for the coalgebra structure (A, Δ, ϵ).*
- *These four diagrams commute:*

(1.3.4)
$$\begin{tikzcd}
 & A \otimes A \arrow[dl, "\Delta\otimes\Delta"'] \arrow[dr, "m"] & \\
A \otimes A \otimes A \otimes A \arrow[d, "\mathrm{id}\otimes T\otimes \mathrm{id}"'] & & A \arrow[dl, "\Delta"] \\
A \otimes A \otimes A \otimes A \arrow[r, "m\otimes m"'] & A \otimes A &
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes A \arrow[r, "\epsilon\otimes\epsilon"] \arrow[d, "m"'] & \mathbf{k} \otimes \mathbf{k} \arrow[d, "m"] \\
A \arrow[r, "\epsilon"'] & \mathbf{k}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{k} \arrow[r, "u"] \arrow[d, "\Delta"'] & A \arrow[d, "\Delta"] \\
\mathbf{k} \otimes \mathbf{k} \arrow[r, "u\otimes u"'] & A \otimes A
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{k} \arrow[rr, "\mathrm{id}"] \arrow[dr, "u"'] & & \mathbf{k} \\
 & A \arrow[ur, "\epsilon"'] &
\end{tikzcd}$$

Exercise 1.3.6. (a) If A, A', B and B' are four $\mathbf{k}$-algebras, and $f : A \to A'$ and $g : B \to B'$ are two $\mathbf{k}$-algebra homomorphisms, then show that $f \otimes g : A \otimes B \to A' \otimes B'$ is a $\mathbf{k}$-algebra homomorphism.

(b) If C, C', D and D' are four $\mathbf{k}$-coalgebras, and $f : C \to C'$ and $g : D \to D'$ are two $\mathbf{k}$-coalgebra homomorphisms, then show that $f \otimes g : C \otimes D \to C' \otimes D'$ is a $\mathbf{k}$-coalgebra homomorphism.

Definition 1.3.7. Call the $\mathbf{k}$-module A a $\mathbf{k}$-*bialgebra* if it is a $\mathbf{k}$-algebra and $\mathbf{k}$-coalgebra satisfying the three equivalent conditions in Proposition 1.3.5.

Example 1.3.8. For a group G, one can make the group algebra $\mathbf{k}G$ a coalgebra with counit $\mathbf{k}G \xrightarrow{\epsilon} \mathbf{k}$ mapping $t_g \mapsto 1$ for all g in G, and with comultiplication $\mathbf{k}G \xrightarrow{\Delta} \mathbf{k}G \otimes \mathbf{k}G$ given by $\Delta(t_g) := t_g \otimes t_g$. Checking the various diagrams in (1.3.4) commute is easy. For example, one can check the pentagonal diagram on each basis element $t_g \otimes t_h$:

$$\begin{tikzcd}
& t_g \otimes t_h \arrow[dl, "\Delta\otimes\Delta"'] \arrow[dr, "m"] & \\
t_g \otimes t_g \otimes t_h \otimes t_h \arrow[d, "\mathrm{id}\otimes T\otimes\mathrm{id}"'] & & t_{gh} \arrow[ddl, "\Delta"] \\
t_g \otimes t_h \otimes t_g \otimes t_h \arrow[dr, "m\otimes m"'] & & \\
& t_{gh} \otimes t_{gh} &
\end{tikzcd}$$

Remark 1.3.9. In fact, one can think of adding a bialgebra structure to a $\mathbf{k}$-algebra A as a way of making A-modules M, N have an A-module structure on their tensor product $M \otimes N$: the algebra $A \otimes A$ already acts naturally on $M \otimes N$, so one can let a in A act via $\Delta(a)$ in $A \otimes A$. In the theory of group representations over $\mathbf{k}$, that is, $\mathbf{k}G$-modules M, this is how one defines the *diagonal action* of G on $M \otimes N$, namely t_g acts as $t_g \otimes t_g$.

Definition 1.3.10. An element x in a coalgebra for which $\Delta(x) = x \otimes x$ and $\epsilon(x) = 1$ is called *group-like*.

An element x in a bialgebra for which $\Delta(x) = 1 \otimes x + x \otimes 1$ is called *primitive*. We shall also sometimes abbreviate "primitive element" as "primitive".

Example 1.3.11. Let V be a $\mathbf{k}$-module. The *tensor algebra* $T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$ is a coalgebra, with counit ϵ equal to the identity on $V^{\otimes 0} = \mathbf{k}$ and the zero map on $V^{\otimes n}$ for $n > 0$, and with comultiplication defined to make the elements x in $V^{\otimes 1} = V$ all primitive:

$$\Delta(x) := 1 \otimes x + x \otimes 1 \text{ for } x \in V^{\otimes 1}.$$

Since the elements of V generate $T(V)$ as a $\mathbf{k}$-algebra, and since $T(V) \otimes T(V)$ is also an associative $\mathbf{k}$-algebra, the universal property of $T(V)$ as the free associative $\mathbf{k}$-algebra on the generators V allows one to define $T(V) \overset{\Delta}{\to} T(V) \otimes T(V)$ arbitrarily on V, and extend it as an algebra morphism.

It may not be obvious that this Δ is coassociative, but one can prove this as follows. Note that

$$((\mathrm{id} \otimes \Delta) \circ \Delta)(x) = x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x = ((\Delta \otimes \mathrm{id}) \circ \Delta)(x)$$

for every x in V. Hence the two maps $(\mathrm{id} \otimes \Delta) \circ \Delta$ and $(\Delta \otimes \mathrm{id}) \circ \Delta$, considered as algebra morphisms $T(V) \to T(V) \otimes T(V) \otimes T(V)$, must coincide on every element of $T(V)$ since they coincide on V. We leave it as an exercise to check the map ϵ defined as above satisfies the counit axioms (1.2.2).

Here is a sample calculation in $T(V)$ when x, y, z are three elements of V:

$$\begin{aligned}\Delta(xyz) &= \Delta(x)\Delta(y)\Delta(z)\\ &= (1\otimes x + x\otimes 1)(1\otimes y + y\otimes 1)(1\otimes z + z\otimes 1)\\ &= (1\otimes xy + x\otimes y + y\otimes x + xy\otimes 1)(1\otimes z + z\otimes 1)\\ &= 1\otimes xyz + x\otimes yz + y\otimes xz + z\otimes xy\\ &\qquad + xy\otimes z + xz\otimes y + yz\otimes x + xyz\otimes 1.\end{aligned}$$

This illustrates the idea that comultiplication "takes basis elements apart" (and, in the case of $T(V)$, not just basis elements, but any decomposable tensors). Here for any $v_1, v_2, \ldots, v_n$ in V one has

$$\Delta\left(v_1 v_2 \cdots v_n\right) = \sum v_{j_1} \cdots v_{j_r} \otimes v_{k_1} \cdots v_{k_{n-r}}$$

where the sum is over ordered pairs $(j_1, j_2, \ldots, j_r), (k_1, k_2, \ldots, k_{n-r})$ of complementary subwords of the word $(1, 2, \ldots, n)$. [16] Equivalently (and in a more familiar language),

$$\Delta\left(v_1 v_2 \cdots v_n\right) = \sum_{I \subset \{1,2,\ldots,n\}} v_I \otimes v_{\{1,2,\ldots,n\}\setminus I}, \tag{1.3.5}$$

where v_J (for J a subset of $\{1, 2, \ldots, n\}$) denotes the product of all v_j with $j \in J$ in the order of increasing j.

We can rewrite the axioms of a $\mathbf{k}$-bialgebra A using Sweedler notation. Indeed, asking for $\Delta : A \to A\otimes A$ to be a $\mathbf{k}$-algebra morphism is equivalent to requiring that

$$\sum_{(ab)} (ab)_1 \otimes (ab)_2 = \sum_{(a)} \sum_{(b)} a_1 b_1 \otimes a_2 b_2 \qquad \text{for all } a, b \in A \tag{1.3.6}$$

and $\sum_{(1)} 1_1 \otimes 1_2 = 1_A \otimes 1_A$. (The other axioms have already been rewritten or don't need Sweedler notation.)

Recall one can quotient a $\mathbf{k}$-algebra A by a two-sided ideal J to obtain a quotient algebra A/J. An analogous construction can be done for coalgebras using the following concept, which is dual to that of a two-sided ideal:

Definition 1.3.12. In a coalgebra C, a *two-sided coideal* is a $\mathbf{k}$-submodule $J \subset C$ for which

$$\begin{aligned}&\Delta(J) \subset J\otimes C + C\otimes J,\\ &\epsilon(J) = 0.\end{aligned}$$

[16] More formally speaking, the sum is over all permutations $(j_1, j_2, \ldots, j_r, k_1, k_2, \ldots, k_{n-r})$ of $(1, 2, \ldots, n)$ satisfying $j_1 < j_2 < \cdots < j_r$ and $k_1 < k_2 < \cdots < k_{n-r}$.

The quotient **k**-module C/J then inherits a coalgebra structure[17]. Similarly, in a bialgebra A, a subset $J \subset A$ which is both a two-sided ideal and two-sided coideal gives rise to a quotient bialgebra A/J.

Exercise 1.3.13. Let A and C be two **k**-coalgebras, and $f : A \to C$ a surjective coalgebra homomorphism.

(a) If f is surjective, then show that $\ker f$ is a two-sided coideal of A.
(b) If **k** is a field, then show that $\ker f$ is a two-sided coideal of A.

Example 1.3.14. Let V be a **k**-module. The *symmetric algebra* $\mathrm{Sym}(V)$ was defined as the quotient of the tensor algebra $T(V)$ by the two-sided ideal J generated by all *commutators* $[x, y] = xy - yx$ for x, y in V (see Example 1.1.3). Note that x, y are primitive elements in $T(V)$, and the following very reusable calculation shows that *the commutator of two primitives is primitive*:

$$\begin{aligned}
\Delta[x,y] &= \Delta(xy - yx) = \Delta(x)\Delta(y) - \Delta(y)\Delta(x) \\
&\qquad \text{(since } \Delta \text{ is an algebra homomorphism)} \\
&= (1 \otimes x + x \otimes 1)(1 \otimes y + y \otimes 1) \\
&\qquad - (1 \otimes y + y \otimes 1)(1 \otimes x + x \otimes 1) \\
&= 1 \otimes xy - 1 \otimes yx + xy \otimes 1 - yx \otimes 1 \\
&\qquad + x \otimes y + y \otimes x - x \otimes y - y \otimes x \\
&= 1 \otimes (xy - yx) + (xy - yx) \otimes 1 \\
&= 1 \otimes [x,y] + [x,y] \otimes 1. \qquad (1.3.7)
\end{aligned}$$

In particular, the commutators $[x, y]$ have $\Delta[x, y]$ in $J \otimes T(V) + T(V) \otimes J$. They also satisfy $\epsilon([x, y]) = 0$. Since they are generators for J as a two-sided ideal, it is not hard to see this implies $\Delta(J) \subset J \otimes T(V) + T(V) \otimes J$, and $\epsilon(J) = 0$. Thus J is also a two-sided coideal, and $\mathrm{Sym}(V) = T(V)/J$ inherits a bialgebra structure.

In fact we will see in Section 3.1 that symmetric algebras are the universal example of bialgebras which are *graded, connected, commutative, cocommutative.* But first we should define some of these concepts.

Definition 1.3.15. (a) A *graded* **k**-*module*[18] is a **k**-module V equipped with a **k**-module direct sum decomposition $V = \bigoplus_{n \geq 0} V_n$. In this case, the addend V_n (for any given $n \in \mathbb{N}$) is called the *n-th homogeneous component* (or the *n-th graded component*) of the graded **k**-module V. Furthermore, elements x in V_n are said to be *homogeneous* of degree n; occasionally, the notation $\deg(x) = n$ is used to signify this[19]. The decomposition $\bigoplus_{n \geq 0} V_n$ of V (that is, the family of submodules $(V_n)_{n \in \mathbb{N}}$) is called the *grading* of V.

[17]Indeed, $J \otimes C + C \otimes J$ is contained in the kernel of the canonical map $C \otimes C \to (C/J) \otimes (C/J)$; therefore, the condition $\Delta(J) \subset J \otimes C + C \otimes J$ shows that the map $C \xrightarrow{\Delta} C \otimes C \twoheadrightarrow (C/J) \otimes (C/J)$ factors through a map $\overline{\Delta} : C/J \to (C/J) \otimes (C/J)$. Likewise, $\epsilon(J) = 0$ shows that the map $\epsilon : C \to \mathbf{k}$ factors through a map $\overline{\epsilon} : C/J \to \mathbf{k}$. Equipping C/J with these maps $\overline{\Delta}$ and $\overline{\epsilon}$, we obtain a coalgebra (as the commutativity of the required diagrams follows from the corresponding property of C).

[18]also known as an "$\mathbb{N}$-*graded* **k**-*module*"

[19]This notation should not be taken too literally, as it would absurdly imply that $\deg(0)$ "equals" every $n \in \mathbb{N}$ at the same time, since $0 \in V_n$ for all n.

(b) The tensor product $V \otimes W$ of two graded $\mathbf{k}$-modules V and W is, by default, endowed with the graded module structure in which

$$(V \otimes W)_n := \bigoplus_{i+j=n} V_i \otimes W_j.$$

(c) A $\mathbf{k}$-linear map $V \xrightarrow{\varphi} W$ between two graded $\mathbf{k}$-modules is called *graded* if $\varphi(V_n) \subset W_n$ for all n. Graded $\mathbf{k}$-linear maps are also called *homomorphisms of graded* $\mathbf{k}$*-modules*. An *isomorphism of graded* $\mathbf{k}$*-modules* means an invertible graded $\mathbf{k}$-linear map whose inverse is also graded.[20]

(d) Say that a $\mathbf{k}$-algebra (or coalgebra, or bialgebra) is *graded* if it is a graded $\mathbf{k}$-module and all of the relevant structure maps (u, ϵ, m, Δ) are graded.

(e) Say that a graded $\mathbf{k}$-module V is *connected* if $V_0 \cong \mathbf{k}$.

(f) Let V be a graded $\mathbf{k}$-module. Then, a *graded* $\mathbf{k}$*-submodule of* V (sometimes also called a *homogeneous* $\mathbf{k}$*-submodule of* V) means a graded $\mathbf{k}$-module W such that $W \subset V$ as sets, and such that the inclusion map $W \hookrightarrow V$ is a graded $\mathbf{k}$-linear map.
Note that if W is a graded $\mathbf{k}$-submodule of V, then the grading of W is uniquely determined by the underlying set of W and the grading of V – namely, the n-th graded component W_n of W is $W_n = W \cap V_n$ for each $n \in \mathbb{N}$. Thus, we can specify a graded $\mathbf{k}$-submodule of V without explicitly specifying its grading. From this point of view, a graded $\mathbf{k}$-submodule of V can also be defined as a $\mathbf{k}$-submodule W of V satisfying $W = \sum_{n \in \mathbb{N}} (W \cap V_n)$. (This sum is automatically a direct sum, and thus defines a grading on W.)

Example 1.3.16. Let $\mathbf{k}$ be a field. A path-connected space X has its homology and cohomology

$$H_*(X; \mathbf{k}) = \bigoplus_{i \geq 0} H_i(X; \mathbf{k}), \qquad H^*(X; \mathbf{k}) = \bigoplus_{i \geq 0} H^i(X; \mathbf{k})$$

carrying the structure of connected graded coalgebras and algebras, respectively. If in addition, X is a topological group, or even less strongly, a *homotopy-associative* H*-space* (e.g. the *loop space* ΩY on some other space Y), the continuous multiplication map $X \times X \to X$ induces an algebra structure on $H_*(X; \mathbf{k})$ and a coalgebra structure on $H^*(X; \mathbf{k})$, so that each become bialgebras in the topologist's sense (i.e., with the twist as in (1.3.3)), and these bialgebras are dual to each other in a sense soon to be discussed. This was Hopf's motivation: the (co-)homology of a compact Lie group carries bialgebra structure that explains why it takes a certain form; see Cartier [35, §2].

Example 1.3.17. Let V be a graded $\mathbf{k}$-module. Then, its tensor algebra $T(V)$ and its symmetric algebra $\mathrm{Sym}(V)$ are graded Hopf algebras. The grading is given as follows: If $v_1, v_2, \ldots, v_k$ are homogeneous elements of V having degrees $i_1, i_2, \ldots, i_k$, respectively, then the elements $v_1 v_2 \cdots v_k$ of

[20] We shall see in Exercise 1.3.18 that the "whose inverse is also graded" requirement is actually superfluous (i.e., it is automatically satisfied for an invertible graded $\mathbf{k}$-linear map); we are imposing it only in order to stick to our tradition of defining "isomorphisms" as invertible morphisms whose inverses are morphisms as well.

$T(V)$ and $\operatorname{Sym}(V)$ are homogeneous of degree $i_1 + i_2 + \cdots + i_k$. That is, we have

$$\deg(v_1 v_2 \cdots v_k) = \deg(v_1) + \deg(v_2) + \cdots + \deg(v_k)$$

for any homogeneous elements $v_1, v_2, \ldots, v_k$ of V.

Assuming that $V_0 = 0$, the graded algebras $T(V)$ and $\operatorname{Sym}(V)$ are connected. This is a fairly common situation in combinatorics. For example, we will often turn a (non-graded) $\mathbf{k}$-module V into a graded $\mathbf{k}$-module by declaring that all elements of V are homogeneous of degree 1, but at other times, it will make sense to have V live in different (positive) degrees.

Exercise 1.3.18. Let V and W be two graded $\mathbf{k}$-modules. Prove that if $f : V \to W$ is an invertible graded $\mathbf{k}$-linear map, then its inverse $f^{-1} : W \to V$ is also graded.

Exercise 1.3.19. Let $A = \bigoplus_{n \geq 0} A_n$ be a graded $\mathbf{k}$-bialgebra. We denote by $\mathfrak{p}$ the set of all primitive elements of A.

(a) Show that $\mathfrak{p}$ is a graded $\mathbf{k}$-submodule of A (that is, we have $\mathfrak{p} = \bigoplus_{n \geq 0} (\mathfrak{p} \cap A_n)$).
(b) Show that $\mathfrak{p}$ is a two-sided coideal of A.

Exercise 1.3.20. Let A be a connected graded $\mathbf{k}$-bialgebra. Show the following:

(a) The $\mathbf{k}$-submodule $\mathbf{k} = \mathbf{k} \cdot 1_A$ of A lies in A_0.
(b) The map u is an isomorphism $\mathbf{k} \xrightarrow{u} A_0$.
(c) We have $A_0 = \mathbf{k} \cdot 1_A$.
(d) The two-sided ideal $\ker \epsilon$ is the $\mathbf{k}$-module of positive degree elements $I = \bigoplus_{n>0} A_n$.
(e) The map ϵ restricted to A_0 is the inverse isomorphism $A_0 \xrightarrow{\epsilon} \mathbf{k}$ to u.
(f) For every $x \in A$, we have
$$\Delta(x) \in x \otimes 1 + A \otimes I.$$
(g) Every x in I satisfies
$$\Delta(x) = 1 \otimes x + x \otimes 1 + \Delta_+(x), \qquad \text{where } \Delta_+(x) \text{ lies in } I \otimes I.$$
(h) Every $n > 0$ and every $x \in A_n$ satisfy

$$\Delta(x) = 1 \otimes x + x \otimes 1 + \Delta_+(x), \qquad \text{where } \Delta_+(x) \text{ lies in } \sum_{k=1}^{n-1} A_k \otimes A_{n-k}.$$

(Use only the gradedness of the unit u and counit ϵ maps, along with commutativity of diagrams (1.2.2), and (1.3.4) and the connectedness of A.)

Having discussed graded $\mathbf{k}$-modules, let us also define the concept of a *graded basis*, which is the analogue of the notion of a basis in the graded context. Roughly speaking, a graded basis of a graded $\mathbf{k}$-module is a basis that comprises bases of all its homogeneous components. More formally:

Definition 1.3.21. Let $V = \bigoplus_{n \geq 0} V_n$ be a graded $\mathbf{k}$-module. A *graded basis* of the graded $\mathbf{k}$-module V means a basis $\{v_i\}_{i \in I}$ of the $\mathbf{k}$-module V whose indexing set I is partitioned into subsets $I_0, I_1, I_2, \ldots$ (which are allowed to be empty) with the property that, for every $n \in \mathbb{N}$, the subfamily $\{v_i\}_{i \in I_n}$ is a basis of the $\mathbf{k}$-module V_n.

Example 1.3.22. Consider the polynomial ring $\mathbf{k}[x]$ in one variable x over $\mathbf{k}$. This is a graded $\mathbf{k}$-module (graded by the degree of a polynomial; thus, each x^n is homogeneous of degree n). Then, the family $(x^n)_{n\in\mathbb{N}} = (x^0, x^1, x^2, \ldots)$ is a graded basis of $\mathbf{k}[x]$ (presuming that its indexing set $\mathbb{N}$ is partitioned into the one-element subsets $\{0\}, \{1\}, \{2\}, \ldots$). The family $((-x)^n)_{n\in\mathbb{N}} = (x^0, -x^1, x^2, -x^3, \ldots)$ is a graded basis of $\mathbf{k}[x]$ as well. But the family $((1+x)^n)_{n\in\mathbb{N}}$ is not, since it contains non-homogeneous elements.

We end this section by discussing morphisms between bialgebras. They are defined as one would expect:

Definition 1.3.23. A *morphism of bialgebras* (also known as a $\mathbf{k}$-*bialgebra homomorphism*) is a $\mathbf{k}$-linear map $A \xrightarrow{\varphi} B$ between two $\mathbf{k}$-bialgebras A and B that is simultaneously a $\mathbf{k}$-algebra homomorphism and a $\mathbf{k}$-coalgebra homomorphism.

For example, any $\mathbf{k}$-linear map $f : V \to W$ between two $\mathbf{k}$-modules V and W induces a $\mathbf{k}$-linear map $T(f) : T(V) \to T(W)$ between their tensor algebras (which sends each $v_1 v_2 \cdots v_k \in T(V)$ to $f(v_1) f(v_2) \cdots f(v_k) \in T(W)$) as well as a $\mathbf{k}$-linear map $\operatorname{Sym}(f) : \operatorname{Sym}(V) \to \operatorname{Sym}(W)$ between their symmetric algebras; both of these maps $T(f)$ and $\operatorname{Sym}(f)$ are morphisms of bialgebras.

Graded bialgebras come with a special family of endomorphisms, as the following exercise shows:

Exercise 1.3.24. Fix $q \in \mathbf{k}$. Let $A = \bigoplus_{n\in\mathbb{N}} A_n$ be a graded $\mathbf{k}$-bialgebra (where the A_n are the homogeneous components of A). Let $D_q : A \to A$ be the $\mathbf{k}$-module endomorphism of A defined by setting

$$D_q(a) = q^n a \qquad \text{for each } n \in \mathbb{N} \text{ and each } a \in A_n.$$

(It is easy to see that this is well-defined; equivalently, D_q could be defined as the direct sum $\bigoplus_{n\in\mathbb{N}} (q^n \cdot \mathrm{id}_{A_n}) : \bigoplus_{n\in\mathbb{N}} A_n \to \bigoplus_{n\in\mathbb{N}} A_n$ of the maps $q^n \cdot \mathrm{id}_{A_n} : A_n \to A_n$.)

Prove that D_q is a $\mathbf{k}$-bialgebra homomorphism.

The tensor product of two bialgebras is canonically a bialgebra, as the following proposition shows:

Proposition 1.3.25. *Let A and B be two $\mathbf{k}$-bialgebras. Then, $A \otimes B$ is both a $\mathbf{k}$-algebra and a $\mathbf{k}$-coalgebra (by Definition 1.3.3). These two structures, combined, turn $A \otimes B$ into a $\mathbf{k}$-bialgebra.*

Exercise 1.3.26. (a) Prove Proposition 1.3.25.

(b) Let G and H be two groups. Show that the $\mathbf{k}$-bialgebra $\mathbf{k}G \otimes \mathbf{k}H$ (defined as in Proposition 1.3.25) is isomorphic to the $\mathbf{k}$-bialgebra $\mathbf{k}[G \times H]$. (The notation $\mathbf{k}[S]$ is a synonym for $\mathbf{k}S$.)

1.4. **Antipodes and Hopf algebras.**

There is one more piece of structure needed to make a bialgebra a Hopf algebra, although it will come for free in the connected graded case.

Definition 1.4.1. For any coalgebra C and algebra A, one can endow the $\mathbf{k}$-module $\mathrm{Hom}(C, A)$ (which consists of all $\mathbf{k}$-linear maps from C to A) with an associative algebra structure called the *convolution algebra*: Define the product $f \star g$ of two maps f, g in $\mathrm{Hom}(C, A)$ by $(f \star g)(c) = \sum f(c_1)g(c_2)$, using the Sweedler notation[21] $\Delta(c) = \sum c_1 \otimes c_2$. Equivalently, $f \star g$ is the composite

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{m} A .$$

The associativity of this multiplication $\star$ is easy to check (see Exercise 1.4.2 below).

The map $u \circ \epsilon$ is a two-sided identity element for $\star$, meaning that every $f \in \mathrm{Hom}(C, A)$ satisfies

$$\sum f(c_1)\epsilon(c_2) = f(c) = \sum \epsilon(c_1)f(c_2)$$

for all $c \in C$. One sees this by adding a top row to (1.2.2):

$$\begin{tikzcd} A \otimes \mathbf{k} \arrow[r] & A & \mathbf{k} \otimes A \arrow[l] \\ C \otimes \mathbf{k} \arrow[r] \arrow[u, "f \otimes \mathrm{id}"] & C \arrow[u, "f"] & \mathbf{k} \otimes C \arrow[l] \arrow[u, "\mathrm{id} \otimes f"'] \\ C \otimes C \arrow[u, "\mathrm{id} \otimes \epsilon"] & C \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[u, "\mathrm{id}"] & C \otimes C \arrow[u, "\epsilon \otimes \mathrm{id}"'] \end{tikzcd} \tag{1.4.1}$$

In particular, when one has a bialgebra A, the convolution product $\star$ gives an associative algebra structure on $\mathrm{End}(A) := \mathrm{Hom}(A, A)$.

Exercise 1.4.2. Let C be a $\mathbf{k}$-coalgebra and A be a $\mathbf{k}$-algebra. Show that the binary operation $\star$ on $\mathrm{Hom}\,(C, A)$ is associative.

The product $f \star g$ of two elements f and g in a convolution algebra $\mathrm{Hom}(C, A)$ is often called their *convolution*.

The following simple (but useful) property of convolution algebras says essentially that the $\mathbf{k}$-algebra $(\mathrm{Hom}\,(C, A)\,, \star)$ is a covariant functor in A and a contravariant functor in C, acting on morphisms by pre- and post-composition:

Proposition 1.4.3. *Let C and C' be two $\mathbf{k}$-coalgebras, and let A and A' be two $\mathbf{k}$-algebras. Let $\gamma : C \to C'$ be a $\mathbf{k}$-coalgebra morphism. Let $\alpha : A \to A'$ be a $\mathbf{k}$-algebra morphism.*

The map

$$\mathrm{Hom}\,(C', A) \to \mathrm{Hom}\,(C, A')\,, \qquad f \mapsto \alpha \circ f \circ \gamma$$

is a $\mathbf{k}$-algebra homomorphism from the convolution algebra $(\mathrm{Hom}\,(C', A)\,, \star)$ *to the convolution algebra* $(\mathrm{Hom}\,(C, A')\,, \star)$.

Proof of Proposition 1.4.3. Denote this map by φ. We must show that φ is a $\mathbf{k}$-algebra homomorphism.

Recall that α is an algebra morphism; thus, $\alpha \circ m_A = m_{A'} \circ (\alpha \otimes \alpha)$ and $\alpha \circ u_A = u_{A'}$. Also, γ is a coalgebra morphism; thus, $\Delta_{C'} \circ \gamma = (\gamma \otimes \gamma) \circ \Delta_C$ and $\epsilon_{C'} \circ \gamma = \epsilon_C$.

[21]See the paragraph around (1.2.3) for the meaning of this notation.

Now, the definition of φ yields $\varphi(u_A \circ \epsilon_{C'}) = \underbrace{\alpha \circ u_A}_{=u_{A'}} \circ \underbrace{\epsilon_{C'} \circ \gamma}_{=\epsilon_C} = u_{A'} \circ \epsilon_C$; in other words, φ sends the unity of the algebra $(\operatorname{Hom}(C', A), \star)$ to the unity of the algebra $(\operatorname{Hom}(C, A'), \star)$.

Furthermore, every $f \in \operatorname{Hom}(C', A)$ and $g \in \operatorname{Hom}(C', A)$ satisfy

$$\begin{aligned}
\varphi(f \star g) &= \alpha \circ \underbrace{(f \star g)}_{=m_A \circ (f \otimes g) \circ \Delta_{C'}} \circ \gamma \\
&= \underbrace{\alpha \circ m_A}_{=m_{A'} \circ (\alpha \otimes \alpha)} \circ (f \otimes g) \circ \underbrace{\Delta_{C'} \circ \gamma}_{=(\gamma \otimes \gamma) \circ \Delta_C} \\
&= m_{A'} \circ \underbrace{(\alpha \otimes \alpha) \circ (f \otimes g) \circ (\gamma \otimes \gamma)}_{=(\alpha \circ f \circ \gamma) \otimes (\alpha \circ g \circ \gamma)} \circ \Delta_C \\
&= m_{A'} \circ ((\alpha \circ f \circ \gamma) \otimes (\alpha \circ g \circ \gamma)) \circ \Delta_C \\
&= \underbrace{(\alpha \circ f \circ \gamma)}_{=\varphi(f)} \star \underbrace{(\alpha \circ g \circ \gamma)}_{=\varphi(g)} = \varphi(f) \star \varphi(g). \qquad (1.4.2)
\end{aligned}$$

Thus, φ is a $\mathbf{k}$-algebra homomorphism (since φ is a $\mathbf{k}$-linear map and sends the unity of the algebra $(\operatorname{Hom}(C', A), \star)$ to the unity of the algebra $(\operatorname{Hom}(C, A'), \star)$). $\square$

Exercise 1.4.4. Let C and D be two $\mathbf{k}$-coalgebras, and let A and B be two $\mathbf{k}$-algebras. Prove that:

(a) If $f : C \to A$, $f' : C \to A$, $g : D \to B$ and $g' : D \to B$ are four $\mathbf{k}$-linear maps, then

$$(f \otimes g) \star (f' \otimes g') = (f \star f') \otimes (g \star g')$$

in the convolution algebra $\operatorname{Hom}(C \otimes D, A \otimes B)$.

(b) Let R be the $\mathbf{k}$-linear map

$$(\operatorname{Hom}(C, A), \star) \otimes (\operatorname{Hom}(D, B), \star) \to (\operatorname{Hom}(C \otimes D, A \otimes B), \star)$$

which sends every tensor $f \otimes g \in (\operatorname{Hom}(C, A), \star) \otimes (\operatorname{Hom}(D, B), \star)$ to the map $f \otimes g : C \otimes D \to A \otimes B$. (Notice that the tensor $f \otimes g$ and the map $f \otimes g$ are different things which happen to be written in the same way.) Then, R is a $\mathbf{k}$-algebra homomorphism.

Exercise 1.4.5. Let C and D be two $\mathbf{k}$-coalgebras. Let A be a $\mathbf{k}$-algebra. Let Φ be the canonical $\mathbf{k}$-module isomorphism

$$\operatorname{Hom}(C \otimes D, A) \to \operatorname{Hom}(C, \operatorname{Hom}(D, A))$$

(defined by $((\Phi(f))(c))(d) = f(c \otimes d)$ for all $f \in \operatorname{Hom}(C \otimes D, A)$, $c \in C$ and $d \in D$). Prove that Φ is a $\mathbf{k}$-algebra isomorphism

$$(\operatorname{Hom}(C \otimes D, A), \star) \to (\operatorname{Hom}(C, (\operatorname{Hom}(D, A), \star)), \star).$$

Definition 1.4.6. A bialgebra A is called a *Hopf algebra* if there is an element S (called an *antipode* for A) in $\operatorname{End}(A)$ which is a 2-sided inverse under $\star$ for the identity map id_A. In other words, this diagram commutes:

(1.4.3)
$$\begin{tikzcd}
 & A \otimes A \arrow[rr, "S \otimes \mathrm{id}_A"] & & A \otimes A \arrow[dr, "m"] & \\
A \arrow[ur, "\Delta"] \arrow[rr, "\epsilon"] \arrow[dr, "\Delta"'] & & \mathbf{k} \arrow[rr, "u"] & & A \\
 & A \otimes A \arrow[rr, "\mathrm{id}_A \otimes S"'] & & A \otimes A \arrow[ur, "m"'] &
\end{tikzcd}$$

Or equivalently, if we follow the Sweedler notation in writing $\Delta(a) = \sum a_1 \otimes a_2$, then

$$\sum_{(a)} S(a_1)a_2 = u(\epsilon(a)) = \sum_{(a)} a_1 S(a_2). \tag{1.4.4}$$

Example 1.4.7. For a group algebra $\mathbf{k}G$, one can define an antipode $\mathbf{k}$-linearly via $S(t_g) = t_{g^{-1}}$. The top pentagon in the above diagram commutes because

$$(S \star \mathrm{id})(t_g) = m((S \otimes \mathrm{id})(t_g \otimes t_g)) = S(t_g)t_g = t_{g^{-1}}t_g = t_e = (u \circ \epsilon)(t_g).$$

Note that when it exists, the antipode S is unique, as with all 2-sided inverses in associative algebras: if S, S' are both 2-sided $\star$-inverses to id_A then

$$S' = (u \circ \epsilon) \star S' = (S \star \mathrm{id}_A) \star S' = S \star (\mathrm{id}_A \star S') = S \star (u \circ \epsilon) = S.$$

Thus, we can speak of "*the antipode*" of a Hopf algebra.

Unlike the comultiplication Δ, the antipode S of a Hopf algebra is not always an algebra homomorphism. It is instead an algebra *anti-homomorphism*, a notion we shall now introduce:

Definition 1.4.8. (a) For any two $\mathbf{k}$-modules U and V, we let $T_{U,V} : U \otimes V \to V \otimes U$ be the $\mathbf{k}$-linear map $U \otimes V \to V \otimes U$ sending every $u \otimes v$ to $v \otimes u$. This map $T_{U,V}$ is called the *twist map* for U and V.

(b) A $\mathbf{k}$-*algebra anti-homomorphism* means a $\mathbf{k}$-linear map $f : A \to B$ between two $\mathbf{k}$-algebras A and B which satisfies $f \circ m_A = m_B \circ (f \otimes f) \circ T_{A,A}$ and $f \circ u_A = u_B$.

(c) A $\mathbf{k}$-*coalgebra anti-homomorphism* means a $\mathbf{k}$-linear map $f : C \to D$ between two $\mathbf{k}$-coalgebras C and D which satisfies $\Delta_D \circ f = T_{D,D} \circ (f \otimes f) \circ \Delta_C$ and $\epsilon_D \circ f = \epsilon_C$.

(d) A $\mathbf{k}$-*algebra anti-endomorphism* of a $\mathbf{k}$-algebra A means a $\mathbf{k}$-algebra anti-homomorphism from A to A.

(e) A $\mathbf{k}$-*coalgebra anti-endomorphism* of a $\mathbf{k}$-coalgebra C means a $\mathbf{k}$-coalgebra anti-homomorphism from C to C.

Parts (b) and (c) of Definition 1.4.8 can be restated in terms of elements:

- A $\mathbf{k}$-linear map $f : A \to B$ between two $\mathbf{k}$-algebras A and B is a $\mathbf{k}$-algebra anti-homomorphism if and only if it satisfies $f(ab) = f(b) f(a)$ for all $a, b \in A$ as well as $f(1) = 1$.
- A $\mathbf{k}$-linear map $f : C \to D$ between two $\mathbf{k}$-coalgebras C and D is a $\mathbf{k}$-coalgebra anti-homomorphism if and only if it satisfies $\sum_{(f(c))} (f(c))_1 \otimes (f(c))_2 = \sum_{(c)} f(c_2) \otimes f(c_1)$ and $\epsilon(f(c)) = \epsilon(c)$ for all $c \in C$.

Example 1.4.9. Let $n \in \mathbb{N}$, and consider the $\mathbf{k}$-algebra $\mathbf{k}^{n\times n}$ of $n \times n$-matrices over $\mathbf{k}$. The map $\mathbf{k}^{n\times n} \to \mathbf{k}^{n\times n}$ that sends each matrix A to its transpose A^T is a $\mathbf{k}$-algebra anti-endomorphism of $\mathbf{k}^{n\times n}$.

We warn the reader that the composition of two $\mathbf{k}$-algebra anti-homomorphisms is not generally a $\mathbf{k}$-algebra anti-homomorphism again, but rather a $\mathbf{k}$-algebra homomorphism. The same applies to coalgebra anti-homomorphisms. Other than that, however, anti-homomorphisms share many of the helpful properties of homomorphisms. In particular, two $\mathbf{k}$-algebra anti-homomorphisms are identical if they agree on a generating set of their domain. Thus, the next proposition is useful when one wants to check that a certain map *is* the antipode in a particular Hopf algebra, by checking it on an algebra generating set.

Proposition 1.4.10. *The antipode S in a Hopf algebra A is an algebra anti-endomorphism: $S(1) = 1$, and $S(ab) = S(b)S(a)$ for all a, b in A.*

Proof. This is surprisingly nontrivial; the following argument comes from [213, proof of Proposition 4.0.1].

Since Δ is an algebra morphism, one has $\Delta(1) = 1 \otimes 1$, and therefore $1 = u\epsilon(1) = S(1) \cdot 1 = S(1)$.

To show $S(ab) = S(b)S(a)$, consider $A \otimes A$ as a coalgebra and A as an algebra. Then $\operatorname{Hom}(A \otimes A, A)$ is an associative algebra with a convolution product $\circledast$ (to be distinguished from the convolution $\star$ on $\operatorname{End}(A)$), having two-sided identity element $u_A\epsilon_{A\otimes A}$. We define three elements f, g, h of $\operatorname{Hom}(A \otimes A, A)$ by

$$\begin{aligned} f(a \otimes b) &= ab, \\ g(a \otimes b) &= S(b)S(a), \\ h(a \otimes b) &= S(ab). \end{aligned}$$

We will show that these three elements have the property that

$$h \circledast f = u_A\epsilon_{A\otimes A} = f \circledast g, \tag{1.4.5}$$

which would then show the desired equality $h = g$ via associativity:

$$h = h \circledast (u_A\epsilon_{A\otimes A}) = h \circledast (f \circledast g) = (h \circledast f) \circledast g = (u_A\epsilon_{A\otimes A}) \circledast g = g.$$

So we evaluate the three elements in (1.4.5) on $a \otimes b$. To do so, we use Sweedler notation – i.e., we assume $\Delta(a) = \sum_{(a)} a_1 \otimes a_2$ and $\Delta(b) =$

$\sum_{(b)} b_1 \otimes b_2$, and hence $\Delta(ab) = \sum_{(a),(b)} a_1 b_1 \otimes a_2 b_2$ (by (1.3.6)); then,

$$(u_A \epsilon_{A \otimes A})(a \otimes b) = u_A(\epsilon_A(a)\epsilon_A(b)) = u_A(\epsilon_A(ab)).$$

$$\begin{aligned}(h \circledast f)(a \otimes b) &= \sum_{(a),(b)} h(a_1 \otimes b_1) f(a_2 \otimes b_2) = \sum_{(a),(b)} S(a_1 b_1) a_2 b_2 \\ &= (S \star \mathrm{id}_A)(ab) = u_A(\epsilon_A(ab)).\end{aligned}$$

$$\begin{aligned}(f \circledast g)(a \otimes b) &= \sum_{(a),(b)} f(a_1 \otimes b_1) g(a_2 \otimes b_2) = \sum_{(a),(b)} a_1 b_1 S(b_2) S(a_2) \\ &= \sum_{(a)} a_1 \cdot (\mathrm{id}_A \star S)(b) \cdot S(a_2) = u_A(\epsilon_A(b)) \sum_{(a)} a_1 S(a_2) \\ &= u_A(\epsilon_A(b)) u_A(\epsilon_A(a)) = u_A(\epsilon_A(ab)).\end{aligned}$$

These results are equal, so that (1.4.5) holds, and we conclude that $h = g$ as explained above. $\square$

Remark 1.4.11. Recall from Remark 1.3.9 that the comultiplication on a bialgebra A allows one to define an A-module structure on the tensor product $M \otimes N$ of two A-modules M, N. Similarly, the anti-endomorphism S in a Hopf algebra allows one to turn *left* A-modules into *right* A-modules, or vice-versa.[22] E.g., left A-modules M naturally have a right A-module structure on the dual $\mathbf{k}$-module $M^* := \mathrm{Hom}(M, \mathbf{k})$, defined via $(fa)(m) := f(am)$ for f in M^* and a in A. The antipode S can be used to turn this back into a left A-module M^*, via $(af)(m) = f(S(a)m)$.

For groups G and left $\mathbf{k}G$-modules (group representations) M, this is how one defines the *contragredient action* of G on M^*, namely t_g acts as $(t_g f)(m) = f(t_{g^{-1}} m)$.

More generally, if A is a Hopf algebra and M and N are two left A-modules, then $\mathrm{Hom}(M, N)$ (the Hom here means $\mathrm{Hom}_{\mathbf{k}}$, not Hom_A) canonically becomes a left A-module by setting

$$(af)(m) = \sum_{(a)} a_1 f(S(a_2) m) \qquad \text{for all } a \in A,\ f \in \mathrm{Hom}(M, N) \text{ and } m \in M.$$

[22] Be warned that these two transformations are not mutually inverse! Turning a left A-module into a right one and then again into a left one using the antipode might lead to a non-isomorphic A-module, unless the antipode S satisfies $S^2 = \mathrm{id}$.

[23] When A is the group algebra $\mathbf{k}G$ of a group G, this leads to

$$(t_g f)(m) = t_g f(t_{g^{-1}} m) \qquad \text{for all } g \in G,\ f \in \operatorname{Hom}(M,N) \text{ and } m \in M.$$

This is precisely how one commonly makes $\operatorname{Hom}(M,N)$ a representation of G for two representations M and N.

Along the same lines, whenever A is a $\mathbf{k}$-bialgebra, we are supposed to think of the counit $A \xrightarrow{\epsilon} \mathbf{k}$ as giving a way to make $\mathbf{k}$ into a *trivial* A-module. This A-module $\mathbf{k}$ behaves as one would expect: the canonical isomorphisms $\mathbf{k} \otimes M \to M$, $M \otimes \mathbf{k} \to M$ and (if A is a Hopf algebra) $\operatorname{Hom}(M, \mathbf{k}) \to M^*$ are A-module isomorphisms for any A-module M.

Corollary 1.4.12. *Let A be a commutative Hopf algebra. Then, its antipode is an involution: $S^2 = \mathrm{id}_A$.*

Proof. One checks that $S^2 = S \circ S$ is a right $\star$-inverse to S, as follows:

$$\begin{aligned}
(S \star S^2)(a) &= \sum_{(a)} S(a_1) S^2(a_2) \\
&= S\left(\sum_{(a)} S(a_2) a_1\right) && \text{(by Proposition 1.4.10)} \\
&= S\left(\sum_{(a)} a_1 S(a_2)\right) && \text{(by commutativity of } A\text{)} \\
&= S\left(u(\epsilon(a))\right) \\
&= u(\epsilon(a)) \qquad \text{(since } S(1) = 1 \text{ by Proposition 1.4.10)}.
\end{aligned}$$

Since S itself is the $\star$-inverse to id_A, this shows that $S^2 = \mathrm{id}_A$. ☐

Remark 1.4.13. We won't need it, but it is easy to adapt the above proof to show that $S^2 = \mathrm{id}_A$ also holds for *cocommutative* Hopf algebras (the dual notion to commutativity; see Definition 1.5.2 below for the precise definition); see [157, Corollary 1.5.12] or [213, Proposition 4.0.1 6)] or Exercise 1.5.13 below. For a general Hopf algebra which is not finite-dimensional over a field $\mathbf{k}$, the antipode S may not even have finite order, even in the connected graded setting. E.g., Aguiar and Sottile [7] show that the Malvenuto-Reutenauer Hopf algebra of permutations has antipode of infinite order. In general, antipodes need not even be invertible [214].

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[23] In more abstract terms, this A-module structure is given by the composition

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{\mathrm{id}_A \otimes S} A \otimes A^{\mathrm{op}} \longrightarrow \operatorname{End}(\operatorname{Hom}(M,N)),$$

where the last arrow is the morphism

$$\begin{aligned}
A \otimes A^{\mathrm{op}} &\longrightarrow \operatorname{End}(\operatorname{Hom}(M,N)), \\
a \otimes b &\longmapsto (f \mapsto (M \to N,\ m \mapsto af(bm))).
\end{aligned}$$

Here, A^{op} denotes the *opposite algebra* of A, which is the $\mathbf{k}$-algebra differing from A only in the multiplication being twisted (the product of a and b in A^{op} is defined to be the product of b and a in A). As $\mathbf{k}$-modules, $A^{\mathrm{op}} = A$, but we prefer to use A^{op} instead of A here to ensure that all morphisms in the above composition are algebra morphisms.

Proposition 1.4.14. *Let A and B be two Hopf algebras. Then, the* $\mathbf{k}$*-bialgebra $A \otimes B$ (defined as in Proposition 1.3.25) is a Hopf algebra. The antipode of this Hopf algebra $A \otimes B$ is the map $S_A \otimes S_B : A \otimes B \to A \otimes B$, where S_A and S_B are the antipodes of the Hopf algebras A and B.*

Exercise 1.4.15. Prove Proposition 1.4.14.

In our frequent setting of connected graded bialgebras, antipodes come for free.

Proposition 1.4.16. *A connected graded bialgebra A has a unique antipode S, which is a graded map $A \xrightarrow{S} A$, endowing it with a Hopf structure.*

Proof. Let us try to define a ($\mathbf{k}$-linear) left $\star$-inverse S to id_A on each homogeneous component A_n, via induction on n.

In the base case $n = 0$, Proposition 1.4.10 and its proof show that one must define $S(1) = 1$ so S is the identity on $A_0 = \mathbf{k}$.

In the inductive step, recall from Exercise 1.3.20(h) that a homogeneous element a of degree $n > 0$ has $\Delta(a) = a \otimes 1 + \sum a'_1 \otimes a'_2$, with each $\deg(a'_1) < n$. (Here $\sum a'_1 \otimes a'_2$ stands for a sum of tensors $a'_{1,k} \otimes a'_{2,k}$, with each $a'_{1,k}$ being homogeneous of degree $\deg(a'_{1,k}) < n$. This is a slight variation on Sweedler notation.) Hence in order to have $S \star \mathrm{id}_A = u\epsilon$, one must define $S(a)$ in such a way that $S(a) \cdot 1 + \sum S(a'_1)a'_2 = u\epsilon(a) = 0$ and hence $S(a) := -\sum S(a'_1)a'_2$, where $S(a'_1)$ have already been uniquely defined by induction (since $\deg(a'_{1,k}) < n$). This does indeed define such a left $\star$-inverse S to id_A, by induction. It is also a graded map by induction.

The same argument shows how to define a right $\star$-inverse S' to id_A. Then $S = S'$ is a two-sided $\star$-inverse to id_A by the associativity of $\star$. $\square$

Here is another consequence of the fact that $S(1) = 1$.

Proposition 1.4.17. *In bialgebras, primitive elements x have $\epsilon(x) = 0$, and in Hopf algebras, they have $S(x) = -x$.*

Proof. In a bialgebra, $\epsilon(1) = 1$. Hence $\Delta(x) = 1 \otimes x + x \otimes 1$ implies via (1.2.2) that $1 \cdot \epsilon(x) + \epsilon(1)x = x$, so $\epsilon(x) = 0$. It also implies via (1.4.3) that $S(x)1 + S(1)x = u\epsilon(x) = u(0) = 0$, so $S(x) = -x$. $\square$

Thus, whenever A is a Hopf algebra generated as an algebra by its primitive elements, S is its unique $\mathbf{k}$-algebra anti-endomorphism that negates all primitive elements.

Example 1.4.18. The tensor and symmetric algebras $T(V)$ and $\mathrm{Sym}(V)$ are each generated by V, and each element of V is primitive when regarded as an element of either of them. Hence one has in $T(V)$ that

$$\begin{aligned} S(x_{i_1} x_{i_2} \cdots x_{i_k}) &= (-x_{i_k}) \cdots (-x_{i_2})(-x_{i_1}) \\ &= (-1)^k x_{i_k} \cdots x_{i_2} x_{i_1} \end{aligned} \tag{1.4.6}$$

for each word $(i_1, \ldots, i_k)$ in the alphabet I if V is a free $\mathbf{k}$-module with basis $\{x_i\}_{i \in I}$. The same holds in $\mathrm{Sym}(V)$ for each multiset $\{i_1, \ldots, i_k\}_{\text{multiset}}$, recalling that the monomials are now commutative. In other words, for a commutative polynomial $f(x_1, x_2, \ldots, x_n)$ in $\mathrm{Sym}(V)$, the antipode S sends $f(x_1, x_2, \ldots, x_n)$ to $f(-x_1, -x_2, \ldots, -x_n)$, negating all the variables.

The antipode for a connected graded Hopf algebra has an interesting formula due to Takeuchi [214], reminiscent of P. Hall's formula for the Möbius function of a poset[24]. For the sake of stating this, consider (for every $k \in \mathbb{N}$) the k-fold *tensor power* $A^{\otimes k} = A \otimes \cdots \otimes A$ (defined in Example 1.1.2) and define *iterated multiplication and comultiplication* maps

$$A^{\otimes k} \xrightarrow{m^{(k-1)}} A \qquad \text{and} \qquad A \xrightarrow{\Delta^{(k-1)}} A^{\otimes k}$$

by induction over k, setting $m^{(-1)} = u$, $\Delta^{(-1)} = \epsilon$, $m^{(0)} = \Delta^{(0)} = \mathrm{id}_A$, and

$$\begin{aligned} m^{(k)} &= m \circ (\mathrm{id}_A \otimes m^{(k-1)}) && \text{for every } k \geq 1; \\ \Delta^{(k)} &= (\mathrm{id}_A \otimes \Delta^{(k-1)}) \circ \Delta && \text{for every } k \geq 1. \end{aligned}$$

Using associativity and coassociativity, one can see that for $k \geq 1$ these maps also satisfy

$$\begin{aligned} m^{(k)} &= m \circ (m^{(k-1)} \otimes \mathrm{id}_A) && \text{for every } k \geq 1; \\ \Delta^{(k)} &= (\Delta^{(k-1)} \otimes \mathrm{id}_A) \circ \Delta && \text{for every } k \geq 1 \end{aligned}$$

(so we could just as well have used $\mathrm{id}_A \otimes m^{(k-1)}$ instead of $m^{(k-1)} \otimes \mathrm{id}_A$ in defining them) and further symmetry properties (see Exercise 1.4.19 and Exercise 1.4.20). They are how one gives meaning to the right sides of these equations:

$$m^{(k)}(a^{(1)} \otimes \cdots \otimes a^{(k+1)}) = a^{(1)} \cdots a^{(k+1)};$$
$$\Delta^{(k)}(b) = \sum b_1 \otimes \cdots \otimes b_{k+1} \text{ in Sweedler notation.}$$

Exercise 1.4.19. Let A be a $\mathbf{k}$-algebra. Let us define, for every $k \in \mathbb{N}$, a $\mathbf{k}$-linear map $m^{(k)} : A^{\otimes(k+1)} \to A$. Namely, we define these maps by induction over k, with the induction base $m^{(0)} = \mathrm{id}_A$, and with the induction step $m^{(k)} = m \circ \left(\mathrm{id}_A \otimes m^{(k-1)}\right)$ for every $k \geq 1$. (This generalizes our definition of $m^{(k)}$ for Hopf algebras A given above, except for $m^{(-1)}$ which we have omitted.)

(a) Show that $m^{(k)} = m \circ \left(m^{(i)} \otimes m^{(k-1-i)}\right)$ for every $k \geq 0$ and $0 \leq i \leq k-1$.
(b) Show that $m^{(k)} = m \circ \left(m^{(k-1)} \otimes \mathrm{id}_A\right)$ for every $k \geq 1$.
(c) Show that $m^{(k)} = m^{(k-1)} \circ (\mathrm{id}_{A^{\otimes i}} \otimes m \otimes \mathrm{id}_{A^{\otimes(k-1-i)}})$ for every $k \geq 0$ and $0 \leq i \leq k-1$.
(d) Show that $m^{(k)} = m^{(k-1)} \circ (\mathrm{id}_{A^{\otimes(k-1)}} \otimes m) = m^{(k-1)} \circ (m \otimes \mathrm{id}_{A^{\otimes(k-1)}})$ for every $k \geq 1$.

Exercise 1.4.20. Let C be a $\mathbf{k}$-coalgebra. Let us define, for every $k \in \mathbb{N}$, a $\mathbf{k}$-linear map $\Delta^{(k)} : C \to C^{\otimes(k+1)}$. Namely, we define these maps by induction over k, with the induction base $\Delta^{(0)} = \mathrm{id}_C$, and with the induction step $\Delta^{(k)} = \left(\mathrm{id}_C \otimes \Delta^{(k-1)}\right) \circ \Delta$ for every $k \geq 1$. (This generalizes our definition of $\Delta^{(k)}$ for Hopf algebras A given above, except for $\Delta^{(-1)}$ which we have omitted.)

(a) Show that $\Delta^{(k)} = \left(\Delta^{(i)} \otimes \Delta^{(k-1-i)}\right) \circ \Delta$ for every $k \geq 0$ and $0 \leq i \leq k-1$.
(b) Show that $\Delta^{(k)} = \left(\Delta^{(k-1)} \otimes \mathrm{id}_C\right) \circ \Delta$ for every $k \geq 1$.

[24]In fact, for incidence Hopf algebras, Takeuchi's formula generalizes Hall's formula– see Corollary 7.2.3.

(c) Show that $\Delta^{(k)} = \left(\mathrm{id}_{C^{\otimes i}} \otimes \Delta \otimes \mathrm{id}_{C^{\otimes (k-1-i)}}\right) \circ \Delta^{(k-1)}$ for every $k \geq 0$ and $0 \leq i \leq k-1$.

(d) Show that $\Delta^{(k)} = \left(\mathrm{id}_{C^{\otimes (k-1)}} \otimes \Delta\right) \circ \Delta^{(k-1)} = \left(\Delta \otimes \mathrm{id}_{C^{\otimes (k-1)}}\right) \circ \Delta^{(k-1)}$ for every $k \geq 1$.

Remark 1.4.21. Exercise 1.4.19 holds more generally for nonunital associative algebras A (that is, $\mathbf{k}$-modules A equipped with a $\mathbf{k}$-linear map $m : A \otimes A \to A$ such that the diagram (1.1.1) is commutative, but not necessarily admitting a unit map u). Similarly, Exercise 1.4.20 holds for non-counital coassociative coalgebras C. The existence of a unit in A, respectively a counit in C, allows slightly extending these two exercises by additionally introducing maps $m^{(-1)} = u : \mathbf{k} \to A$ and $\Delta^{(-1)} = \epsilon : C \to \mathbf{k}$; however, not much is gained from this extension.[25]

Exercise 1.4.22. For every $k \in \mathbb{N}$ and every $\mathbf{k}$-bialgebra H, consider the map $\Delta_H^{(k)} : H \to H^{\otimes (k+1)}$ (this is the map $\Delta^{(k)}$ defined as in Exercise 1.4.20 for $C = H$), and the map $m_H^{(k)} : H^{\otimes (k+1)} \to H$ (this is the map $m^{(k)}$ defined as in Exercise 1.4.19 for $A = H$).

Let H be a $\mathbf{k}$-bialgebra. Let $k \in \mathbb{N}$. Show that:[26]

(a) The map $m_H^{(k)} : H^{\otimes (k+1)} \to H$ is a $\mathbf{k}$-coalgebra homomorphism.

(b) The map $\Delta_H^{(k)} : H \to H^{\otimes (k+1)}$ is a $\mathbf{k}$-algebra homomorphism.

(c) We have $m_{H^{\otimes (k+1)}}^{(\ell)} \circ \left(\Delta_H^{(k)}\right)^{\otimes (\ell+1)} = \Delta_H^{(k)} \circ m_H^{(\ell)}$ for every $\ell \in \mathbb{N}$.

(d) We have $\left(m_H^{(\ell)}\right)^{\otimes (k+1)} \circ \Delta_{H^{\otimes (\ell+1)}}^{(k)} = \Delta_H^{(k)} \circ m_H^{(\ell)}$ for every $\ell \in \mathbb{N}$.

The iterated multiplication and comultiplication maps allow explicitly computing the convolution of multiple maps; the following formula will often be used without explicit mention:

Exercise 1.4.23. Let C be a $\mathbf{k}$-coalgebra, and A be a $\mathbf{k}$-algebra. Let $k \in \mathbb{N}$. Let $f_1, f_2, \ldots, f_k$ be k elements of $\operatorname{Hom}(C, A)$. Show that

$$f_1 \star f_2 \star \cdots \star f_k = m_A^{(k-1)} \circ (f_1 \otimes f_2 \otimes \cdots \otimes f_k) \circ \Delta_C^{(k-1)}.$$

We are now ready to state Takeuchi's formula for the antipode:

Proposition 1.4.24. *In a connected graded Hopf algebra A, the antipode has formula*

$$\begin{aligned} S &= \sum_{k \geq 0} (-1)^k m^{(k-1)} f^{\otimes k} \Delta^{(k-1)} \\ &= u\epsilon - f + m \circ f^{\otimes 2} \circ \Delta - m^{(2)} \circ f^{\otimes 3} \circ \Delta^{(2)} + \cdots \end{aligned} \tag{1.4.7}$$

where $f := \mathrm{id}_A - u\epsilon$ in $\operatorname{End}(A)$.

Proof. We argue as in [214, proof of Lemma 14] or [7, §5]. For any f in $\operatorname{End}(A)$, the following explicit formula expresses its k-fold convolution

[25]The identity $m^{(k)} = m \circ \left(\mathrm{id}_A \otimes m^{(k-1)}\right)$ for a $\mathbf{k}$-algebra A still holds when $k = 0$ if it is interpreted in the right way (viz., if A is identified with $A \otimes \mathbf{k}$ using the canonical homomorphism).

[26]The following statements are taken from [167]; specifically, part (c) is [167, Lem. 1.8].

power $f^{\star k} := f \star \cdots \star f$ in terms of its tensor powers $f^{\otimes k} := f \otimes \cdots \otimes f$ (according to Exercise 1.4.23):

$$f^{\star k} = m^{(k-1)} \circ f^{\otimes k} \circ \Delta^{(k-1)}.$$

Therefore any f annihilating A_0 will be *locally $\star$-nilpotent* on A, meaning that for each n one has that A_n is annihilated by $f^{\star m}$ for every $m > n$: homogeneity forces that for a in A_n, every summand of $\Delta^{(m-1)}(a)$ must contain among its m tensor factors at least one factor lying in A_0, so each summand is annihilated by $f^{\otimes m}$, and $f^{\star m}(a) = 0$.

In particular such f have the property that $u\epsilon + f$ has as two-sided $\star$-inverse

$$\begin{aligned}(u\epsilon + f)^{\star(-1)} &= u\epsilon - f + f \star f - f \star f \star f + \cdots \\ &= \sum_{k \geq 0} (-1)^k f^{\star k} = \sum_{k \geq 0} (-1)^k m^{(k-1)} \circ f^{\otimes k} \circ \Delta^{(k-1)}.\end{aligned}$$

The proposition follows upon taking $f := \mathrm{id}_A - u\epsilon$, which annihilates A_0. $\square$

Remark 1.4.25. In fact, one can see that Takeuchi's formula applies more generally to define an antipode $A \xrightarrow{S} A$ in any (not necessarily graded) bialgebra A where the map $\mathrm{id}_A - u\epsilon$ is locally $\star$-nilpotent.

It is also worth noting that the proof of Proposition 1.4.24 gives an alternate proof of Proposition 1.4.16.

To finish our discussion of antipodes, we mention some properties (taken from [213, Lemma 4.0.3]) relating antipodes to convolutional inverses.

Proposition 1.4.26. *Let H be a Hopf algebra with antipode S.*

(a) *For any algebra A and algebra morphism $H \xrightarrow{\alpha} A$, one has $\alpha \circ S = \alpha^{\star -1}$, the convolutional inverse to α in* $\mathrm{Hom}(H, A)$.
(b) *For any coalgebra C and coalgebra morphism $C \xrightarrow{\gamma} H$, one has $S \circ \gamma = \gamma^{\star -1}$, the convolutional inverse to γ in* $\mathrm{Hom}(C, H)$.

Proof. We prove (a); the proof of (b) is similar.

For assertion (a), note that Proposition 1.4.3 (applied to H, H, H, A, id_H and α instead of C, C', A, A', γ and α) shows that the map

$$\mathrm{Hom}\,(H, H) \to \mathrm{Hom}\,(H, A)\,, \qquad f \mapsto \alpha \circ f$$

is a $\mathbf{k}$-algebra homomorphism from the convolution algebra $(\mathrm{Hom}\,(H, H)\,, \star)$ to the convolution algebra $(\mathrm{Hom}\,(H, A)\,, \star)$. Denoting this homomorphism by φ, we thus have $\varphi\left((\mathrm{id}_H)^{\star -1}\right) = (\varphi(\mathrm{id}_H))^{\star -1}$ (since $\mathbf{k}$-algebra homomorphisms preserve inverses). Now,

$$\alpha \circ S = \varphi(S) = \varphi\left((\mathrm{id}_H)^{\star -1}\right) = (\varphi(\mathrm{id}_H))^{\star -1} = (\alpha \circ \mathrm{id}_H)^{\star -1} = \alpha^{\star -1}.$$

$\square$

A rather useful consequence of Proposition 1.4.26 is the fact ([213, Lemma 4.0.4]) that a bialgebra morphism between Hopf algebras automatically respects the antipodes:

Corollary 1.4.27. *Let H_1 and H_2 be Hopf algebras with antipodes S_1 and S_2, respectively. Then, any bialgebra morphism $H_1 \xrightarrow{\beta} H_2$ is a Hopf* *morphism*[27]*, that is, it commutes with the antipodes (i.e., we have $\beta \circ S_1 = S_2 \circ \beta$).*

Proof. Proposition 1.4.26(a) (applied to $H = H_1$, $S = S_1$, $A = H_2$ and $\alpha = \beta$) yields $\beta \circ S_1 = \beta^{\star -1}$. Proposition 1.4.26(b) (applied to $H = H_2$, $S = S_2$, $C = H_1$ and $\gamma = \beta$) yields $S_2 \circ \beta = \beta^{\star -1}$. Comparing these equalities shows that $\beta \circ S_1 = S_2 \circ \beta$, qed. □

Exercise 1.4.28. Prove that the antipode S of a Hopf algebra A is a coalgebra anti-endomorphism, i.e., that it satisfies $\epsilon \circ S = \epsilon$ and $\Delta \circ S = T \circ (S \otimes S) \circ \Delta$, where $T : A \otimes A \to A \otimes A$ is the twist map sending every $a \otimes b$ to $b \otimes a$.

Exercise 1.4.29. If C is a $\mathbf{k}$-coalgebra and if A is a $\mathbf{k}$-algebra, then a $\mathbf{k}$-linear map $f : C \to A$ is said to be *$\star$-invertible* if it is invertible as an element of the $\mathbf{k}$-algebra $(\operatorname{Hom}(C, A), \star)$. In this case, the multiplicative inverse $f^{\star(-1)}$ of f in $(\operatorname{Hom}(C, A), \star)$ is called the *$\star$-inverse* of f.

Recall the concepts introduced in Definition 1.4.8.

(a) If C is a $\mathbf{k}$-bialgebra, if A is a $\mathbf{k}$-algebra, and if $r : C \to A$ is a $\star$-invertible $\mathbf{k}$-algebra homomorphism, then prove that the $\star$-inverse $r^{\star(-1)}$ of r is a $\mathbf{k}$-algebra anti-homomorphism.

(b) If C is a $\mathbf{k}$-bialgebra, if A is a $\mathbf{k}$-coalgebra, and if $r : A \to C$ is a $\star$-invertible $\mathbf{k}$-coalgebra homomorphism, then prove that the $\star$-inverse $r^{\star(-1)}$ of r is a $\mathbf{k}$-coalgebra anti-homomorphism.

(c) Derive Proposition 1.4.10 from Exercise 1.4.29(a), and derive Exercise 1.4.28 from Exercise 1.4.29(b).

(d) Prove Corollary 1.4.12 again using Proposition 1.4.26.

(e) If C is a graded $\mathbf{k}$-coalgebra, if A is a graded $\mathbf{k}$-algebra, and if $r : C \to A$ is a $\star$-invertible $\mathbf{k}$-linear map that is graded, then prove that the $\star$-inverse $r^{\star(-1)}$ of r is also graded.

Exercise 1.4.30. (a) Let A be a Hopf algebra. If $P : A \to A$ is a $\mathbf{k}$-linear map such that every $a \in A$ satisfies

$$\sum_{(a)} P(a_2) \cdot a_1 = u(\epsilon(a)),$$

then prove that the antipode S of A is invertible and its inverse is P.

(b) Let A be a Hopf algebra. If $P : A \to A$ is a $\mathbf{k}$-linear map such that every $a \in A$ satisfies

$$\sum_{(a)} a_2 \cdot P(a_1) = u(\epsilon(a)),$$

then prove that the antipode S of A is invertible and its inverse is P.

(c) Show that the antipode of a connected graded Hopf algebra is invertible.

(Compare this exercise to [157, Lemma 1.5.11].)

[27] A *Hopf morphism* (or, more officially, a *Hopf algebra morphism*, or *homomorphism of Hopf algebras*) between two Hopf algebras A and B is defined to be a bialgebra morphism $f : A \to B$ that satisfies $f \circ S_A = S_B \circ f$.

Definition 1.4.31. Let C be a $\mathbf{k}$-coalgebra. A *subcoalgebra* of C means a $\mathbf{k}$-coalgebra D such that $D \subset C$ and such that the canonical inclusion map $D \to C$ is a $\mathbf{k}$-coalgebra homomorphism[28]. When $\mathbf{k}$ is a field, we can equivalently define a subcoalgebra of C as a $\mathbf{k}$-submodule D of C such that $\Delta_C(D)$ is a subset of the $\mathbf{k}$-submodule $D \otimes D$ of $C \otimes C$; however, this might no longer be equivalent when $\mathbf{k}$ is not a field[29].

Similarly, a *subbialgebra* of a bialgebra C is a $\mathbf{k}$-bialgebra D such that $D \subset C$ and such that the canonical inclusion map $D \to C$ is a $\mathbf{k}$-bialgebra homomorphism. Also, a *Hopf subalgebra* of a Hopf algebra C is a $\mathbf{k}$-Hopf algebra D such that $D \subset C$ and such that the canonical inclusion map $D \to C$ is a $\mathbf{k}$-Hopf algebra homomorphism.[30]

Exercise 1.4.32. Let C be a $\mathbf{k}$-coalgebra. Let D be a $\mathbf{k}$-submodule of C such that D is a direct summand of C as a $\mathbf{k}$-module (i.e., there exists a $\mathbf{k}$-submodule E of C such that $C = D \oplus E$). (This is automatically satisfied if $\mathbf{k}$ is a field.) Assume that $\Delta(D) \subset C \otimes D$ and $\Delta(D) \subset D \otimes C$. (Here, we are abusing the notation $C \otimes D$ to denote the $\mathbf{k}$-submodule of $C \otimes C$ spanned by tensors of the form $c \otimes d$ with $c \in C$ and $d \in D$; similarly, $D \otimes C$ should be understood.) Show that there is a canonically defined $\mathbf{k}$-coalgebra structure on D which makes D a subcoalgebra of C.

The next exercise is implicit in [4, §5]:

Exercise 1.4.33. Let $\mathbf{k}$ be a field. Let C be a $\mathbf{k}$-coalgebra, and let U be any $\mathbf{k}$-module. Let $f : C \to U$ be a $\mathbf{k}$-linear map. Recall the map $\Delta^{(2)} : C \to C^{\otimes 3}$ from Exercise 1.4.20. Let $K = \ker\left((\mathrm{id}_C \otimes f \otimes \mathrm{id}_C) \circ \Delta^{(2)}\right)$.

(a) Show that K is a $\mathbf{k}$-subcoalgebra of C.
(b) Show that every $\mathbf{k}$-subcoalgebra of C which is a subset of $\ker f$ must be a subset of K.

Exercise 1.4.34.
(a) Let $C = \bigoplus_{n\geq 0} C_n$ be a graded $\mathbf{k}$-coalgebra, and A be any $\mathbf{k}$-algebra. Notice that C_0 itself is a $\mathbf{k}$-subcoalgebra of C. Let $h : C \to A$ be a $\mathbf{k}$-linear map such that the restriction $h\mid_{C_0}$ is a $\star$-invertible map in $\operatorname{Hom}(C_0, A)$. Prove that h is a $\star$-invertible map in $\operatorname{Hom}(C, A)$. (This is a weaker version of Takeuchi's [214, Lemma 14].)
(b) Let $A = \bigoplus_{n\geq 0} A_n$ be a graded $\mathbf{k}$-bialgebra. Notice that A_0 is a subbialgebra of A. Assume that A_0 is a Hopf algebra. Show that A is a Hopf algebra.
(c) Obtain yet another proof of Proposition 1.4.16.

Exercise 1.4.35. Let $A = \bigoplus_{n\geq 0} A_n$ be a connected graded $\mathbf{k}$-bialgebra. Let $\mathfrak{p}$ be the $\mathbf{k}$-submodule of A consisting of the primitive elements of A.

(a) If I is a two-sided coideal of A such that $I \cap \mathfrak{p} = 0$ and such that $I = \bigoplus_{n\geq 0} (I \cap A_n)$, then prove that $I = 0$.

[28]In this definition, we follow [162, p. 55] and [225, §6.7]; other authors may use other definitions.

[29]This is because the $\mathbf{k}$-submodule $D \otimes D$ of $C \otimes C$ is generally not isomorphic to the $\mathbf{k}$-module $D \otimes D$. See [162, p. 56] for specific counterexamples for the non-equivalence of the two notions of a subcoalgebra. Notice that the equivalence is salvaged if D is a direct summand of C as a $\mathbf{k}$-module (see Exercise 1.4.32 for this).

[30]By Corollary 1.4.27, we can also define it as a subbialgebra of C that happens to be a Hopf algebra.

(b) Let $f : A \to C$ be a graded surjective coalgebra homomorphism from A to a graded $\mathbf{k}$-coalgebra C. If $f \mid_{\mathfrak{p}}$ is injective, then prove that f is injective.
(c) Assume that $\mathbf{k}$ is a field. Show that the claim of Exercise 1.4.35(b) is valid even without requiring f to be surjective.

Remark 1.4.36. Exercise 1.4.35 (b) and (c) are often used in order to prove that certain coalgebra homomorphisms are injective.

The word "bialgebra" can be replaced by "coalgebra" in Exercise 1.4.35, provided that the notion of a connected graded coalgebra is defined correctly (namely, as a graded coalgebra such that the restriction of ϵ to the 0-th graded component is an isomorphism), and the notion of the element 1 of a connected graded coalgebra is defined accordingly (namely, as the preimage of $1 \in \mathbf{k}$ under the restriction of ϵ to the 0-th graded component).

1.5. **Commutativity, cocommutativity.** Recall that a $\mathbf{k}$-algebra A is *commutative* if and only if all $a, b \in A$ satisfy $ab = ba$. Here is a way to restate this classical definition using tensors instead of pairs of elements:

Definition 1.5.1. A $\mathbf{k}$-algebra A is said to be *commutative* if the following diagram commutes:

(1.5.1)
$$\begin{tikzcd} A \otimes A \arrow[rr, "T"] \arrow[dr, "m"'] & & A \otimes A \arrow[dl, "m"] \\ & A & \end{tikzcd}$$

where T is the twist map $T_{A,A}$ (see Definition 1.4.8(a) for its definition).

Having thus redefined commutative algebras in terms of tensors and linear maps, we can dualize this definition (reversing all arrows) and obtain the notion of *cocommutative coalgebras*:

Definition 1.5.2. A $\mathbf{k}$-coalgebra C is said to be *cocommutative* if the following diagram commutes:

(1.5.2)
$$\begin{tikzcd} C \otimes C \arrow[rr, "T"] & & C \otimes C \\ & C \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] & \end{tikzcd}$$

where T is the twist map $T_{C,C}$ (see Definition 1.4.8(a) for its definition).

Example 1.5.3. Group algebras $\mathbf{k}G$ are always cocommutative. They are commutative if and only if G is abelian or $\mathbf{k} = 0$.

Tensor algebras $T(V)$ are always cocommutative, but not generally commutative[31].

Symmetric algebras $\mathrm{Sym}(V)$ are always cocommutative and commutative.

Homology and cohomology of H-spaces are always cocommutative and commutative *in the topologist's sense* where one reinterprets that twist map $A \otimes A \xrightarrow{T} A \otimes A$ to have the extra sign as in (1.3.3).

[31]If $\mathbf{k}$ is a field, then $T(V)$ is commutative if and only if $\dim_{\mathbf{k}} V \leq 1$.

Note how the cocommutative Hopf algebras $T(V), \mathrm{Sym}(V)$ have much of their structure controlled by their $\mathbf{k}$-submodules V, which consist of primitive elements only (although, in general, not of all their primitive elements). This is not far from the truth in general, and closely related to Lie algebras.

Exercise 1.5.4. Recall that a *Lie algebra* over $\mathbf{k}$ is a $\mathbf{k}$-module $\mathfrak{g}$ with a $\mathbf{k}$-bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies $[x,x] = 0$ for x in $\mathfrak{g}$, and the *Jacobi identity*

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]], \text{ or equivalently}$$
$$[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0$$

for all $x, y, z \in \mathfrak{g}$. This $\mathbf{k}$-bilinear map $[\cdot,\cdot]$ is called the *Lie bracket* of $\mathfrak{g}$.

(a) Check that any associative algebra A gives rise to a Lie algebra by means of the commutator operation $[a,b] := ab - ba$.
(b) If A is also a bialgebra, show that the $\mathbf{k}$-submodule of primitive elements $\mathfrak{p} \subset A$ is closed under the Lie bracket, that is, $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{p}$, and hence forms a Lie subalgebra.

Conversely, given a Lie algebra $\mathfrak{p}$, one constructs the *universal enveloping algebra* $\mathcal{U}(\mathfrak{p}) := T(\mathfrak{p})/J$ as the quotient of the tensor algebra $T(\mathfrak{p})$ by the two-sided ideal J generated by all elements $xy - yx - [x,y]$ for x, y in $\mathfrak{p}$.

(c) Show that J is also a two-sided coideal in $T(\mathfrak{p})$ for its usual coalgebra structure, and hence the quotient $\mathcal{U}(\mathfrak{p})$ inherits the structure of a cocommutative bialgebra.
(d) Show that the antipode S on $T(\mathfrak{p})$ preserves J, meaning that $S(J) \subset J$, and hence $\mathcal{U}(\mathfrak{p})$ inherits the structure of a (cocommutative) Hopf algebra.

There are theorems, discussed in [35, §3.8], [157, Chap. 5], [60, §3.2] giving various mild hypotheses in addition to cocommutativity which imply that the inclusion of the $\mathbf{k}$-module $\mathfrak{p}$ of primitives in a Hopf algebra A extends to a Hopf isomorphism $\mathcal{U}(\mathfrak{p}) \cong A$.

Exercise 1.5.5. Let C be a cocommutative $\mathbf{k}$-coalgebra. Let A be a commutative $\mathbf{k}$-algebra. Show that the convolution algebra $(\mathrm{Hom}(C,A), \star)$ is commutative (i.e., every $f, g \in \mathrm{Hom}(C,A)$ satisfy $f \star g = g \star f$).

Exercise 1.5.6. (a) Let C be a $\mathbf{k}$-coalgebra. Show that C is cocommutative if and only if its comultiplication $\Delta_C : C \to C \otimes C$ is a $\mathbf{k}$-coalgebra homomorphism.
(b) Let A be a $\mathbf{k}$-algebra. Show that A is commutative if and only if its multiplication $m_A : A \otimes A \to A$ is a $\mathbf{k}$-algebra homomorphism.

Remark 1.5.7. If C is a $\mathbf{k}$-coalgebra, then $\epsilon_C : C \to \mathbf{k}$ is always a $\mathbf{k}$-coalgebra homomorphism. Similarly, $u_A : \mathbf{k} \to A$ is a $\mathbf{k}$-algebra homomorphism whenever A is a $\mathbf{k}$-algebra.

Exercise 1.5.8. (a) Let A and B be two $\mathbf{k}$-algebras, at least one of which is commutative. Prove that the $\mathbf{k}$-algebra anti-homomorphisms from A to B are the same as the $\mathbf{k}$-algebra homomorphisms from A to B.
(b) State and prove the dual of this result.

Exercise 1.5.9. Let A be a commutative $\mathbf{k}$-algebra, and let $k \in \mathbb{N}$. The symmetric group $\mathfrak{S}_k$ acts on the k-fold tensor power $A^{\otimes k}$ by permuting the tensor factors: $\sigma(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}$ for all $v_1, v_2, \ldots, v_k \in A$ and $\sigma \in \mathfrak{S}_k$. For every $\pi \in \mathfrak{S}_k$, denote by $\rho(\pi)$ the action of π on $A^{\otimes k}$ (this is an endomorphism of $A^{\otimes k}$). Show that every $\pi \in \mathfrak{S}_k$ satisfies $m^{(k-1)} \circ (\rho(\pi)) = m^{(k-1)}$. (Recall that $m^{(k-1)} : A^{\otimes k} \to A$ is defined as in Exercise 1.4.19 for $k \geq 1$, and by $m^{(-1)} = u : \mathbf{k} \to A$ for $k = 0$.)

Exercise 1.5.10. State and solve the analogue of Exercise 1.5.9 for cocommutative $\mathbf{k}$-coalgebras.

Exercise 1.5.11. (a) If H is a $\mathbf{k}$-bialgebra and A is a commutative $\mathbf{k}$-algebra, and if f and g are two $\mathbf{k}$-algebra homomorphisms $H \to A$, then prove that $f \star g$ also is a $\mathbf{k}$-algebra homomorphism $H \to A$.

(b) If H is a $\mathbf{k}$-bialgebra and A is a commutative $\mathbf{k}$-algebra, and if $f_1, f_2, \ldots, f_k$ are several $\mathbf{k}$-algebra homomorphisms $H \to A$, then prove that $f_1 \star f_2 \star \cdots \star f_k$ also is a $\mathbf{k}$-algebra homomorphism $H \to A$.

(c) If H is a Hopf algebra and A is a commutative $\mathbf{k}$-algebra, and if $f : H \to A$ is a $\mathbf{k}$-algebra homomorphism, then prove that $f \circ S : H \to A$ (where S is the antipode of H) is again a $\mathbf{k}$-algebra homomorphism, and is a $\star$-inverse to f.

(d) If A is a commutative $\mathbf{k}$-algebra, then show that $m^{(k)}$ is a $\mathbf{k}$-algebra homomorphism for every $k \in \mathbb{N}$. (The map $m^{(k)} : A^{\otimes(k+1)} \to A$ is defined as in Exercise 1.4.19.)

(e) If C' and C are two $\mathbf{k}$-coalgebras, if $\gamma : C \to C'$ is a $\mathbf{k}$-coalgebra homomorphism, if A and A' are two $\mathbf{k}$-algebras, if $\alpha : A \to A'$ is a $\mathbf{k}$-algebra homomorphism, and if $f_1, f_2, \ldots, f_k$ are several $\mathbf{k}$-linear maps $C' \to A$, then prove that

$$\alpha \circ (f_1 \star f_2 \star \cdots \star f_k) \circ \gamma = (\alpha \circ f_1 \circ \gamma) \star (\alpha \circ f_2 \circ \gamma) \star \cdots \star (\alpha \circ f_k \circ \gamma).$$

(f) If H is a commutative $\mathbf{k}$-bialgebra, and k and ℓ are two nonnegative integers, then prove that $\mathrm{id}_H^{\star k} \circ \mathrm{id}_H^{\star \ell} = \mathrm{id}_H^{\star(k\ell)}$.

(g) If H is a commutative $\mathbf{k}$-Hopf algebra, and k and ℓ are two integers, then prove that $\mathrm{id}_H^{\star k} \circ \mathrm{id}_H^{\star \ell} = \mathrm{id}_H^{\star(k\ell)}$. (These powers $\mathrm{id}_H^{\star k}$, $\mathrm{id}_H^{\star \ell}$ and $\mathrm{id}_H^{\star(k\ell)}$ are well-defined since id_H is $\star$-invertible.)

(h) State and prove the duals of parts (a)–(g) of this exercise.

Remark 1.5.12. The maps $\mathrm{id}_H^{\star k}$ for $k \in \mathbb{N}$ are known as the *Adams operators* of the bialgebra H; they are studied, inter alia, in [5]. Particular cases (and variants) of Exercise 1.5.11(f) appear in [167, Corollaire II.9] and [78, Theorem 1]. Exercise 1.5.11(f) and its dual are [135, Prop. 1.6].

Exercise 1.5.13. Prove that the antipode S of a cocommutative Hopf algebra A satisfies $S^2 = \mathrm{id}_A$. (This was a statement made in Remark 1.4.13.)

Exercise 1.5.14. Let A be a cocommutative graded Hopf algebra with antipode S. Define a $\mathbf{k}$-linear map $E : A \to A$ by having $E(a) = (\deg a) \cdot a$ for every homogeneous element a of A.

(a) Prove that for every $a \in A$, the elements $(S \star E)(a)$ and $(E \star S)(a)$ (where $\star$ denotes convolution in $\mathrm{Hom}(A, A)$) are primitive.

(b) Prove that for every primitive $p \in A$, we have

$$(S \star E)(p) = (E \star S)(p) = E(p).$$

(c) Prove that for every $a \in A$ and every primitive $p \in A$, we have

$$(S \star E)(ap) = [(S \star E)(a), p] + \epsilon(a) E(p),$$

where $[u, v]$ denotes the commutator $uv - vu$ of u and v.

(d) If A is connected and $\mathbb{Q}$ is a subring of $\mathbf{k}$, prove that the $\mathbf{k}$-algebra A is generated by the $\mathbf{k}$-submodule $\mathfrak{p}$ consisting of the primitive elements of A.

(e) Assume that A is the tensor algebra $T(V)$ of a $\mathbf{k}$-module V, and that the $\mathbf{k}$-submodule $V = V^{\otimes 1}$ of $T(V)$ is the degree-1 homogeneous component of A. Show that

$$(S \star E)(x_1 x_2 \ldots x_n) = [\ldots[[x_1, x_2], x_3], \ldots, x_n]$$

for any $n \geq 1$ and any $x_1, x_2, \ldots, x_n \in V$.

Remark 1.5.15. Exercise 1.5.14 gives rise to a certain idempotent map $A \to A$ when $\mathbf{k}$ is a commutative $\mathbb{Q}$-algebra and A is a cocommutative connected graded $\mathbf{k}$-Hopf algebra. Namely, the $\mathbf{k}$-linear map $A \to A$ sending every homogeneous $a \in A$ to $\frac{1}{\deg a}(S \star E)(a)$ (or 0 if $\deg a = 0$) is idempotent and is a projection on the $\mathbf{k}$-module of primitive elements of A. It is called the *Dynkin idempotent*; see [168] for more of its properties.[32] Part (c) of the exercise is more or less Baker's identity.

1.6. **Duals.** Recall that for $\mathbf{k}$-modules V, taking the dual $\mathbf{k}$-module $V^* := \operatorname{Hom}(V, \mathbf{k})$ reverses $\mathbf{k}$-linear maps. That is, every $\mathbf{k}$-linear map $V \xrightarrow{\varphi} W$ induces an *adjoint map* $W^* \xrightarrow{\varphi^*} V^*$ defined uniquely by

$$(f, \varphi(v)) = (\varphi^*(f), v)$$

in which (f, v) is the bilinear pairing $V^* \times V \to \mathbf{k}$ sending $(f, v) \mapsto f(v)$. If V and W are finite free $\mathbf{k}$-modules[33], more can be said: When φ is expressed in terms of a basis $\{v_i\}_{i \in I}$ for V and a basis $\{w_j\}_{j \in J}$ for W by some matrix, the map φ^* is expressed by the transpose matrix in terms of the dual bases of these two bases[34].

The correspondence $\varphi \mapsto \varphi^*$ between $\mathbf{k}$-linear maps $V \xrightarrow{\varphi} W$ and $\mathbf{k}$-linear maps $W^* \xrightarrow{\varphi^*} V^*$ is one-to-one when W is finite free. However, this is not the case in many combinatorial situations (in which W is usually free but not finite free). Fortunately, many of the good properties of finite free modules carry over to a certain class of graded modules as long as the dual V^* is replaced by a smaller module V^o called the graded dual. Let us first introduce the latter:

When $V = \bigoplus_{n \geq 0} V_n$ is a graded $\mathbf{k}$-module, note that the dual $V^* = \prod_{n \geq 0} (V_n)^*$ can contain functionals f supported on infinitely many V_n. However, we can consider the $\mathbf{k}$-submodule $V^o := \bigoplus_{n \geq 0} (V_n)^* \subset \prod_{n \geq 0} (V_n)^* =$

[32] We will see another such idempotent in Exercise 5.4.6.

[33] A $\mathbf{k}$-module is said to be *finite free* if it has a finite basis. If $\mathbf{k}$ is a field, then a finite free $\mathbf{k}$-module is the same as a finite-dimensional $\mathbf{k}$-vector space.

[34] If $\{v_i\}_{i \in I}$ is a basis of a finite free $\mathbf{k}$-module V, then the *dual basis* of this basis is defined as the basis $\{f_i\}_{i \in I}$ of V^* that satisfies $(f_i, v_j) = \delta_{i,j}$ for all i and j. (Recall that $\delta_{i,j}$ is the Kronecker delta: $\delta_{i,j} = 1$ if $i = j$ and 0 else.)

V^*, sometimes called the *graded dual*[35], consisting of the functions f that vanish on all but finitely many V_n. Notice that V^o is graded, whereas V^* (in general) is not. If $V \xrightarrow{\varphi} W$ is a graded $\mathbf{k}$-linear map, then the adjoint map $W^* \xrightarrow{\varphi^*} V^*$ restricts to a graded $\mathbf{k}$-linear map $W^o \to V^o$, which we (abusively) still denote by φ^*.

A graded $\mathbf{k}$-module $V = \bigoplus_{n \geq 0} V_n$ is said to be *of finite type* if each V_n is a finite free $\mathbf{k}$-module[36]. When the graded $\mathbf{k}$-module V is of finite type, the graded $\mathbf{k}$-module V^o is again of finite type[37] and satisfies $(V^o)^o \cong V$. Many other properties of finite free modules are salvaged in this situation; most importantly: The correspondence $\varphi \mapsto \varphi^*$ between graded $\mathbf{k}$-linear maps $V \to W$ and graded $\mathbf{k}$-linear maps $W^o \to V^o$ is one-to-one when W is of finite type[38].

Reversing the diagrams should then make it clear that, in the finite free or finite-type situation, duals of algebras are coalgebras, and vice-versa, and duals of bialgebras or Hopf algebras are bialgebras or Hopf algebras. For example, the product in a Hopf algebra A of finite type uniquely defines the coproduct of A^o via adjointness:

$$(\Delta_{A^o}(f), a \otimes b)_{A \otimes A} = (f, ab)_A.$$

Thus if A has a basis $\{a_i\}_{i \in I}$ with *product structure constants* $\{c^i_{j,k}\}$, meaning

$$a_j a_k = \sum_{i \in I} c^i_{j,k} a_i,$$

then the dual basis $\{f_i\}_{i \in I}$ has the same $\{c^i_{j,k}\}$ as its *coproduct structure constants*:

$$\Delta_{A^o}(f_i) = \sum_{(j,k) \in I \times I} c^i_{j,k} f_j \otimes f_k.$$

The assumption that A be of finite type was indispensable here; in general, the dual of a $\mathbf{k}$-algebra does not become a $\mathbf{k}$-coalgebra. However, the dual of a $\mathbf{k}$-coalgebra still becomes a $\mathbf{k}$-algebra, as shown in the following exercise:

Exercise 1.6.1. For any two $\mathbf{k}$-modules U and V, let $\rho_{U,V} : U^* \otimes V^* \to (U \otimes V)^*$ be the $\mathbf{k}$-linear map which sends every tensor $f \otimes g \in U^* \otimes V^*$ to the composition $U \otimes V \xrightarrow{f \otimes g} \mathbf{k} \otimes \mathbf{k} \xrightarrow{m_{\mathbf{k}}} \mathbf{k}$ of the map[39] $f \otimes g$ with the canonical isomorphism $\mathbf{k} \otimes \mathbf{k} \xrightarrow{m_{\mathbf{k}}} \mathbf{k}$. When $\mathbf{k}$ is a field and U is finite-dimensional, this map $\rho_{U,V}$ is a $\mathbf{k}$-vector space isomorphism (and usually

[35]Do not mistake this for the coalgebraic restricted dual A° of [213, §6.0].

[36]This meaning of "finite type" can differ from the standard one.

[37]More precisely: Let $V = \bigoplus_{n \geq 0} V_n$ be of finite type, and let $\{v_i\}_{i \in I}$ be a *graded basis* of V, that is, a basis of the $\mathbf{k}$-module V such that the indexing set I is partitioned into subsets $I_0, I_1, I_2, \ldots$ (which are allowed to be empty) with the property that, for every $n \in \mathbb{N}$, the subfamily $\{v_i\}_{i \in I_n}$ is a basis of the $\mathbf{k}$-module V_n. Then, we can define a family $\{f_i\}_{i \in I}$ of elements of V^o by setting $(f_i, v_j) = \delta_{i,j}$ for all $i, j \in I$. This family $\{f_i\}_{i \in I}$ is a graded basis of the graded $\mathbf{k}$-module V^o. (Actually, for every $n \in \mathbb{N}$, the subfamily $\{f_i\}_{i \in I_n}$ is a basis of the $\mathbf{k}$-submodule $(V_n)^*$ of V^o – indeed the dual basis to the basis $\{v_i\}_{i \in I_n}$ of V_n.) This basis $\{f_i\}_{i \in I}$ is said to be the *dual basis* to the basis $\{v_i\}_{i \in I}$ of V.

[38]Only W has to be of finite type here; V can be any graded $\mathbf{k}$-module.

[39]Keep in mind that the *tensor* $f \otimes g \in U^* \otimes V^*$ is not the same as the *map* $U \otimes V \xrightarrow{f \otimes g} \mathbf{k} \otimes \mathbf{k}$.

regarded as the identity); more generally, it is injective whenever $\mathbf{k}$ is a field[40]. Also, let $s : \mathbf{k} \to \mathbf{k}^*$ be the canonical isomorphism. Prove that:

(a) If C is a $\mathbf{k}$-coalgebra, then C^* becomes a $\mathbf{k}$-algebra if we define its associative operation by $m_{C^*} = \Delta_C^* \circ \rho_{C,C} : C^* \otimes C^* \to C^*$ and its unit map to be $\epsilon_C^* \circ s : \mathbf{k} \to C^*$. [41]

(b) The $\mathbf{k}$-algebra structure defined on C^* in part (a) is precisely the one defined on $\operatorname{Hom}(C, \mathbf{k}) = C^*$ in Definition 1.4.1 applied to $A = \mathbf{k}$.

(c) If C is a graded $\mathbf{k}$-coalgebra, then C^o is a $\mathbf{k}$-subalgebra of the $\mathbf{k}$-algebra C^* defined in part (a).

(d) If $f : C \to D$ is a homomorphism of $\mathbf{k}$-coalgebras, then $f^* : D^* \to C^*$ is a homomorphism of $\mathbf{k}$-algebras.

(e) Let U be a graded $\mathbf{k}$-module (not necessarily of finite type), and let V be a graded $\mathbf{k}$-module of finite type. Then, there is a 1-to-1 correspondence between graded $\mathbf{k}$-linear maps $U \to V$ and graded $\mathbf{k}$-linear maps $V^o \to U^o$ given by $f \mapsto f^*$.

(f) Let C be a graded $\mathbf{k}$-coalgebra (not necessarily of finite type), and let D be a graded $\mathbf{k}$-coalgebra of finite type. Part (e) of this exercise shows that there is a 1-to-1 correspondence between graded $\mathbf{k}$-linear maps $C \to D$ and graded $\mathbf{k}$-linear maps $D^o \to C^o$ given by $f \mapsto f^*$. This correspondence has the property that a given graded $\mathbf{k}$-linear map $f : C \to D$ is a $\mathbf{k}$-coalgebra morphism if and only if $f^* : D^o \to C^o$ is a $\mathbf{k}$-algebra morphism.

Another example of a Hopf algebra is provided by the so-called shuffle algebra. Before we introduce it, let us define the *shuffles* of two words:

Definition 1.6.2. Given two words $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_m)$, the *multiset of shuffles of a and b* is defined as the multiset

$$\left\{ \left(c_{w(1)}, c_{w(2)}, \ldots, c_{w(n+m)}\right) \ : \ w \in \mathrm{Sh}_{n,m} \right\}_{\text{multiset}},$$

where $(c_1, c_2, \ldots, c_{n+m})$ is the concatenation

$$a \cdot b = (a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m),$$

and where $\mathrm{Sh}_{n,m}$ is the subset[42]

$$\left\{ w \in \mathfrak{S}_{n+m} \ : \ w^{-1}(1) < w^{-1}(2) < \cdots < w^{-1}(n); \right.$$
$$\left. w^{-1}(n+1) < w^{-1}(n+2) < \cdots < w^{-1}(n+m) \right\}$$

of the symmetric group $\mathfrak{S}_{n+m}$. Informally speaking, the shuffles of the two words a and b are the words obtained by overlaying the words a and b, after first moving their letters apart so that no letters get superimposed when the words are overlayed[43]. In particular, any shuffle of a and b contains a and

[40] Over arbitrary rings it does not have to be even that!

[41] If C is a finite free $\mathbf{k}$-module, then this $\mathbf{k}$-algebra structure is the same as the one defined above by adjointness. But the advantage of the new definition is that it works even if C is not a finite free $\mathbf{k}$-module.

[42] **Warning:** This definition of $\mathrm{Sh}_{n,m}$ is highly nonstandard, and many authors define $\mathrm{Sh}_{n,m}$ to be the set of the inverses of the permutations belonging to what we call $\mathrm{Sh}_{n,m}$.

[43] For instance, if $a = (1, 3, 2, 1)$ and $b = (2, 4)$, then the shuffle $(1, 2, 3, 2, 4, 1)$ of a and b can be obtained by moving the letters of a and b apart as follows:

$$\begin{array}{rcccccc} a = & 1 & & 3 & 2 & & 1 \\ b = & & 2 & & & 4 & \end{array}$$

b as subsequences. The multiset of shuffles of a and b has $\binom{m+n}{n}$ elements (counted with multiplicity) and is denoted by $a \sqcup\!\sqcup\, b$. For instance, the shuffles of $(1, 2, 1)$ and $(3, 2)$ are

$$\begin{aligned}&(\underline{1}, \underline{2}, \underline{1}, 3, 2), (\underline{1}, \underline{2}, 3, \underline{1}, 2), (\underline{1}, \underline{2}, 3, 2, \underline{1}), (\underline{1}, 3, \underline{2}, \underline{1}, 2), (\underline{1}, 3, \underline{2}, 2, \underline{1}),\\ &(\underline{1}, 3, 2, \underline{2}, \underline{1}), (3, \underline{1}, \underline{2}, \underline{1}, 2), (3, \underline{1}, \underline{2}, 2, \underline{1}), (3, \underline{1}, 2, \underline{2}, \underline{1}), (3, 2, \underline{1}, \underline{2}, \underline{1}),\end{aligned}$$

listed here as often as they appear in the multiset $(1, 2, 1) \sqcup\!\sqcup\, (3, 2)$. Here we have underlined the letters taken from a – that is, the letters at positions $w^{-1}(1)$, $w^{-1}(2)$, $\ldots$, $w^{-1}(n)$.

Example 1.6.3. When $A = T(V)$ is the tensor algebra for a finite free $\mathbf{k}$-module V, having $\mathbf{k}$-basis $\{x_i\}_{i\in I}$, its graded dual A^o is another Hopf algebra whose basis $\left\{y_{(i_1,\ldots,i_\ell)}\right\}$ (the dual basis of the basis $\{x_{i_1} \cdots x_{i_\ell}\}$ of $A = T(V)$) is indexed by words in the alphabet I. This Hopf algebra A^o could be called the *shuffle algebra* of V^*. (To be more precise, it is isomorphic to the shuffle algebra of V^* introduced in Proposition 1.6.7 further below; we prefer not to call A^o itself the shuffle algebra of V^*, since A^o has several disadvantages[44].) Duality shows that the *cut* coproduct in A^o is defined by

$$\Delta y_{(i_1,\ldots,i_\ell)} = \sum_{j=0}^{\ell} y_{(i_1,\ldots,i_j)} \otimes y_{(i_{j+1},i_{j+2},\ldots,i_\ell)}. \tag{1.6.1}$$

For example,

$$\Delta y_{abcb} = y_\varnothing \otimes y_{abcb} + y_a \otimes y_{bcb} + y_{ab} \otimes y_{cb} + y_{abc} \otimes y_b + y_{abcb} \otimes y_\varnothing.$$

Duality also shows that the *shuffle* product in A^o will be given by

$$y_{(i_1,\ldots,i_\ell)} y_{(j_1,\ldots,j_m)} = \sum_{\mathbf{k}=(k_1,\ldots,k_{\ell+m})\in \mathbf{i} \sqcup\!\sqcup\, \mathbf{j}} y_{(k_1,\ldots,k_{\ell+m})} \tag{1.6.2}$$

where $\mathbf{i} \sqcup\!\sqcup\, \mathbf{j}$ (as in Definition 1.6.2) denotes the multiset of the $\binom{\ell+m}{\ell}$ words obtained as *shuffles* of the two words $\mathbf{i} = (i_1, \ldots, i_\ell)$ and $\mathbf{j} = (j_1, \ldots, j_m)$. For example,

$$\begin{aligned}y_{ab} y_{cb} &= y_{abcb} + y_{acbb} + y_{cabb} + y_{cabb} + y_{acbb} + y_{cbab}\\ &= y_{abcb} + 2y_{acbb} + 2y_{cabb} + y_{cbab}.\end{aligned}$$

Equivalently, one has

$$\begin{aligned}&y_{(i_1,i_2,\ldots,i_\ell)} y_{(i_{\ell+1},i_{\ell+2},\ldots,i_{\ell+m})}\\ &= \sum_{\substack{w\in\mathfrak{S}_{\ell+m}:\\ w(1)<\cdots<w(\ell),\\ w(\ell+1)<\cdots<w(\ell+m)}} y_{\left(i_{w^{-1}(1)},i_{w^{-1}(2)},\ldots,i_{w^{-1}(\ell+m)}\right)} &(1.6.3)\\ &= \sum_{\sigma\in \mathrm{Sh}_{\ell,m}} y_{\left(i_{\sigma(1)},i_{\sigma(2)},\ldots,i_{\sigma(\ell+m)}\right)} &(1.6.4)\end{aligned}$$

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and then overlaying them to obtain 1 2 3 2 4 1 . Other ways of moving letters apart lead to further shuffles (not always distinct).

[44]Specifically, A^o has the disadvantages of being defined only when V^* is the dual of a finite free $\mathbf{k}$-module V, and depending on a choice of basis, whereas Proposition 1.6.7 will define shuffle algebras in full generality and canonically.

(using the notations of Definition 1.6.2 again). Lastly, the antipode S of A^o is the adjoint of the antipode of $A = T(V)$ described in (1.4.6):

$$Sy_{(i_1,i_2,\ldots,i_\ell)} = (-1)^\ell y_{(i_\ell,\ldots,i_2,i_1)}.$$

Since the coalgebra $T(V)$ is cocommutative, its graded dual $T(V)^o$ is commutative.

Exercise 1.6.4. Let V be a 1-dimensional free $\mathbf{k}$-module with basis element x, so $\mathrm{Sym}(V) \cong \mathbf{k}[x]$, with $\mathbf{k}$-basis $\{1 = x^0, x^1, x^2, \ldots\}$.

(a) Check that the powers x^i satisfy

$$\begin{aligned} x^i \cdot x^j &= x^{i+j}, \\ \Delta(x^n) &= \sum_{i+j=n} \binom{n}{i} x^i \otimes x^j, \\ S(x^n) &= (-1)^n x^n. \end{aligned}$$

(b) Check that the dual basis elements $\{f^{(0)}, f^{(1)}, f^{(2)}, \ldots\}$ for $\mathrm{Sym}(V)^o$, defined by $f^{(i)}(x^j) = \delta_{i,j}$, satisfy

$$\begin{aligned} f^{(i)} f^{(j)} &= \binom{i+j}{i} f^{(i+j)}, \\ \Delta(f^{(n)}) &= \sum_{i+j=n} f^{(i)} \otimes f^{(j)}, \\ S(f^{(n)}) &= (-1)^n f^{(n)}. \end{aligned}$$

(c) Show that if $\mathbb{Q}$ is a subring of $\mathbf{k}$, then the $\mathbf{k}$-linear map $\mathrm{Sym}(V)^o \to \mathrm{Sym}(V)$ sending $f^{(n)} \mapsto \frac{x^n}{n!}$ is a graded Hopf isomorphism.

For this reason, the Hopf structure on $\mathrm{Sym}(V)^o$ is called a *divided power algebra*.

(d) Show that when $\mathbf{k}$ is a field of characteristic $p > 0$, one has $(f^{(1)})^p = 0$, and hence why there can be no Hopf isomorphism $\mathrm{Sym}(V)^o \to \mathrm{Sym}(V)$.

Exercise 1.6.5. Let V have $\mathbf{k}$-basis $\{x_1, \ldots, x_n\}$, and let $V \oplus V$ have $\mathbf{k}$-basis $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$, so that one has isomorphisms

$$\mathrm{Sym}(V \oplus V) \cong \mathbf{k}[\mathbf{x}, \mathbf{y}] \cong \mathbf{k}[\mathbf{x}] \otimes \mathbf{k}[\mathbf{y}] \cong \mathrm{Sym}(V) \otimes \mathrm{Sym}(V).$$

Here we are using the abbreviations $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$.

(a) Show that our usual coproduct on $\mathrm{Sym}(V)$ can be re-expressed as follows:

$$\begin{array}{ccc} \mathrm{Sym}(V) & & \mathrm{Sym}(V) \otimes \mathrm{Sym}(V) \\ \| & & \| \\ \mathbf{k}[\mathbf{x}] & \xrightarrow{\Delta} & \mathbf{k}[\mathbf{x}, \mathbf{y}], \\ f(x_1, \ldots, x_n) & \longmapsto & f(x_1 + y_1, \ldots, x_n + y_n). \end{array}$$

In other words, it is induced from the diagonal map

$$\begin{array}{ccc} V & \longrightarrow & V \oplus V, \\ x_i & \longmapsto & x_i + y_i. \end{array} \tag{1.6.5}$$

(b) One can similarly define a coproduct on the *exterior algebra* $\wedge V$, which is the quotient $T(V)/J$ where J is the two-sided ideal generated by the elements $\{x^2 (= x \otimes x)\}_{x \in V}$ in $T^2(V)$. The ideal J is a graded $\mathbf{k}$-submodule of $T(V)$ (this is not obvious!), and the quotient $T(V)/J$ becomes a graded commutative algebra

$$\wedge V = \bigoplus_{d=0}^{n} \wedge^d V \left(= \bigoplus_{d=0}^{\infty} \wedge^d V \right),$$

if one views the elements of $V = \wedge^1 V$ as having *odd* degree, and uses the topologist's sign convention (as in (1.3.3)). One again has $\wedge(V \oplus V) = \wedge V \otimes \wedge V$ as graded algebras. Show that one can again let the diagonal map (1.6.5) induce a map

(1.6.6)
$$\begin{array}{ccc} \wedge(V) & \xrightarrow{\Delta} & \wedge V \otimes \wedge V, \\ f(x_1, \ldots, x_n) & \longmapsto & f(x_1 + y_1, \ldots, x_n + y_n) \\ \| & & \| \\ \sum c_{i_1, \ldots, i_d} \cdot x_{i_1} \wedge \cdots \wedge x_{i_d} & & \sum c_{i_1, \ldots, i_d} \cdot (x_{i_1} + y_{i_1}) \wedge \cdots \wedge (x_{i_d} + y_{i_d}), \end{array}$$

which makes $\wedge V$ into a connected graded Hopf algebra.

(c) Show that in the tensor algebra $T(V)$, if one views the elements of $V = V^{\otimes 1}$ as having odd degree, and uses the topologist's sign convention (1.3.3) in the twist map when defining $T(V)$, then for any x in V one has $\Delta(x^2) = 1 \otimes x^2 + x^2 \otimes 1$.

(d) Let us use the convention (1.3.3) as in part (c). Show that the two-sided ideal $J \subset T(V)$ generated by $\{x^2\}_{x \in V}$ is also a two-sided coideal and a graded $\mathbf{k}$-submodule of $T(V)$, and hence the quotient $\wedge V = T(V)/J$ inherits the structure of a graded bialgebra. Check that the coproduct on $\wedge V$ inherited from $T(V)$ is the same as the one defined in part (b).

[**Hint:** The ideal J in part (b) is a graded $\mathbf{k}$-submodule of $T(V)$, but this is not completely obvious (not all elements of V have to be homogeneous!).]

Exercise 1.6.6. Let C be a $\mathbf{k}$-coalgebra. As we know from Exercise 1.6.1(a), this makes C^* into a $\mathbf{k}$-algebra.

Let A be a $\mathbf{k}$-algebra which is finite free as $\mathbf{k}$-module. This makes A^* into a $\mathbf{k}$-coalgebra.

Let $f : C \to A$ and $g : C \to A$ be two $\mathbf{k}$-linear maps. Show that $f^* \star g^* = (f \star g)^*$.

The above arguments might have created the impression that duals of bialgebras have good properties only under certain restrictive conditions (e.g., the dual of a bialgebra H does not generally become a bialgebra unless H is of finite type), and so they cannot be used in proofs and constructions unless one is willing to sacrifice some generality (e.g., we had to require V to be finite free in Example 1.6.3). While the first part of this impression is true, the second is not always; often there is a way to gain back the generality lost from using duals. As an example of this, let us define the shuffle algebra of an arbitrary $\mathbf{k}$-module (not just of a dual of a finite free $\mathbf{k}$-module as in Example 1.6.3):

Proposition 1.6.7. *Let V be a* $\mathbf{k}$*-module. Define a* $\mathbf{k}$*-linear map* $\Delta_{\sqcup\!\sqcup} : T(V) \to T(V) \otimes T(V)$ *by setting*

$$\Delta_{\sqcup\!\sqcup}(v_1 v_2 \cdots v_n) = \sum_{k=0}^{n} (v_1 v_2 \cdots v_k) \otimes (v_{k+1} v_{k+2} \cdots v_n) \qquad \text{for all } n \in \mathbb{N} \text{ and } v_1, v_2, \ldots, v_n \in V.$$

[45] *Define a* $\mathbf{k}$*-bilinear map* $\underline{\sqcup\!\sqcup} : T(V) \times T(V) \to T(V)$*, which will be written in infix notation (that is, we will write* $a \,\underline{\sqcup\!\sqcup}\, b$ *instead of* $\underline{\sqcup\!\sqcup}(a, b)$*), by setting*[46]

$$(v_1 v_2 \cdots v_\ell) \,\underline{\sqcup\!\sqcup}\, (v_{\ell+1} v_{\ell+2} \cdots v_{\ell+m}) = \sum_{\sigma \in \mathrm{Sh}_{\ell,m}} v_{\sigma(1)} v_{\sigma(2)} \cdots v_{\sigma(\ell+m)} \qquad \text{for all } \ell, m \in \mathbb{N} \text{ and } v_1, v_2, \ldots, v_{\ell+m} \in V.$$

[47] *Consider also the comultiplication* ϵ *of the Hopf algebra* $T(V)$*.*

Then, the $\mathbf{k}$*-module* $T(V)$*, endowed with the multiplication* $\underline{\sqcup\!\sqcup}$*, the unit* $1_{T(V)} \in V^{\otimes 0} \subset T(V)$*, the comultiplication* $\Delta_{\sqcup\!\sqcup}$ *and the counit* ϵ*, becomes a commutative Hopf algebra. This Hopf algebra is called the* shuffle algebra *of* V*, and denoted by* $\mathrm{Sh}(V)$*. The antipode of the Hopf algebra* $\mathrm{Sh}(V)$ *is precisely the antipode* S *of* $T(V)$*.*

Exercise 1.6.8. Prove Proposition 1.6.7.

[**Hint:** When V is a finite free $\mathbf{k}$-module, Proposition 1.6.7 follows from Example 1.6.3. The trick is to derive the general case from this specific one. Every $\mathbf{k}$-linear map $f : W \to V$ between two $\mathbf{k}$-modules W and V induces a map $T(f) : T(W) \to T(V)$ which preserves $\Delta_{\sqcup\!\sqcup}$, $\underline{\sqcup\!\sqcup}$, $1_{T(W)}$, ϵ and S (in the appropriate meanings – e.g., preserving $\Delta_{\sqcup\!\sqcup}$ means $\Delta_{\sqcup\!\sqcup} \circ T(f) = (T(f) \otimes T(f)) \circ \Delta_{\sqcup\!\sqcup}$). Show that each of the equalities that need to be proven in order to verify Proposition 1.6.7 can be "transported" along such a map $T(f)$ from a $T(W)$ for a suitably chosen finite free $\mathbf{k}$-module W.]

It is also possible to prove Proposition 1.6.7 "by foot", as long as one is ready to make combinatorial arguments about cutting shuffles.

Remark 1.6.9.
(a) Let V be a finite free $\mathbf{k}$-module. The Hopf algebra $T(V)^o$ (studied in Example 1.6.3) is naturally isomorphic to the shuffle algebra $\mathrm{Sh}(V^*)$ (defined as in Proposition 1.6.7 but for V^* instead of V) as Hopf algebras, by the obvious isomorphism (namely, the direct sum of the isomorphisms $(V^{\otimes n})^* \to (V^*)^{\otimes n}$ over all $n \in \mathbb{N}$). [48]
(b) The same statement applies to the case when V is a graded $\mathbf{k}$-module of finite type satisfying $V_0 = 0$ rather than a finite free

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[45]This is well-defined, because the right hand side is n-multilinear in $v_1, v_2, \ldots, v_n$, and because any n-multilinear map $V^{\times n} \to M$ into a $\mathbf{k}$-module M gives rise to a unique $\mathbf{k}$-linear map $V^{\otimes n} \to M$.

[46]Many authors use the symbol $\sqcup\!\sqcup$ instead of $\underline{\sqcup\!\sqcup}$ here, but we prefer to reserve the former notation for the shuffle product of words.

[47]Again, this is well-defined by the $\ell + m$-multilinearity of the right hand side.

[48]This can be verified by comparing (1.6.1) with the definition of $\Delta_{\sqcup\!\sqcup}$, and comparing (1.6.4) with the definition of $\underline{\sqcup\!\sqcup}$.

$\mathbf{k}$-module, provided that V^* and $(V^{\otimes n})^*$ are replaced by V^o and $(V^{\otimes n})^o$.

We shall return to shuffle algebras in Section 6.3, where we will show that under certain conditions ($\mathbb{Q}$ being a subring of $\mathbf{k}$, and V being a free $\mathbf{k}$-module) the algebra structure on a shuffle algebra $\mathrm{Sh}(V)$ is a polynomial algebra in an appropriately chosen set of generators[49].

1.7. **Infinite sums and Leray's theorem.** In this section (which can be skipped, as it will not be used except in a few exercises), we will see how a Hopf algebra structure on a $\mathbf{k}$-algebra reveals knowledge about the $\mathbf{k}$-algebra itself. Specifically, we will show that if $\mathbf{k}$ is a commutative $\mathbb{Q}$-algebra, and if A is any commutative connected graded $\mathbf{k}$-Hopf algebra, then A as a $\mathbf{k}$-algebra must be (isomorphic to) a symmetric algebra of a $\mathbf{k}$-module[50]. This is a specimen of a class of facts which are commonly called *Leray theorems*; for different specimens, see [156, Theorem 7.5] or [35, p. 17, "Hopf's theorem"] or [35, §2.5, A, B, C] or [35, Theorem 3.8.3].[51] In a sense, these facts foreshadow Zelevinsky's theory of positive self-dual Hopf algebras, which we shall encounter in Chapter 3; however, the latter theory works in a much less general setting (and makes much stronger claims).

We shall first explore the possibilities of applying a formal power series v to a linear map $f : C \to A$ from a coalgebra C to an algebra A. We have already seen an example of this in the proof of Proposition 1.4.7 above (where the power series $\sum_{k\geq 0} (-1)^k T^k \in \mathbf{k}[[T]]$ was applied to the locally $\star$-nilpotent map $\mathrm{id}_A - u_A\epsilon_A : A \to A$); we shall now take a more systematic approach and establish general criteria for when such applications are possible. First, we will have to make sense of infinite sums of maps from a coalgebra to an algebra. This is somewhat technical, but the effort will pay off.

Definition 1.7.1. Let A be an abelian group (written additively).

We say that a family $(a_q)_{q\in Q} \in A^Q$ of elements of A is *finitely supported* if all but finitely many $q \in Q$ satisfy $a_q = 0$. Clearly, if $(a_q)_{q\in Q} \in A^Q$ is a finitely supported family, then the sum $\sum_{q\in Q} a_q$ is well-defined (since all but finitely many of its addends are 0). Sums like this satisfy the usual rules for sums, even though their indexing set Q may be infinite. (For example, if $(a_q)_{q\in Q}$ and $(b_q)_{q\in Q}$ are two finitely supported families in A^Q, then the family $(a_q + b_q)_{q\in Q}$ is also finitely supported, and we have $\sum_{q\in Q} a_q + \sum_{q\in Q} b_q = \sum_{q\in Q} (a_q + b_q)$.)

Definition 1.7.2. Let C and A be two $\mathbf{k}$-modules.

We say that a family $(f_q)_{q\in Q} \in (\mathrm{Hom}(C, A))^Q$ of maps $f_q \in \mathrm{Hom}(C, A)$ is *pointwise finitely supported* if for each $x \in C$, the family $(f_q(x))_{q\in Q} \in A^Q$ of elements of A is finitely supported.[52] If $(f_q)_{q\in Q} \in (\mathrm{Hom}(C, A))^Q$ is a

[49]This says nothing about the coalgebra structure on $\mathrm{Sh}(V)$ – which is much more complicated in these generators.

[50]If $\mathbf{k}$ is a field, then this simply means that A as a $\mathbf{k}$-algebra must be a polynomial ring over $\mathbf{k}$.

[51]Notice that many of these sources assume $\mathbf{k}$ to be a field; some of their proofs rely on this assumption.

[52]Here are some examples of pointwise finitely supported families:

pointwise finitely supported family, then the sum $\sum_{q\in Q} f_q$ is defined to be the map $C \to A$ sending each $x \in C$ to $\sum_{q\in Q} f_q(x)$. [53]

Note that the concept of a "pointwise finitely supported" family $(f_q)_{q\in Q} \in (\operatorname{Hom}(C, A))^Q$ is precisely the concept of a "summable" family in [60, Definition 1].

Definition 1.7.3. For the rest of Section 1.7, we shall use the following conventions:

- Let C be a $\mathbf{k}$-coalgebra. Let A be a $\mathbf{k}$-algebra.
- We shall avoid our standard practice of denoting the unit map $u_A : \mathbf{k} \to A$ of a $\mathbf{k}$-algebra A by u; instead, we will use the letter u (without the subscript A) for other purposes.

Definition 1.7.2 allows us to work with infinite sums in $\operatorname{Hom}(C, A)$, provided that we are summing a pointwise finitely supported family. We shall next state some properties of such sums:[54]

Proposition 1.7.4. *Let $(f_q)_{q\in Q} \in (\operatorname{Hom}(C, A))^Q$ be a pointwise finitely supported family. Then, the map $\sum_{q\in Q} f_q$ belongs to $\operatorname{Hom}(C, A)$.*

Proposition 1.7.5. *Let $(f_q)_{q\in Q}$ and $(g_q)_{q\in Q}$ be two pointwise finitely supported families in $(\operatorname{Hom}(C, A))^Q$. Then, the family $(f_q + g_q)_{q\in Q} \in (\operatorname{Hom}(C, A))^Q$ is also pointwise finitely supported, and satisfies*

$$\sum_{q\in Q} f_q + \sum_{q\in Q} g_q = \sum_{q\in Q} (f_q + g_q).$$

Proposition 1.7.6. *Let $(f_q)_{q\in Q} \in (\operatorname{Hom}(C, A))^Q$ and $(g_r)_{r\in R} \in (\operatorname{Hom}(C, A))^R$ be two pointwise finitely supported families. Then, the family $(f_q \star g_r)_{(q,r)\in Q\times R} \in (\operatorname{Hom}(C, A))^{Q\times R}$ is pointwise finitely supported, and satisfies*

$$\sum_{(q,r)\in Q\times R} (f_q \star g_r) = \left(\sum_{q\in Q} f_q\right) \star \left(\sum_{r\in R} g_r\right).$$

Roughly speaking, the above three propositions say that sums of the form $\sum_{q\in Q} f_q$ (where $(f_q)_{q\in Q}$ is a pointwise finitely supported family) satisfy

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- If Q is a finite set, then any family $(f_q)_{q\in Q} \in (\operatorname{Hom}(C, A))^Q$ is pointwise finitely supported.
- More generally, any finitely supported family $(f_q)_{q\in Q} \in (\operatorname{Hom}(C, A))^Q$ is pointwise finitely supported.
- If C is a graded $\mathbf{k}$-module, and if $(f_n)_{n\in\mathbb{N}} \in (\operatorname{Hom}(C, A))^{\mathbb{N}}$ is a family of maps such that $f_n(C_m) = 0$ whenever $n \neq m$, then the family $(f_n)_{n\in\mathbb{N}}$ is pointwise finitely supported.
- If C is a graded $\mathbf{k}$-coalgebra and A is any $\mathbf{k}$-algebra, and if $f \in \operatorname{Hom}(C, A)$ satisfies $f(C_0) = 0$, then the family $(f^{\star n})_{n\in\mathbb{N}} \in (\operatorname{Hom}(C, A))^{\mathbb{N}}$ is pointwise finitely supported. (This will be proven in Proposition 1.7.11(h).)

[53]This definition of $\sum_{q\in Q} f_q$ generalizes the usual definition of $\sum_{q\in Q} f_q$ when Q is a finite set (because if Q is a finite set, then any family $(f_q)_{q\in Q} \in (\operatorname{Hom}(C, A))^Q$ is pointwise finitely supported).

[54]See Exercise 1.7.9 below for the proofs of these properties.

the usual rules for finite sums. Furthermore, the following properties of pointwise finitely supported families hold:

Proposition 1.7.7. *Let $(f_q)_{q\in Q} \in (\operatorname{Hom}(C,A))^Q$ be a pointwise finitely supported family. Let $(\lambda_q)_{q\in Q} \in \mathbf{k}^Q$ be any family of elements of $\mathbf{k}$. Then, the family $(\lambda_q f_q)_{q\in Q} \in (\operatorname{Hom}(C,A))^Q$ is pointwise finitely supported.*

Proposition 1.7.8. *Let $(f_q)_{q\in Q} \in (\operatorname{Hom}(C,A))^Q$ and $(g_q)_{q\in Q} \in (\operatorname{Hom}(C,A))^Q$ be two families such that $(f_q)_{q\in Q}$ is pointwise finitely supported. Then, the family $(f_q \star g_q)_{q\in Q} \in (\operatorname{Hom}(C,A))^Q$ is also pointwise finitely supported.*

Exercise 1.7.9. Prove Propositions 1.7.4, 1.7.5, 1.7.6, 1.7.7 and 1.7.8.

We can now define the notion of a "pointwise $\star$-nilpotent" map. Roughly speaking, this will mean an element of $(\operatorname{Hom}(C,A),\star)$ that can be substituted into any power series because its powers (with respect to the convolution $\star$) form a pointwise finitely supported family. Here is the definition:

Definition 1.7.10. (a) A map $f \in \operatorname{Hom}(C,A)$ is said to be *pointwise $\star$-nilpotent* if and only if the family $(f^{\star n})_{n\in\mathbb{N}} \in (\operatorname{Hom}(C,A))^{\mathbb{N}}$ is pointwise finitely supported. Equivalently, a map $f \in \operatorname{Hom}(C,A)$ is pointwise $\star$-nilpotent if and only if for each $x \in C$, the family $(f^{\star n}(x))_{n\in\mathbb{N}}$ of elements of A is finitely supported.

(b) If $f \in \operatorname{Hom}(C,A)$ is a pointwise $\star$-nilpotent map, and if $(\lambda_n)_{n\in\mathbb{N}} \in \mathbf{k}^{\mathbb{N}}$ is any family of scalars, then the family $(\lambda_n f^{\star n})_{n\in\mathbb{N}} \in (\operatorname{Hom}(C,A))^{\mathbb{N}}$ is pointwise finitely supported[55], and thus the infinite sum $\sum_{n\geq 0} \lambda_n f^{\star n} = \sum_{n\in\mathbb{N}} \lambda_n f^{\star n}$ is well-defined and belongs to $\operatorname{Hom}(C,A)$ (by Proposition 1.7.4).[56]

(c) We let $\mathfrak{n}(C,A)$ be the set of all pointwise $\star$-nilpotent maps $f \in \operatorname{Hom}(C,A)$. Note that this is not necessarily a $\mathbf{k}$-submodule of $\operatorname{Hom}(C,A)$.

(d) Consider the ring $\mathbf{k}[[T]]$ of formal power series in an indeterminate T over $\mathbf{k}$. For any power series $u \in \mathbf{k}[[T]]$ and any $f \in \mathfrak{n}(C,A)$, we define a map $u^\star(f) \in \operatorname{Hom}(C,A)$ by $u^\star(f) = \sum_{n\geq 0} u_n f^{\star n}$, where u is written in the form $u = \sum_{n\geq 0} u_n T^n$ with $(u_n)_{n\geq 0} \in \mathbf{k}^{\mathbb{N}}$. (This sum $\sum_{n\geq 0} u_n f^{\star n}$ is well-defined in $\operatorname{Hom}(C,A)$, since f is pointwise $\star$-nilpotent.)

The following proposition gathers some properties of pointwise $\star$-nilpotent maps[57]:

[55]This follows easily from Proposition 1.7.7 above. (In fact, the map f is pointwise $\star$-nilpotent, and thus the family $(f^{\star n})_{n\in\mathbb{N}} \in (\operatorname{Hom}(C,A))^{\mathbb{N}}$ is pointwise finitely supported (by the definition of "pointwise $\star$-nilpotent"). Hence, Proposition 1.7.7 (applied to $Q = \mathbb{N}$ and $(f_q)_{q\in Q} = (f^{\star n})_{n\in\mathbb{N}}$ and $(\lambda_q)_{q\in Q} = (\lambda_n)_{n\in\mathbb{N}}$) shows that the family $(\lambda_n f^{\star n})_{n\in\mathbb{N}} \in (\operatorname{Hom}(C,A))^{\mathbb{N}}$ is pointwise finitely supported.)

[56]Notice that the concept of "local $\star$-nilpotence" we used in the proof of Proposition 1.4.24 serves the same function (viz., ensuring that the sum $\sum_{n\in\mathbb{N}} \lambda_n f^{\star n}$ is well-defined). But local $\star$-nilpotence is only defined when a grading is present, whereas pointwise $\star$-nilpotence is defined in the general case. Also, local $\star$-nilpotence is more restrictive (i.e., a locally $\star$-nilpotent map is always pointwise $\star$-nilpotent, but the converse does not always hold).

[57]See Exercise 1.7.13 below for the proofs of these properties.

Proposition 1.7.11. (a) *For any $f \in \mathfrak{n}(C, A)$ and $k \in \mathbb{N}$, we have*

$$\left(T^k\right)^\star (f) = f^{\star k}. \tag{1.7.1}$$

(b) *For any $f \in \mathfrak{n}(C, A)$ and $u, v \in \mathbf{k}[[T]]$, we have*

$$(u+v)^\star (f) = u^\star (f) + v^\star (f) \qquad \text{and} \tag{1.7.2}$$

$$(uv)^\star (f) = u^\star (f) \star v^\star (f). \tag{1.7.3}$$

Also, for any $f \in \mathfrak{n}(C, A)$ and $u \in \mathbf{k}[[T]]$ and $\lambda \in \mathbf{k}$, we have

$$(\lambda u)^\star (f) = \lambda u^\star (f). \tag{1.7.4}$$

Also, for any $f \in \mathfrak{n}(C, A)$, we have

$$0^\star (f) = 0 \qquad \text{and} \tag{1.7.5}$$

$$1^\star (f) = u_A \epsilon_C. \tag{1.7.6}$$

(c) *If $f, g \in \mathfrak{n}(C, A)$ satisfy $f \star g = g \star f$, then $f + g \in \mathfrak{n}(C, A)$.*
(d) *For any $\lambda \in \mathbf{k}$ and $f \in \mathfrak{n}(C, A)$, we have $\lambda f \in \mathfrak{n}(C, A)$.*
(e) *If $f \in \mathfrak{n}(C, A)$ and $g \in \operatorname{Hom}(C, A)$ satisfy $f \star g = g \star f$, then $f \star g \in \mathfrak{n}(C, A)$.*
(f) *If $v \in \mathbf{k}[[T]]$ is a power series whose constant term is 0, then $v^\star (f) \in \mathfrak{n}(C, A)$ for each $f \in \mathfrak{n}(C, A)$.*
(g) *If $u, v \in \mathbf{k}[[T]]$ are two power series such that the constant term of v is 0, and if $f \in \mathfrak{n}(C, A)$ is arbitrary, then*

$$(u[v])^\star (f) = u^\star (v^\star (f)). \tag{1.7.7}$$

Here, $u[v]$ denotes the composition *of u with v; this is the power series obtained by substituting v for T in u. (This power series is well-defined, since v has constant term 0.) Furthermore, notice that the right hand side of (1.7.7) is well-defined, since Proposition 1.7.11(f) shows that $v^\star (f) \in \mathfrak{n}(C, A)$.*
(h) *If C is a graded $\mathbf{k}$-coalgebra, and if $f \in \operatorname{Hom}(C, A)$ satisfies $f(C_0) = 0$, then $f \in \mathfrak{n}(C, A)$.*
(i) *If B is any $\mathbf{k}$-algebra, and if $s : A \to B$ is any $\mathbf{k}$-algebra homomorphism, then every $u \in \mathbf{k}[[T]]$ and $f \in \mathfrak{n}(C, A)$ satisfy*

$$s \circ f \in \mathfrak{n}(C, B) \qquad \text{and} \qquad u^\star (s \circ f) = s \circ (u^\star (f)).$$

(j) *If C is a connected graded $\mathbf{k}$-bialgebra, and if $F : C \to A$ is a $\mathbf{k}$-algebra homomorphism, then $F - u_A \epsilon_C \in \mathfrak{n}(C, A)$.*

Example 1.7.12. Let C be a graded $\mathbf{k}$-coalgebra. Let $f \in \operatorname{Hom}(C, A)$ be such that $f(C_0) = 0$. Then, we claim that the map $u_A \epsilon_C + f : C \to A$ is $\star$-invertible. (This observation has already been made in the proof of Proposition 1.4.24, at least in the particular case when $C = A$.)

Let us see how this claim follows from Proposition 1.7.11. First, Proposition 1.7.11(h) shows that $f \in \mathfrak{n}(C, A)$. Now, define a power series $u \in \mathbf{k}[[T]]$ by $u = 1 + T$. Then, the power series u has constant term 1, and thus has a multiplicative inverse $v = u^{-1} \in \mathbf{k}[[T]]$. Consider this v. (Explicitly, $v = \sum_{n \geq 0} (-1)^n T^n$, but this does not matter for us.) Now, (1.7.3) yields $(uv)^\star (f) = u^\star (f) \star v^\star (f)$. Since $uv = 1$ (because $v = u^{-1}$), we have $(uv)^\star (f) = 1^\star (f) = u_A \epsilon_C$ (by (1.7.6)). Thus, $u^\star (f) \star v^\star (f) = (uv)^\star (f) = u_A \epsilon_C$. Hence, the map $u^\star (f)$ has a right $\star$-inverse.

Also, from $u = 1 + T$, we obtain

$$u^\star(f) = (1+T)^\star(f) = \underbrace{1^\star(f)}_{=u_A\epsilon_C} + \underbrace{T^\star(f)}_{\substack{=f^{\star 1} \\ \text{(by (1.7.1), applied to } k=1)}} \qquad \text{(by (1.7.2))}$$

$$= u_A\epsilon_C + \underbrace{f^{\star 1}}_{=f} = u_A\epsilon_C + f.$$

Thus, the map $u_A\epsilon_C + f$ has a right $\star$-inverse (since the map $u^\star(f)$ has a right $\star$-inverse). A similar argument shows that this map $u_A\epsilon_C + f$ has a left $\star$-inverse. Consequently, the map $u_A\epsilon_C + f$ is $\star$-invertible.

Exercise 1.7.13. Prove Proposition 1.7.11.

Definition 1.7.14. (a) For the rest of Section 1.7, we assume that $\mathbf{k}$ is a commutative $\mathbb{Q}$-algebra. Thus, the two formal power series $\exp = \sum_{n\geq 0} \dfrac{1}{n!} T^n \in \mathbf{k}[[T]]$ and $\log(1+T) = \sum_{n\geq 1} \dfrac{(-1)^{n-1}}{n} T^n \in \mathbf{k}[[T]]$ are well-defined.

(b) Define two power series $\overline{\exp} \in \mathbf{k}[[T]]$ and $\overline{\log} \in \mathbf{k}[[T]]$ by $\overline{\exp} = \exp - 1$ and $\overline{\log} = \log(1+T)$.

(c) If u and v are two power series in $\mathbf{k}[[T]]$ such that v has constant term 0, then $u[v]$ denotes the *composition* of u with v; this is the power series obtained by substituting v for T in u.

The following proposition is just a formal analogue of the well-known fact that the exponential function and the logarithm are mutually inverse (on their domains of definition):[58]

Proposition 1.7.15. *Both power series* $\overline{\exp}$ *and* $\overline{\log}$ *have constant term* 0 *and satisfy* $\overline{\exp}\left[\overline{\log}\right] = T$ *and* $\overline{\log}\left[\overline{\exp}\right] = T$.

For any map $f \in \mathfrak{n}(C,A)$, the power series $\exp$, $\overline{\exp}$ and $\overline{\log}$ give rise to three further maps $\exp^\star f$, $\overline{\exp}^\star f$ and $\overline{\log}^\star f$. We can also define a map $\log^\star g$ whenever g is a map in $\operatorname{Hom}(C,A)$ satisfying $g - u_A\epsilon_C \in \mathfrak{n}(C,A)$ (but we cannot define $\log^\star f$ for $f \in \mathfrak{n}(C,A)$, since $\log$ is not per se a power series); in order to do this, we need a simple lemma:

Lemma 1.7.16. *Let* $g \in \operatorname{Hom}(C,A)$ *be such that* $g - u_A\epsilon_C \in \mathfrak{n}(C,A)$. *Then,* $\overline{\log}^\star(g - u_A\epsilon_C)$ *is a well-defined element of* $\mathfrak{n}(C,A)$.

Definition 1.7.17. If $g \in \operatorname{Hom}(C,A)$ is a map satisfying $g - u_A\epsilon_C \in \mathfrak{n}(C,A)$, then we define a map $\log^\star g \in \mathfrak{n}(C,A)$ by $\log^\star g = \overline{\log}^\star(g - u_A\epsilon_C)$. (This is well-defined, according to Lemma 1.7.16.)

Proposition 1.7.18. (a) *Each* $f \in \mathfrak{n}(C,A)$ *satisfies* $\exp^\star f - u_A\epsilon_C \in \mathfrak{n}(C,A)$ *and*

$$\log^\star(\exp^\star f) = f.$$

(b) *Each* $g \in \operatorname{Hom}(C,A)$ *satisfying* $g - u_A\epsilon_C \in \mathfrak{n}(C,A)$ *satisfies*

$$\exp^\star(\log^\star g) = g.$$

(c) *If* $f, g \in \mathfrak{n}(C,A)$ *satisfy* $f \star g = g \star f$, *then* $f + g \in \mathfrak{n}(C,A)$ *and* $\exp^\star(f+g) = (\exp^\star f) \star (\exp^\star g)$.

[58]See Exercise 1.7.20 below for the proof of this proposition, as well as of the lemma and proposition that follow afterwards.

(d) *The* $\mathbf{k}$*-linear map* $0 : C \to A$ *satisfies* $0 \in \mathfrak{n}(C, A)$ *and* $\exp^{\star} 0 = u_A \epsilon_C$.

(e) *If* $f \in \mathfrak{n}(C, A)$ *and* $n \in \mathbb{N}$*, then* $nf \in \mathfrak{n}(C, A)$ *and* $\exp^{\star}(nf) = (\exp^{\star} f)^{\star n}$.

(f) *If* $f \in \mathfrak{n}(C, A)$*, then*

$$\log^{\star}(f + u_A \epsilon_C) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} f^{\star n}. \tag{1.7.8}$$

Example 1.7.19. Consider again the Hopf algebra $\mathbf{k}[x]$ from Exercise 1.6.4. Let $c_1 : \mathbf{k}[x] \to \mathbf{k}$ be the $\mathbf{k}$-linear map sending each polynomial $p \in \mathbf{k}[x]$ to the coefficient of x^1 in p. (In other words, c_1 sends each polynomial $p \in \mathbf{k}[x]$ to its derivative at 0.)

Then, $c_1((\mathbf{k}[x])_0) = 0$ (as can easily be seen). Hence, Proposition 1.7.11(h) shows that $c_1 \in \mathfrak{n}(\mathbf{k}[x], \mathbf{k})$. Thus, a map $\exp^{\star}(c_1) : \mathbf{k}[x] \to \mathbf{k}$ is well-defined. It is not hard to see that this map is explicitly given by

$$(\exp^{\star}(c_1))(p) = p(1) \qquad \text{for every } p \in \mathbf{k}[x].$$

(In fact, this follows easily after showing that each $n \in \mathbb{N}$ satisfies

$$(c_1)^{\star n}(p) = n! \cdot (\text{the coefficient of } x^n \text{ in } p) \qquad \text{for every } p \in \mathbf{k}[x],$$

which in turn is easily seen by induction.)

Note that the equality $(\exp^{\star}(c_1))(p) = p(1)$ shows that the map $\exp^{\star}(c_1)$ is a $\mathbf{k}$-algebra homomorphism. This is a particular case of a fact that we will soon see (Proposition 1.7.23).

Exercise 1.7.20. Prove Proposition 1.7.15, Lemma 1.7.16 and Proposition 1.7.18.

Next, we state another sequence of facts (some of which have nothing to do with Hopf algebras), beginning with a fact about convolutions which is similar to Proposition 1.4.3:[59]

Proposition 1.7.21. *Let* C *and* C' *be two* $\mathbf{k}$*-coalgebras, and let* A *and* A' *be two* $\mathbf{k}$*-algebras. Let* $\gamma : C \to C'$ *be a* $\mathbf{k}$*-coalgebra morphism. Let* $\alpha : A \to A'$ *be a* $\mathbf{k}$*-algebra morphism.*

(a) *If* $f \in \operatorname{Hom}(C, A)$*,* $g \in \operatorname{Hom}(C, A)$*,* $f' \in \operatorname{Hom}(C', A')$ *and* $g' \in \operatorname{Hom}(C', A')$ *satisfy* $f' \circ \gamma = \alpha \circ f$ *and* $g' \circ \gamma = \alpha \circ g$*, then* $(f' \star g') \circ \gamma = \alpha \circ (f \star g)$.

(b) *If* $f \in \operatorname{Hom}(C, A)$ *and* $f' \in \operatorname{Hom}(C', A')$ *satisfy* $f' \circ \gamma = \alpha \circ f$*, then each* $n \in \mathbb{N}$ *satisfies* $(f')^{\star n} \circ \gamma = \alpha \circ f^{\star n}$.

Proposition 1.7.22. *Let* C *be a* $\mathbf{k}$*-bialgebra. Let* A *be a commutative* $\mathbf{k}$*-algebra. Let* $f \in \operatorname{Hom}(C, A)$ *be such that* $f\left((\ker \epsilon)^2\right) = 0$ *and* $f(1) = 0$*. Then, any* $x, y \in C$ *and* $n \in \mathbb{N}$ *satisfy*

$$f^{\star n}(xy) = \sum_{i=0}^{n} \binom{n}{i} f^{\star i}(x) f^{\star(n-i)}(y).$$

Proposition 1.7.23. *Let* C *be a* $\mathbf{k}$*-bialgebra. Let* A *be a commutative* $\mathbf{k}$*-algebra. Let* $f \in \mathfrak{n}(C, A)$ *be such that* $f\left((\ker \epsilon)^2\right) = 0$ *and* $f(1) = 0$*. Then,* $\exp^{\star} f : C \to A$ *is a* $\mathbf{k}$*-algebra homomorphism.*

[59]See Exercise 1.7.28 below for their proofs.

Lemma 1.7.24. *Let V be any torsionfree abelian group (written additively). Let $N \in \mathbb{N}$. For every $k \in \{0, 1, \ldots, N\}$, let w_k be an element of V. Assume that*

$$\sum_{k=0}^{N} w_k n^k = 0 \qquad \text{for all } n \in \mathbb{N}. \tag{1.7.9}$$

Then, $w_k = 0$ for every $k \in \{0, 1, \ldots, N\}$.

Lemma 1.7.25. *Let V be a torsionfree abelian group (written additively). Let $(w_k)_{k \in \mathbb{N}} \in V^{\mathbb{N}}$ be a finitely supported family of elements of V. Assume that*

$$\sum_{k \in \mathbb{N}} w_k n^k = 0 \qquad \text{for all } n \in \mathbb{N}.$$

Then, $w_k = 0$ for every $k \in \mathbb{N}$.

Proposition 1.7.26. *Let C be a graded $\mathbf{k}$-bialgebra. Let A be a commutative $\mathbf{k}$-algebra. Let $f \in \operatorname{Hom}(C, A)$ be such that $f(C_0) = 0$. Assume that*[60] *$\exp^\star f : C \to A$ is a $\mathbf{k}$-algebra homomorphism. Then, $f\left((\ker \epsilon)^2\right) = 0$.*

Proposition 1.7.27. *Let C be a connected graded $\mathbf{k}$-bialgebra. Let A be a commutative $\mathbf{k}$-algebra. Let $f \in \mathfrak{n}(C, A)$ be such that $f\left((\ker \epsilon)^2\right) = 0$ and $f(1) = 0$. Assume further that $f(C)$ generates the $\mathbf{k}$-algebra A. Then, $\exp^\star f : C \to A$ is a surjective $\mathbf{k}$-algebra homomorphism.*

Exercise 1.7.28. Prove Lemmas 1.7.24 and 1.7.25 and Propositions 1.7.21, 1.7.22, 1.7.23, 1.7.26 and 1.7.27.

[**Hint:** For Proposition 1.7.26, show first that $\exp^\star(nf) = (\exp^\star f)^{\star n}$ is a $\mathbf{k}$-algebra homomorphism for each $n \in \mathbb{N}$. Turn this into an equality between polynomials in n, and use Lemma 1.7.25.]

With these preparations, we can state our version of Leray's theorem:

Theorem 1.7.29. *Let A be a commutative connected graded $\mathbf{k}$-bialgebra.*[61]

(a) *We have $\operatorname{id}_A - u_A \epsilon_A \in \mathfrak{n}(A, A)$; thus, the map $\log^\star(\operatorname{id}_A) \in \mathfrak{n}(A, A)$ is well-defined. We denote this map $\log^\star(\operatorname{id}_A)$ by $\mathfrak{e}$.*

(b) *We have $\ker \mathfrak{e} = \mathbf{k} \cdot 1_A + (\ker \epsilon)^2$ and $\mathfrak{e}(A) \cong (\ker \epsilon) / (\ker \epsilon)^2$ (as $\mathbf{k}$-modules).*

(c) *For each $\mathbf{k}$-module V, let ι_V be the canonical inclusion $V \to \operatorname{Sym} V$. Let $\mathfrak{q}$ be the map*

$$A \xrightarrow{\mathfrak{e}} \mathfrak{e}(A) \xrightarrow{\iota_{\mathfrak{e}(A)}} \operatorname{Sym}(\mathfrak{e}(A)).$$

Then, $\mathfrak{q} \in \mathfrak{n}(A, \operatorname{Sym}(\mathfrak{e}(A)))$[62]*.*

(d) *Let $\mathbf{i}$ be the canonical inclusion $\mathfrak{e}(A) \to A$. Recall the universal property of the symmetric algebra: If V is a $\mathbf{k}$-module, if W is a commutative $\mathbf{k}$-algebra, and if $\varphi : V \to W$ is any $\mathbf{k}$-linear map, then there exists a unique $\mathbf{k}$-algebra homomorphism $\Phi : \operatorname{Sym} V \to W$ satisfying $\varphi = \Phi \circ \iota_V$. Applying this to $V = \mathfrak{e}(A)$, $W = A$ and $\varphi = \mathbf{i}$,*

[60] Notice that $\exp^\star f$ is well-defined, since Proposition 1.7.11(h) yields $f \in \mathfrak{n}(C, A)$.

[61] Keep in mind that $\mathbf{k}$ is assumed to be a commutative $\mathbb{Q}$-algebra.

[62] Do not mistake the map $\mathfrak{q}$ for $\mathfrak{e}$. While every $a \in A$ satisfies $\mathfrak{q}(a) = \mathfrak{e}(a)$, the two maps $\mathfrak{q}$ and $\mathfrak{e}$ have different target sets, and thus we do **not** have $(\exp^\star \mathfrak{q})(a) = (\exp^\star \mathfrak{e})(a)$ for every $a \in A$.

we conclude that there exists a unique $\mathbf{k}$*-algebra homomorphism* $\Phi : \operatorname{Sym}(\mathfrak{e}(A)) \to A$ *satisfying* $\mathbf{i} = \Phi \circ \iota_{\mathfrak{e}(A)}$*. Denote this* Φ *by* $\mathfrak{s}$*. Then, the maps* $\exp^\star \mathfrak{q} : A \to \operatorname{Sym}(\mathfrak{e}(A))$ *and* $\mathfrak{s} : \operatorname{Sym}(\mathfrak{e}(A)) \to A$ *are mutually inverse* $\mathbf{k}$*-algebra isomorphisms.*

(e) *We have* $A \cong \operatorname{Sym}\left((\ker \epsilon) / (\ker \epsilon)^2\right)$ *as* $\mathbf{k}$*-algebras.*

(f) *The map* $\mathfrak{e} : A \to A$ *is a projection (i.e., it satisfies* $\mathfrak{e} \circ \mathfrak{e} = \mathfrak{e}$*).*

Remark 1.7.30. (a) The main upshot of Theorem 1.7.29 is that any commutative connected graded $\mathbf{k}$-bialgebra A (where $\mathbf{k}$ is a commutative $\mathbb{Q}$-algebra) is isomorphic **as a k-algebra** to the symmetric algebra $\operatorname{Sym} W$ of some $\mathbf{k}$-module W. (Specifically, Theorem 1.7.29(e) claims this for $W = (\ker \epsilon) / (\ker \epsilon)^2$, whereas Theorem 1.7.29(d) claims this for $W = \mathfrak{e}(A)$; these two modules W are isomorphic by Theorem 1.7.29(b).) This is a useful statement even without any specific knowledge about W, since symmetric algebras are a far tamer class of algebras than arbitrary commutative algebras. For example, if $\mathbf{k}$ is a field, then symmetric algebras are just polynomial algebras (up to isomorphism). This can be applied, for example, to the case of the shuffle algebra $\operatorname{Sh}(V)$ of a $\mathbf{k}$-module V. The consequence is that the shuffle algebra $\operatorname{Sh}(V)$ of any $\mathbf{k}$-module V (where $\mathbf{k}$ is a commutative $\mathbb{Q}$-algebra) is isomorphic **as a k-algebra** to a symmetric algebra $\operatorname{Sym} W$. When V is a free $\mathbf{k}$-module, one can actually show that $\operatorname{Sh}(V)$ is isomorphic **as a k-algebra** to the symmetric algebra of a **free** $\mathbf{k}$-module W (that is, to a polynomial ring over $\mathbf{k}$); however, this W is not easy to characterize. Such a characterization is given by *Radford's theorem* (Theorem 6.3.4 below) using the concept of *Lyndon words*. Notice that if V has rank ≥ 2, then W is not finitely generated.

(b) The isomorphism in Theorem 1.7.29(e) is generally not an isomorphism of Hopf algebras. However, with a little (rather straightforward) work, it reveals to be an isomorphism of **graded** $\mathbf{k}$-algebras. Actually, all maps mentioned in Theorem 1.7.29 are graded, provided that we use the appropriate gradings for $\mathfrak{e}(A)$ and $\operatorname{Sym}(\mathfrak{e}(A))$. (To define the appropriate grading for $\mathfrak{e}(A)$, we must show that $\mathfrak{e}$ is a graded map, whence $\mathfrak{e}(A)$ is a homogeneous submodule of A; this provides $\mathfrak{e}(A)$ with the grading we seek. The grading on $\operatorname{Sym}(\mathfrak{e}(A))$ then follows from the usual definition of the grading on the symmetric algebra $\operatorname{Sym} V$ of a graded $\mathbf{k}$-module V: Namely, if V is a graded $\mathbf{k}$-module, then the n-th graded component of $\operatorname{Sym} V$ is defined to be the span of all products of the form $v_1 v_2 \cdots v_k \in \operatorname{Sym} V$, where $v_1, v_2, \ldots, v_k \in V$ are homogeneous elements satisfying $\deg(v_1) + \deg(v_2) + \cdots + \deg(v_k) = n$.)

(c) The map $\mathfrak{e} : A \to A$ from Theorem 1.7.29 is called the *Eulerian idempotent* of A.

(d) Theorem 1.7.29 is concerned with commutative bialgebras. Most of its claims have a "dual version", concerning cocommutative bialgebras. Again, the Eulerian idempotent plays a crucial role; but the result characterizes not the $\mathbf{k}$-algebra structure on A, but the $\mathbf{k}$-coalgebra structure on A. This leads to the Cartier-Milnor-Moore theorem; see [35, §3.8] and [60, §3.2]. We shall say a bit about the Eulerian idempotent for a cocommutative bialgebra in Exercises 5.4.6 and 5.4.8.

Example 1.7.31. Consider the symmetric algebra $\operatorname{Sym} V$ of a $\mathbf{k}$-module V. Then, $\operatorname{Sym} V$ is a commutative connected graded $\mathbf{k}$-bialgebra, and thus Theorem 1.7.29 can be applied to $A = \operatorname{Sym} V$. What is the projection $\mathfrak{e} : A \to A$ obtained in this case?

Theorem 1.7.29(b) shows that its kernel is

$$\operatorname{Ker} \mathfrak{e} = \underbrace{\mathbf{k} \cdot 1_A}_{=\operatorname{Sym}^0 V} + \underbrace{(\ker \epsilon)^2}_{=\sum_{n \geq 2} \operatorname{Sym}^n V} = \operatorname{Sym}^0 V + \sum_{n \geq 2} \operatorname{Sym}^n V$$

$$= \sum_{n \neq 1} \operatorname{Sym}^n V. \tag{1.7.10}$$

This does not yet characterize $\mathfrak{e}$ completely, because we have yet to determine the action of $\mathfrak{e}$ on $\operatorname{Sym}^1 V$. Fortunately, the elements of $\operatorname{Sym}^1 V$ are all primitive (recall that $\Delta_{\operatorname{Sym} V}(v) = 1 \otimes v + v \otimes 1$ for each $v \in V$), and it can easily be shown that the map $\mathfrak{e}$ fixes any primitive element of A [63]. Therefore, the map $\mathfrak{e}$ fixes all elements of $\operatorname{Sym}^1 V$. Since we also know that $\mathfrak{e}$ annihilates all elements of $\sum_{n \neq 1} \operatorname{Sym}^n V$ (by (1.7.10)), we thus conclude that $\mathfrak{e}$ is the canonical projection from the direct sum $\operatorname{Sym} V = \bigoplus_{n \in \mathbb{N}} \operatorname{Sym}^n V$ onto its addend $\operatorname{Sym}^1 V$.

Example 1.7.32. For this example, let A be the shuffle algebra $\operatorname{Sh}(V)$ of a $\mathbf{k}$-module V. (See Proposition 1.6.7 for its definition, and keep in mind that its product is being denoted by $\sqcup\!\sqcup$, whereas the notation uv is still being used for the product of two elements u and v in the **tensor** algebra $T(V)$.)

Theorem 1.7.29 can be applied to $A = \operatorname{Sh}(V)$. What is the projection $\mathfrak{e} : A \to A$ obtained in this case?

Let us compute $\mathfrak{e}(v_1 v_2)$ for two elements $v_1, v_2 \in V$. Indeed, define a map $\widetilde{\operatorname{id}} : A \to A$ by $\widetilde{\operatorname{id}} = \operatorname{id}_A - u_A \epsilon_A$. Then, $\widetilde{\operatorname{id}} \in \mathfrak{n}(A, A)$ and $\log^\star \left(\underbrace{\widetilde{\operatorname{id}} + u_A \epsilon_A}_{=\operatorname{id}_A} \right) = \log^\star(\operatorname{id}_A) = \mathfrak{e}$. Hence, (1.7.8) (applied to $C = A$ and $f = \widetilde{\operatorname{id}}$) shows that

$$\mathfrak{e} = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \widetilde{\operatorname{id}}^{\star n}. \tag{1.7.11}$$

Thus, we need to compute $\widetilde{\operatorname{id}}^{\star n}(v_1 v_2)$ for each $n \geq 1$.

Notice that the map $\widetilde{\operatorname{id}}$ annihilates A_0, but fixes any element of A_k for $k > 0$. Thus,

$$\widetilde{\operatorname{id}}(w_1 w_2 \cdots w_k) = \begin{cases} w_1 w_2 \cdots w_k, & \text{if } k > 0; \\ 0, & \text{if } k = 0 \end{cases} \qquad \text{for any } w_1, w_2, \ldots, w_k \in V.$$

But it is easy to see that the map $\widetilde{\operatorname{id}}^{\star n} : A \to A$ annihilates A_k whenever $n > k$. In particular, for every $n > 2$, the map $\widetilde{\operatorname{id}}^{\star n} : A \to A$ annihilates

[63] See Exercise 5.4.6(f) further below for this proof. (While Exercise 5.4.6 requires A to be cocommutative, this requirement is not used in the solution to Exercise 5.4.6(f). That said, this requirement is actually satisfied for $A = \operatorname{Sym} V$, so we do not even need to avoid it here.)

A_2, and therefore satisfies

$$\widetilde{\mathrm{id}}^{\star n}(v_1v_2) = 0 \qquad (\text{since } v_1v_2 \in A_2). \tag{1.7.12}$$

It remains to find $\widetilde{\mathrm{id}}^{\star n}(v_1v_2)$ for $n \in \{1,2\}$.

We have $\widetilde{\mathrm{id}}^{\star 1} = \widetilde{\mathrm{id}}$ and thus

$$\widetilde{\mathrm{id}}^{\star 1}(v_1v_2) = \widetilde{\mathrm{id}}(v_1v_2) = v_1v_2$$

and

$$\begin{aligned}
\widetilde{\mathrm{id}}^{\star 2}(v_1v_2) &= \underbrace{\widetilde{\mathrm{id}}(1)}_{=0} \sqcup\!\sqcup \underbrace{\widetilde{\mathrm{id}}(v_1v_2)}_{=v_1v_2} + \underbrace{\widetilde{\mathrm{id}}(v_1)}_{=v_1} \sqcup\!\sqcup \underbrace{\widetilde{\mathrm{id}}(v_2)}_{=v_2} + \underbrace{\widetilde{\mathrm{id}}(v_1v_2)}_{=v_1v_2} \sqcup\!\sqcup \underbrace{\widetilde{\mathrm{id}}(1)}_{=0} \\
&\qquad (\text{since } \Delta_{\mathrm{Sh}\,V}(v_1v_2) = 1 \otimes v_1v_2 + v_1 \otimes v_2 + v_1v_2 \otimes 1) \\
&= \underbrace{0 \sqcup\!\sqcup (v_1v_2)}_{=0} + \underbrace{v_1 \sqcup\!\sqcup v_2}_{=v_1v_2+v_2v_1} + \underbrace{(v_1v_2) \sqcup\!\sqcup 0}_{=0} \\
&= v_1v_2 + v_2v_1.
\end{aligned}$$

Now, applying both sides of (1.7.11) to v_1v_2, we find

$$\begin{aligned}
&\mathfrak{e}(v_1v_2) \\
&= \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \widetilde{\mathrm{id}}^{\star n}(v_1v_2) \\
&= \underbrace{\frac{(-1)^{1-1}}{1}}_{=1} \underbrace{\widetilde{\mathrm{id}}^{\star 1}(v_1v_2)}_{=v_1v_2} + \underbrace{\frac{(-1)^{2-1}}{2}}_{=\frac{-1}{2}} \underbrace{\widetilde{\mathrm{id}}^{\star 2}(v_1v_2)}_{=v_1v_2+v_2v_1} + \sum_{n\geq 3} \frac{(-1)^{n-1}}{n} \underbrace{\widetilde{\mathrm{id}}^{\star n}(v_1v_2)}_{\substack{=0 \\ (\text{by } (1.7.12))}} \\
&= v_1v_2 + \frac{-1}{2}(v_1v_2 + v_2v_1) + \underbrace{\sum_{n\geq 3} \frac{(-1)^{n-1}}{n} 0}_{=0} = \frac{1}{2}(v_1v_2 - v_2v_1).
\end{aligned}$$

This describes the action of $\mathfrak{e}$ on the graded component A_2 of $A = \mathrm{Sh}(V)$.

Similarly, we can describe $\mathfrak{e}$ acting on any other graded component:

$$\begin{aligned}
\mathfrak{e}(1) &= 0; \\
\mathfrak{e}(v_1) &= v_1 \qquad \text{for each } v_1 \in V; \\
\mathfrak{e}(v_1v_2) &= \frac{1}{2}(v_1v_2 - v_2v_1) \qquad \text{for any } v_1, v_2 \in V; \\
\mathfrak{e}(v_1v_2v_3) &= \frac{1}{6}(2v_1v_2v_3 - v_1v_3v_2 - v_2v_1v_3 - v_2v_3v_1 - v_3v_1v_2 + 2v_3v_2v_1) \\
&\qquad \text{for any } v_1, v_2, v_3 \in V, \\
&\ldots .
\end{aligned}$$

With some more work, one can show the following formula for the action of $\mathfrak{e}$ on any nontrivial pure tensor:

$$\begin{aligned}
&\mathfrak{e}\left(v_1 v_2 \cdots v_n\right) \\
&= \sum_{\sigma \in \mathfrak{S}_n} \left(\sum_{k=1+\operatorname{des}\left(\sigma^{-1}\right)}^{n} \frac{(-1)^{k-1}}{k} \binom{n-1-\operatorname{des}\left(\sigma^{-1}\right)}{k-1-\operatorname{des}\left(\sigma^{-1}\right)} \right) v_{\sigma(1)} v_{\sigma(2)} \cdots v_{\sigma(n)} \\
&= \sum_{\sigma \in \mathfrak{S}_n} \frac{(-1)^{\operatorname{des}\left(\sigma^{-1}\right)}}{\operatorname{des}\left(\sigma^{-1}\right)+1} \binom{n}{\operatorname{des}\left(\sigma^{-1}\right)+1}^{-1} v_{\sigma(1)} v_{\sigma(2)} \cdots v_{\sigma(n)} \\
&\qquad \text{for any } n \geq 1 \text{ and } v_1, v_2, \ldots, v_n \in V,
\end{aligned}$$

where we use the notation $\operatorname{des} \pi$ for the number of descents[64] of any permutation $\pi \in \mathfrak{S}_n$. (A statement essentially dual to this appears in [191, Theorem 9.5].)

Theorem 1.7.29(b) yields $\ker \mathfrak{e} = \mathbf{k} \cdot 1_A + (\ker \epsilon)^2$. Notice, however, that $(\ker \epsilon)^2$ means the square of the ideal $\ker \epsilon$ with respect to the shuffle multiplication $\sqcup\!\sqcup$; thus, $(\ker \epsilon)^2$ is the $\mathbf{k}$-linear span of all shuffle products of the form $a \sqcup\!\sqcup b$ with $a \in \ker \epsilon$ and $b \in \ker \epsilon$.

Exercise 1.7.33. Prove Theorem 1.7.29.

[**Hint:** (a) is easy. For (b), define an element $\widetilde{\operatorname{id}}$ of $\mathfrak{n}(A, A)$ by $\widetilde{\operatorname{id}} = \operatorname{id}_A - u_A \epsilon_A$. Observe that $\mathfrak{e} = \sum_{n \geq 1} \dfrac{(-1)^{n-1}}{n} \widetilde{\operatorname{id}}^{\star n}$, and draw the conclusions that $\mathfrak{e}(1_A) = 0$ and that each $x \in A$ satisfies $\widetilde{\operatorname{id}}(x) - \mathfrak{e}(x) \in (\ker \epsilon)^2$ (because $\widetilde{\operatorname{id}}^{\star n}(x) \in (\ker \epsilon)^2$ for every $n \geq 2$). Use this to prove $\ker \mathfrak{e} \subset \mathbf{k} \cdot 1_A + (\ker \epsilon)^2$. On the other hand, prove $\mathfrak{e}\left((\ker \epsilon)^2\right) = 0$ by applying Proposition 1.7.26. Combine to obtain $\ker \mathfrak{e} = \mathbf{k} \cdot 1_A + (\ker \epsilon)^2$. Finish (b) by showing that $A / \left(\mathbf{k} \cdot 1_A + (\ker \epsilon)^2\right) \cong (\ker \epsilon) / (\ker \epsilon)^2$ as $\mathbf{k}$-modules. Part (c) is easy again. For (d), first apply Proposition 1.7.11(i) to show that $\exp^{\star}(\mathfrak{s} \circ \mathfrak{q}) = \mathfrak{s} \circ (\exp^{\star} \mathfrak{q})$. In light of $\mathfrak{s} \circ \mathfrak{q} = \mathfrak{e}$ and $\exp^{\star} \mathfrak{e} = \operatorname{id}_A$, this becomes $\operatorname{id}_A = \mathfrak{s} \circ (\exp^{\star} \mathfrak{q})$. To obtain part (d), it remains to show that $\exp^{\star} \mathfrak{q}$ is a surjective $\mathbf{k}$-algebra homomorphism; but this follows from Proposition 1.7.27. For (e), combine (d) and (b). For (f), use once again the observation that each $x \in A$ satisfies $\widetilde{\operatorname{id}}(x) - \mathfrak{e}(x) \in (\ker \epsilon)^2$.]

[64] A *descent* of a permutation $\pi \in \mathfrak{S}_n$ means an $i \in \{1, 2, \ldots, n-1\}$ satisfying $\pi(i) > \pi(i+1)$.

2. Review of symmetric functions Λ as Hopf algebra

Here we review the ring of symmetric functions, borrowing heavily from standard treatments, such as Macdonald [142, Chap. I], Sagan [186, Chap. 4], Stanley [206, Chap. 7], and Mendes and Remmel [154], but emphasizing the Hopf structure early on. Other recent references for this subject are [224], [189], [63], [153, Chapters 2–3] and [187, Chapter 7].

2.1. **Definition of Λ.**

As before, $\mathbf{k}$ here is a commutative ring (hence could be a field or the integers $\mathbb{Z}$; these are the usual choices).

Given an infinite variable set $\mathbf{x} = (x_1, x_2, \ldots)$, a monomial $\mathbf{x}^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots$ is indexed by a sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ in $\mathbb{N}^\infty$ having finite support[65]; such sequences α are called *weak compositions*. The nonzero entries of the sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ are called the *parts* of the weak composition α.

The sum $\alpha_1 + \alpha_2 + \alpha_3 + \cdots$ of all entries of a weak composition $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$ (or, equivalently, the sum of all parts of α) is called the *size* of α and denoted by $|\alpha|$.

Consider the $\mathbf{k}$-algebra $\mathbf{k}[[\mathbf{x}]] := \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ of all formal power series in the indeterminates $x_1, x_2, x_3, \ldots$ over $\mathbf{k}$; these series are infinite $\mathbf{k}$-linear combinations $\sum_\alpha c_\alpha \mathbf{x}^\alpha$ (with c_α in $\mathbf{k}$) of the monomials $\mathbf{x}^\alpha$ where α ranges over all weak compositions. The product of two such formal power series is well-defined by the usual multiplication rule.

The *degree* of a monomial $\mathbf{x}^\alpha$ is defined to be the number $\deg(\mathbf{x}^\alpha) := \sum_i \alpha_i \in \mathbb{N}$. Given a number $d \in \mathbb{N}$, we say that a formal power series $f(\mathbf{x}) = \sum_\alpha c_\alpha \mathbf{x}^\alpha \in \mathbf{k}[[\mathbf{x}]]$ (with c_α in $\mathbf{k}$) is *homogeneous of degree* d if every weak composition α satisfying $\deg(\mathbf{x}^\alpha) \neq d$ must satisfy $c_\alpha = 0$. In other words, a formal power series is homogeneous of degree d if it is an infinite $\mathbf{k}$-linear combination of monomials of degree d. Every formal power series $f \in \mathbf{k}[[\mathbf{x}]]$ can be uniquely represented as an infinite sum $f_0 + f_1 + f_2 + \cdots$, where each f_d is homogeneous of degree d; in this case, we refer to each f_d as the *d-th homogeneous component of f*. Note that this does not make $\mathbf{k}[[\mathbf{x}]]$ into a graded $\mathbf{k}$-module, since these sums $f_0 + f_1 + f_2 + \cdots$ can have infinitely many nonzero addends. Nevertheless, if f and g are homogeneous power series of degrees d and e, then fg is homogeneous of degree $d + e$.

A formal power series $f(\mathbf{x}) = \sum_\alpha c_\alpha \mathbf{x}^\alpha \in \mathbf{k}[[\mathbf{x}]]$ (with c_α in $\mathbf{k}$) is said to be *of bounded degree* if there exists some bound $d = d(f) \in \mathbb{N}$ such that every weak composition $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$ satisfying $\deg(\mathbf{x}^\alpha) > d$ must satisfy $c_\alpha = 0$. Equivalently, a formal power series $f \in \mathbf{k}[[\mathbf{x}]]$ is of bounded degree if all but finitely many of its homogeneous components are zero. (For example, $x_1^2 + x_2^2 + x_3^2 + \cdots$ and $1 + x_1 + x_2 + x_3 + \cdots$ are of bounded degree, while $x_1 + x_1x_2 + x_1x_2x_3 + \cdots$ and $1 + x_1 + x_1^2 + x_1^3 + \cdots$ are not.) It is easy to see that the sum and the product of two power series of bounded degree also have bounded degree. Thus, the formal power series of bounded degree form a $\mathbf{k}$-subalgebra of $\mathbf{k}[[\mathbf{x}]]$, which we call $R(\mathbf{x})$. This subalgebra $R(\mathbf{x})$ is graded (by degree).

The symmetric group $\mathfrak{S}_n$ permuting the first n variables $x_1, \ldots, x_n$ acts as a group of automorphisms on $R(\mathbf{x})$, as does the union $\mathfrak{S}_{(\infty)} = \bigcup_{n \geq 0} \mathfrak{S}_n$

[65]The *support* of a sequence $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots) \in \mathbb{N}^\infty$ is defined to be the set of all positive integers i for which $\alpha_i \neq 0$.

of the infinite ascending chain $\mathfrak{S}_0 \subset \mathfrak{S}_1 \subset \mathfrak{S}_2 \subset \cdots$ of symmetric groups[66]. This group $\mathfrak{S}_{(\infty)}$ can also be described as the group of all permutations of the set $\{1, 2, 3, \ldots\}$ which leave all but finitely many elements invariant. It is known as the *finitary symmetric group* on $\{1, 2, 3, \ldots\}$.

The group $\mathfrak{S}_{(\infty)}$ also acts on the set of all weak compositions by permuting their entries:

$$\sigma(\alpha_1, \alpha_2, \alpha_3, \ldots) = \left(\alpha_{\sigma^{-1}(1)}, \alpha_{\sigma^{-1}(2)}, \alpha_{\sigma^{-1}(3)}, \ldots\right) \quad \text{for any weak composition } (\alpha_1, \alpha_2, \alpha_3, \ldots) \text{ and any } \sigma \in \mathfrak{S}_{(\infty)}.$$

These two actions are connected by the equality $\sigma(\mathbf{x}^\alpha) = \mathbf{x}^{\sigma\alpha}$ for any weak composition α and any $\sigma \in \mathfrak{S}_{(\infty)}$.

Definition 2.1.1. The *ring of symmetric functions in* $\mathbf{x}$ *with coefficients in* $\mathbf{k}$, denoted $\Lambda = \Lambda_{\mathbf{k}} = \Lambda(\mathbf{x}) = \Lambda_{\mathbf{k}}(\mathbf{x})$, is the $\mathfrak{S}_{(\infty)}$-invariant subalgebra $R(\mathbf{x})^{\mathfrak{S}_{(\infty)}}$ of $R(\mathbf{x})$:

$$\begin{aligned}\Lambda &:= \left\{f \in R(\mathbf{x}) : \sigma(f) = f \text{ for all } \sigma \in \mathfrak{S}_{(\infty)}\right\} \\ &= \left\{f = \sum_\alpha c_\alpha \mathbf{x}^\alpha \in R(\mathbf{x}) : c_\alpha = c_\beta \text{ if } \alpha, \beta \text{ lie in the same } \mathfrak{S}_{(\infty)}\text{-orbit}\right\}.\end{aligned}$$

We refer to the elements of Λ as *symmetric functions* (over $\mathbf{k}$); however, despite this terminology, they are not functions in the usual sense.[67]

Note that Λ is a graded $\mathbf{k}$-algebra, since $\Lambda = \bigoplus_{n \geq 0} \Lambda_n$ where Λ_n are the symmetric functions $f = \sum_\alpha c_\alpha \mathbf{x}^\alpha$ which are *homogeneous of degree* n, meaning $\deg(\mathbf{x}^\alpha) = n$ for all $c_\alpha \neq 0$.

Exercise 2.1.2. Let $f \in R(\mathbf{x})$. Let A be a commutative $\mathbf{k}$-algebra, and $a_1, a_2, \ldots, a_k$ be finitely many elements of A. Show that substituting $a_1, a_2, \ldots, a_k, 0, 0, 0, \ldots$ for $x_1, x_2, x_3, \ldots$ in f yields an infinite sum in which all but finitely many addends are zero. Hence, this sum has a value in A, which is commonly denoted by $f(a_1, a_2, \ldots, a_k)$.

Definition 2.1.3. A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell, 0, 0, \ldots)$ is a weak composition whose entries weakly decrease: $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$. The (uniquely defined) ℓ is said to be the *length* of the partition λ and denoted by $\ell(\lambda)$. Thus, $\ell(\lambda)$ is the number of parts[68] of λ. One sometimes omits trailing zeroes from a partition: e.g., one can write the partition $(3, 1, 0, 0, 0, \ldots)$ as $(3, 1)$. We will often (but not always) write λ_i for the i-th entry of the partition λ (for instance, if $\lambda = (5, 3, 1, 1)$, then $\lambda_2 = 3$ and $\lambda_5 = 0$). If λ_i is

[66]This ascending chain is constructed as follows: For every $n \in \mathbb{N}$, there is an injective group homomorphism $\iota_n : \mathfrak{S}_n \to \mathfrak{S}_{n+1}$ which sends every permutation $\sigma \in \mathfrak{S}_n$ to the permutation $\iota_n(\sigma) = \tau \in \mathfrak{S}_{n+1}$ defined by

$$\tau(i) = \begin{cases} \sigma(i), & \text{if } i \leq n; \\ i, & \text{if } i = n+1 \end{cases} \qquad \text{for all } i \in \{1, 2, \ldots, n+1\}.$$

These homomorphisms ι_n for all n form a chain $\mathfrak{S}_0 \xrightarrow{\iota_0} \mathfrak{S}_1 \xrightarrow{\iota_1} \mathfrak{S}_2 \xrightarrow{\iota_2} \cdots$, which is often regarded as a chain of inclusions.

[67]Being power series, they can be evaluated at appropriate families of variables. But this does not make them functions (no more than polynomials are functions). The terminology "symmetric function" is thus not well-chosen; but it is standard.

[68]Recall that a *part* of a partition means a nonzero entry of the partition.

nonzero, we will also call it the *i-th part* of λ. The sum $\lambda_1+\lambda_2+\cdots+\lambda_\ell = \lambda_1+\lambda_2+\cdots$ (where $\ell=\ell(\lambda)$) of all entries of λ (or, equivalently, of all parts of λ) is the size $|\lambda|$ of λ. For a given integer n, the partitions of size n are referred to as the *partitions of* n. The empty partition $() = (0,0,0,\ldots)$ is denoted by $\varnothing$.

Partitions (as defined above) are sometimes called *integer partitions* in order to distinguish them from set partitions.

Every weak composition α lies in the $\mathfrak{S}_{(\infty)}$-orbit of a unique partition $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_\ell,0,0,\ldots)$ with $\lambda_1\geq\cdots\geq\lambda_\ell>0$. For any partition λ, define the *monomial symmetric function*

$$m_\lambda := \sum_{\alpha\in\mathfrak{S}_{(\infty)}\lambda} \mathbf{x}^\alpha. \tag{2.1.1}$$

Letting λ run through the set Par of all partitions, this gives the *monomial* $\mathbf{k}$*-basis* $\{m_\lambda\}$ of Λ. Letting λ run only through the set Par_n of partitions of n gives the monomial $\mathbf{k}$-basis for Λ_n.

Example 2.1.4. For $n=3$, one has

$$\begin{aligned} m_{(3)} &= x_1^3+x_2^3+x_3^3+\cdots, \\ m_{(2,1)} &= x_1^2x_2+x_1x_2^2+x_1^2x_3+x_1x_3^2+\cdots, \\ m_{(1,1,1)} &= x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4+x_1x_2x_5+\cdots. \end{aligned}$$

The monomial basis $\{m_\lambda\}_{\lambda\in\mathrm{Par}}$ of Λ is thus a graded basis[69] of the graded $\mathbf{k}$-module Λ. (Here and in the following, when we say that a basis $\{u_\lambda\}_{\lambda\in\mathrm{Par}}$ indexed by Par is a graded basis of Λ, we tacitly understand that Par is partitioned into $\mathrm{Par}_0,\mathrm{Par}_1,\mathrm{Par}_2,\ldots$, so that for each $n\in\mathbb{N}$, the subfamily $\{u_\lambda\}_{\lambda\in\mathrm{Par}_n}$ should be a basis for Λ_n.)

Remark 2.1.5. We have defined the symmetric functions as the elements of $R(\mathbf{x})$ invariant under the group $\mathfrak{S}_{(\infty)}$. However, they also are the elements of $R(\mathbf{x})$ invariant under the group $\mathfrak{S}_\infty$ of *all* permutations of the set $\{1,2,3,\ldots\}$ (which acts on $R(\mathbf{x})$ in the same way as its subgroup $\mathfrak{S}_{(\infty)}$ does).[70]

Remark 2.1.6. It is sometimes convenient to work with finite variable sets $x_1,\ldots,x_n$, which one justifies as follows. Note that the algebra homomorphism

$$R(\mathbf{x}) \to R(x_1,\ldots,x_n) = \mathbf{k}[x_1,\ldots,x_n]$$

[69]See Definition 1.3.21 for the meaning of “graded basis”.

[70]*Proof.* We need to show that $\Lambda = R(\mathbf{x})^{\mathfrak{S}_\infty}$. Since

$$\Lambda = \left\{ f=\sum_\alpha c_\alpha\mathbf{x}^\alpha \in R(\mathbf{x}) : c_\alpha=c_\beta \text{ if } \alpha,\beta \text{ lie in the same } \mathfrak{S}_{(\infty)}\text{-orbit} \right\}$$

and

$$R(\mathbf{x})^{\mathfrak{S}_\infty} = \left\{ f=\sum_\alpha c_\alpha\mathbf{x}^\alpha \in R(\mathbf{x}) : c_\alpha=c_\beta \text{ if } \alpha,\beta \text{ lie in the same } \mathfrak{S}_{\infty}\text{-orbit} \right\},$$

this will follow immediately if we can show that two weak compositions α and β lie in the same $\mathfrak{S}_{(\infty)}$-orbit if and only if they lie in the same $\mathfrak{S}_\infty$-orbit. But this is straightforward to check (in fact, two weak compositions α and β lie in the same orbit under either group if and only if they have the same multiset of nonzero entries).

which sends $x_{n+1}, x_{n+2}, \ldots$ to 0 restricts to an algebra homomorphism

$$\Lambda_{\mathbf{k}}(\mathbf{x}) \to \Lambda_{\mathbf{k}}(x_1, \ldots, x_n) = \mathbf{k}[x_1, \ldots, x_n]^{\mathfrak{S}_n}.$$

Furthermore, this last homomorphism is a $\mathbf{k}$-module isomorphism when restricted to Λ_i for $0 \leq i \leq n$, since it sends the monomial basis elements $m_\lambda(\mathbf{x})$ to the monomial basis elements $m_\lambda(x_1, \ldots, x_n)$. Thus, when one proves identities in $\Lambda_{\mathbf{k}}(x_1, \ldots, x_n)$ for all n, they are valid in Λ, that is, Λ is the inverse limit of the $\Lambda(x_1, \ldots, x_n)$ in the category of graded $\mathbf{k}$-algebras.[71]

This characterization of Λ as an inverse limit of the graded $\mathbf{k}$-algebras $\Lambda(x_1, \ldots, x_n)$ can be used as an alternative definition of Λ. The definitions used by Macdonald [142] and Wildon [224] are closely related (see [142, §1.2, p. 19, Remark 1], [90, §A.11] and [224, §1.7] for discussions of this definition). It also suggests that much of the theory of symmetric functions can be rewritten in terms of the $\Lambda(x_1, \ldots, x_n)$ (at the cost of extra complexity); and this indeed is possible[72].

One can also define a comultiplication on Λ as follows.

Consider the countably infinite set of variables

$$(\mathbf{x}, \mathbf{y}) = (x_1, x_2, \ldots, y_1, y_2, \ldots).$$

Although it properly contains $\mathbf{x}$, there are nevertheless bijections between $\mathbf{x}$ and $(\mathbf{x}, \mathbf{y})$, since these two variable sets have the same cardinality.

Let $R(\mathbf{x}, \mathbf{y})$ denote the $\mathbf{k}$-algebra of formal power series in $(\mathbf{x}, \mathbf{y})$ of bounded degree. Let $\mathfrak{S}_{(\infty,\infty)}$ be the group of all permutations of $\{x_1, x_2, \ldots, y_1, y_2, \ldots\}$ leaving all but finitely many variables invariant. Then, $\mathfrak{S}_{(\infty,\infty)}$ acts on $R(\mathbf{x}, \mathbf{y})$ by permuting variables, in the same way as $\mathfrak{S}_{(\infty)}$ acts on $R(\mathbf{x})$. The fixed space $R(\mathbf{x}, \mathbf{y})^{\mathfrak{S}_{(\infty,\infty)}}$ is a $\mathbf{k}$-algebra, which we denote by $\Lambda(\mathbf{x}, \mathbf{y})$. This $\mathbf{k}$-algebra $\Lambda(\mathbf{x}, \mathbf{y})$ is isomorphic to $\Lambda = \Lambda(\mathbf{x})$, since there is a bijection between the two sets of variables $(\mathbf{x}, \mathbf{y})$ and $\mathbf{x}$. More explicitly: The map

$$\begin{array}{rcl} \Lambda = \Lambda(\mathbf{x}) & \xrightarrow{\Delta} & \Lambda(\mathbf{x}, \mathbf{y}), \\ f(\mathbf{x}) = f(x_1, x_2, \ldots) & \longmapsto & f(\mathbf{x}, \mathbf{y}) = f(x_1, x_2, \ldots, y_1, y_2, \ldots) \end{array} \tag{2.1.2}$$

is a graded $\mathbf{k}$-algebra isomorphism. Here, $f(x_1, x_2, \ldots, y_1, y_2, \ldots)$ means the result of choosing some bijection $\phi : \{x_1, x_2, x_3, \ldots\} \to \{x_1, x_2, \ldots, y_1, y_2, \ldots\}$ and substituting $\phi(x_i)$ for every x_i in f. (The choice of ϕ is irrelevant since f is symmetric.[73])

The group $\mathfrak{S}_{(\infty)} \times \mathfrak{S}_{(\infty)}$ is a subgroup of the group $\mathfrak{S}_{(\infty,\infty)}$ (via the obvious injection, which lets each $(\sigma, \tau) \in \mathfrak{S}_{(\infty)} \times \mathfrak{S}_{(\infty)}$ act by separately permuting the $x_1, x_2, x_3, \ldots$ using σ and permuting the $y_1, y_2, y_3, \ldots$ using

[71] *Warning:* The word "graded" here is crucial. Indeed, Λ is **not** the inverse limit of the $\Lambda(x_1, \ldots, x_n)$ in the category of $\mathbf{k}$-algebras. In fact, the latter limit is the $\mathbf{k}$-algebra of all symmetric power series f in $\mathbf{k}[\mathbf{x}]$ with the following property: For each $g \in \mathbb{N}$, there exists a $d \in \mathbb{N}$ such that every monomial in f that involves exactly g distinct indeterminates has degree at most d. For example, the power series $(1 + x_1)(1 + x_2)(1 + x_3) \cdots$ and $m_{(1)} + m_{(2,2)} + m_{(3,3,3)} + \cdots$ satisfy this property, although they do not lie in Λ (unless $\mathbf{k}$ is a trivial ring).

[72] See, for example, [119, Chapter SYM], [174] and [138, Chapters 10–11] for various results of this present chapter rewritten in terms of symmetric polynomials in finitely many variables.

[73] To be more precise, the choice of ϕ is irrelevant because f is $\mathfrak{S}_\infty$-invariant, with the notations of Remark 2.1.5.

τ), and thus also acts on $R(\mathbf{x},\mathbf{y})$. Hence, we have an inclusion of $\mathbf{k}$-algebras $\Lambda(\mathbf{x},\mathbf{y}) = R(\mathbf{x},\mathbf{y})^{\mathfrak{S}_{(\infty,\infty)}} \subset R(\mathbf{x},\mathbf{y})^{\mathfrak{S}_{(\infty)}\times\mathfrak{S}_{(\infty)}} \subset R(\mathbf{x},\mathbf{y})$. The $\mathbf{k}$-module $R(\mathbf{x},\mathbf{y})^{\mathfrak{S}_{(\infty)}\times\mathfrak{S}_{(\infty)}}$ has $\mathbf{k}$-basis $\{m_\lambda(\mathbf{x})m_\mu(\mathbf{y})\}_{\lambda,\mu\in\mathrm{Par}}$, since $m_\lambda(\mathbf{x})m_\mu(\mathbf{y})$ is just the sum of all monomials in the $\mathfrak{S}_{(\infty)}\times\mathfrak{S}_{(\infty)}$-orbit of $\mathbf{x}^\lambda\mathbf{y}^\mu$ (and since any $\mathfrak{S}_{(\infty)}\times\mathfrak{S}_{(\infty)}$-orbit of monomials has exactly one representative of the form $\mathbf{x}^\lambda\mathbf{y}^\mu$ with $\lambda,\mu\in\mathrm{Par}$). Here, of course, $\mathbf{y}$ stands for the set of variables $(y_1,y_2,y_3,\ldots)$, and we define $\mathbf{y}^\mu$ to be $y_1^{\mu_1}y_2^{\mu_2}\cdots$.

On the other hand, the map

$$\begin{array}{rcl} R(\mathbf{x})\otimes R(\mathbf{x}) & \longrightarrow & R(\mathbf{x},\mathbf{y}), \\ f(\mathbf{x})\otimes g(\mathbf{x}) & \longmapsto & f(\mathbf{x})g(\mathbf{y}) \end{array}$$

is a $\mathbf{k}$-algebra homomorphism. Restricting it to $R(\mathbf{x})^{\mathfrak{S}_{(\infty)}}\otimes R(\mathbf{x})^{\mathfrak{S}_{(\infty)}}$, we obtain a $\mathbf{k}$-algebra homomorphism

$$\Lambda\otimes\Lambda = R(\mathbf{x})^{\mathfrak{S}_{(\infty)}}\otimes R(\mathbf{x})^{\mathfrak{S}_{(\infty)}} \longrightarrow R(\mathbf{x},\mathbf{y})^{\mathfrak{S}_{(\infty)}\times\mathfrak{S}_{(\infty)}}, \tag{2.1.3}$$

which is an isomorphism because it sends the basis $\{m_\lambda\otimes m_\mu\}_{\lambda,\mu\in\mathrm{Par}}$ of the $\mathbf{k}$-module $\Lambda\otimes\Lambda$ to the basis $\{m_\lambda(\mathbf{x})m_\mu(\mathbf{y})\}_{\lambda,\mu\in\mathrm{Par}}$ of the $\mathbf{k}$-module $R(\mathbf{x},\mathbf{y})^{\mathfrak{S}_{(\infty)}\times\mathfrak{S}_{(\infty)}}$. Thus, we get an inclusion of graded $\mathbf{k}$-algebras

$$\Lambda(\mathbf{x},\mathbf{y}) = R(\mathbf{x},\mathbf{y})^{\mathfrak{S}_{(\infty,\infty)}} \hookrightarrow R(\mathbf{x},\mathbf{y})^{\mathfrak{S}_{(\infty)}\times\mathfrak{S}_{(\infty)}} \cong \Lambda\otimes\Lambda$$

where the last isomorphism is the inverse of the one in (2.1.3). This gives a comultiplication

$$\begin{array}{rcl} \Lambda = \Lambda(\mathbf{x}) & \xrightarrow{\Delta} & \Lambda(\mathbf{x},\mathbf{y}) \hookrightarrow \Lambda\otimes\Lambda, \\ f(\mathbf{x}) = f(x_1,x_2,\ldots) & \longmapsto & f(\mathbf{x},\mathbf{y}) = f(x_1,x_2,\ldots,y_1,y_2,\ldots). \end{array}$$

Here, $f(x_1,x_2,\ldots,y_1,y_2,\ldots)$ is understood as in (2.1.2).

Example 2.1.7. One has

$$\begin{aligned} \Delta m_{(2,1)} &= m_{(2,1)}(x_1,x_2,\ldots,y_1,y_2,\ldots) \\ &= x_1^2x_2 + x_1x_2^2 + \cdots \\ &\quad + x_1^2y_1 + x_1^2y_2 + \cdots \\ &\quad + x_1y_1^2 + x_1y_2^2 + \cdots \\ &\quad + y_1^2y_2 + y_1y_2^2 + \cdots \\ &= m_{(2,1)}(\mathbf{x}) + m_{(2)}(\mathbf{x})m_{(1)}(\mathbf{y}) + m_{(1)}(\mathbf{x})m_{(2)}(\mathbf{y}) + m_{(2,1)}(\mathbf{y}) \\ &= m_{(2,1)}\otimes 1 + m_{(2)}\otimes m_{(1)} + m_{(1)}\otimes m_{(2)} + 1\otimes m_{(2,1)}. \end{aligned}$$

This example generalizes easily to the following formula:

$$\Delta m_\lambda = \sum_{\substack{(\mu,\nu): \\ \mu\sqcup\nu=\lambda}} m_\mu\otimes m_\nu, \tag{2.1.4}$$

in which $\mu\sqcup\nu$ is the partition obtained by taking the multiset union of the parts of μ and ν, and then reordering them to make them weakly decreasing.

Checking that Δ is coassociative amounts to checking that

$$(\Delta\otimes\mathrm{id})\circ\Delta f = f(\mathbf{x},\mathbf{y},\mathbf{z}) = (\mathrm{id}\otimes\Delta)\circ\Delta f$$

inside $\Lambda(\mathbf{x},\mathbf{y},\mathbf{z})$ as a subring of $\Lambda\otimes\Lambda\otimes\Lambda$.

The counit $\Lambda\xrightarrow{\epsilon}\mathbf{k}$ is defined in the usual fashion for connected graded coalgebras, namely ϵ annihilates $I = \bigoplus_{n>0}\Lambda_n$, and ϵ is the identity on

$\Lambda_0 = \mathbf{k}$; alternatively ϵ sends a symmetric function $f(\mathbf{x})$ to its constant term $f(0, 0, \ldots)$.

Note that Δ is an algebra morphism $\Lambda \to \Lambda \otimes \Lambda$ because it is a composition of maps which are all algebra morphisms. As the unit and counit axioms are easily checked, Λ becomes a connected graded $\mathbf{k}$-bialgebra of finite type, and hence also a Hopf algebra by Proposition 1.4.16. We will identify its antipode more explicitly in Section 2.4 below.

2.2. **Other Bases.**

We introduce the usual other bases of Λ, and explain their significance later.

Definition 2.2.1. Define the families of *power sum symmetric functions* p_n, *elementary symmetric functions* e_n, and *complete homogeneous symmetric functions* h_n, for $n = 1, 2, 3, \ldots$ by

$$p_n := x_1^n + x_2^n + \cdots = m_{(n)}, \tag{2.2.1}$$

$$e_n := \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n} = m_{(1^n)}, \tag{2.2.2}$$

$$h_n := \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n} = \sum_{\lambda \in \mathrm{Par}_n} m_\lambda. \tag{2.2.3}$$

Here, we are using the *multiplicative notation* for partitions: whenever $(m_1, m_2, m_3, \ldots)$ is a weak composition, $(1^{m_1} 2^{m_2} 3^{m_3} \cdots)$ denotes the partition λ such that for every i, the multiplicity of the part i in λ is m_i. The i^{m_i} satisfying $m_i = 0$ are often omitted from this notation, and so the (1^n) in (2.2.2) means $\left(\underbrace{1, 1, \ldots, 1}_{n \text{ ones}}\right)$. (For another example, $(1^2 3^1 4^3) = (1^2 2^0 3^1 4^3 5^0 6^0 7^0 \cdots)$ means the partition $(4, 4, 4, 3, 1, 1)$.) By convention, also define $h_0 = e_0 = 1$, and $h_n = e_n = 0$ if $n < 0$. Extend these multiplicatively to partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ with $\lambda_1 \geq \cdots \geq \lambda_\ell > 0$ by setting

$$p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_\ell},$$
$$e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_\ell},$$
$$h_\lambda := h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_\ell}.$$

Also define the *Schur function*

$$s_\lambda := \sum_T \mathbf{x}^{\mathrm{cont}(T)} \tag{2.2.4}$$

where T runs through all *column-strict tableaux* of shape λ, that is, T is an assignment of entries in $\{1, 2, 3, \ldots\}$ to the cells of the *Ferrers diagram*[74] for λ, weakly increasing left-to-right in rows, and strictly increasing

[74]The *Ferrers diagram* of a partition λ is defined as the set of all pairs $(i, j) \in \{1, 2, 3, \ldots\}^2$ satisfying $j \leq \lambda_i$. This is a set of cardinality $|\lambda|$. Usually, one visually represents a Ferrers diagram by drawing its elements (i, j) as points on the plane, although (unlike the standard convention for drawing points on the plane) one lets the x-axis go top-to-bottom (i.e., the point $(i + 1, j)$ is one step below the point (i, j)), and the y-axis go left-to-right (i.e., the point $(i, j + 1)$ is one step to the right of the point (i, j)). (This is the so-called *English notation*, also known as the *matrix notation* because it is precisely the way one labels the entries of a matrix. Other notations appear in literature,

top-to-bottom in columns. Here $\operatorname{cont}(T)$ denotes the weak composition $(|T^{-1}(1)|, |T^{-1}(2)|, |T^{-1}(3)|, \ldots)$, so that $\mathbf{x}^{\operatorname{cont}(T)} = \prod_i x_i^{|T^{-1}(i)|}$. For example,[75]

$$T = \begin{array}{ccccc} 1 & 1 & 1 & 4 & 7 \\ 2 & 3 & 3 & & \\ 4 & 4 & 6 & & \\ 6 & 7 & & & \end{array}$$

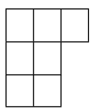
is a column-strict tableau of shape $\lambda = (5,3,3,2)$ with $\mathbf{x}^{\operatorname{cont}(T)} = x_1^3 x_2^1 x_3^2 x_4^3 x_5^0 x_6^2 x_7^2$. If T is a column-strict tableau, then the weak composition $\operatorname{cont}(T)$ is called the *content* of T.

Column-strict tableaux are also known as *semistandard tableaux*, and some authors even omit the adjective and just call them *tableaux* (e.g., Fulton in [73], a book entirely devoted to them).

Example 2.2.2. One has

$$\begin{aligned} m_{(1)} = p_{(1)} = e_{(1)} = h_{(1)} = s_{(1)} &= x_1 + x_2 + x_3 + \cdots, \\ s_{(n)} &= h_n, \\ s_{(1^n)} &= e_n. \end{aligned}$$

such as the French notation used, e.g., in Malvenuto's [145], and the Russian notation used, e.g., in parts of Kerov's [108].) These points are drawn either as dots or as square boxes; in the latter case, the boxes are centered at the points they represent, and they have sidelength 1 so that the boxes centered around (i,j) and $(i,j+1)$ touch each other along a sideline. For example, the Ferrers diagram of the partition $(3,2,2)$ is represented as

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & \bullet & \end{array}$$ (using dots) or as  (using boxes).

The Ferrers diagram of a partition λ uniquely determines λ. One refers to the elements of the Ferrers diagram of λ as the *cells* (or *boxes*) of this diagram (which is particularly natural when one represents them by boxes) or, briefly, as the cells of λ. Notation like "west", "north", "left", "right", "row" and "column" concerning cells of Ferrers diagrams normally refers to their visual representation.

Ferrers diagrams are also known as *Young diagrams*.

One can characterize the Ferrers diagrams of partitions as follows: A finite subset S of $\{1,2,3,\ldots\}^2$ is the Ferrers diagram of some partition if and only if for every $(i,j) \in S$ and every $(i',j') \in \{1,2,3,\ldots\}^2$ satisfying $i' \leq i$ and $j' \leq j$, we have $(i',j') \in S$. In other words, a finite subset S of $\{1,2,3,\ldots\}^2$ is the Ferrers diagram of some partition if and only if it is a lower set of the poset $\{1,2,3,\ldots\}^2$ with respect to the componentwise order.

[75]To visually represent a column-strict tableau T of shape λ, we draw the same picture as when representing the Ferrers diagram of λ, but with a little difference: a cell (i,j) is no longer represented by a dot or box, but instead is represented by the entry of T assigned to this cell. Accordingly, the entry of T assigned to a given cell c is often referred to as *the entry of T in c*.

Example 2.2.3. One has for $\lambda = (2,1)$ that

$$\begin{aligned}
p_{(2,1)} &= p_2 p_1 = (x_1^2 + x_2^2 + \cdots)(x_1 + x_2 + \cdots) \\
&= m_{(2,1)} + m_{(3)}, \\
\\
e_{(2,1)} &= e_2 e_1 = (x_1 x_2 + x_1 x_3 + \cdots)(x_1 + x_2 + \cdots) \\
&= m_{(2,1)} + 3m_{(1,1,1)}, \\
\\
h_{(2,1)} &= h_2 h_1 = (x_1^2 + x_2^2 + \cdots + x_1 x_2 + x_1 x_3 + \cdots)(x_1 + x_2 + \cdots) \\
&= m_{(3)} + 2m_{(2,1)} + 3m_{(1,1,1)},
\end{aligned}$$

and

$$\begin{array}{lcccccccc}
s_{(2,1)} & = x_1^2 x_2 & + x_1^2 x_3 & + x_1 x_2^2 & + x_1 x_3^2 & + x_1 x_2 x_3 & + x_1 x_2 x_3 & + x_1 x_2 x_4 & + \cdots \\
 & 11 & 11 & 12 & 13 & 12 & 13 & 12 & \\
 & 2 & 3 & 2 & 3 & 3 & 2 & 4 & \\
 & \multicolumn{8}{l}{= m_{(2,1)} + 2m_{(1,1,1)}.}
\end{array}$$

In fact, one has these transition matrices for $n = 3$ expressing elements in terms of the monomial basis m_λ:

$$\begin{array}{c} \\ m_{(3)} \\ m_{(2,1)} \\ m_{(1,1,1)} \end{array}
\begin{array}{c} \begin{array}{ccc} p_{(3)} & p_{(2,1)} & p_{(1,1,1)} \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 6 \end{pmatrix} \end{array}, \qquad
\begin{array}{c} \\ m_{(3)} \\ m_{(2,1)} \\ m_{(1,1,1)} \end{array}
\begin{array}{c} \begin{array}{ccc} e_{(3)} & e_{(2,1)} & e_{(1,1,1)} \end{array} \\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & 3 & 6 \end{pmatrix} \end{array},$$

$$\begin{array}{c} \\ m_{(3)} \\ m_{(2,1)} \\ m_{(1,1,1)} \end{array}
\begin{array}{c} \begin{array}{ccc} h_{(3)} & h_{(2,1)} & h_{(1,1,1)} \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix} \end{array}, \qquad
\begin{array}{c} \\ m_{(3)} \\ m_{(2,1)} \\ m_{(1,1,1)} \end{array}
\begin{array}{c} \begin{array}{ccc} s_{(3)} & s_{(2,1)} & s_{(1,1,1)} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \end{array}.$$

Our next goal is to show that $e_\lambda, s_\lambda, h_\lambda$ (and, under some conditions, the p_λ as well) all give bases for Λ. However at the moment it is not yet even clear that s_λ are symmetric!

Proposition 2.2.4. *Schur functions s_λ are symmetric, that is, they lie in Λ.*

Proof. It suffices to show s_λ is symmetric under swapping the variables x_i, x_{i+1}, by providing an involution ι on the set of all column-strict tableaux T of shape λ which switches the $\mathrm{cont}(T)$ for $(i, i+1)\,\mathrm{cont}(T)$. Restrict attention to the entries $i, i+1$ in T, which must look something like this:

$$\begin{array}{cccccccccccccc}
 & & & & & & & & & i & i & i & i & i+1 \quad i+1 \\
 & i & i & i & i & i & i+1 & i+1 & i+1 & i+1 & i+1 & & & \\
i+1 & i+1 & i+1 & & & & & & & & & & &
\end{array}$$

One finds several vertically aligned pairs $\begin{array}{c} i \\ i+1 \end{array}$. If one were to remove all such pairs, the remaining entries would be a sequence of rows, each looking like this:

$$\underbrace{i, i, \ldots, i}_{r \text{ occurrences}}, \underbrace{i+1, i+1, \ldots, i+1}_{s \text{ occurrences}}. \tag{2.2.5}$$

An involution due to Bender and Knuth tells us to leave fixed all the vertically aligned pairs $\begin{smallmatrix} i \\ i+1 \end{smallmatrix}$, but change each sequence of remaining entries as in (2.2.5) to this:

$$\underbrace{i, i, \ldots, i}_{s \text{ occurrences}}, \underbrace{i+1, i+1, \ldots, i+1}_{r \text{ occurrences}} .$$

For example, the above configuration in T would change to

$$\begin{array}{cccccccccccccc}
 & & & & & & & & i & i & i & i & i & i+1 \\
 & i & i & i & i & i+1 & i+1 & i+1 & i+1 & i+1 & i+1 & & & \\
i & i+1 & i+1 & & & & & & & & & & &
\end{array}$$

It is easily checked that this map is an involution, and that it has the effect of swapping $(i, i+1)$ in $\mathrm{cont}(T)$. $\square$

Remark 2.2.5. The symmetry of Schur functions allows one to reformulate them via column-strict tableaux defined with respect to *any* total ordering $\mathcal{L}$ on the positive integers, rather than the usual $1 < 2 < 3 < \cdots$. For example, one can use the *reverse order*[76] $\cdots < 3 < 2 < 1$, or even more exotic orders, such as

$$1 < 3 < 5 < 7 < \cdots < 2 < 4 < 6 < 8 < \cdots .$$

Say that an assignment T of entries in $\{1, 2, 3, \ldots\}$ to the cells of the Ferrers diagram of λ is an *$\mathcal{L}$-column-strict tableau* if it is weakly $\mathcal{L}$-increasing left-to-right in rows, and strictly $\mathcal{L}$-increasing top-to-bottom in columns.

Proposition 2.2.6. *For any total order $\mathcal{L}$ on the positive integers,*

$$s_\lambda = \sum_T \mathbf{x}^{\mathrm{cont}(T)} \tag{2.2.6}$$

as T runs through all $\mathcal{L}$-column-strict tableaux of shape λ.

Proof. Given a weak composition $\alpha = (\alpha_1, \alpha_2, \ldots)$ with $\alpha_{n+1} = \alpha_{n+2} = \cdots = 0$, assume that the integers $1, 2, \ldots, n$ are totally ordered by $\mathcal{L}$ as $w(1) <_{\mathcal{L}} \cdots <_{\mathcal{L}} w(n)$ for some w in $\mathfrak{S}_n$. Then the coefficient of $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ on the right side of (2.2.6) is the same as the coefficient of $\mathbf{x}^{w^{-1}(\alpha)}$ on the right side of (2.2.4) defining s_λ, which by symmetry of s_λ is the same as the coefficient of $\mathbf{x}^\alpha$ on the right side of (2.2.4). $\square$

It is now not hard to show that $p_\lambda, e_\lambda, s_\lambda$ give bases by a triangularity argument[77]. For this purpose, let us introduce a useful partial order on partitions.

Definition 2.2.7. The *dominance* or *majorization* order on Par_n is the partial order on the set Par_n whose greater-or-equal relation $\triangleright$ is defined as follows: For two partitions λ and μ of n, we set $\lambda \triangleright \mu$ (and say that λ *dominates*, or *majorizes*, μ) if and only if

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k \geq \mu_1 + \mu_2 + \cdots + \mu_k \quad \text{for } k = 1, 2, \ldots, n.$$

[76]This reverse order is what one uses when one defines a Schur function as a generating function for *reverse semistandard tableaux* or *column-strict plane partitions*; see Stanley [206, Proposition 7.10.4].

[77]See Section 11.1 for some notions and notations that will be used in this argument.

(The definition of dominance would not change if we would replace "for $k = 1, 2, \ldots, n$" by "for every positive integer k" or by "for every $k \in \mathbb{N}$".)

Definition 2.2.8. For a partition λ, its *conjugate* or *transpose* partition λ^t is the one whose Ferrers diagram is obtained from that of λ by exchanging rows for columns (i.e., by flipping the diagram across the "main", i.e., top-right-to-bottom-left, diagonal)[78]. Alternatively, one has this formula for its i-th entry:

$$(\lambda^t)_i := |\{j : \lambda_j \geq i\}|. \tag{2.2.7}$$

For example, $(4, 3, 1)^t = (3, 2, 2, 1)$, which can be easily verified by flipping the Ferrers diagram of $(4, 3, 1)$ across the "main diagonal":

$$\underbrace{\begin{array}{cccc}\bullet&\bullet&\bullet&\bullet\\ \bullet&\bullet&\bullet&\\ \bullet&&&\end{array}}_{\text{Ferrers diagram of }(4,3,1)} \qquad \longmapsto \qquad \underbrace{\begin{array}{ccc}\bullet&\bullet&\bullet\\ \bullet&\bullet&\\ \bullet&\bullet&\\ \bullet&&\end{array}}_{\text{Ferrers diagram of }(4,2,2,1)}$$

(or simply counting the boxes in each column of this diagram).

Exercise 2.2.9. Let $\lambda, \mu \in \operatorname{Par}_n$. Show that $\lambda \triangleright \mu$ if and only if $\mu^t \triangleright \lambda^t$.

Proposition 2.2.10. *The families $\{e_\lambda\}$ and $\{s_\lambda\}$, as λ runs through all partitions, are graded bases for the graded $\mathbf{k}$-module $\Lambda_{\mathbf{k}}$ whenever $\mathbf{k}$ is a commutative ring. The same holds for the family $\{p_\lambda\}$ when $\mathbb{Q}$ is a subring of $\mathbf{k}$.*

Our proof of this proposition will involve three separate arguments, one for each of the three alleged bases $\{s_\lambda\}$, $\{e_\lambda\}$ and $\{p_\lambda\}$; however, all these three arguments fit the same mold: Each one shows that the alleged basis expands invertibly triangularly[79] in the basis $\{m_\lambda\}$ (possibly after reindexing), with an appropriately chosen partial order on the indexing set. We will simplify our life by restricting ourselves to Par_n for a given $n \in \mathbb{N}$, and by stating the common part of the three arguments in a greater generality (so that we won't have to repeat it thrice):

Lemma 2.2.11. *Let S be a finite poset. We write $\leq$ for the smaller-or-equal relation of S.*

Let M be a free $\mathbf{k}$-module with a basis $(b_\lambda)_{\lambda \in S}$. Let $(a_\lambda)_{\lambda \in S}$ be a further family of elements of M.

For each $\lambda \in S$, let $(g_{\lambda,\mu})_{\mu \in S}$ be the family of the coefficients in the expansion of $a_\lambda \in M$ in the basis $(b_\mu)_{\mu \in S}$; in other words, let $(g_{\lambda,\mu})_{\mu \in S} \in \mathbf{k}^S$ be such that $a_\lambda = \sum_{\mu \in S} g_{\lambda,\mu} b_\mu$. Assume that:

- *Assumption A1: Any $\lambda \in S$ and $\mu \in S$ satisfy $g_{\lambda,\mu} = 0$ unless $\mu \leq \lambda$.*
- *Assumption A2: For any $\lambda \in S$, the element $g_{\lambda,\lambda}$ of $\mathbf{k}$ is invertible.*

Then, the family $(a_\lambda)_{\lambda \in S}$ is a basis of the $\mathbf{k}$-module M.

[78] In more rigorous terms: The cells of the Ferrers diagram of λ^t are the pairs (j, i), where (i, j) ranges over all cells of λ. It is easy to see that this indeed uniquely determines a partition λ^t.

[79] i.e., triangularly, with all diagonal coefficients being invertible

Proof of Lemma 2.2.11. Use the notations of Section 11.1. Assumptions A1 and A2 yield that the $S \times S$-matrix $(g_{\lambda,\mu})_{(\lambda,\mu)\in S\times S} \in \mathbf{k}^{S\times S}$ is invertibly triangular. But the definition of the $g_{\lambda,\mu}$ yields that the family $(a_\lambda)_{\lambda\in S}$ expands in the family $(b_\lambda)_{\lambda\in S}$ through this matrix $(g_{\lambda,\mu})_{(\lambda,\mu)\in S\times S}$. Since the latter matrix is invertibly triangular, this shows that the family $(a_\lambda)_{\lambda\in S}$ expands invertibly triangularly in the family $(b_\lambda)_{\lambda\in S}$. Therefore, Corollary 11.1.19(e) (applied to $(e_s)_{s\in S} = (a_\lambda)_{\lambda\in S}$ and $(f_s)_{s\in S} = (b_\lambda)_{\lambda\in S}$) shows that $(a_\lambda)_{\lambda\in S}$ is a basis of the $\mathbf{k}$-module M (since $(b_\lambda)_{\lambda\in S}$ is a basis of the $\mathbf{k}$-module M). □

Proof of Proposition 2.2.10. We can restrict our attention to each homogeneous component Λ_n and partitions λ of n. Thus, we have to prove that, for each $n \in \mathbb{N}$, the families $(e_\lambda)_{\lambda\in\mathrm{Par}_n}$ and $(s_\lambda)_{\lambda\in\mathrm{Par}_n}$ are bases of the $\mathbf{k}$-module Λ_n, and that the same holds for $(p_\lambda)_{\lambda\in\mathrm{Par}_n}$ if $\mathbb{Q}$ is a subring of $\mathbf{k}$.

Fix $n \in \mathbb{N}$. We already know that $(m_\lambda)_{\lambda\in\mathrm{Par}_n}$ is a basis of the $\mathbf{k}$-module Λ_n.

1. We shall first show that the family $(s_\lambda)_{\lambda\in\mathrm{Par}_n}$ is a basis of the $\mathbf{k}$-module Λ_n.

   For every partition λ, we have $s_\lambda = \sum_{\mu\in\mathrm{Par}} K_{\lambda,\mu} m_\mu$, where the coefficient $K_{\lambda,\mu}$ is the *Kostka number* counting the column-strict tableaux T of shape λ having $\mathrm{cont}(T) = \mu$; this follows because both sides are symmetric functions, and $K_{\lambda,\mu}$ is the coefficient of $\mathbf{x}^\mu$ on both sides[80]. Thus, for every $\lambda \in \mathrm{Par}_n$, one has

$$s_\lambda = \sum_{\mu\in\mathrm{Par}_n} K_{\lambda,\mu} m_\mu \tag{2.2.8}$$

   (since s_λ is homogeneous of degree n). [81] But if λ and μ are partitions satisfying $K_{\lambda,\mu} \neq 0$, then there exists a column-strict tableau T of shape λ having $\mathrm{cont}(T) = \mu$ (since $K_{\lambda,\mu}$ counts such tableaux), and therefore we must have $\lambda_1 + \lambda_2 + \cdots + \lambda_k \geq \mu_1 + \mu_2 + \cdots + \mu_k$ for each positive integer k (since the entries $1, 2, \ldots, k$ in T must all lie within the first k rows of λ); in other words, $\lambda \triangleright \mu$ (if $K_{\lambda,\mu} \neq 0$) [82]. In other words,

$$\text{any } \lambda \in \mathrm{Par}_n \text{ and } \mu \in \mathrm{Par}_n \text{ satisfy } K_{\lambda,\mu} = 0 \text{ unless } \lambda \triangleright \mu. \tag{2.2.9}$$

   One can also check that $K_{\lambda,\lambda} = 1$ for any $\lambda \in \mathrm{Par}_n$ [83]. Hence,

$$\text{for any } \lambda \in \mathrm{Par}_n\text{, the element } K_{\lambda,\lambda} \text{ of } \mathbf{k} \text{ is invertible.} \tag{2.2.10}$$

   Now, let us regard the set Par_n as a poset, whose greater-or-equal relation is $\triangleright$. Lemma 2.2.11 (applied to $S = \mathrm{Par}_n$, $M = \Lambda_n$, $a_\lambda = s_\lambda$, $b_\lambda = m_\lambda$ and $g_{\lambda,\mu} = K_{\lambda,\mu}$) shows that the family $(s_\lambda)_{\lambda\in\mathrm{Par}_n}$ is a basis of the $\mathbf{k}$-module Λ_n (because the Assumptions A1 and A2 of Lemma 2.2.11 are satisfied[84]).

[80] In general, in order to prove that two symmetric functions f and g are equal, it suffices to show that, for every $\mu \in \mathrm{Par}$, the coefficients of $\mathbf{x}^\mu$ in f and in g are equal. (Indeed, all other coefficients are determined by these coefficients because of the symmetry.)

[81] See Exercise 2.2.13(c) below for a detailed proof of (2.2.8).

[82] See Exercise 2.2.13(d) below for a detailed proof of this fact.

[83] See Exercise 2.2.13(e) below for a proof of this.

[84] Indeed, they follow from (2.2.9) and (2.2.10), respectively.

2. Before we show that $(e_\lambda)_{\lambda \in \mathrm{Par}_n}$ is a basis, we define a few notations regarding integer matrices. A $\{0,1\}$-*matrix* means a matrix whose entries belong to the set $\{0,1\}$. If $A \in \mathbb{N}^{\ell \times m}$ is a matrix, then the *row sums* of A means the ℓ-tuple $(r_1, r_2, \ldots, r_\ell)$, where each r_i is the sum of all entries in the i-th row of A; similarly, the *column sums* of A means the m-tuple $(c_1, c_2, \ldots, c_m)$, where each c_j is the sum of all entries in the j-th column of A. (For instance, the row sums of the $\{0,1\}$-matrix $\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$ is $(2,3)$, whereas its column sums is $(1,2,1,1,0)$.) We identify any k-tuple of nonnegative integers $(a_1, a_2, \ldots, a_k)$ with the weak composition $(a_1, a_2, \ldots, a_k, 0, 0, 0, \ldots)$; thus, the row sums and the column sums of a matrix in $\mathbb{N}^{\ell \times m}$ can be viewed as weak compositions. (For example, the column sums of the matrix $\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$ is the 5-tuple $(1,2,1,1,0)$, and can be viewed as the weak composition $(1,2,1,1,0,0,0,\ldots)$.)

For every $\lambda \in \mathrm{Par}_n$, one has

$$e_\lambda = \sum_{\mu \in \mathrm{Par}_n} a_{\lambda,\mu} m_\mu, \tag{2.2.11}$$

where $a_{\lambda,\mu}$ counts $\{0,1\}$-matrices (of size $\ell(\lambda) \times \ell(\mu)$) having row sums λ and column sums μ: indeed, when one expands $e_{\lambda_1} e_{\lambda_2} \cdots$, choosing the monomial $x_{j_1} \ldots x_{j_{\lambda_i}}$ in the e_{λ_i} factor corresponds to putting 1's in the i-th row and columns $j_1, \ldots, j_{\lambda_i}$ of the $\{0,1\}$-matrix [85]. Applying (2.2.11) to λ^t instead of λ, we see that

$$e_{\lambda^t} = \sum_{\mu \in \mathrm{Par}_n} a_{\lambda^t,\mu} m_\mu \tag{2.2.12}$$

for every $\lambda \in \mathrm{Par}_n$.

It is not hard to check[86] that $a_{\lambda,\mu}$ vanishes unless $\lambda^t \triangleright \mu$. Applying this to λ^t instead of λ, we conclude that

$$\text{any } \lambda \in \mathrm{Par}_n \text{ and } \mu \in \mathrm{Par}_n \text{ satisfy } a_{\lambda^t,\mu} = 0 \text{ unless } \lambda \triangleright \mu. \tag{2.2.13}$$

Moreover, one can show that $a_{\lambda^t,\lambda} = 1$ for each $\lambda \in \mathrm{Par}_n$ [87]. Hence,

$$\text{for any } \lambda \in \mathrm{Par}_n \text{, the element } a_{\lambda^t,\lambda} \text{ of } \mathbf{k} \text{ is invertible.} \tag{2.2.14}$$

Now, let us regard the set Par_n as a poset, whose greater-or-equal relation is $\triangleright$. Lemma 2.2.11 (applied to $S = \mathrm{Par}_n$, $M = \Lambda_n$, $a_\lambda = e_{\lambda^t}$, $b_\lambda = m_\lambda$ and $g_{\lambda,\mu} = a_{\lambda^t,\mu}$) shows that the family $(e_{\lambda^t})_{\lambda \in \mathrm{Par}_n}$ is a basis of the $\mathbf{k}$-module Λ_n (because the Assumptions A1 and A2 of Lemma 2.2.11 are satisfied[88]). Hence, $(e_\lambda)_{\lambda \in \mathrm{Par}_n}$ is a basis of Λ_n.

[85] See Exercise 2.2.13(g) below for a detailed proof of (2.2.11).

[86] See Exercise 2.2.13(h) below for a proof of this. This is the easy implication in the *Gale-Ryser Theorem*. (The hard implication is the converse: It says that if $\lambda, \mu \in \mathrm{Par}_n$ satisfy $\lambda^t \triangleright \mu$, then there exists a $\{0,1\}$-matrix having row sums λ and column sums μ, so that $a_{\lambda,\mu}$ is a positive integer. This is proven, e.g., in [114], in [46, Theorem 2.4] and in [224, Section 5.2].)

[87] See Exercise 2.2.13(i) below for a proof of this.

[88] Indeed, they follow from (2.2.13) and (2.2.14), respectively.

3. Assume now that $\mathbb{Q}$ is a subring of $\mathbf{k}$. For every $\lambda \in \operatorname{Par}_n$, one has

$$p_\lambda = \sum_{\mu \in \operatorname{Par}_n} b_{\lambda,\mu} m_\mu, \tag{2.2.15}$$

where $b_{\lambda,\mu}$ counts the ways to partition the nonzero parts $\lambda_1, \ldots, \lambda_\ell$ (where $\ell = \ell(\lambda)$) into blocks such that the sums of the blocks give μ; more formally, $b_{\lambda,\mu}$ is the number of maps $\varphi : \{1, 2, \ldots, \ell\} \to \{1, 2, 3, \ldots\}$ having

$$\mu_j = \sum_{i : \varphi(i) = j} \lambda_i \qquad \text{for all } j = 1, 2, \ldots$$

[89]. Again it is not hard to check that

$$\text{any } \lambda \in \operatorname{Par}_n \text{ and } \mu \in \operatorname{Par}_n \text{ satisfy } b_{\lambda,\mu} = 0 \text{ unless } \mu \triangleright \lambda. \tag{2.2.16}$$

[90] Furthermore, for any $\lambda \in \operatorname{Par}_n$, the element $b_{\lambda,\lambda}$ is a positive integer[91], and thus invertible in $\mathbf{k}$ (since $\mathbb{Q}$ is a subring of $\mathbf{k}$). Thus,

$$\text{for any } \lambda \in \operatorname{Par}_n \text{, the element } b_{\lambda,\lambda} \text{ of } \mathbf{k} \text{ is invertible} \tag{2.2.17}$$

(although we don't always have $b_{\lambda,\lambda} = 1$ this time).

Now, let us regard the set Par_n as a poset, whose smaller-or-equal relation is $\triangleright$. Lemma 2.2.11 (applied to $S = \operatorname{Par}_n$, $M = \Lambda_n$, $a_\lambda = p_\lambda$, $b_\lambda = m_\lambda$ and $g_{\lambda,\mu} = b_{\lambda,\mu}$) shows that the family $(p_\lambda)_{\lambda \in \operatorname{Par}_n}$ is a basis of the $\mathbf{k}$-module Λ_n (because the Assumptions A1 and A2 of Lemma 2.2.11 are satisfied[92]).

□

Remark 2.2.12. When $\mathbb{Q}$ is not a subring of $\mathbf{k}$, the family $\{p_\lambda\}$ is not (in general) a basis of $\Lambda_{\mathbf{k}}$; for instance, $e_2 = \frac{1}{2}\left(p_{(1,1)} - p_2\right) \in \Lambda_{\mathbb{Q}}$ is not in the $\mathbb{Z}$-span of this family. However, if we define $b_{\lambda,\mu}$ as in the above proof, then the $\mathbb{Z}$-linear span of all p_λ equals the $\mathbb{Z}$-linear span of all $b_{\lambda,\lambda} m_\lambda$. Indeed, if $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ with $k = \ell(\mu)$, then $b_{\mu,\mu}$ is the size of the subgroup of $\mathfrak{S}_k$ consisting of all permutations $\sigma \in \mathfrak{S}_k$ having each i satisfy $\mu_{\sigma(i)} = \mu_i$ [93]. As a consequence, $b_{\mu,\mu}$ divides $b_{\lambda,\mu}$ for every partition μ of the same size as λ (because this group acts[94] freely on the set which is enumerated by $b_{\lambda,\mu}$) [95]. Hence, the $\operatorname{Par}_n \times \operatorname{Par}_n$-matrix $\left(\dfrac{b_{\lambda,\mu}}{b_{\mu,\mu}}\right)_{(\lambda,\mu) \in \operatorname{Par}_n \times \operatorname{Par}_n}$ has integer entries. Furthermore, this matrix is unitriangular[96] (indeed, (2.2.16) shows that it is triangular, but its diagonal entries are clearly 1) and thus invertibly triangular. But (2.2.15) shows that the family $(p_\lambda)_{\lambda \in \operatorname{Par}_n}$ expands in the family $(b_{\lambda,\lambda} m_\lambda)_{\lambda \in \operatorname{Par}_n}$ through this matrix. Hence, the family $(p_\lambda)_{\lambda \in \operatorname{Par}_n}$

[89]See Exercise 2.2.13(k) below for a detailed proof of (2.2.15) (and see Exercise 2.2.13(j) for a proof that the numbers $b_{\lambda,\mu}$ are well-defined).

[90]See Exercise 2.2.13(l) below for a proof of this.

[91]This is proven in Exercise 2.2.13(m) below.

[92]Indeed, they follow from (2.2.16) and (2.2.17), respectively.

[93]See Exercise 2.2.13(n) below for a proof of this.

[94]Specifically, an element σ of the group takes $\varphi : \{1, 2, \ldots, \ell\} \to \{1, 2, 3, \ldots\}$ to $\sigma \circ \varphi$.

[95]See Exercise 2.2.13(o) below for a detailed proof of this.

[96]Here, we are using the terminology defined in Section 11.1, and we are regarding Par_n as a poset whose smaller-or-equal relation is $\triangleright$.

expands invertibly triangularly in the family $(b_{\lambda,\lambda}m_\lambda)_{\lambda\in\mathrm{Par}_n}$. Thus, Corollary 11.1.19(b) (applied to $\mathbb{Z}$, Λ_n, Par_n, $(p_\lambda)_{\lambda\in\mathrm{Par}_n}$ and $(b_{\lambda,\lambda}m_\lambda)_{\lambda\in\mathrm{Par}_n}$ instead of $\mathbf{k}$, M, S, $(e_s)_{s\in S}$ and $(f_s)_{s\in S}$) shows that the $\mathbb{Z}$-submodule of Λ_n spanned by $(p_\lambda)_{\lambda\in\mathrm{Par}_n}$ is the $\mathbb{Z}$-submodule of Λ_n spanned by $(b_{\lambda,\lambda}m_\lambda)_{\lambda\in\mathrm{Par}_n}$.

The purpose of the following exercise is to fill in some details omitted from the proof of Proposition 2.2.10.

Exercise 2.2.13. Let $n\in\mathbb{N}$.

(a) Show that every $f\in\Lambda_n$ satisfies
$$f=\sum_{\mu\in\mathrm{Par}_n}\left(\left[\mathbf{x}^\mu\right]f\right)m_\mu.$$
Here, $\left[\mathbf{x}^\mu\right]f$ denotes the coefficient of the monomial $\mathbf{x}^\mu$ in the power series f.

Now, we introduce a notation (which generalizes the notation $K_{\lambda,\mu}$ from the proof of Proposition 2.2.10): For any partition λ and any weak composition μ, we let $K_{\lambda,\mu}$ denote the number of all column-strict tableaux T of shape λ having $\mathrm{cont}\,(T)=\mu$.

(b) Prove that this number $K_{\lambda,\mu}$ is well-defined (i.e., there are only finitely many column-strict tableaux T of shape λ having $\mathrm{cont}\,(T)=\mu$).
(c) Show that $s_\lambda=\sum_{\mu\in\mathrm{Par}_n}K_{\lambda,\mu}m_\mu$ for every $\lambda\in\mathrm{Par}_n$.
(d) Show that $K_{\lambda,\mu}=0$ for any partitions $\lambda\in\mathrm{Par}_n$ and $\mu\in\mathrm{Par}_n$ that don't satisfy $\lambda\triangleright\mu$.
(e) Show that $K_{\lambda,\lambda}=1$ for any $\lambda\in\mathrm{Par}_n$.

Next, we recall a further notation: For any two partitions λ and μ, we let $a_{\lambda,\mu}$ denote the number of all $\{0,1\}$-matrices of size $\ell(\lambda)\times\ell(\mu)$ having row sums λ and column sums μ. (See the proof of Proposition 2.2.10 for the concepts of $\{0,1\}$-matrices and of row sums and column sums.)

(f) Prove that this number $a_{\lambda,\mu}$ is well-defined (i.e., there are only finitely many $\{0,1\}$-matrices of size $\ell(\lambda)\times\ell(\mu)$ having row sums λ and column sums μ).
(g) Show that $e_\lambda=\sum_{\mu\in\mathrm{Par}_n}a_{\lambda,\mu}m_\mu$ for every $\lambda\in\mathrm{Par}_n$.
(h) Show that $a_{\lambda,\mu}=0$ for any partitions $\lambda\in\mathrm{Par}_n$ and $\mu\in\mathrm{Par}_n$ that don't satisfy $\lambda^t\triangleright\mu$.
(i) Show that $a_{\lambda^t,\lambda}=1$ for any $\lambda\in\mathrm{Par}_n$.

Next, we introduce a further notation (which generalizes the notation $b_{\lambda,\mu}$ from the proof of Proposition 2.2.10): For any partition λ and any weak composition μ, we let $b_{\lambda,\mu}$ be the number of all maps $\varphi:\{1,2,\ldots,\ell\}\to\{1,2,3,\ldots\}$ satisfying $\left(\mu_j=\sum_{\substack{i\in\{1,2,\ldots,\ell\};\\ \varphi(i)=j}}\lambda_i\text{ for all }j\geq 1\right)$, where $\ell=\ell(\lambda)$.

(j) Prove that this number $b_{\lambda,\mu}$ is well-defined (i.e., there are only finitely many maps $\varphi:\{1,2,\ldots,\ell\}\to\{1,2,3,\ldots\}$ satisfying
$$\left(\mu_j=\sum_{\substack{i\in\{1,2,\ldots,\ell\};\\ \varphi(i)=j}}\lambda_i\text{ for all }j\geq 1\right)).$$

(k) Show that $p_\lambda = \sum_{\mu \in \operatorname{Par}_n} b_{\lambda,\mu} m_\mu$ for every $\lambda \in \operatorname{Par}_n$.
(l) Show that $b_{\lambda,\mu} = 0$ for any partitions $\lambda \in \operatorname{Par}_n$ and $\mu \in \operatorname{Par}_n$ that don't satisfy $\mu \triangleright \lambda$.
(m) Show that $b_{\lambda,\lambda}$ is a positive integer for any $\lambda \in \operatorname{Par}_n$.
(n) Show that for any partition $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \in \operatorname{Par}_n$ with $k = \ell(\mu)$, the integer $b_{\mu,\mu}$ is the size of the subgroup of $\mathfrak{S}_k$ consisting of all permutations $\sigma \in \mathfrak{S}_k$ having each i satisfy $\mu_{\sigma(i)} = \mu_i$. (In particular, show that this subgroup is indeed a subgroup.)
(o) Show that $b_{\mu,\mu} \mid b_{\lambda,\mu}$ for every $\lambda \in \operatorname{Par}_n$ and $\mu \in \operatorname{Par}_n$.

The bases $\{p_\lambda\}$ and $\{e_\lambda\}$ of Λ are two examples of *multiplicative bases*: these are bases constructed from a sequence $v_1, v_2, v_3, \ldots$ of symmetric functions by taking all possible finite products. We will soon encounter another example. First, let us observe that the finite products of a sequence $v_1, v_2, v_3, \ldots$ of symmetric functions form a basis of Λ if and only if the sequence is an algebraically independent generating set of Λ. This holds more generally for any commutative algebra, as the following simple exercise shows:

Exercise 2.2.14. Let A be a commutative $\mathbf{k}$-algebra. Let $v_1, v_2, v_3, \ldots$ be some elements of A.

For every partition λ, define an element $v_\lambda \in A$ by $v_\lambda = v_{\lambda_1} v_{\lambda_2} \cdots v_{\lambda_{\ell(\lambda)}}$. Prove the following:

(a) The $\mathbf{k}$-subalgebra of A generated by $v_1, v_2, v_3, \ldots$ is the $\mathbf{k}$-submodule of A spanned by the family $(v_\lambda)_{\lambda \in \operatorname{Par}}$.
(b) The elements $v_1, v_2, v_3, \ldots$ generate the $\mathbf{k}$-algebra A if and only if the family $(v_\lambda)_{\lambda \in \operatorname{Par}}$ spans the $\mathbf{k}$-module A.
(c) The elements $v_1, v_2, v_3, \ldots$ are algebraically independent over $\mathbf{k}$ if and only if the family $(v_\lambda)_{\lambda \in \operatorname{Par}}$ is $\mathbf{k}$-linearly independent.

The next exercise states two well-known identities for the *generating functions* of the sequences $(e_0, e_1, e_2, \ldots)$ and $(h_0, h_1, h_2, \ldots)$, which will be used several times further below:

Exercise 2.2.15. In the ring of formal power series $(\mathbf{k}[[\mathbf{x}]])[[t]]$, prove the two identities

$$\begin{aligned} \prod_{i=1}^{\infty} (1 - x_i t)^{-1} &= 1 + h_1(\mathbf{x}) t + h_2(\mathbf{x}) t^2 + \cdots \\ &= \sum_{n \geq 0} h_n(\mathbf{x}) t^n \end{aligned} \tag{2.2.18}$$

and

$$\begin{aligned} \prod_{i=1}^{\infty} (1 + x_i t) &= 1 + e_1(\mathbf{x}) t + e_2(\mathbf{x}) t^2 + \cdots \\ &= \sum_{n \geq 0} e_n(\mathbf{x}) t^n. \end{aligned} \tag{2.2.19}$$

2.3. **Comultiplications.** Thinking about comultiplication $\Lambda \xrightarrow{\Delta} \Lambda \otimes \Lambda$ on Schur functions forces us to immediately confront the following.

Definition 2.3.1. For partitions μ and λ say that $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for $i = 1, 2, \ldots$. In other words, two partitions μ and λ satisfy $\mu \subseteq \lambda$ if and only if the Ferrers diagram for μ is a subset of the Ferrers diagram of λ. In this case, define the *skew (Ferrers) diagram* λ/μ to be their set difference.[97]

Then define the *skew Schur function* $s_{\lambda/\mu}(\mathbf{x})$ to be the sum $s_{\lambda/\mu} := \sum_T \mathbf{x}^{\operatorname{cont}(T)}$, where the sum ranges over all *column-strict tableaux* T of shape λ/μ, that is, assignments of a value in $\{1, 2, 3, \ldots\}$ to each cell of λ/μ, weakly increasing left-to-right in rows, and strictly increasing top-to-bottom in columns.

Example 2.3.2. Let $\lambda = (5, 3, 3, 2)$ and $\mu = (3, 1, 1, 0)$. Then, $\mu \subseteq \lambda$. The Ferrers diagrams for λ and μ and the skew Ferrers diagram for λ/μ look as follows:

$$\underbrace{\begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & & \\ \bullet & \bullet & \bullet & & \\ \bullet & \bullet & & & \end{array}}_{\text{Ferrers diagram of } \lambda} \qquad \underbrace{\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & & \\ \bullet & & \\ & & \end{array}}_{\text{Ferrers diagram of } \mu} \qquad \underbrace{\begin{array}{ccccc} \cdot & \cdot & \cdot & \bullet & \bullet \\ \cdot & \bullet & \bullet & & \\ \cdot & \bullet & \bullet & & \\ \bullet & \bullet & & & \end{array}}_{\text{skew Ferrers diagram of } \lambda/\mu}$$

(where the small dots represent boxes removed from the diagram). The filling

$$T = \begin{array}{ccccc} \cdot & \cdot & \cdot & 2 & 5 \\ \cdot & 1 & 1 & & \\ 2 & 2 & 4 & & \\ 4 & 5 & & & \end{array}$$

is a column-strict tableau of shape $\lambda/\mu = (5, 3, 3, 2)/(3, 1, 0, 0)$ and it has $\mathbf{x}^{\operatorname{cont}(T)} = x_1^2 x_2^3 x_3^0 x_4^2 x_5^2$.

On the other hand, if we took $\lambda = (5, 3, 1)$ and $\mu = (1, 1, 1, 1)$, then we wouldn't have $\mu \subseteq \lambda$, since $\mu_4 = 1 > 0 = \lambda_4$.

Remark 2.3.3. If μ and λ are partitions such that $\mu \subseteq \lambda$, then $s_{\lambda/\mu} \in \Lambda$. (This is proven similarly as Proposition 2.2.4.) Actually, if $\mu \subseteq \lambda$, then $s_{\lambda/\mu} \in \Lambda_{|\lambda/\mu|}$, where $|\lambda/\mu|$ denotes the number of cells of the skew shape λ/μ (so $|\lambda/\mu| = |\lambda| - |\mu|$).

It is customary to define $s_{\lambda/\mu}$ to be 0 if we don't have $\mu \subseteq \lambda$. This can also be seen by a literal reading of the definition $s_{\lambda/\mu} := \sum_T \mathbf{x}^{\operatorname{cont}(T)}$, as long as we understand that there are no column-strict tableaux of shape λ/μ when λ/μ is not defined.

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[97]In other words, the skew Ferrers diagram λ/μ is the set of all $(i, j) \in \{1, 2, 3, \ldots\}^2$ satisfying $\mu_i < j \leq \lambda_i$.

While the Ferrers diagram for a single partition λ uniquely determines λ, the skew Ferrers diagram λ/μ does not uniquely determine μ and λ. (For instance, it is empty whenever $\lambda = \mu$.) When one wants to keep μ and λ in memory, one speaks of the *skew shape* λ/μ; this simply means the pair (μ, λ). Every notion defined for skew Ferrers diagrams also makes sense for skew shapes, because to any skew shape λ/μ we can assign the skew Ferrers diagram λ/μ (even if not injectively). For instance, the *cells* of the skew shape λ/μ are the cells of the skew Ferrers diagram λ/μ.

One can characterize the skew Ferrers diagrams as follows: A finite subset S of $\{1, 2, 3, \ldots\}^2$ is a skew Ferrers diagram (i.e., there exist two partitions λ and μ such that $\mu \subseteq \lambda$ and such that S is the skew Ferrers diagram λ/μ) if and only if for every $(i, j) \in S$, every $(i', j') \in \{1, 2, 3, \ldots\}^2$ and every $(i'', j'') \in S$ satisfying $i'' \leq i' \leq i$ and $j'' \leq j' \leq j$, we have $(i', j') \in S$.

Clearly, every partition λ satisfies $s_\lambda = s_{\lambda/\varnothing}$.

It is easy to see that two partitions λ and μ satisfy $\mu \subseteq \lambda$ if and only if they satisfy $\mu^t \subseteq \lambda^t$.

Exercise 2.3.4. (a) State and prove an analogue of Proposition 2.2.6 for skew Schur functions.

(b) Let λ, μ, λ' and μ' be partitions such that $\mu \subseteq \lambda$ and $\mu' \subseteq \lambda'$. Assume that the skew Ferrers diagram λ'/μ' can be obtained from the skew Ferrers diagram λ/μ by a 180° rotation.[98] Prove that $s_{\lambda/\mu} = s_{\lambda'/\mu'}$.

Exercise 2.3.5. Let λ and μ be two partitions, and let $k \in \mathbb{N}$ be such that[99] $\mu_k \geq \lambda_{k+1}$. Let F be the skew Ferrers diagram λ/μ. Let $F_{\text{rows}\leq k}$ denote the subset of F consisting of all $(i,j) \in F$ satisfying $i \leq k$. Let $F_{\text{rows}>k}$ denote the subset of F consisting of all $(i,j) \in F$ satisfying $i > k$. Let α and β be two partitions such that $\beta \subseteq \alpha$ and such that the skew Ferrers diagram α/β can be obtained from $F_{\text{rows}\leq k}$ by parallel translation. Let γ and δ be two partitions such that $\delta \subseteq \gamma$ and such that the skew Ferrers diagram γ/δ can be obtained from $F_{\text{rows}>k}$ by parallel translation.[100] Prove that $s_{\lambda/\mu} = s_{\alpha/\beta} s_{\gamma/\delta}$.

Proposition 2.3.6. *The comultiplication $\Lambda \overset{\Delta}{\to} \Lambda \otimes \Lambda$ has the following effect on the symmetric functions discussed so far*[101]*:*

(i) $\Delta p_n = 1 \otimes p_n + p_n \otimes 1$ *for every $n \geq 1$, that is, the power sums p_n are primitive.*
(ii) $\Delta e_n = \sum_{i+j=n} e_i \otimes e_j$ *for every $n \in \mathbb{N}$.*
(iii) $\Delta h_n = \sum_{i+j=n} h_i \otimes h_j$ *for every $n \in \mathbb{N}$.*
(iv) $\Delta s_\lambda = \sum_{\mu \subseteq \lambda} s_\mu \otimes s_{\lambda/\mu}$ *for any partition λ.*
(v) $\Delta s_{\lambda/\nu} = \sum_{\substack{\mu \in \mathrm{Par}: \\ \nu \subseteq \mu \subseteq \lambda}} s_{\mu/\nu} \otimes s_{\lambda/\mu}$ *for any partitions λ and ν.*

Proof. Recall that Δ sends $f(\mathbf{x}) \mapsto f(\mathbf{x}, \mathbf{y})$, and one can easily check that

(i) $p_n(\mathbf{x}, \mathbf{y}) = \sum_i x_i^n + \sum_i y_i^n = p_n(\mathbf{x}) \cdot 1 + 1 \cdot p_n(\mathbf{y})$ for every $n \geq 1$;
(ii) $e_n(\mathbf{x}, \mathbf{y}) = \sum_{i+j=n} e_i(\mathbf{x}) e_j(\mathbf{y})$ for every $n \in \mathbb{N}$;
(iii) $h_n(\mathbf{x}, \mathbf{y}) = \sum_{i+j=n} h_i(\mathbf{x}) h_j(\mathbf{y})$ for every $n \in \mathbb{N}$.

For assertion (iv), note that by (2.2.6), one has

$$s_\lambda(\mathbf{x}, \mathbf{y}) = \sum_T (\mathbf{x}, \mathbf{y})^{\mathrm{cont}(T)}, \tag{2.3.1}$$

[98] For example, this happens when $\lambda = (3,2)$, $\mu = (1)$, $\lambda' = (5,4)$ and $\mu' = (3,1)$.

[99] As usual, we write ν_k for the k-th entry of a partition ν.

[100] Here is an example of the situation: $\lambda = (6,5,5,2,2)$, $\mu = (4,4,3,1)$, $k = 3$ (satisfying $\mu_k = \mu_3 = 3 \geq 2 = \lambda_4 = \lambda_{k+1}$), $\alpha = (3,2,2)$, $\beta = (1,1)$, $\gamma = (2,2)$, and $\delta = (1)$.

[101] The abbreviated summation indexing $\sum_{i+j=n} t_{i,j}$ used here is intended to mean

$$\sum_{\substack{(i,j) \in \mathbb{N}^2; \\ i+j=n}} t_{i,j}.$$

where the sum is over column-strict tableaux T of shape λ having entries in the linearly ordered alphabet

$$x_1 < x_2 < \cdots < y_1 < y_2 < \cdots . \tag{2.3.2}$$

[102] For example,

$$T = \begin{matrix} x_1 & x_1 & x_1 & y_2 & y_5 \\ x_2 & y_1 & y_1 & & \\ y_2 & y_2 & y_4 & & \\ y_4 & y_5 & & & \end{matrix}$$

is such a tableau of shape $\lambda = (5, 3, 3, 2)$. Note that the restriction of T to the alphabet $\mathbf{x}$ gives a column-strict tableau $T_{\mathbf{x}}$ of some shape $\mu \subseteq \lambda$, and the restriction of T to the alphabet $\mathbf{y}$ gives a column-strict tableau $T_{\mathbf{y}}$ of shape λ/μ (e.g. for T in the example above, the tableau $T_{\mathbf{y}}$ appeared in Example 2.3.2). Consequently, one has

$$\begin{aligned} s_\lambda(\mathbf{x}, \mathbf{y}) &= \sum_T \mathbf{x}^{\operatorname{cont}(T_{\mathbf{x}})} \cdot \mathbf{y}^{\operatorname{cont}(T_{\mathbf{y}})} \\ &= \sum_{\mu \subseteq \lambda} \left(\sum_{T_{\mathbf{x}}} \mathbf{x}^{\operatorname{cont}(T_{\mathbf{x}})} \right) \left(\sum_{T_{\mathbf{y}}} \mathbf{y}^{\operatorname{cont}(T_{\mathbf{y}})} \right) = \sum_{\mu \subseteq \lambda} s_\mu(\mathbf{x}) s_{\lambda/\mu}(\mathbf{y}). \end{aligned} \tag{2.3.3}$$

Assertion (v) is obvious in the case when we don't have $\nu \subseteq \lambda$ (in fact, in this case, both $s_{\lambda/\nu}$ and $\sum_{\substack{\mu \in \mathrm{Par}: \\ \nu \subseteq \mu \subseteq \lambda}} s_{\mu/\nu} \otimes s_{\lambda/\mu}$ are clearly zero). In the remaining case, the proof of assertion (v) is similar to that of (iv). (Of course, the tableaux T and $T_{\mathbf{x}}$ now have skew shapes λ/ν and μ/ν, and instead of (2.2.6), we need to use the answer to Exercise 2.3.4(a).) ☐

Notice that parts (ii) and (iii) of Proposition 2.3.6 are particular cases of part (iv), since $h_n = s_{(n)}$ and $e_n = s_{(1^n)}$.

Exercise 2.3.7. (a) Show that the Hopf algebra Λ is cocommutative.
(b) Show that $\Delta s_{\lambda/\nu} = \sum_{\substack{\mu \in \mathrm{Par}: \\ \nu \subseteq \mu \subseteq \lambda}} s_{\lambda/\mu} \otimes s_{\mu/\nu}$ for any partitions λ and ν.

Exercise 2.3.8. Let $n \in \mathbb{N}$. Consider the finite variable set $(x_1, x_2, \ldots, x_n)$ as a subset of $\mathbf{x} = (x_1, x_2, x_3, \ldots)$. Recall that $f(x_1, x_2, \ldots, x_n)$ is a well-defined element of $\mathbf{k}[x_1, x_2, \ldots, x_n]$ for every $f \in R(\mathbf{x})$ (and therefore also for every $f \in \Lambda$, since $\Lambda \subset R(\mathbf{x})$), according to Exercise 2.1.2.

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[102] Here, $(\mathbf{x}, \mathbf{y})^{\operatorname{cont}(T)}$ means the monomial $\prod_{a \in \mathfrak{A}} a^{|T^{-1}(a)|}$, where $\mathfrak{A}$ denotes the totally ordered alphabet $x_1 < x_2 < \cdots < y_1 < y_2 < \cdots$. In other words, $(\mathbf{x}, \mathbf{y})^{\operatorname{cont}(T)}$ is the product of all entries of the tableau T (which is a monomial, since the entries of T are not numbers but variables).

The following rather formal argument should allay any doubts as to why (2.3.1) holds: Let $\mathcal{L}$ denote the totally ordered set which is given by the set $\{1, 2, 3, \ldots\}$ of positive integers, equipped with the total order $1 <_{\mathcal{L}} 3 <_{\mathcal{L}} 5 <_{\mathcal{L}} 7 <_{\mathcal{L}} \cdots <_{\mathcal{L}} 2 <_{\mathcal{L}} 4 <_{\mathcal{L}} 6 <_{\mathcal{L}} 8 <_{\mathcal{L}} \cdots$. Then, (2.2.6) yields $s_\lambda = \sum_T \mathbf{x}^{\operatorname{cont}(T)}$ as T runs through all $\mathcal{L}$-column-strict tableaux of shape λ. Substituting the variables $x_1, y_1, x_2, y_2, x_3, y_3, \ldots$ for $x_1, x_2, x_3, x_4, x_5, x_6, \ldots$ (that is, substituting x_i for x_{2i-1} and y_i for x_{2i}) in this equality, we obtain (2.3.1).

(a) Show that any two partitions λ and μ satisfy

$$s_{\lambda/\mu}(x_1, x_2, \ldots, x_n) = \sum_{\substack{T \text{ is a column-strict} \\ \text{tableau of shape } \lambda/\mu; \\ \text{all entries of } T \text{ belong} \\ \text{to } \{1,2,\ldots,n\}}} \mathbf{x}^{\operatorname{cont}(T)}.$$

(b) If λ is a partition having more than n parts[103], then show that $s_\lambda(x_1, x_2, \ldots, x_n) = 0$.

Remark 2.3.9. An analogue of Proposition 2.2.10 holds for symmetric polynomials in finitely many variables: Let $N \in \mathbb{N}$. Then, we have

(a) The family $\{m_\lambda(x_1, x_2, \ldots, x_N)\}$, as λ runs through all partitions having length $\leq N$, is a graded basis of the graded $\mathbf{k}$-module $\Lambda(x_1, x_2, \ldots, x_N) = \mathbf{k}[x_1, x_2, \ldots, x_N]^{\mathfrak{S}_N}$.
(b) For any partition λ having length $> N$, we have $m_\lambda(x_1, x_2, \ldots, x_N) = 0$.
(c) The family $\{e_\lambda(x_1, x_2, \ldots, x_N)\}$, as λ runs through all partitions whose parts are all $\leq N$, is a graded basis of the graded $\mathbf{k}$-module $\Lambda(x_1, x_2, \ldots, x_N)$.
(d) The family $\{s_\lambda(x_1, x_2, \ldots, x_N)\}$, as λ runs through all partitions having length $\leq N$, is a graded basis of the graded $\mathbf{k}$-module $\Lambda(x_1, x_2, \ldots, x_N)$.
(e) If $\mathbb{Q}$ is a subring of $\mathbf{k}$, then the family $\{p_\lambda(x_1, x_2, \ldots, x_N)\}$, as λ runs through all partitions having length $\leq N$, is a graded basis of the graded $\mathbf{k}$-module $\Lambda(x_1, x_2, \ldots, x_N)$.
(f) If $\mathbb{Q}$ is a subring of $\mathbf{k}$, then the family $\{p_\lambda(x_1, x_2, \ldots, x_N)\}$, as λ runs through all partitions whose parts are all $\leq N$, is a graded basis of the graded $\mathbf{k}$-module $\Lambda(x_1, x_2, \ldots, x_N)$.

Indeed, the claims (a) and (b) are obvious, while the claims (c), (d) and (e) are proven similarly to our proof of Proposition 2.2.10. We leave the proof of (f) to the reader; this proof can also be found in [138, Theorem 10.86][104].

Claim (c) can be rewritten as follows: The elementary symmetric polynomials $e_i(x_1, x_2, \ldots, x_N)$, for $i \in \{1, 2, \ldots, N\}$, form an algebraically independent generating set of $\Lambda(x_1, x_2, \ldots, x_N)$. This is precisely the well-known theorem (due to Gauss)[105] that every symmetric polynomial in N variables $x_1, x_2, \ldots, x_N$ can be written uniquely as a polynomial in the N elementary symmetric polynomials.

2.4. **The antipode, the involution ω, and algebra generators.** Since Λ is a connected graded $\mathbf{k}$-bialgebra, it will have an antipode $\Lambda \xrightarrow{S} \Lambda$ making it a Hopf algebra by Proposition 1.4.16. However, we can identify S more explicitly now.

Proposition 2.4.1. *Each of the families $\{e_n\}_{n=1,2,\ldots}$ and $\{h_n\}_{n=1,2,\ldots}$ are algebraically independent, and generate $\Lambda_{\mathbf{k}}$ as a polynomial algebra for any*

[103]Recall that the *parts* of a partition are its nonzero entries.

[104]See [138, Remark 10.76] for why [138, Theorem 10.86] is equivalent to our claim (f).

[105]See, e.g., [40, *Symmetric Polynomials*, Theorem 5 and Remark 17] or [221, §5.3] or [26, Theorem 1]. In a slightly different form, it also appears in [119, Theorem (5.10)].

commutative ring $\mathbf{k}$*. The same holds for* $\{p_n\}_{n=1,2,\ldots}$ *when* $\mathbb{Q}$ *is a subring of* $\mathbf{k}$*.*

Furthermore, the antipode S *acts as follows:*

(i) $S(p_n) = -p_n$ *for every positive integer* n*.*
(ii) $S(e_n) = (-1)^n h_n$ *for every* $n \in \mathbb{N}$*.*
(iii) $S(h_n) = (-1)^n e_n$ *for every* $n \in \mathbb{N}$*.*

Proof. The assertion that $\{e_n\}_{n\geq 1}$ are algebraically independent and generate Λ is equivalent to Proposition 2.2.10 asserting that $\{e_\lambda\}_{\lambda\in\mathrm{Par}}$ is a basis for Λ. (Indeed, this equivalence follows from parts (b) and (c) of Exercise 2.2.14, applied to $v_n = e_n$ and $v_\lambda = e_\lambda$.) Thus, the former assertion is true. If $\mathbb{Q}$ is a subring of $\mathbf{k}$, then a similar argument (using p_n and p_λ instead of e_n and e_λ) shows that $\{p_n\}_{n\geq 1}$ are algebraically independent and generate Λ.

The assertion $S(p_n) = -p_n$ follows from Proposition 1.4.17 since p_n is primitive by Proposition 2.3.6(i).

For the remaining assertions, start with the easy generating function identities[106]

$$\begin{aligned} H(t) &:= \prod_{i=1}^{\infty} (1 - x_i t)^{-1} = 1 + h_1(\mathbf{x})t + h_2(\mathbf{x})t^2 + \cdots \\ &= \sum_{n\geq 0} h_n(\mathbf{x}) t^n; \end{aligned} \tag{2.4.1}$$

$$\begin{aligned} E(t) &:= \prod_{i=1}^{\infty} (1 + x_i t) = 1 + e_1(\mathbf{x})t + e_2(\mathbf{x})t^2 + \cdots \\ &= \sum_{n\geq 0} e_n(\mathbf{x}) t^n. \end{aligned} \tag{2.4.2}$$

These show that

$$1 = E(-t)H(t) = \left(\sum_{n\geq 0} e_n(\mathbf{x})(-t)^n \right) \left(\sum_{n\geq 0} h_n(\mathbf{x}) t^n \right). \tag{2.4.3}$$

Hence, equating coefficients of powers of t, we see that for $n = 0, 1, 2, \ldots$ we have

$$\sum_{i+j=n} (-1)^i e_i h_j = \delta_{0,n}. \tag{2.4.4}$$

This lets us recursively express the e_n in terms of h_n and vice-versa:

$$e_0 = 1 = h_0; \tag{2.4.5}$$

$$e_n = e_{n-1}h_1 - e_{n-2}h_2 + e_{n-3}h_3 - \cdots ; \tag{2.4.6}$$

$$h_n = h_{n-1}e_1 - h_{n-2}e_2 + h_{n-3}e_3 - \cdots \tag{2.4.7}$$

for $n = 1, 2, 3, \ldots$ Now, let us use the algebraic independence of the generators $\{e_n\}$ for Λ to define a $\mathbf{k}$-algebra endomorphism

$$\begin{array}{ccc} \Lambda & \stackrel{\omega}{\to} & \Lambda, \\ e_n & \longmapsto & h_n \end{array} \qquad \text{(for positive integers } n\text{)}.$$

[106] See the solution to Exercise 2.2.15 for the proofs of the identities.

Then,

$$\omega(e_n) = h_n \qquad \text{for each } n \geq 0 \tag{2.4.8}$$

(indeed, this holds for $n > 0$ by definition, and for $n = 0$ because $\omega(e_0) = \omega(1) = 1 = h_0$). Hence, the identical form of the two recursions (2.4.6) and (2.4.7) shows that

$$\omega(h_n) = e_n \qquad \text{for each } n \geq 0 \tag{2.4.9}$$

[107]. Combining this with (2.4.8), we conclude that $(\omega \circ \omega)(e_n) = e_n$ for each $n \geq 0$. Therefore, the two $\mathbf{k}$-algebra homomorphisms $\omega \circ \omega : \Lambda \to \Lambda$ and $\mathrm{id} : \Lambda \to \Lambda$ agree on each element of the generating set $\{e_n\}$ of Λ. Hence, they are equal, i.e., we have $\omega \circ \omega = \mathrm{id}$. Therefore ω is an involution and therefore a $\mathbf{k}$-algebra automorphism of Λ. This, in turn, yields that the $\{h_n\}$ (being the images of the $\{e_n\}$ under this automorphism) are another algebraically independent generating set for Λ.

For the assertion about the antipode S applied to e_n or h_n, note that the coproduct formulas for e_n, h_n in Proposition 2.3.6(ii),(iii) show that the defining relations for their antipodes (1.4.4) will in this case be

$$\sum_{i+j=n} S(e_i)e_j = \delta_{0,n} = \sum_{i+j=n} e_i S(e_j),$$
$$\sum_{i+j=n} S(h_i)h_j = \delta_{0,n} = \sum_{i+j=n} h_i S(h_j)$$

because $u\epsilon(e_n) = u\epsilon(h_n) = \delta_{0,n}$. Comparing these to (2.4.4), one concludes via induction on n that $S(e_n) = (-1)^n h_n$ and $S(h_n) = (-1)^n e_n$. $\square$

The $\mathbf{k}$-algebra endomorphism ω of Λ defined in the proof of Proposition 2.4.1 is sufficiently important that we record its definition and a selection of fundamental properties:

Definition 2.4.2. Let ω be the $\mathbf{k}$-algebra homomorphism

$$\begin{array}{ccc} \Lambda & \to & \Lambda, \\ e_n & \longmapsto & h_n \end{array} \qquad \text{(for positive integers } n). \tag{2.4.10}$$

This homomorphism ω is known as the *fundamental involution* of Λ.

Proposition 2.4.3. *Consider the fundamental involution ω and the antipode S of the Hopf algebra Λ.*

[107]Here is this argument in more detail: We must show that $\omega(h_n) = e_n$ for each $n \geq 0$. We shall prove this by strong induction on n. Thus, we fix an $n \geq 0$, and assume as induction hypothesis that $\omega(h_m) = e_m$ for each $m < n$. We must then prove that $\omega(h_n) = e_n$. If $n = 0$, then this is obvious; thus, assume WLOG that $n > 0$. Hence,

$$\begin{aligned}
\omega(h_n) &= \omega(h_{n-1}e_1 - h_{n-2}e_2 + h_{n-3}e_3 - \cdots) \qquad (\text{by } (2.4.7)) \\
&= \omega(h_{n-1})\,\omega(e_1) - \omega(h_{n-2})\,\omega(e_2) + \omega(h_{n-3})\,\omega(e_3) - \cdots \\
&\qquad (\text{since } \omega \text{ is a } \mathbf{k}\text{-algebra homomorphism}) \\
&= e_{n-1}\omega(e_1) - e_{n-2}\omega(e_2) + e_{n-3}\omega(e_3) - \cdots \\
&\qquad (\text{since } \omega(h_m) = e_m \text{ for each } m < n) \\
&= e_{n-1}h_1 - e_{n-2}h_2 + e_{n-3}h_3 - \cdots \\
&\qquad (\text{since } (2.4.8) \text{ shows that } \omega(e_m) = h_m \text{ for each } m \geq 0) \\
&= e_n \qquad (\text{by } (2.4.6)),
\end{aligned}$$

as desired. This completes the induction step.

(a) *We have*

$$\omega(e_n) = h_n \qquad \text{for each } n \in \mathbb{Z}.$$

(b) *We have*

$$\omega(h_n) = e_n \qquad \text{for each } n \in \mathbb{Z}.$$

(c) *We have*

$$\omega(p_n) = (-1)^{n-1} p_n \qquad \text{for each positive integer } n.$$

(d) *The map ω is a $\mathbf{k}$-algebra automorphism of Λ and an involution.*
(e) *If $n \in \mathbb{N}$, then*

$$S(f) = (-1)^n \omega(f) \qquad \text{for all } f \in \Lambda_n. \tag{2.4.11}$$

(f) *The map ω is a Hopf algebra automorphism of Λ.*
(g) *The map S is a Hopf algebra automorphism of Λ.*
(h) *Every partition λ satisfies the three equalities*

$$\omega(h_\lambda) = e_\lambda; \tag{2.4.12}$$

$$\omega(e_\lambda) = h_\lambda; \tag{2.4.13}$$

$$\omega(p_\lambda) = (-1)^{|\lambda|-\ell(\lambda)} p_\lambda. \tag{2.4.14}$$

(i) *The map ω is an isomorphism of graded $\mathbf{k}$-modules.*
(j) *The family $(h_\lambda)_{\lambda\in\mathrm{Par}}$ is a graded basis of the graded $\mathbf{k}$-module Λ.*

Exercise 2.4.4. Prove Proposition 2.4.3.

[**Hint:** Parts (a), (b) and (d) have been shown in the proof of Proposition 2.4.1 above. For part (e), let $D_{-1} : \Lambda \to \Lambda$ be the $\mathbf{k}$-algebra morphism sending each homogeneous $f \in \Lambda_n$ to $(-1)^n f$; then argue that $\omega \circ D_{-1}$ and S are two $\mathbf{k}$-algebra morphisms that agree on all elements of the generating set $\{e_n\}$. Derive part (c) from (d) and Proposition 2.4.1. Part (h) then follows by multiplicativity. For parts (f) and (g), check the coalgebra homomorphism axioms on the e_n. Parts (i) and (j) are easy consequences.]

Proposition 2.4.3(e) shows that the antipode S on Λ is, up to sign, the same as the fundamental involution ω. Thus, studying ω is essentially equivalent to studying S.

Remark 2.4.5. Up to now we have not yet derived how the involution ω and the antipode S act on (skew) Schur functions, which is quite beautiful: If λ and μ are partitions satisfying $\mu \subseteq \lambda$, then

$$\begin{aligned} \omega(s_{\lambda/\mu}) &= s_{\lambda^t/\mu^t}, \\ S(s_{\lambda/\mu}) &= (-1)^{|\lambda/\mu|} s_{\lambda^t/\mu^t} \end{aligned} \tag{2.4.15}$$

where recall that λ^t is the transpose or conjugate partition to λ, and $|\lambda/\mu|$ is the number of squares in the skew diagram λ/μ, that is, $|\lambda/\mu| = n - k$ if λ, μ lie in $\mathrm{Par}_n, \mathrm{Par}_k$ respectively.

We will deduce this later in three ways (once as an exercise using the Pieri rules in Exercise 2.7.11, once again using skewing operators in Exercise 2.8.7, and for the third time from the action of the antipode in QSym on P-partition enumerators in Corollary 5.2.22). However, one could also deduce it immediately from our knowledge of the action of ω and S on

e_n, h_n, if we were to prove the following famous *Jacobi-Trudi* and *dual Jacobi-Trudi* formulas[108]:

Theorem 2.4.6. *Skew Schur functions are the following polynomials in* $\{h_n\}, \{e_n\}$*:*

$$s_{\lambda/\mu} = \det(h_{\lambda_i-\mu_j-i+j})_{i,j=1,2,\ldots,\ell}, \tag{2.4.16}$$

$$s_{\lambda^t/\mu^t} = \det(e_{\lambda_i-\mu_j-i+j})_{i,j=1,2,\ldots,\ell} \tag{2.4.17}$$

for any two partitions λ *and* μ *and any* $\ell \in \mathbb{N}$ *satisfying* $\ell(\lambda) \leq \ell$ *and* $\ell(\mu) \leq \ell$.

Since we appear not to need these formulas in the sequel, we will not prove them right away. However, a proof is sketched in the solution to Exercise 2.7.13, and various proofs are well-explained in [126, (39) and (41)], [142, §I.5], [184, Thm. 7.1], [186, §4.5], [206, §7.16], [220, Thms. 3.5 and 3.5*]; also, a simultaneous generalization of both formulas is shown in [83, Theorem 11], and three others in [181, 1.9], [88, Thm. 3.1] and [105]. An elegant treatment of Schur polynomials taking the Jacobi-Trudi formula (2.4.16) as the *definition* of s_λ is given by Tamvakis [215].

2.5. **Cauchy product, Hall inner product, self-duality.** The Schur functions, although a bit unmotivated right now, have special properties with regard to the Hopf structure. One property is intimately connected with the following *Cauchy identity*.

Theorem 2.5.1. *In the power series ring* $\mathbf{k}[[\mathbf{x},\mathbf{y}]] := \mathbf{k}[[x_1, x_2, \ldots, y_1, y_2, \ldots]]$, *one has the following expansion:*

$$\prod_{i,j=1}^{\infty} (1 - x_i y_j)^{-1} = \sum_{\lambda \in \mathrm{Par}} s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y}). \tag{2.5.1}$$

Remark 2.5.2. The left hand side of (2.5.1) is known as the *Cauchy product*, or *Cauchy kernel*.

An equivalent version of the equality (2.5.1) is obtained by replacing each x_i by $x_i t$, and writing the resulting identity in the power series ring $R(\mathbf{x},\mathbf{y})[[t]]$:

$$\prod_{i,j=1}^{\infty} (1 - t x_i y_j)^{-1} = \sum_{\lambda \in \mathrm{Par}} t^{|\lambda|} s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y}). \tag{2.5.2}$$

(Recall that $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$ for any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$.)

Proof of Theorem 2.5.1. We follow the standard combinatorial proof (see [186, §4.8],[206, §7.11,7.12]), which rewrites the left and right sides of (2.5.2), and then compares them with the *Robinson-Schensted-Knuth* (RSK) bijection.[109] On the left side, expanding out each geometric series

$$(1 - t x_i y_j)^{-1} = 1 + t x_i y_j + (t x_i y_j)^2 + (t x_i y_j)^3 + \cdots$$

[108]The second of the following identities is also known as the *von Nägelsbach-Kostka identity*.

[109]The RSK bijection has been introduced by Knuth [111], where what we call "biletters" is referred to as "two-line arrays". The most important ingredient of this algorithm – the RS-insertion operation – however goes back to Schensted. The special case of the RSK algorithm where the biword has to be a permutation (written in two-line notation) and the two tableaux have to be *standard* (i.e., each of them has content (1^n), where n is the size of their shape) is the famous *Robinson-Schensted correspondence* [130].

and thinking of $(x_i y_j)^m$ as m occurrences of a *biletter*[110] $\binom{i}{j}$, we see that the left hand side can be rewritten as the sum of $t^\ell\left(x_{i_1}y_{j_1}\right)\left(x_{i_2}y_{j_2}\right)\cdots\left(x_{i_\ell}y_{j_\ell}\right)$ over all multisets $\left\{\binom{i_1}{j_1},\ldots,\binom{i_\ell}{j_\ell}\right\}_{\text{multiset}}$ of biletters. Order the biletters in such a multiset in the lexicographic order $\leq_{lex}$, which is the total order on the set of all biletters defined by

$$\binom{i_1}{j_1} \leq_{lex} \binom{i_2}{j_2} \qquad \Longleftrightarrow \qquad \text{(we have } i_1 \leq i_2\text{, and if } i_1 = i_2\text{, then } j_1 \leq j_2\text{)}.$$

Defining a *biword* to be an array $\binom{\mathbf{i}}{\mathbf{j}} = \binom{i_1 \cdots i_\ell}{j_1 \cdots j_\ell}$ in which the biletters are ordered $\binom{i_1}{j_1} \leq_{lex} \cdots \leq_{lex} \binom{i_\ell}{j_\ell}$, then the left side of (2.5.2) is the sum $\sum t^\ell \mathbf{x}^{\operatorname{cont}(\mathbf{i})}\mathbf{y}^{\operatorname{cont}(\mathbf{j})}$ over all biwords $\binom{\mathbf{i}}{\mathbf{j}}$, where ℓ stands for the number of biletters in the biword. On the right side, expanding out the Schur functions as sums of tableaux gives $\sum_{(P,Q)} t^\ell \mathbf{x}^{\operatorname{cont}(Q)}\mathbf{y}^{\operatorname{cont}(P)}$ in which the sum is over all ordered pairs (P,Q) of column-strict tableaux *having the same shape*[111], with ℓ cells. (We shall refer to such pairs as *tableau pairs* from now on.)

The *Robinson-Schensted-Knuth algorithm* gives us a bijection between the biwords $\binom{\mathbf{i}}{\mathbf{j}}$ and the tableau pairs (P,Q), which has the property that

$$\operatorname{cont}(\mathbf{i}) = \operatorname{cont}(Q),$$
$$\operatorname{cont}(\mathbf{j}) = \operatorname{cont}(P)$$

(and that the length ℓ of the biword $\binom{\mathbf{i}}{\mathbf{j}}$ equals the size $|\lambda|$ of the common shape of P and Q; but this follows automatically from $\operatorname{cont}(\mathbf{i}) = \operatorname{cont}(Q)$). Clearly, once such a bijection is constructed, the equality (2.5.2) will follow.

Before we define this algorithm, we introduce a simpler operation known as *RS-insertion* (short for Robinson-Schensted insertion). RS-insertion takes as input a column-strict tableau P and a letter j, and returns a new column-strict tableau P' along with a corner cell[112] c of P', which is constructed as follows: Start out by setting $P' = P$. The letter j tries to insert itself into the first row of P' by either bumping out the leftmost letter in the first row strictly larger than j, or else placing itself at the right end of the row if no such larger letter exists. If a letter was bumped from the first row, this letter follows the same rules to insert itself into the second row, and so on[113]. This series of bumps must eventually come to an end[114]. At the end of the bumping, the tableau P' created has an extra corner cell

More about these algorithms can be found in [186, Chapter 3], [154, Chapter 5], [206, §7.11-7.12], [138, Sections 10.9–10.22], [73, Chapters 1 and A], [28, §3, §6] and various other places.

[110]A *biletter* here simply means a pair of letters, written as a column vector. A *letter* means a positive integer.

[111]And this shape should be the Ferrers diagram of a partition (not just a skew diagram).

[112]A *corner cell* of a tableau or of a Ferrers diagram is defined to be a cell c which belongs to the tableau (resp. diagram) but whose immediate neighbors to the east and to the south don't. For example, the cell $(3,2)$ is a corner cell of the Ferrers diagram of the partition $(3,2,2,1)$, and thus also of any tableau whose shape is this partition. But the cell $(2,2)$ is not a corner cell of this Ferrers diagram, since its immediate neighbor to the south is still in the diagram.

[113]Here, rows are allowed to be empty – so it is possible that a letter is bumped from the last nonempty row of P' and settles in the next, initially empty, row.

[114]since we can only bump out entries from nonempty rows

not present in P. If we call this corner cell c, then P' (in its final form) and c are what the RS-insertion operation returns. One says that P' is the result of *inserting*[115] j into the tableau P. It is straightforward to see that this resulting filling P' is a column-strict tableau[116].

Example 2.5.3. To give an example of this operation, let us insert the letter $j = 3$ into the column-strict tableau $\begin{array}{ccccc} 1 & 1 & 3 & 3 & 4 \\ 2 & 2 & 4 & 6 & \\ 3 & 4 & 7 & & \\ 5 & & & & \end{array}$ (we are showing all intermediate states of P'; the underlined letter is always the one that is going to be bumped out at the next step):

$$\begin{array}{ccccc} 1 & 1 & 3 & 3 & \underline{4} \\ 2 & 2 & 4 & 6 & \\ 3 & 4 & 7 & & \\ 5 & & & & \end{array} \underset{\substack{\text{insert } 3;\\ \text{bump out } 4}}{\longmapsto} \begin{array}{ccccc} 1 & 1 & 3 & 3 & 3 \\ 2 & 2 & 4 & \underline{6} & \\ 3 & 4 & 7 & & \\ 5 & & & & \end{array} \underset{\substack{\text{insert } 4;\\ \text{bump out } 6}}{\longmapsto} \begin{array}{ccccc} 1 & 1 & 3 & 3 & 3 \\ 2 & 2 & 4 & 4 & \\ 3 & 4 & \underline{7} & & \\ 5 & & & & \end{array}$$

$$\underset{\substack{\text{insert } 6;\\ \text{bump out } 7}}{\longmapsto} \begin{array}{ccccc} 1 & 1 & 3 & 3 & 3 \\ 2 & 2 & 4 & 4 & \\ 3 & 4 & 6 & & \\ 5 & & & & \end{array} \underset{\substack{\text{insert } 7;\\ \text{done}}}{\longmapsto} \begin{array}{ccccc} 1 & 1 & 3 & 3 & 3 \\ 2 & 2 & 4 & 4 & \\ 3 & 4 & 6 & & \\ 5 & 7 & & & \end{array}.$$

The last tableau in this sequence is the column-strict tableau that is returned. The corner cell that is returned is the second cell of the fourth row (the one containing 7).

RS-insertion will be used as a step in the RSK algorithm; the construction will rely on a simple fact known as the *row bumping lemma*. Let us first define the notion of a *bumping path* (or *bumping route*): If P is a column-strict tableau, and j is a letter, then some letters are inserted into some cells when RS-insertion is applied to P and j. The sequence of these cells (in the order in which they see letters inserted into them) is called the *bumping path* for P and j. This bumping path always ends with the corner cell c which is returned by RS-insertion. As an example, when $j = 1$ is inserted into the tableau P shown below, the result P' is shown with all entries on the bumping path underlined:

$$P = \begin{array}{ccccc} 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 4 & 4 & \\ 3 & 4 & 5 & & \\ 4 & 6 & 6 & & \end{array} \qquad \overset{\text{insert}}{\underset{j=1}{\longmapsto}} \qquad P' = \begin{array}{ccccc} 1 & 1 & \underline{1} & 2 & 3 \\ 2 & 2 & \underline{2} & 4 & \\ 3 & 4 & \underline{4} & & \\ 4 & \underline{5} & 6 & & \\ \underline{6} & & & & \end{array}$$

[115]This terminology is reminiscent of insertion into binary search trees, a basic operation in theoretical computer science. This is more than superficial similarity; there are, in fact, various analogies between Ferrers diagrams (and their fillings) and unlabelled plane binary trees (resp. their labellings), and one of them is the analogy between RS-insertion and binary search tree insertion. See [97, §4.1].

[116]Indeed, the reader can check that P' remains a column-strict tableau throughout the algorithm that defines RS-insertion. (The only part of this that isn't obvious is showing that when a letter t bumped out of some row k is inserted into row $k+1$, the property that the letters increase strictly down columns is preserved. Argue that the bumping-out of t from row k was caused by the insertion of another letter $u < t$, and that the cell of row $k+1$ into which t is then being inserted is in the same column as this u, or in a column further left than it.)

A first simple observation about bumping paths is that bumping paths *trend weakly left* – that is, if the bumping path of P and j is $(c_1, c_2, \ldots, c_k)$, then, for each $1 \leq i < k$, the cell c_{i+1} lies in the same column as c_i or in a column further left.[117] A subtler property of bumping paths is the following *row bumping lemma* ([73, p. 9]):

> **Row bumping lemma:** Let P be a column-strict tableau, and let j and j' be two letters. Applying RS-insertion to the tableau P and the letter j yields a new column-strict tableau P' and a corner cell c. Applying RS-insertion to the tableau P' and the letter j' yields a new column-strict tableau P'' and a corner cell c'.
>
> (a) Assume that $j \leq j'$. Then, the bumping path for P' and j' stays strictly to the right, within each row, of the bumping path for P and j. The cell c' (in which the bumping path for P' and j' ends) is in the same row as the cell c (in which the bumping path for P and j ends) or in a row further up; it is also in a column further right than c.
>
> (b) Assume instead that $j > j'$. Then, the bumping path for P' and j' stays weakly to the left, within each row, of the bumping path for P and j. The cell c' (in which the bumping path for P' and j' ends) is in a row further down than the cell c (in which the bumping path for P and j ends); it is also in the same column as c or in a column further left.

This lemma can be easily proven by induction over the row.[118]

We can now define the actual RSK algorithm. Let $\binom{\mathbf{i}}{\mathbf{j}}$ be a biword. Starting with the pair $(P_0, Q_0) = (\varnothing, \varnothing)$ and $m = 0$, the algorithm applies the following steps (see Example 2.5.4 below):

- If i_{m+1} does not exist (that is, m is the length of $\mathbf{i}$), stop.

[117] This follows easily from the preservation of column-strictness during RS-insertion.

[118] We leave the details to the reader, only giving the main idea for (a) (the proof of (b) is similar). To prove the first claim of (a), it is enough to show that for every i, if any letter is inserted into row i during RS-insertion for P' and j', then some letter is also inserted into row i during RS-insertion for P and j, and the former insertion happens in a cell strictly to the right of the cell where the latter insertion happens. This follows by induction over i. In the induction step, we need to show that if, for a positive integer i, we try to consecutively insert two letters k and k', in this order, into the i-th row of a column-strict tableau, possibly bumping out existing letters in the process, and if we have $k \leq k'$, then the cell into which k is inserted is strictly to the left of the cell into which k' is inserted, and the letter bumped out by the insertion of k is $\leq$ to the letter bumped out by the insertion of k' (or else the insertion of k' bumps out no letter at all – but it cannot happen that k' bumps out a letter but k does not). This statement is completely straightforward to check (by only studying the i-th row). This way, the first claim of (a) is proven, and this entails that the cell c' (being the last cell of the bumping path for P' and j') is in the same row as the cell c or in a row further up. It only remains to show that c' is in a column further right than c. This follows by noticing that, if k is the row in which the cell c' lies, then c' is in a column further right than the entry of the bumping path for P and j in row k (by the first claim of (a)), and this latter entry is further right than or in the same column as the ultimate entry c of this bumping path (since bumping paths trend weakly left).

- Apply RS-insertion to the column-strict tableau P_m and the letter j_{m+1} (the bottom letter of $\binom{i_{m+1}}{j_{m+1}}$). Let P_{m+1} be the resulting column-strict tableau, and let c_{m+1} be the resulting corner cell.
- Create Q_{m+1} from Q_m by adding the top letter i_{m+1} of $\binom{i_{m+1}}{j_{m+1}}$ to Q_m in the cell c_{m+1} (which, as we recall, is the extra corner cell of P_{m+1} not present in P_m).
- Set m to $m+1$.

After all of the biletters have been thus processed, the result of the RSK algorithm is $(P_\ell, Q_\ell) =: (P, Q)$.

Example 2.5.4. The term in the expansion of the left side of (2.5.1) corresponding to

$$(x_1y_2)^1(x_1y_4)^1(x_2y_1)^1(x_4y_1)^1(x_4y_3)^2(x_5y_2)^1$$

is the biword $\binom{\mathbf{i}}{\mathbf{j}} = \binom{1124445}{2411332}$, whose RSK algorithm goes as follows:

$$\begin{array}{rcl|rcl}
P_0 & = & \varnothing & Q_0 & = & \varnothing \\[2ex]
P_1 & = & 2 & Q_1 & = & 1 \\[2ex]
P_2 & = & 2\ \ 4 & Q_2 & = & 1\ \ 1 \\[2ex]
P_3 & = & \begin{matrix} 1 & 4 \\ 2 & \end{matrix} & Q_3 & = & \begin{matrix} 1 & 1 \\ 2 & \end{matrix} \\[2ex]
P_4 & = & \begin{matrix} 1 & 1 \\ 2 & 4 \end{matrix} & Q_4 & = & \begin{matrix} 1 & 1 \\ 2 & 4 \end{matrix} \\[2ex]
P_5 & = & \begin{matrix} 1 & 1 & 3 \\ 2 & 4 & \end{matrix} & Q_5 & = & \begin{matrix} 1 & 1 & 4 \\ 2 & 4 & \end{matrix} \\[2ex]
P_6 & = & \begin{matrix} 1 & 1 & 3 & 3 \\ 2 & 4 & & \end{matrix} & Q_6 & = & \begin{matrix} 1 & 1 & 4 & 4 \\ 2 & 4 & & \end{matrix} \\[2ex]
P := P_7 & = & \begin{matrix} 1 & 1 & 2 & 3 \\ 2 & 3 & & \\ 4 & & & \end{matrix} & Q := Q_7 & = & \begin{matrix} 1 & 1 & 4 & 4 \\ 2 & 4 & & \\ 5 & & & \end{matrix}
\end{array}$$

The bumping rule obviously maintains the property that P_m is a column-strict tableau of some Ferrers shape throughout. It should be clear that (P_m, Q_m) have the same shape at each stage. Also, the construction of Q_m shows that it is at least weakly increasing in rows and weakly increasing in columns throughout. What is perhaps least clear is that Q_m remains strictly increasing down columns. That is, when one has a string of equal letters on top $i_m = i_{m+1} = \cdots = i_{m+r}$, so that on bottom one bumps in $j_m \leq j_{m+1} \leq \cdots \leq j_{m+r}$, one needs to know that the new cells form a *horizontal strip*, that is, no two of them lie in the same column[119]. This follows from (the last claim of) part (a) of the row bumping lemma. Hence, the result (P, Q) of the RSK algorithm is a tableau pair.

[119]Actually, each of these new cells (except for the first one) is in a column further right than the previous one. We will use this stronger fact further below.

To see that the RSK map is a bijection, we show how to recover $\binom{\mathsf{i}}{\mathsf{j}}$ from (P, Q). This is done by *reverse bumping* from (P_{m+1}, Q_{m+1}) to recover both the biletter $\binom{i_{m+1}}{j_{m+1}}$ and the tableaux (P_m, Q_m), as follows. Firstly, i_{m+1} is the maximum entry of Q_{m+1}, and Q_m is obtained by removing the rightmost occurrence of this letter i_{m+1} from Q_{m+1}. [120] To produce P_m and j_{m+1}, find the position of the rightmost occurrence of i_{m+1} in Q_{m+1}, and start *reverse bumping* in P_{m+1} from the entry in this same position, where reverse bumping an entry means inserting it into one row higher by having it bump out the rightmost entry which is strictly smaller.[121] The entry bumped out of the first row is j_{m+1}, and the resulting tableau is P_m.

Finally, to see that the RSK map is surjective, one needs to show that the reverse bumping procedure can be applied to any pair (P, Q) of column-strict tableaux of the same shape, and will result in a (lexicographically ordered) biword $\binom{\mathsf{i}}{\mathsf{j}}$. We leave this verification to the reader.[122] □

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[120]It necessarily has to be the rightmost occurrence, since (according to the previous footnote) the cell into which i_{m+1} was filled at the step from Q_m to Q_{m+1} lies further right than any existing cell of Q_m containing the letter i_{m+1}.

[121]Let us give a few more details on this "reverse bumping" procedure. Reverse bumping (also known as *RS-deletion* or *reverse RS-insertion*) is an operation which takes a column-strict tableau P' and a corner cell c of P', and constructs a column-strict tableau P and a letter j such that RS-insertion for P and j yields P' and c. It starts by setting $P = P'$, and removing the entry in the cell c from P. This removed entry is then denoted by k, and is inserted into the row of P above c, bumping out the rightmost entry which is smaller than k. The letter which is bumped out – say, ℓ –, in turn, is inserted into the row above it, bumping out the rightmost entry which is smaller than ℓ. This procedure continues in the same way until an entry is bumped out of the first row (which will eventually happen). The reverse bumping operation returns the resulting tableau P and the entry which is bumped out of the first row.

It is straightforward to check that the reverse bumping operation is well-defined (i.e., P does stay a column-strict tableau throughout the procedure) and is the inverse of the RS-insertion operation. (In fact, these two operations undo each other step by step.)

[122]It is easy to see that repeatedly applying reverse bumping to (P, Q) will result in a sequence $\binom{i_\ell}{j_\ell}, \binom{i_{\ell-1}}{j_{\ell-1}}, \ldots, \binom{i_1}{j_1}$ of biletters such that applying the RSK algorithm to $\binom{i_1 \cdots i_\ell}{j_1 \cdots j_\ell}$ gives back (P, Q). The question is why we have $\binom{i_1}{j_1} \leq_{lex} \cdots \leq_{lex} \binom{i_\ell}{j_\ell}$. Since the chain of inequalities $i_1 \leq i_2 \leq \cdots \leq i_\ell$ is clear from the choice of entry to reverse-bump, it only remains to show that for every string $i_m = i_{m+1} = \cdots = i_{m+r}$ of equal top letters, the corresponding bottom letters weakly increase (that is, $j_m \leq j_{m+1} \leq \cdots \leq j_{m+r}$). One way to see this is the following:

Assume the contrary; i.e., assume that the bottom letters corresponding to some string $i_m = i_{m+1} = \cdots = i_{m+r}$ of equal top letters do not weakly increase. Thus, $j_{m+p} > j_{m+p+1}$ for some $p \in \{0, 1, \ldots, r-1\}$. Consider this p.

Let us consider the cells containing the equal letters $i_m = i_{m+1} = \cdots = i_{m+r}$ in the tableau Q_{m+r}. Label these cells as $c_m, c_{m+1}, \ldots, c_{m+r}$ from left to right (noticing that no two of them lie in the same column, since Q_{m+r} is column-strict). By the definition of reverse bumping, the first entry to be reverse bumped from P_{m+r} is the entry in position c_{m+r} (since this is the rightmost occurrence of the letter i_{m+r} in Q_{m+r}); then, the next entry to be reverse bumped is the one in position c_{m+r-1}, etc., moving further and further left. Thus, for each $q \in \{0, 1, \ldots, r\}$, the tableau P_{m+q-1} is obtained from P_{m+q} by reverse bumping the entry in position c_{m+q}. Hence, conversely, the tableau P_{m+q} is obtained from P_{m+q-1} by RS-inserting the entry j_{m+q}, which creates the corner cell c_{m+q}.

But recall that $j_{m+p} > j_{m+p+1}$. Hence, part (b) of the row bumping lemma (applied to P_{m+p-1}, j_{m+p}, j_{m+p+1}, P_{m+p}, c_{m+p}, P_{m+p+1} and c_{m+p+1} instead of P, j, j', P', c, P'' and c') shows that the cell c_{m+p+1} is in the same column as the cell c_{m+p} or in a

This is by far not the only known proof of Theorem 2.5.1. Two further proofs will be sketched in Exercise 2.7.10 and Exercise 2.7.8.

Before we move on to extracting identities in Λ from Theorem 2.5.1, let us state (as an exercise) a simple technical fact that will be useful:

Exercise 2.5.5. Let $(q_\lambda)_{\lambda\in\mathrm{Par}}$ be a basis of the $\mathbf{k}$-module Λ. Assume that for each partition λ, the element $q_\lambda \in \Lambda$ is homogeneous of degree $|\lambda|$.

(a) If two families $(a_\lambda)_{\lambda\in\mathrm{Par}} \in \mathbf{k}^{\mathrm{Par}}$ and $(b_\lambda)_{\lambda\in\mathrm{Par}} \in \mathbf{k}^{\mathrm{Par}}$ satisfy

$$\sum_{\lambda\in\mathrm{Par}} a_\lambda q_\lambda(\mathbf{x}) = \sum_{\lambda\in\mathrm{Par}} b_\lambda q_\lambda(\mathbf{x}) \tag{2.5.3}$$

in $\mathbf{k}[[\mathbf{x}]]$, then $(a_\lambda)_{\lambda\in\mathrm{Par}} = (b_\lambda)_{\lambda\in\mathrm{Par}}$. [123]

(b) Consider a further infinite family $\mathbf{y} = (y_1, y_2, y_3, \ldots)$ of indeterminates (disjoint from $\mathbf{x}$). If two families $(a_{\mu,\nu})_{(\mu,\nu)\in\mathrm{Par}^2} \in \mathbf{k}^{\mathrm{Par}^2}$ and $(b_{\mu,\nu})_{(\mu,\nu)\in\mathrm{Par}^2} \in \mathbf{k}^{\mathrm{Par}^2}$ satisfy

$$\sum_{(\mu,\nu)\in\mathrm{Par}^2} a_{\mu,\nu} q_\mu(\mathbf{x}) q_\nu(\mathbf{y}) = \sum_{(\mu,\nu)\in\mathrm{Par}^2} b_{\mu,\nu} q_\mu(\mathbf{x}) q_\nu(\mathbf{y}) \tag{2.5.4}$$

in $\mathbf{k}[[\mathbf{x},\mathbf{y}]]$, then $(a_{\mu,\nu})_{(\mu,\nu)\in\mathrm{Par}^2} = (b_{\mu,\nu})_{(\mu,\nu)\in\mathrm{Par}^2}$.

(c) Consider a further infinite family $\mathbf{z} = (z_1, z_2, z_3, \ldots)$ of indeterminates (disjoint from $\mathbf{x}$ and $\mathbf{y}$). If two families $(a_{\lambda,\mu,\nu})_{(\mu,\nu,\lambda)\in\mathrm{Par}^3} \in \mathbf{k}^{\mathrm{Par}^3}$ and $(b_{\lambda,\mu,\nu})_{(\mu,\nu,\lambda)\in\mathrm{Par}^3} \in \mathbf{k}^{\mathrm{Par}^3}$ satisfy

$$\begin{aligned} &\sum_{(\mu,\nu,\lambda)\in\mathrm{Par}^3} a_{\lambda,\mu,\nu} q_\mu(\mathbf{x}) q_\nu(\mathbf{y}) q_\lambda(\mathbf{z}) \\ &= \sum_{(\mu,\nu,\lambda)\in\mathrm{Par}^3} b_{\lambda,\mu,\nu} q_\mu(\mathbf{x}) q_\nu(\mathbf{y}) q_\lambda(\mathbf{z}) \end{aligned} \tag{2.5.5}$$

in $\mathbf{k}[[\mathbf{x},\mathbf{y},\mathbf{z}]]$, then $(a_{\lambda,\mu,\nu})_{(\mu,\nu,\lambda)\in\mathrm{Par}^3} = (b_{\lambda,\mu,\nu})_{(\mu,\nu,\lambda)\in\mathrm{Par}^3}$.

Remark 2.5.6. Clearly, for any $n \in \mathbb{N}$, we can state an analogue of Exercise 2.5.5 for n infinite families $\mathbf{x}_i = (x_{i,1}, x_{i,2}, x_{i,3}, \ldots)$ of indeterminates (with $i \in \{1, 2, \ldots, n\}$). The three parts of Exercise 2.5.5 are the particular cases of this analogue for $n = 1$, for $n = 2$ and for $n = 3$. We have shied away from stating this analogue in full generality because these particular cases are the only ones we will need.

Corollary 2.5.7. *In the Schur function basis $\{s_\lambda\}$ for Λ, the structure constants for multiplication and comultiplication are the same, that is, if*

column further left. But this contradicts the fact that the cell c_{m+p+1} is in a column further right than the cell c_{m+p} (since we have labeled our cells as $c_m, c_{m+1}, \ldots, c_{m+r}$ from left to right, and no two of them lied in the same column). This contradiction completes our proof.

[123] Note that this does not immediately follow from the linear independence of the basis $(q_\lambda)_{\lambda\in\mathrm{Par}}$. Indeed, linear independence would help if the sums in (2.5.3) were finite, but they are not. A subtler argument (involving the homogeneity of the q_λ) thus has to be used.

one defines scalars $c^{\lambda}_{\mu,\nu}, \hat{c}^{\lambda}_{\mu,\nu}$ *via the unique expansions*

$$s_\mu s_\nu = \sum_\lambda c^{\lambda}_{\mu,\nu} s_\lambda, \tag{2.5.6}$$

$$\Delta(s_\lambda) = \sum_{\mu,\nu} \hat{c}^{\lambda}_{\mu,\nu} s_\mu \otimes s_\nu, \tag{2.5.7}$$

then $c^{\lambda}_{\mu,\nu} = \hat{c}^{\lambda}_{\mu,\nu}$.

Proof. Work in the ring $\mathbf{k}[[\mathbf{x},\mathbf{y},\mathbf{z}]]$, where $\mathbf{y} = (y_1, y_2, y_3, \ldots)$ and $\mathbf{z} = (z_1, z_2, z_3, \ldots)$ are two new sets of variables. The identity (2.5.1) lets one interpret both $c^{\lambda}_{\mu,\nu}, \hat{c}^{\lambda}_{\mu,\nu}$ as the coefficient[124] of $s_\mu(\mathbf{x})s_\nu(\mathbf{y})s_\lambda(\mathbf{z})$ in the product

$$\begin{aligned}\prod_{i,j=1}^{\infty}(1-x_iz_j)^{-1}\prod_{i,j=1}^{\infty}(1-y_iz_j)^{-1} &\overset{(2.5.1)}{=} \left(\sum_\mu s_\mu(\mathbf{x})s_\mu(\mathbf{z})\right)\left(\sum_\nu s_\nu(\mathbf{y})s_\nu(\mathbf{z})\right) \\ &= \sum_{\mu,\nu} s_\mu(\mathbf{x})s_\nu(\mathbf{y}) \cdot s_\mu(\mathbf{z})s_\nu(\mathbf{z}) \\ &= \sum_{\mu,\nu} s_\mu(\mathbf{x})s_\nu(\mathbf{y})\left(\sum_\lambda c^{\lambda}_{\mu,\nu} s_\lambda(\mathbf{z})\right)\end{aligned}$$

since, regarding $x_1, x_2, \ldots, y_1, y_2, \ldots$ as lying in a single variable set $(\mathbf{x},\mathbf{y})$, separate from the variables $\mathbf{z}$, the Cauchy identity (2.5.1) expands the same product as

$$\begin{aligned}\prod_{i,j=1}^{\infty}(1-x_iz_j)^{-1}\prod_{i,j=1}^{\infty}(1-y_iz_j)^{-1} &= \sum_\lambda s_\lambda(\mathbf{x},\mathbf{y})s_\lambda(\mathbf{z}) \\ &= \sum_\lambda\left(\sum_{\mu,\nu}\hat{c}^{\lambda}_{\mu,\nu} s_\mu(\mathbf{x})s_\nu(\mathbf{y})\right)s_\lambda(\mathbf{z}).\end{aligned}$$

□

Definition 2.5.8. The coefficients $c^{\lambda}_{\mu,\nu} = \hat{c}^{\lambda}_{\mu,\nu}$ appearing in the expansions (2.5.6) and (2.5.7) are called *Littlewood-Richardson coefficients*.

Remark 2.5.9. We will interpret $c^{\lambda}_{\mu,\nu}$ combinatorially in Section 2.6. By now, however, we can already prove some properties of these coefficients:

We have

$$c^{\lambda}_{\mu,\nu} = c^{\lambda}_{\nu,\mu} \qquad \text{for all } \lambda, \mu, \nu \in \mathrm{Par} \tag{2.5.8}$$

(by comparing coefficients in $\sum_\lambda c^{\lambda}_{\mu,\nu} s_\lambda = s_\mu s_\nu = s_\nu s_\mu = \sum_\lambda c^{\lambda}_{\nu,\mu} s_\lambda$). Furthermore, let λ and μ be two partitions (not necessarily satisfying $\mu \subseteq \lambda$). Comparing the expansion

$$s_\lambda(\mathbf{x},\mathbf{y}) = \Delta(s_\lambda) = \sum_{\mu,\nu} c^{\lambda}_{\mu,\nu} s_\mu(\mathbf{x})s_\nu(\mathbf{y}) = \sum_{\mu\in\mathrm{Par}}\left(\sum_{\nu\in\mathrm{Par}} c^{\lambda}_{\mu,\nu} s_\nu(\mathbf{y})\right)s_\mu(\mathbf{x})$$

[124] Let us explain why speaking of coefficients makes sense here:

We want to use the fact that if a power series $f \in \mathbf{k}[[\mathbf{x},\mathbf{y},\mathbf{z}]]$ is written in the form $f = \sum_{(\mu,\nu,\lambda)\in\mathrm{Par}^3} a_{\lambda,\mu,\nu} s_\mu(\mathbf{x})\, s_\nu(\mathbf{y})\, s_\lambda(\mathbf{z})$ for some coefficients $a_{\lambda,\mu,\nu} \in \mathbf{k}$, then these coefficients $a_{\lambda,\mu,\nu}$ are uniquely determined by f. But this fact is precisely the claim of Exercise 2.5.5(c) above (applied to $q_\lambda = s_\lambda$).

with

$$s_\lambda(\mathbf{x},\mathbf{y}) = \sum_{\mu\subseteq\lambda} s_\mu(\mathbf{x}) s_{\lambda/\mu}(\mathbf{y}) = \sum_{\mu\in\mathrm{Par}} s_\mu(\mathbf{x}) s_{\lambda/\mu}(\mathbf{y})$$

[125], one concludes that

$$\sum_{\mu\in\mathrm{Par}} \left(\sum_{\nu\in\mathrm{Par}} c^\lambda_{\mu,\nu} s_\nu(\mathbf{y}) \right) s_\mu(\mathbf{x}) = \sum_{\mu\in\mathrm{Par}} s_\mu(\mathbf{x}) s_{\lambda/\mu}(\mathbf{y}) = \sum_{\mu\in\mathrm{Par}} s_{\lambda/\mu}(\mathbf{y}) s_\mu(\mathbf{x}).$$

Treating the indeterminates $\mathbf{y}$ as constants, and comparing coefficients before $s_\mu(\mathbf{x})$ on both sides of this equality[126], we arrive at another standard interpretation for $c^\lambda_{\mu,\nu}$:

$$s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu,\nu} s_\nu.$$

In particular, $c^\lambda_{\mu,\nu}$ vanishes unless $\mu \subseteq \lambda$. Consequently, $c^\lambda_{\mu,\nu}$ vanishes unless $\nu \subseteq \lambda$ as well (since $c^\lambda_{\mu,\nu} = c^\lambda_{\nu,\mu}$) and furthermore vanishes unless the equality $|\mu| + |\nu| = |\lambda|$ holds[127]. Altogether, we conclude that $c^\lambda_{\mu,\nu}$ vanishes unless $\mu, \nu \subseteq \lambda$ and $|\mu| + |\nu| = |\lambda|$.

Exercise 2.5.10. Show that any four partitions κ, λ, φ and ψ satisfy

$$\sum_{\rho\in\mathrm{Par}} c^\rho_{\kappa,\lambda} c^\rho_{\varphi,\psi} = \sum_{(\alpha,\beta,\gamma,\delta)\in\mathrm{Par}^4} c^\lambda_{\beta,\delta} c^\varphi_{\alpha,\beta} c^\psi_{\gamma,\delta}.$$

Exercise 2.5.11. (a) For any partition μ, prove that

$$\sum_{\lambda\in\mathrm{Par}} s_\lambda(\mathbf{x})\, s_{\lambda/\mu}(\mathbf{y}) = s_\mu(\mathbf{x}) \cdot \prod_{i,j=1}^{\infty} (1 - x_i y_j)^{-1}$$

in the power series ring $\mathbf{k}[[\mathbf{x},\mathbf{y}]] = \mathbf{k}[[x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots]]$.

(b) Let α and β be two partitions. Show that

$$\sum_{\lambda\in\mathrm{Par}} s_{\lambda/\alpha}(\mathbf{x})\, s_{\lambda/\beta}(\mathbf{y}) = \left(\sum_{\rho\in\mathrm{Par}} s_{\beta/\rho}(\mathbf{x})\, s_{\alpha/\rho}(\mathbf{y}) \right) \cdot \prod_{i,j=1}^{\infty} (1 - x_i y_j)^{-1}$$

in the power series ring $\mathbf{k}[[\mathbf{x},\mathbf{y}]] = \mathbf{k}[[x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots]]$.

[**Hint:** For (b), expand the product

$$\prod_{i,j=1}^{\infty} (1 - x_i y_j)^{-1} \prod_{i,j=1}^{\infty} (1 - x_i w_j)^{-1} \prod_{i,j=1}^{\infty} (1 - z_i y_j)^{-1} \prod_{i,j=1}^{\infty} (1 - z_i w_j)^{-1}$$

in the power series ring

$$\mathbf{k}[[x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots, z_1, z_2, z_3, \ldots, w_1, w_2, w_3, \ldots]]$$

[125] In the last equality, we removed the condition $\mu \subseteq \lambda$ on the addends of the sum; this does not change the value of the sum (because we have $s_{\lambda/\mu} = 0$ whenever we don't have $\mu \subseteq \lambda$).

[126] "Comparing coefficients" means applying Exercise 2.5.5(a) to $q_\lambda = s_\lambda$ in this case (although the base ring $\mathbf{k}$ is now replaced by $\mathbf{k}[[\mathbf{y}]]$, and the index μ is used instead of λ, since λ is already taken).

[127] In fact, this is clear when we don't have $\mu \subseteq \lambda$. When we do have $\mu \subseteq \lambda$, this follows from observing that $s_{\lambda/\mu} \in \Lambda_{|\lambda/\mu|}$ has zero coefficient before s_ν whenever $|\mu| + |\nu| \neq |\lambda|$.

in two ways: once by applying Theorem 2.5.1 to the two variable sets $(\mathbf{z}, \mathbf{x})$ and $(\mathbf{w}, \mathbf{y})$ and then using (2.3.3); once again by applying (2.5.1) to the two variable sets $\mathbf{z}$ and $\mathbf{w}$ and then applying Exercise 2.5.11(a) twice.]

The statement of Exercise 2.5.11(b) is known as the *skew Cauchy identity*, and appears in Sagan-Stanley [188, Cor. 6.12], Stanley [206, exercise 7.27(c)] and Macdonald [142, §I.5, example 26]; it seems to be due to Zelevinsky. It generalizes the statement of Exercise 2.5.11(a), which in turn is a generalization of Theorem 2.5.1.

Definition 2.5.12. Define the *Hall inner product* on Λ to be the $\mathbf{k}$-bilinear form $(\cdot, \cdot)$ which makes $\{s_\lambda\}$ an orthonormal basis, that is, $(s_\lambda, s_\nu) = \delta_{\lambda,\nu}$.

Exercise 2.5.13. (a) If n and m are two distinct nonnegative integers, and if $f \in \Lambda_n$ and $g \in \Lambda_m$, then show that $(f, g) = 0$.

(b) If $n \in \mathbb{N}$ and $f \in \Lambda_n$, then prove that $(h_n, f) = f(1)$ (where $f(1)$ is defined as in Exercise 2.1.2).

(c) Show that $(f, g) = (g, f)$ for all $f \in \Lambda$ and $g \in \Lambda$. (In other words, the Hall inner product is symmetric.)

The Hall inner product induces a $\mathbf{k}$-module homomorphism $\Lambda \to \Lambda^o$ (sending every $f \in \Lambda$ to the $\mathbf{k}$-linear map $\Lambda \to \mathbf{k}$, $g \mapsto (f, g)$). This homomorphism is invertible (since the Hall inner product has an orthonormal basis), so that $\Lambda^o \cong \Lambda$ as $\mathbf{k}$-modules. But in fact, more can be said:

Corollary 2.5.14. *The isomorphism $\Lambda^o \cong \Lambda$ induced by the Hall inner product is an isomorphism of Hopf algebras.*

Proof. We have seen that the orthonormal basis $\{s_\lambda\}$ of Schur functions is *self-dual*, in the sense that its multiplication and comultiplication structure constants are the same. Thus the isomorphism $\Lambda^o \cong \Lambda$ induced by the Hall inner product is an isomorphism of bialgebras[128], and hence also a Hopf algebra isomorphism by Corollary 1.4.27. $\square$

We next identify two other dual pairs of bases, by expanding the Cauchy product in two other ways.

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[128]Here are some details on the proof:

Let $\gamma : \Lambda \to \Lambda^o$ be the $\mathbf{k}$-module isomorphism $\Lambda \to \Lambda^o$ induced by the Hall inner product. We want to show that γ is an isomorphism of bialgebras.

Let $\{s_\lambda^*\}$ be the basis of Λ^o dual to the basis $\{s_\lambda\}$ of Λ. Thus, for any partition λ, we have

$$\gamma(s_\lambda) = s_\lambda^* \tag{2.5.9}$$

(since any partition μ satisfies $(\gamma(s_\lambda))(s_\mu) = (s_\lambda, s_\mu) = \delta_{\lambda,\mu} = s_\lambda^*(s_\mu)$, and thus the two $\mathbf{k}$-linear maps $\gamma(s_\lambda) : \Lambda \to \mathbf{k}$ and $s_\lambda^* : \Lambda \to \mathbf{k}$ are equal to each other on the basis $\{s_\mu\}$ of Λ, which forces them to be identical).

The coproduct structure constants of the basis $\{s_\lambda^*\}$ of Λ^o equal the product structure constants of the basis $\{s_\lambda\}$ of Λ (according to our discussion of duals in Section 1.6). Since the latter are the Littlewood-Richardson numbers $c^\lambda_{\mu,\nu}$ (because of (2.5.6)), we thus conclude that the former are $c^\lambda_{\mu,\nu}$ as well. In other words, every $\lambda \in \mathrm{Par}$ satisfies

$$\Delta_{\Lambda^o} s_\lambda^* = \sum_{\mu,\nu} c^\lambda_{\mu,\nu} s_\mu^* \otimes s_\nu^* \tag{2.5.10}$$

Proposition 2.5.15. *One can also expand*

$$\prod_{i,j=1}^{\infty}(1-x_iy_j)^{-1} = \sum_{\lambda\in\mathrm{Par}} h_\lambda(\mathbf{x})m_\lambda(\mathbf{y}) \tag{2.5.11}$$
$$= \sum_{\lambda\in\mathrm{Par}} z_\lambda^{-1}p_\lambda(\mathbf{x})p_\lambda(\mathbf{y})$$

where $z_\lambda := m_1!\cdot 1^{m_1}\cdot m_2!\cdot 2^{m_2}\cdots$ if λ is written in multiplicative notation as $\lambda = (1^{m_1}, 2^{m_2}, \ldots)$ with multiplicity m_i for the part i. (Here, we assume that $\mathbb{Q}$ is a subring of $\mathbf{k}$ for the last equality.)

Remark 2.5.16. It is relevant later (and explains the notation) that z_λ is the size of the $\mathfrak{S}_n$-centralizer subgroup for a permutation having cycle type[129] λ with $|\lambda| = n$. This is a classical (and fairly easy) result (see, e.g., [186, Prop. 1.1.1] or [206, Prop. 7.7.3] for a proof).

(where the sum is over all pairs (μ,ν) of partitions). On the other hand, applying the map $\gamma\otimes\gamma : \Lambda\otimes\Lambda\to\Lambda^o\otimes\Lambda^o$ to the equality (2.5.7) yields

$$\begin{aligned}(\gamma\otimes\gamma)(\Delta(s_\lambda)) &= (\gamma\otimes\gamma)\left(\sum_{\mu,\nu}\hat{c}^\lambda_{\mu,\nu}s_\mu\otimes s_\nu\right) = \sum_{\mu,\nu}\underbrace{\hat{c}^\lambda_{\mu,\nu}}_{=c^\lambda_{\mu,\nu}}\underbrace{\gamma(s_\mu)}_{\substack{=s^*_\mu\\ \text{(by (2.5.9))}}}\otimes\underbrace{\gamma(s_\nu)}_{\substack{=s^*_\nu\\ \text{(by (2.5.9))}}}\\ &= \sum_{\mu,\nu}c^\lambda_{\mu,\nu}s^*_\mu\otimes s^*_\nu = \Delta_{\Lambda^o}\underbrace{s^*_\lambda}_{\substack{=\gamma(s_\lambda)\\ \text{(by (2.5.9))}}} \qquad \text{(by (2.5.10))}\\ &= \Delta_{\Lambda^o}(\gamma(s_\lambda))\end{aligned}$$

for each $\lambda\in\mathrm{Par}$. In other words, the two $\mathbf{k}$-linear maps $(\gamma\otimes\gamma)\circ\Delta$ and $\Delta_{\Lambda^o}\circ\gamma$ are equal to each other on each s_λ with $\lambda\in\mathrm{Par}$. Hence, these two maps must be identical (since the s_λ form a basis of Λ). Hence, $\Delta_{\Lambda^o}\circ\gamma = (\gamma\otimes\gamma)\circ\Delta$.

Our next goal is to show that $\epsilon_{\Lambda^o}\circ\gamma = \epsilon$. Indeed, each $\lambda\in\mathrm{Par}$ satisfies

$$\begin{aligned}(\epsilon_{\Lambda^o}\circ\gamma)(s_\lambda) &= \epsilon_{\Lambda^o}(\gamma(s_\lambda)) = (\gamma(s_\lambda))(1) \qquad \text{(by the definition of } \epsilon_{\Lambda^o})\\ &= \left(s_\lambda, \underbrace{1}_{=s_\varnothing}\right) = (s_\lambda, s_\varnothing) = \delta_{\lambda,\varnothing} = \epsilon(s_\lambda).\end{aligned}$$

Hence, $\epsilon_{\Lambda^o}\circ\gamma = \epsilon$. Combined with $\Delta_{\Lambda^o}\circ\gamma = (\gamma\otimes\gamma)\circ\Delta$, this shows that γ is a $\mathbf{k}$-coalgebra homomorphism. Similar reasoning can be used to prove that γ is a $\mathbf{k}$-algebra homomorphism. Altogether, we thus conclude that γ is a bialgebra homomorphism. Since γ is a $\mathbf{k}$-module isomorphism, this yields that γ is an isomorphism of bialgebras. Qed.

[129]If σ is a permutation of a finite set X, then the *cycle type* of σ is defined as the list of the lengths of all cycles of σ (that is, of all orbits of σ acting on X) written in decreasing order. This is clearly a partition of $|X|$. (Some other authors write it in increasing order instead, or treat it as a multiset.)

For instance, the permutation of the set $\{0,3,6,9,12\}$ that sends 0 to 3, 3 to 9, 6 to 6, 9 to 0, and 12 to 12 has cycle type $(3,1,1)$, since the cycles of this permutation have lengths 3, 1 and 1.

It is known that two permutations in $\mathfrak{S}_n$ have the same cycle type if and only if they are conjugate. Thus, for a given partition λ with $|\lambda| = n$, any two permutations in $\mathfrak{S}_n$ having cycle type λ are conjugate and therefore their $\mathfrak{S}_n$-centralizer subgroups have the same size.

Proof of Proposition 2.5.15. For the first expansion, note that (2.2.18) shows

$$\begin{aligned}
\prod_{i,j=1}^{\infty}(1-x_iy_j)^{-1} &= \prod_{j=1}^{\infty}\sum_{n\geq 0} h_n(\mathbf{x})y_j^n \\
&= \sum_{\substack{\text{weak}\\ \text{compositions}\\ (n_1,n_2,\ldots)}} (h_{n_1}(\mathbf{x})h_{n_2}(\mathbf{x})\cdots)(y_1^{n_1}y_2^{n_2}\cdots) \\
&= \sum_{\lambda\in\mathrm{Par}} \sum_{\substack{\text{weak}\\ \text{compositions}\\ (n_1,n_2,\ldots)\\ \text{satisfying}\\ (n_1,n_2,\ldots)\in\mathfrak{S}_{(\infty)}\lambda}} \underbrace{(h_{n_1}(\mathbf{x})h_{n_2}(\mathbf{x})\cdots)}_{\substack{=h_\lambda(\mathbf{x})\\ (\text{since } (n_1,n_2,\ldots)\in\mathfrak{S}_{(\infty)}\lambda)}} \underbrace{(y_1^{n_1}y_2^{n_2}\cdots)}_{=\mathbf{y}^{(n_1,n_2,\ldots)}} \\
&= \sum_{\lambda\in\mathrm{Par}} h_\lambda(\mathbf{x}) \underbrace{\sum_{\substack{\text{weak}\\ \text{compositions}\\ (n_1,n_2,\ldots)\\ \text{satisfying}\\ (n_1,n_2,\ldots)\in\mathfrak{S}_{(\infty)}\lambda}} \mathbf{y}^{(n_1,n_2,\ldots)}}_{=m_\lambda(\mathbf{y})} \\
&= \sum_{\lambda\in\mathrm{Par}} h_\lambda(\mathbf{x})m_\lambda(\mathbf{y}).
\end{aligned}$$

For the second expansion (and for later use in the proof of Theorem 4.9.5) note that

$$\begin{aligned}
\log H(t) = \log\prod_{i=1}^{\infty}(1-x_it)^{-1} &= \sum_{i=1}^{\infty} -\log(1-x_it) \\
&= \sum_{i=1}^{\infty}\sum_{m=1}^{\infty}\frac{(x_it)^m}{m} = \sum_{m=1}^{\infty}\frac{1}{m}p_m(\mathbf{x})t^m,
\end{aligned} \tag{2.5.12}$$

so that taking $\frac{d}{dt}$ then shows that

$$P(t) := \sum_{m\geq 0} p_{m+1}t^m = \frac{H'(t)}{H(t)} = H'(t)E(-t). \tag{2.5.13}$$

A similar calculation shows that

$$\log\prod_{i,j=1}^{\infty}(1-x_iy_j)^{-1} = \sum_{m=1}^{\infty}\frac{1}{m}p_m(\mathbf{x})p_m(\mathbf{y}) \tag{2.5.14}$$

and hence

$$\prod_{i,j=1}^{\infty}(1-x_iy_j)^{-1} = \exp\left(\sum_{m=1}^{\infty}\frac{1}{m}p_m(\mathbf{x})p_m(\mathbf{y})\right) = \prod_{m=1}^{\infty}\exp\left(\frac{1}{m}p_m(\mathbf{x})p_m(\mathbf{y})\right)$$

$$= \prod_{m=1}^{\infty}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{m}p_m(\mathbf{x})p_m(\mathbf{y})\right)^k$$

$$= \sum_{\substack{\text{weak compositions} \\ (k_1,k_2,k_3,\ldots)}} \prod_{m=1}^{\infty}\left(\frac{1}{k_m!}\left(\frac{1}{m}p_m(\mathbf{x})p_m(\mathbf{y})\right)^{k_m}\right)$$

(by the product rule)

$$= \sum_{\substack{\text{weak compositions} \\ (k_1,k_2,k_3,\ldots)}} \prod_{m=1}^{\infty}\frac{(p_m(\mathbf{x})p_m(\mathbf{y}))^{k_m}}{k_m!m^{k_m}}$$

$$= \sum_{\substack{\text{weak compositions} \\ (k_1,k_2,k_3,\ldots)}} \frac{\prod_{m=1}^{\infty}(p_m(\mathbf{x}))^{k_m}\prod_{m=1}^{\infty}(p_m(\mathbf{y}))^{k_m}}{\prod_{m=1}^{\infty}(k_m!m^{k_m})}$$

$$= \sum_{\substack{\text{weak compositions} \\ (k_1,k_2,k_3,\ldots)}} \frac{p_{\left(1^{k_1}2^{k_2}3^{k_3}\cdots\right)}(\mathbf{x})\,p_{\left(1^{k_1}2^{k_2}3^{k_3}\cdots\right)}(\mathbf{y})}{z_{\left(1^{k_1}2^{k_2}3^{k_3}\cdots\right)}} = \sum_{\lambda\in\mathrm{Par}}\frac{p_\lambda(\mathbf{x})p_\lambda(\mathbf{y})}{z_\lambda}$$

due to the fact that every partition can be uniquely written in the form $\left(1^{k_1}2^{k_2}3^{k_3}\cdots\right)$ with $(k_1,k_2,k_3,\ldots)$ a weak composition. $\square$

Corollary 2.5.17. (a) *With respect to the Hall inner product on Λ, one also has dual bases $\{h_\lambda\}$ and $\{m_\lambda\}$.*

(b) *If $\mathbb{Q}$ is a subring of $\mathbf{k}$, then $\{p_\lambda\}$ and $\left\{z_\lambda^{-1}p_\lambda\right\}$ are also dual bases with respect to the Hall inner product on Λ.*

(c) *If $\mathbb{R}$ is a subring of $\mathbf{k}$, then $\left\{\frac{p_\lambda}{\sqrt{z_\lambda}}\right\}$ is an orthonormal basis of Λ with respect to the Hall inner product.*

Proof. Since (2.5.1) and (2.5.11) showed

$$\prod_{i,j=1}^{\infty}(1-x_iy_j)^{-1} = \sum_{\lambda\in\mathrm{Par}}s_\lambda(\mathbf{x})s_\lambda(\mathbf{y}) = \sum_{\lambda\in\mathrm{Par}}h_\lambda(\mathbf{x})m_\lambda(\mathbf{y})$$
$$= \sum_{\lambda\in\mathrm{Par}}p_\lambda(\mathbf{x})z_\lambda^{-1}p_\lambda(\mathbf{y}) = \sum_{\lambda\in\mathrm{Par}}\frac{p_\lambda(\mathbf{x})}{\sqrt{z_\lambda}}\frac{p_\lambda(\mathbf{y})}{\sqrt{z_\lambda}},$$

it suffices to show that any pair of graded bases[130] $\{u_\lambda\},\{v_\lambda\}$ of Λ having

$$\sum_{\lambda\in\mathrm{Par}}s_\lambda(\mathbf{x})s_\lambda(\mathbf{y}) = \sum_{\lambda\in\mathrm{Par}}u_\lambda(\mathbf{x})v_\lambda(\mathbf{y})$$

[130] See Definition 1.3.21 for the concept of a "graded basis", and recall our convention that a graded basis of Λ is tacitly assumed to have its indexing set Par partitioned into $\mathrm{Par}_0, \mathrm{Par}_1, \mathrm{Par}_2, \ldots$. Thus, a graded basis of Λ means a basis $\{w_\lambda\}_{\lambda\in\mathrm{Par}}$ of the $\mathbf{k}$-module Λ (indexed by the partitions $\lambda\in\mathrm{Par}$) with the property that, for every $n\in\mathbb{N}$, the subfamily $\{w_\lambda\}_{\lambda\in\mathrm{Par}_n}$ is a basis of the $\mathbf{k}$-module Λ_n.

will be dual with respect to $(\cdot, \cdot)$. To show this, consider such a pair of graded bases. Write transition matrices $A = (a_{\nu,\lambda})_{(\nu,\lambda)\in\mathrm{Par}\times\mathrm{Par}}$ and $B = (b_{\nu,\lambda})_{(\nu,\lambda)\in\mathrm{Par}\times\mathrm{Par}}$ uniquely expressing

$$u_\lambda = \sum_\nu a_{\nu,\lambda} s_\nu, \tag{2.5.15}$$

$$v_\lambda = \sum_\nu b_{\nu,\lambda} s_\nu. \tag{2.5.16}$$

Recall that $\mathrm{Par} = \bigsqcup_{r\in\mathbb{N}} \mathrm{Par}_r$. Hence, we can view A as a block matrix, where the blocks are indexed by pairs of nonnegative integers, and the (r,s)-th block is $(a_{\nu,\lambda})_{(\nu,\lambda)\in\mathrm{Par}_r\times\mathrm{Par}_s}$. For reasons of homogeneity[131], we have $a_{\nu,\lambda} = 0$ for any $(\nu,\lambda) \in \mathrm{Par}^2$ satisfying $|\nu| \neq |\lambda|$. Therefore, the (r,s)-th block of A is zero whenever $r \neq s$. In other words, the block matrix A is block-diagonal. Similarly, B can be viewed as a block-diagonal matrix. The diagonal blocks of A and B are finite square matrices (since Par_r is a finite set for each $r \in \mathbb{N}$); therefore, products such as A^tB, B^tA and AB^t are well-defined (since all sums involved in their definition have only finitely many nonzero addends) and subject to the law of associativity. Moreover, the matrix A is invertible (being a transition matrix between two bases), and its inverse is again block-diagonal (because A is block-diagonal).

The equalities (2.5.15) and (2.5.16) show that $(u_\alpha, v_\beta) = \sum_\nu a_{\nu,\alpha} b_{\nu,\beta}$ (by the orthonormality of the s_λ). Hence, we want to prove that $\sum_\nu a_{\nu,\alpha} b_{\nu,\beta} = \delta_{\alpha,\beta}$. In other words, we want to prove that $A^tB = I$, that is, $B^{-1} = A^t$. On the other hand, one has

$$\sum_\lambda s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y}) = \sum_\lambda u_\lambda(\mathbf{x}) v_\lambda(\mathbf{y}) = \sum_\lambda \sum_\nu a_{\nu,\lambda} s_\nu(\mathbf{x}) \sum_\rho b_{\rho,\lambda} s_\rho(\mathbf{y}).$$

Comparing coefficients[132] of $s_\nu(\mathbf{x}) s_\rho(\mathbf{y})$ forces $\sum_\lambda a_{\nu,\lambda} b_{\rho,\lambda} = \delta_{\nu,\rho}$, or in other words, $AB^t = I$. Since A is invertible, this yields $B^tA = I$, and hence $A^tB = I$, as desired.[133] $\square$

[131] More precisely: The power series u_λ is homogeneous of degree $|\lambda|$, and the power series s_ν is homogeneous of degree $|\nu|$.

[132] Comparing coefficients is legitimate because if a power series $f \in \mathbf{k}[[\mathbf{x}, \mathbf{y}]]$ is written in the form $f = \sum_{(\nu,\rho)\in\mathrm{Par}^2} a_{\rho,\nu} s_\nu(\mathbf{x}) s_\rho(\mathbf{y})$ for some coefficients $a_{\rho,\nu} \in \mathbf{k}$, then these coefficients $a_{\rho,\nu}$ are uniquely determined by f. This is just a restatement of Exercise 2.5.5(b).

[133] In our argument above, we have obtained the invertibility of A from the fact that A is a transition matrix between two bases. Here is an alternative way to prove that A is invertible:

Recall that A and B^t are block-diagonal matrices. Hence, the equality $AB^t = I$ rewrites as $A_{r,r}(B^t)_{r,r} = I$ for all $r \in \mathbb{N}$, where we are using the notation $C_{r,s}$ for the (r,s)-th block of a block matrix C. But this shows that each diagonal block $A_{r,r}$ of A is right-invertible. Therefore, each diagonal block $A_{r,r}$ of A is invertible (because $A_{r,r}$ is a square matrix of finite size, and such matrices are always invertible when they are right-invertible). Consequently, the block-diagonal matrix A is invertible, and its inverse is again a block-diagonal matrix (whose diagonal blocks are the inverses of the $A_{r,r}$).

Corollary 2.5.17 is a known and fundamental fact[134]. However, our definition of the Hall inner product is unusual; most authors (e.g., Macdonald in [142, §I.4, (4.5)], Hazewinkel/Gubareni/Kirichenko in [93, Def. 4.1.21], and Stanley in [206, (7.30)]) *define* the Hall inner product as the bilinear form satisfying $(h_\lambda, m_\mu) = \delta_{\lambda,\mu}$ (or, alternatively, $(m_\lambda, h_\mu) = \delta_{\lambda,\mu}$), and only later prove that the basis $\{s_\lambda\}$ is orthonormal with respect to this scalar product. (Of course, the fact that this definition is equivalent to our Definition 2.5.12 follows either from this orthonormality, or from our Corollary 2.5.17(a).)

The tactic applied in the proof of Corollary 2.5.17 can not only be used to show that certain bases of Λ are dual, but also, with a little help from linear algebra over rings (Exercise 2.5.18), it can be strengthened to show that certain families of symmetric functions are bases to begin with, as we will see in Exercise 2.5.19 and Exercise 2.5.20.

Exercise 2.5.18. (a) Prove that if an endomorphism of a finitely generated $\mathbf{k}$-module is surjective, then this endomorphism is a $\mathbf{k}$-module isomorphism.

(b) Let A be a finite free $\mathbf{k}$-module with finite basis $(\gamma_i)_{i\in I}$. Let $(\beta_i)_{i\in I}$ be a family of elements of A which spans the $\mathbf{k}$-module A. Prove that $(\beta_i)_{i\in I}$ is a $\mathbf{k}$-basis of A.

Exercise 2.5.19. For each partition λ, let v_λ be an element of $\Lambda_{|\lambda|}$. Assume that the family $(v_\lambda)_{\lambda\in\mathrm{Par}}$ spans the $\mathbf{k}$-module Λ. Prove that the family $(v_\lambda)_{\lambda\in\mathrm{Par}}$ is a graded basis of the graded $\mathbf{k}$-module Λ.

Exercise 2.5.20. (a) Assume that for every partition λ, two homogeneous elements u_λ and v_λ of Λ, both having degree $|\lambda|$, are given. Assume further that

$$\sum_{\lambda\in\mathrm{Par}} s_\lambda(\mathbf{x})\, s_\lambda(\mathbf{y}) = \sum_{\lambda\in\mathrm{Par}} u_\lambda(\mathbf{x})\, v_\lambda(\mathbf{y})$$

in $\mathbf{k}[[\mathbf{x},\mathbf{y}]] = \mathbf{k}[[x_1,x_2,x_3,\ldots,y_1,y_2,y_3,\ldots]]$. Show that $(u_\lambda)_{\lambda\in\mathrm{Par}}$ and $(v_\lambda)_{\lambda\in\mathrm{Par}}$ are $\mathbf{k}$-bases of Λ, and actually are dual bases with respect to the Hall inner product on Λ.

(b) Use this to give a new proof of the fact that $(h_\lambda)_{\lambda\in\mathrm{Par}}$ is a $\mathbf{k}$-basis of Λ.

Exercise 2.5.21. Prove that $\sum_{m\geq 0} p_{m+1} t^m = \dfrac{H'(t)}{H(t)}$. (This was proven in (2.5.13) in the case when $\mathbb{Q}$ is a subring of $\mathbf{k}$, but here we make no requirements on $\mathbf{k}$.)

The following exercises give some useful criteria for algebraic independence of families of symmetric functions:

Exercise 2.5.22. Let $v_1, v_2, v_3, \ldots$ be elements of Λ. Assume that $v_n \in \Lambda_n$ for each positive integer n. Assume further that $v_1, v_2, v_3, \ldots$ generate the $\mathbf{k}$-algebra Λ. Then:

(a) Prove that $v_1, v_2, v_3, \ldots$ are algebraically independent over $\mathbf{k}$.

[134]For example, Corollary 2.5.17(a) appears in [126, Corollary 3.3] (though the definition of Schur functions in [126] is different from ours; we will meet this alternative definition later on), and parts (b) and (c) of Corollary 2.5.17 are equivalent to [142, §I.4, (4.7)] (though Macdonald defines the Hall inner product using Corollary 2.5.17(a)).

(b) For every partition λ, define an element $v_\lambda \in \Lambda$ by $v_\lambda = v_{\lambda_1} v_{\lambda_2} \cdots v_{\lambda_{\ell(\lambda)}}$. Prove that the family $(v_\lambda)_{\lambda \in \mathrm{Par}}$ is a graded basis of the graded $\mathbf{k}$-module Λ.

Exercise 2.5.23. For each partition λ, let $a_\lambda \in \mathbf{k}$. Assume that the element $a_{(n)} \in \mathbf{k}$ is invertible for each positive integer n. Let $v_1, v_2, v_3, \ldots$ be elements of Λ such that each positive integer n satisfies $v_n = \sum_{\lambda \in \mathrm{Par}_n} a_\lambda h_\lambda$. Prove that the elements $v_1, v_2, v_3, \ldots$ generate the $\mathbf{k}$-algebra Λ and are algebraically independent over $\mathbf{k}$.

Exercise 2.5.24. Let $v_1, v_2, v_3, \ldots$ be elements of Λ. Assume that $v_n \in \Lambda_n$ for each positive integer n. Assume further that $(p_n, v_n) \in \mathbf{k}$ is invertible for each positive integer n. Prove that the elements $v_1, v_2, v_3, \ldots$ generate the $\mathbf{k}$-algebra Λ and are algebraically independent over $\mathbf{k}$.

Exercise 2.5.25. Let $f \in \Lambda$, and let β be a weak composition. Let $\mu \in \mathrm{Par}$ be the partition consisting of the nonzero entries of β (sorted in decreasing order).[135] Prove that

$$(f, h_\mu) = (h_\mu, f) = \left(\text{the coefficient of } \mathbf{x}^\beta \text{ in } f\right).$$

Exercise 2.5.26. Assume that $\mathbb{Q}$ is a subring of $\mathbf{k}$. Define a positive integer z_λ for each $\lambda \in \mathrm{Par}$ as in Proposition 2.5.15. Prove that every $n \in \mathbb{N}$ satisfies the two equalities

$$h_n = \sum_{\lambda \in \mathrm{Par}_n} z_\lambda^{-1} p_\lambda \tag{2.5.17}$$

and

$$e_n = \sum_{\lambda \in \mathrm{Par}_n} (-1)^{|\lambda| - \ell(\lambda)} z_\lambda^{-1} p_\lambda. \tag{2.5.18}$$

2.6. **Bialternants, Littlewood-Richardson: Stembridge's concise proof.** There is a more natural way in which Schur functions arise as a $\mathbf{k}$-basis for Λ, coming from consideration of polynomials in a finite variable set, and the relation between those which are symmetric and those which are *alternating*.

For the remainder of this section, fix a nonnegative integer n, and let $\mathbf{x} = (x_1, \ldots, x_n)$ be a finite variable set. This means that $s_{\lambda/\mu} = s_{\lambda/\mu}(\mathbf{x}) = \sum_T \mathbf{x}^{\mathrm{cont}(T)}$ is a generating function for column-strict tableaux T as in Definition 2.3.1, but with the extra condition that T have entries in $\{1, 2, \ldots, n\}$. [136] As a consequence, $s_{\lambda/\mu}$ is a polynomial in $\mathbf{k}[x_1, x_2, \ldots, x_n]$ (not just a power series), since there are only finitely many column-strict tableaux T of shape λ/μ having all their entries in $\{1, 2, \ldots, n\}$. We will assume without further mention that all partitions appearing in the section have at most n parts.

Definition 2.6.1. Let $\mathbf{k}$ be the ring $\mathbb{Z}$ or a field of characteristic not equal to 2. (We require this to avoid certain annoyances in the discussion of alternating polynomials in characteristic 2.)

Say that a polynomial $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ is *alternating* if for every permutation w in $\mathfrak{S}_n$ one has that

$$(wf)(\mathbf{x}) = f(x_{w(1)}, \ldots, x_{w(n)}) = \mathrm{sgn}(w) f(\mathbf{x}).$$

[135] For example, if $\beta = (1, 0, 3, 1, 2, 3, 0, 0, 0, \ldots)$, then $\mu = (3, 3, 2, 1, 1)$.

[136] See Exercise 2.3.8(a) for this.

Let $\Lambda^{\mathrm{sgn}} \subset \mathbf{k}[x_1, \ldots, x_n]$ denote the subset of alternating polynomials[137].

As with Λ and its monomial basis $\{m_\lambda\}$, there is an obvious $\mathbf{k}$-basis for Λ^{sgn}, coming from the fact that a polynomial $f = \sum_\alpha c_\alpha \mathbf{x}^\alpha$ is alternating if and only if $c_{w(\alpha)} = \mathrm{sgn}(w) c_\alpha$ for every w in $\mathfrak{S}_n$ and every $\alpha \in \mathbb{N}^n$. This means that every alternating f is a $\mathbf{k}$-linear combination of the following elements.

Definition 2.6.2. For $\alpha = (\alpha_1, \ldots, \alpha_n)$ in $\mathbb{N}^n$, define the *alternant*

$$a_\alpha := \sum_{w \in \mathfrak{S}_n} \mathrm{sgn}(w) w(\mathbf{x}^\alpha) = \det \begin{bmatrix} x_1^{\alpha_1} & \cdots & x_1^{\alpha_n} \\ x_2^{\alpha_1} & \cdots & x_2^{\alpha_n} \\ \vdots & \ddots & \vdots \\ x_n^{\alpha_1} & \cdots & x_n^{\alpha_n} \end{bmatrix}.$$

Example 2.6.3. One has

$a_{(1,5,0)} = x_1^1 x_2^5 x_3^0 - x_1^5 x_2^1 x_3^0 - x_1^1 x_2^0 x_3^5 - x_1^0 x_2^5 x_3^1 + x_1^0 x_2^1 x_3^5 + x_1^5 x_2^0 x_3^1 = -a_{(5,1,0)}$.

Similarly, $a_{w(\alpha)} = \mathrm{sgn}(w) a_\alpha$ for every $w \in \mathfrak{S}_n$ and every $\alpha \in \mathbb{N}^n$.

Meanwhile, $a_{(5,2,2)} = 0$ since the transposition $t = \binom{123}{132}$ fixes $(5, 2, 2)$ and hence

$$a_{(5,2,2)} = t(a_{(5,2,2)}) = \mathrm{sgn}(t) a_{(5,2,2)} = -a_{(5,2,2)}.$$

[138] Alternatively, $a_{(5,2,2)} = 0$ as it is a determinant of a matrix with two equal columns. Similarly, $a_\alpha = 0$ for every n-tuple $\alpha \in \mathbb{N}^n$ having two equal entries.

This example illustrates that, for a $\mathbf{k}$-basis for Λ^{sgn}, one can restrict attention to alternants a_α in which α is a *strict partition*, i.e., in which α satisfies $\alpha_1 > \alpha_2 > \cdots > \alpha_n$. One can therefore uniquely express $\alpha = \lambda + \rho$, where λ is a (weak) partition $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ and where $\rho := (n-1, n-2, \ldots, 2, 1, 0)$ is sometimes called the *staircase partition*[139]. For example $\alpha = (5, 1, 0) = (3, 0, 0) + (2, 1, 0) = \lambda + \rho$.

Proposition 2.6.4. *Let $\mathbf{k}$ be the ring $\mathbb{Z}$ or a field of characteristic not equal to 2.*

The alternants $\{a_{\lambda+\rho}\}$ as λ runs through the partitions with at most n parts form a $\mathbf{k}$-basis for Λ^{sgn}. In addition, the bialternants $\{\frac{a_{\lambda+\rho}}{a_\rho}\}$ *as λ runs through the same set form a $\mathbf{k}$-basis for $\Lambda(x_1, \ldots, x_n) = \mathbf{k}[x_1, \ldots, x_n]^{\mathfrak{S}_n}$.*

[137] When $\mathbf{k}$ has characteristic 2 (or, more generally, is an arbitrary commutative ring), it is probably best to define the alternating polynomials $\Lambda_{\mathbf{k}}^{\mathrm{sgn}}$ as the $\mathbf{k}$-submodule $\Lambda^{\mathrm{sgn}} \otimes_{\mathbb{Z}} \mathbf{k}$ of $\mathbb{Z}[x_1, \ldots, x_n] \otimes_{\mathbb{Z}} \mathbf{k} \cong \mathbf{k}[x_1, \ldots, x_n]$.

[138] One subtlety should be addressed: We want to prove that $a_{(5,2,2)} = 0$ in $\mathbf{k}[x_1, \ldots, x_n]$ for every commutative ring $\mathbf{k}$. It is clearly enough to prove that $a_{(5,2,2)} = 0$ in $\mathbb{Z}[x_1, \ldots, x_n]$. Since 2 is not a zero-divisor in $\mathbb{Z}[x_1, \ldots, x_n]$, we can achieve this by showing that $a_{(5,2,2)} = -a_{(5,2,2)}$. We would not be able to make this argument directly over an arbitrary commutative ring $\mathbf{k}$.

[139] The name is owed to its Ferrers shape. For instance, if $n = 5$, then the Ferrers diagram of ρ (represented using dots) has the form

$$\begin{array}{llll} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & & \\ \bullet & & & \end{array}.$$

Proof. The first assertion should be clear from our previous discussion: the alternants $\{a_{\lambda+\rho}\}$ span Λ^{sgn} by definition, and they are $\mathbf{k}$-linearly independent because they are supported on disjoint sets of monomials $\mathbf{x}^\alpha$.

The second assertion follows from the first, after proving the following **Claim**: $f(\mathbf{x})$ lies in Λ^{sgn} if and only if $f(\mathbf{x}) = a_\rho \cdot g(\mathbf{x})$ where $g(\mathbf{x})$ lies in $\mathbf{k}[\mathbf{x}]^{\mathfrak{S}_n}$ and where

$$a_\rho = \det(x_i^{n-j})_{i,j=1,2,\ldots,n} = \prod_{1 \le i < j \le n} (x_i - x_j)$$

is the *Vandermonde determinant/product*. In other words,

$$\Lambda^{\mathrm{sgn}} = a_\rho \cdot \mathbf{k}[\mathbf{x}]^{\mathfrak{S}_n}$$

is a free $\mathbf{k}[\mathbf{x}]^{\mathfrak{S}_n}$-module of rank 1, with a_ρ as its $\mathbf{k}[\mathbf{x}]^{\mathfrak{S}_n}$-basis element.

To see the Claim, first note the inclusion

$$\Lambda^{\mathrm{sgn}} \supset a_\rho \cdot \mathbf{k}[\mathbf{x}]^{\mathfrak{S}_n}$$

since the product of a symmetric polynomial and an alternating polynomial is an alternating polynomial. For the reverse inclusion, note that since an alternating polynomial $f(\mathbf{x})$ changes sign whenever one exchanges two distinct variables x_i, x_j, it must vanish upon setting $x_i = x_j$, and therefore be divisible by $x_i - x_j$, so divisible by the entire product $\prod_{1 \le i < j \le n}(x_i - x_j) = a_\rho$. But then the quotient $g(\mathbf{x}) = \frac{f(\mathbf{x})}{a_\rho}$ is symmetric, as it is a quotient of two alternating polynomials. $\square$

Let us now return to the general setting, where $\mathbf{k}$ is an arbitrary commutative ring. We are not requiring that the assumptions of Proposition 2.6.4 be valid; we can still study the a_α of Definition 2.6.2, but we cannot use Proposition 2.6.4 anymore. We will show that the fraction $\frac{a_{\lambda+\rho}}{a_\rho}$ is nevertheless a well-defined polynomial in $\Lambda(x_1, \ldots, x_n)$ whenever λ is a partition[140], and in fact equals the Schur function $s_\lambda(\mathbf{x})$. As a consequence, the mysterious bialternant basis $\{\frac{a_{\lambda+\rho}}{a_\rho}\}$ of $\Lambda(x_1, \ldots, x_n)$ defined in Proposition 2.6.4 still exists in the general setting, and is plainly the Schur functions $\{s_\lambda(\mathbf{x})\}$. Stembridge [210] noted that one could give a remarkably concise proof of an even stronger assertion, which simultaneously gives one of the standard combinatorial interpretations for the Littlewood-Richardson coefficients $c^\lambda_{\mu,\nu}$. For the purposes of stating it, we introduce for a tableau T the notation $T|_{\mathrm{cols} \ge j}$ (resp. $T|_{\mathrm{cols} \le j}$) to indicate the subtableau which is the restriction of T to the union of its columns $j, j+1, j+2, \ldots$ (resp. columns $1, 2, \ldots, j$).

Example 2.6.5. If $T = \begin{array}{cccc} & & 1 & 2 \\ & 2 & 2 & 3 \\ 3 & 5 & & \end{array}$, then

$$T|_{\mathrm{cols} \ge 3} = \begin{array}{cc} 1 & 2 \\ 2 & 3 \end{array} \qquad \text{and} \qquad T|_{\mathrm{cols} \le 2} = \begin{array}{cc} & \\ & 2 \\ 3 & 5 \end{array}$$

(note that $T|_{\mathrm{cols} \le 2}$ has an empty first row).

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[140] This can also be deduced by base change from the $\mathbf{k} = \mathbb{Z}$ case of Proposition 2.6.4.

Theorem 2.6.6. *For partitions λ, μ, ν with $\mu \subseteq \lambda$, one has*[141]

$$a_{\nu+\rho} s_{\lambda/\mu} = \sum_T a_{\nu+\mathrm{cont}(T)+\rho}$$

where T runs through all column-strict tableaux with entries in $\{1, 2, \ldots, n\}$ of shape λ/μ with the property that for each $j = 1, 2, 3, \ldots$, the weak composition $\nu + \mathrm{cont}(T|_{\mathrm{cols} \geq j})$ is a partition.

Before proving Theorem 2.6.6, let us see some of its consequences.

Corollary 2.6.7. *For any partition λ, we have*[142]

$$s_\lambda(\mathbf{x}) = \frac{a_{\lambda+\rho}}{a_\rho}.$$

Proof. Fix a partition λ. Take $\nu = \mu = \varnothing$ in Theorem 2.6.6. Note that there is only one column-strict tableau T of shape λ such that each $\mathrm{cont}(T|_{\mathrm{cols} \geq j})$ is a partition, namely the tableau having every entry in row i equal to i:

$$\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & & \\ 3 & 3 & 3 & & \\ 4 & 4 & & & \end{array}$$

[143]. Furthermore, this T has $\mathrm{cont}(T) = \lambda$, so the theorem says $a_\rho s_\lambda = a_{\lambda+\rho}$. ☐

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[141]Again, we can drop the requirement that $\mu \subseteq \lambda$, provided that we understand that there are no column-strict tableaux of shape λ/μ unless $\mu \subseteq \lambda$.

[142]Notice that division by a_ρ is unambiguous in the ring $\mathbf{k}[x_1, \ldots, x_n]$, since a_ρ is not a zero-divisor (in fact, $a_\rho = \prod_{1 \leq i < j \leq n} (x_i - x_j)$ is the product of the binomials $x_i - x_j$, none of which is a zero-divisor).

[143]*Proof.* It is clear that the tableau having every entry in row i equal to i indeed satisfies the condition that each $\mathrm{cont}(T|_{\mathrm{cols} \geq j})$ is a partition. It remains to show that it is the only column-strict tableau (of shape λ) satisfying this condition.

Let T be a column-strict tableau of shape λ satisfying the condition that each $\mathrm{cont}(T|_{\mathrm{cols} \geq j})$ is a partition. We must show that for each i, every entry in row i of T is equal to i. Assume the contrary. Thus, there exists some i such that row i of T contains an entry distinct from i. Consider the smallest such i. Hence, rows $1, 2, \ldots, i-1$ of T are filled with entries $1, 2, \ldots, i-1$, whereas row i has some entry distinct from i. Choose some j such that the j-th entry of row i of T is distinct from i. This entry cannot be smaller than i (since it has $i-1$ entries above it in its column, and the entries of T increase strictly down columns); thus, it has to be larger than i. Therefore, all entries in rows $i, i+1, i+2, \ldots$ of $T|_{\mathrm{cols} \geq j}$ are larger than i as well (since they lie southeast of this entry). Hence, each entry of $T|_{\mathrm{cols} \geq j}$ is either smaller than i (if it is in one of rows $1, 2, \ldots, i-1$) or larger than i (if it is in row i or further down). Thus, i is not an entry of $T|_{\mathrm{cols} \geq j}$. In other words, $\mathrm{cont}_i(T|_{\mathrm{cols} \geq j}) = 0$. Since $\mathrm{cont}(T|_{\mathrm{cols} \geq j})$ is a partition, we thus conclude that $\mathrm{cont}_k(T|_{\mathrm{cols} \geq j}) = 0$ for all $k > i$. In other words, $T|_{\mathrm{cols} \geq j}$ has no entries larger than i. But this contradicts the fact that the j-th entry of row i of T is larger than i. This contradiction completes our proof.

Example 2.6.8. For $n = 2$, so that $\rho = (1, 0)$, if we take $\lambda = (4, 2)$, then one has

$$\begin{aligned}
\frac{a_{\lambda+\rho}}{a_\rho} &= \frac{a_{(4,2)+(1,0)}}{a_{(1,0)}} = \frac{a_{(5,2)}}{a_{(1,0)}} \\
&= \frac{x_1^5 x_2^2 - x_1^2 x_2^5}{x_1 - x_2} \\
&= x_1^4 x_2^2 + x_1^3 x_2^3 + x_1^2 x_2^4 \\
&= \mathbf{x}^{\operatorname{cont}\begin{pmatrix} 1111 \\ 22 \end{pmatrix}} + \mathbf{x}^{\operatorname{cont}\begin{pmatrix} 1112 \\ 22 \end{pmatrix}} + \mathbf{x}^{\operatorname{cont}\begin{pmatrix} 1122 \\ 22 \end{pmatrix}} \\
&= s_{(4,2)} = s_\lambda.
\end{aligned}$$

Some authors use the equality in Corollary 2.6.7 to *define* the Schur polynomial $s_\lambda (x_1, x_2, \ldots, x_n)$ in n variables; this definition, however, has the drawback of not generalizing easily to infinitely many variables or to skew Schur functions[144].

Next divide through by a_ρ on both sides of Theorem 2.6.6 (and use Corollary 2.6.7) to give the following.

Corollary 2.6.9. *For partitions λ, μ, ν having at most n parts, one has*

$$s_\nu s_{\lambda/\mu} = \sum_T s_{\nu+\operatorname{cont}(T)} \tag{2.6.1}$$

where T runs through the same set as in Theorem 2.6.6. In particular, taking $\nu = \varnothing$, we obtain

$$s_{\lambda/\mu} = \sum_T s_{\operatorname{cont}(T)} \tag{2.6.2}$$

where in the sum T runs through all column-strict tableaux of shape λ/μ for which each $\operatorname{cont}(T|_{\text{cols}\geq j})$ is a partition.

Proof of Theorem 2.6.6. Start by rewriting the left side of the theorem:

$$\begin{aligned}
a_{\nu+\rho} s_{\lambda/\mu} &= \sum_{w\in\mathfrak{S}_n} \operatorname{sgn}(w) \mathbf{x}^{w(\nu+\rho)} s_{\lambda/\mu} = \sum_{w\in\mathfrak{S}_n} \operatorname{sgn}(w) \mathbf{x}^{w(\nu+\rho)} w(s_{\lambda/\mu}) \\
&\qquad \big(\text{since } w(s_{\lambda/\mu}) = s_{\lambda/\mu} \text{ for any } w \in \mathfrak{S}_n\big) \\
&= \sum_{w\in\mathfrak{S}_n} \operatorname{sgn}(w) \mathbf{x}^{w(\nu+\rho)} \sum_{\substack{\text{column-strict } T \\ \text{of shape } \lambda/\mu}} \mathbf{x}^{w(\operatorname{cont}(T))} \\
&= \sum_{\substack{\text{column-strict } T \\ \text{of shape } \lambda/\mu}} \sum_{w\in\mathfrak{S}_n} \operatorname{sgn}(w) \mathbf{x}^{w(\nu+\operatorname{cont}(T)+\rho)} \\
&= \sum_{\substack{\text{column-strict } T \\ \text{of shape } \lambda/\mu}} a_{\nu+\operatorname{cont}(T)+\rho}.
\end{aligned}$$

[144] With some effort, it is possible to use Corollary 2.6.7 in order to define the Schur function s_λ in infinitely many variables. Indeed, one can define this Schur function as the unique element of Λ whose evaluation at $(x_1, x_2, \ldots, x_n)$ equals $\frac{a_{\lambda+\rho}}{a_\rho}$ for every $n \in \mathbb{N}$. If one wants to use such a definition, however, one needs to check that such an element exists. This is the approach to defining s_λ taken in [126, Definition 1.4.2] and in [142, §I.3].

We wish to cancel out all the summands indexed by column-strict tableaux T which fail any of the conditions that $\nu + \mathrm{cont}(T|_{\mathrm{cols}\geq j})$ be a partition. Given such a T, find the maximal j for which it fails this condition[145], and then find the minimal k for which

$$\nu_k + \mathrm{cont}_k(T|_{\mathrm{cols}\geq j}) < \nu_{k+1} + \mathrm{cont}_{k+1}(T|_{\mathrm{cols}\geq j}).$$

Maximality of j forces

$$\nu_k + \mathrm{cont}_k(T|_{\mathrm{cols}\geq j+1}) \geq \nu_{k+1} + \mathrm{cont}_{k+1}(T|_{\mathrm{cols}\geq j+1}).$$

Since column-strictness implies that column j of T can contain at most one occurrence of k or of $k+1$ (or neither or both), the previous two inequalities imply that column j must contain an occurrence of $k+1$ and no occurrence of k, so that

$$\nu_k + \mathrm{cont}_k(T|_{\mathrm{cols}\geq j}) + 1 = \nu_{k+1} + \mathrm{cont}_{k+1}(T|_{\mathrm{cols}\geq j}).$$

This implies that the adjacent transposition $t_{k,k+1}$ swapping k and $k+1$ fixes the vector $\nu + \mathrm{cont}(T|_{\mathrm{cols}\geq j}) + \rho$.

Now create a new tableau T^* from T by applying the Bender-Knuth involution (from the proof of Proposition 2.2.4) on letters $k, k+1$, but *only to columns* $1, 2, \ldots, j-1$ of T, leaving columns $j, j+1, j+2, \ldots$ unchanged.[146] One should check that T^* is still column-strict, but this holds because column j of T has no occurrences of letter k. Note that

$$t_{k,k+1}\,\mathrm{cont}(T|_{\mathrm{cols}\leq j-1}) = \mathrm{cont}(T^*|_{\mathrm{cols}\leq j-1})$$

and hence

$$t_{k,k+1}(\nu + \mathrm{cont}(T) + \rho) = \nu + \mathrm{cont}(T^*) + \rho,$$

so that $a_{\nu+\mathrm{cont}(T)+\rho} = -a_{\nu+\mathrm{cont}(T^*)+\rho}$.

Because T and T^* have exactly the same columns $j, j+1, j+2, \ldots$, the tableau T^* is also a violator of at least one of the conditions that $\nu + \mathrm{cont}(T^*|_{\mathrm{cols}\geq j})$ be a partition, and has the same choice of maximal j and minimal k as did T. Hence the map $T \mapsto T^*$ is an involution on the violators that lets one cancel their summands $a_{\nu+\mathrm{cont}(T)+\rho}$ and $a_{\nu+\mathrm{cont}(T^*)+\rho}$ in pairs.[147] □

Example 2.6.10. Here is an example of the construction of T^* in the above proof. Let $n = 6$ and $\lambda = (5,4,4)$ and $\mu = (2,2)$ and $\nu = (1)$. Let T be the column-strict tableau

$$\begin{array}{ccccc} & & 1 & 2 & 2 \\ & & 2 & 3 & \\ 2 & 2 & 3 & 4 & \end{array} \qquad \text{of shape } \lambda/\mu.$$

[145]Such a j exists because $\nu + \mathrm{cont}(T|_{\mathrm{cols}\geq j})$ is a partition for all sufficiently high j (in fact, ν itself is a partition).

[146]See Example 2.6.10 below for an example of this construction.

[147]One remark is in order: The tableaux T and T^* may be equal. In this case, the summands $a_{\nu+\mathrm{cont}(T)+\rho}$ and $a_{\nu+\mathrm{cont}(T^*)+\rho}$ do not cancel, as they are the same summand. However, this summand is zero (because $t_{k,k+1}(\nu+\mathrm{cont}(T)+\rho) = \nu + \mathrm{cont}\left(\underbrace{T^*}_{=T}\right) + \rho = \nu + \mathrm{cont}(T) + \rho$ shows that the n-tuple $\nu + \mathrm{cont}(T) + \rho$ has two equal entries, and thus $a_{\nu+\mathrm{cont}(T)+\rho} = 0$), and thus does not affect the sum.

Then,

$$\operatorname{cont}(T|_{\operatorname{cols}\geq 5}) = (0,1,0,0,0,\ldots)$$
$$\text{(since } T|_{\operatorname{cols}\geq 5} \text{ has a single entry, which is 2),}$$
$$\text{so that} \quad \nu + \operatorname{cont}(T|_{\operatorname{cols}\geq 5}) = (1,1,0,0,0,\ldots) \text{ is a partition.}$$

But

$$\operatorname{cont}(T|_{\operatorname{cols}\geq 4}) = (0,2,1,1,0,0,0,\ldots),$$
$$\text{and thus} \quad \nu + \operatorname{cont}(T|_{\operatorname{cols}\geq 4}) = (1,2,1,1,0,0,0,\ldots) \text{ is not a partition.}$$

Thus, the j in the above proof of Theorem 2.6.6 is 4. Furthermore, the k in the proof is 1, since $\nu_1 + \operatorname{cont}_1(T|_{\operatorname{cols}\geq 4}) = 1 + 0 = 1 < 2 = 0 + 2 = \nu_2 + \operatorname{cont}_2(T|_{\operatorname{cols}\geq 4})$. Thus, T^* is obtained from T by applying the Bender-Knuth involution on letters $1, 2$ to columns $1, 2, 3$ only, leaving columns $4, 5$ unchanged. The result is

$$T^* = \begin{array}{ccccc} & & 1 & 2 & 2 \\ & & 2 & 3 & \\ 1 & 1 & 3 & 4 & \end{array}.$$

So far (in this section) we have worked with a finite set of variables $x_1, x_2, \ldots, x_n$ (where n is a fixed nonnegative integer) and with partitions having at most n parts. We now drop these conventions and restrictions; thus, partitions again mean arbitrary partitions, and $\mathbf{x}$ again means the infinite family $(x_1, x_2, x_3, \ldots)$ of variables. In this setting, we have the following analogue of Corollary 2.6.9:

Corollary 2.6.11. *For partitions λ, μ, ν (of any lengths), one has*

$$s_\nu s_{\lambda/\mu} = \sum_T s_{\nu+\operatorname{cont}(T)} \tag{2.6.3}$$

where T runs through all column-strict tableaux of shape λ/μ with the property that for each $j = 1, 2, 3, \ldots$, the weak composition $\nu + \operatorname{cont}(T|_{\operatorname{cols}\geq j})$ is a partition. In particular, taking $\nu = \varnothing$, we obtain

$$s_{\lambda/\mu} = \sum_T s_{\operatorname{cont}(T)} \tag{2.6.4}$$

where in the sum T runs through all column-strict tableaux of shape λ/μ for which each $\operatorname{cont}(T|_{\operatorname{cols}\geq j})$ is a partition.

Proof of Corollary 2.6.11. Essentially, Corollary 2.6.11 is obtained from Corollary 2.6.9 by "letting n (that is, the number of variables) tend to ∞". This can be formalized in different ways: One way is to endow the ring of power series $\mathbf{k}[[\mathbf{x}]] = \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ with the coefficientwise topology[148], and to show that the left hand side of (2.6.1) tends to the left hand

[148]This topology is defined as follows:

We endow the ring $\mathbf{k}$ with the discrete topology. Then, we can regard the $\mathbf{k}$-module $\mathbf{k}[[\mathbf{x}]]$ as a direct product of infinitely many copies of $\mathbf{k}$ (by identifying every power series in $\mathbf{k}[[\mathbf{x}]]$ with the family of its coefficients). Hence, the product topology is a well-defined topology on $\mathbf{k}[[\mathbf{x}]]$; this topology is denoted as the *coefficientwise topology*. Its name is due to the fact that a sequence $(a_n)_{n\in\mathbb{N}}$ of power series converges to a power series a with respect to this topology if and only if for every monomial $\mathfrak{m}$, all sufficiently high $n \in \mathbb{N}$ satisfy

$$(\text{the coefficient of } \mathfrak{m} \text{ in } a_n) = (\text{the coefficient of } \mathfrak{m} \text{ in } a).$$

side of (2.6.3) when $n \to \infty$, and the same holds for the right hand sides. A different approach proceeds by regarding Λ as the inverse limit of the $\Lambda(x_1, x_2, \ldots, x_n)$. $\square$

Comparing coefficients of a given Schur function s_ν in (2.6.4), we obtain the following version of the Littlewood-Richardson rule.

Corollary 2.6.12. *For partitions λ, μ, ν (of any lengths), the Littlewood-Richardson coefficient $c^\lambda_{\mu,\nu}$ counts column-strict tableaux T of shape λ/μ with $\operatorname{cont}(T) = \nu$ having the property that each $\operatorname{cont}(T|_{\mathrm{cols} \geq j})$ is a partition.*

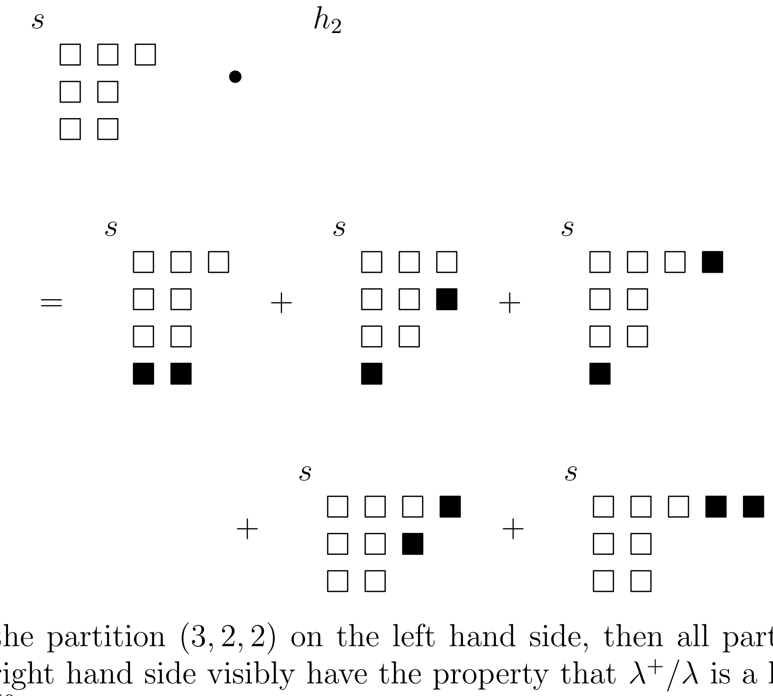
2.7. **The Pieri and Assaf-McNamara skew Pieri rule.** The classical *Pieri rule* refers to two special cases of the Littlewood-Richardson rule. To state them, recall that a skew shape is called a *horizontal (resp. vertical) strip* if no two of its cells lie in the same column (resp. row). A *horizontal (resp. vertical) n-strip* (for $n \in \mathbb{N}$) shall mean a horizontal (resp. vertical) strip of size n (that is, having exactly n cells).

Theorem 2.7.1. *For every partition λ and any $n \in \mathbb{N}$, we have*

$$s_\lambda h_n = \sum_{\substack{\lambda^+ : \lambda^+/\lambda \text{ is a} \\ \text{horizontal } n\text{-strip}}} s_{\lambda^+}; \tag{2.7.1}$$

$$s_\lambda e_n = \sum_{\substack{\lambda^+ : \lambda^+/\lambda \text{ is a} \\ \text{vertical } n\text{-strip}}} s_{\lambda^+}. \tag{2.7.2}$$

Example 2.7.2. In the following equality, we are representing each partition by its Ferrers diagram[149].



If λ is the partition $(3, 2, 2)$ on the left hand side, then all partitions λ^+ on the right hand side visibly have the property that λ^+/λ is a horizontal 2-strip[150], as (2.7.1) predicts.

[149]And we are drawing each Ferrers diagram with its boxes spaced out, in order to facilitate counting the boxes.

[150]We have colored the boxes of λ^+/λ black.

Proof of Theorem 2.7.1. For the first Pieri formula involving h_n, as $h_n = s_{(n)}$ one has

$$s_\lambda h_n = \sum_{\lambda^+} c^{\lambda^+}_{\lambda,(n)} s_{\lambda^+}.$$

Corollary 2.6.12 says $c^{\lambda^+}_{\lambda,(n)}$ counts column-strict tableaux T of shape λ^+/λ having $\operatorname{cont}(T) = (n)$ (i.e. all entries of T are 1's), with an extra condition. Since its entries are all equal, such a T must certainly have shape being a horizontal strip, and more precisely a horizontal n-strip (since it has n cells). Conversely, for any horizontal n-strip, there is a unique such filling, and it will trivially satisfy the extra condition that $\operatorname{cont}(T|_{\text{cols}\geq j})$ is a partition for each j. Hence $c^{\lambda^+}_{\lambda,(n)}$ is 1 if λ^+/λ is a horizontal n-strip, and 0 else.

For the second Pieri formula involving e_n, using $e_n = s_{(1^n)}$ one has

$$s_\lambda e_n = \sum_{\lambda^+} c^{\lambda^+}_{\lambda,(1^n)} s_{\lambda^+}.$$

Corollary 2.6.12 says $c^{\lambda^+}_{\lambda,(1^n)}$ counts column-strict tableaux T of shape λ^+/λ having $\operatorname{cont}(T) = (1^n)$, so its entries are $1, 2, \ldots, n$ each occurring once, with the extra condition that $1, 2, \ldots, n$ appear from right to left. Together with the tableau condition, this forces at most one entry in each row, that is λ^+/λ is a vertical strip, and then there is a unique way to fill it (maintaining column-strictness and the extra condition that $1, 2, \ldots, n$ appear from right to left). Thus $c^{\lambda^+}_{\lambda,(1^n)}$ is 1 if λ^+/λ is a vertical n-strip, and 0 else. $\square$

In 2009, Assaf and McNamara [9] proved an elegant generalization.

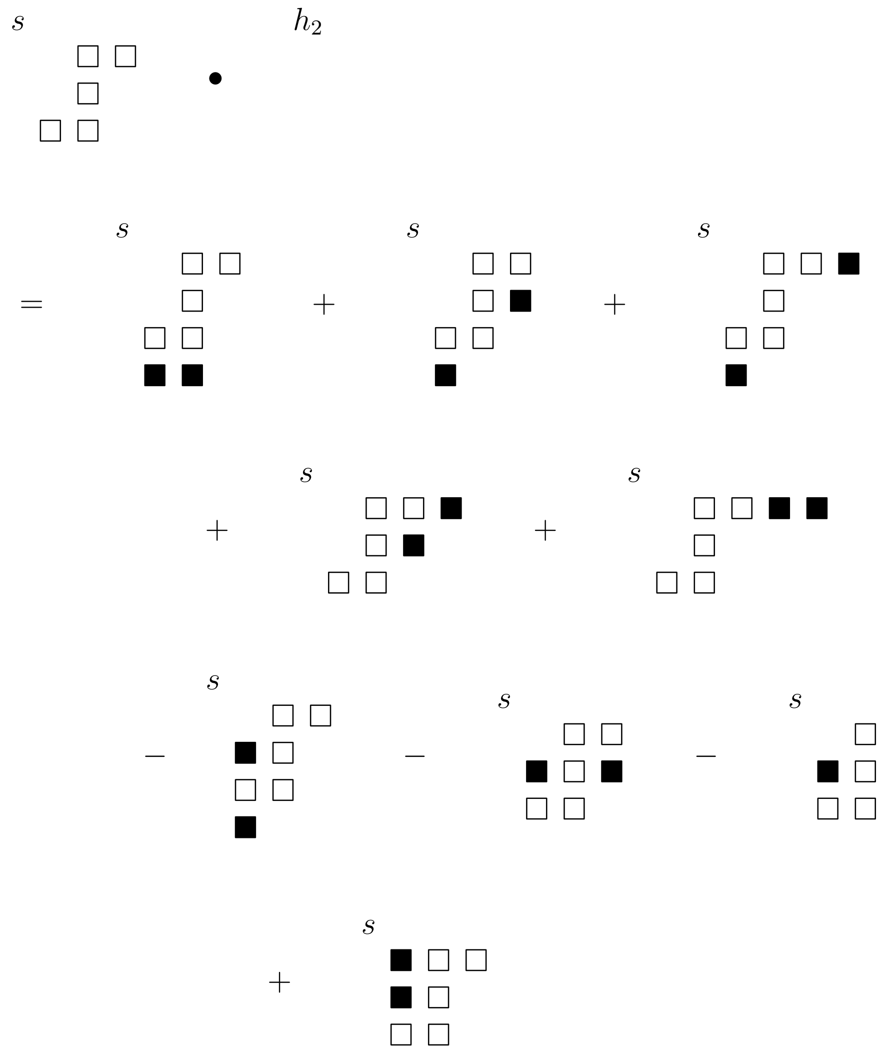
Theorem 2.7.3. *For any partitions λ and μ and any $n \in \mathbb{N}$, we have*[151]

$$s_{\lambda/\mu} h_n = \sum_{\substack{\lambda^+,\mu^-: \\ \lambda^+/\lambda \text{ a horizontal strip;} \\ \mu/\mu^- \text{ a vertical strip;} \\ |\lambda^+/\lambda|+|\mu/\mu^-|=n}} (-1)^{|\mu/\mu^-|} s_{\lambda^+/\mu^-}; \tag{2.7.3}$$

$$s_{\lambda/\mu} e_n = \sum_{\substack{\lambda^+,\mu^-: \\ \lambda^+/\lambda \text{ a vertical strip;} \\ \mu/\mu^- \text{ a horizontal strip;} \\ |\lambda^+/\lambda|+|\mu/\mu^-|=n}} (-1)^{|\mu/\mu^-|} s_{\lambda^+/\mu^-}. \tag{2.7.4}$$

[151]Note that $\mu \subseteq \lambda$ is not required. (The left hand sides are 0 otherwise, but this does not trivialize the equalities.)

Example 2.7.4. With the same conventions as in Example 2.7.2[152], we have



which illustrates the first equality of Theorem 2.7.3.

Theorem 2.7.3 is proven in the next section, using an important Hopf algebra tool.

Exercise 2.7.5. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ and $\mu = (\mu_1, \mu_2, \mu_3, \ldots)$ be two partitions such that $\mu \subseteq \lambda$.

(a) Show that λ/μ is a horizontal strip if and only if every $i \in \{1, 2, 3, \ldots\}$ satisfies $\mu_i \geq \lambda_{i+1}$. [153]

(b) Show that λ/μ is a vertical strip if and only if every $i \in \{1, 2, 3, \ldots\}$ satisfies $\lambda_i \leq \mu_i + 1$.

Exercise 2.7.6. (a) Let λ and μ be two partitions such that $\mu \subseteq \lambda$. Let $n \in \mathbb{N}$. Show that $(h_n, s_{\lambda/\mu})$ equals 1 if λ/μ is a horizontal n-strip, and equals 0 otherwise.

(b) Use part (a) to give a new proof of (2.7.1).

---

[152] but this time coloring both the boxes in λ^+/λ and the boxes in μ/μ^- black

[153] In other words, λ/μ is a horizontal strip if and only if $(\lambda_2, \lambda_3, \lambda_4, \ldots) \subseteq \mu$. This simple observation has been used by Pak and Postnikov [165, §10] for a new approach to RSK-type algorithms.

Exercise 2.7.7. Prove Theorem 2.7.1 again using the ideas of the proof of Theorem 2.5.1.

Exercise 2.7.8. Let A be a commutative ring, and $n \in \mathbb{N}$.

(a) Let $a_1, a_2, \ldots, a_n$ be n elements of A. Let $b_1, b_2, \ldots, b_n$ be n further elements of A. If $a_i - b_j$ is an invertible element of A for every $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, n\}$, then prove that

$$\det\left(\left(\frac{1}{a_i - b_j}\right)_{i,j=1,2,\ldots,n}\right) = \frac{\prod_{1 \le j < i \le n} \left((a_i - a_j)(b_j - b_i)\right)}{\prod_{(i,j) \in \{1,2,\ldots,n\}^2} (a_i - b_j)}.$$

(b) Let $a_1, a_2, \ldots, a_n$ be n elements of A. Let $b_1, b_2, \ldots, b_n$ be n further elements of A. If $1 - a_i b_j$ is an invertible element of A for every $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, n\}$, then prove that

$$\det\left(\left(\frac{1}{1 - a_i b_j}\right)_{i,j=1,2,\ldots,n}\right) = \frac{\prod_{1 \le j < i \le n} \left((a_i - a_j)(b_i - b_j)\right)}{\prod_{(i,j) \in \{1,2,\ldots,n\}^2} (1 - a_i b_j)}.$$

(c) Use the result of part (b) to give a new proof for Theorem 2.5.1.[154]

The determinant on the left hand side of Exercise 2.7.8(a) is known as the *Cauchy determinant*.

Exercise 2.7.9. Prove that $s_{(a,b)} = h_a h_b - h_{a+1} h_{b-1}$ for any two integers $a \ge b \ge 0$ (where we set $h_{-1} = 0$ as usual).

(Note that this is precisely the Jacobi-Trudi formula (2.4.16) in the case when $\lambda = (a, b)$ is a partition with at most two entries and $\mu = \varnothing$.)

Exercise 2.7.10. If λ is a partition and μ is a weak composition, let $K_{\lambda,\mu}$ denote the number of column-strict tableaux T of shape λ having $\operatorname{cont}(T) = \mu$. (This $K_{\lambda,\mu}$ is called the (λ, μ)-*Kostka number*.)

(a) Use Theorem 2.7.1 to show that every partition μ satisfies $h_\mu = \sum_\lambda K_{\lambda,\mu} s_\lambda$, where the sum ranges over all partitions λ.
(b) Use this to give a new proof for Theorem 2.5.1.[155]
(c) Give a new proof of the fact (previously shown as Proposition 2.4.3(j)) that $(h_\lambda)_{\lambda \in \mathrm{Par}}$ is a graded basis of the graded $\mathbf{k}$-module Λ.

Exercise 2.7.11. (a) Define a $\mathbf{k}$-linear map $\mathfrak{Z} : \Lambda \to \Lambda$ by having it send s_λ to s_{λ^t} for every partition λ. (This is clearly well-defined, since $(s_\lambda)_{\lambda \in \mathrm{Par}}$ is a $\mathbf{k}$-basis of Λ.) Show that

$$\mathfrak{Z}(f h_n) = \mathfrak{Z}(f) \cdot \mathfrak{Z}(h_n) \qquad \text{for every } f \in \Lambda \text{ and every } n \in \mathbb{N}.$$

(b) Show that $\mathfrak{Z} = \omega$.
(c) Show that $c^\lambda_{\mu,\nu} = c^{\lambda^t}_{\mu^t,\nu^t}$ for any three partitions λ, μ and ν.
(d) Use this to prove (2.4.15).[156]

[154]This approach to Theorem 2.5.1 is taken in [44, §4] (except that [44] only works with finitely many variables).

[155]Of course, this gives a new proof of Theorem 2.5.1 only when coupled with a proof of Theorem 2.7.1 which does not rely on Theorem 2.5.1. The proof of Theorem 2.7.1 we gave in the text above did not rely on Theorem 2.5.1, whereas the proof of (2.7.1) given in Exercise 2.7.6(b) did.

[156]The first author learned this approach to (2.4.15) from Alexander Postnikov.

Exercise 2.7.12. (a) Show that

$$\prod_{i,j=1}^{\infty}(1+x_iy_j)=\sum_{\lambda\in\mathrm{Par}}s_\lambda(\mathbf{x})\,s_{\lambda^t}(\mathbf{y})=\sum_{\lambda\in\mathrm{Par}}e_\lambda(\mathbf{x})\,m_\lambda(\mathbf{y})$$

in the power series ring $\mathbf{k}[[\mathbf{x},\mathbf{y}]]=\mathbf{k}[[x_1,x_2,x_3,\ldots,y_1,y_2,y_3,\ldots]]$.

(b) Assume that $\mathbb{Q}$ is a subring of $\mathbf{k}$. Show that

$$\prod_{i,j=1}^{\infty}(1+x_iy_j)=\sum_{\lambda\in\mathrm{Par}}(-1)^{|\lambda|-\ell(\lambda)}\,z_\lambda^{-1}p_\lambda(\mathbf{x})\,p_\lambda(\mathbf{y})$$

in the power series ring $\mathbf{k}[[\mathbf{x},\mathbf{y}]]=\mathbf{k}[[x_1,x_2,x_3,\ldots,y_1,y_2,y_3,\ldots]]$, where z_λ is defined as in Proposition 2.5.15.

The first equality of Exercise 2.7.12(a) appears in [206, Thm. 7.14.3], [186, Thm. 4.8.6] and several other references under the name of the *dual Cauchy identity*, and is commonly proven using a "dual" analogue of the Robinson-Schensted-Knuth algorithm.

Exercise 2.7.13. Prove Theorem 2.4.6.

[**Hint:**[157] Switch $\mathbf{x}$ and $\mathbf{y}$ in the formula of Exercise 2.5.11(a), and specialize the resulting equality by replacing $\mathbf{y}$ by a finite set of variables $(y_1,y_2,\ldots,y_\ell)$; then, set $n=\ell$ and $\rho=(n-1,n-2,\ldots,0)$, and multiply with the alternant $a_\rho(y_1,y_2,\ldots,y_\ell)$, using Corollary 2.6.7 to simplify the result; finally, extract the coefficient of $\mathbf{y}^{\lambda+\rho}$.]

Exercise 2.7.14. Prove the following:

(a) We have $(S(f),S(g))=(f,g)$ for all $f\in\Lambda$ and $g\in\Lambda$.

(b) We have $(e_n,f)=(-1)^n\cdot(S(f))(1)$ for any $n\in\mathbb{N}$ and $f\in\Lambda_n$. (See Exercise 2.1.2 for the meaning of $(S(f))(1)$.)

2.8. **Skewing and Lam's proof of the skew Pieri rule.** We codify here the operation $s_\mu^\perp$ of *skewing by* s_μ, acting on Schur functions via

$$s_\mu^\perp(s_\lambda)=s_{\lambda/\mu}$$

(where, as before, one defines $s_{\lambda/\mu}=0$ if $\mu\not\subseteq\lambda$). These operations play a crucial role

- in Lam's proof of the skew Pieri rule,
- in Lam, Lauve, and Sottile's proof [120] of a more general skew Littlewood-Richardson rule that had been conjectured by Assaf and McNamara, and
- in Zelevinsky's structure theory of PSH's to be developed in the next chapter.

We are going to define them in the general setting of any graded Hopf algebra.

Definition 2.8.1. Given a graded Hopf algebra A, and its (graded) dual A^o, let $(\cdot,\cdot)=(\cdot,\cdot)_A:A^o\times A\to\mathbf{k}$ be the pairing defined by $(f,a):=f(a)$

[157] This is the proof given in Stanley [206, §7.16, Second Proof of Thm. 7.16.1] and Macdonald [142, proof of (5.4)].

for f in A^o and a in A. Then define for each f in A^o an operator $A \xrightarrow{f^\perp} A$ as follows[158]: for a in A with $\Delta(a) = \sum a_1 \otimes a_2$, let

$$f^\perp(a) = \sum (f, a_1) a_2.$$

In other words, $f^\perp$ is the composition

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{f \otimes \mathrm{id}} \mathbf{k} \otimes A \xrightarrow{\cong} A,$$

where the rightmost arrow is the canonical isomorphism $\mathbf{k} \otimes A \to A$. This operator $f^\perp$ is called *skewing by* f.

Now, recall that the Hall inner product induces an isomorphism $\Lambda^o \cong \Lambda$ (by Corollary 2.5.14). Hence, we can regard any element $f \in \Lambda$ as an element of Λ^o; this allows us to define an operator $f^\perp : \Lambda \to \Lambda$ for each $f \in \Lambda$ (by regarding f as an element of Λ^o, and applying Definition 2.8.1 to $A = \Lambda$). Explicitly, this operator is given by

$$f^\perp(a) = \sum (f, a_1) a_2 \quad \text{whenever} \quad \Delta(a) = \sum a_1 \otimes a_2, \tag{2.8.1}$$

where the inner product (f, a_1) is now understood as a Hall inner product.

Recall that each partition λ satisfies

$$\Delta s_\lambda = \sum_{\mu \subseteq \lambda} s_\mu \otimes s_{\lambda/\mu} = \sum_{\nu \subseteq \lambda} s_\nu \otimes s_{\lambda/\nu} = \sum_{\nu} s_\nu \otimes s_{\lambda/\nu}$$

(since $s_{\lambda/\nu} = 0$ unless $\nu \subseteq \lambda$). Hence, for any two partitions λ and μ, we have

$$\begin{aligned} s_\mu^\perp (s_\lambda) &= \sum_\nu \underbrace{(s_\mu, s_\nu)}_{=\delta_{\mu,\nu}} s_{\lambda/\nu} \\ &\qquad \text{(by (2.8.1), applied to } f = s_\mu \text{ and } a = s_\lambda) \\ &= \sum_\nu \delta_{\mu,\nu} s_{\lambda/\nu} = s_{\lambda/\mu}. \end{aligned} \tag{2.8.2}$$

Thus, skewing acts on the Schur functions exactly as desired.

Proposition 2.8.2. *Let A be a graded Hopf algebra. The $f^\perp$ operators $A \to A$ have the following properties.*

(i) *For every $f \in A^o$, the map $f^\perp$ is adjoint to left multiplication $A^o \xrightarrow{f\cdot} A^o$ in the sense that*

$$(g, f^\perp(a)) = (fg, a).$$

(ii) *For every $f, g \in A^o$, we have $(fg)^\perp(a) = g^\perp(f^\perp(a))$, that is, A becomes a right A^o-module via the $f^\perp$ action.*[159]

(iii) *The unity 1_{A^o} of the $\mathbf{k}$-algebra A^o satisfies $(1_{A^o})^\perp = \mathrm{id}_A$.*

(iv) *Assume that A is of finite type (so A^o becomes a Hopf algebra, not just an algebra). If an $f \in A^o$ satisfies $\Delta(f) = \sum f_1 \otimes f_2$, then*

$$f^\perp(ab) = \sum f_1^\perp(a) f_2^\perp(b).$$

[158] This $f^\perp(a)$ is called $a \leftharpoonup f$ in Montgomery [157, Example 1.6.5].

[159] This makes sense, since A^o is a $\mathbf{k}$-algebra (by Exercise 1.6.1(c), applied to $C = A$).

In particular, if f is primitive in A^o, so that $\Delta(f) = f \otimes 1 + 1 \otimes f$, then $f^\perp$ is a derivation*:*

$$f^\perp(ab) = f^\perp(a) \cdot b + a \cdot f^\perp(b).$$

Proof. For (i), note that

$$(g, f^\perp(a)) = \sum (f, a_1)(g, a_2) = (f \otimes g, \Delta_A(a)) = (m_{A^o}(f \otimes g), a) = (fg, a).$$

For (ii), using (i) and considering any h in A^o, one has that

$$(h, (fg)^\perp(a)) = (fgh, a) = (gh, f^\perp(a)) = (h, g^\perp(f^\perp(a))).$$

For (iii), we recall that the unity 1_{A^o} of A^o is the counit ϵ of A, and thus every $a \in A$ satisfies

$$\begin{aligned}
(1_{A^o})^\perp(a) = \epsilon^\perp(a) &= \sum_{(a)} \underbrace{(\epsilon, a_1)}_{=\epsilon(a_1)} a_2 && \left(\text{by the definition of } \epsilon^\perp\right) \\
&= \sum_{(a)} \epsilon(a_1) a_2 = a && (\text{by the axioms of a coalgebra}),
\end{aligned}$$

so that $(1_{A^o})^\perp = \mathrm{id}_A$.

For (iv), noting that

$$\Delta(ab) = \Delta(a)\Delta(b) = \left(\sum_{(a)} a_1 \otimes a_2\right)\left(\sum_{(b)} b_1 \otimes b_2\right) = \sum_{(a),(b)} a_1 b_1 \otimes a_2 b_2,$$

one has that

$$\begin{aligned}
f^\perp(ab) &= \sum_{(a),(b)} (f, a_1 b_1)_A \; a_2 b_2 = \sum_{(a),(b)} (\Delta(f), a_1 \otimes b_1)_{A \otimes A} \; a_2 b_2 \\
&= \sum_{(f),(a),(b)} (f_1, a_1)_A (f_2, b_1)_A \; a_2 b_2 \\
&= \sum_{(f)} \left(\sum_{(a)} (f_1, a_1)_A a_2\right)\left(\sum_{(b)} (f_2, b_1)_A b_2\right) = \sum_{(f)} f_1^\perp(a) f_2^\perp(b).
\end{aligned}$$

□

The Pieri rules (Theorem 2.7.1) expressed multiplication by h_n or by e_n in the basis $(s_\lambda)_{\lambda \in \mathrm{Par}}$ of Λ. We can similarly express skewing by h_n or by e_n:

Proposition 2.8.3. *For every partition λ and any $n \in \mathbb{N}$, we have*

$$h_n^\perp s_\lambda = \sum_{\substack{\lambda^- : \lambda/\lambda^- \text{ is a} \\ \text{horizontal } n\text{-strip}}} s_{\lambda^-}; \tag{2.8.3}$$

$$e_n^\perp s_\lambda = \sum_{\substack{\lambda^- : \lambda/\lambda^- \text{ is a} \\ \text{vertical } n\text{-strip}}} s_{\lambda^-}. \tag{2.8.4}$$

Exercise 2.8.4. Prove Proposition 2.8.3.

[**Hint:** Use Theorem 2.7.1 and $(s_{\mu^-}, e_n^\perp s_\mu) = (e_n s_{\mu^-}, s_\mu)$.]

The following interaction between multiplication and $h^\perp$ is the key to deducing the skew Pieri formula from the usual Pieri formulas.

Lemma 2.8.5. *For any f, g in Λ and any $n \in \mathbb{N}$, one has*

$$f \cdot h_n^\perp(g) = \sum_{k=0}^{n} (-1)^k h_{n-k}^\perp (e_k^\perp(f) \cdot g).$$

Proof. Starting with the right side, first apply Proposition 2.8.2(iv):

$$\begin{aligned}
&\sum_{k=0}^{n} (-1)^k \underbrace{h_{n-k}^\perp(e_k^\perp(f) \cdot g)}_{\substack{=\sum_{j=0}^{n-k} h_j^\perp(e_k^\perp(f)) \cdot h_{n-k-j}^\perp(g) \\ \text{(by Proposition 2.8.2(iv), applied} \\ \text{to } h_{n-k},\ e_k^\perp(f) \text{ and } g \text{ instead of } f,\ a \text{ and } b)}} \\
&= \sum_{k=0}^{n} (-1)^k \sum_{j=0}^{n-k} h_j^\perp(e_k^\perp(f)) \cdot h_{n-k-j}^\perp(g) \\
&= \sum_{j=0}^{n} \sum_{k=0}^{n-j} (-1)^k h_j^\perp(e_k^\perp(f)) \cdot h_{n-k-j}^\perp(g) \\
&= \sum_{j=0}^{n} \sum_{i=0}^{n-j} (-1)^{n-i-j} h_j^\perp(e_{n-i-j}^\perp(f)) \cdot h_i^\perp(g) \\
&\qquad \text{(reindexing } i := n-k-j \text{ in the inner sum)} \\
&= \sum_{i=0}^{n} (-1)^{n-i} \left(\sum_{j=0}^{n-i} (-1)^j h_j^\perp(e_{n-i-j}^\perp(f)) \right) \cdot h_i^\perp(g) \\
&= \sum_{i=0}^{n} (-1)^{n-i} \left(\sum_{j=0}^{n-i} (-1)^j e_{n-i-j} h_j \right)^\perp (f) \cdot h_i^\perp(g) \\
&\qquad \text{(by Proposition 2.8.2(ii))} \\
&= 1^\perp(f) \cdot h_n^\perp(g) = f \cdot h_n^\perp(g)
\end{aligned}$$

where the second-to-last equality used (2.4.4). ∎

Proof of Theorem 2.7.3. We prove (2.7.3); the equality (2.7.4) is analogous, swapping $h_i \leftrightarrow e_i$ and swapping the words "vertical" $\leftrightarrow$ "horizontal". For any $f \in \Lambda$, we have

$$\begin{aligned}
\left(s_{\lambda/\mu}, f\right) &= \left(s_\mu^\perp(s_\lambda), f\right) && \text{(by (2.8.2))} \\
&= \left(f, s_\mu^\perp(s_\lambda)\right) && \text{(by symmetry of } (\cdot,\cdot)_\Lambda) \\
&= (s_\mu f, s_\lambda) && \text{(by Proposition 2.8.2(i))} \\
&= (s_\lambda, s_\mu f) && \text{(by symmetry of } (\cdot,\cdot)_\Lambda) .
\end{aligned} \tag{2.8.5}$$

Hence for any g in Λ, one can compute that

$$\begin{aligned}
(h_n s_{\lambda/\mu}\ ,\ g) &\overset{Prop.}{\underset{2.8.2(i)}{=}} (s_{\lambda/\mu}\ ,\ h_n^\perp g) \overset{(2.8.5)}{=} (s_\lambda\ ,\ s_\mu \cdot h_n^\perp g) \\
&\overset{Lemma}{\underset{2.8.5}{=}} \sum_{k=0}^{n} (-1)^k (s_\lambda\ ,\ h_{n-k}^\perp(e_k^\perp(s_\mu) \cdot g)) \\
&\overset{Prop.}{\underset{2.8.2(i)}{=}} \sum_{k=0}^{n} (-1)^k (h_{n-k} s_\lambda\ , e_k^\perp(s_\mu) \cdot g).
\end{aligned} \tag{2.8.6}$$

The first Pieri rule in Theorem 2.7.1 lets one rewrite $h_{n-k}s_\lambda = \sum_{\lambda^+} s_{\lambda^+}$, with the sum running through λ^+ for which λ^+/λ is a horizontal $(n-k)$-strip. Meanwhile, (2.8.4) lets one rewrite $e_k^\perp s_\mu = \sum_{\mu^-} s_{\mu^-}$, with the sum running through μ^- for which μ/μ^- is a vertical k-strip. Thus the right hand side of (2.8.6) becomes

$$\sum_{k=0}^{n}(-1)^k \left(\sum_{\lambda^+} s_{\lambda^+} \ , \ \sum_{\mu^-} s_{\mu^-} \cdot g \right) \overset{(2.8.5)}{=} \left(\sum_{k=0}^{n}(-1)^k \sum_{(\lambda^+,\mu^-)} s_{\lambda^+/\mu^-}, g \right)$$

where the sum is over the pairs (λ^+, μ^-) for which λ^+/λ is a horizontal $(n-k)$-strip and μ/μ^- is a vertical k-strip. This proves (2.7.3). $\square$

Exercise 2.8.6. Let $n \in \mathbb{N}$.

(a) For every $k \in \mathbb{N}$, let $p(n,k)$ denote the number of partitions of n of length k. Let $c(n)$ denote the number of *self-conjugate* partitions of n (that is, partitions λ of n satisfying $\lambda^t = \lambda$). Show that

$$(-1)^n c(n) = \sum_{k=0}^{n} (-1)^k p(n,k).$$

(This application of Hopf algebras was found by Aguiar and Lauve, [5, §5.1]. See also [206, Chapter 1, Exercise 22(b)] for an elementary proof.)

(b) For every partition λ, let $C(\lambda)$ denote the number of corner cells of the Ferrers diagram of λ (these are the cells of the Ferrers diagram whose neighbors to the east and to the south both lie outside of the Ferrers diagram). For every partition λ, let $\mu_1(\lambda)$ denote the number of parts of λ equal to 1. Show that

$$\sum_{\lambda \in \mathrm{Par}_n} C(\lambda) = \sum_{\lambda \in \mathrm{Par}_n} \mu_1(\lambda).$$

(This is also due to Stanley.)

Exercise 2.8.7. The goal of this exercise is to prove (2.4.15) using the skewing operators that we have developed.[160] Recall the involution $\omega : \Lambda \to \Lambda$ defined in (2.4.10).

(a) Show that $\omega(p_\lambda) = (-1)^{|\lambda|-\ell(\lambda)} p_\lambda$ for any $\lambda \in \mathrm{Par}$, where $\ell(\lambda)$ denotes the length of the partition λ.
(b) Show that ω is an isometry.
(c) Show that this same map $\omega : \Lambda \to \Lambda$ is a Hopf automorphism.
(d) Prove that $\omega\left(a^\perp b\right) = (\omega(a))^\perp (\omega(b))$ for every $a \in \Lambda$ and $b \in \Lambda$.
(e) For any partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ with length $\ell(\lambda) = \ell$, prove that

$$e_\ell^\perp s_\lambda = s_{(\lambda_1-1,\lambda_2-1,\ldots,\lambda_\ell-1)}.$$

(f) For any partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, prove that

$$h_{\lambda_1}^\perp s_\lambda = s_{(\lambda_2,\lambda_3,\lambda_4,\ldots)}.$$

(g) Prove (2.4.15).

[160]Make sure not to use the results of Exercise 2.7.11 or Exercise 2.7.12 or Exercise 2.7.14 here, or anything else that relied on (2.4.15), in order to avoid circular reasoning.

Exercise 2.8.8. Let n be a positive integer. Prove the following:

(a) We have $(e_n, p_n) = (-1)^{n-1}$.
(b) We have $(e_m, p_n) = 0$ for each $m \in \mathbb{N}$ satisfying $m \neq n$.
(c) We have $e_n^\perp p_n = (-1)^{n-1}$.
(d) We have $e_m^\perp p_n = 0$ for each positive integer m satisfying $m \neq n$.

2.9. **Assorted exercises on symmetric functions.** Over a hundred exercises on symmetric functions are collected in Stanley's [206, chapter 7], and even more (but without any hints or references) on his website[161]. Further sources for results related to symmetric functions are Macdonald's work, including his monograph [142] and his expository [143]. In this section, we gather a few exercises that are not too difficult to handle with the material given above.

Exercise 2.9.1. (a) Let $m \in \mathbb{Z}$. Prove that, for every $f \in \Lambda$, the infinite sum $\sum_{i\in\mathbb{N}} (-1)^i h_{m+i} e_i^\perp f$ is convergent in the discrete topology (i.e., all but finitely many addends of this sum are zero). Hence, we can define a map $\mathbf{B}_m : \Lambda \to \Lambda$ by setting

$$\mathbf{B}_m(f) = \sum_{i\in\mathbb{N}} (-1)^i h_{m+i} e_i^\perp f \qquad \text{for all } f \in \Lambda.$$

Show that this map $\mathbf{B}_m$ is $\mathbf{k}$-linear.

(b) Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ be a partition, and let $m \in \mathbb{Z}$ be such that $m \geq \lambda_1$. Show that

$$\sum_{i\in\mathbb{N}} (-1)^i h_{m+i} e_i^\perp s_\lambda = s_{(m,\lambda_1,\lambda_2,\lambda_3,\ldots)}.$$

(c) Let $n \in \mathbb{N}$. For every n-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^n$, we define an element $\overline{s}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} \in \Lambda$ by

$$\overline{s}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} = \det\left(\left(h_{\alpha_i - i + j}\right)_{i,j=1,2,\ldots,n}\right).$$

Show that

$$s_\lambda = \overline{s}_{(\lambda_1,\lambda_2,\ldots,\lambda_n)} \tag{2.9.1}$$

for every partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ having at most n parts[162].

Furthermore, show that for every n-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^n$, the symmetric function $\overline{s}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)}$ either is 0 or equals $\pm s_\nu$ for some partition ν having at most n parts.

Finally, show that for any n-tuples $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^n$ and $(\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{N}^n$, we have

$$\overline{s}^\perp_{(\beta_1,\beta_2,\ldots,\beta_n)} \overline{s}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} = \det\left(\left(h_{\alpha_i - \beta_j - i + j}\right)_{i,j=1,2,\ldots,n}\right). \tag{2.9.2}$$

(d) For every $n \in \mathbb{N}$, every $m \in \mathbb{Z}$ and every n-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^n$, prove that

$$\sum_{i\in\mathbb{N}} (-1)^i h_{m+i} e_i^\perp \overline{s}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} = \overline{s}_{(m,\alpha_1,\alpha_2,\ldots,\alpha_n)}, \tag{2.9.3}$$

where we are using the notations of Exercise 2.9.1(c).

[161] `http://math.mit.edu/~rstan/ec/ch7supp.pdf`

[162] Recall that a *part* of a partition means a nonzero entry of the partition.

(e) For every $n \in \mathbb{N}$ and every n-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^n$, prove that

$$\overline{s}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} = (\mathbf{B}_{\alpha_1} \circ \mathbf{B}_{\alpha_2} \circ \cdots \circ \mathbf{B}_{\alpha_n})(1),$$

where we are using the notations of Exercise 2.9.1(c) and Exercise 2.9.1(a).

(f) For every $m \in \mathbb{Z}$ and every positive integer n, prove that $\mathbf{B}_m(p_n) = h_m p_n - h_{m+n}$. Here, we are using the notations of Exercise 2.9.1(a).

Remark 2.9.2. The map $\mathbf{B}_m$ defined in Exercise 2.9.1(a) is the so-called *m-th Bernstein creation operator*; it appears in Zelevinsky [227, §4.20(a)] and has been introduced by J.N. Bernstein, who found the result of Exercise 2.9.1(b). It is called a "Schur row adder" in [74]. Exercise 2.9.1(e) appears in Berg/Bergeron/Saliola/Serrano/Zabrocki [17, Theorem 2.3], where it is used as a prototype for defining *noncommutative* analogues of Schur functions, the so-called *immaculate functions.* The particular case of Exercise 2.9.1(e) for $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ a partition of length n (a restatement of Exercise 2.9.1(b)) is proven in [142, §I.5, example 29].

Exercise 2.9.3. (a) Prove that there exists a unique family $(x_n)_{n\geq 1}$ of elements of Λ such that

$$H(t) = \prod_{n=1}^{\infty} (1 - x_n t^n)^{-1}.$$

Denote this family $(x_n)_{n\geq 1}$ by $(w_n)_{n\geq 1}$. For instance,

$$\begin{aligned} &w_1 = s_{(1)}, \qquad w_2 = -s_{(1,1)}, \qquad w_3 = -s_{(2,1)}, \\ &w_4 = -s_{(1,1,1,1)} - s_{(2,1,1)} - s_{(2,2)} - s_{(3,1)}, \\ &w_5 = -s_{(2,1,1,1)} - s_{(2,2,1)} - s_{(3,1,1)} - s_{(3,2)} - s_{(4,1)}. \end{aligned}$$

(b) Show that w_n is homogeneous of degree n for every positive integer n.

(c) For every partition λ, define $w_\lambda \in \Lambda$ by $w_\lambda = w_{\lambda_1} w_{\lambda_2} \cdots w_{\lambda_\ell}$ (where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ with $\ell = \ell(\lambda)$). Notice that w_λ is homogeneous of degree $|\lambda|$. Prove that $\sum_{\lambda \in \mathrm{Par}_n} w_\lambda = h_n$ for every $n \in \mathbb{N}$.

(d) Show that $\{w_\lambda\}_{\lambda \in \mathrm{Par}}$ is a $\mathbf{k}$-basis of Λ. (This basis is called the *Witt basis*[163]; it is studied in [90, §9-§10].[164])

(e) Prove that $p_n = \sum_{d|n} d w_d^{n/d}$ for every positive integer n. (Here, the summation sign $\sum_{d|n}$ means a sum over all positive divisors d of n.)

(f) We are going to show that $-w_n$ is a sum of Schur functions (possibly with repetitions, but without signs!) for every $n \geq 2$. (For $n = 1$, the opposite is true: w_1 is a single Schur function.) This proof goes back to Doran [55][165].

[163]This is due to its relation with Witt vectors in the appropriate sense. Most of the work on this basis has been done by Reutenauer and Hazewinkel.

[164]It also implicitly appears in [12, §5]. Indeed, the q_n of [12] are our w_n (for $\mathbf{k} = R$).

[165]See also Stanley [206, Exercise 7.46].

For any positive integers n and k, define $f_{n,k} \in \Lambda$ by $f_{n,k} = \sum_{\substack{\lambda \in \operatorname{Par}_n, \\ \min \lambda \geq k}} w_\lambda$, where $\min \lambda$ denotes the smallest part[166] of λ. Show that

$$-f_{n,k} = s_{(n-1,1)} + \sum_{i=2}^{k-1} f_{i,i} f_{n-i,i} \qquad \text{for every } n \geq k \geq 2.$$

Conclude that $-f_{n,k}$ is a sum of Schur functions for every $n \in \mathbb{N}$ and $k \geq 2$. Conclude that $-w_n$ is a sum of Schur functions for every $n \geq 2$.

(g) For every partition λ, define $r_\lambda \in \Lambda$ by $r_\lambda = \prod_{i \geq 1} h_{v_i}\left(x_1^i, x_2^i, x_3^i, \ldots\right)$, where v_i is the number of occurrences of i in λ. Show that

$$\sum_{\lambda \in \operatorname{Par}} w_\lambda(\mathbf{x}) r_\lambda(\mathbf{y}) = \prod_{i,j=1}^{\infty} \left(1 - x_i y_j\right)^{-1}.$$

(h) Show that $\{r_\lambda\}_{\lambda \in \operatorname{Par}}$ and $\{w_\lambda\}_{\lambda \in \operatorname{Par}}$ are dual bases of Λ.

Exercise 2.9.4. For this exercise, set $\mathbf{k} = \mathbb{Z}$, and consider $\Lambda = \Lambda_{\mathbb{Z}}$ as a subring of $\Lambda_{\mathbb{Q}}$. Also, consider $\Lambda \otimes_{\mathbb{Z}} \Lambda$ as a subring of $\Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}}$. [167] Recall that the family $(p_n)_{n \geq 1}$ generates the $\mathbb{Q}$-algebra $\Lambda_{\mathbb{Q}}$, but does not generate the $\mathbb{Z}$-algebra Λ.

(a) Define a $\mathbb{Q}$-linear map $Z : \Lambda_{\mathbb{Q}} \to \Lambda_{\mathbb{Q}}$ by setting

$$Z(p_\lambda) = z_\lambda p_\lambda \qquad \text{for every partition } \lambda,$$

where z_λ is defined as in Proposition 2.5.15.[168] Show that $Z(\Lambda) \subset \Lambda$.

(b) Define a $\mathbb{Q}$-algebra homomorphism $\Delta_\times : \Lambda_{\mathbb{Q}} \to \Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}}$ by setting

$$\Delta_\times(p_n) = p_n \otimes p_n \qquad \text{for every positive integer } n.$$

[169] Show that $\Delta_\times(\Lambda) \subset \Lambda \otimes_{\mathbb{Z}} \Lambda$.

(c) Let $r \in \mathbb{Z}$. Define a $\mathbb{Q}$-algebra homomorphism $\epsilon_r : \Lambda_{\mathbb{Q}} \to \mathbb{Q}$ by setting

$$\epsilon_r(p_n) = r \qquad \text{for every positive integer } n.$$

[170] Show that $\epsilon_r(\Lambda) \subset \mathbb{Z}$.

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[166]Recall that a *part* of a partition means a nonzero entry of the partition.

[167]Here is how this works: We have $\Lambda_{\mathbb{Q}} \cong \mathbb{Q} \otimes_{\mathbb{Z}} \Lambda$. But fundamental properties of tensor products yield

$$\mathbb{Q} \otimes_{\mathbb{Z}} (\Lambda \otimes_{\mathbb{Z}} \Lambda) \cong \underbrace{(\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda)}_{\cong \Lambda_{\mathbb{Q}}} \otimes_{\mathbb{Q}} \underbrace{(\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda)}_{\cong \Lambda_{\mathbb{Q}}} \cong \Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}} \tag{2.9.4}$$

as $\mathbb{Q}$-algebras. But $\Lambda \otimes_{\mathbb{Z}} \Lambda$ is a free $\mathbb{Z}$-module (since Λ is a free $\mathbb{Z}$-module), and so the canonical ring homomorphism $\Lambda \otimes_{\mathbb{Z}} \Lambda \to \mathbb{Q} \otimes_{\mathbb{Z}} (\Lambda \otimes_{\mathbb{Z}} \Lambda)$ sending every u to $1_{\mathbb{Q}} \otimes_{\mathbb{Z}} u$ is injective. Composing this ring homomorphism with the $\mathbb{Q}$-algebra isomorphism of (2.9.4) gives an injective ring homomorphism $\Lambda \otimes_{\mathbb{Z}} \Lambda \to \Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}}$. We use this latter homomorphism to identify $\Lambda \otimes_{\mathbb{Z}} \Lambda$ with a subring of $\Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}}$.

[168]This is well-defined, since $(p_\lambda)_{\lambda \in \operatorname{Par}}$ is a $\mathbb{Q}$-module basis of $\Lambda_{\mathbb{Q}}$.

[169]This is well-defined, since the family $(p_n)_{n \geq 1}$ generates the $\mathbb{Q}$-algebra $\Lambda_{\mathbb{Q}}$ and is algebraically independent.

[170]This is well-defined, since the family $(p_n)_{n \geq 1}$ generates the $\mathbb{Q}$-algebra $\Lambda_{\mathbb{Q}}$ and is algebraically independent.

(d) Let $r \in \mathbb{Z}$. Define a $\mathbb{Q}$-algebra homomorphism $\mathbf{i}_r : \Lambda_{\mathbb{Q}} \to \Lambda_{\mathbb{Q}}$ by setting

$$\mathbf{i}_r(p_n) = rp_n \qquad \text{for every positive integer } n.$$

[171] Show that $\mathbf{i}_r(\Lambda) \subset \Lambda$.

(e) Define a $\mathbb{Q}$-linear map $\operatorname{Sq} : \Lambda_{\mathbb{Q}} \to \Lambda_{\mathbb{Q}}$ by setting

$$\operatorname{Sq}(p_\lambda) = p_\lambda^2 \qquad \text{for every partition } \lambda.$$

[172] Show that $\operatorname{Sq}(\Lambda) \subset \Lambda$.

(f) Let $r \in \mathbb{Z}$. Define a $\mathbb{Q}$-algebra homomorphism $\Delta_r : \Lambda_{\mathbb{Q}} \to \Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}}$ by setting

$$\Delta_r(p_n) = \sum_{i=1}^{n-1} \binom{n}{i} p_i \otimes p_{n-i} + r \otimes p_n + p_n \otimes r \qquad \text{for every positive integer } n.$$

[173] Show that $\Delta_r(\Lambda) \subset \Lambda \otimes_{\mathbb{Z}} \Lambda$.

(g) Consider the map $\Delta_\times$ introduced in Exercise 2.9.4(b) and the map ϵ_1 introduced in Exercise 2.9.4(c). Show that the $\mathbb{Q}$-algebra $\Lambda_{\mathbb{Q}}$, endowed with the comultiplication $\Delta_\times$ and the counit ϵ_1, becomes a cocommutative $\mathbb{Q}$-bialgebra.[174]

(h) Define a $\mathbb{Q}$-bilinear map $* : \Lambda_{\mathbb{Q}} \times \Lambda_{\mathbb{Q}} \to \Lambda_{\mathbb{Q}}$, which will be written in infix notation (that is, we will write $a * b$ instead of $*(a,b)$), by setting

$$p_\lambda * p_\mu = \delta_{\lambda,\mu} z_\lambda p_\lambda \qquad \text{for any partitions } \lambda \text{ and } \mu$$

(where z_λ is defined as in Proposition 2.5.15). [175] Show that $f * g \in \Lambda$ for any $f \in \Lambda$ and $g \in \Lambda$.

(i) Show that $\epsilon_1(f) = f(1)$ for every $f \in \Lambda_{\mathbb{Q}}$ (where we are using the notation ϵ_r defined in Exercise 2.9.4(c)).

[**Hint:**

- For (b), show that, for every $f \in \Lambda_{\mathbb{Q}}$, the tensor $\Delta_\times(f)$ is the preimage of

$$\begin{aligned} &f\left(\left(x_i y_j\right)_{(i,j)\in\{1,2,3,\ldots\}^2}\right) \\ &= f(x_1y_1, x_1y_2, x_1y_3, \ldots, x_2y_1, x_2y_2, x_2y_3, \ldots, \ldots) \in \mathbb{Q}[[\mathbf{x},\mathbf{y}]] \end{aligned}$$

under the canonical injection $\Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}} \to \mathbb{Q}[[\mathbf{x},\mathbf{y}]]$ which maps every $f \otimes g$ to $f(\mathbf{x})\, g(\mathbf{y})$. (This requires making sure that the evaluation $f\left(\left(x_i y_j\right)_{(i,j)\in\{1,2,3,\ldots\}^2}\right)$ is well-defined to begin with, i.e., converges as a formal power series.)

  For an alternative solution to (b), compute $\Delta_\times(h_n)$ or $\Delta_\times(e_n)$.
- For (c), compute $\epsilon_r(e_n)$ or $\epsilon_r(h_n)$.

[171] This is well-defined, since the family $(p_n)_{n\geq 1}$ generates the $\mathbb{Q}$-algebra $\Lambda_{\mathbb{Q}}$ and is algebraically independent.

[172] This is well-defined, since $(p_\lambda)_{\lambda\in\operatorname{Par}}$ is a $\mathbb{Q}$-module basis of $\Lambda_{\mathbb{Q}}$.

[173] This is well-defined, since the family $(p_n)_{n\geq 1}$ generates the $\mathbb{Q}$-algebra $\Lambda_{\mathbb{Q}}$ and is algebraically independent.

[174] But unlike $\Lambda_{\mathbb{Q}}$ with the usual coalgebra structure, it is neither graded nor a Hopf algebra.

[175] This is well-defined, since $(p_\lambda)_{\lambda\in\operatorname{Par}}$ is a $\mathbb{Q}$-module basis of $\Lambda_{\mathbb{Q}}$.

- Reduce (d) to (b) and (c) using Exercise 1.3.6.
- Reduce (e) to (b).
- (f) is the hardest part. It is tempting to try and interpret the definition of Δ_r as a convoluted way of saying that $\Delta_r(f)$ is the preimage of $f\left((x_i + y_j)_{(i,j)\in\{1,2,3,\ldots\}^2}\right)$ under the canonical injection $\Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}} \to \mathbb{Q}[[\mathbf{x}, \mathbf{y}]]$ which maps every $f \otimes g$ to $f(\mathbf{x})\, g(\mathbf{y})$. However, this does not make sense since the evaluation $f\left((x_i + y_j)_{(i,j)\in\{1,2,3,\ldots\}^2}\right)$ is (in general) not well-defined[176] (and even if it was, it would fail to explain the r). So we need to get down to finitely many variables. For every $N \in \mathbb{N}$, define a $\mathbb{Q}$-algebra homomorphism $\mathcal{E}_N : \Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}} \to \mathbb{Q}[x_1, x_2, \ldots, x_N, y_1, y_2, \ldots, y_N]$ by sending each $f \otimes g$ to $f(x_1, x_2, \ldots, x_N)\, g(y_1, y_2, \ldots, y_N)$. Show that $\Delta_N(\Lambda) \subset \mathcal{E}_N^{-1}(\mathbb{Z}[x_1, x_2, \ldots, x_N, y_1, y_2, \ldots, y_N])$. This shows that, at least, the coefficients of $\Delta_r(f)$ in front of the $m_\lambda \otimes m_\mu$ with $\ell(\lambda) \leq r$ and $\ell(\mu) \leq r$ (in the $\mathbb{Q}$-basis $(m_\lambda \otimes m_\mu)_{\lambda,\mu\in\mathrm{Par}}$ of $\Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}}$) are integral for $f \in \Lambda$. Of course, we want all coefficients. Show that $\Delta_a = \Delta_b \star \left(\Delta_{\Lambda_{\mathbb{Q}}} \circ \mathbf{i}_{a-b}\right)$ in $\operatorname{Hom}(\Lambda_{\mathbb{Q}}, \Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}})$ for any integers a and b. This allows "moving" the r. This approach to (f) was partly suggested to the first author by Richard Stanley.
- For (h), notice that Definition 3.1.1(b) (below) allows us to construct a bilinear form $(\cdot,\cdot)_{\Lambda_{\mathbb{Q}}\otimes_{\mathbb{Q}}\Lambda_{\mathbb{Q}}} : (\Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}}) \times (\Lambda_{\mathbb{Q}} \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}}) \to \mathbb{Q}$ from the Hall inner product $(\cdot,\cdot) : \Lambda_{\mathbb{Q}} \times \Lambda_{\mathbb{Q}} \to \mathbb{Q}$. Show that

$$(a * b, c) = (a \otimes b, \Delta_\times(c))_{\Lambda_{\mathbb{Q}}\otimes_{\mathbb{Q}}\Lambda_{\mathbb{Q}}} \qquad \text{for all } a, b, c \in \Lambda_{\mathbb{Q}}, \tag{2.9.5}$$

  and then use (b).

]

Remark 2.9.5. The map $\Delta_\times$ defined in Exercise 2.9.4(b) is known as the *internal comultiplication* (or *Kronecker comultiplication*) on $\Lambda_{\mathbb{Q}}$. Unlike the standard comultiplication $\Delta_{\Lambda_{\mathbb{Q}}}$, it is not a graded map, but rather sends every homogeneous component $(\Lambda_{\mathbb{Q}})_n$ into $(\Lambda_{\mathbb{Q}})_n \otimes (\Lambda_{\mathbb{Q}})_n$. The bilinear map $*$ from Exercise 2.9.4(h) is the so-called *internal multiplication* (or *Kronecker multiplication*), and is similarly not graded but rather takes $(\Lambda_{\mathbb{Q}})_n \times (\Lambda_{\mathbb{Q}})_m$ to $(\Lambda_{\mathbb{Q}})_n$ if $n = m$ and to 0 otherwise.

The analogy between the two internal structures is not perfect: While we saw in Exercise 2.9.4(g) how the internal comultiplication yields another bialgebra structure on $\Lambda_{\mathbb{Q}}$, it is not true that the internal multiplication (combined with the usual coalgebra structure of $\Lambda_{\mathbb{Q}}$) forms a bialgebra structure as well. What is missing is a multiplicative unity; if we would take the closure of $\Lambda_{\mathbb{Q}}$ with respect to the grading, then $1+h_1+h_2+h_3+\cdots$ would be such a unity.

The structure constants of the internal comultiplication on the Schur basis $(s_\lambda)_{\lambda\in\mathrm{Par}}$ are equal to the structure constants of the internal multiplication on the Schur basis[177], and are commonly referred to as the *Kronecker coefficients*. They are known to be nonnegative integers (this follows from

[176] e.g., it involves summing infinitely many x_1's if $f = e_1$

[177] This can be obtained, e.g., from (2.9.5).

Exercise 4.4.8(c)[178]), but no combinatorial proof is known for their nonnegativity. Combinatorial interpretations for these coefficients akin to the Littlewood-Richardson rule have been found only in special cases (cf., e.g., [183] and [23] and [132]).

The map Δ_r of Exercise 2.9.4(f) also has some classical theory behind it, relating to Chern classes of tensor products ([151], [142, §I.4, example 5]).

Parts (b), (c), (d), (e) and (f) of Exercise 2.9.4 are instances of a general phenomenon: Many $\mathbb{Z}$-algebra homomorphisms $\Lambda \to A$ (with A a commutative ring, usually torsionfree) are easiest to define by first defining a $\mathbb{Q}$-algebra homomorphism $\Lambda_{\mathbb{Q}} \to A \otimes \mathbb{Q}$ and then showing that this homomorphism restricts to a $\mathbb{Z}$-algebra homomorphism $\Lambda \to A$. One might ask for general criteria when this is possible; specifically, for what choices of $(b_n)_{n\geq 1} \in A^{\{1,2,3,\ldots\}}$ does there exist a $\mathbb{Z}$-algebra homomorphism $\Lambda \to A$ sending the p_n to b_n ? Such choices are called *ghost-Witt vectors* in Hazewinkel [90], and we can give various equivalent conditions for a family $(b_n)_{n\geq 1}$ to be a ghost-Witt vector:

Exercise 2.9.6. Let A be a commutative ring.

For every $n \in \{1, 2, 3, \ldots\}$, let $\varphi_n : A \to A$ be a ring endomorphism of A. Assume that the following properties hold:

- We have $\varphi_n \circ \varphi_m = \varphi_{nm}$ for any two positive integers n and m.
- We have $\varphi_1 = \mathrm{id}$.
- We have $\varphi_p(a) \equiv a^p \bmod pA$ for every $a \in A$ and every prime number p.

(For example, when $A = \mathbb{Z}$, one can set $\varphi_n = \mathrm{id}$ for all n; this simplifies the exercise somewhat. More generally, setting $\varphi_n = \mathrm{id}$ works whenever A is a binomial ring[179]. However, the results of this exercise are at their most useful when A is a multivariate polynomial ring $\mathbb{Z}[x_1, x_2, x_3, \ldots]$ over $\mathbb{Z}$ and the homomorphism φ_n sends every $P \in A$ to $P(x_1^n, x_2^n, x_3^n, \ldots)$.)

Let μ denote the *number-theoretic Möbius function*; this is the function $\{1, 2, 3, \ldots\} \to \mathbb{Z}$ defined by

$$\mu(m) = \begin{cases} 0, & \text{if } m \text{ is not squarefree;} \\ (-1)^{(\text{number of prime factors of } m)}, & \text{if } m \text{ is squarefree} \end{cases}$$

for every positive integer m.

Let ϕ denote the *Euler totient function*; this is the function $\{1, 2, 3, \ldots\} \to \mathbb{N}$ which sends every positive integer m to the number of elements of $\{1, 2, \ldots, m\}$ coprime to m.

[178] Their integrality can also be easily deduced from Exercise 2.9.4(b).

[179] A *binomial ring* is defined to be a torsionfree (as an additive group) commutative ring A which has one of the following equivalent properties:

- For every $n \in \mathbb{N}$ and $a \in A$, we have $a(a-1)\cdots(a-n+1) \in n! \cdot A$. (That is, binomial coefficients $\binom{a}{n}$ with $a \in A$ and $n \in \mathbb{N}$ are defined in A.)
- We have $a^p \equiv a \bmod pA$ for every $a \in A$ and every prime number p.

See [226] and the references therein for studies of these rings. It is not hard to check that $\mathbb{Z}$ and every localization of $\mathbb{Z}$ are binomial rings, and so is any commutative $\mathbb{Q}$-algebra as well as the ring

$$\{P \in \mathbb{Q}[X] \mid P(n) \in \mathbb{Z} \text{ for every } n \in \mathbb{Z}\}$$

(but not the ring $\mathbb{Z}[X]$ itself).

Let $(b_n)_{n\geq 1} \in A^{\{1,2,3,\ldots\}}$ be a family of elements of A. Prove that the following seven assertions are equivalent:

- *Assertion $\mathcal{C}$:* For every positive integer n and every prime factor p of n, we have
$$\varphi_p\left(b_{n/p}\right) \equiv b_n \operatorname{mod} p^{v_p(n)} A.$$
Here, $v_p(n)$ denotes the exponent of p in the prime factorization of n.
- *Assertion $\mathcal{D}$:* There exists a family $(\alpha_n)_{n\geq 1} \in A^{\{1,2,3,\ldots\}}$ of elements of A such that every positive integer n satisfies
$$b_n = \sum_{d|n} d\alpha_d^{n/d}.$$
[180]
- *Assertion $\mathcal{E}$:* There exists a family $(\beta_n)_{n\geq 1} \in A^{\{1,2,3,\ldots\}}$ of elements of A such that every positive integer n satisfies
$$b_n = \sum_{d|n} d\varphi_{n/d}\left(\beta_d\right).$$
- *Assertion $\mathcal{F}$:* Every positive integer n satisfies
$$\sum_{d|n} \mu(d)\,\varphi_d\left(b_{n/d}\right) \in nA.$$
- *Assertion $\mathcal{G}$:* Every positive integer n satisfies
$$\sum_{d|n} \phi(d)\,\varphi_d\left(b_{n/d}\right) \in nA.$$
- *Assertion $\mathcal{H}$:* Every positive integer n satisfies
$$\sum_{i=1}^{n} \varphi_{n/\gcd(i,n)}\left(b_{\gcd(i,n)}\right) \in nA.$$
- *Assertion $\mathcal{J}$:* There exists a ring homomorphism $\Lambda_{\mathbb{Z}} \to A$ which, for every positive integer n, sends p_n to b_n.

[**Hint:** The following identities hold for every positive integer n:

$$\sum_{d|n} \phi(d) = n; \tag{2.9.6}$$

$$\sum_{d|n} \mu(d) = \delta_{n,1}; \tag{2.9.7}$$

$$\sum_{d|n} \mu(d)\frac{n}{d} = \phi(n); \tag{2.9.8}$$

$$\sum_{d|n} d\mu(d)\,\phi\left(\frac{n}{d}\right) = \mu(n). \tag{2.9.9}$$

Furthermore, the following simple lemma is useful: If k is a positive integer, and if $p \in \mathbb{N}$, $a \in A$ and $b \in A$ are such that $a \equiv b \operatorname{mod} p^k A$, then $a^{p^\ell} \equiv b^{p^\ell} \operatorname{mod} p^{k+\ell} A$ for every $\ell \in \mathbb{N}$.]

[180]Here and in the following, summations of the form $\sum_{d|n}$ range over all **positive** divisors of n.

Remark 2.9.7. Much of Exercise 2.9.6 is folklore, but it is hard to pinpoint concrete appearances in literature. The equivalence $\mathcal{C} \iff \mathcal{D}$ appears in Hesselholt [95, Lemma 1] and [96, Lemma 1.1] (in slightly greater generality), where it is referred to as Dwork's lemma and used in the construction of the Witt vector functor. This equivalence is also [90, Lemma 9.93]. The equivalence $\mathcal{D} \iff \mathcal{F} \iff \mathcal{G} \iff \mathcal{H}$ in the case $A = \mathbb{Z}$ is [57, Corollary on p. 10], where it is put into the context of Burnside rings and necklace counting. The equivalence $\mathcal{C} \iff \mathcal{F}$ for finite families $(b_n)_{n\in\{1,2,\ldots,m\}}$ in lieu of $(b_n)_{n\geq 1}$ is [206, Exercise 5.2 **a**]. One of the likely oldest relevant sources is Schur's [195], which proves the equivalence $\mathcal{C} \iff \mathcal{D} \iff \mathcal{F}$ for finite families $(b_n)_{n\in\{1,2,\ldots,m\}}$, as well as a "finite version" of $\mathcal{C} \iff \mathcal{J}$ (Schur did not have Λ, but was working with actual power sums of roots of polynomials).

Exercise 2.9.8. Let A denote the ring $\mathbb{Z}$. For every $n \in \{1, 2, 3, \ldots\}$, let φ_n denote the identity endomorphism id of A. Prove that the seven equivalent assertions $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{J}$ of Exercise 2.9.6 are satisfied for each of the following families $(b_n)_{n\geq 1} \in \mathbb{Z}^{\{1,2,3,\ldots\}}$:

- the family $(b_n)_{n\geq 1} = (q^n)_{n\geq 1}$, where q is a given integer.
- the family $(b_n)_{n\geq 1} = (q)_{n\geq 1}$, where q is a given integer.
- the family $(b_n)_{n\geq 1} = \left(\dbinom{qn}{rn}\right)_{n\geq 1}$, where $r \in \mathbb{Q}$ and $q \in \mathbb{Z}$ are given. (Here, a binomial coefficient $\dbinom{a}{b}$ has to be interpreted as 0 when $b \notin \mathbb{N}$.)
- the family $(b_n)_{n\geq 1} = \left(\dbinom{qn-1}{rn-1}\right)_{n\geq 1}$, where $r \in \mathbb{Z}$ and $q \in \mathbb{Z}$ are given.

Exercise 2.9.9. For every $n \in \{1, 2, 3, \ldots\}$, define a map $\mathbf{f}_n : \Lambda \to \Lambda$ by setting

$$\mathbf{f}_n(a) = a(x_1^n, x_2^n, x_3^n, \ldots) \qquad \text{for every } a \in \Lambda.$$

(So what $\mathbf{f}_n$ does to a symmetric function is replacing all variables $x_1, x_2, x_3, \ldots$ by their n-th powers.)

(a) Show that $\mathbf{f}_n : \Lambda \to \Lambda$ is a $\mathbf{k}$-algebra homomorphism for every $n \in \{1, 2, 3, \ldots\}$.
(b) Show that $\mathbf{f}_n \circ \mathbf{f}_m = \mathbf{f}_{nm}$ for any two positive integers n and m.
(c) Show that $\mathbf{f}_1 = \mathrm{id}$.
(d) Prove that $\mathbf{f}_n : \Lambda \to \Lambda$ is a Hopf algebra homomorphism for every $n \in \{1, 2, 3, \ldots\}$.
(e) Prove that $\mathbf{f}_2(h_m) = \sum_{i=0}^{2m} (-1)^i h_i h_{2m-i}$ for every $m \in \mathbb{N}$.
(f) Assume that $\mathbf{k} = \mathbb{Z}$. Prove that $\mathbf{f}_p(a) \equiv a^p \bmod p\Lambda$ for every $a \in \Lambda$ and every prime number p.
(g) Use Exercise 2.9.6 to obtain new solutions to parts (b), (c), (d), (e) and (f) of Exercise 2.9.4.

The maps $\mathbf{f}_n$ constructed in Exercise 2.9.9 are known as the *Frobenius endomorphisms* of Λ. They are a (deceptively) simple particular case of the notion of *plethysm* ([206, Chapter 7, Appendix 2] and [142, Section I.8]), and are often used as intermediate steps in computing more complicated plethysms[181].

Exercise 2.9.10. For every $n \in \{1, 2, 3, \ldots\}$, define a $\mathbf{k}$-algebra homomorphism $\mathbf{v}_n : \Lambda \to \Lambda$ by

$$\mathbf{v}_n (h_m) = \begin{cases} h_{m/n}, & \text{if } n \mid m; \\ 0, & \text{if } n \nmid m \end{cases} \qquad \text{for every positive integer } m$$

[182].

(a) Show that any positive integers n and m satisfy

$$\mathbf{v}_n (p_m) = \begin{cases} n p_{m/n}, & \text{if } n \mid m; \\ 0, & \text{if } n \nmid m \end{cases} .$$

(b) Show that any positive integers n and m satisfy

$$\mathbf{v}_n (e_m) = \begin{cases} (-1)^{m-m/n} e_{m/n}, & \text{if } n \mid m; \\ 0, & \text{if } n \nmid m \end{cases} .$$

(c) Prove that $\mathbf{v}_n \circ \mathbf{v}_m = \mathbf{v}_{nm}$ for any two positive integers n and m.
(d) Prove that $\mathbf{v}_1 = \mathrm{id}$.
(e) Prove that $\mathbf{v}_n : \Lambda \to \Lambda$ is a Hopf algebra homomorphism for every $n \in \{1, 2, 3, \ldots\}$.

Now, consider also the maps $\mathbf{f}_n : \Lambda \to \Lambda$ defined in Exercise 2.9.9. Fix a positive integer n.

(f) Prove that the maps $\mathbf{f}_n : \Lambda \to \Lambda$ and $\mathbf{v}_n : \Lambda \to \Lambda$ are adjoint with respect to the Hall inner product on Λ.
(g) Show that $\mathbf{v}_n \circ \mathbf{f}_n = \mathrm{id}_\Lambda^{\star n}$.
(h) Prove that $\mathbf{f}_n \circ \mathbf{v}_m = \mathbf{v}_m \circ \mathbf{f}_n$ whenever m is a positive integer coprime to n.

Finally, recall the $w_m \in \Lambda$ defined in Exercise 2.9.3.

(i) Show that any positive integer m satisfies

$$\mathbf{v}_n (w_m) = \begin{cases} w_{m/n}, & \text{if } n \mid m; \\ 0, & \text{if } n \nmid m \end{cases} .$$

The homomorphisms $\mathbf{v}_n : \Lambda \to \Lambda$ defined in Exercise 2.9.10 are called the *Verschiebung endomorphisms* of Λ; this name comes from German, where "Verschiebung" means "shift". This terminology, as well as that of Frobenius endomorphisms, originates in the theory of Witt vectors, and the connection between the Frobenius and Verschiebung endomorphisms of Λ and the identically named operators on Witt vectors is elucidated in [90, Chapter 13][183].

Exercise 2.9.11. Fix $n \in \mathbb{N}$. For any n-tuple $w = (w_1, w_2, \ldots, w_n)$ of integers, define the *descent set* $\mathrm{Des}(w)$ of w to be the set

$$\{i \in \{1, 2, \ldots, n-1\} : w_i > w_{i+1}\} .$$

[181] In the notations of [206, (A2.160)], the value $\mathbf{f}_n (a)$ for an $a \in \Lambda$ can be written as $a [p_n]$ or (when $\mathbf{k} = \mathbb{Z}$) as $p_n [a]$.

[182] This is well-defined, since the family $(h_m)_{m \geq 1}$ generates the $\mathbf{k}$-algebra Λ and is algebraically independent.

[183] which is also where most of the statements of Exercises 2.9.9 and 2.9.10 come from

(a) We say that an n-tuple $(w_1, w_2, \ldots, w_n)$ is *Smirnov* if every $i \in \{1, 2, \ldots, n-1\}$ satisfies $w_i \neq w_{i+1}$.

Fix $k \in \mathbb{N}$, and let $X_{n,k} \in \mathbf{k}[[\mathbf{x}]]$ denote the sum of the monomials $x_{w_1} x_{w_2} \cdots x_{w_n}$ over all Smirnov n-tuples $w = (w_1, w_2, \ldots, w_n) \in \{1, 2, 3, \ldots\}^n$ satisfying $|\operatorname{Des}(w)| = k$. Prove that $X_{n,k} \in \Lambda$.

(b) For any n-tuple $w = (w_1, w_2, \ldots, w_n)$, define the *stagnation set* $\operatorname{Stag}(w)$ of w to be the set $\{i \in \{1, 2, \ldots, n-1\} : w_i = w_{i+1}\}$. (Thus, an n-tuple is Smirnov if and only if its stagnation set is empty.)

For any $d \in \mathbb{N}$ and $s \in \mathbb{N}$, define a power series $X_{n,d,s} \in \mathbf{k}[[\mathbf{x}]]$ as the sum of the monomials $x_{w_1} x_{w_2} \cdots x_{w_n}$ over all n-tuples $w = (w_1, w_2, \ldots, w_n) \in \{1, 2, 3, \ldots\}^n$ satisfying $|\operatorname{Des}(w)| = d$ and $|\operatorname{Stag}(w)| = s$. Prove that $X_{n,d,s} \in \Lambda$ for any nonnegative integers d and s.

(c) Assume that n is positive. For any $d \in \mathbb{N}$ and $s \in \mathbb{N}$, define three further power series $U_{n,d,s}$, $V_{n,d,s}$ and $W_{n,d,s}$ in $\mathbf{k}[[\mathbf{x}]]$ by the following formulas:

$$U_{n,d,s} = \sum_{\substack{w=(w_1,w_2,\ldots,w_n)\in\{1,2,3,\ldots\}^n; \\ |\operatorname{Des}(w)|=d;\ |\operatorname{Stag}(w)|=s; \\ w_1<w_n}} x_{w_1} x_{w_2} \cdots x_{w_n}; \tag{2.9.10}$$

$$V_{n,d,s} = \sum_{\substack{w=(w_1,w_2,\ldots,w_n)\in\{1,2,3,\ldots\}^n; \\ |\operatorname{Des}(w)|=d;\ |\operatorname{Stag}(w)|=s; \\ w_1=w_n}} x_{w_1} x_{w_2} \cdots x_{w_n}; \tag{2.9.11}$$

$$W_{n,d,s} = \sum_{\substack{w=(w_1,w_2,\ldots,w_n)\in\{1,2,3,\ldots\}^n; \\ |\operatorname{Des}(w)|=d;\ |\operatorname{Stag}(w)|=s; \\ w_1>w_n}} x_{w_1} x_{w_2} \cdots x_{w_n}. \tag{2.9.12}$$

Prove that these three power series $U_{n,d,s}$, $V_{n,d,s}$ and $W_{n,d,s}$ belong to Λ.

Remark 2.9.12. The function $X_{n,k}$ in Exercise 2.9.11(a) is a simple example ([199, Example 2.5, Theorem C.3]) of a chromatic quasisymmetric function that happens to be symmetric. See Shareshian/Wachs [199] for more general criteria for such functions to be symmetric, as well as deeper results. For example, [199, Theorem 6.3] gives an expansion for a wide class of chromatic quasisymmetric functions in the Schur basis of Λ, which, in particular, shows that our $X_{n,k}$ satisfies

$$X_{n,k} = \sum_{\lambda \in \operatorname{Par}_n} a_{\lambda,k} s_\lambda,$$

where $a_{\lambda,k}$ is the number of all assignments T of entries in $\{1, 2, \ldots, n\}$ to the cells of the Ferrers diagram of λ such that the following four conditions are satisfied:

- Every element of $\{1, 2, \ldots, n\}$ is used precisely once in the assignment (i.e., we have $\operatorname{cont}(T) = (1^n)$).
- Whenever a cell y of the Ferrers diagram lies immediately to the right of a cell x, we have $T(y) - T(x) \geq 2$.
- Whenever a cell y of the Ferrers diagram lies immediately below a cell x, we have $T(y) - T(x) \geq -1$.

- There exist precisely k elements $i \in \{1, 2, \ldots, n-1\}$ such that the cell $T^{-1}(i)$ lies in a row below $T^{-1}(i+1)$.

Are there any such rules for the $X_{n,d,s}$ of part (b)?

Smirnov n-tuples are more usually called Smirnov words, or (occasionally) Carlitz words.

See [68, Chapter 6] for further properties of the symmetric functions $U_{n,d,0}$, $V_{n,d,0}$ and $W_{n,d,0}$ from Exercise 2.9.11(c) (or, more precisely, of their generating functions $\sum_d U_{n,d,0} t^d$ etc.).

Exercise 2.9.13. (a) Let $n \in \mathbb{N}$. Define a matrix $A_n = (a_{i,j})_{i,j=1,2,\ldots,n} \in \Lambda^{n \times n}$ by

$$a_{i,j} = \begin{cases} p_{i-j+1}, & \text{if } i \geq j; \\ i, & \text{if } i = j-1; \\ 0, & \text{if } i < j-1 \end{cases} \qquad \text{for all } (i,j) \in \{1, 2, \ldots, n\}^2 .$$

This matrix A_n looks as follows:

$$A_n = \begin{pmatrix} p_1 & 1 & 0 & \cdots & 0 & 0 \\ p_2 & p_1 & 2 & \cdots & 0 & 0 \\ p_3 & p_2 & p_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{n-1} & p_{n-2} & p_{n-3} & \cdots & p_1 & n-1 \\ p_n & p_{n-1} & p_{n-2} & \cdots & p_2 & p_1 \end{pmatrix}.$$

Show that $\det(A_n) = n! e_n$.

(b) Let n be a positive integer. Define a matrix $B_n = (b_{i,j})_{i,j=1,2,\ldots,n} \in \Lambda^{n \times n}$ by

$$b_{i,j} = \begin{cases} i e_i, & \text{if } j = 1; \\ e_{i-j+1}, & \text{if } j > 1 \end{cases} \qquad \text{for all } (i,j) \in \{1, 2, \ldots, n\}^2 .$$

The matrix B_n looks as follows:

$$B_n = \begin{pmatrix} e_1 & e_0 & e_{-1} & \cdots & e_{-n+3} & e_{-n+2} \\ 2e_2 & e_1 & e_0 & \cdots & e_{-n+4} & e_{-n+3} \\ 3e_3 & e_2 & e_1 & \cdots & e_{-n+5} & e_{-n+4} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (n-1) e_{n-1} & e_{n-2} & e_{n-3} & \cdots & e_1 & e_0 \\ n e_n & e_{n-1} & e_{n-2} & \cdots & e_2 & e_1 \end{pmatrix}$$
$$= \begin{pmatrix} e_1 & 1 & 0 & \cdots & 0 & 0 \\ 2e_2 & e_1 & 1 & \cdots & 0 & 0 \\ 3e_3 & e_2 & e_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (n-1) e_{n-1} & e_{n-2} & e_{n-3} & \cdots & e_1 & 1 \\ n e_n & e_{n-1} & e_{n-2} & \cdots & e_2 & e_1 \end{pmatrix}.$$

Show that $\det(B_n) = p_n$.

The formulas of Exercise 2.9.13, for finitely many variables, appear in Prasolov's [171, §4.1][184]. In [171, §4.2], Prasolov gives four more formulas,

[184] where our symmetric functions e_k, h_k, p_k, evaluated in finitely many indeterminates, are denoted σ_k, p_k, s_k, respectively

which express e_n as a polynomial in the $h_1, h_2, h_3, \ldots$, or h_n as a polynomial in the $e_1, e_2, e_3, \ldots$, or p_n as a polynomial in the $h_1, h_2, h_3, \ldots$, or $n! h_n$ as a polynomial in the $p_1, p_2, p_3, \ldots$. These are not novel for us, since the first two of them are particular cases of Theorem 2.4.6, whereas the latter two can be derived from Exercise 2.9.13 by applying ω. (Note that ω is only well-defined on symmetric functions in infinitely many indeterminates, so we need to apply ω **before** evaluating at finitely many indeterminates; this explains why Prasolov has to prove the latter two identities separately.)

Exercise 2.9.14. In the following, if $k \in \mathbb{N}$, we shall use the notation 1^k for $\underbrace{1, 1, \ldots, 1}_{k \text{ times}}$ (in contexts such as $(n, 1^m)$). So, for example, $(3, 1^4)$ is the partition $(3, 1, 1, 1, 1)$.

(a) Show that $e_n h_m = s_{(m+1,1^{n-1})} + s_{(m,1^n)}$ for any two positive integers n and m.

(b) Show that

$$\sum_{i=0}^{b} (-1)^i h_{a+i+1} e_{b-i} = s_{\left(a+1,1^b\right)}$$

for any $a \in \mathbb{N}$ and $b \in \mathbb{N}$.

(c) Show that

$$\sum_{i=0}^{b} (-1)^i h_{a+i+1} e_{b-i} = (-1)^b \delta_{a+b,-1}$$

for any negative integer a and every $b \in \mathbb{N}$. (As usual, we set $h_j = 0$ for $j < 0$ here.)

(d) Show that

$$\begin{aligned} \Delta s_{\left(a+1,1^b\right)} = 1 \otimes s_{\left(a+1,1^b\right)} &+ s_{\left(a+1,1^b\right)} \otimes 1 \\ &+ \sum_{\substack{(c,d,e,f) \in \mathbb{N}^4; \\ c+e=a-1; \\ d+f=b}} s_{\left(c+1,1^d\right)} \otimes s_{\left(e+1,1^f\right)} \\ &+ \sum_{\substack{(c,d,e,f) \in \mathbb{N}^4; \\ c+e=a; \\ d+f=b-1}} s_{\left(c+1,1^d\right)} \otimes s_{\left(e+1,1^f\right)} \end{aligned}$$

for any $a \in \mathbb{N}$ and $b \in \mathbb{N}$.

Our next few exercises survey some results on Littlewood-Richardson coefficients.

Exercise 2.9.15. Let $m \in \mathbb{N}$ and $k \in \mathbb{N}$. Let λ and μ be two partitions such that $\ell(\lambda) \leq k$ and $\ell(\mu) \leq k$. Assume that all parts of λ and all parts of μ are $\leq m$. (It is easy to see that this assumption is equivalent to requiring $\lambda_i \leq m$ and $\mu_i \leq m$ for every positive integer i. [185]). Let $\lambda^\vee$ and $\mu^\vee$ denote the k-tuples $(m - \lambda_k, m - \lambda_{k-1}, \ldots, m - \lambda_1)$ and $(m - \mu_k, m - \mu_{k-1}, \ldots, m - \mu_1)$, respectively.

(a) Show that $\lambda^\vee$ and $\mu^\vee$ are partitions, and that $s_{\lambda/\mu} = s_{\mu^\vee/\lambda^\vee}$.

(b) Show that $c^{\lambda}_{\mu,\nu} = c^{\mu^\vee}_{\lambda^\vee,\nu}$ for any partition ν.

[185] As usual, we are denoting by ν_i the i-th entry of a partition ν here.

(c) Let ν be a partition such that $\ell(\nu) \leq k$, and such that all parts of ν are $\leq m$. Let $\nu^\vee$ denote the k-tuple $(m-\nu_k, m-\nu_{k-1}, \ldots, m-\nu_1)$. Show that $\nu^\vee$ is a partition, and satisfies

$$c^\lambda_{\mu,\nu} = c^\lambda_{\nu,\mu} = c^{\mu^\vee}_{\lambda^\vee,\nu} = c^{\mu^\vee}_{\nu,\lambda^\vee} = c^{\nu^\vee}_{\mu,\lambda^\vee} = c^{\nu^\vee}_{\lambda^\vee,\mu}.$$

(d) Show that

$$s_{\lambda^\vee}(x_1, x_2, \ldots, x_k) = (x_1 x_2 \cdots x_k)^m \cdot s_\lambda\left(x_1^{-1}, x_2^{-1}, \ldots, x_k^{-1}\right)$$

in the Laurent polynomial ring $\mathbf{k}\left[x_1, x_2, \ldots, x_k, x_1^{-1}, x_2^{-1}, \ldots, x_k^{-1}\right]$.

(e) Let r be a nonnegative integer. Show that $(r+\lambda_1, r+\lambda_2, \ldots, r+\lambda_k)$ is a partition and satisfies

$$s_{(r+\lambda_1, r+\lambda_2, \ldots, r+\lambda_k)}(x_1, x_2, \ldots, x_k) = (x_1 x_2 \cdots x_k)^r \cdot s_\lambda(x_1, x_2, \ldots, x_k)$$

in the polynomial ring $\mathbf{k}[x_1, x_2, \ldots, x_k]$.

Exercise 2.9.16. Let $m \in \mathbb{N}$, $n \in \mathbb{N}$ and $k \in \mathbb{N}$. Let μ and ν be two partitions such that $\ell(\mu) \leq k$ and $\ell(\nu) \leq k$. Assume that all parts of μ are $\leq m$ (that is, $\mu_i \leq m$ for every positive integer i) [186], and that all parts of ν are $\leq n$ (that is, $\nu_i \leq n$ for every positive integer i). Let $\mu^{\vee\{m\}}$ denote the k-tuple $(m-\mu_k, m-\mu_{k-1}, \ldots, m-\mu_1)$, and let $\nu^{\vee\{n\}}$ denote the k-tuple $(n-\nu_k, n-\nu_{k-1}, \ldots, n-\nu_1)$.

(a) Show that $\mu^{\vee\{m\}}$ and $\nu^{\vee\{n\}}$ are partitions.

Now, let λ be a further partition such that $\ell(\lambda) \leq k$.

(b) If not all parts of λ are $\leq m+n$, then show that $c^\lambda_{\mu,\nu} = 0$.

(c) If all parts of λ are $\leq m+n$, then show that $c^\lambda_{\mu,\nu} = c^{\lambda^{\vee\{m+n\}}}_{\mu^{\vee\{m\}},\nu^{\vee\{n\}}}$, where $\lambda^{\vee\{m+n\}}$ denotes the k-tuple

$$(m+n-\lambda_k, m+n-\lambda_{k-1}, \ldots, m+n-\lambda_1).$$

The results of Exercise 2.7.11(c) and Exercise 2.9.15(c) are two *symmetries of Littlewood-Richardson coefficients*[187]; combining them yields further such symmetries. While these symmetries were relatively easy consequences of our algebraic definition of the Littlewood-Richardson coefficients, it is a much more challenging task to derive them bijectively from a combinatorial definition of these coefficients (such as the one given in Corollary 2.6.12). Some such derivations appear in [218], in [11], in [16, Example 3.6, Proposition 5.11 and references therein], [73, §5.1, §A.1, §A.4] and [109, (2.12)] (though a different combinatorial interpretation of $c^\lambda_{\mu,\nu}$ is used in the latter three).

Exercise 2.9.17. Recall our usual notations: For every partition λ and every positive integer i, the i-th entry of λ is denoted by λ_i. The sign $\triangleright$ stands for dominance order. We let λ^t denote the conjugate partition of a partition λ.

For any two partitions μ and ν, we define two new partitions $\mu+\nu$ and $\mu \sqcup \nu$ of $|\mu|+|\nu|$ as follows:

- The partition $\mu+\nu$ is defined as $(\mu_1+\nu_1, \mu_2+\nu_2, \mu_3+\nu_3, \ldots)$.

[186] As usual, we are denoting by ν_i the i-th entry of a partition ν here.

[187] The result of Exercise 2.9.16(c) can also be regarded as a symmetry of Littlewood-Richardson coefficients; see [10, §3.3].

- The partition $\mu \sqcup \nu$ is defined as the result of sorting the list
$$\left(\mu_1, \mu_2, \ldots, \mu_{\ell(\mu)}, \nu_1, \nu_2, \ldots, \nu_{\ell(\nu)}\right)$$
in decreasing order.

(a) Show that any two partitions μ and ν satisfy $(\mu + \nu)^t = \mu^t \sqcup \nu^t$ and $(\mu \sqcup \nu)^t = \mu^t + \nu^t$.

(b) Show that any two partitions μ and ν satisfy $c^{\mu+\nu}_{\mu,\nu} = 1$ and $c^{\mu\sqcup\nu}_{\mu,\nu} = 1$.

(c) If $k \in \mathbb{N}$ and $n \in \mathbb{N}$ satisfy $k \leq n$, and if $\mu \in \operatorname{Par}_k$, $\nu \in \operatorname{Par}_{n-k}$ and $\lambda \in \operatorname{Par}_n$ are such that $c^{\lambda}_{\mu,\nu} \neq 0$, then prove that $\mu + \nu \rhd \lambda \rhd \mu \sqcup \nu$.

(d) If $n \in \mathbb{N}$ and $m \in \mathbb{N}$ and $\alpha, \beta \in \operatorname{Par}_n$ and $\gamma, \delta \in \operatorname{Par}_m$ are such that $\alpha \rhd \beta$ and $\gamma \rhd \delta$, then show that $\alpha + \gamma \rhd \beta + \delta$ and $\alpha \sqcup \gamma \rhd \beta \sqcup \delta$.

(e) Let $m \in \mathbb{N}$ and $k \in \mathbb{N}$, and let λ be the partition
$$\left(m^k\right) = \left(\underbrace{m, m, \ldots, m}_{k \text{ times}}\right).$$
Show that any two partitions μ and ν satisfy $c^{\lambda}_{\mu,\nu} \in \{0, 1\}$.

(f) Let $a \in \mathbb{N}$ and $b \in \mathbb{N}$, and let λ be the partition $\left(a + 1, 1^b\right)$ (using the notation of Exercise 2.9.14). Show that any two partitions μ and ν satisfy $c^{\lambda}_{\mu,\nu} \in \{0, 1\}$.

(g) If λ is any partition, and if μ and ν are two rectangular partitions[188], then show that $c^{\lambda}_{\mu,\nu} \in \{0, 1\}$.

Exercise 2.9.17(g) is part of Stembridge's [211, Thm. 2.1]; we refer to that article for further results of its kind.

The Littlewood-Richardson rule comes in many different forms, whose equivalence is not always immediate. Our version (Corollary 2.6.12) has the advantage of being the simplest to prove and one of the simplest to state. Other versions can be found in [206, appendix 1 to Ch. 7], Fulton's [73, Ch. 5] and van Leeuwen's [129]. We restrict ourselves to proving some very basic equivalences that allow us to restate parts of Corollary 2.6.12:

Exercise 2.9.18. We shall use the following notations:

- If T is a column-strict tableau and j is a positive integer, then we use the notation $T|_{\mathrm{cols}\geq j}$ for the restriction of T to the union of its columns $j, j+1, j+2, \ldots$. (This notation has already been used in Section 2.6.)
- If T is a column-strict tableau and S is a set of cells of T, then we write $T|_S$ for the restriction of T to the set S of cells.[189]
- If T is a column-strict tableau, then an *NE-set* of T means a set S of cells of T such that whenever $s \in S$, every cell of T which lies northeast[190] of s must also belong to S.

[188]A partition is called *rectangular* if it has the form $\left(m^k\right) = \left(\underbrace{m, m, \ldots, m}_{k \text{ times}}\right)$ for some $m \in \mathbb{N}$ and $k \in \mathbb{N}$.

[189]This restriction $T|_S$ is not necessarily a tableau of skew shape; it is just a map from S to $\{1, 2, 3, \ldots\}$. The content $\operatorname{cont}(T|_S)$ is nevertheless well-defined (in the usual way: $(\operatorname{cont}(T|_S))_i = \left|(T|_S)^{-1}(i)\right|$).

[190]A cell (r, c) is said to lie *northeast* of a cell (r', c') if and only if we have $r \leq r'$ and $c \geq c'$.

- The *Semitic reading word*[191] of a column-strict tableau T is the concatenation[192] $r_1 r_2 r_3 \cdots$, where r_i is the word obtained by reading the i-th row of T from right to left.[193]
- If $w = (w_1, w_2, \ldots, w_n)$ is a word, then a *prefix* of w means a word of the form $(w_1, w_2, \ldots, w_i)$ for some $i \in \{0, 1, \ldots, n\}$. (In particular, both w and the empty word are prefixes of w.)

  A word w over the set of positive integers is said to be *Yamanouchi* if for any prefix v of w and any positive integer i, there are at least as many i's among the letters of v as there are $(i+1)$'s among them.[194]

Prove the following two statements:

(a) Let μ be a partition. Let $b_{i,j}$ be a nonnegative integer for every two positive integers i and j. Assume that $b_{i,j} = 0$ for all but finitely many pairs (i, j).

The following two assertions are equivalent:

- *Assertion $\mathcal{A}$:* There exist a partition λ and a column-strict tableau T of shape λ/μ such that all $(i, j) \in \{1, 2, 3, \ldots\}^2$ satisfy

$$b_{i,j} = (\text{the number of all entries } i \text{ in the } j\text{-th row of } T)\,. \tag{2.9.13}$$

- *Assertion $\mathcal{B}$:* The inequality

$$\begin{aligned} &\mu_{j+1} + (b_{1,j+1} + b_{2,j+1} + \cdots + b_{i+1,j+1}) \\ &\leq \mu_j + (b_{1,j} + b_{2,j} + \cdots + b_{i,j}) \end{aligned} \tag{2.9.14}$$

  holds for all $(i, j) \in \mathbb{N} \times \{1, 2, 3, \ldots\}$.

(b) Let λ and μ be two partitions, and let T be a column-strict tableau of shape λ/μ. Then, the following five assertions are equivalent:

- *Assertion $\mathcal{C}$:* For every positive integer j, the weak composition $\operatorname{cont}(T|_{\operatorname{cols} \geq j})$ is a partition.

---

[191] The notation comes from [129] and is a reference to the Arabic and Hebrew way of writing.

[192] If $s_1, s_2, s_3, \ldots$ are several words (finitely or infinitely many), then the *concatenation* $s_1 s_2 s_3 \cdots$ is defined as the word which is obtained by starting with the empty word, then appending s_1 to its end, then appending s_2 to the end of the result, then appending s_3 to the end of the result, etc.

[193] For example, the Semitic reading word of the tableau

$$\begin{array}{cccc} & 3 & 4 & 4 & 5 \\ 1 & 4 & 6 & & \\ 3 & 5 & & & \end{array}$$

is 544364153.

The Semitic reading word of a tableau T is what is called the *reverse reading word* of T in [206, §A.1.3].

[194] For instance, the words 11213223132 and 1213 are Yamanouchi, while the words 132, 21 and 1121322332111 are not. The Dyck words (defined as in [206, Example 6.6.6], and written using 1's and 2's instead of x's and y's) are precisely the Yamanouchi words whose letters are 1's and 2's and in which the letter 1 appears as often as the letter 2.

Yamanouchi words are often called lattice permutations.

- *Assertion $\mathcal{D}$:* For every positive integers j and i, the number of entries $i+1$ in the first j rows[195] of T is $\leq$ to the number of entries i in the first $j-1$ rows of T.
- *Assertion $\mathcal{E}$:* For every NE-set S of T, the weak composition $\operatorname{cont}(T|_S)$ is a partition.
- *Assertion $\mathcal{F}$:* The Semitic reading word of T is Yamanouchi.
- *Assertion $\mathcal{G}$:* There exists a column-strict tableau S whose shape is a partition and which satisfies the following property: For any positive integers i and j, the number of entries i in the j-th row of T equals the number of entries j in the i-th row of S.

Remark 2.9.19. The equivalence of Assertions $\mathcal{C}$ and $\mathcal{F}$ in Exercise 2.9.18(b) is the "not-too-difficult exercise" mentioned in [210]. It yields the equivalence between our version of the Littlewood-Richardson rule (Corollary 2.6.12) and that in [206, A1.3.3].

In the next exercises, we shall restate Corollary 2.6.11 in a different form. While Corollary 2.6.11 provided a decomposition of the product of a skew Schur function with a Schur function into a sum of Schur functions, the different form that we will encounter in Exercise 2.9.21(b) will give a combinatorial interpretation for the Hall inner product between two skew Schur functions. Let us first generalize Exercise 2.9.18(b):

Exercise 2.9.20. Let us use the notations of Exercise 2.9.18. Let κ, λ and μ be three partitions, and let T be a column-strict tableau of shape λ/μ.

(a) Prove that the following five assertions are equivalent:
   - *Assertion $\mathcal{C}^{(\kappa)}$:* For every positive integer j, the weak composition $\kappa + \operatorname{cont}(T|_{\mathrm{cols}\geq j})$ is a partition.
   - *Assertion $\mathcal{D}^{(\kappa)}$:* For every positive integers j and i, we have

$$\begin{aligned}&\kappa_{i+1} + (\text{the number of entries } i+1 \text{ in the first } j \text{ rows of } T)\\ &\leq \kappa_i + (\text{the number of entries } i \text{ in the first } j-1 \text{ rows of } T)\,.\end{aligned}$$

   - *Assertion $\mathcal{E}^{(\kappa)}$:* For every NE-set S of T, the weak composition $\kappa + \operatorname{cont}(T|_S)$ is a partition.
   - *Assertion $\mathcal{F}^{(\kappa)}$:* For every prefix v of the Semitic reading word of T, and for every positive integer i, we have

$$\begin{aligned}&\kappa_i + (\text{the number of } i\text{'s among the letters of } v)\\ &\geq \kappa_{i+1} + (\text{the number of } (i+1)\text{'s among the letters of } v)\,.\end{aligned}$$

   - *Assertion $\mathcal{G}^{(\kappa)}$:* There exist a partition ζ and a column-strict tableau S of shape ζ/κ which satisfies the following property: For any positive integers i and j, the number of entries i in the j-th row of T equals the number of entries j in the i-th row of S.

(b) Let τ be a partition such that $\tau = \kappa + \operatorname{cont} T$. Consider the five assertions $\mathcal{C}^{(\kappa)}$, $\mathcal{D}^{(\kappa)}$, $\mathcal{E}^{(\kappa)}$, $\mathcal{F}^{(\kappa)}$ and $\mathcal{G}^{(\kappa)}$ introduced in Exercise 2.9.20(a). Let us also consider the following assertion:

[195]The "first j rows" mean the 1-st row, the 2-nd row, etc., the j-th row (even if some of these rows are empty).

– *Assertion* $\mathcal{H}^{(\kappa)}$*:* There exists a column-strict tableau S of shape τ/κ which satisfies the following property: For any positive integers i and j, the number of entries i in the j-th row of T equals the number of entries j in the i-th row of S.

Prove that the six assertions $\mathcal{C}^{(\kappa)}$, $\mathcal{D}^{(\kappa)}$, $\mathcal{E}^{(\kappa)}$, $\mathcal{F}^{(\kappa)}$, $\mathcal{G}^{(\kappa)}$ and $\mathcal{H}^{(\kappa)}$ are equivalent.

Clearly, Exercise 2.9.18(b) is the particular case of Exercise 2.9.20 when $\kappa = \varnothing$.

Using Exercise 2.9.20, we can restate Corollary 2.6.11 in several ways:

Exercise 2.9.21. Let λ, μ and κ be three partitions.

(a) Show that
$$s_\kappa s_{\lambda/\mu} = \sum_T s_{\kappa+\operatorname{cont} T},$$
where the sum ranges over all column-strict tableaux T of shape λ/μ satisfying the five equivalent assertions $\mathcal{C}^{(\kappa)}$, $\mathcal{D}^{(\kappa)}$, $\mathcal{E}^{(\kappa)}$, $\mathcal{F}^{(\kappa)}$ and $\mathcal{G}^{(\kappa)}$ introduced in Exercise 2.9.20(a).

(b) Let τ be a partition. Show that $\left(s_{\lambda/\mu}, s_{\tau/\kappa}\right)_\Lambda$ is the number of all column-strict tableaux T of shape λ/μ satisfying $\tau = \kappa + \operatorname{cont} T$ and also satisfying the six equivalent assertions $\mathcal{C}^{(\kappa)}$, $\mathcal{D}^{(\kappa)}$, $\mathcal{E}^{(\kappa)}$, $\mathcal{F}^{(\kappa)}$, $\mathcal{G}^{(\kappa)}$ and $\mathcal{H}^{(\kappa)}$ introduced in Exercise 2.9.20.

Exercise 2.9.21(a) is merely Corollary 2.6.11, rewritten in light of Exercise 2.9.20. Various parts of it appear in the literature. For instance, [126, (53)] easily reveals to be a restatement of the fact that $s_\kappa s_{\lambda/\mu} = \sum_T s_{\nu+\operatorname{cont} T}$, where the sum ranges over all column-strict tableaux T of shape λ/μ satisfying Assertion $\mathcal{D}^{(\kappa)}$.

Exercise 2.9.21(b) is one version of a "skew Littlewood-Richardson rule" that goes back to Zelevinsky [228] (although Zelevinsky's version uses both a different language and a combinatorial interpretation which is not obviously equivalent to ours). It appears in various sources; for instance, [126, Theorem 5.2, second formula] says that $\left(s_{\lambda/\mu}, s_{\tau/\kappa}\right)_\Lambda$ is the number of all column-strict tableaux T of shape λ/μ satisfying $\tau = \kappa + \operatorname{cont} T$ and the assertion $\mathcal{H}^{(\kappa)}$, whereas [75, Theorem 1.2] says that $\left(s_{\lambda/\mu}, s_{\tau/\kappa}\right)_\Lambda$ is the number of all all column-strict tableaux T of shape λ/μ satisfying $\tau = \kappa + \operatorname{cont} T$ and the assertion $\mathcal{F}^{(\kappa)}$. (Notice that Gasharov's proof of [75, Theorem 1.2] uses the same involutions as Stembridge's proof of Theorem 2.6.6; it can thus be regarded as a close precursor to Stembridge's proof. However, it uses the Jacobi-Trudi identities, while Stembridge's does not.)

Exercise 2.9.22. Let $\mathbb{K}$ be a field.[196] If $N \in \mathbb{K}^{n\times n}$ is a nilpotent matrix, then the *Jordan type* of N is defined to be the list of the sizes of the Jordan blocks in the Jordan normal form of N, sorted in decreasing order[197]. This Jordan type is a partition of n, and uniquely determines N up to similarity (i.e., two nilpotent $n \times n$-matrices N and N' are similar if and only if the Jordan types of N and N' are equal). If f is a nilpotent endomorphism of a

[196]This field has no relation to the ring $\mathbf{k}$, over which our symmetric functions are defined.

[197]The Jordan normal form of N is well-defined even if $\mathbb{K}$ is not algebraically closed, because N is nilpotent (so the characteristic polynomial of N is X^n).

finite-dimensional $\mathbb{K}$-vector space V, then we define the *Jordan type* of f as the Jordan type of any matrix representing f (the choice of the matrix does not matter, since the Jordan type of a matrix remains unchanged under conjugation).

(a) Let $n \in \mathbb{N}$. Let $N \in \mathbb{K}^{n \times n}$ be a nilpotent matrix. Let $\lambda \in \operatorname{Par}_n$. Show that the matrix N has Jordan type λ if and only if every $k \in \mathbb{N}$ satisfies

$$\dim \left(\ker \left(N^k\right)\right) = \left(\lambda^t\right)_1 + \left(\lambda^t\right)_2 + \ldots + \left(\lambda^t\right)_k .$$

(Here, we are using the notation λ^t for the transpose of a partition λ, and the notation ν_i for the i-th entry of a partition ν.)

(b) Let f be a nilpotent endomorphism of a finite-dimensional $\mathbb{K}$-vector space V. Let U be an f-stable $\mathbb{K}$-vector subspace of V (that is, a $\mathbb{K}$-vector subspace of V satisfying $f(U) \subset U$). Then, restricting f to U gives a nilpotent endomorphism $f \mid U$ of U, and the endomorphism f also induces a nilpotent endomorphism $\overline{f}$ of the quotient space V/U. Let λ, μ and ν be the Jordan types of f, $f \mid U$ and $\overline{f}$, respectively. Show that $c^{\lambda}_{\mu,\nu} \neq 0$ (if $\mathbb{Z}$ is a subring of $\mathbf{k}$).

[**Hint:** For (b), Exercise 2.7.11(c) shows that it is enough to prove that $c^{\lambda^t}_{\mu^t,\nu^t} \neq 0$. Due to Corollary 2.6.12, this only requires constructing a column-strict tableau T of shape λ^t/μ^t with $\operatorname{cont} T = \nu^t$ which has the property that each $\operatorname{cont}(T|_{\text{cols}\geq j})$ is a partition. Construct this tableau by defining $a_{i,j} = \dim\left(\left(f^i\right)^{-1}(U) \cap \ker\left(f^j\right)\right)$ for all $(i,j) \in \mathbb{N}^2$, and requiring that the number of entries i in the j-th row of T be $a_{i,j} - a_{i,j-1} - a_{i-1,j} + a_{i-1,j-1}$ for all $(i,j) \in \{1,2,3,\ldots\}^2$. Use Exercise 2.9.18(a) to prove that this indeed defines a column-strict tableau, and Exercise 2.9.18(b) to verify that it satisfies the condition on $\operatorname{cont}(T|_{\text{cols}\geq j})$.]

Remark 2.9.23. Exercise 2.9.22 is a taste of the connections between the combinatorics of partitions and the Jordan normal form. Much more can, and has, been said. Marc van Leeuwen's [127] is dedicated to some of these connections; in particular, our Exercise 2.9.22(a) is [127, Proposition 1.1], and a far stronger version of Exercise 2.9.22(b) appears in [127, Theorem 4.3 (2)], albeit only for the case of an infinite $\mathbb{K}$. One can prove a converse to Exercise 2.9.22(b) as well: If $c^{\lambda}_{\mu,\nu} \neq 0$, then there exist V, f and U satisfying the premises of Exercise 2.9.22(b). When $\mathbb{K}$ is a finite field, we can ask enumerative questions, such as how many U's are there for given V, f, λ, μ and ν; we will see a few answers in Section 4.9 (specifically, Proposition 4.9.4), and a more detailed treatment is given in [142, Ch. 2].

The relationship between partitions and Jordan normal forms can be exploited to provide linear-algebraic proofs of purely combinatorial facts. See [28, Sections 6 and 9] for some examples. Note that [28, Lemma 9.10] is the statement that, under the conditions of Exercise 2.9.22(b), we have $\nu \subseteq \lambda$. This is a direct consequence of Exercise 2.9.22(b) (since $c^{\lambda}_{\mu,\nu} \neq 0$ can happen only if $\nu \subseteq \lambda$).

Exercise 2.9.24. Let $a \in \Lambda$. Prove the following:

(a) The set $\left\{ g \in \Lambda \;\mid\; g^{\perp} a = (\omega(g))^{\perp} a \right\}$ is a $\mathbf{k}$-subalgebra of Λ.

(b) Assume that $e_k^\perp a = h_k^\perp a$ for each positive integer k. Then, $g^\perp a = (\omega(g))^\perp a$ for each $g \in \Lambda$.

Exercise 2.9.25. Let $n \in \mathbb{N}$. Let ρ be the partition $(n-1, n-2, \ldots, 1)$. Prove that $s_{\rho/\mu} = s_{\rho/\mu^t}$ for every $\mu \in \operatorname{Par}$.

Remark 2.9.26. Exercise 2.9.25 appears in [180, Corollary 7.32], and is due to John Stembridge. Using Remark 2.5.9, we can rewrite it as yet another equality between Littlewood-Richardson coefficients: Namely, $c^\rho_{\mu,\nu} = c^\rho_{\mu^t,\nu}$ for any $\mu \in \operatorname{Par}$ and $\nu \in \operatorname{Par}$.

3. Zelevinsky's structure theory of positive self-dual Hopf algebras

Chapter 2 showed that, as a $\mathbb{Z}$-basis for the Hopf algebra $\Lambda = \Lambda_{\mathbb{Z}}$, the Schur functions $\{s_\lambda\}$ have two special properties: they have the *same* structure constants $c^\lambda_{\mu,\nu}$ for their multiplication as for their comultiplication (Corollary 2.5.7), and these structure constants are all *nonnegative* integers (Corollary 2.6.12). Zelevinsky [227, §2,3] isolated these two properties as crucial.

Definition 3.0.1. Say that a connected graded Hopf algebra A over $\mathbf{k} = \mathbb{Z}$ with a distinguished $\mathbb{Z}$-basis $\{\sigma_\lambda\}$ consisting of homogeneous elements[198] is a *positive self-dual Hopf algebra* (or *PSH*) if it satisfies the two further axioms

- **(self-duality)** The same structure constants $a^\lambda_{\mu,\nu}$ appear for the product $\sigma_\mu \sigma_\nu = \sum_\lambda a^\lambda_{\mu,\nu} \sigma_\lambda$ and the coproduct $\Delta \sigma_\lambda = \sum_{\mu,\nu} a^\lambda_{\mu,\nu} \sigma_\mu \otimes \sigma_\nu$.
- **(positivity)** The $a^\lambda_{\mu,\nu}$ are all nonnegative (integers).

Call $\{\sigma_\lambda\}$ the *PSH-basis* of A.

He then developed a beautiful structure theory for PSH's, explaining how they can be uniquely expressed as tensor products of copies of PSH's each isomorphic to Λ after rescaling their grading. The next few sections explain this, following his exposition closely.

3.1. Self-duality implies polynomiality.

We begin with a property that forces a Hopf algebra to have algebra structure which is a *polynomial* algebra, specifically the symmetric algebra $\operatorname{Sym}(\mathfrak{p})$, where $\mathfrak{p}$ is the $\mathbf{k}$-submodule of primitive elements.

Recall from Exercise 1.3.20(g) that for a connected graded Hopf algebra $A = \bigoplus_{n=0}^\infty A_n$, every x in the two-sided ideal $I := \ker \epsilon = \bigoplus_{n>0} A_n$ has the property that its comultiplication takes the form

$$\Delta(x) = 1 \otimes x + x \otimes 1 + \Delta_+(x)$$

where $\Delta_+(x)$ lies in $I \otimes I$. Recall also that the elements x for which $\Delta_+(x) = 0$ are called the *primitives*. Denote by $\mathfrak{p}$ the $\mathbf{k}$-submodule of primitive elements inside A.

Given a PSH A (over $\mathbf{k} = \mathbb{Z}$) with a PSH-basis $\{\sigma_\lambda\}$, we consider the bilinear form $(\cdot,\cdot)_A : A \times A \to \mathbb{Z}$ on A that makes this basis orthonormal. Similarly, the elements $\{\sigma_\lambda \otimes \sigma_\mu\}$ give an orthonormal basis for a form $(\cdot,\cdot)_{A\otimes A}$ on $A \otimes A$. The bilinear form $(\cdot,\cdot)_A$ on the PSH A gives rise to a $\mathbb{Z}$-linear map $A \to A^o$, which is easily seen to be injective and a $\mathbb{Z}$-algebra homomorphism. We thus identify A with a subalgebra of A^o. When A is of finite type, this map is a Hopf algebra isomorphism, thus allowing us to identify A with A^o. This is an instance of the following notion of self-duality.

Definition 3.1.1. (a) If $(\cdot,\cdot) : V \times W \to \mathbf{k}$ is a bilinear form on the product $V \times W$ of two graded $\mathbf{k}$-modules $V = \bigoplus_{n\geq 0} V_n$ and $W = \bigoplus_{n\geq 0} W_n$, then we say that this form $(\cdot,\cdot)$ is *graded* if every two distinct nonnegative integers n and m satisfy $(V_n, W_m) = 0$ (that

[198] not necessarily indexed by partitions

is, if every two homogeneous elements $v \in V$ and $w \in W$ having distinct degrees satisfy $(v, w) = 0$).

(b) If $(\cdot, \cdot)_V : V \times V \to \mathbf{k}$ and $(\cdot, \cdot)_W : W \times W \to \mathbf{k}$ are two symmetric bilinear forms on some $\mathbf{k}$-modules V and W, then we can canonically define a symmetric bilinear form $(\cdot, \cdot)_{V \otimes W}$ on the $\mathbf{k}$-module $V \otimes W$ by letting

$$(v \otimes w, v' \otimes w')_{V \otimes W} = (v, v')_V (w, w')_W \qquad \text{for all } v, v' \in V \text{ and } w, w' \in W.$$

This new bilinear form is graded if the original two forms $(\cdot, \cdot)_V$ and $(\cdot, \cdot)_W$ were graded (presuming that V and W are graded).

(c) Say that a bialgebra A is *self-dual* with respect to a given symmetric bilinear form $(\cdot, \cdot) : A \times A \to \mathbf{k}$ if one has $(a, m(b \otimes c))_A = (\Delta(a), b \otimes c)_{A \otimes A}$ and $(1_A, a) = \epsilon(a)$ for a, b, c in A. If A is a graded Hopf algebra of finite type, and this form $(\cdot, \cdot)$ is graded, then this is equivalent to the $\mathbf{k}$-module map $A \to A^o$ induced by $(\cdot, \cdot)_A$ giving a Hopf algebra homomorphism.

Thus, any PSH A is self-dual with respect to the bilinear form $(\cdot, \cdot)_A$ that makes its PSH-basis orthonormal.

Notice also that the injective $\mathbb{Z}$-algebra homomorphism $A \to A^o$ obtained from the bilinear form $(\cdot, \cdot)_A$ on a PSH A allows us to regard each $f \in A$ as an element of A^o. Thus, for any PSH A and any $f \in A$, an operator $f^\perp : A \to A$ is well-defined (indeed, regard f as an element of A^o, and apply Definition 2.8.1).

Proposition 3.1.2. *Let A be a Hopf algebra over $\mathbf{k} = \mathbb{Z}$ or $\mathbf{k} = \mathbb{Q}$ which is graded, connected, and self-dual with respect to a positive definite graded*[199] *bilinear form. Then:*

(a) *Within the ideal I, the $\mathbf{k}$-submodule of primitives $\mathfrak{p}$ is the orthogonal complement to the $\mathbf{k}$-submodule I^2.*

(b) *In particular, $\mathfrak{p} \cap I^2 = 0$.*

(c) *When $\mathbf{k} = \mathbb{Q}$, one has $I = \mathfrak{p} \oplus I^2$.*

Proof. (a) Note that $I^2 = m(I \otimes I)$. Hence an element x in I lies in the perpendicular space to I^2 if and only if one has for all y in $I \otimes I$ that

$$0 = (x, m(y))_A = (\Delta(x), y)_{A \otimes A} = (\Delta_+(x), y)_{A \otimes A}$$

where the second equality uses self-duality, while the third equality uses the fact that y lies in $I \otimes I$ and the form $(\cdot, \cdot)_{A \otimes A}$ makes distinct homogeneous components orthogonal. Since y was arbitrary, this means x is perpendicular to I^2 if and only if $\Delta_+(x) = 0$, that is, x lies in $\mathfrak{p}$.

(b) This follows from (a), since the form $(\cdot, \cdot)_A$ is positive definite.

(c) This follows from (a) using some basic linear algebra[200] when A is of finite type (which is the only case we will ever encounter in practice). See Exercise 3.1.6 for the general proof. $\square$

[199] That is, $(A_i, A_j) = 0$ for $i \neq j$.

[200] Specifically, either the existence of an orthogonal projection on a subspace of a finite-dimensional inner-product space over $\mathbb{Q}$, or the fact that $\dim \left(W^\perp\right) = \dim V - \dim W$ for a subspace W of a finite-dimensional inner-product space V over $\mathbb{Q}$ can be used.

Remark 3.1.3. One might wonder why we didn't just say $I = \mathfrak{p} \oplus I^2$ even when $\mathbf{k} = \mathbb{Z}$ in Proposition 3.1.2(c). However, this is false even for $A = \Lambda_{\mathbb{Z}}$: the second homogeneous component $(\mathfrak{p} \oplus I^2)_2$ is the index 2 sublattice of Λ_2 which is $\mathbb{Z}$-spanned by $\{p_2, e_1^2\}$, containing $2e_2$, but not containing e_2 itself.

Already the fact that $\mathfrak{p} \cap I^2 = 0$ has a strong implication.

Lemma 3.1.4. *A connected graded Hopf algebra A over any ring $\mathbf{k}$ having $\mathfrak{p} \cap I^2 = 0$ must necessarily be commutative (as an algebra).*

Proof. The component $A_0 = \mathbf{k}$ commutes with all of A. This forms the base case for an induction on $i+j$ in which one shows that any elements x in A_i and y in A_j with $i, j > 0$ will have $[x, y] := xy - yx = 0$. Since $[x, y]$ lies in I^2, it suffices to show that $[x, y]$ also lies in $\mathfrak{p}$:

$$\begin{aligned}
\Delta[x,y] &= [\Delta(x), \Delta(y)] \\
&= [1 \otimes x + x \otimes 1 + \Delta_+(x), 1 \otimes y + y \otimes 1 + \Delta_+(y)] \\
&= [1 \otimes x + x \otimes 1, 1 \otimes y + y \otimes 1] \\
&\qquad + [1 \otimes x + x \otimes 1, \Delta_+(y)] + [\Delta_+(x), 1 \otimes y + y \otimes 1] \\
&\qquad + [\Delta_+(x), \Delta_+(y)] \\
&= [1 \otimes x + x \otimes 1, 1 \otimes y + y \otimes 1] \\
&= 1 \otimes [x,y] + [x,y] \otimes 1
\end{aligned}$$

showing that $[x, y]$ lies in $\mathfrak{p}$. Here the second-to-last equality used the inductive hypotheses: homogeneity implies that $\Delta_+(x)$ is a sum of homogeneous tensors of the form $z_1 \otimes z_2$ satisfying $\deg(z_1), \deg(z_2) < i$, so that by induction they will commute with $1 \otimes y, y \otimes 1$, thus proving that $[\Delta_+(x), 1 \otimes y + y \otimes 1] = 0$; a symmetric argument shows $[1 \otimes x + x \otimes 1, \Delta_+(y)] = 0$, and a similar argument shows $[\Delta_+(x), \Delta_+(y)] = 0$. The last equality is an easy calculation, and was done already in the process of proving (1.3.7). $\square$

Remark 3.1.5. Zelevinsky actually shows [227, Proof of A.1.3, p. 150] that the assumption of $\mathfrak{p} \cap I^2 = 0$ (along with hypotheses of unit, counit, graded, connected, and Δ being a morphism for multiplication) already implies the *associativity* of the multiplication in A ! One shows by induction on $i+j+k$ that any x, y, z in A_i, A_j, A_k with $i, j, k > 0$ have vanishing *associator* $\operatorname{assoc}(x, y, z) := x(yz) - (xy)z$. In the inductive step, one first notes that $\operatorname{assoc}(x, y, z)$ lies in I^2, and then checks that $\operatorname{assoc}(x, y, z)$ also lies in $\mathfrak{p}$, by a calculation very similar to the one above, repeatedly using the fact that $\operatorname{assoc}(x, y, z)$ is multilinear in its three arguments.

Exercise 3.1.6. Prove Proposition 3.1.2(c) in the general case.

This leads to a general structure theorem.

Theorem 3.1.7. *If a connected graded Hopf algebra A over a field $\mathbf{k}$ of characteristic zero has $I = \mathfrak{p} \oplus I^2$, then the inclusion $\mathfrak{p} \hookrightarrow A$ extends to a Hopf algebra isomorphism from the symmetric algebra $\operatorname{Sym}_{\mathbf{k}}(\mathfrak{p}) \to A$. In particular, A is both commutative and cocommutative.*

Note that the hypotheses of Theorem 3.1.7 are valid, using Proposition 3.1.2(c), whenever A is obtained from a PSH (over $\mathbb{Z}$) by tensoring with $\mathbb{Q}$.

Proof of Theorem 3.1.7. Since Lemma 3.1.4 implies that A is commutative, the universal property of $\mathrm{Sym}_{\mathbf{k}}(\mathfrak{p})$ as a free commutative algebra on generators $\mathfrak{p}$ shows that the inclusion $\mathfrak{p} \hookrightarrow A$ at least extends to an algebra morphism $\mathrm{Sym}_{\mathbf{k}}(\mathfrak{p}) \overset{\varphi}{\to} A$. Since the Hopf structure on $\mathrm{Sym}_{\mathbf{k}}(\mathfrak{p})$ makes the elements of $\mathfrak{p}$ primitive (see Example 1.3.14), this φ is actually a coalgebra morphism (since $\Delta \circ \varphi = (\varphi \otimes \varphi) \circ \Delta$ and $\epsilon \circ \varphi = \epsilon$ need only to be checked on algebra generators), hence a bialgebra morphism, hence a Hopf algebra morphism (by Corollary 1.4.27). It remains to show that φ is surjective, and injective.

For the surjectivity of φ, note that the hypothesis $I = \mathfrak{p} \oplus I^2$ implies that the composite $\mathfrak{p} \hookrightarrow I \to I/I^2$ gives a $\mathbf{k}$-vector space isomorphism. What follows is a standard argument to deduce that $\mathfrak{p}$ generates A as a commutative graded $\mathbf{k}$-algebra. One shows by induction on n that any homogeneous element a in A_n lies in the $\mathbf{k}$-subalgebra generated by $\mathfrak{p}$. The base case $n = 0$ is trivial as a lies in $A_0 = \mathbf{k} \cdot 1_A$. In the inductive step where a lies in I, write $a \equiv p \bmod I^2$ for some p in $\mathfrak{p}$. Thus $a = p + \sum_i b_i c_i$, where b_i, c_i lie in I but have strictly smaller degree, so that by induction they lie in the subalgebra generated by $\mathfrak{p}$, and hence so does a.

Note that the surjectivity argument did not use the assumption that $\mathbf{k}$ has characteristic zero, but we will now use it in the injectivity argument for φ, to establish the following

(3.1.1) **Claim:** Every primitive element of $\mathrm{Sym}(\mathfrak{p})$ lies in $\mathfrak{p} = \mathrm{Sym}^1(\mathfrak{p})$.

Note that this claim fails in positive characteristic, e.g. if $\mathbf{k}$ has characteristic 2 then x^2 lies in $\mathrm{Sym}^2(\mathfrak{p})$, however

$$\Delta(x^2) = 1 \otimes x^2 + 2x \otimes x + x^2 \otimes 1 = 1 \otimes x^2 + x^2 \otimes 1.$$

To prove the claim (3.1.1), assume not, so that by gradedness, there must exist some primitive element $y \neq 0$ lying in some $\mathrm{Sym}^n(\mathfrak{p})$ with $n \geq 2$. This would mean that $f(y) = 0$, where the map f is defined as the composition

$$\mathrm{Sym}^n(\mathfrak{p}) \xrightarrow{\Delta} \bigoplus_{i+j=n} \mathrm{Sym}^i(\mathfrak{p}) \otimes \mathrm{Sym}^j(\mathfrak{p}) \overset{\text{projection}}{\longrightarrow} \mathrm{Sym}^1(\mathfrak{p}) \otimes \mathrm{Sym}^{n-1}(\mathfrak{p})$$

of the coproduct Δ with the component projection of $\bigoplus_{i+j=n} \mathrm{Sym}^i(\mathfrak{p}) \otimes \mathrm{Sym}^j(\mathfrak{p})$ onto $\mathrm{Sym}^1(\mathfrak{p}) \otimes \mathrm{Sym}^{n-1}(\mathfrak{p})$. However, one can check on a basis that the multiplication backward $\mathrm{Sym}^1(\mathfrak{p}) \otimes \mathrm{Sym}^{n-1}(\mathfrak{p}) \overset{m}{\to} \mathrm{Sym}^n(\mathfrak{p})$ has the property that $m \circ f = n \cdot \mathrm{id}_{\mathrm{Sym}^n(\mathfrak{p})}$: Indeed,

$$(m \circ f)(x_1 \cdots x_n) = m\left(\sum_{j=1}^n x_j \otimes x_1 \cdots \widehat{x_j} \cdots x_n\right) = n \cdot x_1 \cdots x_n$$

for $x_1, \ldots, x_n$ in $\mathfrak{p}$. Then $n \cdot y = m(f(y)) = m(0) = 0$ leads to the contradiction that $y = 0$, since $\mathbf{k}$ has characteristic zero. Thus, (3.1.1) is proven.

Now one can argue the injectivity of the (graded) map[201] φ by assuming that one has a nonzero homogeneous element u in $\ker(\varphi)$ of minimum degree. In particular, $\deg(u) \geq 1$. Also since $\mathfrak{p} \hookrightarrow A$, one has that u is not in $\mathrm{Sym}^1(\mathfrak{p}) = \mathfrak{p}$, and hence u is not primitive by (3.1.1). Consequently

[201] The grading on $\mathrm{Sym}(\mathfrak{p})$ is induced from the grading on $\mathfrak{p}$, a homogeneous subspace of $I \subset A$ as it is the kernel of the graded map $I \xrightarrow{\Delta_+} A \otimes A$.

$\Delta_+(u) \neq 0$, and one can find a nonzero component $u^{(i,j)}$ of $\Delta_+(u)$ lying in $\mathrm{Sym}(\mathfrak{p})_i \otimes \mathrm{Sym}(\mathfrak{p})_j$ for some $i, j > 0$. Since this forces $i, j < \deg(u)$, one has that φ maps both $\mathrm{Sym}(\mathfrak{p})_i, \mathrm{Sym}(\mathfrak{p})_j$ injectively into A_i, A_j. Hence the tensor product map

$$\mathrm{Sym}(\mathfrak{p})_i \otimes \mathrm{Sym}(\mathfrak{p})_j \xrightarrow{\varphi\otimes\varphi} A_i \otimes A_j$$

is also injective[202]. This implies $(\varphi\otimes\varphi)(u^{(i,j)}) \neq 0$, giving the contradiction that

$$0 = \Delta_+^A(0) = \Delta_+^A(\varphi(u)) = (\varphi \otimes \varphi)(\Delta_+^{\mathrm{Sym}(\mathfrak{p})}(u))$$

contains the nonzero $A_i \otimes A_j$-component $(\varphi \otimes \varphi)(u^{(i,j)})$.

(An alternative proof of the injectivity of φ proceeds as follows: By (3.1.1), the subspace of primitive elements of $\mathrm{Sym}(\mathfrak{p})$ is $\mathfrak{p}$, and clearly $\varphi\mid_{\mathfrak{p}}$ is injective. Hence, Exercise 1.4.35(b) (applied to the homomorphism φ) shows that φ is injective.) ☐

Before closing this section, we mention one nonobvious corollary of the Claim (3.1.1), when applied to the ring of symmetric functions $\Lambda_{\mathbb{Q}}$ with $\mathbb{Q}$-coefficients, since Proposition 2.4.1 says that $\Lambda_{\mathbb{Q}} = \mathbb{Q}[p_1, p_2, \ldots] = \mathrm{Sym}(V)$ where $V = \mathbb{Q}\{p_1, p_2, \ldots\}$.

Corollary 3.1.8. *The subspace $\mathfrak{p}$ of primitives in $\Lambda_{\mathbb{Q}}$ is one-dimensional in each degree $n = 1, 2, \ldots$, and spanned by $\{p_1, p_2, \ldots\}$.*

We note in passing that this corollary can also be obtained in a simpler fashion and a greater generality:

Exercise 3.1.9. Let $\mathbf{k}$ be any commutative ring. Show that the primitive elements of Λ are precisely the elements of the $\mathbf{k}$-linear span of $p_1, p_2, p_3, \ldots$.

3.2. **The decomposition theorem.** Our goal here is Zelevinsky's theorem [227, Theorem 2.2] giving a canonical decomposition of any PSH as a tensor product into PSH's that each have only one primitive element in their PSH-basis. For the sake of stating it, we introduce some notation.

Definition 3.2.1. Given a PSH A with PSH-basis Σ, let $\mathcal{C} := \Sigma \cap \mathfrak{p}$ be the primitive elements in Σ. For each ρ in $\mathcal{C}$, let $A(\rho) \subset A$ be the $\mathbb{Z}$-span of

$$\Sigma(\rho) := \{\sigma \in \Sigma : \text{ there exists } n \geq 0 \text{ with } (\sigma, \rho^n) \neq 0\}.$$

Definition 3.2.2. The tensor product of two PSHs A_1 and A_2 with PSH-bases Σ_1 and Σ_2 is defined as the graded Hopf algebra $A_1 \otimes A_2$ with PSH-basis $\{\sigma_1 \otimes \sigma_2\}_{(\sigma_1,\sigma_2)\in\Sigma_1\times\Sigma_2}$. It is easy to see that this is again a PSH. The tensor product of any finite family of PSHs is defined similarly[203].

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[202] One needs to know that for two injective maps $V_i \xrightarrow{\varphi_i} W_i$ of $\mathbf{k}$-vector spaces V_i, W_i with $i = 1, 2$, the tensor product $\varphi_1 \otimes \varphi_2$ is also injective. Factoring it as $\varphi_1 \otimes \varphi_2 = (\mathrm{id} \otimes \varphi_2) \circ (\varphi_1 \otimes \mathrm{id})$, one sees that it suffices to show that for an injective map $V \overset{\varphi}{\hookrightarrow} W$ of free $\mathbf{k}$-modules, and any free $\mathbf{k}$-module U, the map $V \otimes U \xrightarrow{\varphi\otimes\mathrm{id}} W \otimes U$ is also injective. Since tensor products commute with direct sums, and U is (isomorphic to) a direct sum of copies of $\mathbf{k}$, this reduces to the easy-to-check case where $U = \mathbf{k}$.

Note that some kind of freeness or flatness hypothesis on U is needed here since, e.g. the injective $\mathbb{Z}$-module maps $\mathbb{Z} \xrightarrow{\varphi_1=(\cdot\times 2)} \mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \xrightarrow{\varphi_2=\mathrm{id}} \mathbb{Z}/2\mathbb{Z}$ have $\varphi_1 \otimes \varphi_2 = 0$ on $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \neq 0$.

[203] For the empty family, it is the connected graded Hopf algebra $\mathbb{Z}$ with PSH-basis $\{1\}$.

Theorem 3.2.3. *Any PSH A has a canonical tensor product decomposition*

$$A = \bigotimes_{\rho \in \mathcal{C}} A(\rho)$$

with $A(\rho)$ a PSH, and ρ the only primitive element in its PSH-basis $\Sigma(\rho)$.

Although in all the applications, $\mathcal{C}$ will be finite, when $\mathcal{C}$ is infinite one should interpret the tensor product in the theorem as the inductive limit of tensor products over finite subsets of $\mathcal{C}$, that is, linear combinations of basic tensors $\bigotimes_\rho a_\rho$ in which there are only finitely many factors $a_\rho \neq 1$.

The first step toward the theorem uses a certain unique factorization property.

Lemma 3.2.4. *Let $\mathcal{P}$ be a set of pairwise orthogonal primitives in a PSH A. Then,*

$$(\rho_1 \cdots \rho_r, \pi_1 \cdots \pi_s) = 0$$

for ρ_i, π_j in $\mathcal{P}$ unless $r = s$ and one can reindex so that $\rho_i = \pi_i$.

Proof. Induct on r. For $r > 0$, one has

$$\begin{aligned}(\rho_1 \cdots \rho_r, \pi_1 \cdots \pi_s) &= (\rho_2 \cdots \rho_r, \rho_1^\perp(\pi_1 \cdots \pi_s)) \\ &= (\rho_2 \cdots \rho_r, \sum_{j=1}^{s} (\pi_1 \cdots \pi_{j-1} \cdot \rho_1^\perp(\pi_j) \cdot \pi_{j+1} \cdots \pi_s))\end{aligned}$$

from Proposition 2.8.2(iv) because ρ_1 is primitive[204]. On the other hand, since each π_j is primitive, one has $\rho_1^\perp(\pi_j) = (\rho_1, 1) \cdot \pi_j + (\rho_1, \pi_j) \cdot 1 = (\rho_1, \pi_j)$ which vanishes unless $\rho_1 = \pi_j$. Hence $(\rho_1 \cdots \rho_r, \pi_1 \cdots \pi_s) = 0$ unless $\rho_1 \in \{\pi_1, \ldots, \pi_s\}$, in which case after reindexing so that $\pi_1 = \rho_1$, it equals

$$n \cdot (\rho_1, \rho_1) \cdot (\rho_2 \cdots \rho_r, \pi_2 \cdots \pi_s)$$

if there are exactly n occurrences of ρ_1 among $\pi_1, \ldots, \pi_s$. Now apply induction. $\square$

So far the positivity hypothesis for a PSH has played little role. Now we use it to introduce a certain partial order on the PSH A, and then a semigroup grading.

Definition 3.2.5. For a subset S of an abelian group, let $\mathbb{Z}S$ (resp. $\mathbb{N}S$) denote the subgroup of $\mathbb{Z}$-linear combinations (resp. submonoid of $\mathbb{N}$-linear combinations[205]) of the elements of S.

In a PSH A with PSH-basis Σ, the subset $\mathbb{N}\Sigma$ forms a submonoid, and lets one define a partial order on A via $a \leq b$ if $b - a$ lies in $\mathbb{N}\Sigma$.

We note a few trivial properties of this partial order:

- The positivity hypothesis implies that $\mathbb{N}\Sigma \cdot \mathbb{N}\Sigma \subset \mathbb{N}\Sigma$.
- Hence multiplication by an element $c \geq 0$ (meaning c lies in $\mathbb{N}\Sigma$) preserves the order: $a \leq b$ implies $ac \leq bc$ since $(b-a)c$ lies in $\mathbb{N}\Sigma$.
- Thus $0 \leq c \leq d$ and $0 \leq a \leq b$ together imply $ac \leq bc \leq bd$.

[204]Strictly speaking, this argument needs further justification since A might not be of finite type (and if it is not, Proposition 2.8.2(iv) cannot be applied). It is more adequate to refer to the proof of Proposition 2.8.2(iv), which indeed goes through with ρ_1 taking the role of f.

[205]Recall that $\mathbb{N} := \{0, 1, 2, \ldots\}$.

This allows one to introduce a semigroup grading on A.

Definition 3.2.6. Let $\mathbb{N}^{\mathcal{C}}_{\text{fin}}$ denote the additive submonoid of $\mathbb{N}^{\mathcal{C}}$ consisting of those $\alpha = (\alpha_\rho)_{\rho\in\mathcal{C}}$ with finite support.

Note that for any α in $\mathbb{N}^{\mathcal{C}}_{\text{fin}}$, one has that the product $\prod_{\rho\in\mathcal{C}} \rho^{\alpha_\rho} \geq 0$. Define

$$\Sigma(\alpha) := \{\sigma \in \Sigma : \sigma \leq \prod_{\rho\in\mathcal{C}} \rho^{\alpha_\rho}\},$$

that is, the subset of Σ on which $\prod_{\rho\in\mathcal{C}} \rho^{\alpha_\rho}$ has support. Also define

$$A_{(\alpha)} := \mathbb{Z}\Sigma(\alpha) \subset A.$$

Proposition 3.2.7. *The PSH A has an $\mathbb{N}^{\mathcal{C}}_{\text{fin}}$-semigroup-grading: one has an orthogonal direct sum decomposition*

$$A = \bigoplus_{\alpha\in\mathbb{N}^{\mathcal{C}}_{\text{fin}}} A_{(\alpha)}$$

for which

$$A_{(\alpha)}A_{(\beta)} \subset A_{(\alpha+\beta)}, \tag{3.2.1}$$

$$\Delta A_{(\alpha)} \subset \bigoplus_{\alpha=\beta+\gamma} A_{(\beta)} \otimes A_{(\gamma)}. \tag{3.2.2}$$

Proof. We will make free use of the fact that a PSH A is commutative, since it embeds in $A \otimes_{\mathbb{Z}} \mathbb{Q}$, which is commutative by Theorem 3.1.7.

Note that the orthogonality $\big(A_{(\alpha)}, A_{(\beta)}\big) = 0$ for $\alpha \neq \beta$ is equivalent to the assertion that

$$\left(\prod_{\rho\in\mathcal{C}} \rho^{\alpha_\rho}, \prod_{\rho\in\mathcal{C}} \rho^{\beta_\rho}\right) = 0,$$

which follows from Lemma 3.2.4.

Next let us deal with the assertion (3.2.1). It suffices to check that when τ, ω in Σ lie in $A_{(\alpha)}, A_{(\beta)}$, respectively, then $\tau\omega$ lies in $A_{(\alpha+\beta)}$. But note that any σ in Σ having $\sigma \leq \tau\omega$ will then have

$$\sigma \leq \tau\omega \leq \prod_{\rho\in\mathcal{C}} \rho^{\alpha_\rho} \cdot \prod_{\rho\in\mathcal{C}} \rho^{\beta_\rho} = \prod_{\rho\in\mathcal{C}} \rho^{\alpha_\rho+\beta_\rho}$$

so that σ lies in $A_{(\alpha+\beta)}$. This means that $\tau\omega$ lies in $A_{(\alpha+\beta)}$.

This lets us check that $\bigoplus_{\alpha\in\mathbb{N}^{\mathcal{C}}_{\text{fin}}} A_{(\alpha)}$ exhaust A. It suffices to check that any σ in Σ lies in some $A_{(\alpha)}$. Proceed by induction on $\deg(\sigma)$, with the case $\sigma = 1$ being trivial; the element 1 always lies in Σ, and hence lies in $A_{(\alpha)}$ for $\alpha = 0$. For σ lying in I, one either has $(\sigma, a) \neq 0$ for some a in I^2, or else σ lies in $(I^2)^\perp = \mathfrak{p}$ (by Proposition 3.1.2(a)), so that σ is in $\mathcal{C}$ and we are done. If $(\sigma, a) \neq 0$ with a in I^2, then σ appears in the support of some $\mathbb{Z}$-linear combination of elements $\tau\omega$ where τ, ω lie in Σ and have strictly smaller degree than σ has. There exists at least one such pair τ, ω for which $(\sigma, \tau\omega) \neq 0$, and therefore $\sigma \leq \tau\omega$. Then by induction τ, ω lie in some $A_{(\alpha)}, A_{(\beta)}$, respectively, so $\tau\omega$ lies in $A_{(\alpha+\beta)}$, and hence σ lies in $A_{(\alpha+\beta)}$ also.

Self-duality shows that (3.2.1) implies (3.2.2): if a, b, c lie in $A_{(\alpha)}, A_{(\beta)}, A_{(\gamma)}$, respectively, then $(\Delta a, b \otimes c)_{A\otimes A} = (a, bc)_A = 0$ unless $\alpha = \beta + \gamma$. $\square$

Proposition 3.2.8. *For α, β in $\mathbb{N}_{\text{fin}}^{\mathcal{C}}$ with disjoint support, one has a bijection*

$$\begin{array}{rcl} \Sigma(\alpha) \times \Sigma(\beta) & \longrightarrow & \Sigma(\alpha+\beta), \\ (\sigma, \tau) & \longmapsto & \sigma\tau. \end{array}$$

Thus, the multiplication map $A_{(\alpha)} \otimes A_{(\beta)} \to A_{(\alpha+\beta)}$ is an isomorphism.

Proof. We first check that for σ_1, σ_2 in $\Sigma(\alpha)$ and τ_1, τ_2 in $\Sigma(\beta)$, one has

$$(\sigma_1\tau_1, \sigma_2\tau_2) = \delta_{(\sigma_1,\tau_1),(\sigma_2,\tau_2)}. \tag{3.2.3}$$

Note that this is equivalent to showing both

- that $\sigma\tau$ lie in $\Sigma(\alpha+\beta)$ so that the map is well-defined, since it shows $(\sigma\tau, \sigma\tau) = 1$, and
- that the map is injective.

One calculates

$$\begin{aligned} (\sigma_1\tau_1, \sigma_2\tau_2)_A &= (\sigma_1\tau_1, m(\sigma_2 \otimes \tau_2))_A \\ &= (\Delta(\sigma_1\tau_1), \sigma_2 \otimes \tau_2)_{A\otimes A} \\ &= (\Delta(\sigma_1)\Delta(\tau_1), \sigma_2 \otimes \tau_2)_{A\otimes A}. \end{aligned}$$

Note that due to (3.2.2), $\Delta(\sigma_1)\Delta(\tau_1)$ lies in $\sum A_{(\alpha'+\beta')} \otimes A_{(\alpha''+\beta'')}$ where

$$\begin{aligned} \alpha' + \alpha'' &= \alpha, \\ \beta' + \beta'' &= \beta. \end{aligned}$$

Since $\sigma_2 \otimes \tau_2$ lies in $A_{(\alpha)} \otimes A_{(\beta)}$, the only nonvanishing terms in the inner product come from those with

$$\begin{aligned} \alpha' + \beta' &= \alpha, \\ \alpha'' + \beta'' &= \beta. \end{aligned}$$

As α, β have disjoint support, this can only happen if

$$\alpha' = \alpha, \ \alpha'' = 0, \ \beta' = 0, \ \beta'' = \beta;$$

that is, the only nonvanishing term comes from $(\sigma_1 \otimes 1)(1 \otimes \tau_1) = \sigma_1 \otimes \tau_1$. Hence

$$(\sigma_1\tau_1, \sigma_2\tau_2)_A = (\sigma_1 \otimes \tau_1, \sigma_2 \otimes \tau_2)_{A\otimes A} = \delta_{(\sigma_1,\tau_1),(\sigma_2,\tau_2)}.$$

To see that the map is surjective, express

$$\begin{aligned} \prod_{\rho\in\mathcal{C}} \rho^{\alpha_\rho} &= \sum_i \sigma_i, \\ \prod_{\rho\in\mathcal{C}} \rho^{\beta_\rho} &= \sum_j \tau_j \end{aligned}$$

with $\sigma_i \in \Sigma(\alpha)$ and $\tau_j \in \Sigma(\beta)$. Then each product $\sigma_i\tau_j$ is in $\Sigma(\alpha+\beta)$ by (3.2.3), and

$$\prod_{\rho\in\mathcal{C}} \rho^{\alpha_\rho+\beta_\rho} = \sum_{i,j} \sigma_i\tau_j$$

shows that $\{\sigma_i\tau_j\}$ exhausts $\Sigma(\alpha+\beta)$. This gives surjectivity. $\square$

Proof of Theorem 3.2.3. Recall from Definition 3.2.1 that for each ρ in $\mathcal{C}$, one defines $A(\rho) \subset A$ to be the $\mathbb{Z}$-span of

$$\Sigma(\rho) := \{\sigma \in \Sigma : \text{ there exists } n \geq 0 \text{ with } (\sigma, \rho^n) \neq 0\}.$$

In other words, $A(\rho) := \bigoplus_{n \geq 0} A_{(n \cdot e_\rho)}$ where e_ρ in $\mathbb{N}^{\mathcal{C}}_{\text{fin}}$ is the standard basis element indexed by ρ. Proposition 3.2.7 then shows that $A(\rho)$ is a Hopf subalgebra of A. Since every α in $\mathbb{N}^{\mathcal{C}}_{\text{fin}}$ can be expressed as the (finite) sum $\sum_\rho \alpha_\rho e_\rho$, and the e_ρ have disjoint support, iterating Proposition 3.2.8 shows that $A = \bigotimes_{\rho \in \mathcal{C}} A(\rho)$. Lastly, $\Sigma(\rho)$ is clearly a PSH-basis for $A(\rho)$, and if σ is any primitive element in $\Sigma(\rho)$ then $(\sigma, \rho^n) \neq 0$ lets one conclude via Lemma 3.2.4 that $\sigma = \rho$ (and $n = 1$). $\square$

3.3. **Λ is the unique indecomposable PSH.** The goal here is to prove the rest of Zelevinsky's structure theory for PSH's. Namely, if A has only one primitive element ρ in its PSH-basis Σ, then A must be isomorphic as a PSH to the ring of symmetric functions Λ, after one rescales the grading of A. Note that every σ in Σ has $\sigma \leq \rho^n$ for some n, and hence has degree divisible by the degree of ρ. Thus one can divide all degrees by that of ρ and assume ρ has degree 1.

The idea is to find within A and Σ a set of elements that play the role of

$$\{h_n = s_{(n)}\}_{n=0,1,2,\ldots}, \qquad \{e_n = s_{(1^n)}\}_{n=0,1,2,\ldots}$$

within $A = \Lambda$ and its PSH-basis of Schur functions $\Sigma = \{s_\lambda\}$. Zelevinsky's argument does this by isolating some properties that turn out to characterize these elements:

(a) $h_0 = e_0 = 1$, and $h_1 = e_1 =: \rho$ has ρ^2 a sum of two elements of Σ, namely
$$\rho^2 = h_2 + e_2.$$
(b) For all $n = 0, 1, 2, \ldots$, there exist unique elements h_n, e_n in $A_n \cap \Sigma$ that satisfy
$$h_2^\perp e_n = 0,$$
$$e_2^\perp h_n = 0$$
with h_2, e_2 being the two elements of Σ introduced in (a).
(c) For $k = 0, 1, 2, \ldots, n$ one has
$$h_k^\perp h_n = h_{n-k} \text{ and } \sigma^\perp h_n = 0 \text{ for } \sigma \in \Sigma \setminus \{h_0, h_1, \ldots, h_n\},$$
$$e_k^\perp e_n = e_{n-k} \text{ and } \sigma^\perp e_n = 0 \text{ for } \sigma \in \Sigma \setminus \{e_0, e_1, \ldots, e_n\}.$$

In particular, $e_k^\perp h_n = 0 = h_k^\perp e_n$ for $k \geq 2$.
(d) Their coproducts are
$$\Delta(h_n) = \sum_{i+j=n} h_i \otimes h_j,$$
$$\Delta(e_n) = \sum_{i+j=n} e_i \otimes e_j.$$

We will prove Zelevinsky's result [227, Theorem 3.1] as a combination of the following two theorems.

Theorem 3.3.1. *Let A be a PSH with PSH-basis Σ containing only one primitive ρ, and assume that the grading has been rescaled so that ρ has degree 1. Then, after renaming $\rho = e_1 = h_1$, one can find unique sequences*

$\{h_n\}_{n=0,1,2,\ldots}$, $\{e_n\}_{n=0,1,2,\ldots}$ of elements of Σ having properties (a),(b),(c),(d) listed above.

The second theorem uses the following notion.

Definition 3.3.2. A *PSH-morphism* $A \xrightarrow{\varphi} A'$ between two PSH's A, A' having PSH-bases Σ, Σ' is a graded Hopf algebra morphism for which $\varphi(\mathbb{N}\Sigma) \subset \mathbb{N}\Sigma'$. If $A = A'$ and $\Sigma = \Sigma'$ it will be called a *PSH-endomorphism*. If φ is an isomorphism and restricts to a bijection $\Sigma \to \Sigma'$, it will be called a *PSH-isomorphism*[206]; if it is both a PSH-isomorphism and an endomorphism, it is a *PSH-automorphism*.[207]

Theorem 3.3.3. *The elements $\{h_n\}_{n=0,1,2,\ldots}$, $\{e_n\}_{n=0,1,2,\ldots}$ in Theorem 3.3.1 also satisfy the following.*

(e) *The elements h_n, e_n in A satisfy the same relation* (2.4.4)
$$\sum_{i+j=n} (-1)^i e_i h_j = \delta_{0,n}$$
as their counterparts in Λ, along with the property that
$$A = \mathbb{Z}[h_1, h_2, \ldots] = \mathbb{Z}[e_1, e_2, \ldots].$$

(f) *There is exactly one nontrivial automorphism $A \xrightarrow{\omega} A$ as a PSH, swapping $h_n \leftrightarrow e_n$.*

(g) *There are exactly two PSH-isomorphisms $A \to \Lambda$:*
- *one sending h_n to the complete homogeneous symmetric functions $h_n(\mathbf{x})$, while sending e_n to the elementary symmetric functions $e_n(\mathbf{x})$,*
- *the second one (obtained by composing the first with ω) sending $h_n \mapsto e_n(\mathbf{x})$ and $e_n \mapsto h_n(\mathbf{x})$.*

Before embarking on the proof, we mention one more bit of convenient terminology: say that an element σ in Σ is a *constituent* of a in $\mathbb{N}\Sigma$ when $\sigma \leq a$, that is, σ appears with nonzero coefficient c_σ in the unique expansion $a = \sum_{\tau \in \Sigma} c_\tau \tau$.

Proof of Theorem 3.3.1. One fact that occurs frequently is this:

$$\text{Every } \sigma \text{ in } \Sigma \cap A_n \text{ is a constituent of } \rho^n. \tag{3.3.1}$$

This follows from Theorem 3.2.3, since ρ is the only primitive element of Σ: one has $A = A(\rho)$ and $\Sigma = \Sigma(\rho)$, so that σ is a constituent of some ρ^m, and homogeneity considerations force $m = n$.

Notice that A is of finite type (due to (3.3.1)). Thus, A^o is a graded Hopf algebra isomorphic to A.

Assertion (a). Note that
$$(\rho^2, \rho^2) = (\rho^\perp(\rho^2), \rho) = (2\rho, \rho) = 2$$

[206] This definition is easily seen to be equivalent to saying that a PSH-isomorphism is an invertible PSH-morphism whose inverse is again a PSH-morphism.

[207] The reader should be warned that not every invertible PSH-endomorphism is necessarily a PSH-automorphism. For instance, it is an easy exercise to check that $\Lambda \otimes \Lambda \to \Lambda \otimes \Lambda$, $f \otimes g \mapsto \sum_{(f)} f_1 \otimes f_2 g$ is a well-defined invertible PSH-endomorphism of the PSH $\Lambda \otimes \Lambda$ with PSH-basis $(s_\lambda \otimes s_\mu)_{(\lambda,\mu) \in \mathrm{Par} \times \mathrm{Par}}$, but not a PSH-automorphism.

using the fact that $\rho^\perp$ is a derivation since ρ is primitive (Proposition 2.8.2(iv)). On the other hand, expressing $\rho^2 = \sum_{\sigma \in \Sigma} c_\sigma \sigma$ with c_σ in $\mathbb{N}$, one has $(\rho^2, \rho^2) = \sum_\sigma c_\sigma^2$. Hence exactly two of the $c_\sigma = 1$, so ρ^2 has exactly two distinct constituents. Denote them by h_2 and e_2. One concludes that $\Sigma \cap A_2 = \{h_2, e_2\}$ from (3.3.1).

Note also that the same argument shows $\Sigma \cap A_1 = \{\rho\}$, so that $A_1 = \mathbb{Z}\rho$. Since $\rho^\perp h_2$ lies in $A_1 = \mathbb{Z}\rho$ and $(\rho^\perp h_2, \rho) = (h_2, \rho^2) = 1$, we have $\rho^\perp h_2 = \rho$. Similarly $\rho^\perp e_2 = \rho$.

Assertion (b). We will show via induction on n the following three assertions for $n \geq 1$:

(3.3.2)
- There exists an element h_n in $\Sigma \cap A_n$ with $e_2^\perp h_n = 0$.
- This element h_n is unique.
- Furthermore $\rho^\perp h_n = h_{n-1}$.

In the base cases $n = 1, 2$, it is not hard to check that our previously labelled elements, h_1, h_2 (namely $h_1 := \rho$, and h_2 as named in part (a)) really *are* the unique elements satisfying these hypotheses.

In the inductive step, it turns out that we will find h_n as a constituent of ρh_{n-1}. Thus we again use the derivation property of $\rho^\perp$ to compute that ρh_{n-1} has exactly two constituents:

$$\begin{aligned}(\rho h_{n-1}, \rho h_{n-1}) &= (\rho^\perp(\rho h_{n-1}), h_{n-1}) \\ &= (h_{n-1} + \rho \cdot \rho^\perp h_{n-1}, h_{n-1}) \\ &= (h_{n-1} + \rho h_{n-2}, h_{n-1}) \\ &= 1 + (h_{n-2}, \rho^\perp h_{n-1}) \\ &= 1 + (h_{n-2}, h_{n-2}) = 1 + 1 = 2\end{aligned}$$

where the inductive hypothesis $\rho^\perp h_{n-1} = h_{n-2}$ was used twice. We next show that exactly one of the two constituents of ρh_{n-1} is annihilated by $e_2^\perp$. Note that since e_2 lies in A_2, and A_1 has $\mathbb{Z}$-basis element ρ, there is a constant c in $\mathbb{Z}$ such that

$$\Delta(e_2) = e_2 \otimes 1 + c\rho \otimes \rho + 1 \otimes e_2. \tag{3.3.3}$$

On the other hand, (a) showed

$$1 = (e_2, \rho^2)_A = (\Delta(e_2), \rho \otimes \rho)_{A \otimes A}$$

so one must have $c = 1$. Therefore by Proposition 2.8.2(iv) again,

$$\begin{array}{rcccccc} e_2^\perp(\rho h_{n-1}) & = & e_2^\perp(\rho) h_{n-1} & + & \rho^\perp(\rho)\rho^\perp(h_{n-1}) & + & \rho e_2^\perp(h_{n-1}) \\ & = & 0 & + & h_{n-2} & + & 0 \\ & = & h_{n-2}, & & & & \end{array} \tag{3.3.4}$$

where the first term vanished due to degree considerations and the last term vanished by the inductive hypothesis. Bearing in mind that ρh_{n-1} lies in $\mathbb{N}\Sigma$, and in a PSH with PSH-basis Σ, any skewing operator $\sigma^\perp$ for σ in Σ will preserve $\mathbb{N}\Sigma$, one concludes from (3.3.4) that

- one of the two distinct constituents of the element ρh_{n-1} must be sent by $e_2^\perp$ to h_{n-2}, and
- the other constituent of ρh_{n-1} must be annihilated by $e_2^\perp$; call this second constituent h_n.

Lastly, to see that this h_n is unique, it suffices to show that any element σ of $\Sigma \cap A_n$ which is killed by $e_2^\perp$ must be a constituent of ρh_{n-1}. This holds for the following reason. We know $\sigma \leq \rho^n$ by (3.3.1), and hence $0 \neq (\rho^n, \sigma) = (\rho^{n-1}, \rho^\perp \sigma)$, implying that $\rho^\perp \sigma \neq 0$. On the other hand, since $0 = \rho^\perp e_2^\perp \sigma = e_2^\perp \rho^\perp \sigma$, one has that $\rho^\perp \sigma$ is annihilated by $e_2^\perp$, and hence $\rho^\perp \sigma$ must be a (positive) multiple of h_{n-1} by part of our inductive hypothesis. Therefore $(\sigma, \rho h_{n-1}) = (\rho^\perp \sigma, h_{n-1})$ is positive, that is, σ is a constituent of ρh_{n-1}.

The preceding argument, applied to $\sigma = h_n$, shows that $\rho^\perp h_n = c h_{n-1}$ for some c in $\{1, 2, \ldots\}$. Since $(\rho^\perp h_n, h_{n-1}) = (h_n, \rho h_{n-1}) = 1$, this c must be 1, so that $\rho^\perp h_n = h_{n-1}$. This completes the induction step in the proof of (3.3.2).

One can then argue, swapping the roles of e_n, h_n in the above argument, the existence and uniqueness of a sequence $\{e_n\}_{n=0}^\infty$ in Σ satisfying the properties analogous to (3.3.2), with $e_0 := 1, e_1 := \rho$.

Assertion (c). Iterating the property from (b) that $\rho^\perp h_n = h_{n-1}$ shows that $(\rho^k)^\perp h_n = h_{n-k}$ for $0 \leq k \leq n$. However one also has an expansion

$$\rho^k = c h_k + \sum_{\substack{\sigma \in \Sigma \cap A_k: \\ \sigma \neq h_k}} c_\sigma \sigma$$

for some integers $c, c_\sigma > 0$, since every σ in $\Sigma \cap A_k$ is a constituent of ρ^k. Hence

$$1 = (h_{n-k}, h_{n-k}) = ((\rho^k)^\perp h_n, (\rho^k)^\perp h_n) \geq c^2 (h_k^\perp h_n, h_k^\perp h_n)$$

using Proposition 2.8.2(ii). Hence if we knew that $h_k^\perp h_n \neq 0$ this would force

$$h_k^\perp h_n = (\rho^k)^\perp h_n = h_{n-k}$$

as well as $\sigma^\perp h_n = 0$ for all $\sigma \notin \{h_0, h_1, \ldots, h_n\}$. But

$$(\rho^{n-k})^\perp h_k^\perp h_n = h_k^\perp (\rho^{n-k})^\perp h_n = h_k^\perp h_k = 1 \neq 0$$

so $h_k^\perp h_n \neq 0$, as desired. The argument for $e_k^\perp e_n = e_{n-k}$ is symmetric.

The last assertion in (c) follows if one checks that $e_n \neq h_n$ for each $n \geq 2$, but this holds since $e_2^\perp(h_n) = 0$ but $e_2^\perp(e_n) = e_{n-2}$.

Assertion (d). Part (c) implies that

$$(\Delta h_n, \sigma \otimes \tau)_{A \otimes A} = (h_n, \sigma\tau)_A = (\sigma^\perp h_n, \tau)_A = 0$$

unless $\sigma = h_k$ for some $k = 0, 1, 2, \ldots, n$ and $\tau = h_{n-k}$. Also one can compute

$$(\Delta h_n, h_k \otimes h_{n-k}) = (h_n, h_k h_{n-k}) = (h_k^\perp h_n, h_{n-k}) \overset{(c)}{=} (h_{n-k}, h_{n-k}) = 1.$$

This is equivalent to the assertion for Δh_n in (d). The argument for Δe_n is symmetric. $\square$

Before proving Theorem 3.3.3, we note some consequences of Theorem 3.3.1. Define for each partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell)$ the following two elements of A:

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_\ell} = h_{\lambda_1} h_{\lambda_2} \cdots ,$$
$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_\ell} = e_{\lambda_1} e_{\lambda_2} \cdots .$$

Also, define the *lexicographic order* on Par_n by saying $\lambda <_{\mathrm{lex}} \mu$ if $\lambda \neq \mu$ and the smallest index i for which $\lambda_i \neq \mu_i$ has $\lambda_i < \mu_i$. Recall also that λ^t denotes the *conjugate* or *transpose* partition to λ, obtained by swapping rows and columns in the Ferrers diagram.

The following unitriangularity lemma will play a role in the proof of Theorem 3.3.3(e).

Lemma 3.3.4. *Under the hypotheses of Theorem 3.3.1, for λ, μ in Par_n, one has*

$$e_\mu^\perp h_\lambda = \begin{cases} 1, & \text{if } \mu = \lambda^t; \\ 0, & \text{if } \mu >_{\mathrm{lex}} \lambda^t. \end{cases} \tag{3.3.5}$$

Consequently

$$\det \left[(e_{\mu^t}, h_\lambda) \right]_{\lambda,\mu \in \mathrm{Par}_n} = 1. \tag{3.3.6}$$

Proof. Notice that A is of finite type (as shown in the proof of Theorem 3.3.1). Thus, A^o is a graded Hopf algebra isomorphic to A.

Also, notice that any $m \in \mathbb{N}$ and any $a_1, a_2, \ldots, a_\ell \in A$ satisfy

$$e_m^\perp (a_1 a_2 \cdots a_\ell) = \sum_{i_1 + \cdots + i_\ell = m} e_{i_1}^\perp (a_1) \cdots e_{i_\ell}^\perp (a_\ell). \tag{3.3.7}$$

Indeed, this follows by induction over ℓ using Proposition 2.8.2(iv) (and the coproduct formula for $\Delta(e_n)$ in Theorem 3.3.1(d)).

In order to prove (3.3.5), induct on the length of μ. If λ has length ℓ, so that $\lambda_1^t = \ell$, then

$$\begin{aligned}
e_\mu^\perp h_\lambda &= e_{(\mu_2,\mu_3,\ldots)}^\perp \left(e_{\mu_1}^\perp (h_{\lambda_1} \cdots h_{\lambda_\ell}) \right) \\
&\qquad \left(\text{since } e_\mu = e_{\mu_1} e_{(\mu_2,\mu_3,\ldots)} \text{ and thus } e_\mu^\perp = e_{(\mu_2,\mu_3,\ldots)}^\perp \circ e_{\mu_1}^\perp \right) \\
&= e_{(\mu_2,\mu_3,\ldots)}^\perp \sum_{i_1+\cdots+i_\ell=\mu_1} e_{i_1}^\perp(h_{\lambda_1}) \cdots e_{i_\ell}^\perp(h_{\lambda_\ell}) \qquad (\text{by (3.3.7)}) \\
&= e_{(\mu_2,\mu_3,\ldots)}^\perp \sum_{\substack{i_1+\cdots+i_\ell=\mu_1; \\ \text{each of } i_1,\ldots,i_\ell \text{ is } \leq 1}} e_{i_1}^\perp(h_{\lambda_1}) \cdots e_{i_\ell}^\perp(h_{\lambda_\ell}) \\
&\qquad \left(\text{since } e_k^\perp h_n = 0 \text{ for } k \geq 2\right) \\
&= \begin{cases} 0, & \text{if } \mu_1 > \ell = \lambda_1^t; \\ e_{(\mu_2,\mu_3,\ldots)}^\perp h_{(\lambda_1-1,\ldots,\lambda_\ell-1)}, & \text{if } \mu_1 = \ell = \lambda_1^t \end{cases}
\end{aligned}$$

where the last equality used

$$e_k^\perp(h_n) = \begin{cases} h_{n-1}, & \text{if } k = 1; \\ 0, & \text{if } k \geq 2. \end{cases}$$

Now apply the induction hypothesis, since $(\lambda_1 - 1, \ldots, \lambda_\ell - 1)^t = (\lambda_2^t, \lambda_3^t, \ldots)$.

To prove (3.3.6), note that any λ, μ in Par_n satisfy $(e_{\mu^t}, h_\lambda) = (e_{\mu^t}^\perp(h_\lambda), 1) = e_{\mu^t}^\perp(h_\lambda)$ (since degree considerations enforce $e_{\mu^t}^\perp(h_\lambda) \in A_0 = \mathbf{k} \cdot 1$), and thus

$$(e_{\mu^t}, h_\lambda) = e_{\mu^t}^\perp(h_\lambda) = \begin{cases} 1, & \text{if } \mu^t = \lambda^t; \\ 0, & \text{if } \mu^t >_{\mathrm{lex}} \lambda^t \end{cases}$$

(by (3.3.5)). This means that the matrix $[(e_{\mu^t}, h_\lambda)]_{\lambda,\mu\in\mathrm{Par}_n}$ is unitriangular with respect to some total order on Par_n (namely, the lexicographic order on the conjugate partitions), and hence has determinant 1. $\square$

The following proposition will be the crux of the proof of Theorem 3.3.3(f) and (g), and turns out to be closely related to Kerov's *asymptotic theory of characters of the symmetric groups* [108].

Proposition 3.3.5. *Given a PSH A with PSH-basis Σ containing only one primitive ρ, the two maps $A \to \mathbb{Z}$ defined on $A = \bigoplus_{n\geq 0} A_n$ via*

$$\delta_h = \bigoplus_n h_n^\perp,$$
$$\delta_e = \bigoplus_n e_n^\perp$$

are characterized as the only two $\mathbb{Z}$-linear maps $A \xrightarrow{\delta} \mathbb{Z}$ with the three properties of being

- ***positive***: $\delta(\mathbb{N}\Sigma) \subset \mathbb{N}$,
- ***multiplicative***: $\delta(a_1 a_2) = \delta(a_1)\delta(a_2)$ *for all* $a_1, a_2 \in A$, *and*
- ***normalized***: $\delta(\rho) = 1$.

Proof. Notice that A is of finite type (as shown in the proof of Theorem 3.3.1). Thus, A^o is a graded Hopf algebra isomorphic to A.

It should be clear from their definitions that δ_h, δ_e are $\mathbb{Z}$-linear, positive and normalized. To see that δ_h is multiplicative, by $\mathbb{Z}$-linearity, it suffices to check that for a_1, a_2 in A_{n_1}, A_{n_2} with $n_1 + n_2 = n$, one has

$$\delta_h(a_1 a_2) = h_n^\perp(a_1 a_2) = \sum_{i_1+i_2=n} h_{i_1}^\perp(a_1) h_{i_2}^\perp(a_2) = h_{n_1}^\perp(a_1) h_{n_2}^\perp(a_2) = \delta_h(a_1)\delta_h(a_2)$$

in which the second equality used Proposition 2.8.2(iv) and Theorem 3.3.1(d). The argument for δ_e is symmetric.

Conversely, given $A \xrightarrow{\delta} \mathbb{Z}$ which is $\mathbb{Z}$-linear, positive, multiplicative, and normalized, note that

$$\delta(h_2) + \delta(e_2) = \delta(h_2 + e_2) = \delta(\rho^2) = \delta(\rho)^2 = 1^2 = 1$$

and hence positivity implies that either $\delta(h_2) = 0$ or $\delta(e_2) = 0$. Assume the latter holds, and we will show that $\delta = \delta_h$.

Given any σ in $\Sigma \cap A_n \setminus \{h_n\}$, note that $e_2^\perp \sigma \neq 0$ by Theorem 3.3.1(b), and hence $0 \neq (e_2^\perp \sigma, \rho^{n-2}) = (\sigma, e_2 \rho^{n-2})$. Thus σ is a constituent of $e_2\rho^{n-2}$, so positivity implies

$$0 \leq \delta(\sigma) \leq \delta(e_2 \rho^{n-2}) = \delta(e_2)\delta(\rho^{n-2}) = 0.$$

Thus $\delta(\sigma) = 0$ for σ in $\Sigma \cap A_n \setminus \{h_n\}$. Since $\delta(\rho^n) = \delta(\rho)^n = 1^n = 1$, this forces $\delta(h_n) = 1$, for each $n \geq 0$ (including $n = 0$, as $1 = \delta(\rho) = \delta(\rho \cdot 1) = \delta(\rho)\delta(1) = 1 \cdot \delta(1) = \delta(1)$). Thus $\delta = \delta_h$. The argument when $\delta(h_2) = 0$ showing $\delta = \delta_e$ is symmetric. $\square$

Proof of Theorem 3.3.3. Many of the assertions of parts (e) and (f) will come from constructing the unique nontrivial PSH-automorphism ω of A from the antipode S: for homogeneous a in A_n, define $\omega(a) := (-1)^n S(a)$. We now study some of the properties of S and ω.

Notice that A is of finite type (as shown in the proof of Theorem 3.3.1). Thus, A^o is a graded Hopf algebra isomorphic to A.

Since A is a PSH, it is commutative by Theorem 3.1.7 (applied to $A \otimes_{\mathbb{Z}} \mathbb{Q}$). This implies both that S is an algebra endomorphism by Proposition 1.4.10 (since Exercise 1.5.8(a) shows that the algebra anti-endomorphisms of a commutative algebra are the same as its algebra endomorphisms), and that $S^2 = \mathrm{id}_A$ by Corollary 1.4.12. Thus, ω is an algebra endomorphism and satisfies $\omega^2 = \mathrm{id}_A$.

Since A is self-dual and the defining diagram (1.4.3) satisfied by the antipode S is sent to itself when one replaces A by A^o and all maps by their adjoints, one concludes that $S = S^*$ (where S^* means the restricted adjoint $S^* : A^o \to A^o$), i.e., S is self-adjoint. Since S is an algebra endomorphism, and $S = S^*$, in fact S is also a coalgebra endomorphism, a bialgebra endomorphism, and a Hopf endomorphism (by Corollary 1.4.27). The same properties are shared by ω.

Since $\mathrm{id}_A = S^2 = SS^*$, one concludes that S is an isometry, and hence so is ω.

Since ρ is primitive, one has $S(\rho) = -\rho$ and $\omega(\rho) = \rho$. Therefore $\omega(\rho^n) = \rho^n$ for $n = 1, 2, \ldots$. Use this as follows to check that ω is a PSH-automorphism, which amounts to checking that every σ in Σ has $\omega(\sigma)$ in Σ:

$$(\omega(\sigma), \omega(\sigma)) = (\sigma, \sigma) = 1$$

so that $\pm\omega(\sigma)$ lies in Σ, but also if σ lies in A_n, then

$$(\omega(\sigma), \rho^n) = (\sigma, \omega(\rho^n)) = (\sigma, \rho^n) > 0.$$

In summary, ω is a PSH-automorphism of A, an isometry, and an involution.

Let us try to determine the action of ω on the $\{h_n\}$. By similar reasoning as in (3.3.3), one has

$$\Delta(h_2) = h_2 \otimes 1 + \rho \otimes \rho + 1 \otimes h_2.$$

Thus $0 = S(h_2) + S(\rho)\rho + h_2$, and combining this with $S(\rho) = -\rho$, one has $S(h_2) = e_2$. Thus also $\omega(h_2) = (-1)^2 S(h_2) = e_2$.

We claim that this forces $\omega(h_n) = e_n$, because $h_2^\perp \omega(h_n) = 0$ via the following calculation: for any a in A one has

$$\begin{aligned}
(h_2^\perp \omega(h_n), a) &= (\omega(h_n), h_2 a) \\
&= (h_n, \omega(h_2 a)) \\
&= (h_n, e_2 \omega(a)) \\
&= (e_2^\perp h_n, \omega(a)) = (0, \omega(a)) = 0.
\end{aligned}$$

Consequently the involution ω swaps h_n and e_n, while the antipode S has $S(h_n) = (-1)^n e_n$ and $S(e_n) = (-1)^n h_n$. Thus the coproduct formulas in (d) and definition of the antipode S imply the relation (2.4.4) between $\{h_n\}$ and $\{e_n\}$.

This relation (2.4.4) also lets one recursively express the h_n as polynomials with integer coefficients in the $\{e_n\}$, and vice-versa, so that $\{h_n\}$ and $\{e_n\}$ each generate the same $\mathbb{Z}$-subalgebra A' of A. We wish to show that A' exhausts A.

We argue that Lemma 3.3.4 implies that the *Gram matrix* $[(h_\mu, h_\lambda)]_{\mu,\lambda\in\mathrm{Par}_n}$ has determinant ± 1 as follows. Since $\{h_n\}$ and $\{e_n\}$ both generate A', there exists a $\mathbb{Z}$-matrix $(a_{\mu,\lambda})$ expressing $e_{\mu^t} = \sum_\lambda a_{\mu,\lambda} h_\lambda$, and one has

$$[(e_{\mu^t}, h_\lambda)] = [a_{\mu,\lambda}] \cdot [(h_\mu, h_\lambda)] .$$

Taking determinants of these three $\mathbb{Z}$-matrices, and using the fact that the determinant on the left is 1 (by (3.3.6)), both determinants on the right must also be ± 1.

Now we will show that every $\sigma \in \Sigma \cap A_n$ lies in A'_n. Uniquely express $\sigma = \sigma' + \sigma''$ in which σ' lies in the $\mathbb{R}$-span $\mathbb{R}A'_n$ and σ'' lies in the real perpendicular space $(\mathbb{R}A'_n)^\perp$ inside $\mathbb{R} \otimes_{\mathbb{Z}} A_n$. One can compute $\mathbb{R}$-coefficients $(c_\mu)_{\mu\in\mathrm{Par}_n}$ that express $\sigma' = \sum_\mu c_\mu h_\mu$ by solving the system

$$\left(\sum_\mu c_\mu h_\mu, h_\lambda\right) = (\sigma, h_\lambda) \text{ for } \lambda \in \mathrm{Par}_n .$$

This linear system is governed by the Gram matrix $[(h_\mu, h_\lambda)]_{\mu,\lambda\in\mathrm{Par}_n}$ with determinant ± 1, and its right side has $\mathbb{Z}$-entries since σ, h_λ lie in A. Hence the solution $(c_\mu)_{\mu\in\mathrm{Par}_n}$ will have $\mathbb{Z}$-entries, so σ' lies in A'. Furthermore, $\sigma'' = \sigma - \sigma'$ will lie in A, and hence by the orthogonality of σ', σ'',

$$1 = (\sigma, \sigma) = (\sigma', \sigma') + (\sigma'', \sigma'').$$

One concludes that either $\sigma'' = 0$, or $\sigma' = 0$. The latter cannot occur since it would mean that $\sigma = \sigma''$ is perpendicular to all of A'. But $\rho^n = h_1^n$ lies in A', and $(\sigma, \rho^n) \neq 0$. Thus $\sigma'' = 0$, meaning $\sigma = \sigma'$ lies in A'. This completes the proof of assertion (e). Note that in the process, having shown $\det(h_\mu, h_\lambda)_{\lambda,\mu\in\mathrm{Par}_n} = \pm 1$, one also knows that $\{h_\lambda\}_{\lambda\in\mathrm{Par}_n}$ are $\mathbb{Z}$-linearly independent, so that $\{h_1, h_2, \ldots\}$ are algebraically independent[208], and $A = \mathbb{Z}[h_1, h_2, \ldots]$ is the polynomial algebra generated by $\{h_1, h_2, \ldots\}$.

For assertion (f), we have seen that ω gives such a PSH-automorphism $A \to A$, swapping $h_n \leftrightarrow e_n$. Conversely, given a PSH-automorphism $A \xrightarrow{\varphi} A$, consider the positive, multiplicative, normalized $\mathbb{Z}$-linear map $\delta := \delta_h \circ \varphi : A \to \mathbb{Z}$. Proposition 3.3.5 shows that either

- $\delta = \delta_h$, which then forces $\varphi(h_n) = h_n$ for all n, so $\varphi = \mathrm{id}_A$, or
- $\delta = \delta_e$, which then forces $\varphi(e_n) = h_n$ for all n, so $\varphi = \omega$.

For assertion (g), given a PSH A with PSH-basis Σ having exactly one primitive ρ, since we have seen $A = \mathbb{Z}[h_1, h_2, \ldots]$, where h_n in A is as defined in Theorem 3.3.1, one can uniquely define an algebra morphism $A \xrightarrow{\varphi} \Lambda$ that sends the element h_n to the complete homogeneous symmetric function $h_n(\mathbf{x})$. Assertions (d) and (e) show that φ is a bialgebra isomorphism, and hence it is a Hopf isomorphism. To show that it is a PSH-isomorphism, we first note that it is an isometry because one can iterate Proposition 2.8.2(iv) together with assertions (c) and (d) to compute all inner products

$$(h_\mu, h_\lambda)_A = (1, h_\mu^\perp h_\lambda)_A = (1, h_{\mu_1}^\perp h_{\mu_2}^\perp \cdots (h_{\lambda_1} h_{\lambda_2} \cdots))_A$$

for μ, λ in Par_n. Hence

$$(h_\mu, h_\lambda)_A = (h_\mu(\mathbf{x}), h_\lambda(\mathbf{x}))_\Lambda = (\varphi(h_\mu), \varphi(h_\lambda))_\Lambda.$$

[208] by Exercise 2.2.14(c)

Once one knows φ is an isometry, then elements ω in $\Sigma \cap A_n$ are characterized in terms of the form $(\cdot, \cdot)$ by $(\omega, \omega) = 1$ and $(\omega, \rho^n) > 0$. Hence φ sends each σ in Σ to a Schur function s_λ, and is a PSH-isomorphism. $\square$

4. Complex representations for $\mathfrak{S}_n$, wreath products, $GL_n(\mathbb{F}_q)$

After reviewing the basics that we will need from representation and character theory of finite groups, we give Zelevinsky's three main examples of PSH's arising as spaces of virtual characters for three towers of finite groups:

- *symmetric* groups,
- their *wreath products* with any finite group, and
- the finite *general linear* groups.

Much in this chapter traces its roots to Zelevinsky's book [227]. The results concerning the symmetric groups, however, are significantly older and spread across the literature: see, e.g., [206, §7.18], [73, §7.3], [142, §I.7], [186, §4.7], [113], for proofs using different tools.

4.1. **Review of complex character theory.**

We shall now briefly discuss some basics of representation (and character) theory that will be used below. A good source for this material, including the crucial Mackey formula, is Serre [197, Chaps. 1-7].[209]

4.1.1. *Basic definitions, Maschke, Schur.*

For a group G, a *representation of G* is a homomorphism $G \xrightarrow{\varphi} GL(V)$ for some vector space V over a field. We will take the field to be $\mathbb{C}$ from now on, and we will also assume that V is finite-dimensional over $\mathbb{C}$. Thus a representation of G is the same as a finite-dimensional (left) $\mathbb{C}G$-module V. (We use the notations $\mathbb{C}G$ and $\mathbb{C}[G]$ synonymously for the group algebra of G over $\mathbb{C}$. More generally, if S is a set, then $\mathbb{C}S = \mathbb{C}[S]$ denotes the free $\mathbb{C}$-module with basis S.)

We also assume that G is finite, so that Maschke's Theorem[210] says that $\mathbb{C}G$ is semisimple, meaning that every $\mathbb{C}G$-module $U \subset V$ has a $\mathbb{C}G$-module complement U' with $V = U \oplus U'$. Equivalently, *indecomposable* $\mathbb{C}G$-modules are the same thing as *simple* (=*irreducible*) $\mathbb{C}G$-modules.

Schur's Lemma implies that for two simple $\mathbb{C}G$-modules V_1, V_2, one has

$$\operatorname{Hom}_{\mathbb{C}G}(V_1, V_2) \cong \begin{cases} \mathbb{C}, & \text{if } V_1 \cong V_2; \\ 0, & \text{if } V_1 \not\cong V_2. \end{cases}$$

4.1.2. *Characters and Hom spaces.*

A $\mathbb{C}G$-module V is completely determined up to isomorphism by its *character*

$$\begin{array}{rcl} G & \xrightarrow{\chi_V} & \mathbb{C}, \\ g & \longmapsto & \chi_V(g) := \operatorname{trace}(g : V \to V). \end{array}$$

This character χ_V is a *class function*, meaning it is constant on G-conjugacy classes. The space $R_{\mathbb{C}}(G)$ of class functions $G \to \mathbb{C}$ has a Hermitian, positive definite form

$$(f_1, f_2)_G := \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}.$$

For any two $\mathbb{C}G$-modules V_1, V_2,

$$(\chi_{V_1}, \chi_{V_2})_G = \dim_{\mathbb{C}} \operatorname{Hom}_{\mathbb{C}G}(V_1, V_2). \tag{4.1.1}$$

[209] More advanced treatments of representation theory can be found in [222] and [69].

[210] ... which has a beautiful generalization to finite-dimensional Hopf algebras due to Larson and Sweedler; see Montgomery [157, §2.2].

The set of all *irreducible characters*

$$\mathrm{Irr}(G) = \{\chi_V : V \text{ is a simple } \mathbb{C}G\text{-module}\}$$

forms an orthonormal basis of $R_{\mathbb{C}}(G)$ with respect to this form, and spans a $\mathbb{Z}$-sublattice

$$R(G) := \mathbb{Z}\,\mathrm{Irr}(G) \subset R_{\mathbb{C}}(G)$$

sometimes called the *virtual characters* of G. For every $\mathbb{C}G$-module V, the character χ_V belongs to $R(G)$.

Instead of working with the Hermitian form $(\cdot,\cdot)_G$ on G, we could also (and some authors do) define a $\mathbb{C}$-bilinear form $\langle\cdot,\cdot\rangle_G$ on $R_{\mathbb{C}}(G)$ by

$$\langle f_1, f_2\rangle_G := \frac{1}{|G|}\sum_{g\in G} f_1(g) f_2(g^{-1}).$$

This form is not identical with $(\cdot,\cdot)_G$ (indeed, $\langle\cdot,\cdot\rangle_G$ is bilinear while $(\cdot,\cdot)_G$ is Hermitian), but it still satisfies (4.1.1), and thus is identical with $(\cdot,\cdot)_G$ on $R(G)\times R(G)$. Hence, for all we are going to do until Section 4.9, we could just as well use the form $\langle\cdot,\cdot\rangle_G$ instead of $(\cdot,\cdot)_G$.

4.1.3. *Tensor products.* Given two groups G_1, G_2 and $\mathbb{C}G_i$-modules V_i for $i = 1, 2$, their tensor product $V_1 \otimes_{\mathbb{C}} V_2$ becomes a $\mathbb{C}[G_1 \times G_2]$-module via $(g_1, g_2)(v_1 \otimes v_2) = g_1(v_1) \otimes g_2(v_2)$. This module is called the *(outer) tensor product* of V_1 and V_2. When V_1, V_2 are both simple, then so is $V_1 \otimes V_2$, and every simple $\mathbb{C}[G_1 \times G_2]$-module arises this way (with V_1 and V_2 determined uniquely up to isomorphism).[211] Thus one has identifications and isomorphisms

$$\mathrm{Irr}(G_1 \times G_2) = \mathrm{Irr}(G_1) \times \mathrm{Irr}(G_2),$$
$$R(G_1 \times G_2) \cong R(G_1) \otimes_{\mathbb{Z}} R(G_2);$$

here, $\chi_{V_1} \otimes \chi_{V_2} \in R(G_1) \otimes_{\mathbb{Z}} R(G_2)$ is being identified with $\chi_{V_1 \otimes V_2} \in R(G_1 \times G_2)$ for all $\mathbb{C}G_1$-modules V_1 and all $\mathbb{C}G_2$-modules V_2. The latter isomorphism is actually a restriction of the isomorphism $R_{\mathbb{C}}(G_1 \times G_2) \cong R_{\mathbb{C}}(G_1) \otimes_{\mathbb{C}} R_{\mathbb{C}}(G_2)$ under which every pure tensor $\phi_1 \otimes \phi_2 \in R_{\mathbb{C}}(G_1) \otimes_{\mathbb{C}} R_{\mathbb{C}}(G_2)$ corresponds to the class function $G_1 \times G_2 \to \mathbb{C}$, $(g_1, g_2) \mapsto \phi_1(g_1) \otimes \phi_2(g_2)$.

Given two $\mathbb{C}G_1$-modules V_1 and W_1 and two $\mathbb{C}G_2$-modules V_2 and W_2, we have

$$(\chi_{V_1\otimes V_2}, \chi_{W_1\otimes W_2})_{G_1\times G_2} = (\chi_{V_1}, \chi_{W_1})_{G_1} (\chi_{V_2}, \chi_{W_2})_{G_2}. \tag{4.1.2}$$

4.1.4. *Induction and restriction.* Given a subgroup $H < G$ and $\mathbb{C}H$-module U, one can use the fact that $\mathbb{C}G$ is a $(\mathbb{C}G, \mathbb{C}H)$-bimodule to form the *induced $\mathbb{C}G$-module*

$$\mathrm{Ind}_H^G U := \mathbb{C}G \otimes_{\mathbb{C}H} U.$$

The fact that $\mathbb{C}G$ is free as a (right-)$\mathbb{C}H$-module[212] on basis elements $\{t_g\}_{gH\in G/H}$ makes this tensor product easy to analyze. For example one

[211] This is proven in [197, §3.2, Thm. 10]. The fact that $\mathbb{C}$ is algebraically closed is essential for this!

[212] ... which also has a beautiful generalization to finite-dimensional Hopf algebras due to Nichols and Zoeller; see [157, §3.1].

can compute its character

$$\chi_{\operatorname{Ind}_H^G U}(g) = \frac{1}{|H|} \sum_{\substack{k \in G: \\ kgk^{-1} \in H}} \chi_U(kgk^{-1}). \tag{4.1.3}$$

[213] One can also recognize when a $\mathbb{C}G$-module V is isomorphic to $\operatorname{Ind}_H^G U$ for some $\mathbb{C}H$-module U: this happens if and only if there is an H-stable subspace $U \subset V$ having the property that $V = \bigoplus_{gH \in G/H} gU$.

The above construction of a $\mathbb{C}G$-module $\operatorname{Ind}_H^G U$ corresponding to any $\mathbb{C}H$-module U is part of a functor Ind_H^G from the category of $\mathbb{C}H$-modules to the category of $\mathbb{C}G$-modules[214]; this functor is called *induction*.

Besides induction on $\mathbb{C}H$-modules, one can define induction on class functions of H:

Exercise 4.1.1. Let G be a finite group, and H a subgroup of G. Let $f \in R_{\mathbb{C}}(H)$ be a class function. We define the *induction* $\operatorname{Ind}_H^G f$ of f to be the function $G \to \mathbb{C}$ given by

$$\left(\operatorname{Ind}_H^G f\right)(g) = \frac{1}{|H|} \sum_{\substack{k \in G: \\ kgk^{-1} \in H}} f\left(kgk^{-1}\right) \qquad \text{for all } g \in G. \tag{4.1.4}$$

(a) Prove that this induction $\operatorname{Ind}_H^G f$ is a class function on G, hence belongs to $R_{\mathbb{C}}(G)$.
(b) Let J be a system of right coset[215] representatives for $H \backslash G$, so that $G = \bigsqcup_{j \in J} Hj$. Prove that

$$\left(\operatorname{Ind}_H^G f\right)(g) = \sum_{\substack{j \in J: \\ jgj^{-1} \in H}} f\left(jgj^{-1}\right) \qquad \text{for all } g \in G.$$

The induction Ind_H^G defined in Exercise 4.1.1 is a $\mathbb{C}$-linear map $R_{\mathbb{C}}(H) \to R_{\mathbb{C}}(G)$. Since every $\mathbb{C}H$-module U satisfies

$$\chi_{\operatorname{Ind}_H^G U} = \operatorname{Ind}_H^G(\chi_U) \tag{4.1.5}$$

[216], this $\mathbb{C}$-linear map Ind_H^G restricts to a $\mathbb{Z}$-linear map $R(H) \to R(G)$ (also denoted Ind_H^G) which sends the character χ_U of any $\mathbb{C}H$-module U to the character $\chi_{\operatorname{Ind}_H^G U}$ of the induced $\mathbb{C}G$-module $\operatorname{Ind}_H^G U$.

Exercise 4.1.2. Let G, H and I be three finite groups such that $I < H < G$. Let U be a $\mathbb{C}I$-module. Prove that $\operatorname{Ind}_H^G \operatorname{Ind}_I^H U \cong \operatorname{Ind}_I^G U$. (This fact is often referred to as the *transitivity of induction*.)

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[213] See [197, §7.2, Prop. 20(ii)] for the proof of this equality. (Another proof is given in [69, Remark 5.9.2 (the Remark after Theorem 4.32 in the arXiv version)], but [69] uses a different definition of $\operatorname{Ind}_H^G U$; see Remark 4.1.5 for why it is equivalent to ours. Yet another proof of (4.1.3) is given in Exercise 4.1.14(k).)

[214] On morphisms, it sends any $f : U \to U'$ to $\operatorname{id}_{\mathbb{C}G} \otimes_{\mathbb{C}H} f : \mathbb{C}G \otimes_{\mathbb{C}H} U \to \mathbb{C}G \otimes_{\mathbb{C}H} U'$.

[215] A *right coset* of a subgroup H in a group G is defined to be a subset of G having the form Hj for some $j \in G$. Similarly, a *left coset* has the form jH for some $j \in G$.

[216] This follows by comparing the value of $\chi_{\operatorname{Ind}_H^G U}(g)$ obtained from (4.1.3) with the value of $\left(\operatorname{Ind}_H^G(\chi_U)\right)(g)$ found using (4.1.4).

Exercise 4.1.3. Let G_1 and G_2 be two groups. Let $H_1 < G_1$ and $H_2 < G_2$ be two subgroups. Let U_1 be a $\mathbb{C}H_1$-module, and U_2 be a $\mathbb{C}H_2$-module. Show that

$$\operatorname{Ind}_{H_1 \times H_2}^{G_1 \times G_2} (U_1 \otimes U_2) \cong \left(\operatorname{Ind}_{H_1}^{G_1} U_1\right) \otimes \left(\operatorname{Ind}_{H_2}^{G_2} U_2\right) \tag{4.1.6}$$

as $\mathbb{C}[G_1 \times G_2]$-modules.

The *restriction* operation $V \mapsto \operatorname{Res}_H^G V$ restricts a $\mathbb{C}G$-module V to a $\mathbb{C}H$-module. *Frobenius reciprocity* asserts the adjointness between Ind_H^G and Res_H^G

$$\operatorname{Hom}_{\mathbb{C}G}(\operatorname{Ind}_H^G U, V) \cong \operatorname{Hom}_{\mathbb{C}H}(U, \operatorname{Res}_H^G V), \tag{4.1.7}$$

as a special case ($S = A = \mathbb{C}G, R = \mathbb{C}H, B = U, C = V$) of the general *adjoint associativity*

$$\operatorname{Hom}_S(A \otimes_R B, C) \cong \operatorname{Hom}_R(B, \operatorname{Hom}_S(A, C)) \tag{4.1.8}$$

for S, R two rings, A an (S, R)-bimodule, B a left R-module, C a left S-module.

We can define not just the restriction of a $\mathbb{C}G$-module, but also the *restriction of a class function* $f \in R_{\mathbb{C}}(G)$. When H is a subgroup of G, the restriction $\operatorname{Res}_H^G f$ of an $f \in R_{\mathbb{C}}(G)$ is defined as the result of restricting the map $f : G \to \mathbb{C}$ to H. This $\operatorname{Res}_H^G f$ is easily seen to belong to $R_{\mathbb{C}}(H)$, and so Res_H^G is a $\mathbb{C}$-linear map $R_{\mathbb{C}}(G) \to R_{\mathbb{C}}(H)$. This map restricts to a $\mathbb{Z}$-linear map $R(G) \to R(H)$, since we have $\operatorname{Res}_H^G \chi_V = \chi_{\operatorname{Res}_H^G V}$ for any $\mathbb{C}G$-module V. Taking characters in (4.1.7) (and recalling $\operatorname{Res}_H^G \chi_V = \chi_{\operatorname{Res}_H^G V}$ and (4.1.5)), we obtain

$$(\operatorname{Ind}_H^G \chi_U, \chi_V)_G = (\chi_U, \operatorname{Res}_H^G \chi_V)_H. \tag{4.1.9}$$

By bilinearity, this yields the equality

$$\left(\operatorname{Ind}_H^G \alpha, \beta\right)_G = \left(\alpha, \operatorname{Res}_H^G \beta\right)_H$$

for any class functions $\alpha \in R_{\mathbb{C}}(H)$ and $\beta \in R_{\mathbb{C}}(G)$ (since $R(G)$ spans $R_{\mathbb{C}}(G)$ as a $\mathbb{C}$-vector space).

Exercise 4.1.4. Let G be a finite group, and let $H < G$. Let U be a $\mathbb{C}H$-module. If A and B are two algebras, P is a (B, A)-bimodule and Q is a left B-module, then $\operatorname{Hom}_B(P, Q)$ is a left A-module (since $\mathbb{C}G$ is a $(\mathbb{C}H, \mathbb{C}G)$-bimodule). As a consequence, $\operatorname{Hom}_{\mathbb{C}H}(\mathbb{C}G, U)$ is a $\mathbb{C}G$-module. Prove that this $\mathbb{C}G$-module is isomorphic to $\operatorname{Ind}_H^G U$.

Remark 4.1.5. Some texts *define* the induction $\operatorname{Ind}_H^G U$ of a $\mathbb{C}H$-module U to be $\operatorname{Hom}_{\mathbb{C}H}(\mathbb{C}G, U)$ (rather than to be $\mathbb{C}G \otimes_{\mathbb{C}H} U$, as we did).[217] As Exercise 4.1.4 shows, this definition is equivalent to ours as long as G is finite (but not otherwise).

Exercise 4.1.4 yields the following "wrong-way" version of Frobenius reciprocity:

Exercise 4.1.6. Let G be a finite group; let $H < G$. Let U be a $\mathbb{C}G$-module, and let V be a $\mathbb{C}H$-module. Prove that $\operatorname{Hom}_{\mathbb{C}G}\left(U, \operatorname{Ind}_H^G V\right) \cong \operatorname{Hom}_{\mathbb{C}H}\left(\operatorname{Res}_H^G U, V\right)$.

[217] Or they define it as a set of morphisms of H-sets from G to U (this is how [69, Def. 5.8.1 (Def. 4.28 in the arXiv version)] defines it); this is easily seen to be equivalent to $\operatorname{Hom}_{\mathbb{C}H}(\mathbb{C}G, U)$.

4.1.5. *Mackey's formula.* Mackey gave an alternate description of a module which has been induced and then restricted. To state it, for a subgroup $H < G$ and g in G, let $H^g := g^{-1}Hg$ and ${}^gH := gHg^{-1}$. Given a $\mathbb{C}H$-module U, say defined by a homomorphism $H \xrightarrow{\varphi} GL(U)$, let U^g denote the $\mathbb{C}[gHg^{-1}]$-module on the same $\mathbb{C}$-vector space U defined by the composite homomorphism

$$\begin{array}{ccccc} {}^gH & \longrightarrow & H & \xrightarrow{\varphi} & GL(U), \\ h & \longmapsto & g^{-1}hg. & & \end{array}$$

Theorem 4.1.7. *(Mackey's formula) Consider subgroups $H, K < G$, and any $\mathbb{C}H$-module U. If $\{g_1, \ldots, g_t\}$ are double coset representatives for $K\backslash G/H$, then*

$$\operatorname{Res}^G_K \operatorname{Ind}^G_H U \cong \bigoplus_{i=1}^t \operatorname{Ind}^K_{{}^{g_i}H\cap K} \left(\left(\operatorname{Res}^H_{H\cap K^{g_i}} U \right)^{g_i} \right).$$

Proof. In this proof, all tensor product symbols $\otimes$ should be interpreted as $\otimes_{\mathbb{C}H}$. Recall $\mathbb{C}G$ has $\mathbb{C}$-basis $\{t_g\}_{g\in G}$. For subsets $S \subset G$, let $\mathbb{C}[S]$ denote the $\mathbb{C}$-span of $\{t_g\}_{g\in S}$ in $\mathbb{C}G$.

Note that each double coset KgH gives rise to a sub-(K, H)-bimodule $\mathbb{C}[KgH]$ within $\mathbb{C}G$, and one has a $\mathbb{C}K$-module direct sum decomposition

$$\operatorname{Ind}^G_H U = \mathbb{C}G \otimes U = \bigoplus_{i=1}^t \mathbb{C}[Kg_iH] \otimes U.$$

Hence it suffices to check for any element g in G that

$$\mathbb{C}[KgH] \otimes U \cong \operatorname{Ind}^K_{{}^gH\cap K} \left(\left(\operatorname{Res}^H_{H\cap K^g} U \right)^g \right).$$

Note that ${}^gH \cap K$ is the subgroup of K consisting of the elements k in K for which $kgH = gH$. Hence by picking $\{k_1, \ldots, k_s\}$ to be coset representatives for $K/({}^gH \cap K)$, one disjointly decomposes the double coset

$$KgH = \bigsqcup_{j=1}^s k_j({}^gH \cap K)gH,$$

giving a $\mathbb{C}$-vector space direct sum decomposition

$$\begin{aligned} \mathbb{C}[KgH] \otimes U &= \bigoplus_{j=1}^s \mathbb{C}\left[k_j\left({}^gH \cap K\right) gH\right] \otimes U \\ &\cong \operatorname{Ind}^K_{{}^gH\cap K} \left(\mathbb{C}[({}^gH \cap K)\, gH] \otimes U\right). \end{aligned}$$

So it remains to check that one has a $\mathbb{C}[{}^gH \cap K]$-module isomorphism

$$\mathbb{C}[({}^gH \cap K)\, gH] \otimes U \cong \left(\operatorname{Res}^H_{H\cap K^g} U \right)^g.$$

Bearing in mind that, for each k in ${}^gH \cap K$ and h in H, one has $g^{-1}kg$ in H and hence

$$t_{kgh} \otimes u = t_g \cdot t_{g^{-1}kg\cdot h} \otimes u = t_g \otimes g^{-1}kgh \cdot u,$$

one sees that this isomorphism can be defined by mapping

$$t_{kgh} \otimes u \longmapsto g^{-1}kgh \cdot u. \qquad \square$$

4.1.6. *Inflation and fixed points.* There are two (adjoint) constructions on representations that apply when one has a normal subgroup $K \triangleleft G$. Given a $\mathbb{C}[G/K]$-module U, say defined by the homomorphism $G/K \overset{\varphi}{\to} GL(U)$, the *inflation* of U to a $\mathbb{C}G$-module $\operatorname{Infl}_{G/K}^{G} U$ has the same underlying space U, and is defined by the composite homomorphism $G \to G/K \overset{\varphi}{\to} GL(U)$. We will later use the easily-checked fact that when $H < G$ is any other subgroup, one has

$$\operatorname{Res}_H^G \operatorname{Infl}_{G/K}^G U = \operatorname{Infl}_{H/H\cap K}^H \operatorname{Res}_{H/H\cap K}^{G/K} U. \tag{4.1.10}$$

(We regard $H/H \cap K$ as a subgroup of G/K, since the canonical homomorphism $H/H \cap K \to G/K$ is injective.)

Inflation turns out to be adjoint to the K*-fixed space construction* sending a $\mathbb{C}G$-module V to the $\mathbb{C}[G/K]$-module

$$V^K := \{v \in V : k(v) = v \text{ for } k \in K\}.$$

Note that V^K is indeed a G-stable subspace: for any v in V^K and g in G, one has that $g(v)$ lies in V^K since an element k in K satisfies $kg(v) = g \cdot g^{-1}kg(v) = g(v)$ as $g^{-1}kg$ lies in K. One has this adjointness

$$\operatorname{Hom}_{\mathbb{C}G}(\operatorname{Infl}_{G/K}^G U, V) = \operatorname{Hom}_{\mathbb{C}[G/K]}(U, V^K), \tag{4.1.11}$$

because any $\mathbb{C}G$-module homomorphism φ on the left must have the property that $k\varphi(u) = \varphi(k(u)) = \varphi(u)$ for all k in K, so that φ actually lies on the right.

We will also need the following formula for the character χ_{V^K} in terms of the character χ_V:

$$\chi_{V^K}(gK) = \frac{1}{|K|} \sum_{k \in K} \chi_V(gk). \tag{4.1.12}$$

To see this, note that when one has a $\mathbb{C}$-linear endomorphism φ on a space V that preserves some $\mathbb{C}$-subspace $W \subset V$, if $V \overset{\pi}{\to} W$ is any idempotent projection onto W, then the trace of the restriction $\varphi|_W$ equals the trace of $\varphi \circ \pi$ on V. Applying this to $W = V^K$ and $\varphi = g$, with $\pi = \frac{1}{|K|} \sum_{k \in K} k$, gives (4.1.12).[218]

Another way to restate (4.1.12) is:

$$\chi_{V^K}(gK) = \frac{1}{|K|} \sum_{h \in gK} \chi_V(h). \tag{4.1.13}$$

Inflation and K-fixed space construction can also be defined on class functions. For inflation, this is particularly easy: Inflation $\operatorname{Infl}_{G/K}^G f$ of an $f \in R_{\mathbb{C}}(G/K)$ is defined as the composition $G \twoheadrightarrow G/K \xrightarrow{f} \mathbb{C}$. This is a class function of G and thus lies in $R_{\mathbb{C}}(G)$. Thus, inflation $\operatorname{Infl}_{G/K}^G$ is a $\mathbb{C}$-linear map $R_{\mathbb{C}}(G/K) \to R_{\mathbb{C}}(G)$. It restricts to a $\mathbb{Z}$-linear map $R(G/K) \to R(G)$, since it is clear that every $\mathbb{C}(G/K)$-module U satisfies $\operatorname{Infl}_{G/K}^G \chi_U = \chi_{\operatorname{Infl}_{G/K}^G U}$.

We can also use (4.1.12) (or (4.1.13)) as inspiration for defining a "K-fixed space construction" on class functions. Explicitly, for every class

[218] For another proof of (4.1.12), see Exercise 4.1.14(l).

function $f \in R_{\mathbb{C}}(G)$, we define a class function $f^K \in R_{\mathbb{C}}(G/K)$ by

$$f^K(gK) = \frac{1}{|K|} \sum_{k \in K} f(gk) = \frac{1}{|K|} \sum_{h \in gK} f(h).$$

The map $(\cdot)^K : R_{\mathbb{C}}(G) \to R_{\mathbb{C}}(G/K)$, $f \mapsto f^K$ is $\mathbb{C}$-linear, and restricts to a $\mathbb{Z}$-linear map $R(G) \to R(G/K)$. Again, we have a compatibility with the K-fixed point construction on modules: We have $\chi_{V^K} = (\chi_V)^K$ for every $\mathbb{C}G$-module V.

Taking characters in (4.1.11), we obtain

$$(\operatorname{Infl}_{G/K}^G \chi_U, \chi_V)_G = (\chi_U, \chi_V^K)_{G/K} \tag{4.1.14}$$

for any $\mathbb{C}[G/K]$-module U and any $\mathbb{C}G$-module V (since $\chi_{\operatorname{Infl}_{G/K}^G U} = \operatorname{Infl}_{G/K}^G \chi_U$ and $\chi_{V^K} = (\chi_V)^K$). By $\mathbb{Z}$-linearity, this implies that

$$\left(\operatorname{Infl}_{G/K}^G \alpha, \beta\right)_G = \left(\alpha, \beta^K\right)_{G/K}$$

for any class functions $\alpha \in R_{\mathbb{C}}(G/K)$ and $\beta \in R_{\mathbb{C}}(G)$.

There is also an analogue of (4.1.6):

Lemma 4.1.8. *Let G_1 and G_2 be two groups, and $K_1 < G_1$ and $K_2 < G_2$ be two respective subgroups. Let U_i be a $\mathbb{C}G_i$-module for each $i \in \{1, 2\}$. Then,*

$$(U_1 \otimes U_2)^{K_1 \times K_2} = U_1^{K_1} \otimes U_2^{K_2} \tag{4.1.15}$$

(as subspaces of $U_1 \otimes U_2$).

Proof. The subgroup $K_1 = K_1 \times 1$ of $G_1 \times G_2$ acts on $U_1 \otimes U_2$, and its fixed points are $(U_1 \otimes U_2)^{K_1} = U_1^{K_1} \otimes U_2$ (because for a $\mathbb{C}K_1$-module, tensoring with U_2 is the same as taking a direct power, which clearly commutes with taking fixed points). Similarly, $(U_1 \otimes U_2)^{K_2} = U_1 \otimes U_2^{K_2}$. Now,

$$\begin{aligned}(U_1 \otimes U_2)^{K_1 \times K_2} &= (U_1 \otimes U_2)^{K_1} \cap (U_1 \otimes U_2)^{K_2} = \left(U_1^{K_1} \otimes U_2\right) \cap \left(U_1 \otimes U_2^{K_2}\right) \\ &= U_1^{K_1} \otimes U_2^{K_2}\end{aligned}$$

according to the known linear-algebraic fact stating that if P and Q are subspaces of two vector spaces U and V, respectively, then $(P \otimes V) \cap (U \otimes Q) = P \otimes Q$. $\square$

Exercise 4.1.9. (a) Let G_1 and G_2 be two groups. Let V_i and W_i be finite-dimensional $\mathbb{C}G_i$-modules for every $i \in \{1, 2\}$. Prove that the $\mathbb{C}$-linear map

$$\operatorname{Hom}_{\mathbb{C}G_1}(V_1, W_1) \otimes \operatorname{Hom}_{\mathbb{C}G_2}(V_2, W_2) \to \operatorname{Hom}_{\mathbb{C}[G_1 \times G_2]}(V_1 \otimes V_2, W_1 \otimes W_2)$$

sending each tensor $f \otimes g$ to the tensor product $f \otimes g$ of homomorphisms is a vector space isomorphism.

(b) Use part (a) to give a new proof of (4.1.2).

As an aside, (4.1.10) has a "dual" analogue:

Exercise 4.1.10. Let G be a finite group, and let $K \triangleleft G$ and $H < G$. Let U be a $\mathbb{C}H$-module. As usual, regard $H/(H \cap K)$ as a subgroup of G/K. Show that $\left(\operatorname{Ind}_H^G U\right)^K \cong \operatorname{Ind}_{H/(H\cap K)}^{G/K} \left(U^{H \cap K}\right)$ as $\mathbb{C}[G/K]$-modules.

Inflation also "commutes" with induction:

Exercise 4.1.11. Let G be a finite group, and let $K < H < G$ be such that $K \triangleleft G$. Thus, automatically, $K \triangleleft H$, and we regard the quotient H/K as a subgroup of G/K. Let V be a $\mathbb{C}[H/K]$-module. Show that $\operatorname{Infl}_{G/K}^{G} \operatorname{Ind}_{H/K}^{G/K} V \cong \operatorname{Ind}_{H}^{G} \operatorname{Infl}_{H/K}^{H} V$ as $\mathbb{C}G$-modules.

Exercise 4.1.12. Let G be a finite group, and let $K \triangleleft G$. Let V be a $\mathbb{C}G$-module. Let $I_{V,K}$ denote the $\mathbb{C}$-vector subspace of V spanned by all elements of the form $v - kv$ for $k \in K$ and $v \in V$.

(a) Show that $I_{V,K}$ is a $\mathbb{C}G$-submodule of V.
(b) Let V_K denote the quotient $\mathbb{C}G$-module $V/I_{V,K}$. (This module is occasionally called the *K-coinvariant module of V*, a name it sadly shares with at least two other non-equivalent constructions in algebra.) Show that $V_K \cong \operatorname{Infl}_{G/K}^{G}\left(V^K\right)$ as $\mathbb{C}G$-modules. (Use $\operatorname{char}\mathbb{C} = 0$.)

In the remainder of this subsection, we shall briefly survey generalized notions of induction and restriction, defined in terms of a group homomorphism ρ rather than in terms of a group G and a subgroup H. These generalized notions (defined by van Leeuwen in [128, §2.2]) will not be used in the rest of these notes, but they shed some new light on the facts about induction, restriction, inflation and fixed point construction discussed above. (In particular, they reveal that some of said facts have common generalizations.)

The reader might have noticed that the definitions of inflation and of restriction (both for characters and for modules) are similar. In fact, they both are particular cases of the following construction:

Remark 4.1.13. Let G and H be two finite groups, and let $\rho : H \to G$ be a group homomorphism.

- If $f \in R_{\mathbb{C}}(G)$, then the *ρ-restriction* $\operatorname{Res}_\rho f$ of f is defined as the map $f \circ \rho : H \to \mathbb{C}$. This map is easily seen to belong to $R_{\mathbb{C}}(H)$.
- If V is a $\mathbb{C}G$-module, then the *ρ-restriction* $\operatorname{Res}_\rho V$ of V is the $\mathbb{C}H$-module with ground space V and action given by

$$h \cdot v = \rho(h) \cdot v \qquad \text{for every } h \in H \text{ and } v \in V.$$

This construction generalizes both inflation and restriction: If H is a subgroup of G, and if $\rho : H \to G$ is the inclusion map, then $\operatorname{Res}_\rho f = \operatorname{Res}_H^G f$ (for any $f \in R_{\mathbb{C}}(G)$) and $\operatorname{Res}_\rho V = \operatorname{Res}_H^G V$ (for any $\mathbb{C}G$-module V). If, instead, we have $G = H/K$ for a normal subgroup K of H, and if $\rho : H \to G$ is the projection map, then $\operatorname{Res}_\rho f = \operatorname{Infl}_{H/K}^{H} f$ (for any $f \in R_{\mathbb{C}}(H/K)$) and $\operatorname{Res}_\rho V = \operatorname{Infl}_{H/K}^{H} V$ (for any $\mathbb{C}[H/K]$-module V).

A subtler observation is that induction and fixed point construction can be generalized by a common notion. This is the subject of Exercise 4.1.14 below.

Exercise 4.1.14. Let G and H be two finite groups, and let $\rho : H \to G$ be a group homomorphism. We introduce the following notations:

- If $f \in R_{\mathbb{C}}(H)$, then the *ρ-induction* $\operatorname{Ind}_\rho f$ of f is a map $G \to \mathbb{C}$ which is defined as follows:
$$\left(\operatorname{Ind}_\rho f\right)(g) = \frac{1}{|H|} \sum_{\substack{(h,k) \in H \times G; \\ k\rho(h)k^{-1} = g}} f(h) \qquad \text{for every } g \in G.$$
- If U is a $\mathbb{C}H$-module, then the *ρ-induction* $\operatorname{Ind}_\rho U$ of U is defined as the $\mathbb{C}G$-module $\mathbb{C}G \otimes_{\mathbb{C}H} U$, where $\mathbb{C}G$ is regarded as a $(\mathbb{C}G, \mathbb{C}H)$-bimodule according to the following rule: The left $\mathbb{C}G$-module structure on $\mathbb{C}G$ is plain multiplication inside $\mathbb{C}G$; the right $\mathbb{C}H$-module structure on $\mathbb{C}G$ is induced by the $\mathbb{C}$-algebra homomorphism $\mathbb{C}[\rho] : \mathbb{C}H \to \mathbb{C}G$ (thus, it is explicitly given by $\gamma\eta = \gamma \cdot (\mathbb{C}[\rho])\eta$ for all $\gamma \in \mathbb{C}G$ and $\eta \in \mathbb{C}H$).

Prove the following properties of this construction:

(a) For every $f \in R_{\mathbb{C}}(H)$, we have $\operatorname{Ind}_\rho f \in R_{\mathbb{C}}(G)$.
(b) For any finite-dimensional $\mathbb{C}H$-module U, we have $\chi_{\operatorname{Ind}_\rho U} = \operatorname{Ind}_\rho \chi_U$.
(c) If H is a subgroup of G, and if $\rho : H \to G$ is the inclusion map, then $\operatorname{Ind}_\rho f = \operatorname{Ind}_H^G f$ for every $f \in R_{\mathbb{C}}(H)$.
(d) If H is a subgroup of G, and if $\rho : H \to G$ is the inclusion map, then $\operatorname{Ind}_\rho U = \operatorname{Ind}_H^G U$ for every $\mathbb{C}H$-module U.
(e) If $G = H/K$ for some normal subgroup K of H, and if $\rho : H \to G$ is the projection map, then $\operatorname{Ind}_\rho f = f^K$ for every $f \in R_{\mathbb{C}}(H)$.
(f) If $G = H/K$ for some normal subgroup K of H, and if $\rho : H \to G$ is the projection map, then $\operatorname{Ind}_\rho U \cong U^K$ for every $\mathbb{C}H$-module U.
(g) Any class functions $\alpha \in R_{\mathbb{C}}(H)$ and $\beta \in R_{\mathbb{C}}(G)$ satisfy
$$\left(\operatorname{Ind}_\rho \alpha, \beta\right)_G = \left(\alpha, \operatorname{Res}_\rho \beta\right)_H \tag{4.1.16}$$
and
$$\left\langle \operatorname{Ind}_\rho \alpha, \beta\right\rangle_G = \left\langle \alpha, \operatorname{Res}_\rho \beta\right\rangle_H . \tag{4.1.17}$$
(See Remark 4.1.13 for the definition of $\operatorname{Res}_\rho \beta$.)
(h) We have $\operatorname{Hom}_{\mathbb{C}G}\left(\operatorname{Ind}_\rho U, V\right) \cong \operatorname{Hom}_{\mathbb{C}H}\left(U, \operatorname{Res}_\rho V\right)$ for every $\mathbb{C}H$-module U and every $\mathbb{C}G$-module V. (See Remark 4.1.13 for the definition of $\operatorname{Res}_\rho V$.)
(i) Similarly to how we made $\mathbb{C}G$ into a $(\mathbb{C}G, \mathbb{C}H)$-bimodule, let us make $\mathbb{C}G$ into a $(\mathbb{C}H, \mathbb{C}G)$-bimodule (so the right $\mathbb{C}G$-module structure is plain multiplication inside $\mathbb{C}G$, whereas the left $\mathbb{C}H$-module structure is induced by the $\mathbb{C}$-algebra homomorphism $\mathbb{C}[\rho] : \mathbb{C}H \to \mathbb{C}G$). If U is any $\mathbb{C}H$-module, then the $\mathbb{C}G$-module $\operatorname{Hom}_{\mathbb{C}H}(\mathbb{C}G, U)$ (defined as in Exercise 4.1.4 using the $(\mathbb{C}H, \mathbb{C}G)$-bimodule structure on $\mathbb{C}G$) is isomorphic to $\operatorname{Ind}_\rho U$.
(j) We have $\operatorname{Hom}_{\mathbb{C}G}\left(U, \operatorname{Ind}_\rho V\right) \cong \operatorname{Hom}_{\mathbb{C}H}\left(\operatorname{Res}_\rho U, V\right)$ for every $\mathbb{C}G$-module U and every $\mathbb{C}H$-module V. (See Remark 4.1.13 for the definition of $\operatorname{Res}_\rho V$.)

Furthermore:

(k) Use the above to prove the formula (4.1.3).
(l) Use the above to prove the formula (4.1.12).

[**Hint:** Part (b) of this exercise is hard. To solve it, it is useful to have a way of computing the trace of a linear operator without knowing a basis of the vector space it is acting on. There is a way to do this using a "finite dual generating system", which is a somewhat less restricted notion than that of a basis[219]. Try to create a finite dual generating system for $\operatorname{Ind}_\rho U$ from one for U (and from the group G), and then use it to compute $\chi_{\operatorname{Ind}_\rho U}$.

The solution of part (i) is a modification of the solution of Exercise 4.1.4, but complicated by the fact that H is no longer (necessarily) a subgroup of G. Part (f) can be solved by similar arguments, or using part (i), or using Exercise 4.1.12(b).]

The result of Exercise 4.1.14(h) generalizes (4.1.7) (because of Exercise 4.1.14(d)), but also generalizes (4.1.11) (due to Exercise 4.1.14(f)). Similarly, Exercise 4.1.14(g) generalizes both (4.1.9) and (4.1.14). Similarly, Exercise 4.1.14(i) generalizes Exercise 4.1.4, and Exercise 4.1.14(j) generalizes Exercise 4.1.6.

Similarly, Exercise 4.1.3 is generalized by the following exercise:

Exercise 4.1.15. Let G_1, G_2, H_1 and H_2 be four finite groups. Let $\rho_1 : H_1 \to G_1$ and $\rho_2 : H_2 \to G_2$ be two group homomorphisms. These two homomorphisms clearly induce a group homomorphism $\rho_1 \times \rho_2 : H_1 \times H_2 \to G_1 \times G_2$. Let U_1 be a $\mathbb{C}H_1$-module, and U_2 be a $\mathbb{C}H_2$-module. Show that

$$\operatorname{Ind}_{\rho_1 \times \rho_2} (U_1 \otimes U_2) \cong (\operatorname{Ind}_{\rho_1} U_1) \otimes (\operatorname{Ind}_{\rho_2} U_2)$$

as $\mathbb{C}[G_1 \times G_2]$-modules.

The Ind_ρ and Res_ρ operators behave "functorially" with respect to composition. Here is what this means:

Exercise 4.1.16. Let G, H and I be three finite groups. Let $\rho : H \to G$ and $\tau : I \to H$ be two group homomorphisms.

(a) We have $\operatorname{Ind}_\rho \operatorname{Ind}_\tau U \cong \operatorname{Ind}_{\rho\circ\tau} U$ for every $\mathbb{C}I$-module U.
(b) We have $\operatorname{Ind}_\rho \operatorname{Ind}_\tau f = \operatorname{Ind}_{\rho\circ\tau} f$ for every $f \in R_{\mathbb{C}}(I)$.
(c) We have $\operatorname{Res}_\tau \operatorname{Res}_\rho V = \operatorname{Res}_{\rho\circ\tau} V$ for every $\mathbb{C}G$-module V.
(d) We have $\operatorname{Res}_\tau \operatorname{Res}_\rho f = \operatorname{Res}_{\rho\circ\tau} f$ for every $f \in R_{\mathbb{C}}(G)$.

Exercise 4.1.16(a), of course, generalizes Exercise 4.1.2.

4.1.7. *Semidirect products.* Recall that a *semidirect product* is a group $G \ltimes K$ having two subgroups G, K with

- $K \triangleleft (G \ltimes K)$ is a normal subgroup,
- $G \ltimes K = GK = KG$, and

[219]More precisely: Let $\mathbb{K}$ be a field, and V be a $\mathbb{K}$-vector space. A *finite dual generating system* for V means a triple $\left(I, (a_i)_{i\in I}, (f_i)_{i\in I}\right)$, where

- I is a finite set;
- $(a_i)_{i\in I}$ is a family of elements of V;
- $(f_i)_{i\in I}$ is a family of elements of V^* (where V^* means $\operatorname{Hom}_{\mathbb{K}}(V, \mathbb{K})$)

such that every $v \in V$ satisfies $v = \sum_{i\in I} f_i(v)\, a_i$. For example, if $(e_j)_{j\in J}$ is a finite basis of the vector space V, and if $\left(e_j^*\right)_{j\in J}$ is the basis of V^* dual to this basis $(e_j)_{j\in J}$, then $\left(J, (e_j)_{j\in J}, \left(e_j^*\right)_{j\in J}\right)$ is a finite dual generating system for V; however, most finite dual generating systems are not obtained this way.

The crucial observation is now that if $\left(I, (a_i)_{i\in I}, (f_i)_{i\in I}\right)$ is a finite dual generating system for a vector space V, and if T is an endomorphism of V, then

$$\operatorname{trace} T = \sum_{i\in I} f_i(Ta_i).$$

Prove this!

- $G \cap K = \{e\}$.

In this setting one has two interesting adjoint constructions, applied in Section 4.5.

Proposition 4.1.17. *Fix a $\mathbb{C}[G \ltimes K]$-module V.*

(i) *For any $\mathbb{C}G$-module U, one has $\mathbb{C}[G \ltimes K]$-module structure*
$$\Phi(U) := U \otimes V,$$
determined via
$$\begin{aligned} k(u \otimes v) &= u \otimes k(v), \\ g(u \otimes v) &= g(u) \otimes g(v). \end{aligned}$$

(ii) *For any $\mathbb{C}[G \ltimes K]$-module W, one has $\mathbb{C}G$-module structure*
$$\Psi(W) := \operatorname{Hom}_{\mathbb{C}K}(\operatorname{Res}^{G \ltimes K}_K V, \operatorname{Res}^{G \ltimes K}_K W),$$
determined via $g(\varphi) = g \circ \varphi \circ g^{-1}$.

(iii) *The maps*
$$\mathbb{C}G - \text{mods} \underset{\Psi}{\overset{\Phi}{\rightleftharpoons}} \mathbb{C}[G \ltimes K] - \text{mods}$$
are adjoint in the sense that one has an isomorphism
$$\begin{array}{ccc} \operatorname{Hom}_{\mathbb{C}G}(U, \Psi(W)) & \longrightarrow & \operatorname{Hom}_{\mathbb{C}[G \ltimes K]}(\Phi(U), W) \\ \| & & \| \\ \operatorname{Hom}_{\mathbb{C}G}(U, \operatorname{Hom}_{\mathbb{C}K}(\operatorname{Res}^{G \ltimes K}_K V, \operatorname{Res}^{G \ltimes K}_K W)) & & \operatorname{Hom}_{\mathbb{C}[G \ltimes K]}(U \otimes V, W), \\ & & \\ \varphi & \longmapsto & \overline{\varphi}(u \otimes v) := \varphi(u)(v). \end{array}$$

(iv) *One has a $\mathbb{C}G$-module isomorphism*
$$(\Psi \circ \Phi)(U) \cong U \otimes \operatorname{End}_{\mathbb{C}K}(\operatorname{Res}^{G \ltimes K}_K V).$$
In particular, if $\operatorname{Res}^{G \ltimes K}_K V$ is a simple $\mathbb{C}K$-module, then $(\Psi \circ \Phi)(U) \cong U$.

Proof. These are mostly straightforward exercises in the definitions. To check assertion (iv), for example, note that K acts only in the right tensor factor in $\operatorname{Res}^{G \ltimes K}_K (U \otimes V)$, and hence as $\mathbb{C}G$-modules one has
$$\begin{aligned} (\Psi \circ \Phi)(U) &= \operatorname{Hom}_{\mathbb{C}K}(\operatorname{Res}^{G \ltimes K}_K V,\ \operatorname{Res}^{G \ltimes K}_K (U \otimes V)) \\ &= \operatorname{Hom}_{\mathbb{C}K}(\operatorname{Res}^{G \ltimes K}_K V,\ U \otimes \operatorname{Res}^{G \ltimes K}_K V) \\ &= U \otimes \operatorname{Hom}_{\mathbb{C}K}(\operatorname{Res}^{G \ltimes K}_K V,\ \operatorname{Res}^{G \ltimes K}_K V) \\ &= U \otimes \operatorname{End}_{\mathbb{C}K}(\operatorname{Res}^{G \ltimes K}_K V). \end{aligned}$$
$\square$

4.2. **Three towers of groups.** Here we consider three towers of groups
$$G_* = (G_0 < G_1 < G_2 < G_3 < \cdots)$$
where either

- $G_n = \mathfrak{S}_n$, the *symmetric group*[220], or
- $G_n = \mathfrak{S}_n[\Gamma]$, the *wreath product* of the symmetric group with some arbitrary finite group Γ, or

[220]The symmetric group $\mathfrak{S}_0$ is the group of all permutations of the empty set $\{1, 2, \ldots, 0\} = \varnothing$. It is a trivial group. (Note that $\mathfrak{S}_1$ is also a trivial group.)

- $G_n = GL_n(\mathbb{F}_q)$, the finite general linear group[221].

Here the wreath product $\mathfrak{S}_n[\Gamma]$ can be thought of informally as the group of *monomial* $n \times n$ matrices whose nonzero entries lie in Γ, that is, $n \times n$ matrices having exactly one nonzero entry in each row and column, and that entry is an element of Γ. E.g.

$$\begin{bmatrix} 0 & g_2 & 0 \\ g_1 & 0 & 0 \\ 0 & 0 & g_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & g_6 \\ 0 & g_5 & 0 \\ g_4 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & g_2 g_5 & 0 \\ 0 & 0 & g_1 g_6 \\ g_3 g_4 & 0 & 0 \end{bmatrix}.$$

More formally, $\mathfrak{S}_n[\Gamma]$ is the semidirect product $\mathfrak{S}_n \ltimes \Gamma^n$ in which $\mathfrak{S}_n$ acts on Γ^n via $\sigma(\gamma_1, \ldots, \gamma_n) = (\gamma_{\sigma^{-1}(1)}, \ldots, \gamma_{\sigma^{-1}(n)})$.

For each of the three towers G_*, there are embeddings $G_i \times G_j \hookrightarrow G_{i+j}$ and we introduce maps $\operatorname{ind}_{i,j}^{i+j}$ taking $\mathbb{C}[G_i \times G_j]$-modules to $\mathbb{C}G_{i+j}$-modules, as well as maps $\operatorname{res}_{i,j}^{i+j}$ carrying modules in the reverse direction which are adjoint:

$$\operatorname{Hom}_{\mathbb{C}G_{i+j}}(\operatorname{ind}_{i,j}^{i+j} U, V) = \operatorname{Hom}_{\mathbb{C}[G_i \times G_j]}(U, \operatorname{res}_{i,j}^{i+j} V). \tag{4.2.1}$$

Definition 4.2.1. For $G_n = \mathfrak{S}_n$, one embeds $\mathfrak{S}_i \times \mathfrak{S}_j$ into $\mathfrak{S}_{i+j}$ as the permutations that permute $\{1, 2, \ldots, i\}$ and $\{i+1, i+2, \ldots, i+j\}$ separately. Here one defines

$$\operatorname{ind}_{i,j}^{i+j} := \operatorname{Ind}_{\mathfrak{S}_i \times \mathfrak{S}_j}^{\mathfrak{S}_{i+j}},$$
$$\operatorname{res}_{i,j}^{i+j} := \operatorname{Res}_{\mathfrak{S}_i \times \mathfrak{S}_j}^{\mathfrak{S}_{i+j}}.$$

For $G_n = \mathfrak{S}_n[\Gamma]$, similarly embed $\mathfrak{S}_i[\Gamma] \times \mathfrak{S}_j[\Gamma]$ into $\mathfrak{S}_{i+j}[\Gamma]$ as block monomial matrices whose two diagonal blocks have sizes i, j respectively, and define

$$\operatorname{ind}_{i,j}^{i+j} := \operatorname{Ind}_{\mathfrak{S}_i[\Gamma] \times \mathfrak{S}_j[\Gamma]}^{\mathfrak{S}_{i+j}[\Gamma]},$$
$$\operatorname{res}_{i,j}^{i+j} := \operatorname{Res}_{\mathfrak{S}_i[\Gamma] \times \mathfrak{S}_j[\Gamma]}^{\mathfrak{S}_{i+j}[\Gamma]}.$$

For $G_n = GL_n(\mathbb{F}_q)$, which we will denote just GL_n, similarly embed $GL_i \times GL_j$ into GL_{i+j} as block diagonal matrices whose two diagonal blocks have sizes i, j respectively. However, one also introduces as an intermediate the *parabolic subgroup* $P_{i,j}$ consisting of the block upper-triangular matrices of the form

$$\begin{bmatrix} g_i & \ell \\ 0 & g_j \end{bmatrix}$$

where g_i, g_j lie in GL_i, GL_j, respectively, and ℓ in $\mathbb{F}_q^{i \times j}$ is arbitrary. One has a quotient map $P_{i,j} \to GL_i \times GL_j$ whose kernel $K_{i,j}$ is the set of matrices of the form

$$\begin{bmatrix} I_i & \ell \\ 0 & I_j \end{bmatrix}$$

with ℓ again arbitrary. Here one defines

$$\operatorname{ind}_{i,j}^{i+j} := \operatorname{Ind}_{P_{i,j}}^{GL_{i+j}} \operatorname{Infl}_{GL_i \times GL_j}^{P_{i,j}},$$
$$\operatorname{res}_{i,j}^{i+j} := \left(\operatorname{Res}_{P_{i,j}}^{GL_{i+j}}(-)\right)^{K_{i,j}}.$$

[221]The group $GL_0(\mathbb{F}_q)$ is a trivial group, consisting of the empty 0×0 matrix.

In the case $G_n = GL_n$, the operation $\mathrm{ind}_{i,j}^{i+j}$ is sometimes called *parabolic induction* or *Harish-Chandra induction*. The operation $\mathrm{res}_{i,j}^{i+j}$ is essentially just the $K_{i,j}$-fixed point construction $V \mapsto V^{K_{i,j}}$. However writing it as the above two-step composite makes it more obvious, (via (4.1.7) and (4.1.11)) that $\mathrm{res}_{i,j}^{i+j}$ is again adjoint to $\mathrm{ind}_{i,j}^{i+j}$.

Definition 4.2.2. For each of the three towers G_*, define a graded $\mathbb{Z}$-module

$$A := A(G_*) = \bigoplus_{n \geq 0} R(G_n)$$

with a bilinear form $(\cdot, \cdot)_A$ whose restriction to $A_n := R(G_n)$ is the usual form $(\cdot, \cdot)_{G_n}$, and such that $\Sigma := \bigsqcup_{n \geq 0} \mathrm{Irr}(G_n)$ gives an orthonormal $\mathbb{Z}$-basis. Notice that $A_0 = \mathbb{Z}$ has its basis element 1 equal to the unique irreducible character of the trivial group G_0.

Bearing in mind that $A_n = R(G_n)$ and

$$A_i \otimes A_j = R(G_i) \otimes R(G_j) \cong R(G_i \times G_j),$$

one then has candidates for product and coproduct defined by

$$\begin{array}{rrcl} m := \mathrm{ind}_{i,j}^{i+j} : & A_i \otimes A_j & \longrightarrow & A_{i+j}, \\ \Delta := \bigoplus_{i+j=n} \mathrm{res}_{i,j}^{i+j} : & A_n & \longrightarrow & \bigoplus_{i+j=n} A_i \otimes A_j. \end{array}$$

The coassociativity of Δ is an easy consequence of transitivity of the constructions of restriction and fixed points[222]. We could derive the associativity of m from the transitivity of induction and inflation, but this would be more complicated[223]; we will instead prove it differently.

We first show that the maps m and Δ are adjoint with respect to the forms $(\cdot, \cdot)_A$ and $(\cdot, \cdot)_{A \otimes A}$. In fact, if U, V, W are modules over $\mathbb{C}G_i$, $\mathbb{C}G_j$, $\mathbb{C}G_{i+j}$, respectively, then we can write the $\mathbb{C}[G_i \times G_j]$-module $\mathrm{res}_{i,j}^{i+j} W$ as a direct sum $\bigoplus_k X_k \otimes Y_k$ with X_k being $\mathbb{C}G_i$-modules and Y_k being $\mathbb{C}G_j$-modules; we then have

$$\mathrm{res}_{i,j}^{i+j} \chi_W = \sum_k \chi_{X_k} \otimes \chi_{Y_k} \tag{4.2.2}$$

and

$$\begin{aligned} &(m(\chi_U \otimes \chi_V), \chi_W)_A \\ &= \left(\mathrm{ind}_{i,j}^{i+j}(\chi_{U \otimes V}), \chi_W\right)_A = \left(\mathrm{ind}_{i,j}^{i+j}(\chi_{U \otimes V}), \chi_W\right)_{G_{i+j}} \\ &= \left(\chi_{U \otimes V}, \mathrm{res}_{i,j}^{i+j} \chi_W\right)_{G_i \times G_j} = \left(\chi_{U \otimes V}, \sum_k \chi_{X_k} \otimes \chi_{Y_k}\right)_{G_i \times G_j} \\ &= \sum_k (\chi_{U \otimes V}, \chi_{X_k \otimes Y_k})_{G_i \times G_j} = \sum_k (\chi_U, \chi_{X_k})_{G_i} (\chi_V, \chi_{Y_k})_{G_j} \end{aligned}$$

[222]More precisely, using this transitivity, it is easily reduced to proving that $K_{i+j,k} \cdot (K_{i,j} \times \{I_k\}) = K_{i,j+k} \cdot (\{I_i\} \times K_{j,k})$ (an equality between subgroups of GL_{i+j+k}) for any three nonnegative integers i, j, k. But this equality can be proven by realizing that both of its sides equal the set of all block matrices of the form $\begin{pmatrix} I_i & \ell & \ell' \\ 0 & I_j & \ell'' \\ 0 & 0 & I_k \end{pmatrix}$ with ℓ, ℓ' and ℓ'' being matrices of sizes $i \times j$, $i \times k$ and $j \times k$, respectively.

[223]See Exercise 4.3.11(c) for such a derivation.

(the third equality sign follows by taking dimensions in (4.2.1) and recalling (4.1.1); the fourth equality sign follows from (4.2.2); the sixth one follows from (4.1.2)) and

$$\begin{aligned}
&(\chi_U \otimes \chi_V, \Delta(\chi_W))_{A\otimes A} \\
&= \left(\chi_U \otimes \chi_V, \operatorname{res}^{i+j}_{i,j} \chi_W\right)_{A\otimes A} = \left(\chi_U \otimes \chi_V, \sum_k \chi_{X_k} \otimes \chi_{Y_k}\right)_{A\otimes A} \\
&= \sum_k (\chi_U, \chi_{X_k})_A (\chi_V, \chi_{Y_k})_A = \sum_k (\chi_U, \chi_{X_k})_{G_i} (\chi_V, \chi_{Y_k})_{G_j}
\end{aligned}$$

(the first equality sign follows by removing all terms in $\Delta(\chi_W)$ whose scalar product with $\chi_U \otimes \chi_V$ vanishes for reasons of gradedness; the second equality sign follows from (4.2.2)), which in comparison yield

$$(m(\chi_U \otimes \chi_V), \chi_W)_A = (\chi_U \otimes \chi_V, \Delta(\chi_W))_{A\otimes A},$$

thus showing that m and Δ are adjoint maps. Therefore, m is associative (since Δ is coassociative).

Endowing $A = \bigoplus_{n\geq 0} R(G_n)$ with the obvious unit and counit maps, it thus becomes a graded, finite-type $\mathbb{Z}$-algebra and $\mathbb{Z}$-coalgebra.

The next section addresses the issue of why they form a bialgebra. However, assuming this for the moment, it should be clear that each of these algebras A is a PSH having $\Sigma = \bigsqcup_{n\geq 0} \operatorname{Irr}(G_n)$ as its PSH-basis. Σ is self-dual because m, Δ are defined by adjoint maps, and it is positive because m, Δ take irreducible representations to genuine representations not just virtual ones, and hence have characters which are nonnegative sums of irreducible characters.

Exercise 4.2.3. Let i, j and k be three nonnegative integers. Let U be a $\mathbb{C}\mathfrak{S}_i$-module, let V be a $\mathbb{C}\mathfrak{S}_j$-module, and let W be a $\mathbb{C}\mathfrak{S}_k$-module. Show that there are canonical $\mathbb{C}[\mathfrak{S}_i \times \mathfrak{S}_j \times \mathfrak{S}_k]$-module isomorphisms

$$\begin{aligned}
&\operatorname{Ind}^{\mathfrak{S}_{i+j+k}}_{\mathfrak{S}_{i+j}\times\mathfrak{S}_k} \left(\operatorname{Ind}^{\mathfrak{S}_{i+j}}_{\mathfrak{S}_i\times\mathfrak{S}_j}(U \otimes V) \otimes W\right) \\
&\cong \operatorname{Ind}^{\mathfrak{S}_{i+j+k}}_{\mathfrak{S}_i\times\mathfrak{S}_j\times\mathfrak{S}_k}(U \otimes V \otimes W) \\
&\cong \operatorname{Ind}^{\mathfrak{S}_{i+j+k}}_{\mathfrak{S}_i\times\mathfrak{S}_{j+k}} \left(U \otimes \operatorname{Ind}^{\mathfrak{S}_{j+k}}_{\mathfrak{S}_j\times\mathfrak{S}_k}(V \otimes W)\right).
\end{aligned}$$

(Similar statements hold for the other two towers of groups and their respective ind functors, although the one for the GL_* tower is harder to prove. See Exercise 4.3.11(a) for a more general result.)

4.3. **Bialgebra and double cosets.** To show that the algebra and coalgebras $A = A(G_*)$ are bialgebras, the central issue is checking the pentagonal diagram in (1.3.4), that is, as maps $A \otimes A \to A \otimes A$, one has

$$\Delta \circ m = (m \otimes m) \circ (\mathrm{id} \otimes T \otimes \mathrm{id}) \circ (\Delta \otimes \Delta). \tag{4.3.1}$$

In checking this, it is convenient to have a lighter notation for various subgroups of the groups G_n corresponding to compositions α.

Definition 4.3.1. (a) An *almost-composition* is a (finite) tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ of nonnegative integers. Its *length* is defined to be ℓ and denoted by $\ell(\alpha)$; its *size* is defined to be $\alpha_1 + \alpha_2 + \cdots + \alpha_\ell$ and denoted by $|\alpha|$; its *parts* are its entries $\alpha_1, \alpha_2, \ldots, \alpha_\ell$. The almost-compositions of size n are called the *almost-compositions of* n.

(b) A *composition* is a finite tuple of positive integers. Of course, any composition is an almost-composition, and so all notions defined for almost-compositions (like size and length) make sense for compositions.

Note that any partition of n (written without trailing zeroes) is a composition of n. We write $\varnothing$ (and sometimes, sloppily, (0), when there is no danger of mistaking it for the almost-composition (0)) for the empty composition $()$.

Definition 4.3.2. Given an almost-composition $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ of n, define a subgroup

$$G_\alpha \cong G_{\alpha_1} \times \cdots \times G_{\alpha_\ell} < G_n$$

via the block-diagonal embedding with diagonal blocks of sizes $(\alpha_1, \ldots, \alpha_\ell)$. This G_α is called a *Young subgroup* $\mathfrak{S}_\alpha$ when $G_n = \mathfrak{S}_n$, and a *Levi subgroup* when $G_n = GL_n$. In the case when $G_n = \mathfrak{S}_n[\Gamma]$, we also denote G_α by $\mathfrak{S}_\alpha[\Gamma]$. In the case where $G_n = GL_n$, also define the *parabolic subgroup* P_α to be the subgroup of G_n consisting of block-upper triangular matrices whose diagonal blocks have sizes $(\alpha_1, \ldots, \alpha_\ell)$, and let K_α be the kernel of the obvious surjection $P_\alpha \to G_\alpha$ which sends a block upper-triangular matrix to the tuple of its diagonal blocks whose sizes are $\alpha_1, \alpha_2, \ldots, \alpha_\ell$. Notice that $P_{(i,j)} = P_{i,j}$ for any i and j with $i+j = n$; similarly, $K_{(i,j)} = K_{i,j}$ for any i and j with $i + j = n$. We will also abbreviate $G_{(i,j)} = G_i \times G_j$ by $G_{i,j}$.

When $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ is an almost-composition, we abbreviate $G_{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)}$ by $G_{\alpha_1,\alpha_2,\ldots,\alpha_\ell}$ (and similarly for the P's).

Definition 4.3.3. Let K and H be two groups, $\tau : K \to H$ a group homomorphism, and U a $\mathbb{C}H$-module. Then, U^τ is defined as the $\mathbb{C}K$-module with ground space U and action given by $k \cdot u = \tau(k) \cdot u$ for all $k \in K$ and $u \in U$. [224] This very simple construction generalizes the definition of U^g for an element $g \in G$, where G is a group containing H as a subgroup; in fact, in this situation we have $U^g = U^\tau$, where $K = {}^gH$ and $\tau : K \to H$ is the map $k \mapsto g^{-1}kg$.

Using homogeneity, checking the bialgebra condition (4.3.1) in the homogeneous component $(A \otimes A)_n$ amounts to the following: for each pair of representations U_1, U_2 of G_{r_1}, G_{r_2} with $r_1 + r_2 = n$, and for each (c_1, c_2) with $c_1 + c_2 = n$, one must verify that

$$\begin{aligned} &\operatorname{res}^n_{c_1,c_2} \left(\operatorname{ind}^n_{r_1,r_2} (U_1 \otimes U_2)\right) \\ &\quad \cong \bigoplus_A \left(\operatorname{ind}^{c_1}_{a_{11},a_{21}} \otimes \operatorname{ind}^{c_2}_{a_{12},a_{22}}\right) \left(\left(\operatorname{res}^{r_1}_{a_{11},a_{12}} U_1 \otimes \operatorname{res}^{r_2}_{a_{21},a_{22}} U_2\right)^{\tau_A^{-1}}\right) \end{aligned} \tag{4.3.2}$$

where the direct sum is over all matrices $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ in $\mathbb{N}^{2\times 2}$ with row sums (r_1, r_2) and column sums (c_1, c_2), and where τ_A is the obvious

[224] We have already met this $\mathbb{C}K$-module U^τ in Remark 4.1.13, where it was called $\operatorname{Res}_\tau U$.

isomorphism between the subgroups

$$\begin{array}{lll} G_{a_{11},a_{12},a_{21},a_{22}} & (< G_{r_1,r_2}) & \text{and} \\ G_{a_{11},a_{21},a_{12},a_{22}} & (< G_{c_1,c_2}) & \end{array} \tag{4.3.3}$$

(we are using the inverse τ_A^{-1} of this isomorphism τ_A to identify modules for the first subgroup with modules for the second subgroup, according to Definition 4.3.3).

As one might guess, (4.3.2) comes from the Mackey formula (Theorem 4.1.7), once one identifies the appropriate double coset representatives. This is just as easy to do in a slightly more general setting.

Definition 4.3.4. Given almost-compositions α, β of n having lengths ℓ, m and a matrix A in $\mathbb{N}^{\ell \times m}$ with row sums α and column sums β, define a permutation w_A in $\mathfrak{S}_n$ as follows. Disjointly decompose $[n] = \{1, 2, \ldots, n\}$ into consecutive intervals of numbers

$$[n] = I_1 \sqcup \cdots \sqcup I_\ell \qquad \text{such that } |I_i| = \alpha_i$$

(so the smallest α_1 elements of $[n]$ go into I_1, the next-smallest α_2 elements of $[n]$ go into I_2, and so on). Likewise, disjointly decompose $[n]$ into consecutive intervals of numbers

$$[n] = J_1 \sqcup \cdots \sqcup J_m \qquad \text{such that } |J_j| = \beta_j.$$

For every $j \in [m]$, disjointly decompose J_j into consecutive intervals of numbers $J_j = J_{j,1} \sqcup J_{j,2} \sqcup \cdots \sqcup J_{j,\ell}$ such that every $i \in [\ell]$ satisfies $|J_{j,i}| = a_{ij}$. For every $i \in [\ell]$, disjointly decompose I_i into consecutive intervals of numbers $I_i = I_{i,1} \sqcup I_{i,2} \sqcup \cdots \sqcup I_{i,m}$ such that every $j \in [m]$ satisfies $|I_{i,j}| = a_{ij}$. Now, for every $i \in [\ell]$ and $j \in [m]$, let $\pi_{i,j}$ be the increasing bijection from $J_{j,i}$ to $I_{i,j}$ (this is well-defined since these two sets both have cardinality a_{ij}). The disjoint union of these bijections $\pi_{i,j}$ over all i and j is a bijection $[n] \to [n]$ (since the disjoint union of the sets $J_{j,i}$ over all i and j is $[n]$, and so is the disjoint union of the sets $I_{i,j}$), that is, a permutation of $[n]$; this permutation is what we call w_A.

Example 4.3.5. Taking $n = 9$ and $\alpha = (4, 5), \beta = (3, 4, 2)$, one has

$$\begin{array}{lll} I_1 = \{1, 2, 3, 4\}, & I_2 = \{5, 6, 7, 8, 9\}, & \\ J_1 = \{1, 2, 3\}, & J_2 = \{4, 5, 6, 7\}, & J_3 = \{8, 9\}. \end{array}$$

Then one possible matrix A having row and column sums α, β is $A = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 2 & 2 \end{bmatrix}$, and its associated permutation w_A written in two-line notation is

$$\begin{pmatrix} 1 & 2 & 3 & | & 4 & 5 & 6 & 7 & | & 8 & 9 \\ \underline{1} & \underline{2} & \underline{\underline{5}} & | & \underline{3} & \underline{4} & \underline{\underline{6}} & \underline{\underline{7}} & | & \underline{\underline{8}} & \underline{\underline{9}} \end{pmatrix}$$

with vertical lines dividing the sets J_j on top, and with elements of I_i underlined i times on the bottom.

Remark 4.3.6. Given almost-compositions α and β of n having lengths ℓ and m, and a permutation $w \in \mathfrak{S}_n$. It is easy to see that there exists a matrix $A \in \mathbb{N}^{\ell \times m}$ satisfying $w_A = w$ if and only if the restriction of w to each J_j and the restriction of w^{-1} to each I_i are increasing. In this case, the matrix A is determined by $a_{ij} = |w(J_j) \cap I_i|$.

Among our three towers G_* of groups, the symmetric group tower ($G_n = \mathfrak{S}_n$) is the simplest one. We will now see that it also embeds into the two others, in the sense that $\mathfrak{S}_n$ embeds into $\mathfrak{S}_n[\Gamma]$ for every Γ and into $GL_n(\mathbb{F}_q)$ for every q.

First, for every $n \in \mathbb{N}$ and any group Γ, we embed the group $\mathfrak{S}_n$ into $\mathfrak{S}_n[\Gamma]$ by means of the canonical embedding $\mathfrak{S}_n \to \mathfrak{S}_n \ltimes \Gamma^n = \mathfrak{S}_n[\Gamma]$. If we regard elements of $\mathfrak{S}_n[\Gamma]$ as $n \times n$ monomial matrices with nonzero entries in Γ, then this boils down to identifying every $\pi \in \mathfrak{S}_n$ with the permutation matrix of π (in which the 1's are read as the neutral element of Γ). If α is an almost-composition of n, then this embedding $\mathfrak{S}_n \to \mathfrak{S}_n[\Gamma]$ makes the subgroup $\mathfrak{S}_\alpha$ of $\mathfrak{S}_n$ become a subgroup of $\mathfrak{S}_n[\Gamma]$, more precisely a subgroup of $\mathfrak{S}_\alpha[\Gamma] < \mathfrak{S}_n[\Gamma]$.

For every $n \in \mathbb{N}$ and every q, we embed the group $\mathfrak{S}_n$ into $GL_n(\mathbb{F}_q)$ by identifying every permutation $\pi \in \mathfrak{S}_n$ with its permutation matrix in $GL_n(\mathbb{F}_q)$. If α is an almost-composition of n, then this embedding makes the subgroup $\mathfrak{S}_\alpha$ of $\mathfrak{S}_n$ become a subgroup of $GL_n(\mathbb{F}_q)$. If we let $G_n = GL_n(\mathbb{F}_q)$, then $\mathfrak{S}_\alpha < G_\alpha < P_\alpha$.

The embeddings we have just defined commute with the group embeddings $G_n < G_{n+1}$ on both sides.

Proposition 4.3.7. *The permutations $\{w_A\}$, as A runs over all matrices in $\mathbb{N}^{\ell \times m}$ having row sums α and column sums β, give*

(a) *a system of double coset representatives for $\mathfrak{S}_\alpha \backslash \mathfrak{S}_n / \mathfrak{S}_\beta$;*
(b) *a system of double coset representatives for $\mathfrak{S}_\alpha[\Gamma] \backslash \mathfrak{S}_n[\Gamma] / \mathfrak{S}_\beta[\Gamma]$;*
(b) *a system of double coset representatives for $P_\alpha \backslash GL_n / P_\beta$.*

Proof. (a) We give an algorithm to show that every double coset $\mathfrak{S}_\alpha w \mathfrak{S}_\beta$ contains some w_A. Start by altering w within its coset $w\mathfrak{S}_\beta$, that is, by permuting the *positions* within each set J_j, to obtain a representative w' for $w\mathfrak{S}_\beta$ in which each set $w'(J_j)$ appears in increasing order in the second line of the two-line notation for w'. Then alter w' within its coset $\mathfrak{S}_\alpha w'$, that is, by permuting the *values* within each set I_i, to obtain a representative w_A having the elements of each set I_i appearing in increasing order in the second line; because the values within each set I_i are consecutive, this alteration will not ruin the property that one had each set $w'(J_j)$ appearing in increasing order. For example, one might have

$$\begin{aligned}
w &= \begin{pmatrix} 1 & 2 & 3 & | & 4 & 5 & 6 & 7 & | & 8 & 9 \\ \underline{4} & \underline{\underline{8}} & \underline{2} & | & \underline{\underline{5}} & \underline{3} & \underline{\underline{9}} & \underline{1} & | & \underline{\underline{7}} & \underline{\underline{6}} \end{pmatrix}, \\
w' &= \begin{pmatrix} 1 & 2 & 3 & | & 4 & 5 & 6 & 7 & | & 8 & 9 \\ \underline{2} & \underline{4} & \underline{\underline{8}} & | & \underline{1} & \underline{3} & \underline{\underline{5}} & \underline{\underline{9}} & | & \underline{\underline{6}} & \underline{\underline{7}} \end{pmatrix} \in w\mathfrak{S}_\beta, \\
w_A &= \begin{pmatrix} 1 & 2 & 3 & | & 4 & 5 & 6 & 7 & | & 8 & 9 \\ \underline{1} & \underline{2} & \underline{\underline{5}} & | & \underline{3} & \underline{4} & \underline{\underline{6}} & \underline{\underline{7}} & | & \underline{\underline{8}} & \underline{\underline{9}} \end{pmatrix} \in \mathfrak{S}_\alpha w' \subset \mathfrak{S}_\alpha w' \mathfrak{S}_\beta = \mathfrak{S}_\alpha w \mathfrak{S}_\beta.
\end{aligned}$$

Next note that $\mathfrak{S}_\alpha w_A \mathfrak{S}_\beta = \mathfrak{S}_\alpha w_B \mathfrak{S}_\beta$ implies $A = B$, since the quantities

$$a_{i,j}(w) := |w(J_j) \cap I_i|$$

are easily seen to be constant on double cosets $\mathfrak{S}_\alpha w \mathfrak{S}_\beta$.

(b) Double coset representatives for $\mathfrak{S}_\alpha \backslash \mathfrak{S}_n / \mathfrak{S}_\beta$ should also provide double coset representatives for $\mathfrak{S}_\alpha[\Gamma] \backslash \mathfrak{S}_n[\Gamma] / \mathfrak{S}_\beta[\Gamma]$, since

$$\mathfrak{S}_\alpha[\Gamma] = \mathfrak{S}_\alpha \Gamma^n = \Gamma^n \mathfrak{S}_\alpha.$$

Thus, part (b) follows from part (a).

(c) In our proof of part (a) above, we showed that $\mathfrak{S}_\alpha w_A \mathfrak{S}_\beta = \mathfrak{S}_\alpha w_B \mathfrak{S}_\beta$ implies $A = B$. A similar argument shows that $P_\alpha w_A P_\beta = P_\alpha w_B P_\beta$ implies $A = B$: for g in GL_n, the rank $r_{ij}(g)$ of the matrix obtained by restricting g to rows $I_i \sqcup I_{i+1} \sqcup \cdots \sqcup I_\ell$ and columns $J_1 \sqcup J_2 \sqcup \cdots \sqcup J_j$ is constant on double cosets $P_\alpha g P_\beta$, and for a permutation matrix w one can recover $a_{i,j}(w)$ from the formula

$$a_{i,j}(w) = r_{i,j}(w) - r_{i,j-1}(w) - r_{i+1,j}(w) + r_{i+1,j-1}(w).$$

Thus it only remains to show that every double coset $P_\alpha g P_\beta$ contains some w_A. Since $\mathfrak{S}_\alpha < P_\alpha$, and we have seen already that every double coset $\mathfrak{S}_\alpha w \mathfrak{S}_\beta$ contains some w_A, it suffices to show that every double coset $P_\alpha g P_\beta$ contains some permutation w. However, we claim that this is already true for the smaller double cosets BgB where $B = P_{1^n}$ is the *Borel subgroup* of upper triangular invertible matrices, that is, one has the usual *Bruhat decomposition*

$$GL_n = \bigsqcup_{w \in \mathfrak{S}_n} BwB.$$

To prove this decomposition, we show how to find a permutation w in each double coset BgB. The freedom to alter g within its coset gB allows one to scale columns and add scalar multiples of earlier columns to later columns. We claim that using such column operations, one can always find a representative g' for coset gB in which

- the bottommost nonzero entry of each column is 1 (call this entry a *pivot*),
- the entries to right of each pivot within its row are all 0, and
- there is one pivot in each row and each column, so that their positions are the positions of the 1's in some permutation matrix w.

In fact, we will see below that $BgB = BwB$ in this case. The algorithm which produces g' from g is simple: starting with the leftmost column, find its bottommost nonzero entry, and scale the column to make this entry a 1, creating the pivot in this column. Now use this pivot to clear out all entries in its row to its right, using column operations that subtract multiples of this column from later columns. Having done this, move on to the next column to the right, and repeat, scaling to create a pivot, and using it to eliminate entries to its right.[225]

[225] To see that this works, we need to check three facts:

(a) We will find a nonzero entry in every column during our algorithm.
(b) Our column operations preserve the zeroes lying to the right of already existing pivots.
(c) Every row contains exactly one pivot at the end of the algorithm.

But fact **(a)** simply says that our matrix can never have an all-zero column during the algorithm; this is clear (since the rank of the matrix remains constant during the algorithm and was n at its beginning). Fact **(b)** holds because all our operations either scale columns (which clearly preserves zero entries) or subtract a multiple of the column c containing the current pivot from a later column d (which will preserve every zero lying to the right of an already existing pivot, because any already existing pivot must

For example, the typical matrix g lying in the double coset BwB where

$$w = \begin{pmatrix} 1 & 2 & 3 & | & 4 & 5 & 6 & 7 & | & 8 & 9 \\ \underline{4} & \underline{\underline{8}} & \underline{2} & | & \underline{\underline{5}} & \underline{3} & \underline{\underline{9}} & \underline{1} & | & \underline{\underline{7}} & \underline{\underline{6}} \end{pmatrix}$$

from before is one that can be altered within its coset gB to look like this:

$$g' = \begin{bmatrix} * & * & * & * & * & * & 1 & 0 & 0 \\ * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & * & 0 & * & 1 \\ 0 & * & 0 & 0 & 0 & * & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \in gB.$$

Having found this g' in gB, a similar algorithm using left multiplication by B shows that w lies in $Bg' \subset Bg'B = BgB$. This time no scalings are required to create the pivot entries: starting with the bottom row, one uses its pivot to eliminate all the entries above it in the same column (shown by stars $*$ above) by adding multiples of the bottom row to higher rows. Then do the same using the pivot in the next-to-bottom row, etc. The result is the permutation matrix for w. $\square$

Remark 4.3.8. The Bruhat decomposition $GL_n = \bigsqcup_{w \in \mathfrak{S}_n} BwB$ is related to the so-called *LPU factorization* – one of a myriad of matrix factorizations appearing in linear algebra.[226] It is actually a fairly general phenomenon, and requires neither the finiteness of $\mathbb{F}$, nor the invertibility, nor even the squareness of the matrices (see Exercise 4.3.9(b) for an analogue holding in a more general setup).

Exercise 4.3.9. Let $\mathbb{F}$ be any field.

(a) For any $n \in \mathbb{N}$ and any $A \in GL_n(\mathbb{F})$, prove that there exist a lower-triangular matrix $L \in GL_n(\mathbb{F})$, an upper-triangular matrix $U \in GL_n(\mathbb{F})$ and a permutation matrix $P \in \mathfrak{S}_n \subset GL_n(\mathbb{F})$ (here, we identify permutations with the corresponding permutation matrices) such that $A = LPU$.

(b) Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Let $F_{n,m}$ denote the set of all $n \times m$-matrices $B \in \{0,1\}^{n \times m}$ such that each row of B contains at most one 1 and each column of B contains at most one 1. We regard $F_{n,m}$ as a subset of $\mathbb{F}^{n \times m}$ by means of regarding $\{0,1\}$ as a subset of $\mathbb{F}$.

For every $k \in \mathbb{N}$, we let B_k denote the subgroup of $GL_k(\mathbb{F})$ consisting of all upper-triangular matrices.

lie in a column $b < c$ and therefore both columns c and d have zeroes in its row). Fact **(c)** follows from noticing that there are n pivots altogether at the end of the algorithm, but no row can contain two of them (since the entries to the right of a pivot in its row are 0).

[226]Specifically, an *LPU factorization* of a matrix $A \in GL_n(\mathbb{F})$ (for an arbitrary field $\mathbb{F}$) means a way to write A as a product $A = LPU$ with $L \in GL_n(\mathbb{F})$ being lower-triangular, $U \in GL_n(\mathbb{F})$ being upper-triangular, and $P \in \mathfrak{S}_n \subset GL_n(\mathbb{F})$ being a permutation matrix. Such a factorization always exists (although it is generally not unique). This can be derived from the Bruhat decomposition (see Exercise 4.3.9(a) for a proof). See also [212] for related discussion.

Prove that

$$\mathbb{F}^{n\times m} = \bigsqcup_{f\in F_{n,m}} B_n f B_m.$$

Corollary 4.3.10. *For each of the three towers of groups G_*, the product and coproduct structures on $A = A(G_*)$ endow it with a bialgebra structure, and hence they form PSH's.*

Proof. The first two towers $G_n = \mathfrak{S}_n$ and $G_n = \mathfrak{S}_n[\Gamma]$ have product, coproduct defined by induction, restriction along embeddings $G_i \times G_j < G_{i+j}$. Hence the desired bialgebra equality (4.3.2) follows from Mackey's Theorem 4.1.7, taking $G = G_n, H = G_{(r_1,r_2)}, K = G_{(c_1,c_2)}, U = U_1 \otimes U_2$ with double coset representatives[227]

$$\begin{aligned}&\{g_1, \ldots, g_t\}\\ &= \left\{w_{A^t} \ : \ A \in \mathbb{N}^{2\times 2},\ A \text{ has row sums } (r_1, r_2) \text{ and column sums } (c_1, c_2)\right\}\end{aligned}$$

and checking for a given double coset

$$KgH = (G_{c_1,c_2})w_{A^t}(G_{r_1,r_2})$$

indexed by a matrix A in $\mathbb{N}^{2\times 2}$ with row sums (r_1, r_2) and column sums (c_1, c_2), that the two subgroups appearing on the left in (4.3.3) are exactly

$$\begin{aligned} H \cap K^{w_{A^t}} &= G_{r_1,r_2} \cap (G_{c_1,c_2})^{w_{A^t}},\\ {}^{w_{A^t}}H \cap K &= {}^{w_{A^t}}(G_{r_1,r_2}) \cap G_{c_1,c_2}, \end{aligned}$$

respectively. One should also apply (4.1.6) and check that the isomorphism τ_A between the two subgroups in (4.3.3) is the conjugation isomorphism by w_{A^t} (that is, $\tau_A(g) = w_{A^t} g w_{A^t}^{-1}$ for every $g \in H \cap K^{w_{A^t}}$). We leave all of these bookkeeping details to the reader to check. [228]

[227] Proposition 4.3.7 gives as a system of double coset representatives for $G_{(c_1,c_2)}\backslash G_n / G_{(r_1,r_2)}$ the elements

$$\begin{aligned}&\left\{w_A \ : \ A \in \mathbb{N}^{2\times 2},\ A \text{ has row sums } (c_1, c_2) \text{ and column sums } (r_1, r_2)\right\}\\ &= \left\{w_{A^t} \ : \ A \in \mathbb{N}^{2\times 2},\ A \text{ has row sums } (r_1, r_2) \text{ and column sums } (c_1, c_2)\right\}\end{aligned}$$

where A^t denotes the transpose matrix of A.

[228] It helps to recognize w_{A^t} as the permutation written in two-line notation as

$$\left(\begin{array}{cccc|cccc|cccc|cccc} 1 & 2 & \ldots & a_{11} & a_{11}+1 & a_{11}+2 & \ldots & r_1 & r_1+1 & r_1+2 & \ldots & a'_{22} & a'_{22}+1 & a'_{22}+2 & \ldots & n \\ 1 & 2 & \ldots & a_{11} & c_1+1 & c_1+2 & \ldots & a'_{22} & a_{11}+1 & a_{11}+2 & \ldots & c_1 & a'_{22}+1 & a'_{22}+2 & \ldots & n \end{array}\right),$$

where $a'_{22} = r_1 + a_{21} = c_1 + a_{12} = n - a_{22}$. In matrix form, w_{A^t} is the block matrix $\begin{bmatrix} I_{a_{11}} & 0 & 0 & 0 \\ 0 & 0 & I_{a_{21}} & 0 \\ 0 & I_{a_{12}} & 0 & 0 \\ 0 & 0 & 0 & I_{a_{22}} \end{bmatrix}$.

For the tower with $G_n = GL_n$, there is slightly more work to be done to check the equality (4.3.2). Via Mackey's Theorem 4.1.7 and Proposition 4.3.7(c), the left side is

$$\operatorname{res}^n_{c_1,c_2}\left(\operatorname{ind}^n_{r_1,r_2}(U_1 \otimes U_2)\right)$$
$$= \left(\operatorname{Res}^{G_n}_{P_{c_1,c_2}} \operatorname{Ind}^{G_n}_{P_{r_1,r_2}} \operatorname{Infl}^{P_{r_1,r_2}}_{G_{r_1,r_2}}(U_1 \otimes U_2)\right)^{K_{c_1,c_2}}$$

(4.3.4)
$$= \bigoplus_A \left(\operatorname{Ind}^{P_{c_1,c_2}}_{{}^{w_{A^t}}P_{r_1,r_2} \cap P_{c_1,c_2}} \left(\left(\operatorname{Res}^{P_{r_1,r_2}}_{P_{r_1,r_2} \cap P^{w_{A^t}}_{c_1,c_2}} \operatorname{Infl}^{P_{r_1,r_2}}_{G_{r_1,r_2}}(U_1 \otimes U_2)\right)^{\tau_A^{-1}}\right)\right)^{K_{c_1,c_2}}$$

where A runs over the usual 2×2 matrices. The right side is a direct sum over this same set of matrices A:

$$\bigoplus_A \left(\operatorname{ind}^{c_1}_{a_{11},a_{21}} \otimes \operatorname{ind}^{c_2}_{a_{12},a_{22}}\right)\left(\left(\operatorname{res}^{r_1}_{a_{11},a_{12}} U_1 \otimes \operatorname{res}^{r_2}_{a_{21},a_{22}} U_2\right)^{\tau_A^{-1}}\right)$$
$$= \bigoplus_A \left(\operatorname{Ind}^{G_{c_1}}_{P_{a_{11},a_{21}}} \otimes \operatorname{Ind}^{G_{c_2}}_{P_{a_{12},a_{22}}}\right) \circ \left(\operatorname{Infl}^{P_{a_{11},a_{21}}}_{G_{a_{11},a_{21}}} \otimes \operatorname{Infl}^{P_{a_{12},a_{22}}}_{G_{a_{12},a_{22}}}\right)$$
$$\left(\left(\left(\operatorname{Res}^{G_{r_1}}_{P_{a_{11},a_{12}}} U_1\right)^{K_{a_{11},a_{12}}} \otimes \left(\operatorname{Res}^{G_{r_2}}_{P_{a_{21},a_{22}}} U_2\right)^{K_{a_{21},a_{22}}}\right)^{\tau_A^{-1}}\right)$$
$$= \bigoplus_A \operatorname{Ind}^{G_{c_1,c_2}}_{P_{a_{11},a_{21}} \times P_{a_{12},a_{22}}} \operatorname{Infl}^{P_{a_{11},a_{21}} \times P_{a_{12},a_{22}}}_{G_{a_{11},a_{21},a_{12},a_{22}}}$$

(4.3.5)
$$\left(\left(\left(\operatorname{Res}^{G_{r_1,r_2}}_{P_{a_{11},a_{12}} \times P_{a_{21},a_{22}}}(U_1 \otimes U_2)\right)^{K_{a_{11},a_{12}} \times K_{a_{21},a_{22}}}\right)^{\tau_A^{-1}}\right)$$

(by (4.1.6), (4.1.15) and their obvious analogues for restriction and inflation). Thus it suffices to check for each 2×2 matrix A that any $\mathbb{C}G_{c_1,c_2}$-module of the form $V_1 \otimes V_2$ has the same inner product with the A-summands of (4.3.4) and (4.3.5). Abbreviate $w := w_{A^t}$ and $\tau := \tau_A^{-1}$.

Notice that ${}^w P_{r_1,r_2}$ is the group of all matrices having the block form

(4.3.6)
$$\begin{bmatrix} g_{11} & h & i & j \\ 0 & g_{21} & 0 & k \\ d & e & g_{12} & \ell \\ 0 & f & 0 & g_{22} \end{bmatrix}$$

in which the diagonal blocks g_{ij} for $i, j = 1, 2$ are invertible of size $a_{ij} \times a_{ij}$, while the blocks $h, i, j, k, \ell, d, e, f$ are all arbitrary matrices[229] of the appropriate (rectangular) block sizes. Hence, ${}^w P_{r_1,r_2} \cap P_{c_1,c_2}$ is the group of all matrices having the block form

(4.3.7)
$$\begin{bmatrix} g_{11} & h & i & j \\ 0 & g_{21} & 0 & k \\ 0 & 0 & g_{12} & \ell \\ 0 & 0 & 0 & g_{22} \end{bmatrix}$$

in which the diagonal blocks g_{ij} for $i, j = 1, 2$ are invertible of size $a_{ij} \times a_{ij}$, while the blocks h, i, j, k, ℓ are all arbitrary matrices of the appropriate (rectangular) block sizes; then ${}^w P_{r_1,r_2} \cap G_{c_1,c_2}$ is the subgroup where the

[229] The blocks i and j have nothing to do with the indices i, j in g_{ij}.

blocks i, j, k all vanish. The canonical projection ${}^wP_{r_1,r_2} \cap P_{c_1,c_2} \to {}^wP_{r_1,r_2} \cap G_{c_1,c_2}$ (obtained by restricting the projection $P_{c_1,c_2} \to G_{c_1,c_2}$) has kernel ${}^wP_{r_1,r_2} \cap P_{c_1,c_2} \cap K_{c_1,c_2}$. Consequently,

$$\begin{aligned} &({}^wP_{r_1,r_2} \cap P_{c_1,c_2}) \,/\, ({}^wP_{r_1,r_2} \cap P_{c_1,c_2} \cap K_{c_1,c_2}) \\ &= {}^wP_{r_1,r_2} \cap G_{c_1,c_2}. \end{aligned} \tag{4.3.8}$$

Similarly,

$$\begin{aligned} &\left(P_{r_1,r_2} \cap P^w_{c_1,c_2}\right) \,/\, \left(P_{r_1,r_2} \cap P^w_{c_1,c_2} \cap K_{r_1,r_2}\right) \\ &= G_{r_1,r_2} \cap P^w_{c_1,c_2}. \end{aligned} \tag{4.3.9}$$

Computing first the inner product of $V_1 \otimes V_2$ with the A-summand of (4.3.4), and using adjointness properties, one gets

$$\begin{aligned} &\left(\left(\operatorname{Res}^{P_{r_1,r_2}}_{P_{r_1,r_2} \cap P^w_{c_1,c_2}} \operatorname{Infl}^{P_{r_1,r_2}}_{G_{r_1,r_2}} (U_1 \otimes U_2)\right)^\tau,\right. \\ &\qquad\qquad \left.\operatorname{Res}^{P_{c_1,c_2}}_{{}^wP_{r_1,r_2} \cap P_{c_1,c_2}} \operatorname{Infl}^{P_{c_1,c_2}}_{G_{c_1,c_2}} (V_1 \otimes V_2)\right)_{{}^wP_{r_1,r_2} \cap P_{c_1,c_2}} \\ &\overset{(4.1.10)}{=} \left(\left(\operatorname{Infl}^{P_{r_1,r_2} \cap P^w_{c_1,c_2}}_{G_{r_1,r_2} \cap P^w_{c_1,c_2}} \operatorname{Res}^{G_{r_1,r_2}}_{G_{r_1,r_2} \cap P^w_{c_1,c_2}} (U_1 \otimes U_2)\right)^\tau,\right. \\ &\qquad\qquad \left.\operatorname{Infl}^{{}^wP_{r_1,r_2} \cap P_{c_1,c_2}}_{{}^wP_{r_1,r_2} \cap G_{c_1,c_2}} \operatorname{Res}^{G_{c_1,c_2}}_{{}^wP_{r_1,r_2} \cap G_{c_1,c_2}} (V_1 \otimes V_2)\right)_{{}^wP_{r_1,r_2} \cap P_{c_1,c_2}} \end{aligned}$$

(by (4.3.9) and (4.3.8)). One can compute this inner product by first recalling that ${}^wP_{r_1,r_2} \cap P_{c_1,c_2}$ is the group of matrices having the block form (4.3.7) in which the diagonal blocks g_{ij} for $i, j = 1, 2$ are invertible of size $a_{ij} \times a_{ij}$, while the blocks h, i, j, k, ℓ are all arbitrary matrices of the appropriate (rectangular) block sizes; then ${}^wP_{r_1,r_2} \cap G_{c_1,c_2}$ is the subgroup where the blocks i, j, k all vanish. The inner product above then becomes

$$\begin{aligned} &\frac{1}{|{}^wP_{r_1,r_2} \cap P_{c_1,c_2}|} \sum_{\substack{(g_{ij}) \\ (h,i,j,k,\ell)}} \chi_{U_1}\begin{pmatrix} g_{11} & i \\ 0 & g_{12} \end{pmatrix} \chi_{U_2}\begin{pmatrix} g_{21} & k \\ 0 & g_{22} \end{pmatrix} \\ &\qquad\qquad\qquad\qquad \overline{\chi}_{V_1}\begin{pmatrix} g_{11} & h \\ 0 & g_{21} \end{pmatrix} \overline{\chi}_{V_2}\begin{pmatrix} g_{12} & \ell \\ 0 & g_{22} \end{pmatrix}. \end{aligned} \tag{4.3.10}$$

If one instead computes the inner product of $V_1 \otimes V_2$ with the A-summand of (4.3.5), using adjointness properties and (4.1.13) one gets

$$\begin{aligned} &\left(\left(\left(\operatorname{Res}^{G_{r_1,r_2}}_{P_{a_{11},a_{12}} \times P_{a_{21},a_{22}}} (U_1 \otimes U_2)\right)^{K_{a_{11},a_{12}} \times K_{a_{21},a_{22}}}\right)^\tau,\right. \\ &\qquad \left.\left(\operatorname{Res}^{G_{c_1,c_2}}_{P_{a_{11},a_{21}} \times P_{a_{12},a_{22}}} (V_1 \otimes V_2)\right)^{K_{a_{11},a_{21}} \times K_{a_{12},a_{22}}}\right)_{G_{a_{11},a_{21},a_{12},a_{22}}} \\ &= \frac{1}{|G_{a_{11},a_{21},a_{12},a_{22}}|} \sum_{(g_{ij})} \frac{1}{|K_{a_{11},a_{12}} \times K_{a_{21},a_{22}}|} \sum_{(i,k)} \chi_{U_1}\begin{pmatrix} g_{11} & i \\ 0 & g_{12} \end{pmatrix} \chi_{U_2}\begin{pmatrix} g_{21} & k \\ 0 & g_{22} \end{pmatrix} \\ &\qquad\qquad \frac{1}{|K_{a_{11},a_{21}} \times K_{a_{12},a_{22}}|} \sum_{(h,\ell)} \overline{\chi}_{V_1}\begin{pmatrix} g_{11} & h \\ 0 & g_{21} \end{pmatrix} \overline{\chi}_{V_2}\begin{pmatrix} g_{12} & \ell \\ 0 & g_{22} \end{pmatrix}. \end{aligned}$$

But this right hand side can be seen to equal (4.3.10), after one notes that

$$\begin{aligned} &|{}^wP_{r_1,r_2} \cap P_{c_1,c_2}| \\ &= |G_{a_{11},a_{21},a_{12},a_{22}}| \cdot |K_{a_{11},a_{12}} \times K_{a_{21},a_{22}}| \cdot |K_{a_{11},a_{21}} \times K_{a_{12},a_{22}}| \cdot \#\{j \in \mathbb{F}_q^{a_{11} \times a_{22}}\} \end{aligned}$$

and that the summands in (4.3.10) are independent of the matrix j in the summation. $\square$

We can also define a $\mathbb{C}$-vector space $A_{\mathbb{C}}$ as the direct sum $\bigoplus_{n\geq 0} R_{\mathbb{C}}(G_n)$. In the same way as we have made $A = \bigoplus_{n\geq 0} R(G_n)$ into a $\mathbb{Z}$-bialgebra, we can turn $A_{\mathbb{C}} = \bigoplus_{n\geq 0} R_{\mathbb{C}}(G_n)$ into a $\mathbb{C}$-bialgebra[230]. There is a $\mathbb{C}$-bilinear form $(\cdot,\cdot)_{A_{\mathbb{C}}}$ on $A_{\mathbb{C}}$ which can be defined either as the $\mathbb{C}$-bilinear extension of the $\mathbb{Z}$-bilinear form $(\cdot,\cdot)_A : A \times A \to \mathbb{Z}$ to $A_{\mathbb{C}}$, or (equivalently) as the $\mathbb{C}$-bilinear form on $A_{\mathbb{C}}$ which restricts to $\langle\cdot,\cdot\rangle_{\mathfrak{S}_n}$ on every homogeneous component $R_{\mathbb{C}}(G_n)$ and makes different homogeneous components mutually orthogonal. The obvious embedding of A into the $\mathbb{C}$-bialgebra $A_{\mathbb{C}}$ (obtained from the embeddings $R(G_n) \to R_{\mathbb{C}}(G_n)$ for all n) respects the bialgebra operations[231], and the $\mathbb{C}$-bialgebra $A_{\mathbb{C}}$ can be identified with $A \otimes_{\mathbb{Z}} \mathbb{C}$ (the result of extending scalars to $\mathbb{C}$ in A), because every finite group G satisfies $R_{\mathbb{C}}(G) \cong R(G) \otimes_{\mathbb{Z}} \mathbb{C}$. The embedding of A into $A_{\mathbb{C}}$ also respects the bilinear forms.

Exercise 4.3.11. Let G_* be one of the three towers.

For every almost-composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ of $n \in \mathbb{N}$, let us define a map $\operatorname{ind}_\alpha^n$ which takes $\mathbb{C}G_\alpha$-modules to $\mathbb{C}G_n$-modules as follows: If $G_* = \mathfrak{S}_*$ or $G_* = \mathfrak{S}_*[\Gamma]$, we set

$$\operatorname{ind}_\alpha^n := \operatorname{Ind}_{G_\alpha}^{G_n}.$$

If $G_* = GL_*$, then we set

$$\operatorname{ind}_\alpha^n := \operatorname{Ind}_{P_\alpha}^{G_n} \operatorname{Infl}_{G_\alpha}^{P_\alpha}.$$

(Note that $\operatorname{ind}_\alpha^n = \operatorname{ind}_{i,j}^n$ if α has the form (i, j).)

Similarly, for every almost-composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ of $n \in \mathbb{N}$, let us define a map $\operatorname{res}_\alpha^n$ which takes $\mathbb{C}G_n$-modules to $\mathbb{C}G_\alpha$-modules as follows: If $G_* = \mathfrak{S}_*$ or $G_* = \mathfrak{S}_*[\Gamma]$, we set

$$\operatorname{res}_\alpha^n := \operatorname{Res}_{G_\alpha}^{G_n}.$$

If $G_* = GL_*$, then we set

$$\operatorname{res}_\alpha^n := \left(\operatorname{Res}_{P_\alpha}^{G_n}(-)\right)^{K_\alpha}.$$

(Note that $\operatorname{res}_\alpha^n = \operatorname{res}_{i,j}^n$ if α has the form (i, j).)

(a) If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ is an almost-composition of an integer $n \in \mathbb{N}$ satisfying $\ell \geq 1$, and if V_i is a $\mathbb{C}G_{\alpha_i}$-module for every $i \in$

[230]The definitions of m and Δ for this $\mathbb{C}$-bialgebra look the same as for A: For instance, m is still defined to be $\operatorname{ind}_{i,j}^{i+j}$ on $(A_{\mathbb{C}})_i \otimes (A_{\mathbb{C}})_j$, where $\operatorname{ind}_{i,j}^{i+j}$ is defined by the same formulas as in Definition 4.2.1. However, the operators of induction, restriction, inflation and K-fixed space construction appearing in these formulas now act on class functions as opposed to modules.

The fact that these maps m and Δ satisfy the axioms of a $\mathbb{C}$-bialgebra is easy to check: they are merely the $\mathbb{C}$-linear extensions of the maps m and Δ of the $\mathbb{Z}$-bialgebra A (this is because, for instance, induction of class functions and induction of modules are related by the identity (4.1.5)), and thus satisfy the same axioms as the latter.

[231]This is because, for example, induction of class functions harmonizes with induction of modules (i.e., the equality (4.1.5) holds).

$\{1,2,\ldots,\ell\}$, then show that

$$\begin{aligned}
&\operatorname{ind}^n_{\alpha_1+\alpha_2+\cdots+\alpha_{\ell-1},\alpha_\ell}\left(\operatorname{ind}^{\alpha_1+\alpha_2+\cdots+\alpha_{\ell-1}}_{(\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1})}\left(V_1\otimes V_2\otimes\cdots\otimes V_{\ell-1}\right)\otimes V_\ell\right)\\
&\cong \operatorname{ind}^n_\alpha\left(V_1\otimes V_2\otimes\cdots\otimes V_\ell\right)\\
&\cong \operatorname{ind}^n_{\alpha_1,\alpha_2+\alpha_3+\cdots+\alpha_\ell}\left(V_1\otimes \operatorname{ind}^{\alpha_2+\alpha_3+\cdots+\alpha_\ell}_{(\alpha_2,\alpha_3,\ldots,\alpha_\ell)}\left(V_2\otimes V_3\otimes\cdots\otimes V_\ell\right)\right).
\end{aligned}$$

(b) Solve Exercise 4.2.3 again using Exercise 4.3.11(a).

(c) We proved above that the map $m : A\otimes A\to A$ (where $A = A(G_*)$) is associative, by using the adjointness of m and Δ. Give a new proof of this fact, which makes no use of Δ.

(d) If $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_\ell)$ is an almost-composition of an $n\in\mathbb{N}$, and if $\chi_i\in R(G_{\alpha_i})$ for every $i\in\{1,2,\ldots,\ell\}$, then show that

$$\chi_1\chi_2\cdots\chi_\ell = \operatorname{ind}^n_\alpha\left(\chi_1\otimes\chi_2\otimes\cdots\otimes\chi_\ell\right)$$

in $A = A(G_*)$.

(e) If $n\in\mathbb{N}$, $\ell\in\mathbb{N}$ and $\chi\in R(G_n)$, then show that

$$\Delta^{(\ell-1)}\chi = \sum \operatorname{res}^n_\alpha \chi$$

in $A^{\otimes\ell}$, where $A = A(G_*)$. Here, the sum on the right hand side runs over all almost-compositions α of n having length ℓ.

4.4. **Symmetric groups.** Finally, some payoff. Consider the tower of symmetric groups $G_n = \mathfrak{S}_n$, and $A = A(G_*) =: A(\mathfrak{S})$. Denote by $\underline{1}_{\mathfrak{S}_n}, \operatorname{sgn}_{\mathfrak{S}_n}$ the *trivial* and *sign* characters on $\mathfrak{S}_n$. For a partition λ of n, denote by $\underline{1}_{\mathfrak{S}_\lambda}, \operatorname{sgn}_{\mathfrak{S}_\lambda}$ the trivial and sign characters restricted to the Young subgroup $\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1}\times\mathfrak{S}_{\lambda_2}\times\cdots$, and denote by $\underline{1}_\lambda$ the class function which is the characteristic function for the $\mathfrak{S}_n$-conjugacy class of permutations of cycle type λ.

Theorem 4.4.1. (a) *Irreducible complex characters $\{\chi^\lambda\}$ of $\mathfrak{S}_n$ are indexed by partitions λ in Par_n, and one has a PSH-isomorphism, the* Frobenius characteristic map[232],

$$A = A(\mathfrak{S}) \xrightarrow{\mathrm{ch}} \Lambda$$

that for $n\geq 0$ and $\lambda\in\mathrm{Par}_n$ sends

$$\begin{array}{rcl}
\underline{1}_{\mathfrak{S}_n} & \longmapsto & h_n,\\
\operatorname{sgn}_{\mathfrak{S}_n} & \longmapsto & e_n,\\
\chi^\lambda & \longmapsto & s_\lambda,\\
\operatorname{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_\lambda}\underline{1}_{\mathfrak{S}_\lambda} & \longmapsto & h_\lambda,\\
\operatorname{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_\lambda}\operatorname{sgn}_{\mathfrak{S}_\lambda} & \longmapsto & e_\lambda,\\
\underline{1}_\lambda & \longmapsto & \frac{p_\lambda}{z_\lambda}
\end{array}$$

(where ch *is extended to a $\mathbb{C}$-linear map $A_{\mathbb{C}}\to\Lambda_{\mathbb{C}}$), and for $n\geq 1$ sends*

$$\underline{1}_{(n)} \longmapsto \frac{p_n}{n}.$$

Here, z_λ is defined as in Proposition 2.5.15.

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[232]It is unrelated to the Frobenius endomorphisms from Exercise 2.9.9.

(b) *For each $n \geq 0$, the involution on class functions $f : \mathfrak{S}_n \to \mathbb{C}$ sending $f \longmapsto \operatorname{sgn}_{\mathfrak{S}_n} * f$ where*

$$(\operatorname{sgn}_{\mathfrak{S}_n} * f)(g) := \operatorname{sgn}(g) f(g)$$

preserves the $\mathbb{Z}$-sublattice $R(\mathfrak{S}_n)$ of genuine characters. The direct sum of these involutions induces an involution on $A = A(\mathfrak{S}) = \bigoplus_{n \geq 0} R(\mathfrak{S}_n)$ that corresponds under ch *to the involution ω on Λ.*

Proof. (a) Corollary 4.3.10 implies that the set $\Sigma = \bigsqcup_{n \geq 0} \operatorname{Irr}(\mathfrak{S}_n)$ gives a PSH-basis for A. Since a character χ of $\mathfrak{S}_n$ has

$$\Delta(\chi) = \bigoplus_{i+j=n} \operatorname{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_i \times \mathfrak{S}_j} \chi, \tag{4.4.1}$$

such an element $\chi \in \Sigma \cap A_n$ is never primitive for $n \geq 2$. Hence the unique irreducible character $\rho = \underline{1}_{\mathfrak{S}_1}$ of $\mathfrak{S}_1$ is the only element of $\mathcal{C} = \Sigma \cap \mathfrak{p}$.

Thus Theorem 3.3.3(g) tells us that there are two PSH-isomorphisms $A \to \Lambda$, each of which sends Σ to the PSH-basis of Schur functions $\{s_\lambda\}$ for Λ. It also tells us that we can pin down one of the two isomorphisms to call ch, by insisting that it map the two characters $\underline{1}_{\mathfrak{S}_2}, \operatorname{sgn}_{\mathfrak{S}_2}$ in $\operatorname{Irr}(\mathfrak{S}_2)$ to h_2, e_2 (and not e_2, h_2).

Bearing in mind the coproduct formula (4.4.1), and the fact that $\underline{1}_{\mathfrak{S}_n}, \operatorname{sgn}_{\mathfrak{S}_n}$ restrict, respectively, to trivial and sign characters of $\mathfrak{S}_i \times \mathfrak{S}_j$ for $i + j = n$, one finds that for $n \geq 2$ one has $\operatorname{sgn}^\perp_{\mathfrak{S}_2}$ annihilating $\underline{1}_{\mathfrak{S}_n}$, and $\underline{1}^\perp_{\mathfrak{S}_2}$ annihilating $\operatorname{sgn}_{\mathfrak{S}_n}$. Therefore Theorem 3.3.1(b) (applied to Λ) implies $\underline{1}_{\mathfrak{S}_n}, \operatorname{sgn}_{\mathfrak{S}_n}$ are sent under ch to h_n, e_n. Then the fact that $\operatorname{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_\lambda} \underline{1}_{\mathfrak{S}_\lambda}, \operatorname{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_\lambda} \operatorname{sgn}_{\mathfrak{S}_\lambda}$ are sent to h_λ, e_λ follows via induction products.

Recall that the $\mathbb{C}$-vector space $A_\mathbb{C} = \bigoplus_{n \geq 0} R_\mathbb{C}(\mathfrak{S}_n)$ is a $\mathbb{C}$-bialgebra, and can be identified with $A \otimes_\mathbb{Z} \mathbb{C}$. The multiplication and the comultiplication of $A_\mathbb{C}$ are $\mathbb{C}$-linear extensions of those of A, and are still given by the same formulas $m = \operatorname{ind}^{i+j}_{i,j}$ and $\Delta = \bigoplus_{i+j=n} \operatorname{res}^{i+j}_{i,j}$ as those of A (but now, induction and restriction are defined for class functions, not just for representations). The $\mathbb{C}$-bilinear form $(\cdot, \cdot)_{A_\mathbb{C}}$ on $A_\mathbb{C}$ extends both the $\mathbb{Z}$-bilinear form $(\cdot, \cdot)_A$ on A and the $\mathbb{C}$-bilinear forms $\langle \cdot, \cdot \rangle_{\mathfrak{S}_n}$ on all $R_\mathbb{C}(\mathfrak{S}_n)$.

For the assertion about $\underline{1}_{(n)}$, note that it is primitive in $A_\mathbb{C}$ for $n \geq 1$, because as a class function, the indicator function of n-cycles vanishes upon restriction to $\mathfrak{S}_i \times \mathfrak{S}_j$ for $i + j = n$ if both $i, j \geq 1$; these subgroups contain no n-cycles. Hence Corollary 3.1.8 implies that $\operatorname{ch}(\underline{1}_{(n)})$ is a scalar multiple of p_n. To pin down the scalar, note $p_n = m_{(n)}$ so $(h_n, p_n)_\Lambda = (h_n, m_n)_\Lambda = 1$, while $\operatorname{ch}^{-1}(h_n) = \underline{1}_{\mathfrak{S}_n}$ has

$$(\underline{1}_{\mathfrak{S}_n}, \underline{1}_{(n)}) = \frac{1}{n!} \cdot (n-1)! = \frac{1}{n}.$$

[233] Thus $\operatorname{ch}(\underline{1}_{(n)}) = \frac{p_n}{n}$. The fact that $\operatorname{ch}(\underline{1}_\lambda) = \frac{p_\lambda}{z_\lambda}$ then follows via induction product calculations[234]. Part (b) follows from Exercise 4.4.4 below. $\square$

[233] The first equality sign in this computation uses the fact that the number of all n-cycles in $\mathfrak{S}_n$ is $(n-1)!$. This is because any n-cycle in $\mathfrak{S}_n$ can be uniquely written in the form $(i_1, i_2, \ldots, i_{n-1}, n)$ (in cycle notation) with $(i_1, i_2, \ldots, i_{n-1})$ being a permutation in $\mathfrak{S}_{n-1}$ (written in one-line notation).

[234] For instance, one can use (4.1.3) to show that $z_\lambda \underline{1}_\lambda = \lambda_1 \lambda_2 \cdots \lambda_\ell \cdot \underline{1}_{(\lambda_1)} \underline{1}_{(\lambda_2)} \cdots \underline{1}_{(\lambda_\ell)}$ if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ with $\ell = \ell(\lambda)$. See Exercise 4.4.3(d) for the details.

Remark 4.4.2. The paper of Liulevicius [133] gives a very elegant alternate approach to the Frobenius map as a Hopf isomorphism $A(\mathfrak{S}) \xrightarrow{\mathrm{ch}} \Lambda$, inspired by equivariant K-theory and vector bundles over spaces which are finite sets of points!

Exercise 4.4.3. If P is a subset of a group G, we denote by $\underline{1}_P$ the map $G \to \mathbb{C}$ which sends every element of P to 1 and all remaining elements of G to 0. [235] For any finite group G and any $h \in G$, we introduce the following notations:

- Let $Z_G(h)$ denote the centralizer of h in G.
- Let $\mathrm{Conj}_G(h)$ denote the conjugacy class of h in G.
- Define a map $\alpha_{G,h} : G \to \mathbb{C}$ by $\alpha_{G,h} = |Z_G(h)| \underline{1}_{\mathrm{Conj}_G(h)}$. This map $\alpha_{G,h}$ is a class function[236].

(a) Prove that $\alpha_{G,h}(g) = \sum_{k \in G} [khk^{-1} = g]$ for every finite group G and any $h \in G$ and $g \in G$. Here, we are using the Iverson bracket notation (that is, for any statement $\mathcal{A}$, we define $[\mathcal{A}]$ to be the integer 1 if $\mathcal{A}$ is true, and 0 otherwise).

(b) Prove that if H is a subgroup of a finite group G, and if $h \in H$, then $\mathrm{Ind}_H^G \alpha_{H,h} = \alpha_{G,h}$.

(c) Prove that if G_1 and G_2 are finite groups, and if $h_1 \in G_1$ and $h_2 \in G_2$, then the canonical isomorphism $R_{\mathbb{C}}(G_1) \otimes R_{\mathbb{C}}(G_2) \to R_{\mathbb{C}}(G_1 \times G_2)$ sends $\alpha_{G_1,h_1} \otimes \alpha_{G_2,h_2}$ to $\alpha_{G_1 \times G_2,(h_1,h_2)}$.

(d) Fill in the details of the proof of $\mathrm{ch}(\underline{1}_\lambda) = \frac{p_\lambda}{z_\lambda}$ in the proof of Theorem 4.4.1.

(e) Obtain an alternative proof of Remark 2.5.16.

(f) If G and H are two finite groups, and if $\rho : H \to G$ is a group homomorphism, then prove that $\mathrm{Ind}_\rho \alpha_{H,h} = \alpha_{G,\rho(h)}$ for every $h \in H$, where $\mathrm{Ind}_\rho \alpha_{H,h}$ is defined as in Exercise 4.1.14.

Exercise 4.4.4. If G is a group and U_1 and U_2 are two $\mathbb{C}G$-modules, then the tensor product $U_1 \otimes U_2$ is a $\mathbb{C}[G \times G]$-module, which can be made into a $\mathbb{C}G$-module by letting $g \in G$ act as $(g, g) \in G \times G$. This $\mathbb{C}G$-module $U_1 \otimes U_2$ is called the *inner tensor product*[237] of U_1 and U_2, and is a restriction of the outer tensor product $U_1 \otimes U_2$ using the inclusion map $G \to G \times G,\ g \mapsto (g, g)$.

Let $n \geq 0$, and let $\mathrm{sgn}_{\mathfrak{S}_n}$ be the 1-dimensional $\mathbb{C}\mathfrak{S}_n$-module $\mathbb{C}$ on which every $g \in \mathfrak{S}_n$ acts as multiplication by $\mathrm{sgn}(g)$. If V is a $\mathbb{C}\mathfrak{S}_n$-module, show that the involution on $A(\mathfrak{S}) = \bigoplus_{n \geq 0} R(\mathfrak{S}_n)$ defined in Theorem 4.4.1(b) sends $\chi_V \mapsto \chi_{\mathrm{sgn}_{\mathfrak{S}_n} \otimes V}$ where $\mathrm{sgn}_{\mathfrak{S}_n} \otimes V$ is the inner tensor product of $\mathrm{sgn}_{\mathfrak{S}_n}$ and V. Use this to show that this involution is a nontrivial PSH-automorphism of $A(\mathfrak{S})$, and deduce Theorem 4.4.1(b).

[235] This is not in conflict with the notation $\underline{1}_G$ for the trivial character of G, since $\underline{1}_P = \underline{1}_G$ for $P = G$. Note that $\underline{1}_P$ is a class function when P is a union of conjugacy classes of G.

[236] In fact, $\underline{1}_{\mathrm{Conj}_G(h)}$ is a class function (since $\mathrm{Conj}_G(h)$ is a conjugacy class), and so $\alpha_{G,h}$ (being the scalar multiple $|Z_G(h)| \underline{1}_{\mathrm{Conj}_G(h)}$ of $\underline{1}_{\mathrm{Conj}_G(h)}$) must also be a class function.

[237] Do not confuse this with the inner product of characters.

Exercise 4.4.5. Let $n \in \mathbb{N}$. For every permutation $\sigma \in \mathfrak{S}_n$, we let $\operatorname{type} \sigma$ denote the cycle type of σ. Extend $\operatorname{ch} : A = A(\mathfrak{S}) \to \Lambda$ to a $\mathbb{C}$-linear map $A_{\mathbb{C}} \to \Lambda_{\mathbb{C}}$. We shall call the latter map ch, too.

(a) Prove that every class function $f \in R_{\mathbb{C}}(\mathfrak{S}_n)$ satisfies

$$\operatorname{ch}(f) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} f(\sigma) p_{\operatorname{type} \sigma}.$$

(b) Let H be a subgroup of $\mathfrak{S}_n$. Prove that every class function $f \in R_{\mathbb{C}}(H)$ satisfies

$$\operatorname{ch}\left(\operatorname{Ind}_H^{\mathfrak{S}_n} f\right) = \frac{1}{|H|} \sum_{h \in H} f(h) p_{\operatorname{type} h}.$$

Exercise 4.4.6. (a) Show that for every $n \geq 0$, every $g \in \mathfrak{S}_n$ and every finite-dimensional $\mathbb{C}\mathfrak{S}_n$-module V, we have $\chi_V(g) \in \mathbb{Z}$.

(b) Show that for every $n \geq 0$ and every finite-dimensional $\mathbb{C}\mathfrak{S}_n$-module V, there exists a $\mathbb{Q}\mathfrak{S}_n$-module W such that $V \cong \mathbb{C} \otimes_{\mathbb{Q}} W$. (In the representation theorists' parlance, this says that all representations of $\mathfrak{S}_n$ are *defined over* $\mathbb{Q}$. This part of the exercise requires some familiarity with representation theory.)

Remark 4.4.7. Parts (a) and (b) of Exercise 4.4.6 both follow from an even stronger result: For every $n \geq 0$ and every finite-dimensional $\mathbb{C}\mathfrak{S}_n$-module V, there exists a $\mathbb{Z}\mathfrak{S}_n$-module W which is finitely generated and free as a $\mathbb{Z}$-module and satisfies $V \cong \mathbb{C} \otimes_{\mathbb{Z}} W$ as $\mathbb{C}\mathfrak{S}_n$-modules. This follows from the combinatorial approach to the representation theory of $\mathfrak{S}_n$, in which the irreducible representations of $\mathbb{C}\mathfrak{S}_n$ (the *Specht modules*) are constructed using Young tableaux and tabloids. See the literature on the symmetric group, e.g., [186], [73, §7], [223] or [115, Section 2.2] for this approach.

The connection between Λ and $A(\mathfrak{S})$ as established in Theorem 4.4.1 benefits both the study of Λ and that of $A(\mathfrak{S})$. The following two exercises show some applications to Λ:

Exercise 4.4.8. If G is a group and U_1 and U_2 are two $\mathbb{C}G$-modules, then let $U_1 \boxtimes U_2$ denote the inner tensor product of U_1 and U_2 (as defined in Exercise 4.4.4). Consider also the binary operation $*$ on $\Lambda_{\mathbb{Q}}$ defined in Exercise 2.9.4(h).

(a) Show that $\operatorname{ch}(\chi_{U_1 \boxtimes U_2}) = \operatorname{ch}(\chi_{U_1}) * \operatorname{ch}(\chi_{U_2})$ for any $n \in \mathbb{N}$ and any two $\mathbb{C}\mathfrak{S}_n$-modules U_1 and U_2.
(b) Use this to obtain a new solution for Exercise 2.9.4(h).
(c) Show that $s_\mu * s_\nu \in \sum_{\lambda \in \operatorname{Par}} \mathbb{N} s_\lambda$ for any two partitions μ and ν.

[**Hint:** For any group G, introduce a binary operation $*$ on $R_{\mathbb{C}}(G)$ which satisfies $\chi_{U_1 \boxtimes U_2} = \chi_{U_1} * \chi_{U_2}$ for any two $\mathbb{C}G$-modules U_1 and U_2.]

Exercise 4.4.9. Define a $\mathbb{Q}$-bilinear map $\boxdot : \Lambda_{\mathbb{Q}} \times \Lambda_{\mathbb{Q}} \to \Lambda_{\mathbb{Q}}$, which will be written in infix notation (that is, we will write $a \boxdot b$ instead of $\boxdot(a, b)$), by setting

$$p_\lambda \boxdot p_\mu = \prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\ell(\mu)} p_{\operatorname{lcm}(\lambda_i, \mu_j)}^{\gcd(\lambda_i, \mu_j)} \qquad \text{for any partitions } \lambda \text{ and } \mu.$$

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(a) Show that $\Lambda_{\mathbb{Q}}$, equipped with the binary operation $\boxdot$, becomes a commutative $\mathbb{Q}$-algebra with unity p_1.
(b) For every $r \in \mathbb{Z}$, define the $\mathbb{Q}$-algebra homomorphism $\epsilon_r : \Lambda_{\mathbb{Q}} \to \mathbb{Q}$ as in Exercise 2.9.4(c). Show that $1 \boxdot f = \epsilon_1(f)\, 1$ for every $f \in \Lambda_{\mathbb{Q}}$ (where 1 denotes the unity of Λ).
(c) Show that $s_\mu \boxdot s_\nu \in \sum_{\lambda \in \mathrm{Par}} \mathbb{N} s_\lambda$ for any two partitions μ and ν.
(d) Show that $f \boxdot g \in \Lambda$ for any $f \in \Lambda$ and $g \in \Lambda$.

[**Hint:** For every set X, let $\mathfrak{S}_X$ denote the group of all permutations of X. For two sets X and Y, there is a canonical group homomorphism $\mathfrak{S}_X \times \mathfrak{S}_Y \to \mathfrak{S}_{X \times Y}$, which is injective if X and Y are nonempty. For positive integers n and m, this yields an embedding $\mathfrak{S}_n \times \mathfrak{S}_m \to \mathfrak{S}_{\{1,2,\ldots,n\} \times \{1,2,\ldots,m\}}$, which, once $\mathfrak{S}_{\{1,2,\ldots,n\} \times \{1,2,\ldots,m\}}$ is identified with $\mathfrak{S}_{nm}$ (using an arbitrary but fixed bijection $\{1,2,\ldots,n\} \times \{1,2,\ldots,m\} \to \{1,2,\ldots,nm\}$), can be regarded as an embedding $\mathfrak{S}_n \times \mathfrak{S}_m \to \mathfrak{S}_{nm}$ and thus allows defining a $\mathbb{C}\mathfrak{S}_{nm}$-module $\mathrm{Ind}_{\mathfrak{S}_n \times \mathfrak{S}_m}^{\mathfrak{S}_{nm}} (U \otimes V)$ for any $\mathbb{C}\mathfrak{S}_n$-module U and any $\mathbb{C}\mathfrak{S}_m$-module V. This gives a binary operation on $A(\mathfrak{S})$. Show that this operation corresponds to $\boxdot$ under the PSH-isomorphism $\mathrm{ch} : A(\mathfrak{S}) \to \Lambda$.]

Remark 4.4.10. The statements (and the idea of the solution) of Exercise 4.4.9 are due to Manuel Maia and Miguel Méndez (see [144] and, more explicitly, [155]), who call the operation $\boxdot$ the *arithmetic product*. Li [131, Thm. 3.5] denotes it by $\boxtimes$ and relates it to the enumeration of unlabelled graphs.

4.5. **Wreath products.** Next consider the tower of groups $G_n = \mathfrak{S}_n[\Gamma]$ for a finite group Γ, and the Hopf algebra $A = A(G_*) =: A(\mathfrak{S}[\Gamma])$. Recall (from Theorem 4.4.1) that irreducible complex representations χ^λ of $\mathfrak{S}_n$ are indexed by partitions λ in Par_n. Index the irreducible complex representations of Γ as $\mathrm{Irr}(\Gamma) = \{\rho_1, \ldots, \rho_d\}$.

Definition 4.5.1. Define for a partition λ in Par_n and ρ in $\mathrm{Irr}(\Gamma)$ a representation $\chi^{\lambda,\rho}$ of $\mathfrak{S}_n[\Gamma]$ in which σ in $\mathfrak{S}_n$ and $\gamma = (\gamma_1, \ldots, \gamma_n)$ in Γ^n act on the space $\chi^\lambda \otimes (\rho^{\otimes n})$ as follows:

$$\begin{aligned} \sigma(u \otimes (v_1 \otimes \cdots \otimes v_n)) &= \sigma(u) \otimes (v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}); \\ \gamma(u \otimes (v_1 \otimes \cdots \otimes v_n)) &= u \otimes (\gamma_1 v_1 \otimes \cdots \otimes \gamma_n v_n). \end{aligned} \tag{4.5.1}$$

Theorem 4.5.2. *The irreducible $\mathbb{C}\mathfrak{S}_n[\Gamma]$-modules are the induced characters*

$$\chi^{\underline{\lambda}} := \mathrm{Ind}_{\mathfrak{S}_{\mathrm{degs}(\underline{\lambda})}[\Gamma]}^{\mathfrak{S}_n[\Gamma]} \left(\chi^{\lambda^{(1)},\rho_1} \otimes \cdots \otimes \chi^{\lambda^{(d)},\rho_d} \right)$$

as $\underline{\lambda}$ runs through all functions

$$\begin{array}{ccc} \mathrm{Irr}(\Gamma) & \xrightarrow{\underline{\lambda}} & \mathrm{Par}, \\ \rho_i & \longmapsto & \lambda^{(i)} \end{array}$$

with the property that $\sum_{i=1}^d |\lambda^{(i)}| = n$. Here, $\mathrm{degs}(\underline{\lambda})$ denotes the d-tuple $\left(\left|\lambda^{(1)}\right|, \left|\lambda^{(2)}\right|, \ldots, \left|\lambda^{(d)}\right|\right) \in \mathbb{N}^d$, and $\mathfrak{S}_{\mathrm{degs}(\underline{\lambda})}$ is defined as the subgroup $\mathfrak{S}_{\left|\lambda^{(1)}\right|} \times \mathfrak{S}_{\left|\lambda^{(2)}\right|} \times \cdots \times \mathfrak{S}_{\left|\lambda^{(d)}\right|}$ of $\mathfrak{S}_n$.

[238]This is well-defined, since $(p_\lambda)_{\lambda \in \mathrm{Par}}$ is a $\mathbb{Q}$-module basis of $\Lambda_{\mathbb{Q}}$.

Furthermore, one has a PSH-isomorphism

$$\begin{array}{rcl} A(\mathfrak{S}[\Gamma]) & \longrightarrow & \Lambda^{\otimes d}, \\ \chi^{\underline{\lambda}} & \longmapsto & s_{\lambda^{(1)}} \otimes \cdots \otimes s_{\lambda^{(d)}}. \end{array}$$

Proof. We know from Corollary 4.3.10 that $A(\mathfrak{S}[\Gamma])$ is a PSH, with PSH-basis Σ given by the union of all irreducible characters of all groups $\mathfrak{S}_n[\Gamma]$. Therefore Theorem 3.2.3 tells us that $A(\mathfrak{S}[\Gamma]) \cong \bigotimes_{\rho \in \mathcal{C}} A(\mathfrak{S}[\Gamma])(\rho)$ where $\mathcal{C}$ is the set of irreducible characters which are also primitive. Just as in the case of $\mathfrak{S}_n$, it is clear from the definition of the coproduct that an irreducible character ρ of $\mathfrak{S}_n[\Gamma]$ is primitive if and only if $n = 1$, that in this case $\mathfrak{S}_n[\Gamma] = \Gamma$, and ρ lies in $\operatorname{Irr}(\Gamma) = \{\rho_1, \ldots, \rho_d\}$.

The remaining assertions of the theorem will then follow from the definition of the induction product algebra structure on $A(\mathfrak{S}[\Gamma])$, once we have shown that, for every $\rho \in \operatorname{Irr}(\Gamma)$, there is a PSH-isomorphism sending

$$\begin{array}{rcl} A(\mathfrak{S}) & \longrightarrow & A(\mathfrak{S}[\Gamma])(\rho), \\ \chi^{\lambda} & \longmapsto & \chi^{\lambda,\rho}. \end{array} \tag{4.5.2}$$

Such an isomorphism comes from applying Proposition 4.1.17 to the semidirect product $\mathfrak{S}_n[\Gamma] = \mathfrak{S}_n \ltimes \Gamma^n$, so that $K = \Gamma^n, G = \mathfrak{S}_n$, and fixing $V = \rho^{\otimes n}$ as $\mathbb{C}\mathfrak{S}_n[\Gamma]$-module with structure as defined in (4.5.1) (but with λ set to (n), so that χ^{λ} is the trivial 1-dimensional $\mathbb{C}\mathfrak{S}_n$-module). One obtains for each n, maps

$$R(\mathfrak{S}_n) \underset{\Psi}{\overset{\Phi}{\rightleftharpoons}} R(\mathfrak{S}_n[\Gamma])$$

where

$$\begin{array}{rcl} \chi & \overset{\Phi}{\longmapsto} & \chi \otimes (\rho^{\otimes n}), \\ \alpha & \overset{\Psi}{\longmapsto} & \operatorname{Hom}_{\mathbb{C}\Gamma^n}(\rho^{\otimes n}, \alpha). \end{array}$$

Taking the direct sum of these maps for all n gives maps $A(\mathfrak{S}) \underset{\Psi}{\overset{\Phi}{\rightleftharpoons}} A(\mathfrak{S}[\Gamma])$.

These maps are coalgebra morphisms because of their interaction with restriction to $\mathfrak{S}_i \times \mathfrak{S}_j$. Since Proposition 4.1.17(iii) gives the adjointness property that

$$(\chi, \Psi(\alpha))_{A(\mathfrak{S})} = (\Phi(\chi), \alpha)_{A(\mathfrak{S}[\Gamma])},$$

one concludes from the self-duality of $A(\mathfrak{S}), A(\mathfrak{S}[\Gamma])$ that Φ, Ψ are also algebra morphisms. Since they take genuine characters to genuine characters, they are PSH-morphisms. Since ρ being a simple $\mathbb{C}\Gamma$-module implies that $V = \rho^{\otimes n}$ is a simple $\mathbb{C}\Gamma^n$-module, Proposition 4.1.17(iv) shows that

$$(\Psi \circ \Phi)(\chi) = \chi \tag{4.5.3}$$

for all $\mathfrak{S}_n$-characters χ. Hence Φ is an injective PSH-morphism. Using adjointness, (4.5.3) also shows that Φ sends $\mathbb{C}\mathfrak{S}_n$-simples χ to $\mathbb{C}[\mathfrak{S}_n[\Gamma]]$-simples $\Phi(\chi)$:

$$(\Phi(\chi), \Phi(\chi))_{A(\mathfrak{S}[\Gamma])} = ((\Psi \circ \Phi)(\chi), \chi)_{A(\mathfrak{S})} = (\chi, \chi)_{A(\mathfrak{S})} = 1.$$

Since $\Phi(\chi) = \chi \otimes (\rho^{\otimes n})$ has $V = \rho^{\otimes n}$ as a constituent upon restriction to Γ^n, Frobenius Reciprocity shows that the irreducible character $\Phi(\chi)$ is a constituent of $\operatorname{Ind}_{\Gamma^n}^{\mathfrak{S}_n[\Gamma]} \rho^{\otimes n} = \rho^n$. Hence the entire image of Φ lies in $A(\mathfrak{S}[\Gamma])(\rho)$ (due to how we defined $A(\rho)$ in the proof of Theorem 3.2.3), and so Φ must restrict to an isomorphism as desired in (4.5.2). $\square$

One of Zelevinsky's sample applications of the theorem is this branching rule.

Corollary 4.5.3. *Given* $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(d)})$ *with* $\sum_{i=1}^{d} |\lambda^{(i)}| = n$, *one has*

$$\operatorname{Res}^{\mathfrak{S}_n[\Gamma]}_{\mathfrak{S}_{n-1}[\Gamma]\times\Gamma}\left(\chi^{\underline{\lambda}}\right) = \sum_{i=1}^{d} \sum_{\substack{\lambda^{(i)}_{-}\subseteq\lambda^{(i)}:\\ |\lambda^{(i)}/\lambda^{(i)}_{-}|=1}} \chi^{(\lambda^{(1)},\ldots,\lambda^{(i)}_{-},\ldots,\lambda^{(d)})} \otimes \rho_i.$$

(We are identifying functions $\underline{\lambda} : \operatorname{Irr}(\Gamma) \to \operatorname{Par}$ *with the corresponding* d*-tuples* $\left(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(d)}\right)$ *here.)*

Example 4.5.4. For Γ a two-element group, so $\operatorname{Irr}(\Gamma) = \{\rho_1, \rho_2\}$ and $d = 2$, then

$$\operatorname{Res}^{\mathfrak{S}_6[\Gamma]}_{\mathfrak{S}_5[\Gamma]\times\Gamma}\left(\chi^{((3,1),(1,1))}\right) = \chi^{((3),(1,1))} \otimes \rho_1 + \chi^{((2,1),(1,1))} \otimes \rho_1 + \chi^{((3,1),(1))} \otimes \rho_2.$$

Proof of Corollary 4.5.3. By Theorem 4.5.2, this is equivalent to computing in the Hopf algebra $A := \Lambda^{\otimes d}$ the component of the coproduct of $s_{\lambda^{(1)}} \otimes \cdots \otimes s_{\lambda^{(d)}}$ that lies in $A_{n-1} \otimes A_1$. Working within each tensor factor Λ, we conclude from Proposition 2.3.6(iv) that the $\Lambda_{|\lambda|-1} \otimes \Lambda_1$-component of $\Delta(s_\lambda)$ is

$$\sum_{\substack{\lambda_{-}\subseteq\lambda:\\ |\lambda/\lambda_{-}|=1}} s_{\lambda_{-}} \otimes \rho.$$

One must apply this in each of the d tensor factors of $A = \Lambda^{\otimes d}$, then sum on i. $\square$

4.6. **General linear groups.** We now consider the tower of finite general linear groups $G_n = GL_n = GL_n(\mathbb{F}_q)$ and $A = A(G_*) =: A(GL)$. Corollary 4.3.10 tells us that $A(GL)$ is a PSH, with PSH-basis Σ given by the union of all irreducible characters of all groups GL_n. Therefore Theorem 3.2.3 tells us that

$$A(GL) \cong \bigotimes_{\rho\in\mathcal{C}} A(GL)(\rho) \tag{4.6.1}$$

where $\mathcal{C} = \Sigma \cap \mathfrak{p}$ is the set of primitive irreducible characters.

Definition 4.6.1. Call an irreducible representation ρ of GL_n *cuspidal* for $n \geq 1$ if it lies in $\mathcal{C}$, that is, its restriction to proper parabolic subgroups $P_{i,j}$ with $i + j = n$ and $i, j > 0$ contain no nonzero vectors which are $K_{i,j}$-invariant. Given an irreducible character σ of GL_n, say that $d(\sigma) = n$, and let $\mathcal{C}_n := \{\rho \in \mathcal{C} : d(\rho) = n\}$ for $n \geq 1$ denote the subset of cuspidal characters of GL_n.

Just as was the case for $\mathfrak{S}_1$ and $\mathfrak{S}_1[\Gamma] = \Gamma$, *every* irreducible character ρ of $GL_1(\mathbb{F}_q) = \mathbb{F}_q^\times$ is cuspidal. However, this does not exhaust the cuspidal characters. In fact, one can predict the number of cuspidal characters in $\mathcal{C}_n$, using knowledge of the number of conjugacy classes in GL_n. Let $\mathcal{F}$ denote the set of all nonconstant monic irreducible polynomials $f(x) \neq x$ in $\mathbb{F}_q[x]$. Let $\mathcal{F}_n := \{f \in \mathcal{F} : \deg(f) = n\}$ for $n \geq 1$.

Proposition 4.6.2. *The number* $|\mathcal{C}_n|$ *of cuspidal characters of* $GL_n(\mathbb{F}_q)$ *is the number of* $|\mathcal{F}_n|$ *of irreducible monic degree* n *polynomials* $f(x) \neq x$ *in* $\mathbb{F}_q[x]$ *with nonzero constant term.*

Proof. We show $|\mathcal{C}_n| = |\mathcal{F}_n|$ for $n \geq 1$ by strong induction on n. For the base case[239] $n = 1$, just as with the families $G_n = \mathfrak{S}_n$ and $G_n = \mathfrak{S}_n[\Gamma]$, when $n = 1$ any irreducible character χ of $G_1 = GL_1(\mathbb{F}_q)$ gives a primitive element of $A = A(GL)$, and hence is cuspidal. Since $GL_1(\mathbb{F}_q) = \mathbb{F}_q^\times$ is abelian, there are $|\mathbb{F}_q^\times| = q-1$ such cuspidal characters in $\mathcal{C}_1$, which agrees with the fact that there are $q-1$ monic (irreducible) linear polynomials $f(x) \neq x$ in $\mathbb{F}_q[x]$, namely $\mathcal{F}_1 := \{f(x) = x - c : c \in \mathbb{F}_q^\times\}$.

In the inductive step, use the fact that the number $|\Sigma_n|$ of irreducible complex characters χ of $GL_n(\mathbb{F}_q)$ equals its number of conjugacy classes. These conjugacy classes are uniquely represented by *rational canonical forms*, which are parametrized by functions $\underline{\lambda} : \mathcal{F} \to \text{Par}$ with the property that $\sum_{f\in\mathcal{F}} \deg(f)|\underline{\lambda}(f)| = n$. On the other hand, (4.6.1) tells us that $|\Sigma_n|$ is similarly parametrized by the functions $\underline{\lambda} : \mathcal{C} \to \text{Par}$ having the property that $\sum_{\rho\in\mathcal{C}} d(\rho)|\underline{\lambda}(\rho)| = n$. Thus we have parallel disjoint decompositions

$$\begin{array}{lll} \mathcal{C} & = \bigsqcup_{n\geq 1} \mathcal{C}_n & \text{where } \mathcal{C}_n = \{\rho \in \mathcal{C} : d(\rho) = n\}\,, \\ \mathcal{F} & = \bigsqcup_{n\geq 1} \mathcal{F}_n & \text{where } \mathcal{F}_n = \{f \in \mathcal{F} : \deg(f) = n\}\,, \end{array}$$

and hence an equality for all $n \geq 1$

$$\begin{aligned} \left| \left\{ \mathcal{C} \xrightarrow{\underline{\lambda}} \text{Par} : \quad \sum_{\rho\in\mathcal{C}} d(\rho)|\underline{\lambda}(\rho)| = n \right\} \right| &= |\Sigma_n| \\ &= \left| \left\{ \mathcal{F} \xrightarrow{\underline{\lambda}} \text{Par} : \quad \sum_{f\in\mathcal{F}} \deg(f)|\underline{\lambda}(f)| = n \right\} \right|. \end{aligned}$$

Since there is only one partition λ having $|\lambda| = 1$ (namely, $\lambda = (1)$), this leads to parallel recursions

$$|\mathcal{C}_n| = |\Sigma_n| - \left| \left\{ \bigsqcup_{i=1}^{n-1} \mathcal{C}_i \xrightarrow{\underline{\lambda}} \text{Par} : \quad \sum_{\rho\in\mathcal{C}} d(\rho)|\underline{\lambda}(\rho)| = n \right\} \right|,$$

$$|\mathcal{F}_n| = |\Sigma_n| - \left| \left\{ \bigsqcup_{i=1}^{n-1} \mathcal{F}_i \xrightarrow{\underline{\lambda}} \text{Par} : \quad \sum_{f\in\mathcal{F}} \deg(f)|\underline{\lambda}(f)| = n \right\} \right|,$$

and induction implies that $|\mathcal{C}_n| = |\mathcal{F}_n|$. $\square$

We shall use the notation $\underline{1}_H$ for the trivial character of a group H whenever H is a finite group. This generalizes the notations $\underline{1}_{\mathfrak{S}_n}$ and $\underline{1}_{\mathfrak{S}_\lambda}$ introduced above.

Example 4.6.3. Taking $q = 2$, let us list the sets $\mathcal{F}_n$ of monic irreducible polynomials $f(x) \neq x$ in $\mathbb{F}_2[x]$ of degree n for $n \leq 3$, so that we know how many cuspidal characters of $GL_n(\mathbb{F}_q)$ in $\mathcal{C}_n$ to expect:

$$\begin{aligned} \mathcal{F}_1 &= \{x+1\}; \\ \mathcal{F}_2 &= \{x^2+x+1\}; \\ \mathcal{F}_3 &= \{x^3+x+1, x^3+x^2+1\}. \end{aligned}$$

Thus we expect

- one cuspidal character of $GL_1(\mathbb{F}_2)$, namely $\rho_1 (= \underline{1}_{GL_1(\mathbb{F}_2)})$,

[239] Actually, we don't need any base case for our strong induction. We nevertheless handle the case $n = 1$ as a warmup.

- one cuspidal character ρ_2 of $GL_2(\mathbb{F}_2)$, and
- two cuspidal characters ρ_3, ρ_3' of $GL_3(\mathbb{F}_2)$.

We will say more about ρ_2, ρ_3, ρ_3' in the next section.

Exercise 4.6.4. Let $\mu : \{1, 2, 3, \ldots\} \to \mathbb{Z}$ denote the *number-theoretic Möbius function*, defined by setting $\mu(m) = (-1)^d$ if $m = p_1 \cdots p_d$ for d distinct primes $p_1, p_2, \ldots, p_d$, and $\mu(m) = 0$ if m is not squarefree.

(a) Show that for $n \geq 2$, we have

$$|\mathcal{C}_n| (= |\mathcal{F}_n|) = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) q^d. \tag{4.6.2}$$

(Here, the summation sign $\sum_{d|n}$ means a sum over all positive divisors d of n.)

(b) Show that (4.6.2) also counts the *necklaces* with n beads of q colors (= the equivalence classes under the $\mathbb{Z}/n\mathbb{Z}$-action of cyclic rotation on sequences $(a_1, \ldots, a_n)$ in $\mathbb{F}_q^n$) which are *primitive* in the sense that no nontrivial rotation fixes any of the sequences within the equivalence class. For example, when $q = 2$, here are systems of distinct representatives of these primitive necklaces for $n = 2, 3, 4$:

$$\begin{aligned} n = 2: &\quad \{(0,1)\}; \\ n = 3: &\quad \{(0,0,1), (0,1,1)\}; \\ n = 4: &\quad \{(0,0,0,1), (0,0,1,1), (0,1,1,1)\}. \end{aligned}$$

The result of Exercise 4.6.4(a) was stated by Gauss for prime q, and by Witt for general q; it is discussed in [37], [182, Section 7.6.2] and (for prime q) [84, (4.12.3)]. Exercise 4.6.4(b) is also well-known. See [182, Section 7.6.2] for a bijection explaining why the answers to both parts of Exercise 4.6.4 are the same.

4.7. **Steinberg's unipotent characters.** Not surprisingly, the (cuspidal) character $\iota := \underline{1}_{GL_1}$ of $GL_1(\mathbb{F}_q)$ plays a distinguished role. The parabolic subgroup $P_{(1^n)}$ of $GL_n(\mathbb{F}_q)$ is the Borel subgroup B of upper triangular matrices, and we have $\iota^n = \operatorname{Ind}_B^{GL_n} \underline{1}_B = \mathbb{C}[GL_n/B]$ (identifying representations with their characters as usual)[240]. The subalgebra $A(GL)(\iota)$ of $A(GL)$ is the $\mathbb{Z}$-span of the irreducible characters σ that appear as constituents of $\iota^n = \operatorname{Ind}_B^{GL_n} \underline{1}_B = \mathbb{C}[GL_n/B]$ for some n.

Definition 4.7.1. An irreducible character σ of GL_n appearing as a constituent of $\operatorname{Ind}_B^{GL_n} \underline{1}_B = \mathbb{C}[GL_n/B]$ is called a *unipotent character*. Equivalently, by Frobenius reciprocity, σ is unipotent if it contains a nonzero B-invariant vector.

---

[240] *Proof.* Exercise 4.3.11(d) (applied to $G_* = GL_*$, $\ell = n$, $\alpha = (1^n) = \left(\underbrace{1, 1, \ldots, 1}_{n \text{ times}}\right)$ and $\chi_i = \iota$) gives

$$\iota^n = \operatorname{ind}_{(1^n)}^n \iota^{\otimes n} = \underbrace{\operatorname{Ind}_{P_{(1^n)}}^{G_n}}_{=\operatorname{Ind}_B^{GL_n}} \underbrace{\operatorname{Infl}_{G_{(1^n)}}^{P_{(1^n)}} \iota^{\otimes n}}_{=\underline{1}_{P_{(1^n)}} = \underline{1}_B} = \operatorname{Ind}_B^{GL_n} \underline{1}_B = \mathbb{C}\left[GL_n/B\right],$$

where the last equality follows from the general fact that if G is a finite group and H is a subgroup of G, then $\operatorname{Ind}_H^G \underline{1}_H \cong \mathbb{C}\left[G/H\right]$ as $\mathbb{C}G$-modules.

In particular, $\underline{1}_{GL_n}$ is a unipotent character of GL_n for each n.

Proposition 4.7.2. *One can choose $\Lambda \cong A(GL)(\iota)$ in Theorem 3.3.3(g) so that $h_n \longmapsto \underline{1}_{GL_n}$.*

Proof. Theorem 3.3.1(a) tells us $\iota^2 = \mathrm{Ind}_B^{GL_2} \underline{1}_B$ must have exactly two irreducible constituents, one of which is $\underline{1}_{GL_2}$; call the other one St_2. Choose the isomorphism so as to send $h_2 \longmapsto \underline{1}_{GL_2}$. Then $h_n \mapsto \underline{1}_{GL_n}$ follows from the claim that $\mathrm{St}_2^\perp(\underline{1}_{GL_n}) = 0$ for $n \geq 2$: one has

$$\Delta(\underline{1}_{GL_n}) = \sum_{i+j=n} \left(\mathrm{Res}^{G_n}_{P_{i,j}} \underline{1}_{GL_n} \right)^{K_{i,j}} = \sum_{i+j=n} \underline{1}_{GL_i} \otimes \underline{1}_{GL_j}$$

so that $\mathrm{St}_2^\perp(\underline{1}_{GL_n}) = (\mathrm{St}_2, \underline{1}_{GL_2})\underline{1}_{GL_{n-2}} = 0$ since $\mathrm{St}_2 \neq \underline{1}_{GL_2}$.

$\square$

This subalgebra $A(GL)(\iota)$, and the unipotent characters χ_q^λ corresponding under this isomorphism to the Schur functions s_λ, were introduced by Steinberg [208]. He wrote down χ_q^λ as a virtual sum of induced characters $\mathrm{Ind}_{P_\alpha}^{GL_n} \underline{1}_{P_\alpha} (= \underline{1}_{G_{\alpha_1}} \cdots \underline{1}_{G_{\alpha_\ell}})$, modelled on the Jacobi-Trudi determinantal expression for $s_\lambda = \det(h_{\lambda_i - i + j})$. Note that $\mathrm{Ind}_{P_\alpha}^{GL_n} \underline{1}_{P_\alpha}$ is the transitive permutation representation $\mathbb{C}[G/P_\alpha]$ for GL_n permuting the *finite partial flag variety* G/P_α, that is, the set of *α-flags of subspaces*

$$\{0\} \subset V_{\alpha_1} \subset V_{\alpha_1+\alpha_2} \subset \cdots \subset V_{\alpha_1+\alpha_2+\cdots+\alpha_{\ell-1}} \subset \mathbb{F}_q^n$$

where $\dim_{\mathbb{F}_q} V_d = d$ in each case. This character has dimension equal to $|G/P_\alpha|$, with formula given by the *q-multinomial coefficient* (see e.g. Stanley [206, §1.7]):

$$\begin{bmatrix} n \\ \alpha \end{bmatrix}_q = \frac{[n]!_q}{[\alpha_1]!_q \cdots [\alpha_\ell]!_q}$$

where $[n]!_q := [n]_q[n-1]_q \cdots [2]_q[1]_q$ and $[n]_q := 1 + q + \cdots + q^{n-1} = \frac{q^n-1}{q-1}$.

Our terminology St_2 is motivated by the $n = 2$ special case of the *Steinberg character* St_n, which is the unipotent character corresponding under the isomorphism in Proposition 4.7.2 to $e_n = s_{(1^n)}$. It can be defined by the virtual sum

$$\mathrm{St}_n := \chi_q^{(1^n)} = \sum_\alpha (-1)^{n-\ell(\alpha)} \mathrm{Ind}_{P_\alpha}^{GL_n} \underline{1}_{P_\alpha}$$

in which the sum runs through all compositions α of n. This turns out to be the genuine character for $GL_n(\mathbb{F}_q)$ acting on the top homology group of its *Tits building*: the simplicial complex whose vertices are nonzero proper subspaces V of $\mathbb{F}_q^n$, and whose simplices correspond to flags of nested subspaces. One needs to know that this Tits building has only top homology, so that one can deduce the above character formula from the Hopf trace formula; see Björner [22].

4.8. **Examples: $GL_2(\mathbb{F}_2)$ and $GL_3(\mathbb{F}_2)$.** Let's get our hands dirty.

Example 4.8.1. For $n = 2$, there are two unipotent characters, $\chi_q^{(2)} = \underline{1}_{GL_2}$ and

$$\mathrm{St}_2 := \chi_q^{(1,1)} = \underline{1}_{GL_1}^2 - \underline{1}_{GL_2} = \mathrm{Ind}_B^{GL_2} \underline{1}_B - \underline{1}_{GL_2} \tag{4.8.1}$$

since the Jacobi-Trudi formula (2.4.16) gives $s_{(1,1)} = \det \begin{bmatrix} h_1 & h_2 \\ 1 & h_1 \end{bmatrix} = h_1^2 - h_2$. The description (4.8.1) for this Steinberg character St_2 shows that it has dimension

$$|GL_2/B| - 1 = (q+1) - 1 = q$$

and that one can think of it as follows: consider the permutation action of GL_2 on the $q+1$ lines $\{\ell_0, \ell_1, \ldots, \ell_q\}$ in the projective space $\mathbb{P}^1_{\mathbb{F}_q} = GL_2(\mathbb{F}_q)/B$, and take the invariant subspace perpendicular to the sum of basis elements $e_{\ell_0} + \cdots + e_{\ell_q}$.

Example 4.8.2. Continuing the previous example, but taking $q = 2$, we find that we have constructed two unipotent characters: $\underline{1}_{GL_2} = \chi^{(2)}_{q=2}$ of dimension 1, and $\mathrm{St}_2 = \chi^{(1,1)}_{q=2}$ of dimension $q = 2$. This lets us identify the unique cuspidal character ρ_2 of $GL_2(\mathbb{F}_2)$, using knowledge of the character table of $GL_2(\mathbb{F}_2) \cong \mathfrak{S}_3$:

		$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$
$\underline{1}_{GL_2} = \chi^{(2)}_{q=2}$	unipotent	1	1	1
$\mathrm{St}_2 = \chi^{(1,1)}_{q=2}$	unipotent	2	0	-1
ρ_2	cuspidal	1	-1	1

In other words, the cuspidal character ρ_2 of $GL_2(\mathbb{F}_2)$ corresponds under the isomorphism $GL_2(\mathbb{F}_2) \cong \mathfrak{S}_3$ to the sign character $\mathrm{sgn}_{\mathfrak{S}_3}$.

Example 4.8.3. Continuing the previous example to $q = 2$ and $n = 3$ lets us analyze the irreducible characters of $GL_3(\mathbb{F}_2)$. Recalling our labelling $\rho_1, \rho_2, \rho_3, \rho_3'$ from Example 4.6.3 of the cuspidal characters of $GL_n(\mathbb{F}_2)$ for $n = 1, 2, 3$, Zelevinsky's Theorem 3.2.3 tells us that the $GL_3(\mathbb{F}_2)$-irreducible characters should be labelled by functions $\{\rho_1, \rho_2, \rho_3, \rho_3'\} \xrightarrow{\underline{\lambda}} \mathrm{Par}$ for which

$$1 \cdot |\underline{\lambda}(\rho_1)| + 2 \cdot |\underline{\lambda}(\rho_2)| + 3 \cdot |\underline{\lambda}(\rho_3)| + 3 \cdot |\underline{\lambda}(\rho_3')| = 3.$$

We will label such an irreducible character $\chi^{\underline{\lambda}} = \chi^{(\underline{\lambda}(\rho_1), \underline{\lambda}(\rho_2), \underline{\lambda}(\rho_3), \underline{\lambda}(\rho_3'))}$.

Three of these irreducibles will be the unipotent characters, mapping under the isomorphism from Proposition 4.7.2 as follows:

- $s_{(3)} = h_3 \longmapsto \chi^{((3), \varnothing, \varnothing, \varnothing)} = \underline{1}_{GL_3}$ of dimension 1.
-

$$s_{(2,1)} = \det \begin{bmatrix} h_2 & h_3 \\ 1 & h_1 \end{bmatrix} = h_2 h_1 - h_3 \longmapsto \chi^{((2,1), \varnothing, \varnothing, \varnothing)} = \mathrm{Ind}_{P_{2,1}}^{GL_3} \underline{1}_{P_{2,1}} - \underline{1}_{GL_3},$$

$$\text{of dimension } \begin{bmatrix} 3 \\ 2,1 \end{bmatrix}_q - \begin{bmatrix} 3 \\ 3 \end{bmatrix}_q = [3]_q - 1 = q^2 + q \overset{q=2}{\rightsquigarrow} 6.$$

- Lastly,

$$\begin{aligned}s_{(1,1,1)} &= \det \begin{bmatrix} h_1 & h_2 & h_3 \\ 1 & h_1 & h_2 \\ 0 & 1 & h_1 \end{bmatrix} = h_1^3 - h_2h_1 - h_1h_2 + h_3 \\ &\longmapsto \mathrm{St}_3 = \chi^{((1,1,1),\varnothing,\varnothing,\varnothing)} \\ &= \mathrm{Ind}_B^{GL_3} \underline{1}_B - \mathrm{Ind}_{P_{2,1}}^{GL_3} \underline{1}_{P_{2,1}} - \mathrm{Ind}_{P_{1,2}}^{GL_3} \underline{1}_{P_{1,2}} + \underline{1}_{GL_3}\end{aligned}$$

of dimension

$$\begin{aligned}&\begin{bmatrix} 3 \\ 1,1,1 \end{bmatrix}_q - \begin{bmatrix} 3 \\ 2,1 \end{bmatrix}_q - \begin{bmatrix} 3 \\ 1,2 \end{bmatrix}_q + \begin{bmatrix} 3 \\ 3 \end{bmatrix}_q \\ &= [3]!_q - [3]_q - [3]_q + 1 = q^3 \overset{q=2}{\rightsquigarrow} 8.\end{aligned}$$

There should also be one non-unipotent, non-cuspidal character, namely

$$\chi^{((1),(1),\varnothing,\varnothing)} = \rho_1\rho_2 = \mathrm{Ind}_{P_{1,2}}^{GL_3} \mathrm{Infl}_{GL_1\times GL_2}^{P_{1,2}} \left(\underline{1}_{GL_1} \otimes \rho_2\right)$$

having dimension $\begin{bmatrix} 3 \\ 1,2 \end{bmatrix}_q \cdot 1 \cdot 1 = [3]_q \overset{q=2}{\rightsquigarrow} 7$.

Finally, we expect cuspidal characters $\rho_3 = \chi^{(\varnothing,\varnothing,(1),\varnothing)}, \rho_3' = \chi^{(\varnothing,\varnothing,\varnothing,(1))}$, whose dimensions d_3, d_3' can be deduced from the equation

$$\begin{aligned}1^2 + 6^2 + 8^2 + 7^2 + d_3^2 + (d_3')^2 = |GL_3(\mathbb{F}_2)| &= \left[(q^3 - q^0)(q^3 - q^1)(q^3 - q^2)\right]_{q=2} \\ &= 168.\end{aligned}$$

This forces $d_3^2 + (d_3')^2 = 18$, whose only solution in positive integers is $d_3 = d_3' = 3$.

We can check our predictions of the dimensions for the various $GL_3(\mathbb{F}_2)$-irreducible characters since $GL_3(\mathbb{F}_2)$ is the finite simple group of order 168 (also isomorphic to $PSL_2(\mathbb{F}_7)$), with known character table (see James and Liebeck [104, p. 318]):

	centralizer order	168	8	4	3	7	7
	unipotent?/cuspidal?						
$\underline{1}_{GL_3} = \chi^{((3),\varnothing,\varnothing,\varnothing)}$	unipotent	1	1	1	1	1	1
$\chi^{((2,1),\varnothing,\varnothing,\varnothing)}$	unipotent	6	2	0	0	-1	-1
$\mathrm{St}_3 = \chi^{((1,1,1),\varnothing,\varnothing,\varnothing)}$	unipotent	8	0	0	-1	1	1
$\chi^{((1),(1),\varnothing,\varnothing)}$		7	-1	-1	1	0	0
$\rho_3 = \chi^{(\varnothing,\varnothing,(1),\varnothing)}$	cuspidal	3	-1	1	0	α	$\overline{\alpha}$
$\rho_3' = \chi^{(\varnothing,\varnothing,\varnothing,(1))}$	cuspidal	3	-1	1	0	$\overline{\alpha}$	α

Here $\alpha := -1/2 + i\sqrt{7}/2$.

Remark 4.8.4. It is known (see e.g. Bump [30, Cor. 7.4]) that, for $n \geq 2$, the dimension of any cuspidal irreducible character ρ of $GL_n(\mathbb{F}_q)$ is

$$(q^{n-1} - 1)(q^{n-2} - 1) \cdots (q^2 - 1)(q - 1).$$

Note that when $q = 2$,

- for $n = 2$ this gives $2^1 - 1 = 1$ for the dimension of ρ_2, and
- for $n = 3$ it gives $(2^2 - 1)(2 - 1) = 3$ for the dimensions of ρ_3, ρ_3',

agreeing with our calculations above. Much more is known about the character table of $GL_n(\mathbb{F}_q)$; see Remark 4.9.14 below, Zelevinsky [227, Chap. 11], and Macdonald [142, Chap. IV].

4.9. **The Hall algebra.** There is another interesting Hopf subalgebra (and quotient Hopf algebra) of $A(GL)$, related to unipotent conjugacy classes in $GL_n(\mathbb{F}_q)$.

Definition 4.9.1. Say that an element g in $GL_n(\mathbb{F}_q)$ is *unipotent* if its eigenvalues are all equal to 1. Equivalently, $g \in GL_n(\mathbb{F}_q)$ is unipotent if and only if $g - \mathrm{id}_{\mathbb{F}_q^n}$ is nilpotent. A conjugacy class in $GL_n(\mathbb{F}_q)$ is *unipotent* if its elements are unipotent.

Denote by $\mathcal{H}_n$ the $\mathbb{C}$-subspace of $R_{\mathbb{C}}(GL_n)$ consisting of those class functions which are supported only on unipotent conjugacy classes, and let $\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$ as a $\mathbb{C}$-subspace of $A_{\mathbb{C}}(GL) = \bigoplus_{n\geq 0} R_{\mathbb{C}}(GL_n)$.

Proposition 4.9.2. *The subspace $\mathcal{H}$ is a Hopf subalgebra of $A_{\mathbb{C}}(GL)$, which is graded, connected, and of finite type, and self-dual with respect to the inner product on class functions inherited from $A_{\mathbb{C}}(GL)$. It is also a quotient Hopf algebra of $A_{\mathbb{C}}(GL)$, as the $\mathbb{C}$-linear surjection $A_{\mathbb{C}}(GL) \twoheadrightarrow \mathcal{H}$ restricting class functions to unipotent classes is a Hopf algebra homomorphism. This surjection has kernel $\mathcal{H}^\perp$, which is both an ideal and a two-sided coideal.*

Proof. It is immediately clear that $\mathcal{H}^\perp$ is a graded $\mathbb{C}$-vector subspace of $A_{\mathbb{C}}(GL)$, whose n-th homogeneous component consists of those class functions on GL_n whose values on all unipotent classes are 0. (This holds no matter whether the perpendicular space is taken with respect to the Hermitian form $(\cdot,\cdot)_G$ or with respect to the bilinear form $\langle\cdot,\cdot\rangle_G$.) In other words, $\mathcal{H}^\perp$ is the kernel of the surjection $A_{\mathbb{C}}(GL) \twoheadrightarrow \mathcal{H}$ defined in the proposition.

Given two class functions χ_i, χ_j on GL_i, GL_j and g in GL_{i+j}, one has

$$(\chi_i \cdot \chi_j)(g) = \frac{1}{|P_{i,j}|} \sum_{\substack{h\in GL_{i+j}: \\ h^{-1}gh = \begin{bmatrix} g_i & * \\ 0 & g_j \end{bmatrix} \in P_{i,j}}} \chi_i(g_i)\chi_j(g_j). \tag{4.9.1}$$

Since g is unipotent if and only if $h^{-1}gh$ is unipotent if and only if both g_i, g_j are unipotent, the formula (4.9.1) shows both that $\mathcal{H}$ is a subalgebra[241] and that $\mathcal{H}^\perp$ is a two-sided ideal[242]. It also shows that the surjection $A_{\mathbb{C}}(GL) \twoheadrightarrow \mathcal{H}$ restricting every class function to unipotent classes is an algebra homomorphism[243].

Similarly, for class functions χ on GL_n and (g_i, g_j) in $GL_{i,j} = GL_i \times GL_j$, one has

$$\Delta(\chi)(g_i, g_j) = \frac{1}{q^{ij}} \sum_{k\in\mathbb{F}_q^{i\times j}} \chi \begin{bmatrix} g_i & k \\ 0 & g_j \end{bmatrix}$$

using (4.1.13). This shows both that $\mathcal{H}$ is a sub-coalgebra of $A = A_{\mathbb{C}}(GL)$ (that is, it satisfies $\Delta\mathcal{H} \subset \mathcal{H} \otimes \mathcal{H}$) and that $\mathcal{H}^\perp$ is a two-sided coideal (that is, we have $\Delta(\mathcal{H}^\perp) \subset \mathcal{H}^\perp \otimes A + A \otimes \mathcal{H}^\perp$), since it shows that if

[241]Indeed, if χ_i and χ_j are both supported only on unipotent classes, then the same holds for $\chi_i \cdot \chi_j$.

[242]In fact, if one of χ_i and χ_j annihilates all unipotent classes, then so does $\chi_i \cdot \chi_j$.

[243]because if g is unipotent, then the only values of χ_i and χ_j appearing on the right hand side of (4.9.1) are those on unipotent elements

χ is supported only on unipotent classes, then $\Delta(\chi)$ vanishes on (g_1, g_2) that have either g_1 or g_2 non-unipotent. It also shows that the surjection $A_{\mathbb{C}}(GL) \twoheadrightarrow \mathcal{H}$ restricting every class function to unipotent classes is a coalgebra homomorphism. The rest follows. $\square$

The subspace $\mathcal{H}$ is called the *Hall algebra*. It has an obvious orthogonal $\mathbb{C}$-basis, with interesting structure constants.

Definition 4.9.3. Given a partition λ of n, let J_λ denote the GL_n-conjugacy class of unipotent matrices whose *Jordan type* (that is, the list of the sizes of the Jordan blocks, in decreasing order) is given by λ. Furthermore, let $z_\lambda(q)$ denote the size of the centralizer of any element of this conjugacy class J_λ.

The indicator class functions[244] $\{\underline{1}_{J_\lambda}\}_{\lambda \in \mathrm{Par}}$ form a $\mathbb{C}$-basis for $\mathcal{H}$ whose multiplicative structure constants are called the *Hall coefficients* $g^\lambda_{\mu,\nu}(q)$:

$$\underline{1}_{J_\mu} \underline{1}_{J_\nu} = \sum_\lambda g^\lambda_{\mu,\nu}(q) \; \underline{1}_{J_\lambda}.$$

Because the dual basis to $\{\underline{1}_{J_\lambda}\}$ is $\{z_\lambda(q) \underline{1}_{J_\lambda}\}$, self-duality of $\mathcal{H}$ shows that the Hall coefficients are (essentially) also structure constants for the comultiplication:

$$\Delta \underline{1}_{J_\lambda} = \sum_{\mu,\nu} g^\lambda_{\mu,\nu}(q) \frac{z_\mu(q) z_\nu(q)}{z_\lambda(q)} \cdot \underline{1}_{J_\mu} \otimes \underline{1}_{J_\nu}.$$

The Hall coefficient $g^\lambda_{\mu,\nu}(q)$ has the following interpretation.

Proposition 4.9.4. *Fix any g in $GL_n(\mathbb{F}_q)$ acting unipotently on $\mathbb{F}_q^n$ with Jordan type λ. Then $g^\lambda_{\mu,\nu}(q)$ counts the g-stable $\mathbb{F}_q$-subspaces $V \subset \mathbb{F}_q^n$ for which the restriction $g|V$ acts with Jordan type μ, and the induced map $\bar{g}$ on the quotient space $\mathbb{F}_q^n / V$ has Jordan type ν.*

Proof. Given μ, ν partitions of i, j with $i + j = n$, taking χ_i, χ_j equal to $\underline{1}_{J_\mu}, \underline{1}_{J_\nu}$ in (4.9.1) shows that for any g in GL_n, the value of $\left(\underline{1}_{J_\mu} \cdot \underline{1}_{J_\nu}\right)(g)$ is given by

(4.9.2)

$$\frac{1}{|P_{i,j}|} \left| \left\{ h \in GL_n : h^{-1} g h = \begin{bmatrix} g_i & * \\ 0 & g_j \end{bmatrix} \text{ with } g_i \in J_\mu, g_j \in J_\nu \right\} \right|.$$

Let S denote the set appearing in (4.9.2), and let $\mathbb{F}_q^i$ denote the i-dimensional subspace of $\mathbb{F}_q^n$ spanned by the first i standard basis vectors. Note that the condition on an element h in S saying that $h^{-1}gh$ is in block upper-triangular form can be re-expressed by saying that the subspace $V := h(\mathbb{F}_q^i)$ is g-stable. One then sees that the map $h \overset{\varphi}{\longmapsto} V = h(\mathbb{F}_q^i)$ surjects S onto the set of i-dimensional g-stable subspaces V of $\mathbb{F}_q^n$ for which $g|V$ and $\bar{g}$ are unipotent of types μ, ν, respectively. Furthermore, for any particular such V, its fiber $\varphi^{-1}(V)$ in S is a coset of the stabilizer within GL_n of V, which is conjugate to $P_{i,j}$, and hence has cardinality $|\varphi^{-1}(V)| = |P_{i,j}|$. This proves the assertion of the proposition. $\square$

[244] Here we use the following notation: Whenever P is a subset of a group G, we denote by $\underline{1}_P$ the map $G \to \mathbb{C}$ which sends every element of P to 1 and all remaining elements of G to 0. This is not in conflict with the notation $\underline{1}_G$ for the trivial character of G, since $\underline{1}_P = \underline{1}_G$ for $P = G$. Note that $\underline{1}_P$ is a class function when P is a union of conjugacy classes of G.

The Hall algebra $\mathcal{H}$ will turn out to be isomorphic to the ring $\Lambda_{\mathbb{C}}$ of symmetric functions with $\mathbb{C}$ coefficients, via a composite φ of three maps

$$\Lambda_{\mathbb{C}} \longrightarrow A(GL)(\iota)_{\mathbb{C}} \longrightarrow A(GL)_{\mathbb{C}} \longrightarrow \mathcal{H}$$

in which the first map is the isomorphism from Proposition 4.7.2, the second is inclusion, and the third is the quotient map from Proposition 4.9.2.

Theorem 4.9.5. *The above composite φ is a Hopf algebra isomorphism, sending*

$$\begin{array}{rcll} h_n & \longmapsto & \sum_{\lambda \in \operatorname{Par}_n} \underline{1}_{J_\lambda}, & \\ e_n & \longmapsto & q^{\binom{n}{2}} \underline{1}_{J_{(1^n)}}, & \\ p_n & \longmapsto & \sum_{\lambda \in \operatorname{Par}_n} (q;q)_{\ell(\lambda)-1} \underline{1}_{J_\lambda} & \qquad (\text{for } n > 0), \end{array}$$

where we are using the notation

$$(x;q)_m := (1-x)(1-qx)(1-q^2x)\cdots(1-q^{m-1}x)$$
$$\text{for all } m \in \mathbb{N} \text{ and } x \text{ in any ring.}$$

Proof. That φ is a graded Hopf morphism follows because it is a composite of three such morphisms. We claim that once one shows the formula for the (nonzero) image $\varphi(p_n)$ given above is correct, then this will already show φ is an isomorphism, by the following argument. Note first that $\Lambda_{\mathbb{C}}$ and $\mathcal{H}$ both have dimension $|\operatorname{Par}_n|$ for their n-th homogeneous components, so it suffices to show that the graded map φ is injective. On the other hand, both $\Lambda_{\mathbb{C}}$ and $\mathcal{H}$ are (graded, connected, finite type) *self-dual* Hopf algebras (although with respect to a sesquilinear form), so Theorem 3.1.7 says that each is the symmetric algebra on its space of primitive elements. Thus it suffices to check that φ is injective when restricted to their subspaces of primitives.[245] For $\Lambda_{\mathbb{C}}$, by Corollary 3.1.8 the primitives are spanned by $\{p_1, p_2, \ldots\}$, with only one basis element in each degree $n \geq 1$. Hence φ is injective on the subspace of primitives if and only if it does not annihilate any p_n.

Thus it only remains to show the above formulas for the images of h_n, e_n, p_n under φ. This is clear for h_n, since Proposition 4.7.2 shows that it maps under the first two composites to the indicator function $\underline{1}_{GL_n}$ which then restricts to the sum of indicators $\sum_{\lambda \in \operatorname{Par}_n} \underline{1}_{J_\lambda}$ in $\mathcal{H}$. For e_n, p_n, we resort to generating functions. Let $\tilde{h}_n, \tilde{e}_n, \tilde{p}_n$ denote the three putative images in $\mathcal{H}$ of h_n, e_n, p_n, appearing on the right side in the theorem, and define generating functions

$$\tilde{H}(t) := \sum_{n \geq 0} \tilde{h}_n t^n, \quad \tilde{E}(t) := \sum_{n \geq 0} \tilde{e}_n t^n, \quad \tilde{P}(t) := \sum_{n \geq 0} \tilde{p}_{n+1} t^n \qquad \text{in } \mathcal{H}[[t]].$$

We wish to show that the map $\varphi[[t]] : \Lambda_{\mathbb{C}}[[t]] \to \mathcal{H}[[t]]$ (induced by φ) maps $H(t), E(t), P(t)$ in $\Lambda[[t]]$ to these three generating functions[246]. Since we have already shown this is correct for $H(t)$, by (2.4.3), (2.5.13), it suffices to check that in $\mathcal{H}[[t]]$ one has

$$\begin{array}{rcl} \tilde{H}(t)\tilde{E}(-t) = 1, & \text{or equivalently,} & \sum_{k=0}^{n} (-1)^k \tilde{e}_k \tilde{h}_{n-k} = \delta_{0,n}; \\ \tilde{H}'(t)\tilde{E}(-t) = \tilde{P}(t), & \text{or equivalently,} & \sum_{k=0}^{n} (-1)^k (n-k) \tilde{e}_k \tilde{h}_{n-k} = \tilde{p}_n. \end{array}$$

[245] An alternative way to see that it suffices to check this is by recalling Exercise 1.4.35(c).

[246] See (2.4.1), (2.4.2), (2.5.13) for the definitions of $H(t), E(t), P(t)$.

Thus it would be helpful to evaluate the class function $\tilde{e}_k\tilde{h}_{n-k}$. Note that a unipotent g in GL_n having ℓ Jordan blocks has an ℓ-dimensional 1-eigenspace, so that the number of k-dimensional g-stable $\mathbb{F}_q$-subspaces of $\mathbb{F}_q^n$ on which g has Jordan type $\left(1^k\right)$ (that is, on which g acts as the identity) is the *q-binomial coefficient*

$$\begin{bmatrix} \ell \\ k \end{bmatrix}_q = \frac{(q;q)_\ell}{(q;q)_k (q;q)_{\ell-k}},$$

counting k-dimensional $\mathbb{F}_q$-subspaces V of an ℓ-dimensional $\mathbb{F}_q$-vector space; see, e.g., [206, §1.7]. Hence, for a unipotent g in GL_n having ℓ Jordan blocks, we have

$$\begin{aligned}(\tilde{e}_k\tilde{h}_{n-k})(g) &= q^{\binom{k}{2}} \cdot \left(\underline{1}_{J_{(1^k)}} \cdot \tilde{h}_{n-k}\right)(g) = q^{\binom{k}{2}} \cdot \sum_{\nu \in \operatorname{Par}_{n-k}} \left(\underline{1}_{J_{(1^k)}} \cdot \underline{1}_{J_\nu}\right)(g) \\ &= q^{\binom{k}{2}} \begin{bmatrix} \ell \\ k \end{bmatrix}_q\end{aligned}$$

(by Proposition 4.9.4). Thus one needs for $\ell \geq 1$ that

$$\sum_{k=0}^{\ell} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} \ell \\ k \end{bmatrix}_q = 0, \tag{4.9.3}$$

$$\sum_{k=0}^{\ell} (-1)^k (n-k) q^{\binom{k}{2}} \begin{bmatrix} \ell \\ k \end{bmatrix}_q = (q;q)_{\ell-1}. \tag{4.9.4}$$

Identity (4.9.3) comes from setting $x = 1$ in the *q-binomial theorem* [206, Exer. 3.119]:

$$\begin{aligned}&\sum_{k=0}^{\ell} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} \ell \\ k \end{bmatrix}_q x^{\ell-k} \\ &= (x-1)(x-q)(x-q^2)\cdots(x-q^{\ell-1}).\end{aligned} \tag{4.9.5}$$

Identity (4.9.4) comes from applying $\frac{d}{dx}$ to (4.9.5), then setting $x = 1$, and finally adding $(n-\ell)$ times (4.9.3). $\square$

Exercise 4.9.6. Fix a prime power q. For any $k \in \mathbb{N}$, and any k partitions $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)}$, we define a family $\left(g^\lambda_{\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(k)}}(q)\right)_{\lambda \in \operatorname{Par}}$ of elements of $\mathbb{C}$ by the equation

$$\underline{1}_{J_{\lambda^{(1)}}} \underline{1}_{J_{\lambda^{(2)}}} \cdots \underline{1}_{J_{\lambda^{(k)}}} = \sum_{\lambda \in \operatorname{Par}} g^\lambda_{\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(k)}}(q)\, \underline{1}_{J_\lambda}$$

in $\mathcal{H}$. This notation generalizes the notation $g^\lambda_{\mu,\nu}(q)$ we introduced in Definition 4.9.3. Note that $g^\lambda_\mu(q) = \delta_{\lambda,\mu}$ for any two partitions λ and μ, and that $g^\lambda(q) = \delta_{\lambda,\varnothing}$ for any partition λ (where $g^\lambda(q)$ is to be understood as $g^\lambda_{\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(k)}}(q)$ for $k = 0$).

(a) Let $\lambda \in \operatorname{Par}$, and let $n = |\lambda|$. Let V be an n-dimensional $\mathbb{F}_q$-vector space, and let g be a unipotent endomorphism of V having Jordan type λ. Let $k \in \mathbb{N}$, and let $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)}$ be k partitions. A $\left(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)}\right)$*-compatible g-flag* will mean a sequence $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_k = V$ of g-invariant $\mathbb{F}_q$-vector subspaces

V_i of V such that for every $i \in \{1, 2, \ldots, k\}$, the endomorphism of V_i / V_{i-1} induced by g [247] has Jordan type $\lambda^{(i)}$.

Show that $g^{\lambda}_{\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(k)}}(q)$ is the number of $\left(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)}\right)$-compatible g-flags.[248]

(b) Let $\lambda \in \mathrm{Par}$. Let $k \in \mathbb{N}$, and let $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)}$ be k partitions. Show that $g^{\lambda}_{\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(k)}}(q) = 0$ unless $\left|\lambda^{(1)}\right| + \left|\lambda^{(2)}\right| + \cdots + \left|\lambda^{(k)}\right| = |\lambda|$ and $\lambda^{(1)} + \lambda^{(2)} + \cdots + \lambda^{(k)} \triangleright \lambda$. (Here and in the following, we are using the notations of Exercise 2.9.17).

(c) Let $\lambda \in \mathrm{Par}$, and let us write the transpose partition λ^t as $\lambda^t = \left((\lambda^t)_1, (\lambda^t)_2, \ldots, (\lambda^t)_\ell\right)$. Show that $g^{\lambda}_{\left(1^{(\lambda^t)_1}\right),\left(1^{(\lambda^t)_2}\right),\ldots,\left(1^{(\lambda^t)_\ell}\right)}(q) \neq 0$.

(d) Let $n \in \mathbb{N}$ and $\lambda \in \mathrm{Par}_n$. Show that

$$\varphi(e_\lambda) = \sum_{\mu \in \mathrm{Par}_n;\ \lambda^t \triangleright \mu} \alpha_{\lambda,\mu} \underline{1}_{J_\mu}$$

for some coefficients $\alpha_{\lambda,\mu} \in \mathbb{C}$ satisfying $\alpha_{\lambda,\lambda^t} \neq 0$.

(e) Give another proof of the fact that the map φ is injective.

[**Hint:** For (b), use Exercise 2.9.22(b).]

We next indicate, without proof, how $\mathcal{H}$ relates to the classical Hall algebra.

Definition 4.9.7. Let p be a prime. The usual *Hall algebra*, or what Schiffmann [190, §2.3] calls *Steinitz's classical Hall algebra* (see also Macdonald [142, Chap. II]), has $\mathbb{Z}$-basis elements $\{u_\lambda\}_{\lambda \in \mathrm{Par}}$, with the multiplicative structure constants $g^{\lambda}_{\mu,\nu}(p)$ in

$$u_\mu u_\nu = \sum_\lambda g^{\lambda}_{\mu,\nu}(p)\ u_\lambda$$

defined as follows: fix a finite abelian p-group L of *type* λ, meaning that

$$L \cong \bigoplus_{i=1}^{\ell(\lambda)} \mathbb{Z}/p^{\lambda_i}\mathbb{Z},$$

and let $g^{\lambda}_{\mu,\nu}(p)$ be the number of subgroups M of L of type μ, for which the quotient $N := L/M$ is of type ν. In other words, $g^{\lambda}_{\mu,\nu}(p)$ counts, for a fixed abelian p-group L of type λ, the number of short exact sequences $0 \to M \to L \to N \to 0$ in which M, N have types μ, ν, respectively (modulo isomorphism of short exact sequences restricting to the identity on L).

[247] This is well-defined. In fact, both V_i and V_{i-1} are g-invariant, so that g restricts to an endomorphism of V_i, which further restricts to an endomorphism of V_{i-1}, and thus gives rise to an endomorphism of V_i/V_{i-1}.

[248] This can be seen as a generalization of Proposition 4.9.4. In fact, if μ and ν are two partitions, then a (μ, ν)-compatible g-flag is a sequence $0 = V_0 \subset V_1 \subset V_2 = V$ of g-invariant $\mathbb{F}_q$-vector subspaces V_i of V such that the endomorphism of $V_1/V_0 \cong V_1$ induced by g has Jordan type μ, and the endomorphism of $V_2/V_1 \cong V/V_1$ induced by g has Jordan type ν. Choosing such a sequence amounts to choosing V_1 (since there is only one choice for each of V_0 and V_2), and the conditions on this V_1 are precisely the conditions on V in Proposition 4.9.4.

We claim that when one takes the finite field $\mathbb{F}_q$ of order $q = p$ a *prime*, the $\mathbb{Z}$-linear map

$$u_\lambda \longmapsto \underline{1}_{J_\lambda} \tag{4.9.6}$$

gives an isomorphism from this classical Hall algebra to the $\mathbb{Z}$-algebra $\mathcal{H}_\mathbb{Z} \subset \mathcal{H}$. The key point is *Hall's Theorem*, a non-obvious statement for which Macdonald includes two proofs in [142, Chap. II], one of them due to Zelevinsky[249]. To state it, we first recall some notions about discrete valuation rings.

Definition 4.9.8. A *discrete valuation ring* (short *DVR*) $\mathfrak{o}$ is a principal ideal domain having only one maximal ideal $\mathfrak{m} \neq 0$, with quotient $k = \mathfrak{o}/\mathfrak{m}$ called its *residue field*.

The structure theorem for finitely generated modules over a PID implies that an $\mathfrak{o}$-module L with finite composition series of composition length n must have $L \cong \bigoplus_{i=1}^{\ell(\lambda)} \mathfrak{o}/\mathfrak{m}^{\lambda_i}$ for some partition λ of n; say L has *type* λ in this situation.

Here are the two crucial examples for us.

Example 4.9.9. For any field $\mathbb{F}$, the power series ring $\mathfrak{o} = \mathbb{F}[[t]]$ is a DVR with maximal ideal $\mathfrak{m} = (t)$ and residue field $k = \mathfrak{o}/\mathfrak{m} = \mathbb{F}[[t]]/(t) \cong \mathbb{F}$. An $\mathfrak{o}$-module L of type λ is an $\mathbb{F}$-vector space together with an $\mathbb{F}$-linear transformation $T \in \operatorname{End} L$ that acts on L nilpotently (so that $g := T + 1$ acts unipotently, where $1 = \operatorname{id}_L$) with Jordan blocks of sizes given by λ: each summand $\mathfrak{o}/\mathfrak{m}^{\lambda_i} = \mathbb{F}[[t]]/(t^{\lambda_i})$ of L has an $\mathbb{F}$-basis $\{1, t, t^2, \ldots, t^{\lambda_i - 1}\}$ on which the map T that multiplies by t acts as a nilpotent Jordan block of size λ_i. Note also that, in this setting, $\mathfrak{o}$-submodules are the same as T-stable (or g-stable) $\mathbb{F}$-subspaces.

Example 4.9.10. The ring of p-adic integers $\mathfrak{o} = \mathbb{Z}_p$ is a DVR with maximal ideal $\mathfrak{m} = (p)$ and residue field $k = \mathfrak{o}/\mathfrak{m} = \mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$. An $\mathfrak{o}$-module L of type λ is an abelian p-group of type λ: for each summand, $\mathfrak{o}/\mathfrak{m}^{\lambda_i} = \mathbb{Z}_p/p^{\lambda_i}\mathbb{Z}_p \cong \mathbb{Z}/p^{\lambda_i}\mathbb{Z}$. Note also that, in this setting, $\mathfrak{o}$-submodules are the same as subgroups.

One last notation: $n(\lambda) := \sum_{i \geq 1}(i-1)\lambda_i$, for λ in Par. Hall's Theorem is as follows.

Theorem 4.9.11. *Assume $\mathfrak{o}$ is a DVR with maximal ideal $\mathfrak{m}$, and that its residue field $k = \mathfrak{o}/\mathfrak{m}$ is finite of cardinality q. Fix an $\mathfrak{o}$-module L of type λ. Then the number of $\mathfrak{o}$-submodules M of type μ for which the quotient $N = L/M$ is of type ν can be written as the specialization*

$$[g^\lambda_{\mu,\nu}(t)]_{t=q}$$

of a polynomial $g^\lambda_{\mu,\nu}(t)$ in $\mathbb{Z}[t]$, called the Hall polynomial.

Furthermore, the Hall polynomial $g^\lambda_{\mu,\nu}(t)$ has degree at most $n(\lambda) - (n(\mu) + n(\nu))$, and its coefficient of $t^{n(\lambda)-(n(\mu)+n(\nu))}$ is the Littlewood-Richardson coefficient $c^\lambda_{\mu,\nu}$.

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[249] See also [190, Thm. 2.6, Prop. 2.7] for quick proofs of part of it, similar to Zelevinsky's. Another proof, based on a recent category-theoretical paradigm, can be found in [61, Theorem 3.53].

Comparing what Hall's Theorem says in Examples 4.9.9 and 4.9.10, shows that the map (4.9.6) gives the desired isomorphism from the classical Hall algebra to $\mathcal{H}_{\mathbb{Z}}$.

We close this section with some remarks on the vast literature on Hall algebras that we will *not* discuss here.

Remark 4.9.12. Macdonald's version of Hall's Theorem [142, (4.3)] is stronger than Theorem 4.9.11, and useful for certain applications: he shows that $g^{\lambda}_{\mu,\nu}(t)$ is the zero polynomial whenever the Littlewood-Richardson coefficient $c^{\lambda}_{\mu,\nu}$ is zero.

Remark 4.9.13. In general, not all coefficients of the Hall polynomials $g^{\lambda}_{\mu,\nu}(t)$ are nonnegative (see Butler/Hales [32] for a study of when they are); it often happens that $g^{\lambda}_{\mu,\nu}(1) = 0$ despite $g^{\lambda}_{\mu,\nu}(t)$ not being the zero polynomial[250]. However, in [110, Thm. 4.2], Klein showed that the polynomial values $g^{\lambda}_{\mu,\nu}(p)$ for p prime are always positive when $c^{\lambda}_{\mu,\nu} \neq 0$. (This easily yields the same result for p a prime power.)

Remark 4.9.14. Zelevinsky in [227, Chaps 10, 11] uses the isomorphism $\Lambda_{\mathbb{C}} \to \mathcal{H}$ to derive J. Green's formula for the value of any irreducible character χ of GL_n on any unipotent class J_λ. The answer involves values of irreducible characters of $\mathfrak{S}_n$ along with *Green's polynomials* $Q^{\lambda}_{\mu}(q)$ (see Macdonald [142, §III.7]; they are denoted $Q(\lambda, \mu)$ by Zelevinsky), which express the images under the isomorphism of Theorem 4.9.5 of the symmetric function basis $\{p_\mu\}$ in terms of the basis $\{\underline{1}_{J_\lambda}\}$.

Remark 4.9.15. The Hall polynomials $g^{\lambda}_{\mu,\nu}(t)$ also essentially give the multiplicative structure constants for $\Lambda(\mathbf{x})[t]$ with respect to its basis of *Hall-Littlewood symmetric functions* $P_\lambda = P_\lambda(\mathbf{x}; t)$:

$$P_\mu P_\nu = \sum_{\lambda} t^{n(\lambda)-(n(\mu)+n(\nu))} g^{\lambda}_{\mu,\nu}(t^{-1}) P_\lambda.$$

See Macdonald [142, §III.3].

Remark 4.9.16. Schiffmann [190] discusses self-dual Hopf algebras which vastly generalize the classical Hall algebra called *Ringel-Hall algebras*, associated to abelian categories which are hereditary. Examples come from categories of nilpotent representations of quivers; the quiver having exactly one node and one arc recovers the classical Hall algebra $\mathcal{H}_{\mathbb{Z}}$ discussed above.

Remark 4.9.17. The general linear groups $GL_n(\mathbb{F}_q)$ are one of four families of so-called *classical groups*. Progress has been made on extending Zelevinsky's PSH theory to the other families:

(a) Work of Thiem and Vinroot [217] shows that the tower $\{G_*\}$ of *finite unitary groups* $U_n(\mathbb{F}_{q^2})$ give rise to another positive self-dual Hopf algebra $A = \bigoplus_{n \geq 0} R(U_n(\mathbb{F}_{q^2}))$, in which the role of Harish-Chandra induction is played by *Deligne-Lusztig induction*. In this theory, character and degree

[250] Actually, Butler/Hales show in [32, proof of Prop. 2.4] that the values $g^{\lambda}_{\mu,\nu}(1)$ are the structure constants of the ring Λ with respect to its basis $(m_\lambda)_{\lambda \in \mathrm{Par}}$: we have

$$m_\mu m_\nu = \sum_{\lambda \in \mathrm{Par}} g^{\lambda}_{\mu,\nu}(1) m_\lambda$$

for all partitions μ and ν.

formulas for $U_n(\mathbb{F}_{q^2})$ are related to those of $GL_n(\mathbb{F}_q)$ by substituting $q \mapsto -q$, along with appropriate scalings by ± 1, a phenomenon sometimes called *Ennola duality*. See also [207, §4].

(b) van Leeuwen [128] has studied $\bigoplus_{n \geq 0} R\left(Sp_{2n}\left(\mathbb{F}_q\right)\right)$, $\bigoplus_{n \geq 0} R\left(O_{2n}\left(\mathbb{F}_q\right)\right)$ and $\bigoplus_{n \geq 0} R\left(U_n\left(\mathbb{F}_{q^2}\right)\right)$ not as Hopf algebras, but rather as so-called *twisted PSH-modules* over the PSH $A(GL)$ (a "deformed" version of the older notion of Hopf modules). He classified these PSH-modules axiomatically similarly to Zelevinsky's above classification of PSH's.

(c) In a recent honors thesis [201], Shelley-Abrahamson defined yet another variation of the concept of Hopf modules, named 2-*compatible Hopf modules*, and identified $\bigoplus_{n \geq 0} R\left(Sp_{2n}\left(\mathbb{F}_q\right)\right)$ and $\bigoplus_{n \geq 0} R\left(O_{2n+1}\left(\mathbb{F}_q\right)\right)$ as such modules over $A(GL)$.