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Partial Differential Equations

Organized by
Guido De Philippis, Trieste
Richard M. Schoen, Irvine
Peter M. Topping, Warwick

21 July – 27 July 2019

Abstract. The workshop dealt with nonlinear partial differential equations and some applications in geometry, touching several different topics such as geometric flows, minimal surfaces, semi-linear equations and calculus of variations.

Mathematics Subject Classification (2010): 35JXX, 35KXX, 49QXX.

Introduction by the Organizers

The workshop *Partial differential equations*, organized by Guido De Philippis (SISSA) Rick Schoen (Irvine) and Peter Topping (Warwick) was held July 22–July 27, 2019. The meeting was attended by 54 participants with broad geographic representation. The program consisted of 21 talks and left sufficient time for discussions.

As in the tradition of the workshop, several results concerning applications of nonlinear PDE to geometric problems have been presented. Geometric flows have been the topic of several presentations. The study and the classification of ancient solutions was discussed in one talk for Ricci flow and in another for mean curvature flow. A talk was dedicated to the study of translating solution of mean curvature flow and another one dealt with the regularity of Ricci flows that attain their initial data weakly. A study of Yamabe flow on non compact manifolds has also been presented.

The interaction between PDEs and geometric problems has also been discussed in the "stationary" case. In particular there has been a talk about the equation of prescribed scalar curvature. A "local" version of the positive mass theorem and its relation to a "weak" definition of positive scalar curvature has been the subject of another talk, where a partial answer to the dihedral rigidity conjecture of Gromov has been presented.

Minimal surfaces have been the the topic of various presentation. In particular a complete proof of the “multiplicity one conjecture” for min-max minimal surfaces has been presented. Another talk dealt with the relation between the index and the topology of minimal surfaces.

The relation between classical minimal surfaces and several “weak” notions has been considered in various talk, also thanks to the presence of several experts in geometric measure theory. A new existence theorem for co-dimension one Brakke flow was presented. The physical validation of the use of minimal surfaces in surface tension driven model has been discussed via a capillarity approximation of the classical Plateau problem. Eventually a new approach to the construction of co-dimension two stationary varifolds via phase-field approximation has been shown.

The link between minimal surfaces and semi-linear PDE both of local and non-local type has been the subject of another talk, where a proof of De Giorgi conjecture for the half-laplacian in dimension 4 has been shown.

The solution of Brezis conjecture concerning boundedness of stable solution of quasi linear equation up to dimension 9 has been presented.

Glueing techniques for construction of special solutions of non-linear PDEs have been the topics of two talks, one concerning the Keller-Segel equation and one concerning the Euler equation.

Links between PDEs and the Calculus of Variations have been the subject of various talks, in particular concerning regularity and/or singularity of solutions of variational problems with convex but not strictly convex Lagrangians and the study of Cheeger constants of planar domains. A new variational technique to establish regularity of solution of the obstacle problem based on the logarithmic epiperimetric inequality has been presented.

Eventually preliminary results on a “compensated integrability” approach to the study of solutions of the (non-elliptic) equation $\mathrm{div}\mu = \sigma$ have been discussed.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, “US Junior Oberwolfach Fellows”.

Workshop: Partial Differential Equations

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Abstracts

Translating Solutions of Mean Curvature Flow

Brian White

(joint work with David Hoffman, Tom Ilmanen, Francisco Martín)

A **translator** with velocity $\mathbf{v}$ is a hypersurface M in $\mathbf{R}^{n+1}$ such that

$$t \mapsto M + t\,\mathbf{v}$$

is a mean curvature flow, i.e., such that normal component of the velocity at each point is equal to the mean curvature vector at that point:

$$\overrightarrow{H} = \mathbf{v}^{\perp}.$$

By rotating and scaling, we can make the velocity equal to $-\mathbf{e}_{n+1}$. **Unless otherwise specified, I will assume that the velocity has been so normalized**.

If a translator M (with velocity $-\mathbf{e}_{n+1}$) is the graph of function $u : \Omega \subset \mathbf{R}^n \to \mathbf{R}$, we will say that M is a **translating graph**; in that case, we also refer to the function u as a translator, and we say that u is complete if its graph is a complete submanifold of $\mathbf{R}^{n+1}$. Thus $u : \Omega \subset \mathbf{R}^n \to \mathbf{R}$ is a translator if and only if it solves the translator equation (the nonparametric form of $\overrightarrow{H} = -\mathbf{e}_{n+1}^{\perp}$):

$$D_i \left(\frac{D_i u}{\sqrt{1 + |Du|^2}} \right) = -\frac{1}{\sqrt{1 + |Du|^2}}.$$

An example is the **grim reaper curve**:

$$\{(x, y) : y = \log(\cos x)),\ x \in (-\pi/2, \pi/2)\}.$$

Translators are interesting for a number of reasons:

(1) They provide simple examples of mean curvature flows.
(2) They provide possible models for singularity formation in mean curvature flow. For example, consider a figure 8 curve $M(t)$ in the plane moving by (mean) curvature flow. It will develop a singularity at some finite time T. Let $p(t)$ be the point of maximum curvature $\kappa(t)$. Then $\kappa(t)(M(t) - p(t))$ converges smoothly as $t \to T$ to the grim reaper curve (modulo a rotation of $\mathbf{R}^2$).
(3) They are interesting as examples in minimal surface theory. Ilmanen observed that M is a translator with velocity $\mathbf{v}$ if and only if M is a minimal surface (i.e., critical point of the area functional) with respect to the Riemannian metric
$$g_{ij}(x) = (e^{x \cdot \mathbf{v}})^{2/n} \delta_{ij}.$$
(4) Ilmanen's elliptic regularization [7] scheme lets one get general mean curvature flows as limits of translators (as the speed of translation tends to infinity).

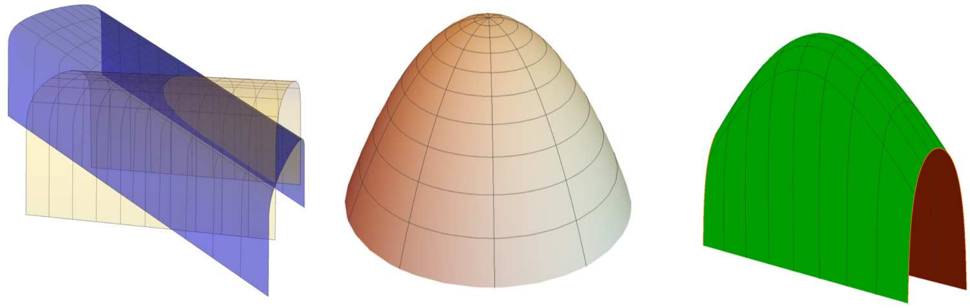


FIGURE 1. Complete graphical translators: grim reaper surface and tilted grim reaper surface (left), bowl soliton (center), and Δ-wing (right). Pictures by F. Martín.

In this talk, I described recent joint work with David Hoffman, Tom Ilmanen, and Francisco Martín: (1) a classification of all the complete translating graphs in $\mathbf{R}^3$, and (2) new families of examples of non-graphical translators in $\mathbf{R}^3$. (These and other results are discussed in the survey [5].)

Before stating the classification theorem, I recall the known examples of translating graphs in $\mathbf{R}^3$. First, the Cartesian product of the grim reaper curve with $\mathbf{R}$ is a translator:

$$(x, y) \in \mathbf{R} \times (-\pi/2, \pi/2) \mapsto \log(\cos y).$$

It is called the **grim reaper surface**.

Second, if we rotate the grim reaper surface by an angle $\theta \in (0, \pi/2)$ about the y-axis and dilate by $1/\cos\theta$, the resulting surface is again a translator, given by

$$(x, y) \in \mathbf{R} \times (-b, b) \mapsto \frac{\log(\cos(y\cos\theta))}{\cos^2\theta} + x\tan\theta,$$

where $b = \pi/(2\cos\theta)$. Note that as θ goes from 0 to $\pi/2$, the width $2b$ of the strip goes from π to ∞. These examples are called **tilted grim reaper surfaces**.

Every translator in $\mathbf{R}^3$ with zero Gauss curvature is (up to translations and up to rotations about a vertical axis) a grim reaper surface, a tilted grim reaper surface, or a vertical plane.

In [3], J. Clutterbuck, O. Schnürer and F. Schulze (see also [1]) proved for each $n \geq 2$ that there is a unique (up to vertical translation) entire, rotationally invariant function $u : \mathbf{R}^n \to \mathbf{R}$ whose graph is a translator. It is called the **bowl soliton**.

The following theorem provides a full classification of all complete translating graphs:

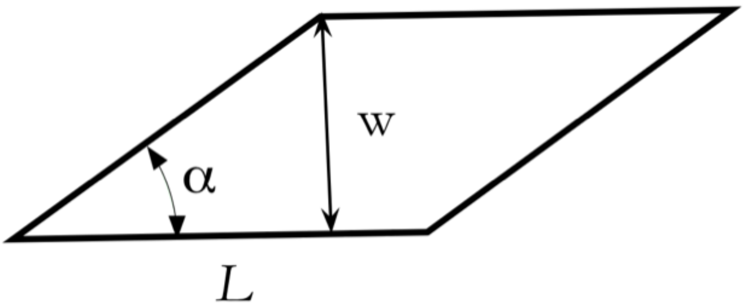
Theorem 1. [4] *For every $b > \pi/2$, there is (up to translation) a unique complete, strictly convex translator $u^b : \mathbf{R} \times (-b, b) \to \mathbf{R}$. Up to isometries of $\mathbf{R}^2$, the only other complete translating graphs in $\mathbf{R}^3$ are the grim reaper surface, the tilted grim reaper surfaces, and the bowl soliton.*

The strictly convex examples over strips are called "Δ-wings". See Figure 1.

I remark that Bourni, Langford, and Tinaglia [2] gave a different proof of part of this theorem: they proved existence (but not uniqueness) of strictly convex translating graphs defined over strips.

In the lecture, I also described new families of examples of complete, non-graphical translators. I began by posing the following problem:

> Consider the parallelogram $P(\alpha, w, L)$ in $\mathbf{R}^2$ whose lower left corner is at the origin, whose base is the segment $0 \le x \le L$ in the x-axis, whose interior angle at the origin is α, and whose width (in the y-direction) is w. Is there a solution of the minimal surface equation or of the translator equation on the domain $P(\alpha, w, L)$ that has boundary values $-\infty$ on the horizontal edges and $+\infty$ on the non-horizontal edges?



For the minimal surface equation, a classical theorem of Jenkins and Serrins asserts that there is a solution if and only if the parallelogram is a rhombus. Note that when there is a solution, its graph is a minimal surface bounded by the four vertical lines corresponding to the vertices of the rhombus. Repeated Schwarz reflection produces a complete, embedded, doubly periodic minimal surface (without boundary). In fact, the resulting surfaces are precisely the doubly periodic minimal surfaces discoved by Scherk in the 1800s.

In the case of the translator equation, we have the following analog of the Jenkins-Serrin Theorem:

Theorem 2. [6] *For each $\alpha \in (0, \pi)$ and $w \in (0, \infty)$, there is a unique $L = L(\alpha, w)$ in $(0, \infty]$ for which the boundary value problem above has a solution.*

(1) The length $L(\alpha, w)$ is finite if and only if $w < \pi$.

(2) The solution is unique up to an additive constant. There is a unique solution $u_{\alpha,w}$ satisfying the additional condition: $(\cos(\alpha/2), \sin(\alpha/2), 0)$ is tangent to the graph of u at the origin.

(3) The graph extends by repeated Schwarz reflection to a periodic surface $\mathscr{S}_{\alpha,w}$.

- *If $w < \pi$, then $\mathscr{S}_{\alpha,w}$ is doubly periodic and we call it a* **Scherk translator**.
- *If $w \ge \pi$, then $\mathscr{S}_{\alpha,w}$ is singly periodic and we call it a* **Scherkenoid**.

See Figure 2.

We also describe the limits of these examples as $\alpha \to 0$ or $\alpha \to \pi$:

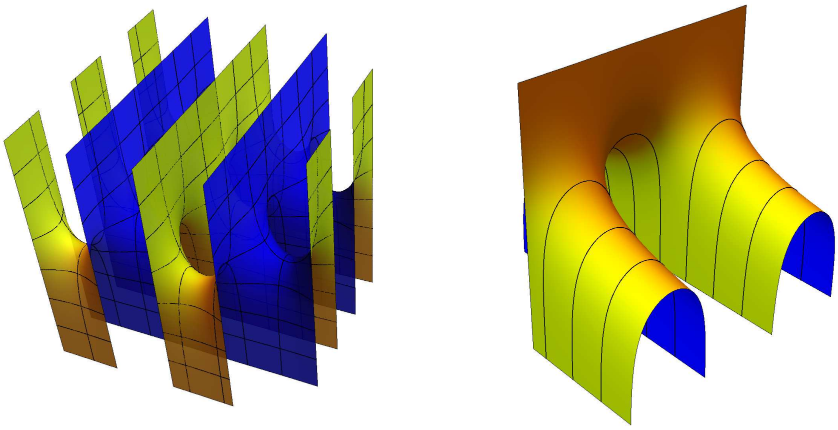


FIGURE 2. Scherk Translator (left) and Scherkenoid (right). Pictures by F. Martín.

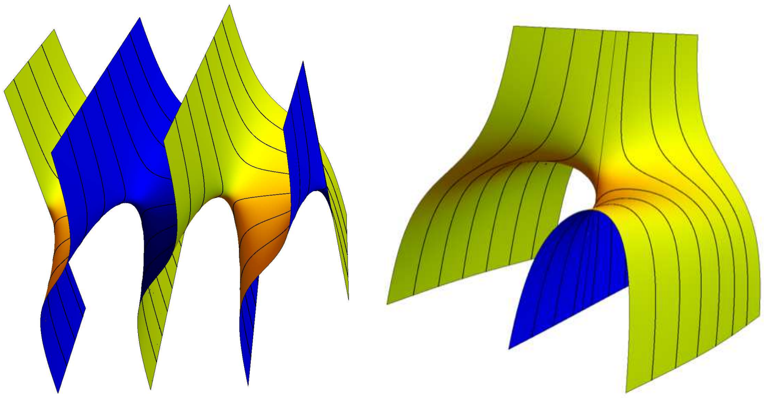


FIGURE 3. Helicoidal-like translator (left) and pitchfork (right). Pictures by F. Martín.

Theorem 3. [6] *As* $\alpha \to 0$*, the surface* $\mathscr{S}_{\alpha,w}$ *converges smoothly to the parallel vertical planes* $y = nw$*,* $n \in \mathbf{Z}$*. As* $\alpha \to \pi$*, the surface* $\mathscr{S}_{\alpha,w}$ *converges smoothly, perhaps after passing to a subsequence, to a limit surface* M*. (We do not know whether the limit depends on the choice of subsequence.) Furthermore,*

- *If* $w < \pi$*, then* M *is helicoid-like. (See Figure 3.)*
- *If* $w > \pi$*, then* M *is a complete, simply connected translator such that* M *contains* Z *and such that* $M \setminus Z$ *projects diffeomorphically onto* $\{-w < y < 0\} \cup \{0 < y < w\}$*. We call such a translator a* **pitchfork** *of width* w*.*

Do these translators arise as blowups of mean curvature flows? Any such blowup must have finite entropy (according to Huisken's monotonicity formula). For that reason, the Scherk-like translators, Scherkenoids, and helicoid-like translators cannot arise as blowups. However, the Δ-wings and pitchforks have finite entropy. Whether they occur as blowups is a fascinating open problem. (The other surfaces mentioned in this article – the grim reaper surface and the bowl solition – do occur as blowups.)

References

[1] Steven J. Altschuler and Lang F. Wu, *Translating surfaces of the non-parametric mean curvature flow with prescribed contact angle*, Calc. Var. Partial Differential Equations **2** (1994), no. 1, 101–111.

[2] T. Bourni, M. Langford, and G. Tinaglia, *On the existence of translating solutions of mean curvature flow in slab regions*, arXiv:1805.05173 (2018).

[3] Julie Clutterbuck, Oliver C. Schnürer, and Felix Schulze, *Stability of translating solutions to mean curvature flow*, Calc. Var. Partial Differential Equations **29** (2007), no. 3, 281–293.

[4] D. Hoffman, T. Ilmanen, F. Martín, and B. White, *Graphical Translators for Mean Curvature Flow*, Calc. Var. and Partial Differential Equations **58:117** (2019).

[5] D. Hoffman, T. Ilmanen, F. Martín, and B. White, *Notes on translating solitons for Mean Curvature Flow*, arXiv:1901.09101 (2019).

[6] D. Hoffman, F. Martín, and B. White, *Scherk-like Translators for Mean Curvature Flow.*, arXiv:1903.04617 (2019).

[7] Tom Ilmanen, *Elliptic regularization and partial regularity for motion by mean curvature*, Mem. Amer. Math. Soc. **108** (1994), no. 520, x+90.

Complexities of bounded area minimal hypersurfaces

Antoine Song

Let (M^{n+1}, g) be a closed Riemannian manifold. We study the geometry of closed minimal hypersurfaces smoothly embedded inside M outside a subset of Hausdorff dimension at most $n-7$. The singularity set is in general not avoidable as the case of area minimizing minimal hypersurfaces already suggests. Apart from the area (i.e. n-dimensional volume), there are several natural measures of complexity associated to a minimal hypersurface: the total Betti number, the size of the singular set and the Morse index. The main question is the following: how do these quantities relate to each other?

Before stating the results, let us give some motivations. Firstly families of bounded area minimal hypersurfaces with an a priori index bound are well understood and enjoy various compactness and diffeomorphism finiteness theorems [11, 3, 2]. It is then natural to wonder what happens when the index bound assumption is dropped. A second motivation comes from the general existence theory for minimal surfaces. Works by F. C. Marques, A. Neves and others in min-max theory revealed recently that minimal hypersurfaces are plentiful in closed Riemannian manifolds. For instance, building on works of Marques-Neves, we showed that any closed manifold of dimension between 3 and 7 has infinitely many closed smooth embedded minimal hypersurfaces [12] and that concluded the solution to a conjecture of S.-T. Yau from the 80's [14]. The result is expected to hold in higher dimensions if one allows the minimal hypersurfaces to have a codimension 7 singular set. Since these minimal hypersurfaces come from variational arguments, they have natural Morse index bounds and one would like to bound the geometry of these hypersurfaces from above by their index. We achieve one step in that direction by studying bounded area minimal hypersurfaces of high Morse index.

Let (M^{n+1}, g) be a Riemannian manifold. Our first result is about ambient dimensions $n+1$ between 3 and 7.

Theorem 1. *For any $A > 0$, there exists a constant C_0 depending on A and the ambient metric g so that for any closed embedded minimal hypersurface $\Sigma \subset M$ with area at most A:*

$$\sum_{i=0}^{n} b^i(\Sigma) \leq C_0(index(\Sigma) + 1),$$

where $b^i(\Sigma)$ denotes the i-th Betti number of Σ. Moreover when $n+1=3$, the constant C_0 can be chosen to depend linearly on the area bound A.

In higher dimensions, we cannot bound the topology in terms of the index (we actually conjecture that there are counterexamples). However, since we consider integral stationary varifolds with support smooth outside a codimension 7 subset, we give a bound on the size of the singular set, which in general is not empty.

Theorem 2. *For any $A > 0$, there exists a constant C_1 depending on A and the ambient metric g so that for any closed embedded minimal hypersurface $\Sigma \subset M$ smooth outside a codimension 7 subset and with area at most A:*

$$\mathcal{H}^{n-7}(Sing(\Sigma)) \leq C_1(index(\Sigma) + 1)^{7/n},$$

where $Sing(\Sigma)$ denotes the singular set of Σ.

In both theorems, we expect the inequalities to be optimal up to the non-explicit constants C_0, C_1. These results are lower bounds for the index. Previously N. Ejiri-M. Micallef [5] proved a general upper bound for the index of minimal 2-surfaces in terms of the area and genus. Such a bound is false for higher dimensional minimal hypersurfaces. As for lower bounds, there have been many works for symmetric ambient metrics, for instance [7, 8, 13, 1, 4]. These bounds are independent of the area but only treat the first Betti number and require the metric to be very symmetric. They are proved using variations on an argument of A. Ros, who understood that harmonic 1-forms can help constructing area-decreasing second variations and hence he could relate the first Betti number to the Morse index of a minimal hypersurface.

We use instead a new quantified covering argument: we find a family of extrinsic geodesic balls $\{b_i\}_{i=0}^{L}$ such that the balls $2b_i$ are disjoint and for any $s < 2$, $\Sigma \cap sb_i$ is stable but for any $s > 2$, $\Sigma \cap sb_i$ is unstable. It is simple to check that the index is bounded below by L. In dimensions 3 to 7 the curvature bounds of R. Schoen-L. Simon [10] (or of R. Schoen in dimension 3 [9]) show that each region $\Sigma \cap b_i$ is geometrically controlled. The crux of the proof is to argue that the number of balls L can be chosen to be at least comparable to $\sum_{i=0}^{n} b^i(\Sigma)$. In higher dimensions, A. Naber-D. Valtorta [6] show that each $Sing(\Sigma) \cap b_i$ has controlled size and similarly we have to find "enough" of these balls b_i to prove the second theorem.

References

[1] L. Ambrozio, A. Carlotto and B. Sharp, *Comparing the Morse index and the first Betti number of minimal hypersurfaces*, J. Differential Geom. **108** (2018), 379–410.

[2] R. Buzano and B. Sharp, *Qualitative and quantitative estimates for minimal hypersurfaces with bounded index and area*, Trans. Amer. Math. Soc. **370** (2018), 4373–4399.

[3] O. Chodosh, D. Ketover and D. Maximo, *Minimal hypersurfaces with bounded index*, Invent. Math. **209** (2017), 617–664.

[4] O. Chodosh, D. Maximo, *On the topology and index of minimal surfaces*, J. Differential Geom. **104** (2016), 399–418.

[5] N. Ejiri and M. Micallef, *Comparison between second variation of area and second variation of energy of a minimal surface*, Adv. Calc. Var. **1** (2008), 223–239.

[6] A. Naber and D. Valtorta, *The Singular Structure and Regularity of Stationary and Minimizing Varifolds*, arXiv:1505.03428v4 [math.DG] (1981).

[7] A. Ros, *One-sided complete stable minimal surfaces*, J. Differential Geom., **74** (2006), 69–92.

[8] A. Savo, *Index bounds for minimal hypersurfaces of the sphere*, Indiana Univ. Math. J. **59** (2010), 823–837.

[9] R. Schoen, *Estimates for stable minimal surfaces in three-dimensional manifolds*, Seminar on Minimal Submanifolds, Ann. of Math. Studies **103**, Princeton University Press (1983), 111–126.

[10] R. Schoen and L. Simon, *Regularity of stable minimal hypersurfaces*, Comm. Pure Appl. Math **34** (1981), 741–797.

[11] B. Sharp, *Compactness of minimal hypersurfaces with bounded index*, J. Differential Geom. **106** (2017), 317–339.

[12] A. Song, *Existence of infinitely many minimal hypersurfaces in closed manifolds*, arXiv:1806.08816 [math.DG] (2018).

[13] F. Urbano, *Second variation of one-sided complete minimal surfaces*, Rev. Mat. Iberoam **29** (2013), 479–494.

[14] S.-T. Yau, *Seminar in Differential Geometry, Problem Section*, Ann. of Math. Stud., **102**, Princeton Univ. Press, Princeton, N.J (1982), 669–706.

Positive scalar curvature and the dihedral rigidity conjecture

Chao Li

A fundamental question in differential geometry is to understand metric/measure properties of Riemannian manifolds under global curvature conditions, and study notions of curvature lower bounds in spaces with low regularity. Such goals are usually achieved via geometric comparison theorems. The quest started with Alexandrov [1], who introduced the notion of *sectional* curvature lower bounds for metric spaces via geometric comparison theorems for geodesic triangles. Similar questions for *Ricci* curvature have also attracted a wide wealth of research recently (Cheeger-Colding-Naber theory; see, e.g., [2, 3, 4, 5, 6]; for an optimal transport approach, see, e.g., [7] [8, 9, 10]).

The case of scalar curvature lower bounds, however, is not as well established, possibly due to a lack of satisfactory geometric comparison theory. In a paper [11] from 2014, Gromov proposed the first steps towards understanding Riemannian manifolds with scalar curvature bounded below. He suggested that a polyhedron comparison theorem should play a role analogous to that of the Alexandrov's triangle comparisons for spaces with sectional curvature lower bounds. Precisely, let M^n be a convex polyhedron in Euclidean space, and g a metric on M. Denote the Euclidean metric by g_0. Gromov made the following conjecture (see section 2.2 of [11], and section 7, Question F_1 of [12]):

Conjecture (The dihedral rigidity conjecture)**.** *Suppose (M, g) has nonnegative scalar curvature and weakly mean convex faces, and along the intersection of any two adjacent faces, the dihedral angle of (M, g) is not larger than the (constant) dihedral angle of (M, g_0). Then (M, g) is isometric to a flat Euclidean polyhedron.*

When $n = 2$, the conjecture is a straightforward consequence of the Gauss-Bonnet formula. In fact, given a Riemann surface (M^2, g), the Gauss curvature $K_g > 0$ everywhere if and only if there exists no geodesic triangle with total inner angle smaller than π.

In [13] and [14], the author verified the dihedral rigidity conjecture for some polytopes:

Theorem 1. *The dihedral rigidity conjecture holds for all 3-dimensional simplices and n-dimensional cubes, $3 \leq n \leq 7$.*

The proof of Theorem 1 is based on considering a related geometric variational problem. Suppose (M^3, g) is a 3-dimensional simplex. Let p be a vertex of M, and F_j, $j = 1, 2, 3$, be the faces of M contains p, and F_4 be the unique face that is disjoint from p. Let P be a flat simplex in $\mathbb{R}^3$, denote its corresponding vertex and faces by p' and F'_j. Suppose that $R(g) \geq 0$, each face of M is weakly mean convex, and $\measuredangle(F_i, F_j) \leq \measuredangle(F'_i, F'_j)$. Denote γ_j, $j = 1, 2, 3$, be the inner angle between F'_j and F_4. We consider the energy functional

$$\mathcal{F}(E) = |\partial E \cap \mathring{M}| - \sum_{j=1}^{3} \cos \gamma_j |\partial E \cap F_j|.$$

Here E is a subset of M with finite perimeter, $p \in E$, and $F_4 \cap E = \emptyset$. We further consider the variational problem

$$I = \inf\{\mathcal{F}(E) : E \subset M \text{ is a set of finite perimeter, } p \in E, F_4 \cap E = \emptyset\}.$$

In the case for cubes, we simply consider

$$\mathcal{F}(E) = |\partial E \cap \mathring{M}|$$

for E a subset of M of finite perimeter, and the corresponding variational problem.

A first observation is that the variational problem for E satisfies a strong maximum principle in the interior (see [15]). We then establish a maximum principle on the boundary and corners, which was based on earlier work of Li-Zhou [16]. As a conclusion, the minimizer E does not touch the barrier face F_4, unless $E = M$.

Thus, any critical point for the functional $\mathcal{F}$ satisfies that $\Sigma = \partial E \cap \mathring{M}$ is a minimal surface that meets the face F_j at constant angle γ_j. In the case where M is a cube, Σ is simply a free boundary minimal surface. We then establish the regularity of Σ at the corners. In a joint work with N. Edelen [17], we establish the conclusion in the case where M is a cube:

Theorem 2. *Let Σ^{n-1} be an area minimizing hypersurface in a cube M, such that the dihedral angles of M is everywhere not larger than $\pi/2$. Then Σ^{n-1} is a $C^{2,\alpha}$ graph over its tangent plane everywhere.*

This regularity result enables us to apply the Schoen-Yau minimal slicing technique [18]. Precisely, using the first eigenfunction of the Jacobi operator as a conformal factor, the surface Σ^{n-1} becomes an overcubic manifold of dimension $(n-1)$ that satisfies all the assumptions of Conjecture. Inductively, we obtain a slicing

$$\Sigma^2 \subset \cdots \subset \Sigma^{n-1} \subset M$$

of M. By the Gauss-Bonnet formula, Σ^2 is isometric to a flat square. We therefore conclude that Σ^{n-1} is isometric to an Euclidean rectangular solid.

To conclude that M is isometric to an Euclidean rectangular solid, we extend an idea by Carlotto-Chodosh-Eichmair [19]. Precisely, we can prove that there exists a collection $\{\Sigma_\rho\}_{\rho\in\mathcal{A}}$, such that $\{\Sigma_\rho\}$ is dense in M, and $\{\partial\Sigma_\rho\}$ is dense on ∂M. This concludes that M is isometric to an Euclidean rectangular solid.

References

[1] A.D. Aleksandrov, *A theorem on triangles in a metric space and some of its applications*, Trudy Mat. Inst. Steklov., v 38, Izdat. Akad Nauk SSSR, Moscow, 1951, pp. 5-23.

[2] J. Cheeger and T.H. Colding, *On the structure of spaces with Ricci curvature bounded below. I*, J. Differential Geom. **46** (1997), no. 3, 406-480.

[3] J. Cheeger and T.H. Colding, *On the structure of spaces with Ricci curvature bounded below. II*, J. Differential Geom. **54** (2000), no. 1, 13-35.

[4] J. Cheeger and T.H. Colding, *On the structure of spaces with Ricci curvature bounded below. III*, J. Differential Geom. **54** (2000), no. 1, 37-74.

[5] T.H. Colding and A. Naber, *Sharp Hölder continuity of tangent cones for spaces with a lower Ricci curvature bound and applications*, Ann. of Math. (2) **176** (2012), no. 2, 1173-1229.

[6] T.H. Colding and A. Naber, *Lower bounds on Ricci curvature and quantitative behavior of singular sets*, Invent. Math. **191** (2013), no. 2, 321-339.

[7] J. Lott and C. Villani, *Ricci curvature for metric-measure spaces via optimal transport*, Ann. of Math. (2) **169** (2009), no. 3, 903-991.

[8] K.-T. Sturm, *A curvature-dimension condition for metric measure spaces*, C. R. Math. Acad. Sci. Paris **342** (2006), no. 3, 197-200.

[9] K.-T. Sturm, *On the geometry of metric measure spaces. I*, Acta Math. **196** (2006), no. 1, 65-131.

[10] K.-T. Sturm, *On the geometry of metric measure spaces. II*, Acta Math. **196** (2006), no. 1, 133-177.

[11] M. Gromov, *Dirac and Plateau billiards in domains with corners*, Cent. Eur. J. Math. **12** (2014), no. 8, 1109-1156.

[12] M. Gromov, *A dozen problems, questions and conjectures about positive scalar curvature*, Foundations of mathematics and physics one century after Hilbert, Springer, Cham, 2018, pp. 135-158.

[13] C. Li, *A polyhedron comparison theorem for 3-manifolds with positive scalar curvature*, to appear in Invent. Math.

[14] C. Li, *The dihedral rigidity conjecture for n-cubes*, ArXiv 1907.03855.

[15] B. Solomon and B. White, *A strong maximum principle for varifolds that are stationary with respect to even parametric elliptic functions*, Indiana Univ. Math. J. **38** (1989), no. 3, 683-691.

[16] M. Li and X. Zhou, *A maximum principle for free boundary minimal varieties of arbitrary codimension*, to appear in Comm. Anal. Geom.
[17] N. Edelen and C. Li, *Regularity of free boundary varifolds in locally convex domains*, in preparation.
[18] R. Schoen and S.T. Yau, *On the structure of manifolds with positive scalar curvature*, Manuscripta Math. **28** (1979), no. 1-3, 159-183.
[19] A. Carlotto, O. Chodosh, M. Eichmair, *Effective versions of the positive mass theorem*, Invent. Math. **206** (2016), no. 3, 975-1016.

Chararcterizing minimizers of a constrained planar isoperimetric problem

Robin Neumayer
(joint work with G. P. Leonardi and G. Saracco)

Given $n \geq 2$ and an open bounded domain $\Omega \subset \mathbb{R}^n$, consider the minimization problem

$$h(\Omega) = \inf \left\{ \frac{P(E)}{|E|} : E \subset \Omega,\ |E| > 0 \right\}, \tag{1}$$

where $P(E)$ and $|E|$ denote the perimeter and volume of E respectively. This constrained isoperimetric problem is known as the *Cheeger problem*, so named for an analogous problem on compact Riemannian manifolds considered by Jeff Cheeger in [2] to establish a lower bound on the first nontrivial eigenvalue of the Laplacian. In the Euclidean setting, this classical isoperimetric problem has roots in the work of Steiner in 1841 in [7].

The infimum value $h(\Omega)$ is known as the Cheeger constant, and a set achieving the infimum is called a Cheeger set. A Cheeger set E exists, and $\partial E \cap \Omega$ has constant mean curvature equal to $h(\Omega)$ and is smooth outside a set of codimension 8. In particular, if $n = 2$, then $\partial E \cap \Omega$ is the countable union of circular arcs of radius $r = 1/h(\Omega)$. While uniqueness fails in general, the union of all Cheeger sets, called the *maximal Cheeger set*, is itself a minimizer of (1).

The Cheeger problem has generated interest in recent years, in part stemming from its connections to numerous other fields including capillarity theory, image processing, and landslide modeling. In each of these settings, it is useful to obtain explicit information about either the Cheeger sets or the value of the Cheeger constant. With this in mind, we are interested in the following general questions:

Given a domain Ω, can one obtain an explicit description of Cheeger sets E in terms of Ω? Can one compute the value of the Cheeger constant?

Some numerical methods based on duality theory have been employed to address these questions, but until recently, Cheeger sets had been precisely characterized for only two classes of domains: convex planar sets [1, 6, 3] and planar strips [5]. In both settings, the Cheeger set E is unique and given by

$$E = \Omega^r \oplus B(0, r). \tag{2}$$

Here, $\Omega^r = \{x \in \Omega : \operatorname{dist}(x, \partial E) \geq r\}$ is the inner parallel set of radius r, and $A \oplus B = \{x + y : x \in A, y \in B\}$ is the Minkowski sum. The value r is given by the unique solution to the equation

$$|\Omega^r| = \pi r^2. \tag{3}$$

As noted above, $h(\Omega) = 1/r$, and thus (3) provides the precise value of the Cheeger constant as well.

The characterization of Cheeger sets given by (2) and (3) cannot be expected to hold for all planar domains. It is not hard to construct counterexamples that fail to be simply connected (for instance, a ball of radius one with a small ball near the boundary removed). Among simply connected domains, one can still construct counterexamples that contain thin necks (for instance, a ball of radius 1 and a ball of radius 2/3 joined by a thin tube). It turns out that, as we show in Theorem 1 below, the presence of necks is essentially the only thing that can go wrong for a simply connected planar domain.

To state this property more precisely, we say that a domain Ω has *no necks of radius* r if, given any points $x_1, x_2 \in \Omega$ such that $B(x_i, r) \subset \Omega$ for $i = 1, 2$, there is a continuous path $\gamma\colon [0,1] \to \Omega$ with endpoints $\gamma(0) = x_1$ and $\gamma(1) = x_2$ such that $B(\gamma(t), r) \subset \Omega$ for all $t \in (0,1)$. The main result of [4] is the following:

Theorem 1 (Leonardi, Neumayer, Saracco). *Let $\Omega \subset \mathbb{R}^2$ be a Jordan domain with $|\partial\Omega| = 0$, and suppose that Ω has no necks of radius $1/h(\Omega)$. Then the maximal Cheeger set is given by* (2) *and* (3).

It is somewhat unfavorable to have a hypothesis in Theorem 1 that involves $h(\Omega)$, as this may not be a priori checkable. What is more useful in practice is the following alternative version of the theorem, with slightly stronger but more easily checkable hypotheses.

Theorem 2 (Leonardi, Neumayer, Saracco). *Let $\Omega \subset \mathbb{R}^2$ be a Jordan domain with $|\partial\Omega| = 0$, and suppose that Ω has no necks of radius r for*

$$\frac{\operatorname{inr}(\Omega)}{2} \leq r \leq \frac{(|\Omega|/\pi)^{1/2}}{2}. \tag{4}$$

Then the maximal Cheeger set is given by (2) *and* (3).

Here $\operatorname{inr}(\Omega)$ denotes the inradius of Ω, or the radius of the largest ball contained in Ω.

As an application of the theorem, we compute the Cheeger constant of the Koch snowflake K. This fractal is constructed, starting from an equilateral triangle (of side length 3 in our normalization), by iteratively replacing the middle third of each edge by an equilateral triangle. Its boundary is a Jordan curve with infinite length, but with zero Lebesgue measure. It has no necks of any radius. Therefore, Theorem 1 applies to K. By computing the Cheeger constant for the polygons in the iterative construction of K and proving error estimates, we establish the following.

Theorem 3 (Leonardi, Neumayer, Saracco)**.** *The Cheeger constant of the Koch snowflake is given by* $h(K) = 1.89124548...$

References

[1] A. S. Besicovitch, *Variants of a classical isoperimetric problem*, Quart. J. Math. Oxford **3** (1952), no. 2, 42–49

[2] J. Cheeger, *A lower bound for the smallest eigenvalue of the Laplacian*, Problems in analysis (Papers dedicated to Salomon Bochner, 1969), Princeton Univ. Press, Princeton, N. J., 195–199

[3] B. Kawohl and T. Lachand-Robert, *Characterization of Cheeger sets for convex subsets of the plane*, Pacific J. Math. **225** (2006), no. 1, 103–118

[4] G. P. Leonardi, R. Neumayer, and G. Saracco, *The Cheeger constant of a Jordan domain without necks*, Calc. Var. Partial Differential Equations **56** (2017), no. 6., Art. 164, 29

[5] G. P. Leonardi and A. Pratelli,*On the Cheeger sets in strips and non-convex domains*, Calc. Var. Partial Differential Equations **55** (2016), no. 1, 1–28

[6] D. Singmaster and D. J. Souppuris, *A constrained isoperimetric problem*, Math. Proc. Cambridge Phil. Soc. **83** (1978), no. 1, 73–82

[7] J. Steiner, *Sur le maximum et le minimum des figures dans le plan, sur la sphere et dans l'espace en general*, J. Math. Pures Appl. **6** (1841), 105–170

Stable phase transitions: from nonlocal to local

Joaquim Serra

(joint work with X. Cabré, E. Cinti, S. Dipierro, A. Figalli)

The Peierls-Nabarro model [11, 10] was introduced in the early 1940's in the context of crystal dislocations and also arises in the study of phase transitions with line-tension effects and boundary vortices in thin magnetic films. This model considers the energy functional

$$I_\varepsilon(u) := \frac{\varepsilon}{4}[u]^2_{H^{1/2}(\mathbb{R}^n)} + \int_{\mathbb{R}^n} W(u), \quad [u]^2_{H^{1/2}(\mathbb{R}^n)} := \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\big|u(x)-u(\bar{x})\big|^2}{|x-\bar{x}|^{n+1}}dxdy,$$

where $u : \mathbb{R}^n \to (-1,1)$ and $W(u) := 1 + \cos(\pi u)$.

The very related Allen-Cahn functional, introduced in the late 1950's in the context of the Van Der Walls-Cahn-Hilliard theory for phase transitions in fluids, is also very connected to the Ginzburg-Landau theory of superconductivity. It is defined as

$$J_\varepsilon(u) := \frac{\varepsilon^2}{2}[u]^2_{H^1(\mathbb{R}^n)} + \int_{\mathbb{R}^n} W(u), \qquad [u]^2_{H^1(\mathbb{R}^n)} := \int_{\mathbb{R}^n} |\nabla u|^2,$$

where $u : \mathbb{R}^n \to (-1,1)$ and $W(u) := \frac{1}{4}(1-u^2)^2$.

In both models $\varepsilon > 0$ is a parameter and $W(u)$ is a so-called *double well potential*, namely, a function with two minima (or wells) at the values $u = -1$ and $u = +1$ which correspond to two "stable phases".

Critical points $u \in C^2(\mathbb{R}^n)$ of I_ε and J_ε solve, respectively the Peierls-Nabarro and Allen-Cahn equations:

$$\varepsilon(-\Delta)^{1/2}u + W'(u) = 0 \qquad \text{and} \qquad \varepsilon^2(-\Delta)u + W'(u) = 0.$$

A deep link between the previous models and minimal surfaces is found when investigating the asymptotic behaviour of a suitable renormalized version of J_ε and I_ε as $\varepsilon \downarrow 0$. Indeed, as a consequence of Γ-convergence results of Modica and Mortola [9] and Alberti, Bouchitté and Seppecher [1] the following holds:

If $u_{\varepsilon_k} : \mathbb{R}^n \to \mathbb{R}$ is a sequence of minimizers (in every bounded set) of either J_{ε_k} or I_{ε_k} and $\varepsilon_k \downarrow 0$, then (up to a subsequence)

$$(1) \quad u_{\varepsilon_k} \xrightarrow{L^1_{\mathrm{loc}}} \chi_E - \chi_{\mathbb{R}^n \setminus E}, \qquad \textit{where } E \subset \mathbb{R}^n \textit{ is a minimizer of the perimeter.}$$

In other words, for every $\lambda \in (-1,1)$ the level sets $\{u_\varepsilon = \lambda\}$ converge to an area minimizing (in particular minimal) hypersurfaces as $\varepsilon \downarrow 0$.

For each of the three functionals, (J_ε, I_ε, and Perimeter) we are interested in studying *stable solutions*, defined as the critical points with nonnegative second variation. More informaly, one can as well say that stable solutions are "minimizers of the energies with respect to tiny perturbations". Heuristically, they stable solutions are "the ones observable in nature", since noise would make unstable critical points immediately decay towards stable ones.

The so-called "stability conjectures" (for the three models above) state that in dimensions $3 \le n \le 7$ the only "entire" stable solutions, i.e. stable objects in the whole $\mathbb{R}^n$, must be flat (for minimal surfaces "entire" is here interpreted as imbedded, connected, complete hypersurface). These are very challenging questions and a positive answer was only known in the case of minimal surfaces in $\mathbb{R}^3$ ([6, 8]).

In a joint work with A. Figalli [7] we prove that every stable solution of

$$(-\Delta)^{1/2} u + f(u) = 0 \qquad \text{in } \mathbb{R}^3$$

is a 1D profile, i.e., $u(x) = \phi(e \cdot x)$ for some $e \in \mathbb{S}^2$, where $\phi : \mathbb{R} \to \mathbb{R}$ is a nondecreasing bounded stable solution in dimension one.

This gives in particular a positive answer of the stability conjecture for Peierls-Nabarro for $n = 3$ (the case of Allen-Cahn remains open for $n = 3$). This result can be regarded as a PDE version of the fact that stable embedded minimal surfaces in $\mathbb{R}^3$ are planes.

Our strategy of proof has is roots in the study of stable critical points for the so-called "nonlocal phase transition models". Such nonlocal models arise as an extrapolation to $s \in (0, 1/2)$ of the 1-parameter family of functionals

$$\mathcal{J}_\varepsilon(u) := \frac{\varepsilon^{2s}}{2}[u]^2_{H^s(\mathbb{R}^n)} + \int_{\mathbb{R}^n} W(u)dx, \quad s \in [1/2, 1],$$

which arises as a natural interpolation between J_ε and I_ε. For $s \in (0, \frac{1}{2})$ the new nonlocal models are no longer asymptotic to the classical perimeter as $\varepsilon \downarrow 0$ but are instead asymptotic to the so-called nonlocal perimeters [3]. For such nonlocal functionals stability gives stronger consequences [4, 5, 2] than for the classical models and, interestingly, some of the new informations obtained from the analysis of nonlocal models lead to interesting conclusions even in the limiting (asymptotically local) case $s = \frac{1}{2}$.

References

[1] G. Alberti, G. Bouchitté, and S. Seppecher, *Phase transition with the line-tension effect*, Arch. Rational Mech. Anal. 144 (1998), 1–46.
[2] X. Cabré, E. Cinti, and J. Serra, *On stable nonlocal minimal cones in* $\mathbb{R}^3$, preprint.
[3] L. Caffarelli, J.-M. Roquejoffre, O. Savin *Nonlocal minimal surfaces*, Comm. Pure Appl. Math. 63 (2010), 1111-1144.
[4] E. Cinti, J. Serra, and E. Valdinoci, *Quantitative flatness results and BV-estimates for stable nonlocal minimal surfaces*, J. Diff. Geom., to appear.
[5] S. Dipierro, J. Serra, and E. Valdinoci, *Improvement of flatness for nonlocal phase transitions*, preprint.
[6] M. do Carmo and C. K. Peng, *Stable complete minimal surfaces in* $\mathbb{R}^3$ *are planes*, Bull. Amer. Math. Soc. (N.S.) 1 (1979), 903–906.
[7] A. Figalli and J. Serra, *On stable solutions for boundary reactions: a De Giorgi-type result in dimension 4+1*, Invent. Math, to appear.
[8] D. Fischer-Colbrie and R. Schoen, *The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature*, Comm. Pure Appl. Maths. 33 (1980), 199–211.
[9] L. Modica and S. Mortola, *Un esempio di* Γ*-convergenza* (Italian), Boll. Un. Mat. Ital. B (5) 14 (1977), 285–299.
[10] F.R.N. Nabarro, *Dislocations in a simple cubic lattice* Proc. Phys. Soc., 59 (1947), 256–272.
[11] R. Peierls, *The size of a dislocation*, Proc. Phys. Soc., 52 (1940), 34–37.

On the regularity of stable solutions to semilinear elliptic PDEs

Alessio Figalli
(joint work with Xavier Cabré, Xavier Ros-Oton, Joaquim Serra)

Given $\Omega \subset \mathbb{R}^n$ a bounded domain and $f : \mathbb{R} \to \mathbb{R}$, we consider $u : \Omega \to \mathbb{R}$ a solution to the semilinear equation

$$-\Delta u = f(u) \qquad \text{in } \Omega \subset \mathbb{R}^n. \tag{1}$$

If we define $F(t) := \int_0^t f(s)\,ds$, then (1) corresponds to the Euler-Lagrange equation for the energy functional

$$\mathcal{E}[u] := \int_\Omega \Big(\frac{|\nabla u|^2}{2} - F(u) \Big)\,dx.$$

In other words, u is a critical point of $\mathcal{E}$, namely

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{E}[u+\epsilon\xi] = 0 \qquad \text{for all } \xi \in C_c^\infty(\Omega).$$

Consider the second variation of $\mathcal{E}$, that is,

$$\frac{d^2}{d\epsilon^2}\Big|_{\epsilon=0} \mathcal{E}[u+\epsilon\xi] = \int_\Omega \Big(|\nabla \xi|^2 - f'(u)\xi^2 \Big)\,dx.$$

Then, one says that u is a stable solution of equation (1) in Ω if the second variation among compactly supported variations is nonnegative, namely

$$\int_\Omega f'(u)\xi^2\,dx \le \int_\Omega |\nabla \xi|^2\,dx \qquad \text{for all } \xi \in C_c^\infty(\Omega).$$

Our interest is in nonnegative nonlinearities f that grow at $+\infty$ faster than linearly. In this case it is well-known that, independently of the Dirichlet boundary conditions that one imposes on (1), the energy $\mathcal{E}$ admits no absolute minimizer.

However, in many instances there exist nonconstant stable solutions, such as local minimizers. The regularity of stable solutions to semilinear elliptic equations is a very classical topic in elliptic equations, initiated in the seminal paper of Crandall and Rabinowitz, which has given rise to a huge literature on the topic.

Note that the dimension plays a key role in this problem. Indeed, when

$$n \geq 10, \quad u = \log \frac{1}{|x|^2}, \quad \text{and} \quad f(u) = 2(n-2)e^u,$$

using Hardy's inequality one can show that u is a singular $W_0^{1,2}(B_1)$ stable solution of (1) in $\Omega = B_1$.

On the other hand, if $f(t) = e^t$ or $f(t) = (1+t)^p$ with $p > 1$, then it is well-known since the 1970's (thanks to Crandall and Rabinowitz) that $W_0^{1,2}(\Omega)$ stable solutions are bounded (and therefore smooth, by classical elliptic regularity theory) when $n \leq 9$.

All these results motivated the following long-standing:

Conjecture. *Let $u \in W_0^{1,2}(\Omega)$ be a stable solution to* (1). *Assume that f is positive, nondecreasing, convex, and superlinear at $+\infty$, and let $n \leq 9$. Then u is bounded.*

In the last 25 years, several attempts have been made in order to prove this result. In particular, partial positive answers to the conjecture above have been given (chronologically):

- by Nedev when $n \leq 3$ (2001);
- by Cabré and Capella when $\Omega = B_1$ and $n \leq 9$ (2006);
- by Cabré when $n = 4$ and Ω is convex (2010);
- by Villegas when $n = 4$ (2013);
- by Cabré and Ros-Oton when $n \leq 7$ and Ω is a convex domain "of double revolution" (2013);
- by Cabré, Sanchón, and Spruck when $n = 5$ and $\limsup_{t\to+\infty} \frac{f'(t)}{f(t)^{1+\epsilon}} < +\infty$ for every $\epsilon > 0$ (2016).

In the recent paper [1] we give a full proof of the conjecture stated above. Actually, as we shall see below, the interior boundedness of solutions requires no convexity or monotonicity of f. This fact was only known in dimension $n \leq 4$, by a result of Cabré. In addition, even more surprisingly, both in the interior and in the global settings we can prove that $W^{1,2}$ *stable solutions are universally bounded for $n \leq 9$*, namely they are bounded in terms only of their L^1 norm, with a constant that is independent of the nonlinearity f.

Our first main result deals with the interior problem. It suffices to prove an a priori estimate for classical solutions.

Theorem 1. *Let B_1 denote the unit ball of $\mathbb{R}^n$. Assume that $u \in C^2(B_1)$ is a stable solution of*

$$-\Delta u = f(u) \quad \textit{in } B_1,$$

with $f : \mathbb{R} \to \mathbb{R}$ locally Lipschitz and nonnegative. Then

$$\|\nabla u\|_{L^{2+\gamma}(B_{1/2})} \leq C\|u\|_{L^1(B_1)},$$

where $\gamma > 0$ and C are dimensional constants. In addition, if $n \leq 9$ then

$$\|u\|_{C^\alpha(\overline{B}_{1/2})} \leq C\|u\|_{L^1(B_1)},$$

where $\alpha > 0$ and C are dimensional constants.

Combining the previous interior bound with the moving planes method, one obtains a universal bound on u when Ω is convex.

Corollary 1. *Let $n \leq 9$ and let $\Omega \subset \mathbb{R}^n$ be any bounded convex C^1 domain. Assume that $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and nonnegative. Let $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ be a stable solution of*

$$\begin{cases} -\Delta u = f(u) & \textit{in } \Omega \\ \quad u = 0 & \textit{on } \partial\Omega. \end{cases}$$

Then there exists a constant C, depending only on Ω, such that

$$\|u\|_{L^\infty(\Omega)} \leq C\|u\|_{L^1(\Omega)}.$$

We now state our second main result, which concerns the global regularity of stable solutions in general C^3 domains when the nonlinearity is convex and nondecreasing. This result completely solves two open problems posed by Brezis and Brezis-Vázquez concerning the so-called Gelfand problem.

Again, we work with classical solutions and prove an a priori estimate. In this case it is crucial for us to assume f to be convex and nondecreasing. Indeed, the proof of regularity up to the boundary relies on a new and very general closedness result for stable solutions with convex nondecreasing nonlinearities.

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^3. Assume that $f : \mathbb{R} \to \mathbb{R}$ is nonnegative, nondecreasing, and convex. Let $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ be a stable solution of*

$$\begin{cases} -\Delta u = f(u) & \textit{in } \Omega \\ \quad u = 0 & \textit{on } \partial\Omega. \end{cases}$$

Then

$$\|\nabla u\|_{L^{2+\gamma}(\Omega)} \leq C\|u\|_{L^1(\Omega)},$$

where $\gamma > 0$ is a dimensional constant and C depends only on Ω. In addition, if $n \leq 9$ then

$$\|u\|_{C^\alpha(\overline{\Omega})} \leq C\|u\|_{L^1(\Omega)},$$

where $\alpha > 0$ is a dimensional constant and C depends only on Ω.

As an immediate consequence of such a priori estimates, we deduce the validity of the long-standing conjecture stated above.

Corollary 2. *Let $\Omega \subset \mathbb{R}^n$ be any bounded domain of class C^3. Assume that $f : \mathbb{R} \to \mathbb{R}$ is nonnegative, nondecreasing, convex, and satisfies*

$$\frac{f(t)}{t} \geq \sigma(t) \longrightarrow +\infty \quad as \quad t \to +\infty$$

for some function $\sigma : \mathbb{R} \to \mathbb{R}$. Let $u \in W_0^{1,2}(\Omega)$ be any stable weak solution of (1) *and assume that $n \leq 9$. Then*

$$\|u\|_{L^\infty(\Omega)} \leq C,$$

where C is a constant depending only on σ and Ω.

References

[1] X. Cabré, A. Figalli, X. Ros-Oton, J. Serra, *Stable solutions to semilinear elliptic equations are smooth up to dimension* 9, preprint, 2019.

Singularities for the Keller-Segel system in $\mathbb{R}^2$

Manuel del Pino
(joint work with J. Dávila, J. Dolbeault, M. Musso and J. Wei)

We consider the Keller-Segel system [7] in $\mathbb{R}^2$:

$$(1) \qquad \begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) & \text{in } \mathbb{R}^2 \times (0,T), \\ v = (-\Delta)^{-1} u := \dfrac{1}{2\pi} \displaystyle\int_{\mathbb{R}^2} \log \frac{1}{|x-z|} u(z,t)\, dz \\ u(\cdot, 0) = u_0 \geq 0 \quad \text{in } \mathbb{R}^2. \end{cases}$$

is a classical diffusion model for *chemotaxis*, the motion of a population of bacteria driven by standard diffusion and a nonlocal drift given by the gradient of a *chemoatractant*, a chemical produced by the bacteria. Population density is represented by $u(x,t)$ and the chemoattractant concentration at each point is computed as a weighted average for the population density.

It is well known that mass $M = \int_{\mathbb{R}^2} u(x,t)\, dx$ is constant along this flow and that solutions blows-up in finite time or approach zero as $t \to +\infty$ according to $M > 8\pi$ or $M < 8\pi$ [6, 1]. The following *second moment identity* holds:

$$\frac{d}{dt} \int_{\mathbb{R}^2} |x|^2 u(x,t)\, dx = 4M \Big(1 - \frac{M}{8\pi}\Big).$$

The most interesting regime is the threshold $M = 8\pi$ in which second moment is preserved (if finite). It is worth noticing that steady states with finite mass for (1) are given by the family of functions (u, v) with

$$-\Delta v = e^v = u \quad \text{in } \mathbb{R}^2,$$

the Liouville equation. All solutions with finite mass $\int_{\mathbb{R}^2} u < +\infty$ are well-known to have mass 8π and are given by

$$U_{\lambda,\xi}(x) = \lambda^{-2} U_0\left(\frac{x-\xi}{\lambda}\right), \quad U_0(x) = \frac{8}{(1+|x|^2)^2}.$$

These functions have infinite second moment. They are stable for the flow (1), see [4]. It is known that if the second moment is finite then the solution of (1) is defined at all times ($T = +\infty$) and it blows-up in infinite time as an asymptotically singular time-dependent scaling of a steady state [2]. We establish the following results.

Theorem 1. *There exists a function $u_0^*(x)$ with*

$$\int_{\mathbb{R}^2} u_0^*(x)\, dx = 8\pi, \quad \int_{\mathbb{R}^2} |x|^2 u_0^*(x)\, dx < +\infty$$

such that for any initial condition in (1) *that is a small perturbation of u_0^* and has mass 8π, the solution has the form*

$$\begin{aligned} u(x,t) &= \frac{1}{\lambda(t)^2} U_0\left(\frac{x-q}{\lambda(t)}\right) + o(1) \\ \lambda(t) &= \frac{1}{\sqrt{\log t}}\,(1+o(1)) \quad \text{as } t \to +\infty. \end{aligned}$$

This result has been established in the radial class by Ghoul and Masmoudi [5] with an entirely different approach, and answers the open question of stability of the phenomenon. The method applies to construct solutions that blow up in finite time simultaneously at several given points in the plane. The following result holds.

Theorem 2. *Given points $q_1, \dots, q_k \in \mathbb{R}^2$, there exists an initial condition $u_0(x)$ with*

$$\int_{\mathbb{R}^2} u_0(x) dx > 8k\pi,$$

such that the solution of (1) *satisfies for some $T > 0$*

$$\begin{aligned} u(x,t) &= \sum_{j=1}^{k} \frac{1}{\lambda_j(t)^2} U_0\left(\frac{x-q_j}{\lambda_j(t)}\right) + O(1) \\ \lambda_j(t) &= \beta_j (T-t)^{\frac{1}{2}} e^{-\frac{1}{2}\sqrt{|\log(T-t)|}}\,(1+o(1)). \end{aligned}$$

as $t \to T$.

The radial case was treated in [8]. A geometric flow with closely connected phenomena is the harmonic map flow from $\mathbb{R}^2$ into S^2,

$$u_t = \Delta u + |\nabla u|^2 u \text{ in } \mathbb{R}^2 \times (0,T)$$

where $u\colon \mathbb{R}^2 \times (0,T) \to S^2$, see [3].

References

[1] A. Blanchet, J. Dolbeault, B. Perthame. *Two-dimensional Keller-Segel model: optimal critical mass and qualitative properties of the solutions.* Electron. J. Differential Equations, pages No. **44**, 32 (2006).
[2] A. Blanchet; J. Carrillo; N. Masmoudi,*Infinite time aggregation for the critical Patlak-Keller-Segel model in $\mathbb{R}^2$*. Comm. Pure Appl. Math. **61** (2008), no. 10, 1449–1481.
[3] J. Dávila, M. del Pino, J. Wei, *Singularity formation for the two-dimensional harmonic map flow into S^2* Preprint ArXiv:1702.05801. To appear in Invent. Math.
[4] E. Carlen; A. Figalli, *Stability for a GNS inequality and the log-HLS inequality, with application to the critical mass Keller-Segel equation.* Duke Math. J. **162** (2013), no. 3, 579–625.
[5] T. Ghoul; N. Masmoudi, *Stability of infinite time blow up for the Patlak Keller Segel system.* Preprint arXiv:1610.00456 (2016).
[6] W. Jäger; S. Luckhaus, *On explosions of solutions to a system of partial differential equations modelling chemotaxis.* Trans. Amer. Math. Soc. **329** (1992), no. 2, 819–824.
[7] E. F. Keller; L. A. Segel, *Initiation of slide mold aggregation viewed as an instability.* J. Theor. Biol. **26** (1970), 399–415.
[8] P. Raphael, R. Schweyer, *On the stability of critical chemotactic aggregation.* Math. Ann. **359** (2014), no. 1–2, 267–377.

Ancient gradient flows of elliptic functionals and Morse index

Christos Mantoulidis
(joint work with Kyeongsu Choi)

The mean curvature flow is a one-parameter family of submanifolds Σ_t of a Riemannian manifold $(M, \overline{g})$ satisfying the evolution equation

$$\tfrac{\partial}{\partial t}x = \mathbf{H}(x,t),\ x \in \Sigma_t,$$

where $\mathbf{H}(x,t)$ denotes the mean curvature vector of the Σ_t at x, and which is the negative gradient of the area element of Σ_t. As a gradient flow of the area functional, the mean curvature flow describes a natural area minimizing process. In our work [10], we studied closed ancient solutions of the mean curvature flow in Riemannian manifolds; that is, flows of closed submanifolds Σ_t that exist for all $t \in (-\infty, T)$. (We also treated the general case of ancient gradient flows of elliptic functionals, but our strongest and most interesting geometric conclusions were for mean curvature flow.)

Ancient solutions of a gradient flow with uniformly bounded energy are quite rare, and their classification is generally studied as a type of parabolic Liouville theory. There have been a number of important classification results for ancient mean curvature flows inside Euclidean space under assumptions on the convexity or the entropy of the flow. See: X.-J. Wang [16], Huisken–Sinestrari [12], Daskalopoulos–Hamilton–Sesum [11], Angenent–Daskalopoulos–Sesum [2, 3], Brendle–Choi [5, 6], Choi–Haslhofer–Hershkovits [9]. Much less is known in the Riemannian setting. See: Bryan–Louie [8], Bryan–Ivaki–Scheuer [7], Huisken–Sinestrari [12].

In [10] we set up a framework for the characterization of certain types of ancient mean curvature flows in Riemannian manifolds as arising from the "unstable manifold" (i.e., the space of unstable directions for the area functional) of a given closed minimal submanifold. While the framework alone is of independent interest, we also applied it to classify ancient solutions in $\mathbf{S}^n$ under certain natural *area* assumptions:

Theorem 1. *There exists a $\delta = \delta(n) > 0$ such that if $(\Sigma_t)_{t\leq 0}$ is an ancient mean curvature flow of closed m-dimensional surfaces in a round n-sphere $\mathbf{S}^n$, with*

$$\lim_{t\to-\infty} \operatorname{Area}(\Sigma_t) < (1+\delta)\operatorname{Area}(\mathbf{S}^m),$$

then $(\Sigma_t)_{t\leq 0}$ is a steady or a canonically shrinking equatorial $\mathbf{S}^m$ along one of $n-m$ directions parallel to the equator.

We emphasize that our result holds true in *arbitrary codimension*, while the previously mentioned results were for codimension-1.

As a corollary to this theorem, we obtained a full classification of ancient embedded flows of curves with uniformly bounded length in $\mathbf{S}^2$ (and, similarly, that there are no nonsteady ancient embedded curve shortening flows with bounded length in flat tori or closed hyperbolic surfaces):

Corollary 1. *Let $(\Gamma_t)_{t\leq 0}$ be an ancient curve shortening flow of embedded closed curves with uniformly bounded length inside a round $\mathbf{S}^2$. Then $(\Gamma_t)_{t\leq 0}$ is a steady or a shrinking equator along circles of latitude.*

We also obtained a stronger classification in the 3-sphere, for which we need to recall the Clifford torus

$$\{(x,y,z,w) \in \mathbf{R}^2 \times \mathbf{R}^2 : x^2+y^2 = z^2+w^2 = \tfrac{1}{2}\} \subset \mathbf{S}^3.$$

This is a smoothly embedded minimal submanifold of $\mathbf{S}^3$ with area $2\pi^2$. By the resolution of the Willmore conjecture by Marques–Neves [13], this is the second smallest area among smooth minimal surfaces, following equatorial $\mathbf{S}^2$ that have area 4π. We showed:

Corollary 2. *Let $(\Sigma_t)_{t\leq 0}$ be an ancient mean curvature flow of closed surfaces in a round 3-sphere, with*

$$A_{-\infty} := \lim_{t\to-\infty} \operatorname{Area}(\Sigma_t) < (1+\delta)2\pi^2.$$

If $\delta > 0$ is sufficiently small, then either $A_{-\infty} = 4\pi$ and $(\Sigma_t)_{t\leq 0}$ is a steady or shrinking equator along spheres of latitude, or $A_{-\infty} = 2\pi^2$ and $(\Sigma_t)_{t\leq 0}$ is a steady or shrinking Clifford torus along a canonical 5-parameter family of ancient flows.

Our classification made use of the following new "canonical family existence" and "strong uniqueness" theorems, which we proved for general ancient gradient flows of elliptic functionals on closed Riemannian manifolds. (We recall that the Morse index of a minimal submanifold is the number of negative eigenvalues of its second variation operator. The Morse index of an equatorial $\mathbf{S}^m$ in $\mathbf{S}^n$ is $n-m$ and of a Clifford torus in $\mathbf{S}^3$ is 5.)

Theorem 2 (Canonical family existence)**.** *Let S be a closed, smoothly embedded minimal submanifold in a Riemannian manifold $(M, \overline{g})$. Let $I \in \mathbf{N}$ denote its*

Morse index. Then there exists an I-parameter family of ancient mean curvature flows on $(-\infty, 0]$ that converge exponentially to S as $t \to -\infty$, and are determined uniquely by their trace at time $t = 0$.

By a delicate analysis of the dynamics of an ancient flow using an ODE lemma of Merle–Zaag [14], we were able to obtain a strong characterization for ancient mean curvature flows:

Theorem 3 (Strong uniqueness). *Let S be a closed, smoothly embedded minimal submanifold of a Riemannian manifold $(M, \overline{g})$. There exists an $\varepsilon > 0$ such that if $(\Sigma_t)_{t \leq 0}$ is an ancient mean curvature flow which stays uniformly ε-close to S in the sense of measures, and*

$$\int_{-\infty}^{0} \operatorname{dist}_{\overline{g}}(\Sigma_t, S)\, dt < \infty,$$

then there exists $\tau \geq 0$ so that $(\Sigma_{t-\tau})_{t \leq 0}$ is one of the flows in the I-parameter canonical family.

We briefly remark that this canonical family existence theorem and strong uniqueness theorem do readily give the aforementioned classification results for area pinched ancient mean curvature flows in round n-spheres. Indeed, the compactness theorem of weak mean curvature flows of K. Brakke [4], a modification of the Łojasiewicz–Simon inequality (pioneered by L. Simon in [15]), and the "integrability" of equators and Clifford tori (a concept pioneered by Allard–Almgren [1]), put together guarantee that the decay rate assumption of the strong uniqueness theorem is satisfied, thus giving the necessary canonical classification.

References

[1] William K. Allard and Frederick J. Almgren, Jr. On the radial behavior of minimal surfaces and the uniqueness of their tangent cones. *Ann. of Math. (2)*, 113(2):215–265, 1981.

[2] Sigurd Angenent, Panagiota Daskalopoulos, and Natasa Sesum. Unique asymptotics of ancient convex mean curvature flow solutions. *J. Differential Geom.*, 111(3):381–455, 2019.

[3] Sigurd B. Angenent, Panagiota Daskalopoulos, and Natasa Sesum. Uniqueness of two-convex closed ancient solutions to the mean curvature flow. *arXiv e-prints*, page arXiv:1804.07230, Apr 2018.

[4] Kenneth A. Brakke. *The motion of a surface by its mean curvature*, volume 20 of *Mathematical Notes*. Princeton University Press, Princeton, N.J., 1978.

[5] S. Brendle and K. Choi. Uniqueness of convex ancient solutions to mean curvature flow in higher dimensions. *arXiv e-prints*, page arXiv:1804.00018, Mar 2018.

[6] Simon Brendle and Kyeongsu Choi. Uniqueness of convex ancient solutions to mean curvature flow in $\mathbb{R}^3$. *Invent. Math.*, 217(1):35–76, 2019.

[7] Paul Bryan, Mohammad N. Ivaki, and Julian Scheuer. On the classification of ancient solutions to curvature flows on the sphere. *arXiv e-prints*, page arXiv:1604.01694, April 2016.

[8] Paul Bryan and Janelle Louie. Classification of convex ancient solutions to curve shortening flow on the sphere. *J. Geom. Anal.*, 26(2):858–872, 2016.

[9] Kyeongsu Choi, Robert Haslhofer, and Or Hershkovits. Ancient low entropy flows, mean convex neighborhoods, and uniqueness. *arXiv e-prints*, page arXiv:1810.08467, Oct 2018.

[10] Kyeongsu Choi, and Christos Mantoulidis. Ancient gradient flows of elliptic functionals and Morse index. *arXiv e-prints*, page arXiv:1902.07697, March 2019.

[11] Panagiota Daskalopoulos, Richard Hamilton, and Natasa Sesum. Classification of compact ancient solutions to the curve shortening flow. *J. Differential Geom.*, 84(3):455–464, 2010.

[12] Gerhard Huisken and Carlo Sinestrari. Convex ancient solutions of the mean curvature flow. *J. Differential Geom.*, 101(2):267–287, 2015.

[13] Fernando C. Marques and André Neves. Min-max theory and the Willmore conjecture. *Ann. of Math. (2)*, 179(2):683–782, 2014.

[14] Frank Merle and Hatem Zaag. Optimal estimates for blowup rate and behavior for nonlinear heat equations. *Comm. Pure Appl. Math.*, 51(2):139–196, 1998.

[15] Leon Simon. Asymptotics for a class of nonlinear evolution equations, with applications to geometric problems. *Ann. of Math. (2)*, 118(3):525–571, 1983.

[16] Xu-Jia Wang. Convex solutions to the mean curvature flow. *Ann. of Math. (2)*, 173(3):1185–1239, 2011.

Classification of 3d κ-solutions

Natasa Sesum

(joint work with Sigurd Angenent, Simon Brendle, Panagiota Daskalopoulos)

Consider an ancient compact 3-dimensional solution to the Ricci flow

$$\frac{\partial}{\partial} g_{ij} = -2R_{ij} \tag{1}$$

existing for $t \in (-\infty, 0)$ so that it shrinks to a round point at T. The goal is to provide the classification of such solutions under natural geometric assumptions.

In [3], G. Perelman established the existence of a rotationally symmetric ancient κ-noncollapsed solution on S^3 which is not a soliton. This is a type II ancient solution backward in time, namely its scalar curvature satisfies $\sup_{M\times(-\infty,0)} |t||R(x,t)| = \infty$ and forms a type I singularity forward in time, since it shrinks to a round point. Perelman's ancient solution has backward in time limits which are the Bryant soliton and the round cylinder $S^2 \times \mathbb{R}$, depending on how the sequence of points and times about which one rescales are chosen. These are the only backward in time limits of the Perelman ancient solution. Let us remark that the Perelman ancient solution is *noncollapsed*.

The well known Hamilton-Ivey pinching estimate tells us that any two or three dimensional Ricci flow ancient solution, with bounded curvature at each time slice, has nonnegative sectional curvature. Since our solution $(S^3, g(t))$ is closed, the strong maximum principle implies that the sectional curvatures, and hence the entire curvature operator, are strictly positive. It follows by Hamilton's Harnack estimate that $R_t \geq 0$, yielding the existence of a uniform constant $C > 0$ so that $\|\mathrm{Rm}\|_{g(t)} \leq C$, for all $t \in (-\infty, t_0)$. The above discussion yields that any closed 3-dimensional κ-noncollapsed ancient solution is actually a *κ-solution*, in the sense that was defined by Perelman in [3].

In a joint work with Brendle and Daskalopoulos we show the following result that was conjectured by Perelman in [3].

Theorem 1 (Brendle, Daskalopoulos, Sesum)**.** *Let $(S^3, g(t))$ be a compact, ancient κ-noncollapsed solution to the Ricci flow* (1) *on S^3. Then $g(t)$ is either a family of contracting spheres or Perelman's solution.*

In a recent important paper by S. Brendle ([2]), the author proved that a 3-dimensional non-compact ancient κ-solution is isometric to either a family of shrinking cylinders or their quotients, or to the Bryant soliton. The author first shows that all 3-dimensional ancient κ-solutions which are non-compact have to be rotationally symmetric. After that he shows that such a rotationally symmetric solution, if not a cylinder or its quotient, must be a steady Ricci soliton and hence the Bryant soliton by one of his earlier works about classification of steady Ricci solitons.

The techniques of Brendle in [2] can be also applied to show rotation symmetry of ancient compact and κ-noncollapsed solution to the Ricci flow (1) on S^3. However, since the rotationally symmetric solutions discovered by Perelman are not solitons, the classification of rotationally symmetric ancient compact and κ-noncollapsed solutions is a difficult problem.

We may assume that our 3d κ solution is rotationally symmetric. By the work of Perelman we know that the asymptotic soliton of $(S^3, g(t))$ is either a round cylinder or a sphere. We can understand this that every κ-solution has a gradient shrinking soliton buried inside of it, in an asymptotic sense as time approaches $-\infty$ (for more details on asymptotic solitons see [3]). In our recent work [1] we show that if the asymptotic soliton is the sphere, then the solution $(S^3, g(t))$ must be the round sphere itself. Hence, from now on *we may assume that the asymptotic soliton of our closed κ-noncollapsed solution is the round cylinder $S^2 \times \mathbb{R}$.*

Since at each time slice, the metric is $SO(3)$-invariant, it can can be written as

$$g = \phi^2 \, dx^2 + \psi^2 \, g_{can}, \qquad \text{on } (-1,1) \times S^2$$

where $(-1,1) \times S^2$ may be naturally identified with the sphere S^3 with its north and south poles removed. The function $\psi(x,t) > 0$ may be regarded as the radius of the hypersurface $\{x\} \times S^2$ at time t. The distance function from the equator is given by

$$s(x,t) = \int_0^x \phi(x', t) \, dx'.$$

and abbreviating $ds = \phi(x,t) \, dx$, we write our metric as $g = ds^2 + \psi^2 \, g_{can}$.

Under the Ricci flow, the profile function $\psi\colon (s_-(t), s_+(t)) \times (-\infty, 0) \to \mathbb{R}$ evolves by

$$\psi_t = \psi_{ss} - \frac{1 - \psi_s^2}{\psi}. \tag{2}$$

Consider next a type I scaling of our metric, which leads to the rescaled profile $u(\sigma, \tau)$ defined by

$$u(\sigma, \tau) := \frac{\psi(s,t)}{\sqrt{-t}}, \qquad \text{with } \sigma := \frac{s}{\sqrt{-t}}, \ \tau = -\log(-t). \tag{3}$$

A direct calculation shows that $u : (\sigma_-(\tau), \sigma_+(\tau)) \times (-\infty, 0) \to \mathbb{R}$ satisfies the equation

$$u_\tau = u_{\sigma\sigma} + \frac{u_\sigma^2}{u} - \frac{1}{u} + \frac{u}{2} \tag{4}$$

with boundary conditions at the tips $u_\sigma(\sigma_-(\tau), \tau) = 1$, $u_\sigma(\sigma_+(\tau), \tau) = -1$.

A crucial first step in showing Theorem 1, in the case of rotational symmetry is to establish the (unique up to scaling) asymptotic behavior of any compact rotationally symmetric κ-noncollased ancient solution to the Ricci flow on S^3 which is not isometric to a sphere. This was recently established by the authors in [1] and is summarized in the next theorem.

Theorem 2 (Angenent, Daskalopoulos, Sesum in [1]). *Let $(S^3, g(t))$ be any reflection and rotationally symmetric compact κ-noncollapsed ancient solution to the Ricci flow on S^3 which is not isometric to a round sphere. Then the rescaled profile $u(\sigma, \tau)$ solution to (4) has the following asymptotic expansions:*

(i) *For every $L > 0$,*

$$u(\sigma,\tau) = \sqrt{2}\Big(1 - \frac{\sigma^2 - 2}{8|\tau|}\Big) + o(|\tau|^{-1}), \qquad \text{on } |\sigma| \le L$$

as $\tau \to -\infty$.

(ii) *Define $z := \sigma/\sqrt{|\tau|}$ and $\bar{u}(\sigma,\tau) := u(z\sqrt{|\tau|}, \tau)$. Then,*

$$\lim_{\tau \to -\infty} \bar{u}(z,\tau) = \sqrt{2 - \frac{z^2}{2}}$$

uniformly on compact subsets of $|z| < 2$.

(iii) *Let $k(t) := R(p_t, t)$ be the maximal scalar curvature which is attained at each one of the two tips p_t, for $t \ll -1$. Then the rescaled Ricci flow solutions $(S^3, \bar{g}_t(s), p_t)$, with $\bar{g}_t(\cdot, s) = k(t)\, g(\cdot, t + k(t)^{-1}\, s)$, converge to the unique Bryant translating soliton with maximal scalar curvature one. Furthermore, $k(t)$ and the diameter $d(t)$ satisfy the asymptotics*

$$k(t) = \frac{\log|t|}{|t|}(1 + o(1)) \quad \text{and} \quad d(t) = 4\sqrt{|t|\log|t|}\,(1 + o(1))$$

as $t \to -\infty$.

In the previous theorem, the authors jointly with Brendle removed the assumption on reflection symmetry and this is in process of writing.

In the proof of Theorem 1, in showing that every 3d κ-solution has to be rotationally symmetric, some of the crucial ingredients were: Perelman's Canonical Neighborhood Theorem for ancient κ-solutions, classification of κ-noncollapsed steady gradient Ricci solitons and the Neck Improvement Theorem which assers that a neck becomes more and more rotationally symmetric under evolution. Starting point in the proof of rotational symmetry is showing that any 3d κ-solution roughly looks like a neck, which is on both sides capped off with a Bryant soliton.

References

[1] S. Angenent, P. Daskalopoulos, N. Sesum *Unique asymptotics of ancient compact non-collapsed solutions to the 3-dimensional Ricci flow.*

[2] S. Brendle, *Ancient solutions to the Ricci flow in dimension three*; arXiv: 1811.02559.

[3] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*; arXiv: 0211159.

Minimizers of strictly convex functionals

Connor Mooney

A central problem in the calculus of variations is to determine the regularity of Lipschitz minimizers of $\int_{B_1} F(\nabla u)\,dx$, where $F : \mathbb{R}^n \to \mathbb{R}$ is convex. Such integrals arise in many applications, e.g. in models of crystal surfaces and traffic congestion. Minimizers solve the Euler-Lagrange equation $\operatorname{div}(\nabla F(\nabla u)) = 0$ in the distribution sense. In non-divergence form we have $F_{ij}(\nabla u)u_{ij} = 0$. The equation is quasilinear, elliptic, and degenerate elliptic if the eigenvalues of D^2F tend to zero or infinity in the image of ∇u. Furthermore, if ∇u is continuous, then the equation has nearly constant coefficients on small scales. Thus, proving C^1 regularity for solutions is a key step towards understanding their higher regularity properties. Let F^* denote the Legendre transform of F. The relation $\operatorname{div}(\nabla F(\nabla F^*)) = n$ suggests that minimizers have the same regularity as F^*. Important examples support a positive answer, including $F = |x|^2$ (Laplace equation), $F = |x|^p$ (p-Laplace equation, for which the positive answer is known when $p > 2$ and $n = 2$), and the case that F is linear on a line segment (the gradient of a minimizer can oscillate in the segment). Since F^* is C^1 when F is strictly convex, we ask:

If F is strictly convex, are Lipschitz minimizers C^1?

Assume now that $F \in C^2(\mathbb{R}^n \backslash D_F)$, where D_F is compact and $D^2F > 0$ away from D_F. When $D_F = \emptyset$, Lipschitz minimizers are C^1 by a fundamental theorem of De Giorgi and Nash ([3], [7]). When D_F consists of a single point, minimizers are C^1 by the techniques developed for the p-Laplace equation, which was studied by Uraltseva, Uhlenbeck, Evans, Lewis, Tolksdorf, and many others. Finally, when $n = 2$ and D_F is finite, minimizers are C^1 by work of De Silva-Savin [4]. We investigated the problem in higher dimensions. Our main results are [6]:

Theorem 1. *The function $u(z_1, z_2) = (|z_1|^2 - |z_2|^2)/(|z_1|^2 + |z_2|^2)^{1/2}$ on $\mathbb{C}^2 \cong \mathbb{R}^4$ is a Lipschitz singular minimizer of a functional of the type $\int_{B_1} F(\nabla u)\,dx$, where F is uniformly convex, D_F is the Clifford torus $\mathbb{S}^3 \cap \{|z_1| = |z_2|\}$, and exactly one eigenvalue of D^2F tends to infinity on D_F.*

Theorem 2. *Assume that $n \geq 2$ and that D_F is finite and contained in some 2-plane (e.g. three points). Then Lipschitz minimizers are C^1.*

Theorem 1 gives a negative answer to the question above, at least in dimension $n \geq 4$. It also shows that that high codimension of the set D_F does not prevent singularities. The example exploits the connection between the differential geometry of hypersurfaces in $\mathbb{R}^n$ with injective Gauss map, and the Hessians of their (one-homogeneous) support functions. Theorem 2 can be viewed as a generalization of the result of De Silva-Savin to higher dimensions. The first step in the proof is to show that ∇u localizes to the convex hull of D_F, using the well-known fact that convex functions of ∇u are sub-solutions to the linearized Euler-Lagrange equation. Colombo-Figalli used this observation to study minimizers when F vanishes on a convex set [2]. When D_F lies in a two-plane, we can then use a higher-dimensional version of the key observation by De Silva-Savin that slightly non-convex functions of ∇u are sub-solutions to the linearized equation.

Theorems 1 and 2 leave several interesting questions open. We list two:

Problem 1. *Prove or disprove that Lipschitz minimizers are C^1 in dimension $n \geq 3$ when D_F consists of finitely many points.*

Problem 2. *Prove or disprove that Lipschitz minimizers are C^1 in dimension $n = 2$ when F is strictly convex.*

Natural candidates for singular minimizers are support functions of hypersurfaces with injective Gauss map that are saddle-shaped away from their singularities. Alexandrov proved that the only such functions on $\mathbb{R}^3$ that are *analytic* outside the origin are linear [1]. Thus, this approach cannot produce singular examples in $\mathbb{R}^3$ that are analytic outside the origin (unlike in $\mathbb{R}^4$, see Theorem 1). Alexandrov also conjectured that his result should hold in the C^2 setting. However, in 2001, Martinez-Maure constructed a striking counterexample [5]. The example consists of four "cross-caps" with figure-eight cross sections that shrink to cusps, glued together so that the cusps form four non-coplanar points. We conjecture:

Conjecture 1. *The support function of the example in* [5] *is a Lipschitz singular minimizer to a functional of the type $\int_{B_1} F(\nabla u)\,dx$, where D_F consists exactly of the four non-coplanar cusps in the example.*

Confirming Conjecture 1 would give a negative answer to Problem 1, and show that Theorem 2 is optimal. Apart from the result of De Silva-Savin, the case $n = 2$ (Problem 2) remains mysterious.

References

[1] Alexandrov, A. D. On uniqueness theorem for closed surfaces. *Doklady Akad. Nauk. SSSR* **22** (1939), 99-102.

[2] Colombo, M.; Figalli, A. Regularity results for very degenerate elliptic equations. *J. Math. Pures Appl.* (9) **101** (2014), no. 1, 94-117.

[3] De Giorgi, E.: Sulla differenziabilità e l'analicità delle estremali degli integrali multipli regolari. *Mem. Accad. Sci. Torino cl. Sci. Fis. Fat. Nat.* 3, 25-43 (1957).

[4] De Silva, D.; Savin O. Minimizers of convex functionals arising in random surfaces. *Duke Math. J.* **151**, no. 3 (2010), 487-532.

[5] Martinez-Maure, Y. Contre-exemple à une caractérisation conjecturée de la sphère. *C. R. Acad. Sci. Paris* **332** (2001), 41-44.

[6] Mooney, C. Minimizers of convex functionals with small degeneracy set. Preprint 2019, arXiv:1903.03103. Submitted.

[7] Nash, J. Continuity of solutions of parabolic and elliptic equations. *Amer. J. Math.* 80, 931-954 (1958).

Prescribing scalar curvature in high dimensions

ANDREA MALCHIODI
(joint work with Martin Mayer)

We deal with the classical problem of prescribing the scalar curvature of closed manifolds, whose study initiated systematically in papers by Kazdan and Warner from the 70's: we will consider in particular *conformal* changes of the metric. On (M^n, g_0), with $n \geq 3$, using the following convenient notation for a metric g conformal to g_0, $g = g_u = u^{\frac{4}{n-2}} g_0$ (with u positive smooth function on M), the scalar curvature transforms according to the formula

$$(1) \qquad R_{g_u} u^{\frac{n+2}{n-2}} = L_{g_0} u := -c_n \Delta_{g_0} u + R_{g_0} u; \qquad c_n = \frac{4(n-1)}{(n-2)},$$

where Δ_{g_0} is the Laplace-Beltrami operator of g_0. The elliptic operator L_{g_0} is known as the *conformal Laplacian.* If one wishes to prescribe the scalar curvature of M as a given function $K : M \to \mathbb{R}$, by (1) one would then need to find positive solutions of the nonlinear elliptic problem

$$(2) \qquad L_{g_0} u = K u^{\frac{n+2}{n-2}} \qquad \text{on} \qquad (M, g_0).$$

When K is zero or negative (in which case (M, g_0) has to be of zero or negative *Yamabe class*, respectively) the nonlinear term in the equation makes the Euler-Lagrange energy for (2) more coercive and solutions always exist, as proved by Kazdan-Warner via the method of sub- and super-solutions. They also showed that for K positive there are obstructions to existence: indeed, if $f : S^n \to \mathbb{R}$ is the restriction to the sphere of a coordinate function in $\mathbb{R}^{n+1}$, then one has

$$(3) \qquad \int_{S^n} \langle \nabla K, \nabla f \rangle_{g_{S^n}} u^{\frac{2n}{n-2}} \, d\mu_{g_{S^n}} = 0,$$

for all solutions u to (2). This forbids e.g. the prescription of affine functions, or in general of functions K on S^n that are monotone in one Euclidean direction.

A general existence result was proved in [1] for the case of S^3, assuming that $K \colon S^3 \to \mathbb{R}_+$ is a Morse function satisfying the generic condition

$$(4) \qquad \{\nabla K = 0\} \cap \{\Delta K = 0\} = \emptyset,$$

together with the *index formula*

$$(5) \qquad \sum_{\{x \in M \,:\, \nabla K(x) = 0, \Delta K(x) < 0\}} (-1)^{m(K,x)} \neq (-1)^n,$$

where $m(K, x)$ stands for the Morse index of K at x (see also [3], [4] for the two-dimensional case).

Theorem 1 ([7]). *Suppose (M^n, g_0) is an Einstein manifold of positive Yamabe class with $n \geq 5$, and that K is a positive Morse function on M satisfying* (4). *Assume we are in one of the following two situations:*

i): *K satisfies $\frac{K_{\max}}{K_{\min}} \leq 2^{\frac{1}{n-2}}$ and* (5)*;*

ii): *K satisfies $\frac{K_{\max}}{K_{\min}} \leq \left(\frac{3}{2}\right)^{\frac{1}{n-2}}$ and has at least two critical points with negative Laplacian.*

Then (2) *has a positive solution.*

The restriction on the dimension in the above theorem is optimal: building on some non-existence result in [8] for the Nirenberg's problem on S^2, it is possible to manufacture curvature functions on S^3 and on S^4 such that, under condition **ii)** (and arbitrarily pinched, indeed), problem (2) has no solution.

The proof of Theorem 1 uses a subcritical approximation of (2). In [5] it was proven that solutions of such approximation with uniformly bounded energy and zero weak limit develop only *isolated simple blow-ups* at critical points of K with negative Laplacian. For $n = 3$, this holds true without assuming any energy bound. In [6] blowing-up solutions of this type were constructed, and their Morse index was determined. Such results, together with the pinching condition and a rigidity result from [2] allowed to prove the theorem by an index-counting formula.

One may wonder whether stronger pinching assumptions might induce existence under weaker conditions than the second one in **ii)**. In view of the Kazdan-Warner obstruction it is tempting to think that when $n \geq 5$ and when $K : S^n \to \mathbb{R}_+$ has more than just one local maximum and minimum, solutions may always exist. We show that in fact this is not the case, and that critical points of K with positive Laplacian are less relevant. For K Morse on S^n we define

$$\mathcal{M}_j(K) = \sharp\{x \,:\, \nabla K(x) = 0, m(K,x) = j\} : \tag{6}$$

we have then the following result.

Theorem 2 ([7])**.** *For any Morse function $\tilde{K}\colon S^n \to \mathbb{R}_+$ ($n \geq 3$) with only one local maximum point, there exists a Morse function $K\colon S^n \to \mathbb{R}$ such that $\mathcal{M}_j(\tilde{K}) = \mathcal{M}_j(K)$ for all j, with positive Laplacian at all critical points of K with the exception of its local maximum, and such that there is no conformal metric on S^n with scalar curvature K. The function K can be chosen so that $\frac{K_{\max}}{K_{\min}}$ is arbitrarily close to* 1*.*

References

[1] A. Bahri, J. M. Coron, *The Scalar-Curvature problem on the standard three-dimensional sphere*, Journal of Functional Analysis, 95 (1991), 106–172.

[2] M. F. Bidaut-Véron, L. Véron, *Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations.* Invent. Math. 106 (1991), no. 3, 489–539.

[3] S. A. Chang, P. Yang, *Prescribing Gaussian curvature on S^2*, Acta Math. 159 (1987), 215–259.

[4] S. A. Chang, P. Yang, *Conformal deformation of metrics on S^2*, J. Diff. Geom. 27 (1988), 256–296.

[5] A. Malchiodi, M. Mayer, *Prescribing Morse scalar curvature: blow-up analysis*, arxiv:1812.09457.

[6] A. Malchiodi, M. Mayer, *Prescribing Morse scalar curvatures: subcritical blowing-up solutions*, arXiv:1812.09461.

[7] A. Malchiodi, M. Mayer, *Prescribing Morse scalar curvature: pinching and Morse theory*, to appear.

[8] M. Struwe, *A flow approach to Nirenberg's problem.* Duke Math. J. 128 (2005), no. 1, 19–64.

On the logarithmic epiperimetric inequality

Bozhidar Velichkov
(joint work with Luca Spolaor and Maria Colombo)

This talk is focused on the so-called *logarithmic epiperimetric inequality*, which is a new variational technique for the study of the singular part of the free boundary. It was first introduced in [3] in the context of the classical obstacle problem, but the approach was then refined and applied to several different free boundary problems: the thin-obstacle problem [4, 5], the Bernoulli (Alt-Caffarelli) one-phase problem [6], (almost-)area-minimizing surfaces [7], and parabolic obstacle problems [12]). The aim of this talk is to give a brief overview of this new tool and to give a new proof of the log-epiperimetric inequality for the obstacle problem, which is based on ideas from [13], [6] and [5].

1. The obstacle problem

Given a domain $D \subset \mathbb{R}^d$, an obstacle $\varphi : D \to \mathbb{R}$, and a boundary datum $g : \partial D \to \mathbb{R}$ ($g \geq \varphi$ on ∂D), the classical obstacle problem is the following:

$$\min\Big\{ \int_D |\nabla w|^2 \, dx \ : \ w = g \text{ on } \partial D \ \text{ and } \ w \geq \varphi \text{ in } D \Big\}.$$

Setting $u := w - \varphi$ and $f := 2\Delta\varphi$, we can rewrite the above problem as

$$\min\Big\{ \int_D \Big(|\nabla u|^2 + fu\Big) \, dx \ : \ u = g - \varphi \text{ on } \partial D \ \text{ and } \ u \geq 0 \text{ in } D \Big\}.$$

For the sake of simplicity we take $D = B_1$ and $f \equiv 1$, and we set $\Omega_u := \{u > 0\}$.

1.1. **Blow-up limits.** If $r_n \to 0$ and $x_0 \in \partial\Omega_u \cap B_1$, then the sequence of functions $u_{r_n,x_0}(x) := r_n^{-2} u(x_0 + r_n x)$ is called a blow-up sequence. It is well-known that, up to a subsequence, every blow-up sequence u_{r_n,x_0} converges (in $C^{1,\alpha}(B_R)$, for every $R > 0$) to a *blow-up limit* $u_0 : \mathbb{R}^d \to \mathbb{R}$. In [2] Caffarelli showed that the free boundary $\partial\Omega_u \cap B_1$ can be decomposed into a regular and a singular part according to the type of blow-up limits at each point. Precisely, we have that $\partial\Omega_u \cap B_1 = \text{Reg}(\partial\Omega_u) \cup \text{Sing}(\partial\Omega_u)$, where:

- $x_0 \in \text{Reg}(\partial\Omega_u)$ if every blow-up limit of u at x_0 is of the form $h_\nu(x) = \frac{1}{4}\big(\sup\{0, x \cdot \nu\}\big)^2$, for some unit vector $\nu \in \mathbb{R}^d$.
- $x_0 \in \text{Sing}(\partial\Omega_u)$ if every blow-up limit of u at x_0 is of the form $Q_A(x) = x \cdot Ax$, for a symmetric positive matrix A with $trace(A) = \frac{1}{4}$.

1.2. **Structure of** $\text{Reg}(\partial\Omega_u)$ **and** $\text{Sing}(\partial\Omega_u)$**.** It is well-known that $\text{Reg}(\partial\Omega_u)$ is an analytic manifold (see [1, 2]) The singular set, on the other hand, is well understood only in dimension two; in dimension $d \geq 3$, the following does hold:

- At every $x_0 \in \text{Sing}$ the blow-up is **unique** (Caffarelli, [2]): $u_{x_0} = Q_{A_{x_0}}$.
- Every $x_0 \in \text{Sing}$ has a **rank**: $\text{Rank}\,(x_0) = \dim \text{Ker}\, A_{x_0}$.
- The m-th **stratum** Σ_m of Sing is defined as: $x_0 \in \Sigma_m \Leftrightarrow \text{Rank}\,(x_0) = m$.

Caffarelli proved in [2] that Σ_m is locally contained in an m-dimensional C^1-regular manifold. In [3] we proved that the regularity of Σ_m can be improved to $C^{1,\log}$. Recently, in [8], Figalli and Serra showed that the logarithmic modulus of continuity is, in general, optimal, but can be (significantly) improved outside a set of zero m-Hausdorff measure. In all these result, the regularity of Σ_m is inherited from the rate of convergence of the sequence u_{r_n,x_0} to the blow-up limit u_{x_0}. The aim of this talk is to prove the $C^{1,\log}$ regularity of Σ_m.

2. (Log-)epiperimetric inequality

In [11], Reifenberg introduced a new *epiperimetric inequality* approach to the regularity of the area-minimizing surfaces, which found several applications in geometric analysis (see for instance [17] and [15]). Weiss was the first to apply this technique to a free boundary problem in [16], precisely for the obstacle problem. The Weiss' approach was then used by Focardi and Spadaro [9] and Garofalo-Petrosyan-Vega-Garcia [10] in the context of the thin-obstacle problem. In [13], together with Luca Spolaor, we proved for the first time an epiperimetric inequality for the one-phase Bernoulli (Alt-Caffarelli) problem. Later on, we used the technique from [13] in the context of the obstacle problem, which led to the *logarithmic epiperimetric inequality* (see [3]), which is also the topic of this talk.

We briefly describe the Reifenberg epiperimetric inequality approach in the context of the obstacle problem. For any function $u : B_1 \to \mathbb{R}$, we set

$$W(u) = \int_{B_1} |\nabla u|^2 \, dx - 2 \int_{\partial B_1} u^2 \, d\mathcal{H}^{d-1} + \int_{B_1} u \, dx,$$

and $\mathcal{E}(u) := W(u) - W(Q_A)$, where Q_A is any singular blow-up limit. Weiss showed in [16] that, if u solves the obstacle problem and $x_0 \in \mathrm{Sing}(\partial\Omega_u)$, then $\mathcal{E}(u_{r,x_0})$ is non-negative and monotone in r, and $\lim_{r\to 0} \mathcal{E}(u_{r,x_0}) = 0$. The key observation of Reifenberg is the following.

Lemma 1 (Reifenberg). *If there is $\varepsilon > 0$ such that the following epiperimetric inequality holds, for every $r > 0$:*

$$\mathcal{E}(u_{x_0,r}) - \mathcal{E}(z_{x_0,r}) \leq -\varepsilon\, \mathcal{E}(z_{x_0,r}), \tag{1}$$

where $z_{x_0,r} : B_1 \to \mathbb{R}$ is the two-homogeneous extension of $u_{x_0,r} : \partial B_1 \to \mathbb{R}$ in B_1, then both the energy $\mathcal{E}(u_{x_0,r})$ and the convergence rate $\|u_{x_0,r} - u_{x_0}\|_{L^2(\partial B_1)}$ are controlled (from above) by r^α, where $\alpha \simeq \varepsilon$.

In the last part of the talk we try to prove an epiperimetric inequality for the energy $\mathcal{E}$. Precisely, starting from a two-homogeneous function $z : B_1 \to \mathbb{R}$, we construct a competitor $h : B_1 \to \mathbb{R}$, which is non-negative, has the same trace (on ∂B_1) as z and satisfies the epiperimetric inequality

$$\mathcal{E}(h) - \mathcal{E}(z) \leq -\varepsilon\, \mathcal{E}(z), \tag{2}$$

which trivially implies (1). Notice that, as a consequence of the recent result [9] and the Reifenberg Lemma, this cannot be true for any z, otherwise the logarithmic modulus of continuity for the singular set wouldn't be optimal. In fact, we try to find at once a good competitor h for z and an estimate for $\mathcal{E}(h) - \mathcal{E}(z)$. Our starting point is the solution of $\Delta h = \frac{1}{2}$ in B_1 with $h = z$ on ∂B_1, which is the best competitor for the energy $\mathcal{E}$. Since this competitor might not satisfy the non-negativity constraint, we suitably modify it and we construct another (non-negative) competitor h, which turns out to satisfy the estimate

$$\mathcal{E}(h) - \mathcal{E}(z) \leq -\mathcal{E}(z)^{1+\gamma} \quad \text{for some} \quad \gamma \in (0,1). \tag{3}$$

Finally, we notice that a suitable adaptation of the Reifenberg lemma implies that, if there is $\gamma > 0$ such that the following *log-epiperimetric inequality* holds

$$\mathcal{E}(u_{x_0,r}) - \mathcal{E}(z_{x_0,r}) \leq -\mathcal{E}(z_{x_0,r})^{1+\gamma} \quad \text{for every} \quad r > 0, \tag{4}$$

then both the energy $\mathcal{E}(u_{x_0,r})$ and the convergence rate $\|u_{x_0,r} - u_{x_0}\|_{L^2(\partial B_1)}$ are controlled (from above) by $|\log r|^{-\alpha}$, where $\alpha \simeq \gamma^{-1}$. In particular, this proves that each stratum Σ_m of the singular part of the free boundary is contained in a $C^{1,\log}$-regular manifold.

References

[1] L. Caffarelli, *The regularity of free boundaries in higher dimensions*, Acta Math. **139** (1977).

[2] L. Caffarelli, *The obstacle problem revisited*, J. Fourier Anal. Appl. **4** (4-5) (1998), 383–402.

[3] M. Colombo, L. Spolaor, B. Velichkov, *A logarithmic epiperimetric inequality for the obstacle problem*, Geom. Funct. Anal. **28** (4) (2018), 1029–1061.

[4] M. Colombo, L. Spolaor, B. Velichkov, *Direct epiperimetric inequalities for the thin obstacle problem and applications*, Comm. Pure Appl. Math., to appear.

[5] M. Colombo, L. Spolaor, B. Velichkov, *On the asymptotic behavior of the solutions to parabolic variational inequalities*, Preprint arXiv:1809.06075

[6] M. Engelstein, L. Spolaor, B. Velichkov, *Uniqueness of the blow-up at isolated singularities for the Alt-Caffarelli functional*, Preprint arXiv:1801.09276.

[7] M. Engelstein, L. Spolaor, B. Velichkov : *(Log-)epiperimetric inequality and regularity over smooth cones for almost area-minimizing currents*, Geometry & Topology, to appear.

[8] A. Figalli, J. Serra, *On the fine structure of the free boundary for the classical obstacle problem*, Invent. Math. **215** (1) (2019), 311–366.

[9] M. Focardi, E. Spadaro, *An epiperimetric inequality for the thin obstacle problem*, Adv. Diff. Eq. **21** (2016), 153–200.

[10] N. Garofalo, A. Petrosyan, M. S. Vega Garcia, *An epiperimetric inequality approach to the regularity of the free boundary in the Signorini problem with variable coefficients*, J. Math. Pures Appl., **105** (6) (2016), 745–787.

[11] E. R. Reifenberg, *An epiperimetric inequality related to the analyticity of minimal surfaces*, Ann. of Math. **80** (1964), 1–14.

[12] W. Shi, *An epiperimetric inequality approach to the parabolic Signorini problem*, Preprint ArXiv:1810.11791 (2018).

[13] L. Spolaor, B. Velichkov, *An epiperimetric inequality for the regularity of some free boundary problems: the 2-dimensional case*, Comm. Pure Appl. Math. **72** (2) (2018), 375–421.

[14] L. Spolaor, B. Trey, B. Velichkov, *Free boundary regularity for a multiphase shape optimization problem*, Comm. PDE, to appear.

[15] J. E. Taylor, *The structure of singularities in soap-bubble-like and soap-film-like minimal surfaces*, Ann. of Math. **103** (3) (1976), 489–539.

[16] G. S. Weiss, *A homogeneity improvement approach to the obstacle problem*, Invent. Math.**138** (1) (1999), 23–50.

[17] B. White, *Tangent cones to two-dimensional area-minimizing integral currents are unique*, Duke Math. J. **50** (1) (1983), 143–160.

Gluig methods for vortices in Euler equations

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(joint work with Juan Dávila, Manuel del Pino, Juncheng Wei)

A fascinating field within the theory of nonlinear PDEs is the analysis of the motion of fluids. The Euler equations (1755) define a system of non-linear PDEs that models the dynamics of an inviscid, incompressible fluid confined to a region Ω in $\mathbb{R}^d$, $d \geq 2$,

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p, \quad \nabla \cdot u = 0 \tag{E}$$

where $u(x,t)$ and $p(x,t)$ represent the vector-valued velocity field and pressure. For a solution $u(x,t)$ its vorticity is defined as $\omega = \nabla \times u$.

If $d = 2$, a classical problem is the *desingularized N-vortex problem*, namely the existence of true smooth solutions of Euler equations with highly concentrated vorticities around N points. Let $W(y) = \frac{8}{(1+|y|^2)^2}$ be the standard Liouville bubble which has mass 8π and, up to translations and scalings, is the only finite-mass solution to equation $-\Delta \log W = W$. We show [1] that, for any sufficiently small $\varepsilon > 0$ there exists a solution u_ε to (E) with vorticity of the form $\omega_\varepsilon(x,t) \approx \sum_{j=1}^N \frac{\kappa_j}{\varepsilon^2} W(\frac{x-\xi_j(t)}{\varepsilon})$ where $\kappa_j \in \mathbb{R}$ and the centers $\xi_j(t)$ solve the Kirchoff-Routh law of motion. This result is obtained by using the inner-outer gluing scheme, which provides explicit and precise information also on the velocity fields, $\frac{1}{|\log \varepsilon|}|u_\varepsilon(\cdot,t)|^2 \rightharpoonup \sum_{j=1}^N 8\pi\kappa_j^2 \delta_{\xi_j(t)}$, as $\varepsilon \to 0$. This refines a previous construction by Marchioro and Pulvirenti [4].

If $d = 3$ and the initial vorticity is concentrated along a smooth curve in space, a long standing question is whether the associated solution exhibits a vorticity still very concentrated around a curve on finite times. It is convenient to scale time in the $3d$ Euler equations in vorticity form

$$|\log \varepsilon| \frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u, \qquad u = \nabla \times \psi, \qquad \psi = (-\Delta)^{-1}\omega.$$

The formal derivation of the motion of the curve was first computed by Da Rios in 1903, and it approximately evolves by the *bi-normal flow of curves* described by the equation $\frac{\partial \gamma}{\partial t} = \frac{\partial \gamma}{\partial s} \times \frac{\partial^2 \gamma}{\partial s^2}$, where $s \mapsto \gamma(s,t)$ is a parametrisation by arclength of the curve at time $t \in [0,T)$, called *the filament*. Jerrard and Seis [3] used refined energy estimates to prove the validity of the asymptotic law under the assumption that vorticity is indeed concentrated at all time. The big open problem is whether one can find solutions of the Euler equations for which the vorticity remains close for a significant period of time to a filament evolving by binormal flow.

In [2], we prove that this is the case when the curve is an helix evolving by bi-normal flow. Using helical symmetries and looking for rotating solutions, this problem can be reduced to finding a concentrating solution for a $2d$ elliptic problem in divergence form.

In [2], we also show that, given a curve $\gamma = \gamma(s,t)$ evolving by bi-normal flow, for any k there exist an approximate solution for which the largest term in the error is of size $O(\frac{\varepsilon}{|\log \varepsilon|^k})$. In a tiny tubular neighborhood of the curve γ consider an ansatz of the form $\omega(x,t) \approx \frac{1}{\varepsilon^2} W(\frac{x-p}{\varepsilon})\, \hat{t}_p$, where p is the closest point on the curve $\Gamma(t)$ to x and $\hat{t}_p$ a corresponding unit tangent vector to $\Gamma(t)$. On each plane orthogonal to the curve, the approximation is a two-dimensional vortex. The largest term in the error, of $\varepsilon|\log \varepsilon|$ size, gets automatically improved to order ε thanks to the bi-normal flow for the curve. Through a combination of elliptic and transport equations we get successive improvements in power of $|\log \varepsilon|^{-1}$ of the approximation.

References

[1] J. Dávila, M. del Pino, M. Musso, J. Wei. *Gluing methods for vortex dynamics in Euler flows.* Preprint 2018, arXiv:1803.00066.

[2] J. Dávila, M. del Pino, M. Musso, J. Wei. In preparation.

[3] R. L. Jerrard, C. Seis *On the vortex filament conjecture for Euler flows.* Preprint arXiv:1603.00227

[4] C. Marchioro, M. Pulvirenti, *Euler evolution for singular initial data and vortex theory.* Comm. Math. Phys. **91**, 4 (1983), 563–572.

Theme & variations on div $\boldsymbol{\mu} = \boldsymbol{\sigma}$

Filip Rindler

(joint work with Adolfo Arroyo-Rabasa, Guido De Philippis, Jonas Hirsch, Anna Skorobogatova)

The following result was proved in [7]:

Theorem 1 ([7, Corollary 1.13]). *Let $\Omega \subset \mathbb{R}^d$ be open and let $\mu \in \mathcal{M}(\Omega; \mathbb{R}^{d\times d})$ be a matrix-valued measure such that*

$$\operatorname{div} \mu = \sigma \in \mathcal{M}(\Omega; \mathbb{R}^d) \quad \textit{in the sense of distributions.}$$

Then,

$$\operatorname{rank}\left(\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}(x)\right) \le d-1 \qquad \textit{for } |\mu|^s\textit{-a.e. } x \in \Omega.$$

In fact, this result is an (easy) corollary to the main result of *loc. cit.*: Let $\mathcal{A}$ be a k^{th}-order linear constant-coefficient PDE operator acting on $\mathbb{R}^m$-valued functions on $\mathbb{R}^d$ via

$$\mathcal{A}\varphi := \sum_{|\alpha|\le k} A_\alpha \partial^\alpha \phi \qquad \text{for all } \varphi \in \mathrm{C}^\infty(\mathbb{R}^d; \mathbb{R}^m),$$

where $A_\alpha \in \mathbb{R}^{n\times m}$ are (constant) matrices, $\alpha = (\alpha_1, \dots, \alpha_d) \in (\mathbb{N} \cup \{0\})^d$ is a multi-index, and $\partial^\alpha := \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d}$. We also assume that at least one A_α with $|\alpha| = k$ is non-zero.

The prototypical situations are $\mathcal{A} = \operatorname{div}, \operatorname{curl}, \operatorname{curl}\operatorname{curl}$ and the boundary operator ∂ for normal currents.

In [7] it was shown that for any measure satisfying a linear PDE in the sense of distributions, there is a strong constraint on the directions of the polar at singular points:

Theorem 2 ([7, Theorem 1.1 and Remark 1.3])**.** *Let $\Omega \subset \mathbb{R}^d$ be open and let $\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$ be an $\mathbb{R}^m$-valued Radon measure on Ω satisfying*

$$\mathcal{A}\mu = \sigma \in \mathcal{M}(\Omega;\mathbb{R}^n) \quad \textit{in the sense of distributions.}$$

Then,

$$\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}(x) \in \Lambda_{\mathcal{A}} \qquad \textit{for } |\mu|^s\textit{-a.e. } x \in \Omega,$$

where $\Lambda_{\mathcal{A}}$ is the wave cone *associated to $\mathcal{A}$, namely*

$$\Lambda_{\mathcal{A}} := \bigcup_{\xi \in \mathbb{R}^d \setminus \{0\}} \ker \mathbb{A}^k(\xi), \qquad \mathbb{A}^k(\xi) := \sum_{|\alpha|=k} A_\alpha \xi^\alpha.$$

In [5] this result was extended to yield *dimensional information* as well:

Definition. Let $\mathrm{Gr}(\ell,d)$ be the Grassmannian of ℓ-planes in $\mathbb{R}^d$. For $\ell = 1,\dots,d$ we define the *ℓ-dimensional wave cone* as

$$\Lambda_{\mathcal{A}}^\ell := \bigcap_{\pi \in \mathrm{Gr}(\ell,d)} \bigcup_{\xi \in \pi \setminus \{0\}} \ker \mathbb{A}^k(\xi).$$

Equivalently, $\Lambda_{\mathcal{A}}^\ell$ can be defined by the following analytical property:

$$\lambda \notin \Lambda_{\mathcal{A}}^\ell \quad \iff \quad (\mathcal{A} \llcorner \pi)\lambda \text{ is elliptic for some } \pi \in \mathrm{Gr}(\ell,d),$$

where $(\mathcal{A} \llcorner \pi)$ is the partial differential operator $\phi \mapsto (\mathcal{A} \llcorner \pi)(\phi) := \mathcal{A}(\phi \circ \mathcal{P}_\pi)$ with $\mathcal{P}_\pi$ the orthogonal projection onto π.

The main result of [5] establishes that on $\mathcal{I}^\ell$-null sets the polar of an $\mathcal{A}$-free measure is constrained to lie in $\Lambda_{\mathcal{A}}^\ell$. Here, $\mathcal{I}^\ell$ denotes the ℓ-dimensional integral-geometric measure (recall in particular that $\mathcal{I}^\ell \ll \mathcal{H}^\ell$, so in the following one can use the ℓ-dimensional Hausdorff measure instead).

Theorem 3 ([5, Theorem 1.3])**.** *Let $\Omega \subset \mathbb{R}^d$ be open and let $\mu \in \mathcal{M}(\Omega;\mathbb{R}^m)$ be an $\mathcal{A}$-free measure on Ω. If $E \subset \Omega$ is a Borel set with $\mathcal{I}^\ell(E) = 0$ for some $\ell \in \{0,\dots,d\}$, then*

$$\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}(x) \in \Lambda_{\mathcal{A}}^\ell \qquad \textit{for } |\mu|\textit{-a.e. } x \in E.$$

Corollary 1. *Let $\mathcal{A}$ and μ be as in Theorem 3 and assume that $\Lambda_{\mathcal{A}}^\ell = \{0\}$ for some $\ell \in \{0,\dots,d\}$. Then,*

$$E \subset \Omega \textit{ Borel with } \mathcal{I}^\ell(E) = 0 \quad \Longrightarrow \quad |\mu|(E) = 0.$$

In particular,

$$\mu \ll \mathcal{I}^\ell \ll \mathcal{H}^\ell$$

and thus

$$\dim_{\mathcal{H}} \mu := \sup\{\, \ell \;:\; \mu \ll \mathcal{H}^\ell \,\} \geq \ell_{\mathcal{A}},$$

where

$$\ell_{\mathcal{A}} := \max\{\, \ell \;:\; \Lambda^{\ell}_{\mathcal{A}} = \{0\}\,\}.$$

Combining Theorem 3 with the Besicovitch–Federer rectifiability theorem (see, e.g., [10, Chapter 18]) one obtains the following rectifiability result. Recall that for a positive measure $\sigma \in \mathcal{M}(\Omega)$ the *upper ℓ-dimensional density* is defined as

$$\theta^*_\ell(\sigma)(x) := \limsup_{r\to 0} \frac{\sigma(B_r(x))}{(2r)^\ell} = \limsup_{r\to 0} \frac{\sigma(B_r(x))}{\mathcal{H}^\ell(B^\ell_r)}, \qquad x \in \Omega.$$

Theorem 4 ([5, Theorem 1.5])**.** *Let $\mathcal{A}$ and μ be as in Theorem 3, and assume that $\Lambda^\ell_{\mathcal{A}} = \{0\}$ for some $\ell \in \{0,\dots,d\}$. Then, the set $\{\theta^*_\ell(|\mu|) = +\infty\}$ is $|\mu|$-negligible. Moreover, $\mu \llcorner \{\theta^*_\ell(|\mu|) > 0\}$ is concentrated on an ℓ-rectifiable set R and*

$$\mu \llcorner R = \theta^*_\ell(|\mu|)\lambda\, \mathcal{H}^\ell \llcorner R,$$

where $\lambda\colon R \to \mathbb{S}^{m-1}$ is $\mathcal{H}^\ell$-measurable; for $\mathcal{H}^\ell$-almost every $x_0 \in R$ (or, equivalently, for $|\mu|$-almost every $x_0 \in R$),

$$(2r)^{-\ell}(T^{x_0,r})_{\#}\mu \overset{*}{\rightharpoonup} \theta^*_\ell(|\mu|)(x_0)\lambda(x_0)\mathcal{H}^\ell \llcorner (T_{x_0}R) \qquad \text{as } r \downarrow 0;$$

and

$$\lambda(x_0) \in \bigcap_{\xi \in (T_{x_0}R)^\perp} \ker \mathbb{A}^k(\xi).$$

Here, $T^{x_0,r}(x) := (x - x_0)/r$ and $T_{x_0}R$ is the the approximate tangent plane to R at x_0.

We remark that Corollary 1 in conjunction with Theorem 4 imply the well-known rectifiability results for functions of bounded variation (see [3]) and functions of bounded deformation [8, 2] (see in particular [2, Proposition 3.5]).

More generally, it can be seen that the above results give optimal dimensionality and rectifiability results for first-order operators and for second-order *scalar* operators ($n = 1$), which contains most of the interesting examples (in particular, it applies to $\mathcal{A} = \operatorname{div}, \operatorname{curl}, \operatorname{curl}\operatorname{curl}, \partial$). On the other hand, for higher-order operators the optimality is not clear. Even the following seems to be open:

Conjecture. *For the 3^{rd}-order scalar operator defined on $\mathrm{C}^\infty(\mathbb{R}^3)$ by*

$$\mathcal{A} := \partial^3_{x_1} + \partial^3_{x_2} + \partial^3_{x_3}$$

we have $\ell_{\mathcal{A}} = 1$ since its characteristic set $\{\, \xi \in \mathbb{R}^3 \;:\; \xi_1^3 + \xi_2^3 + \xi_3^3 = 0\,\}$ is a ruled surface (and hence it contains lines) but since the characteristic set does not contain planes, it is conjectured that every measure $\mu \in \mathcal{M}(\mathbb{R}^3;\mathbb{R})$ with $\mathcal{A}\mu = 0$ satisfies $\mu \ll \mathcal{H}^2$. See [5] for a more general version of this conjecture.

Finally, we can strengthen Theorem 1 as follows:

Theorem 5 ([5, Proposition 3.1])**.** *Let $\Omega \subset \mathbb{R}^d$ be open and let $\mu \in \mathcal{M}(\Omega;\mathbb{R}^{d\times d})$ be a matrix-valued measure such that*

$$\operatorname{div}\mu = \sigma \in \mathcal{M}(\Omega;\mathbb{R}^d) \quad \textit{in the sense of distributions.}$$

Assume that for $|\mu|$-almost every $x \in \Omega$,

$$\operatorname{rank}\left(\frac{\mathrm{d}\mu}{\mathrm{d}|\mu|}(x)\right) \geq \ell.$$

Then, $|\mu| \ll \mathcal{I}^\ell \ll \mathcal{H}^\ell$ and there exists an ℓ-rectifiable set R and a $\mathcal{H}^\ell$-measurable map $\lambda\colon R \to \mathbb{R}^{d\times d}$ satisfying

$$\mu \llcorner \{\theta^*_\ell(|\mu|) > 0\} = \lambda(x)\, \mathcal{H}^\ell_x \llcorner R, \qquad \operatorname{rank}\lambda(x) = \ell \quad \mathcal{H}^\ell\text{-a.e.}$$

The above proposition allows, for instance, to reprove (and in some cases to slightly improve) the results of [1, 6, 4, 9, 11].

References

[1] W. K. Allard, *On the first variation of a varifold*, Ann. of Math., 95 (1972), pp. 417–491.

[2] L. Ambrosio, A. Coscia, and G. Dal Maso, *Fine properties of functions with bounded deformation*, Arch. Ration. Mech. Anal., 139 (1997), pp. 201–238.

[3] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, Oxford University Press, 2000.

[4] L. Ambrosio and H. M. Soner, *A measure-theoretic approach to higher codimension mean curvature flows*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 25 (1997), pp. 27–49 (1998).

[5] A. Arroyo-Rabasa, G. De Philippis, J. Hirsch, and F. Rindler, *Dimensional estimates and rectifiability for measures satisfying linear PDE constraints*, Geom. Funct. Anal., 29 (2019), pp. 639–658.

[6] G. De Philippis, A. De Rosa, and F. Ghiraldin, *Rectifiability of varifolds with locally bounded first variation with respect to anisotropic surface energies*, Comm. Pure Appl. Math., 71 (2018), pp. 1123–1148.

[7] G. De Philippis and F. Rindler, *On the structure of $\mathcal{A}$-free measures and applications*, Ann. of Math., 184 (2016), pp. 1017–1039.

[8] R. V. Kohn, *New estimates for deformations in terms of their strains*, PhD thesis, Princeton University, 1979.

[9] F.-H. Lin, *Gradient estimates and blow-up analysis for stationary harmonic maps*, Ann. of Math., 149 (1999), pp. 785–829.

[10] P. Mattila, *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability*, vol. 44 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1995.

[11] R. Moser, *Stationary measures and rectifiability*, Calc. Var. Partial Differential Equations, 17 (2003), pp. 357–368.

Ancient mean curvature flow of low entropy

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(joint work with Robert Haslhofer, Or Hershkovits)

A family of surfaces $M_t \subset \mathbb{R}^3$ moves by mean curvature flow if the normal velocity at each point is given by the mean curvature vector,

$$(\partial_t x)^\perp = \mathbf{H}(x) \qquad (x \in M_t). \tag{1}$$

Given any closed, embedded, and smooth initial surface $M \subset \mathbb{R}^3$, there exists a unique smooth solution $\mathcal{M} = \{M_t\}_{t\in[0,T)}$ with initial condition M defined on a maximal time interval $[0,T)$. The first singular time $T<\infty$ is characterized by the fact that the curvature blows up, i.e.

$$\lim_{t\nearrow T}\max_{x\in M_t}|A|(x,t)=\infty, \tag{2}$$

where $|A|$ denotes the norm of the second fundamental form.

In the mean convex case, there is a highly developed theory. On the one hand, the flow can be continued smoothly as a surgical solution as constructed by Brendle-Huisken [BH16] and Haslhofer-Kleiner [HK17b]. This in turn facilitates topological and geometric applications, see e. g. [BHH16, HK18]. On the other hand, the flow can also be continued uniquely as a weak (generalized) solution. Weak solutions can be described either as level set solutions as in Evans-Spruck [ES91] and Chen-Giga-Goto [CGG91], or in the framework of geometric measure theory using Brakke solutions [Bra78]. By the deep structure theory of White [Whi00, Whi03] (see also [HK17a]) the space-time dimension of the singular set is at most one, and all blowup limits are smooth and convex. In fact, by a result of Colding-Minicozzi [CM16] the space-time singular set is contained in finitely many compact embedded Lipschitz curves together with a countable set of point singularities. Moreover, the recent work of Brendle-Choi [BC17] and Angenent-Daskalopoulos-Sesum [ADS18] provides a short list of all potential blowup limits (singularity models) in the flow of mean convex surfaces: the round shrinking sphere, the round shrinking cylinder, the translating bowl soliton [AW94], and the ancient ovals [Whi00, HH16].

In stark contrast to the above, when the initial surface is not mean convex, the theory is much more rudimentary. This is, to some extent, an unavoidable feature of the equation. In particular, as already pointed out in the pioneering work of Brakke [Bra78] and Evans-Spruck [ES91] there is the phenomenon of non-uniqueness or fattening. Angenent-Ilmanen-Chopp [AIC95] and Ilmanen-White [Whi02] gave examples of smooth embedded surfaces $M\subset\mathbb{R}^3$ whose level set flow $F_t(M)$ develops a non-empty interior at some positive time. In particular, $F_t(M)$ does not look at all like a two-dimensional evolving surface. These examples also illustrate, in a striking way, the non-uniqueness of (enhanced) Brakke flows.

We prove the mean convex neighborhood conjecture, in order to decrease the gap between the theory in the mean convex case and the theory in the general case without curvature assumptions.

Theorem 1 (mean convex neighborhoods at the first singular time)**.** *Let* $\mathcal{M}=\{M_t\}_{t\in[0,T)}$ *be a mean curvature flow of closed embedded surfaces in* $\mathbb{R}^3$*, where* T *is the first singular time. If some tangent flow at* $X=(x,T)$ *is a sphere or cylinder with multilicity one, then there exists an* $\varepsilon=\varepsilon(X)>0$ *such that, possibly after flipping the orientation, the flow* $\mathcal{M}$ *is mean convex in the parabolic ball* $P(X,\varepsilon)$*. Moreover, any limit flow at* X *is either a round shrinking sphere, a round shrinking cylinder, or a translating bowl soliton.*[1]

[1]It is easy to see that ancient ovals cannot arise as limit flows at the first singular time. It is unknown, whether or not ancient ovals can arise as limit flows at subsequent singularities. This is related to potential accumulations of neckpinches, see e.g. [CM17].

For the mean convex neighborhood theorem at the all singular time, see [CHH]. In order to prove the theorem, we recall the entropy introduced by Colding-Minicozzi [CM12];

$$\text{Ent}[M] = \sup_{y\in\mathbb{R}^3,\lambda>0} \int_M \frac{1}{4\pi\lambda} e^{-\frac{|x-y|^2}{4\lambda}} dA(x). \tag{3}$$

The entropy measures, in a certain sense, the complexity of the surface. For example, the values for a plane, sphere and cylinder are

$$\text{Ent}[\mathbb{R}^2] = 1, \quad \text{Ent}[\mathbb{S}^2] = \frac{4}{e} \sim 1.47, \quad \text{Ent}[\mathbb{S}^1 \times \mathbb{R}] = \sqrt{\frac{2\pi}{e}} \sim 1.52. \tag{4}$$

If $\mathcal{M} = \{M_t\}_{t\in I}$ evolves by mean curvature flow, then $t \mapsto \text{Ent}[M_t]$ is nonincreasing by Husiken's monotonicity formula [Hui90], hence

$$\text{Ent}[\mathcal{M}] := \sup_{t\in I} \text{Ent}[M_t] = \lim_{t\to\inf(I)} \text{Ent}[M_t]. \tag{5}$$

For example, the entropy of a flat plane $\mathcal{P}$, a round shrinking sphere $\mathcal{S}$, a round shrinking cylinder $\mathcal{Z}$, a translating bowl soliton $\mathcal{B}$, and an ancient oval $\mathcal{O}$, are given by

$$\begin{aligned} &\text{Ent}[\mathcal{P}] = 1, \quad \text{Ent}[\mathcal{S}] = \frac{4}{e} \sim 1.47 \\ &\text{Ent}[\mathcal{Z}] = \text{Ent}[\mathcal{B}] = \text{Ent}[\mathcal{O}] = \sqrt{\frac{2\pi}{e}} \sim 1.52. \end{aligned} \tag{6}$$

We consider the following class of Brakke flows:

Definition (ancient low entropy flows)**.** The class of *ancient low entropy flows* consists of all ancient, unit-regular, cyclic, integral Brakke flows $\mathcal{M} = \{\mu_t\}_{t\in(-\infty,T_E(\mathcal{M})]}$ in $\mathbb{R}^3$ with

$$\text{Ent}[\mathcal{M}] \leq \sqrt{\frac{2\pi}{e}}, \tag{7}$$

where $T_E(\mathcal{M}) \leq \infty$ denotes the extinction time.

Then, we can classify the low entropy flow as follows.

Theorem 2 (classification of ancient low entropy flows)**.** *Any ancient low entropy flow in* $\mathbb{R}^3$ *is either*

- *a flat plane, or*
- *a round shrinking sphere, or*
- *a round shrinking cylinder, or*
- *a translating bowl soliton, or*
- *an ancient oval.*

Namely, any ancient low entropy flow in $\mathbb{R}^3$ is convex, and this leads to prove Theorem 1.

References

[ADS15] S. Angenent, P. Daskalopoulos, and N. Sesum. Unique asymptotics of ancient convex mean curvature flow solutions. *arXiv:1503.01178*, 2015.

[ADS18] S. Angenent, P. Daskalopoulos, and N. Sesum. Uniqueness of two-convex closed ancient solutions to the mean curvature flow. *arXiv:1804.07230*, 2018.

[AIC95] S. Angenent, T. Ilmanen, and D. Chopp. A computed example of nonuniqueness of mean curvature flow in $\mathbb{R}^3$. *Comm. Partial Differential Equations*, 20(11-12):1937–1958, 1995.

[CHH] K. Choi, R. Haslhofer, and O. Hershkovits. Ancient low entropy flows, mean convex neighborhoods, and uniqueness. *arXiv:1810.08467*, 2018.

[AW94] S. Altschuler and L. Wu. Translating surfaces of the non-parametric mean curvature flow with prescribed contact angle. *Calc. Var. Partial Differential Equations*, 2(1):101–111, 1994.

[BC17] S. Brendle and K. Choi. Uniqueness of convex ancient solutions to mean curvature flow in $\mathbb{R}^3$. *arXiv:1711.00823v2*, 2017.

[BH16] S. Brendle and G. Huisken. Mean curvature flow with surgery of mean convex surfaces in $\mathbb{R}^3$. *Invent. Math.*, 203(2):615–654, 2016.

[BHH16] R. Buzano, R. Haslhofer, and O. Hershkovits. The moduli space of two-convex embedded spheres. *arXiv:1607.05604*, 2016.

[BK17] R. Bamler and B. Kleiner. Uniqueness and stability of Ricci flow through singularities. *arXiv:1709.04122*, 2017.

[Bra78] K. Brakke. *The motion of a surface by its mean curvature*, volume 20 of *Mathematical Notes*. Princeton University Press, Princeton, N.J., 1978.

[Bre13] S. Brendle. Rotational symmetry of self-similar solutions to the Ricci flow. *Invent. Math.*, 194(3):731–764, 2013.

[Bre16] S. Brendle. Embedded self-similar shrinkers of genus 0. *Ann. of Math. (2)*, 183(2):715–728, 2016.

[BW17] J. Bernstein and L. Wang. A topological property of asymptotically conical self-shrinkers of small entropy. *Duke Math. J.*, 166(3):403–435, 2017.

[CGG91] Y.G. Chen, Y. Giga, and S. Goto. Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *J. Differential Geom.*, 33(3):749–786, 1991.

[CM12] T. Colding and W. Minicozzi. Generic mean curvature flow I; generic singularities. *Ann. of Math. (2)*, 175(2):755–833, 2012.

[CM15] T. Colding and W. Minicozzi. Uniqueness of blowups and Lojasiewicz inequalities. *Ann. of Math. (2)*, 182(1):221–285, 2015.

[CM16] T. Colding and W. Minicozzi. The singular set of mean curvature flow with generic singularities. *Invent. Math.*, 204(2):443–471, 2016.

[CM17] T. Colding and W. Minicozzi. Arnold-Thom gradient conjecture for the arrival time. *arXiv:1712.05381*, 2017.

[ES91] L. Evans and J. Spruck. Motion of level sets by mean curvature. I. *J. Differential Geom.*, 33(3):635–681, 1991.

[Gan18] Z. Gang. A description of a space-and-time neighborhood of generic singularities formed by mean curvature flow. *arXiv:1803.10903v2*, 2018.

[GKS18] Z. Gang, D. Knopf, and M. Sigal. Neckpinch dynamics for asymmetric surfaces evolving by mean curvature flow. *Mem. Amer. Math. Soc.*, 253(1210):v+78, 2018.

[Has15] R. Haslhofer. Uniqueness of the bowl soliton. *Geom. Topol.*, 19(4):2393–2406, 2015.

[Her18] O. Hershkovits. Translators asymptotic to cylinders. *arXiv:1805.10553*, 2018.

[HH16] R. Haslhofer and O. Hershkovits. Ancient solutions of the mean curvature flow. *Comm. Anal. Geom.*, 24(3):593–604, 2016.

[HK17a] R. Haslhofer and B. Kleiner. Mean curvature flow of mean convex hypersurfaces. *Comm. Pure Appl. Math.*, 70(3):511–546, 2017.

[HK17b] R. Haslhofer and B. Kleiner. Mean curvature flow with surgery. *Duke Math. J.*, 166(9):1591–1626, 2017.
[HK18] R. Haslhofer and D. Ketover. Minimal two-spheres in three-spheres. *arXiv:1708.06567*, 2018.
[Hui84] G. Huisken. Flow by mean curvature of convex surfaces into spheres. *J. Differential Geom.*, 20(1):237–266, 1984.
[Hui90] G. Huisken. Asymptotic behavior for singularities of the mean curvature flow. *J. Differential Geom.*, 31(1):285–299, 1990.
[HW18] O. Hershkovits and B. White. Non-fattening of mean curvature flow at singularities of mean convex type. *Comm. Pure Appl. Math. (to appear)*, 2018.
[Ilm94] T. Ilmanen. Elliptic regularization and partial regularity for motion by mean curvature. *Mem. Amer. Math. Soc.*, 108(520):x+90, 1994.
[Ilm95] T. Ilmanen. Singularities of mean curvature flow of surfaces. *https://people.math.ethz.ch/ ilmanen/papers/sing.ps*, 1995.
[Ilm03] T. Ilmanen. Problems in mean curvature flow. *https://people.math.ethz.ch/ ilmanen/classes/eil03/problems03.ps*, 2003.
[SX17] J. Spruck and L. Xiao. Complete translating solitons to the mean curvature flow in $\mathbb{R}^3$ with nonnegative mean curvature. *arXiv:1703.01003*, 2017.
[Wan11] X. Wang. Convex solutions to the mean curvature flow. *Ann. of Math. (2)*, 173(3):1185–1239, 2011.
[Wan16] L. Wang. Asymptotic structure of self-shrinkers. *arXiv:1610.04904*, 2016.
[Whi95] B. White. The topology of hypersurfaces moving by mean curvature. *Comm. Anal. Geom.*, 3(1-2):317–333, 1995.
[Whi97] B. White. Stratification of minimal surfaces, mean curvature flows, and harmonic maps. *J. Reine Angew. Math.*, 488:1–35, 1997.
[Whi00] B. White. The size of the singular set in mean curvature flow of mean convex sets. *J. Amer. Math. Soc.*, 13(3):665–695, 2000.
[Whi02] B. White. Evolution of curves and surfaces by mean curvature. In *Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002)*, pages 525–538, Beijing, 2002. Higher Ed. Press.
[Whi03] B. White. The nature of singularities in mean curvature flow of mean-convex sets. *J. Amer. Math. Soc.*, 16(1):123–138, 2003.
[Whi05] B. White. A local regularity theorem for mean curvature flow. *Ann. of Math. (2)*, 161(3):1487–1519, 2005.

Yamabe flow on noncompact manifolds

Mario B. Schulz

Let (M, g_0) be any smooth, possibly noncompact and geodesically incomplete Riemannian manifold. A *Yamabe flow* on (M, g_0) is a family $(g(t))_{t\in[0,T[}$ of Riemannian metrics on M satisfying

$$(1)\qquad \begin{cases} \frac{\partial}{\partial t} g(t) = -\mathrm{R}_{g(t)}\, g(t) & \text{in } M \times [0, T[\,, \\ \quad g(0) = g_0 & \text{on } M, \end{cases}$$

where R_g denotes the scalar curvature of (M, g). The Yamabe flow was introduced by Richard Hamilton [4] in 1989 who proved that if M is closed (i.e. compact without boundary), then problem (1) has a unique solution for some $T > 0$. The original motivation was that the Yamabe flow on a closed manifold converges after a rescaling of space and time for $t \to \infty$ to a conformal metric of constant scalar curvature and thereby provides an alternative approach to the Yamabe Theorem, which was proven in 1984 by Richard Schoen [10] after work of Yamabe, Trudinger and Aubin.

Here, we focus on the well-posedness of problem (1) in the noncompact setting. In dimension $\dim(M) = 2$, where the Yamabe flow coincides with the Ricci flow, Gregor Giesen and Peter Topping [15, 2, 3, 16] showed that on any smooth, connected surface (M, g_0) there exists a *unique instantaneously complete* solution $(g(t))_{t\in[0,T[}$ to problem (1). Instantaneous completeness means that the Riemannian manifold $(M, g(t))$ is geodesically complete for all $0 < t < T$ even if the initial manifold is incomplete. For example, let g_0 be the restriction of the Euclidean metric to

(a) the flat unit disc $M = B_1 \subset \mathbb{R}^2$, or
(b) the punctured plane $M = \mathbb{R}^2 \setminus \{0\}$.

In both cases, the constant flow $g(t) = g_0$ for all $t \geq 0$ is a geodesically incomplete solution to problem (1), but there also exists an instantaneously complete Yamabe flow with the same initial data which is unique in this class of solutions.

Most remarkably the results of Giesen and Topping allow the initial surface to be incomplete with unbounded curvature. Does their theory generalise to the Yamabe flow on noncompact manifolds of higher dimension? Inspired by example (a), we prove the following existence result [13].

Theorem 1. *Let g_0 be any conformal metric on hyperbolic space $(\mathbb{H}, g_{\mathbb{H}})$ of dimension $m \geq 3$. Then, there exists a Yamabe flow $(g(t))_{t\in[0,\infty[}$ on $\mathbb{H}$ satisfying*

(i) *$g(0) = g_0$,*
(ii) *$g(t) \geq m(m-1)t\, g_{\mathbb{H}}$ for all $t > 0$.*

As in the work of Giesen and Topping, the initial manifold $(\mathbb{H}, g_0)$ is allowed to be incomplete with unbounded curvature. Previous results about the existence of Yamabe flows on general noncompact manifolds of arbitrary dimension typically required initial completeness and curvature bounds [6, 7].

If g_0 is bounded by a multiple of the (conformally equivalent) Euclidean metric on $\mathbb{H}$, then we can prove that the solution constructed in Theorem 1 is unique in the class of Yamabe flows satisfying (i) and (ii) (see [12]). Property (ii) implies instantaneous completeness of the Yamabe flow. Whether (ii) is in fact equivalent to instantaneous completeness of conformally hyperbolic Yamabe flows is an open question which we are able to confirm in the rotationally symmetric case [12].

In view of example (b) it is surprising that punctured manifolds of higher dimension do *not* allow instantaneously complete solutions to the Yamabe flow. We obtain the following result [14].

Theorem 2. *Let (M, g_0) be a closed Riemannian manifold of dimension $m \geq 3$ and let $\emptyset \neq N \subset M$ be a closed submanifold of dimension $n \geq 0$.*

(i) *If $n > \frac{m-2}{2}$ then an instantaneously complete Yamabe flow $(g(t))_{t\in[0,\infty[}$ on $M \setminus N$ with $g(0) = g_0$ exists.*
(ii) *If $n < \frac{m-2}{2}$ then any Yamabe flow on $(M \setminus N, g_0)$ is incomplete and uniquely given by the restriction of the Yamabe flow on (M, g_0).*

To prove that Theorem 2 (ii) extends to the borderline case $n = \frac{m-2}{2}$ for even $m \geq 4$ is work in progress. Theorem 2 can be interpreted as parabolic analogue of several known result about elliptic (Yamabe-type) equations in which the threshold $\frac{m-2}{2}$ plays a similar role [5, 11, 9, 8]. Most relevant for the proof of Theorem 2 (i) is a result by Aviles and McOwen [1] which states, that if (M, g) is a compact Riemannian manifold of dimension $m \geq 3$ and $N \subset M$ a closed submanifold of dimension n, then there exists a complete, conformal metric $\hat{g}$ on $M \setminus N$ with constant negative scalar curvature if and only if $n > \frac{m-2}{2}$. In fact, we may replace $(\mathbb{H}, g_{\mathbb{H}})$ by $(M \setminus N, \hat{g})$ in the proof of Theorem 1 and follow the same approach to construct an instantaneously complete Yamabe flow on $M \setminus N$ provided that $n > \frac{m-2}{2}$. The proof of Theorem 2 (ii) on the other hand, is based on the following observations:

There exists a conformal metric $\overline{g}$ on M conformal and comparable to g_0 such that $\mathrm{R}_{\overline{g}} \equiv 0$ in some neighbourhood Ω of N. Let $r(x) = \mathrm{dist}_{\overline{g}}(x, N)$ and $\tilde{g} = r^{-2}\overline{g}$ on $\Omega \setminus N$. Then, as $r \to 0$,

$$\mathrm{R}_{\tilde{g}} = 4\tfrac{m-1}{m-2} r^{\frac{2-m}{2}} \Delta_{\tilde{g}} r^{\frac{m-2}{2}} = (m-1)(m-2-2n) + o(1) > 0 \text{ in } \Omega. \tag{2}$$

Any Yamabe flow $(g(t))_{t\in[0,T]}$ on $\Omega \setminus N$ is of the form $g(t) = U(\cdot, t)^{\frac{4}{m-2}}\tilde{g}$ with

$$\begin{aligned} \frac{1}{m-1}\frac{\partial U}{\partial t} &= \Big(-\tfrac{1}{c_m}\mathrm{R}_{\tilde{g}} U + \Delta_{\tilde{g}} U\Big) U^{-\frac{4}{m-2}} \\ &= \Big(-\big(r^{-\frac{m-2}{2}}\Delta_{\tilde{g}} r^{\frac{m-2}{2}}\big)\big(U - cr^{\frac{m-2}{2}}\big) + \Delta_{\tilde{g}}\big(U - cr^{\frac{m-2}{2}}\big)\Big) U^{-\frac{4}{m-2}} \end{aligned} \tag{3}$$

in $(\Omega \setminus N) \times [0, T]$ for any $c \in \mathbb{R}$. By (2) the first factor in (3) has the right sign to apply a version of the maximum principle for noncompact manifolds which yields $U(\cdot, t) \leq cr^a$ for all $t \in [0, T]$ and some $c > 0$. Hence, $g(t) \leq c^{\frac{2}{a}} r^2 \tilde{g} = c^{\frac{2}{a}}\overline{g} \leq Cg_0$, which already implies that the Yamabe flow must be geodesically incomplete.

The uniqueness in Theorem 2 (ii) is proven based on this upper bound by means of an energy approach similar to the one Topping [16] used on surfaces.

References

[1] P. Aviles and R. C. McOwen, *Complete conformal metrics with negative scalar curvature in compact Riemannian manifolds*, Duke Math. J. 56.2 (1988), 395–398.

[2] G. Giesen and P. M. Topping, *Ricci flow of negatively curved incomplete surfaces*, Calc. Var. Partial Differential Equations 38.3-4 (2010), 357–367.

[3] G. Giesen and P. M. Topping, *Existence of Ricci flows of incomplete surfaces*, Comm. Partial Differential Equations 36.10 (2011), 1860–1880.

[4] R. Hamilton, *Lectures on geometric flows*, unpublished (1989).

[5] C. Loewner and L. Nirenberg, *Partial differential equations invariant under conformal or projective transformations*, Contributions to analysis (a collection of papers dedicated to Lipman Bers) (1974), 245–272.

[6] L. Ma and Y. An, *The maximum principle and the Yamabe flow*, Partial differential equations and their applications (1999), 211–224.

[7] L. Ma, *Gap theorems for locally conformally flat manifolds*, J. Differential Equations 260.2 (2016), 1414–1429.

[8] R. Mazzeo and F. Pacard, *Constant scalar curvature metrics with isolated singularities*, Duke Math. J. 99.3 (1999), 353–418.

[9] F. Pacard, *The Yamabe problem on subdomains of even-dimensional spheres*, Topol. Methods Nonlinear Anal. 6.1 (1995), 137–150.
[10] R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geom. 20.2 (1984), 479–495.
[11] R. Schoen and S.-T. Yau, *Conformally flat manifolds, Kleinian groups and scalar curvature*, Invent. Math. 92.1 (1988), 47–71.
[12] M. B. Schulz, *Instantaneously complete Yamabe flow on hyperbolic space*, ArXiv e-prints (2016), arXiv: 1612.02745
[13] M. B. Schulz, *Unconditional existence of conformally hyperbolic Yamabe flows*, Analysis & PDE (to appear).
[14] M. B. Schulz, *Incomplete Yamabe flows and removable singularities*, ArXiv e-prints (2019), arXiv: 1907.02059
[15] P. M. Topping, *Ricci flow compactness via pseudolocality, and flows with incomplete initial metrics*, J. Eur. Math. Soc. 12.6 (2010), 1429–1451.
[16] P. M. Topping, *Uniqueness of instantaneously complete Ricci flows*, Geom. Topol. 19.3 (2015), 1477–1492.

On the regularity of Ricci flows coming out of metric spaces

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(joint work with A. Deruelle, F. Schulze)

We consider solutions to Ricci flow defined on manifolds M for a time interval $(0,T]$ satisfying the following:

$$\text{Ricci}(g(t)) \geq -c_0^2 \tag{a_t}$$

$$|\text{Riem}|(g(t)) \leq \frac{c_0^2}{t}. \tag{b_t}$$

for all $t \in (0,T]$. From previous works, [1, 5, 4] it is known that if such a solution is complete for all times $t > 0$ (in [4] the local situation was considered for balls compactly contained in M), then there is a (local in the setting of [4]) limit metric space (M, d_0), as time t approaches zero. In this setting, the metrics $d_t := d(g(t))$ induced by the Riemannian metric at time t, satisfy ([7, 4])

$$e^{c_0^2 t} d_t(\cdot,\cdot) \geq d_0(\cdot,\cdot) \geq d_0(\cdot,\cdot) - \gamma(n) c_0 \sqrt{t}. \tag{c_t}$$

Such solutions occur naturally as pointed limits in i of smooth solutions, starting at smooth initial data $(M_i, g_i(0), p_i)$, $i \in \mathbb{N}$ which satisfy

$$\mathcal{C}(g_i(0)) \geq -K_0 > -\infty \tag{i}$$

$$\text{vol}(B_{g_i(0)}(\cdot,1)) \geq v_0 > 0 \tag{ii}$$

where $\mathcal{C}$ is an appropriate curvature condition: See [1, 5, 2, 4]. For example, in dimension three, $\mathcal{C} = \text{Rc}$ is sufficient, as shown in [1, 4]. The initial value of a flow satisfying $(\mathrm{a}_t), (\mathrm{b}_t)$ is achieved in the Gromov-Hausdorff metric space sense, and (M, d_0) may be non-smooth, as is for example the case for an expanding soliton coming out of a euclidean non-trivial cone of non-negative curvature which is non-smooth at the tip. In the case that a neighbourhood V satisfies (V, d_0) is regular, in the sense that d_0 is *smooth* in a neighbourhood of x for all $x \in V$, then we show that the solution $(V, g(t))_{t\in(0,T)}$ can be extended smoothly to $(V, g(t))_{t\in[0,T)}$ for some smooth Riemannian metric $g(0)$ on V. Here we use a natural definition of *local smoothness* of a metric space. We say d_0 is *smooth* in a neighbourhood of $x \in M$, if there exist $r \geq \tilde{r} > 0$ such that $B_{d_0}(x, r) \subset\subset M$, and a metric space isometry $\varphi : (B_{d_0}(x, r), d) \to (X, \tilde{d}_0)$, and a smooth Riemannian metric $\tilde{g}_0$ defined on X such that $B_{\tilde{d}_0}(x, \tilde{r}) \subset\subset X$ and $(B_{\tilde{d}_0}(x, \tilde{r}), \tilde{d}_0) = (B_{d(\tilde{g}_0)}(x, \tilde{r}), d(\tilde{g}_0))$ where $d(\tilde{g}_0)$ is the distance induced by the Riemannian metric $\tilde{g}_0$ on X, X is a smooth connected manifold. Notice that we have no guarantee that the isometry φ is smooth w.r.t to the structures of M and X. However, by introducing *distance coordinates* $\psi : B_{\tilde{d}_0}(x, \tilde{r}) \to \mathbb{R}^n$, $\psi(\cdot) = (\tilde{d}_0(\cdot, \tilde{a}_1), \ldots, \tilde{d}_0(\cdot, \tilde{a}_n))$, we obtain a smooth $(1+\varepsilon)$ Bi-Lipschitz map from $B_{\tilde{g}_0}(x, \hat{r})$ to $\mathbb{R}^n$, for small enough $\hat{r}$ and appropriately chosen points $\tilde{a}_1, \ldots, \tilde{a}_n$. We may pull these points $\tilde{a}_1, \ldots, \tilde{a}_n$, back to $a_1, \ldots, a_n$ in M using the isometry φ. The map, F_0 we obtain, by pulling ψ back to M using φ,

$$F_0(\cdot) := (d_0(\cdot, a_1), \ldots, d_0(\cdot, a_n)),$$

is a distance coordinate map on M. Although we now have no guarantee that F_0 is smooth, we do know that F_0 is $(1+\varepsilon)$ Bi-Lipschitz, since this property is preserved under isometries. That is

$$|F_0(x) - F_0(y)| \in ((1-\varepsilon)d_0(x, y), (1+\varepsilon)d_0(x, y))$$

In the setting being considered, we can show that in fact we must have

$$|\mathrm{Riem}|(g(t)) \leq \frac{\varepsilon_0(\varepsilon)}{t}$$

on V, if we restrict the time interval to $[0, T(\varepsilon)]$ which is to be expected in view of the Pseudolocality Theorem of G. Perelman [8]. After scaling once, we may also assume that the bound from below on the Ricci curvature is $\varepsilon_0(\varepsilon)$, where $\varepsilon_0(\varepsilon) \to 0$ as $\varepsilon \to 0$. That is, we can replace the c_0 appearing in the estimates of $(\mathrm{c})_t$ by ε_0. Considering the distance coordinates

$$F_t(\cdot) := (d_t(\cdot, a_1), \ldots, d_t(\cdot, a_n)),$$

for $t > 0$, for the same $a_1, \ldots, a_n$ from above, we see, using (c_t) that F_t satisfies

$$|F_t(x) - F_t(y)| \in ((1-\varepsilon_0)d_0(x, y) - \varepsilon_0\sqrt{t}, (1+\varepsilon)d_0(x, y) + \varepsilon_0\sqrt{t}).$$

We call such a map an ε_0 *almost isometry.* We use initial and boundary data given by $F_t(\cdot)$ for any fixed small $t > 0$ and solve the Ricci-Harmonic map heat flow $\frac{\partial}{\partial s} Z_s(\cdot) = \Delta_{g(s)} Z_s(\cdot)$ for $s \in [t, T]$. The regularising properties of the flow guarantee that, after flowing for a time t, that is at time $s = 2t$, the resulting map $Z(\cdot) := Z_t(\cdot, 2t)$ is a $1 + \alpha$ Bi-Lipschitz map where $\alpha \to 0$ as ε_0 to 0. The proof of this (and other estimates) can be found in Theorem 3.7 in [3]. By taking a limit $t \to 0$ we obtain a smooth solution $Z(\cdot, s)$ to the Ricci-Harmonic map heat flow with Dirichlet boundary conditions. This solution is smooth for all time $s > 0$ and $1 + \alpha$ Bi-Lipschitz for all $s \in (0, S(\varepsilon_0))$. In the setting of bounded Ricci curvature and almost isometrically split almost euclidean regions, R. Hochard also considered the Ricci harmonic map heat flow of distance coordinates (on the almost euclidean region), cf. Lemma II.3.10 of [6]. We push the Ricci flow forward using Z. It is well known, see Section 6 of [7], that the resulting family of metrics $\tilde{g}(\cdot,s) := Z(\cdot,s)_*(g(s))$ solves the δ- Ricci DeTurck flow (that is the Ricci DeTurck flow with background metric δ, where δ is the standard metric on $\mathbb{R}^n$), which is a parabolic PDE. In the setting we are considering, the $1+\alpha$ Bi-Lipschitz estimate on Z_t guarantees that the solution $\tilde{g}(s)$ is, without loss of generality bounded above by $(1+\varepsilon_0)\delta$ respectively below by $(1-\varepsilon_0)\delta$ for all s. Furthermore, applications of Theorem 3.7 in [3] at different scales show that $\tilde{g}(s) \to \tilde{g}_0$ in the C^0 sense as $s \searrow 0$, as is shown in Theorem 4.3 of [3]. We construct a comparison solution $\hat{g}$ to the δ- Ricci DeTurck flow with initial values on $B_v(0)$ given by $\tilde{g}_0$ and boundary values on $\partial B_v(0)$ at time s given by $\tilde{g}(\cdot,s)$: without loss of generality $F_0(x) = 0$. $\hat{g}$ is smooth on $B_v(0)\times[0,T]\cup\overline{B_v(0)}\times(0,T]$ and continuous on $\overline{B_v(0)}\times[0,T]$ and also bounded above by $(1+\varepsilon_0)\delta$ respectively below by $(1-\varepsilon_0)\delta$ by construction : See Section 5 of [3] for details.

Using the L^2 *Lemma*, Lemma 6.1 of [3], we show that in fact $\hat{g} = \tilde{g}$. This means that $\tilde{g}(s)$ must have (locally in space) bounded curvature, and that all covariant derivatives thereof are bounded, independent of $s \in (0,T]$. Pulling back with $Z(s)$, we see that the same must be true for the original solution $g(s)$ for $s \in (0,T]$. Using the evolution equation $\frac{\partial}{\partial s} g(s) = -2\mathrm{Rc}(g(s))$, one can show, Section 8 of [7], that the solution $g(s)_{s\in(0,T]}$ can also be smoothly extended back to time zero.

The L^2 Lemma is formulated in the setting of the Ricci DeTurck flow, and does not necessarily assume that the solutions being considered come from the setting above. The content of the L^2 Lemma, Lemma 6.1 of [3], is as follows: given any two solutions $\tilde{g}$ and $\hat{g}$ to the δ - Ricci DeTurck flow which are smooth on $\overline{B_v(0)}\times(0,T]$, bounded above by $(1+\varepsilon_0)\delta$ respectively below by $(1-\varepsilon_0)\delta$, then

$$\frac{\partial}{\partial s}\int_{B_v(0)} \varphi(x,s)|\tilde{g}(x,s)-\hat{g}(x,s)|^2 dx \le 0 \tag{1}$$

for all $s>0$ where $\varphi(\cdot,s) := (1+\lambda|\tilde{g}(s)-\delta|^2+\lambda|\hat{g}(s)-\delta|^2)$ is bounded between $1/2$ and 2 and ε_0 is sufficiently small and λ sufficiently large, depending on n. In the case that solutions are continuous on $\overline{B_v(0)}\times[0,T]$ and equal at time zero, then the lemma implies that they are equal for all time.

References

[1] M. Simon, *Ricci flow of non-collapsed three manifolds whose Ricci curvature is bounded from below* Journal für die reine und angewandte Mathematik. **662** (2012) 59–94.

[2] Bamler, Richard and Cabezas-Rivas, Esther and Wilking Burkhard, *The Ricci flow under almost non-negative curvature conditions* , arXiv:1707.03002, (2017)

[3] Deruelle, Alix and Schulze, Felix and Simon, Miles, *On the regularity of Ricci flows coming out of metric spaces*

[4] Simon, Miles and Topping, Peter, *Local control on the geometry in 3D Ricci flow* , arXiv:1611.06137, (2016),

[5] Schulze, Felix and Simon, Miles, *Expanding solitons with non-negative curvature operator coming out of cones*, Math. Z. **275**, (2013)

[6] Hochard, Raphaël, Théorèmes d'existence en temps court du flot de Ricci pour des variétés non-complètes, non-éffondrées, à courbure minorée,Ph.D Thesis (Université de Bordeaux), (2019),

[7] Hamilton, Richard, *The formation of singularities in the Ricci flow*, Surveys in differential geometry, Vol. II Cambridge, (1995)
[8] Perelman, Grisha, *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math/0211159 (2002)

Codimension two min-max minimal submanifolds from PDEs

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(joint work with Daniel Stern)

The variational construction of minimal submanifolds has received great attention in the last decades—with recent spectacular developments, especially in codimension one.

In order to find min-max minimal hypersurfaces, a (rather involved) discretized method was developed by Almgren and Pitts, in the setting of geometric measure theory. On the other hand, starting from the work of De Giorgi, Modica–Mortola and Sternberg for minimizers, a "level set" approach was proposed, based on (rescalings of) the Allen–Cahn functional

$$F_\epsilon(v) := \int_M \Big(\epsilon|dv|^2 + \frac{1}{4\epsilon}(1-v^2)^2\Big),$$

whose minimizers model a phase transition concentrating on a minimal codimension one interface as $\epsilon \to 0$.

In their pioneering work, Hutchinson–Tonegawa [3] studied families of critical points v_ϵ of F_ϵ with bounded energy and showed, in particular, that their energy measures concentrate along a stationary, integral $(n-1)$-varifold, given by the limit of the level sets $v_\epsilon^{-1}(0)$.

These developments, together with the deep regularity work by Tonegawa–Wickramasekera on stable solutions and subsequent work by Guaraco and Gaspar–Guaraco, provided a PDE alternative to the Almgren–Pitts method, used successfully to attack some profound questions concerning the structure of min-max minimal hypersurfaces.

In this talk, following [4], we explore a natural way to construct minimal varieties of codimension two through PDE methods. Recently, attempts in this direction have been made by Cheng [1] and Stern [5], based on the study of the Ginzburg–Landau functionals

$$F_\epsilon(v) := \frac{1}{|\log\epsilon|}\int_M \Big(|dv|^2 + \frac{1}{4\epsilon^2}(1-|v|^2)^2\Big)$$

on complex-valued maps $v\colon M \to \mathbb{C}$. While the Ginzburg–Landau approach can be employed successfully to produce nontrivial stationary *rectifiable* $(n-2)$-varifolds—based also on works by Lin–Rivière and Bethuel–Brezis–Orlandi, it is not yet known whether the varifolds produced in this way are *integral*, nor is it known whether the full energies $F_\epsilon(v_\epsilon)$ of the min-max critical points converge to the mass of the limiting minimal variety in the case $b_1(M) \neq 0$.

These difficulties point to the deeper fact that the Ginzburg–Landau functionals, though related to the $(n-2)$-area, do *not* provide a straightforward regularization of the codimension-two area functional. Indeed, they should be understood first and foremost as a relaxation of the Dirichlet energy for singular maps to S^1, and while the limiting singularities of critical points may coincide with minimal varieties, the associated variational problems exhibit substantial qualitative differences at both large and small scales.

In [4] we consider instead the self-dual Yang–Mills–Higgs energy

$$E(u,\nabla) := \int_M \Big(|\nabla u|^2 + |F_\nabla|^2 + W(u)\Big) \tag{1}$$

and its rescalings

$$E_\epsilon(u,\nabla) := \int_M \Big(|\nabla u|^2 + \epsilon^2|F_\nabla|^2 + \epsilon^{-2}W(u)\Big), \tag{2}$$

for couples (u,∇) consisting of a section u of a given Hermitian line bundle $L \to M$, and a metric connection ∇ on L. Here, the potential $W : L \to \mathbb{R}$ is given by

$$W(u) := \frac{1}{4}(1-|u|^2)^2, \tag{3}$$

while $F_\nabla \in \Omega^2(\mathrm{End}(L))$ denotes the curvature of ∇. These functionals have a natural $U(1)$-*gauge invariance.*

Taubes [6, 7] studied critical points of (1) for the trivial bundle $L = \mathbb{C} \times \mathbb{R}^2$ on the plane: he gave a complete classification, showing in particular that all finite-energy critical points (u,∇) solve the first order system

$$\nabla_{\partial_1} u \pm i\nabla_{\partial_2} u = 0; \quad *F_\nabla = \pm\frac{1}{2}(1-|u|^2) \tag{4}$$

known as the *vortex equations.* Such solutions minimize energy among pairs (u,∇) with fixed vortex number $N := \frac{1}{2\pi}\int_{\mathbb{R}^2} *F_\nabla \in \mathbb{Z}$, and carry energy exactly $E(u,\nabla) = 2\pi|N|$.

Hong–Jost–Struwe [2] initiated the asymptotic study of (2) for line bundles $L \to \Sigma$ over a closed Riemann surface Σ, showing that the curvature $*\frac{1}{2\pi}F_{\nabla_\epsilon}$ converges as $\epsilon \to 0$ to a finite sum of Dirac masses of total mass $|\deg(L)|$, away from which ∇_ϵ converges to a flat connection ∇_0, and u_ϵ to a unit section u_0 with $\nabla_0 u_0 = 0$, up to change of gauge.

In [4] we develop the asymptotic analysis as $\epsilon \to 0$ for critical points of E_ϵ associated to line bundles $L \to M$ over Riemannian manifolds M^n of arbitrary dimension $n \geq 2$. The main result is the following, which describes the limiting behavior as $\epsilon \to 0$ of the energy measures and curvatures F_{∇_ϵ} for critical points $(u_\epsilon,\nabla_\epsilon)$ satisfying a uniform energy bound.

Theorem. *Let $L \to M$ be a Hermitian line bundle over a closed, oriented Riemannian manifold M^n of dimension $n \geq 2$, and let $(u_\epsilon,\nabla_\epsilon)$ be a family of critical points for E_ϵ satisfying a uniform energy bound*

$$E_\epsilon(u_\epsilon,\nabla_\epsilon) \leq \Lambda < \infty.$$

Then, as $\epsilon \to 0$, the energy measures

$$\mu_\epsilon := \frac{1}{2\pi} e_\epsilon(u_\epsilon, \nabla_\epsilon)\,\mathrm{vol}_g$$

converge subsequentially, in duality with $C^0(M)$, to the weight measure μ of a stationary, integral $(n-2)$-varifold V. Also, for all $0 \le \delta < 1$,

$$\mathrm{spt}(V) = \lim_{\epsilon\to 0} \{|u_\epsilon| \le \delta\}$$

in the Hausdorff topology. The $(n-2)$-currents dual to the curvature forms $\frac{1}{2\pi}F_{\nabla_\epsilon}$ converge subsequentially to an integral $(n-2)$-cycle Γ, with $|\Gamma| \le \mu$.

Roughly speaking, this result says that the energy of the critical points concentrates near the zero sets $u_\epsilon^{-1}(0)$ of u_ϵ as $\epsilon \to 0$, which converge to a (possibly rather singular) minimal submanifold of codimension two.

Note that unit sections of a Hermitian line bundle are indistinguishable up to change of gauge: for a given unit section u of L, one can always choose locally a connection with respect to which u appears constant. Thus, while most of the energy of solutions v_ϵ to the complex Ginzburg–Landau equations falls on annular regions—relatively far from the zero set—where v_ϵ resembles a harmonic S^1-valued map, the energy $e_\epsilon(u_\epsilon, \nabla_\epsilon)$ of a critical pair $(u_\epsilon, \nabla_\epsilon)$ for (2) instead concentrates near the zero set $u_\epsilon^{-1}(0)$, with $|\nabla_\epsilon u_\epsilon|$ vanishing *rapidly* outside this region, allowing for an easier and more effective blow-up analysis.

The obvious advantages of this theorem over analogous results for the complex Ginzburg–Landau equations are the *integrality* of the limit varifold V—due ultimately to the aforementioned quantization of the energy of entire planar solutions—and the concentration of the *full energy measure* to V, independent of the topology of M. Also, the analysis of (2) aligns much more closely with the work of Hutchinson–Tonegawa on the Allen–Cahn equations.

However, while the analysis of the Allen–Cahn functionals does not depend on the precise choice of the double-well potential W, the analysis of the abelian Yang–Mills–Higgs functionals (1)–(2) seems to depend *quite strongly* on the choice $W(u) = \frac{1}{4}(1-|u|^2)^2$. Indeed, already in two dimensions, replacing W with a potential $W_\lambda(u) := \frac{\lambda}{4}(1-|u|^2)^2$ for some $\lambda \neq 1$ yields a dramatically different qualitative behavior, breaking the symmetry which leads to the first-order equations (4), and introducing interactions between disjoint components of the zero set.

We also have the following general existence result, showing that nontrivial families satisfying the hypotheses of our main theorem arise naturally on any line bundle (including, importantly, the trivial bundle) over any Riemannian manifold M^n, from variational constructions.

Theorem. *For any Hermitian line bundle $L \to M$ over an arbitrary closed base manifold M^n, there exists a family of critical points $(u_\epsilon, \nabla_\epsilon)$ with bounded energies $E_\epsilon(u_\epsilon, \nabla_\epsilon)$ and nonempty zero sets $u_\epsilon^{-1}(0) \neq \emptyset$. In particular, the energy μ_ϵ of these families concentrates (subsequentially) on a nontrivial stationary integral $(n-2)$-varifold V as $\epsilon \to 0$.*

For nontrivial bundles $L \to M$, this is obtained by looking at minimizers $(u_\epsilon, \nabla_\epsilon)$ of E_ϵ. In this case, we expect moreover that the limiting minimal variety, together with its multiplicity, coincides with the weight measure $|\Gamma|$ of the limiting $(n-2)$-cycle $\Gamma = \lim_{\epsilon\to 0} * \frac{1}{2\pi} F_{\nabla_\epsilon}$, and that Γ minimizes $(n-2)$-area in its homology class, which is Poincaré dual to the first Chern class $c_1(L)$.

For the trivial bundle $L \cong \mathbb{C} \times M$, we use instead min-max methods. While in [4] we consider only one min-max construction, we mention that many more may be carried out in principle, due to the rich topology of the space

$$\mathcal{M} := \{(u, \nabla) : 0 \not\equiv u \in \Gamma(\mathbb{C} \times M),\ \nabla \text{ a Hermitian connection}\}/\mathcal{G},$$

where $\mathcal{G} := \mathrm{Maps}(M, S^1)$ is the gauge group. It may be of interest to note that the homotopy groups $\pi_i(\mathcal{M})$ are isomorphic to those of the space $\mathcal{Z}_{n-2}(M; \mathbb{Z})$ of integral $(n-2)$-cycles considered by Almgren.

As an application of our results, we obtain a new proof of the existence of stationary integral $(n-2)$-varifolds in an arbitrary Riemannian manifold—a result first proved by Almgren using his geometric measure theory framework.

References

[1] D. R. Cheng. *Geometric Variational Problems: Regular and Singular behavior.* PhD thesis, Stanford University, 2017.

[2] M.-C. Hong, J. Jost and M. Struwe. *Asymptotic limits of a Ginzburg–Landau type functional*, chapter in *Geometric analysis and the calculus of variations*, 99–123. Int. Press, Cambridge, MA, 1996.

[3] J. E. Hutchinson and Y. Tonegawa. Convergence of phase interfaces in the van der Waals–Cahn–Hilliard theory. *Calc. Var. Partial Differential Equations* 10 (2000), no. 1, 49–84.

[4] A. Pigati and D. Stern. Minimal submanifolds from the abelian Higgs model. *ArXiv preprint* 1905.13726, 2019.

[5] D. Stern. Existence and limiting behavior of min-max solutions of the Ginzburg–Landau equations on compact manifolds. To appear in *J. Differential Geom.*

[6] C. H. Taubes. Arbitrary N-vortex solutions to the first order Ginzburg–Landau equations. *Comm. Math. Phys.* 72 (1980), no. 3, 277–292.

[7] C. H. Taubes. On the equivalence of the first and second order equations for gauge theories. *Comm. Math. Phys.* 75 (1980), no. 3, 207–227.

Multiplicity One Conjecture in Min-max theory

Xin Zhou

The min-max theory is a powerful tool to find minimal surfaces, which are the mathematical models for soap films. Motivated by Yau's conjecture on minimal surfaces [15], Marques and Neves proposed a program to establish the Morse theory for the Area functional [6, 7, 8], in which they explored the notion of "volume spectrum" introduced by Gromov in 1980s [4]. One of their goals is to understand the key feature of the min-max theory, that is, the Morse index. The longstanding challenge of min-max theory, especially for Marques-Neves's program, was the "*Multiplicity One Conjecture*" [8, 1.2]. The conjecture said that minimal hypersurfaces produced by the min-max theory are always two-sided and have multiplicity one a generic scenario. This conjecture is a natural nonlinear analog of a famous result by Uhlenbeck [12] for the linear "Laplacian spectrum" in 1960s. This conjecture was proved by the author in [16].

Now we start to state the precise result. Let (M^{n+1}, g) be a closed orientable Riemannian manifold of dimension $3 \leq (n+1) \leq 7$. In [1], Almgren proved that the space of mod-2 cycles $\mathcal{Z}_n(M, \mathbb{Z}_2)$ is weakly homotopic the Eilenberg-MacLane space $K(\mathbb{Z}_2, 1) = \mathbb{RP}^\infty$; (see also [8] for a simpler proof). Later, Gromov [4], Guth [5], Marque-Neves [7] introduced the notion of volume spectrum as a nonlinear version of spectrum for the area functional in $\mathcal{Z}_n(M, \mathbb{Z}_2)$. In particular, the volume spectrum is a non-decreasing sequence of positive numbers

$$0 < \omega_1(M, g) \leq \cdots \leq \omega_k(M, g) \leq \cdots \to +\infty,$$

which is uniquely determined by the metric g in a given closed manifold M.

By adapting the celebrated min-max theory developed by Almgren [2], Pitts [9] (for $3 \leq (n+1) \leq 6$), and Schoen-Simon [11] (for $n+1 = 7$), Marques-Neves [7, 6] proved that each $\omega_k(M, g)$ is associated with an integral varifold V_k whose support is a disjoint collection of smooth, connected, closed, embedded, minimal hypersurfaces $\{\Sigma_1^k, \cdots, \Sigma_{l_k}^k\}$, such that

$$\omega_k(M, g) = \sum_{i=1}^{l_k} m_i^k \cdot \mathrm{Area}(\Sigma_i^k), \tag{1}$$

where $\{m_1^k, \cdots, m_{l_k}^k\} \subset \mathbb{N}$ is a set of positive integers, usually called *multiplicities*.

Our main theorem states that if a component Σ_i^k is not *weakly stable*, then Σ_i^k has to be two-sided and the associated integer multiplicity is identically equal to one, i.e. $m_i^k = 1$. Note that a closed minimal hypersurface Σ is said to be *weakly stable* if it has a 0 as the lowest eigenvalue for the second variation of area; (when Σ is one-sided, one has to pass to its two-sided double cover).

Theorem 1. *Given a closed manifold (M^{n+1}, g) of dimension $3 \leq (n+1) \leq 7$, denote $\{\Sigma_i^k : k \in \mathbb{N}, i = 1, \cdots, l_k\}$ as the min-max minimal hypersurfaces associated with volume spectrum. Then every connected component of $\{\Sigma_i^k : k \in \mathbb{N}, i = 1, \cdots, l_k\}$ which is not weakly stable is two-sided and has multiplicity one. That is, if Σ_i^k is not weakly stable, $k \in \mathbb{N}$, $1 \leq i \leq l_k$, then Σ_i^k is two-sided and $m_i^k = 1$, and*

$$\sum_{i=1}^{l_k} \mathrm{index}(\Sigma_i^k) \leq k.$$

Remark 2. Theorem 1 is an equivalent formulation of the *Multiplicity One Conjecture* of Marques-Neves [8, 1.2] proved by the author in [16, Theorem A]. Indeed, [16, Theorem A] asserts that for a bumpy metric g, all connected components of $\{\Sigma_i^k : k \in \mathbb{N}, i = 1, \cdots, l_k\}$ are two-sided and have multiplicity one. Theorem 1 directly implies [16, Theorem A], as weakly stable minimal hypersurfaces are degenerate and hence do not exist in a bumpy metric. Now we argue that that [16, Theorem A] implies Theorem 1. A metric g is called *bumpy* if every closed immersed minimal hypersurface is non-degenerate. White proved that the set of bumpy metrics is generic in Baire sense [13, 14]. For an arbitrary metric g, we can take a sequence of bumpy metrics $\{g_j\}_{j\in\mathbb{N}}$ such that $g_j \to g$ smoothly. We also know that the k-widths $\{\omega_k(M, g_j)\}_{j\in\mathbb{N}}$ converges to $\omega_k(M, g)$ as $j \to \infty$ for each $k \in \mathbb{N}$. Now fix $k \in \mathbb{N}$; for each g_j, the associated min-max minimal hypersurfaces $V_{k,j}$ are all two-sided and have multiplicity one by [16, Theorem A]. By the compactness theorem [10, Theorem A.6], V_k converges up to a subsequence to a limit integral varifold V, such that the support $\mathrm{spt}(V)$ of V is smooth embedded minimal hypersurfaces. Now using [10, Theorem A.6] again, if a connected component of $\mathrm{spt}(V)$ either has multiplicity greater than one or is one-sided, it (or its two-sided double cover when one-sided) has to carry a positive Jacobi field for the second variation of area, and hence it is weakly stable.

Remark 3. Recently, Chodosh-Mantoulidis [3] proved this conjecture in dimension three $(n+1) = 3$ for the Allen-Cahn setting; they also proved that the total index is exactly k for their k-min-max solutions when $(n+1) = 3$. After our results were posted, Marques-Neves finished their program and also proved the same optimal index estimates for $3 \leq (n+1) \leq 7$ [8, Addendum].

References

[1] F. Almgren. The homotopy groups of the integral cycle groups. *Topology*, 1:257–299, 1962.

[2] F. Almgren. *The theory of varifolds.* Mimeographed notes. Princeton, 1965.

[3] Otis Chodosh and Christos Mantoulidis. Minimal surfaces and the allen-cahn equation on 3-manifolds: index, multiplicity, and curvature estimates. *arXiv preprint arXiv 1803.02716*, 2018.

[4] M. Gromov. Dimension, nonlinear spectra and width. In *Geometric aspects of functional analysis (1986/87)*, volume 1317 of *Lecture Notes in Math.*, pages 132–184. Springer, Berlin, 1988.

[5] Larry Guth. Minimax problems related to cup powers and Steenrod squares. *Geom. Funct. Anal.*, 18(6):1917–1987, 2009.

[6] F. C. Marques and A. Neves. Morse index and multiplicity of min-max minimal hypersurfaces. *Camb. J. Math.*, 4(4):463-511, 2016.

[7] F. C. Marques and A. Neves. Existence of infinitely many minimal hypersurfaces in positive Ricci curvature. *Invent. Math.*, 209(2):577–616, 2017.

[8] F. C. Marques and A. Neves. Morse index of multiplicity one min-max minimal hypersurfaces. *arXiv preprint arXiv:1803.04273v1*, 2018.

[9] J. Pitts. *Existence and regularity of minimal surfaces on Riemannian manifolds*, volume 27 of *Mathematical Notes.* Princeton University Press, Princeton, N.J., 1981.

[10] Ben Sharp. Compactness of minimal hypersurfaces with bounded index. *J. Differential Geom.*, 106(2):317–339, 2017.

[11] R. Schoen and L. Simon. Regularity of stable minimal hypersurfaces. *Comm. Pure Appl. Math.*, 34(6):741–797, 1981.

[12] K. Uhlenbeck. Generic properties of eigenfunctions. *Amer. J. Math.*, 98(4):1059–1078, 1976.

[13] Brian White. The space of minimal submanifolds for varying Riemannian metrics. *Indiana Univ. Math. J.*, 40(1):161–200, 1991.

[14] Brian White. On the bumpy metrics theorem for minimal submanifolds. *Amer. J. Math.*, 139(4):1149–1155, 2017.

[15] Shing Tung Yau. Problem section. In *Seminar on Differential Geometry*, volume 102 of *Ann. of Math. Stud.*, pages 669–706. Princeton Univ. Press, Princeton, N.J., 1982.

[16] Xin Zhou. On the Multiplicity One Conjecture in min-max theory. *arXiv preprint arXiv:1901.01173*, 2019.

An existence theorem for Brakke flow with fixed boundary condition

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(joint work with Salvatore Stuvard, University of Texas at Austin)

A family of hypersurfaces $\{\Gamma_t\}_{t\geq 0}$ in $\mathbb{R}^{n+1}$ is called a mean curvature flow (MCF) if the mean curvature vector of Γ_t is equal to the normal velocity at each point and time. One can formulate a generalized notion of MCF in the setting of varifold called the Brakke flow. The main new result reported in this talk is the global-in-time existence of Brakke flow in a strictly convex domain, starting from given set and fixing the boundary condition. We work under the following assumptions.

Assumption 1. *Let $\Omega_0 \subset \mathbb{R}^{n+1}$ be a strictly convex domain with C^2 boundary $\partial\Omega_0$. Assume that*

(a) $\Gamma_0 \subset \Omega_0$ is a relatively closed countably n-rectifiable set with $\mathcal{H}^n(\Gamma_0) < \infty$, where $\mathcal{H}^n$ is the n-dimensional Hausdorff measure.

(b) $E_{1,0}, \ldots, E_{N,0}$ are mutually disjoint non-empty open sets in Ω_0 such that

$$\bigcup_{i=1}^{N} E_{i,0} = \Omega_0 \setminus \Gamma_0,$$

where $N \geq 2$.

(c) Define

$$\partial\Gamma_0 := \operatorname{clos}\Gamma_0 \setminus \Omega_0$$

where $\operatorname{clos}\Gamma_0$ is the closure of Γ_0 in $\mathbb{R}^{n+1}$. Then, for each $x \in \partial\Gamma_0$, there exist $i, i' \in \{1, \ldots, N\}$ with $i \neq i'$, sequences $x_j, x'_j \in \partial\Omega_0$ with $\lim_{j\to\infty} x_j = \lim_{j\to\infty} x'_j = x$, open balls $B_{r_j}(x_j), B_{r'_j}(x'_j) \subset \mathbb{R}^{n+1}$ with $B_{r_j}(x_j) \cap \Omega_0 = B_{r_j}(x_j) \cap E_{i,0}$ and $B_{r'_j}(x'_j) \cap \Omega_0 = B_{r'_j}(x'_j) \cap E_{i',0}$.

The set Γ_0 is the initial data. Non-empty open sets $E_{1,0} \ldots, E_{N,0}$ make up the complement of Γ_0 and we may think them to be a "labeling of domains". Since $N \geq 2$, we are implicitly assuming that $\Omega_0 \setminus \Gamma_0$ is not connected. The different components need not have different label and the assignment is arbitrary, even though it is most "canonical" if we assume that there are only a finite number of connected components and we assign different labels to each of them. If there are infinitely many components, we may just fix some large N and group them into N open sets suitably. Different labeling typically results in different MCF. The condition (c) asks that each boundary point of $\partial\Gamma_0$ is a genuine boundary point of some $\operatorname{clos} E_i \cap \partial\Omega_0$. The set $\partial\Gamma_0$ can be rather irregular and may be a fractal-like set with $\mathcal{H}^{n-1}(\partial\Gamma_0) = \infty$. The main result is that there exists a Brakke flow $\{\mu_t\}_{t\geq 0}$ starting from Γ_0 having the fixed boundary $\partial\Gamma_0$. More precisely,

Theorem 1. *Under the Assumption 1, there exists a family of Radon measures $\{\mu_t\}_{t\geq 0}$ defined on Ω_0 with the following properties:*

(1) $\mu_0 = \mathcal{H}^n \lfloor_{\Gamma_0}$.

(2) For almost all $t \in [0,\infty)$, there exist an $\mathcal{H}^n$-measurable countably n-rectifiable set $\Gamma_t \subset \Omega_0$ and an $\mathcal{H}^n$-measurable function $\theta_t : \Gamma_t \to \mathbb{N}$ such that $\mu_t = \theta_t \mathcal{H}^n\lfloor_{\Gamma_t}$.

(3) For almost all $t \in [0,\infty)$, there exists a generalized mean curvature vector $h(\mu_t, x)$ of μ_t such that, for all $T > 0$, we have

$$\mu_T(\Omega_0) + \int_0^T dt \int_{\Omega_0} |h(\mu_t, x)|^2 \, d\mu_t(x) \leq \mathcal{H}^n(\Gamma_0).$$

(4) For all $0 \leq t_1 < t_2 < \infty$ and for all $\phi \in C^1_c(\Omega_0 \times [0,\infty); \mathbb{R}_+)$, we have

$$\int_{\Omega_0} \phi(x,t) \, d\mu_t(x) \Big|_{t=t_1}^{t_2} \leq \int_{t_1}^{t_2} dt \int_{\Omega_0} (\nabla\phi(x,t) - \phi(x,t) h(\mu_t, x)) \cdot h(\mu_t, x) + \frac{\partial \phi}{\partial t} \, d\mu_t(x).$$

(5) For all $t \geq 0$, $\partial\Omega_0 \cap \operatorname{clos} \operatorname{spt} \mu_t = \partial\Gamma_0$.

The property (2) says that μ_t is an integral varifold for almost all t and θ_t is the multiplicity function of μ_t. The generalized mean curvature vector $h(\mu_t, x)$ in (3) is defined in the usual sense of varifold, and the implicit claim here is that the first variation of μ_t is bounded and absolutely continuous with respect to μ_t for almost all t. Note that we are not particularly assuming that $\mathcal{H}^n\lfloor_{\Gamma_0}$ has a bounded first variation and one can see that an instantaneous regularization is taking place under the MCF. The inequality in (4) is a weak formulation of "normal velocity = mean curvature vector". If given a smoothly moving family of surfaces $\{\Gamma_t\}_{t\geq 0}$, and if $\mu_t := \mathcal{H}^n\lfloor_{\Gamma_t}$ satisfies the inequality in (4) for all non-negative test function ϕ, then it must be a MCF and vice versa. The claim (5) says in a rigorous term that the boundary of μ_t is fixed and equal to $\partial\Gamma_0$ for all time. We also have accompanying existence of "partitions" which make up the complement of the support of $d\mu_t dt$ in the space-time $\Omega_0 \times [0,\infty)$ which are continuous in time in a suitable sense. For the proof, we modify the time-discrete construction of [1] so that we may fix the boundary data. To do so, in each discrete time step, smoothed mean curvature vector is truncated by a cut-off function which is exponentially small near $\partial\Gamma_0$. To make sure that the surfaces stay inside Ω_0 throughout the construction, we need to insert an extra retraction step in addition to the Lipschitz deformation and motion by the smoothed (and truncated) mean curvature vector.

References

[1] L. Kim, Y. Tonegawa *On the mean curvature flow of grain boundaries*, Annales de l'Institut Fourier (Grenoble) **67** (2017), no. 1, 43–142.

Plateau's problem as a singular limit of capillarity problems

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(joint work with Darren King, Antonello Scardicchio, Salvatore Stuvard)

Minimal surfaces with prescribed boundary provide the basic model for soap films hanging from a wire frame: given a boundaryless $(n-1)$-dimensional surface $\Gamma \subset \mathbb{R}^{n+1}$, one looks for n-dimensional surfaces M with $\partial M = \Gamma$ and with zero mean curvature $H_M = 0$; here $n \geq 1$, with $n = 2$ in the physical case. A limitation in the descriptive power of this model is that it has no length scale (e.g., $t\,M$ is a minimal surface with boundary $t\,\Gamma$, no matter how large $t > 0$ is). Since the most basic length scale involved in this problem is the volume ε of the soap film, we are led to the question of framing Plateau's problem in the context of capillarity theory (area minimization at fixed volume).

As a reference formulation of Plateau's problem we adopt the one introduced by Harrison and Pugh in [1], in its slight generalization considered in [2],

$$(1) \qquad \ell = \inf\Big\{\mathcal{H}^n(S) : S \text{ relatively closed in } \Omega,\ S \text{ is } \mathcal{C}\text{-spanning } W\Big\}.$$

Here W is the region occupied by the wire (think of a δ-neighborhood of Γ for a small $\delta > 0$), $\Omega = \mathbb{R}^{n+1} \setminus W$ is the region accessible to the candidate surfaces, $\mathcal{C}$ is a homotopically non-trivial and homotopically closed class of smooth embeddings of $\mathbb{S}^1$ into Ω, and S is said to be $\mathcal{C}$-spanning W iff $S \cap \gamma \neq \emptyset$ for every $\gamma \in \mathcal{C}$. By the results in [1, 2], minimizers of (1) exist as soon as $\ell < \infty$. Moreover, they satisfy Plateau's laws in the physical case $n = 2$ by [5].

The capillarity problem we want to consider is thus

$$(2) \qquad \psi(\varepsilon) = \inf\Big\{\mathcal{H}^n(\Omega \cap \partial E) : |E| = \varepsilon,\ E \subset \Omega,\ \Omega \cap \partial E \text{ is } \mathcal{C}\text{-spanning } W\Big\},$$

where the sets E are assumed to be open and such that ∂E is $\mathcal{H}^n$-rectifiable. We notice that the spanning condition on $\Omega \cap \partial E$ is imposed to exclude that global minimizers look like round droplets sitting a points of high curvature of ∂W, and to force them to actually resemble soap films. Given a minimizing sequence $\{E_j\}_j$ converging to a limit set E, an obvious difficulty is that $\Omega \cap \partial E_j$ may converge with multiplicity larger than 1 towards a surface K which strictly contains ∂E. Denoting by $\partial^* E$ the reduced boundary of a set of finite perimeter, we first prove the following existence result:

Theorem 1. *Assume that $\ell < \infty$, that ∂W is smooth, that there exists $\tau_0 > 0$ such that $\mathbb{R}^{n+1} \setminus I_\tau(W)$ is connected for every $\tau < \tau_0$, and that there exists a minimizer S of ℓ and $\eta_0 > 0$ such that $I_{\eta_0}(S) \cap \gamma \neq \emptyset$ for every $\gamma \in \mathcal{C}$.*

If $\{E_j\}_j$ is minimizing sequence for $\psi(\varepsilon)$, then, up to possibly extracting a subsequence and up to possibly modify each E_j outside of a large ball containing W (where both operations are still defining a minimizing sequence, which, for simplicity, is still denoted by $\{E_j\}_j$), we have

$$(3) \qquad E_j \to E \text{ in } L^1(\mathbb{R}^{n+1}),$$

$$(4) \qquad \mathcal{H}^n \llcorner (\Omega \cap \partial E_j) \stackrel{*}{\rightharpoonup} 2\,\mathcal{H}^n \llcorner (K \setminus \partial^* E) + \mathcal{H}^n \llcorner \partial^* E\,, \text{ as Radon measures},$$

as $j \to \infty$, where $E \subset \Omega$ is an open set with $\Omega \cap \partial E = \Omega \cap \mathrm{closure}(\partial^* E)$, $|E| = \varepsilon$, $\Omega \cap \partial E \subset K$, and K is an $\mathcal{H}^n$-rectifiable and relatively compact subset of Ω such that K is $\mathcal{C}$-spanning W. Moreover, $\mathcal{F}(K, E) = \psi(\varepsilon)$, where

$$\mathcal{F}(K, E) = 2\,\mathcal{H}^n(K \setminus \partial^* E) + \mathcal{H}^n(\partial^* E)\,,$$

is the relaxed surface tension energy of (K, E).

Because of the identity $\mathcal{F}(K, E) = \psi(\varepsilon)$, a pair (K, E) as in Theorem 1 is called a *generalized minimizer of $\psi(\varepsilon)$*. When $K = \Omega \cap \partial E$, then E is a minimizer of $\psi(\varepsilon)$, but in general K could be strictly larger than $\Omega \cap \partial E$, and in the latter situation we say that *collapsing* occurs.

When W consists of two small disjoint disks in the plane, then $K = \Omega \cap \partial E$ consists of two very flat circular arcs touching W orthogonally. In this case the region E has indeed a small thickness, proportional to ε. At fixed ε, by moving the two disks far away we see that this thickness becomes increasingly smaller. Below a certain thickness threshold, punctured configurations becomes energetically very close to the minimizer, and the probability transition towards such unstable states becomes increasingly consistent. This is an example of a physical feature of actual soap films which is unaccessible to a formulation via minimal surfaces.

When W is obtained by taking a δ-neighborhood of the three vertexes of an equilateral triangle, then the unique minimizer in ℓ consists of a triple junction at the center of the triangle. For small ε, collapsing occurs in $\psi(\varepsilon)$, and the generalized minimizer (K, E) consists of a central circular curvilinear triangle of area ε, joined to the boundary W by three segments of multiplicity 2. This result indicates that, in the presence of singularities, the volume and the thickness of an actual soap film are independent physical parameters. This is another physical feature of soap films which cannot be described relying just on minimal surfaces.

Theorem 2. *Under the assumptions of Theorem 1, if (K, E) is a generalized minimizer of $\psi(\varepsilon)$ and f is a diffeomorphism of Ω into Ω such that $|f(E)| = \varepsilon$, then $\mathcal{F}(K, E) \le \mathcal{F}(f(K), f(E))$. In particular there exists $\lambda \in \mathbb{R}$ such that*

$$\lambda \int_{\partial^* E} X \cdot \nu_E \, d\mathcal{H}^n = \int_{\partial^* E} \mathrm{div}^T X \, d\mathcal{H}^n + 2 \int_{K \setminus \partial^* E} \mathrm{div}^T X \, d\mathcal{H}^n$$

whenever $X \in C^\infty_c(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$ with $X \cdot \nu_\Omega = 0$ along $\partial\Omega$, and where div^T denotes the tangential divergence operator.

Thanks to Theorem 2 and to Allard's regularity theorem for integer rectifiable varifolds, we find the existence of a closed set $\Sigma \subset K$, relatively meager in K, such that $K \setminus \Sigma$ is a smooth hypersurface. In fact, $K \setminus (\Sigma \cup \partial E)$ is a smooth minimal surface, $\partial^* E$ is a smooth hypersurface with constant mean curvature equal to λ, $\mathcal{H}^n(\Sigma \setminus \partial E) = 0$ and $\partial E \setminus \partial^* E$ is meager in K and contained in Σ. We thus have two interesting free boundary problems: (i) on the transition region $\partial E \setminus \partial^* E$; (ii) on the wetted region of the wire $\partial W \cap \mathrm{closure}(K \cup E)$.

Finally, we discuss the convergence towards Plateau's problem.

Theorem 3. *Under the assumptions of Theorem 1, $\psi(\varepsilon) \to 2\,\ell$ as $\varepsilon \to 0^+$. More precisely, if (K_j, E_j) is a sequence of generalized minimizers of $\psi(\varepsilon_j)$ for some $\varepsilon_j \to 0^+$ as $j \to \infty$, then, up to possibly extracting a subsequence, there exists a minimizer S of ℓ such that, as $j \to \infty$,*

$$2\,\mathcal{H}^n \llcorner (K_j \setminus \partial^* E_j) + \mathcal{H}^n \llcorner \partial^* E_j \overset{*}{\rightharpoonup} 2\,\mathcal{H}^n \llcorner S, \quad \text{as Radon measures.} \tag{5}$$

Thus Plateau's problem ℓ is the singular limit as $\varepsilon \to 0^+$ of the capillarity problems $\psi(\varepsilon)$, and this limit provides a selection principle for minimizers in ℓ based on the size of their singular sets. For example, in the planar case, simple examples show that generalized minimizers of $\psi(\varepsilon)$ will necessarily converge to those minimizers of ℓ with the largest number of singular points.

For a more complete discussion on the physical and mathematical meaning of this singular limit we refer to the two papers [4, 3]. In particular, the three theorems above are proved in [3].

References

[1] J. Harrison and H. Pugh. Existence and soap film regularity of solutions to Plateau's problem. *Adv. Calc. Var.*, 9(4):357–394, 2016.

[2] C. De Lellis, F. Ghiraldin, and F. Maggi. A direct approach to Plateau's problem. *J. Eur. Math. Soc. (JEMS)*, 19(8):2219–2240, 2017.

[3] D. King, F. Maggi, and S. Stuvard. Plateau's problem as a singular limit of capillarity problems. arXiv:1907.00551.

[4] F. Maggi, A. Scardicchio, and S. Stuvard. Soap films with gravity and almost-minimal surfaces. *Discrete Contin. Dyn. Syst.*, 2019.

[5] J. E. Taylor. The structure of singularities in soap-bubble-like and soap-film-like minimal surfaces. *Ann. of Math. (2)*, 103(3):489–539, 1976.

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