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Mathematical Aspects of Hydrodynamics

Organized by
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Abstract. The workshop dealt with the partial differential equations that describe fluid motion and related topics. These topics included both inviscid and viscous fluids in two and three dimensions. Some talks addressed aspects of fluid dynamics such as the construction of wild weak solutions, compressible shock formation, inviscid limit and behavior of boundary layers, as well as both polymer/fluid and structure/fluid interaction.

Mathematics Subject Classification (2010): 76xx, 35xx.

Introduction by the Organizers

The workshop *Mathematical Aspects of Hydrodynamics*, organised by Peter Constantin (Princeton) Anna Mazzucato (University Park), Gregory Seregin (Oxford), Edriss S. Titi (Rehovot/College Station) featured 47 participants with broad geographic representation. The primary aim of the workshop was to bring together leading experts working on the mathematical theory of hydrodynamics models, and to have extensive discussions of recent developments and possible future directions of research in this area. The study of hydrodynamics leads to a variety of challenging mathematical issues, which touch different areas of mathematics, partial differential equations, harmonic analysis, and dynamical systems, among others.

The program of the workshop consisted in 24 talks, presented by international specialists in fluid dynamics and partial differential equations coming from Canada, Czech Republic, Brazil, England, France, Germany, Italy, Poland, South Korea, Spain, U.S.A.. Moreover, several doctoral and post-doctoral fellows participated

in the workshop and benefited from the unique academic atmosphere at the Oberwolfach Institute.

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Workshop: Mathematical Aspects of Hydrodynamics

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Abstracts

Onsager Conjecture, and the Kolmogorov 1/3 law

Claude Bardos
(joint work with Edriss Titi, Agneska A. Swierczewska-Gwiazda, Piotr Gwiadzka, and Emil Weidemann)

1. Introduction

Motivations for issues considered in the present talk go back to the Kolmogorov 1/3 law: *In some averaged sense the absence of anomalous energy dissipation in the zero viscosity limit for solutions of Navier−Stokes equations is equivalent to the existence of some averaged $\alpha \geq \frac{1}{3}$ Hölder regularity.* Reduced to solutions of the incompressible Euler equations this turns out to be the Onsager conjecture: *Weak solutions of the incompressible Euler equations conserve the energy as long as they belong to the Hölder space $C^{0,\alpha}$ with $\alpha > \frac{1}{3}$.*

Complete mathematical proofs of this conjecture were given (after first partial results Eyink [7]Â (1994)) by Constantin, E and Titi [5] and followed by several extensions.

Recently Buckmaster, Isett, De Lellis, Székelyhidi and Vlad Vicol (cf. for instance [4]) have shown, for any $\alpha < \frac{1}{3}$, the existence of **some** "admissible wild solutions" Hence the $\alpha > \frac{1}{3}$ regularity is a necessary and sufficient condition **for all solutions** to conserve energy.

On the other hand it is easy to construct cf. [3] Â **some other** solutions of the incompressible Euler equations with almost no regularity hypothesis that do conserve the energy and that are, with no anomalous energy dissipation , the limit of solutions of the Navier−Stokes equations.

However in the presence of a smooth solution of the Euler equations in a domain with boundary a basic theorem of Kato [8] shows that the absence of anomalous energy dissipation is equivalent to the convergence to the smooth solution of the Euler equations.

2. Two Theorems

To underline the role of the boundary condition, first a local, in space time, version of the Onsager has been proposed not only for the Euler equation but also for any system of conservation laws with an extra entropy. Observing that this may not imply global in time conservation of entropy cf. [1] an extra condition is required. For instance for the 0 viscosity limit of solutions of Naviers-Stokes equations this leads to the following.

Theorem 1. *Let u_ν be in $(0,T)\times\Omega$ a family of solutions of Leray-Hopf Navier-Stokes equations with the no slip boundary condition: $u_\nu = 0$ on $\partial\Omega$: And assume the following hypothesis:*

There exists an open subset $V_{\eta_0} = \{x \in \Omega\,, d(x) < \eta_0\}$, *and* $\beta < \infty$ *(both being independent of* ν*):*

(2.1) *No Hölder hypothesis on* p *but* $\sup_\nu \|p_\nu\|_{L^{3/2}((0,T);H^{-\beta}(V_{\eta_0}))} < \infty;$.

For any $\tilde{\Omega} \subset\subset \Omega$ *there exists* $\alpha = \alpha(\tilde{\Omega}) > \frac{1}{3}$ *and a constant* $M(\tilde{\Omega})$ *such that for any* $\nu > 0$ *one has:*

$$\|u_\nu\|_{L^3((0,T);C^{0,\alpha}(\overline{\tilde{\Omega}}))} \leq M(\tilde{\Omega})\,. \tag{2.2}$$

$$\lim_{\eta\to 0}\lim_{\nu\to 0}\int_0^T \frac{1}{\eta}\int_{\{x\in\Omega:\, \frac{\eta}{4}<d(x)<\frac{\eta}{2}<\frac{\eta_0}{2}\}} (\frac{|u_\nu|^2}{2} + p_\nu)u_\nu(t,x)\cdot\vec{n}(\sigma(x))\,dt \leq 0\,. \tag{2.3}$$

Then modulo subsequences u_ν *converges weak*$-*$ in* $L^\infty((0,T);L^2(\Omega))$ *to a weak solution of the Euler equations* $\overline{u_\nu} \in C_{\mathrm{weak}}([0,T);L^2(\Omega))$ *with the same initial data* $u_0(\cdot)$. *Moreover,* $\overline{u_\nu}$ *belongs to* $C([0,T);L^2(\Omega))$ *and conserves the energy. Eventually, there is no anomalous energy dissipation in the vanishing viscosity limit, i.e., for every* $T^* \in (0,T)$ *one has:*

$$\lim_{\nu\to 0}\nu\int_0^{T^*}\int_\Omega |\nabla_x u_\nu(t,x)|^2 dxdt = 0. \tag{2.4}$$

Replacing the above hypothesis by the existence of smooth solution of the Euler equations one extends the Kato theorem as follows.

Theorem 2. *Let* $u(x,t) \in C^1((0,T)\times\Omega)$ *be a "smooth" solution of the incompressible Euler equations, then for vanishing viscosity limit of solutions of the Navier*$-$*Stokes equations with the same initial data* $u_\nu(x,0) = u(x,0)$ *and denoting by* $(x,t) \mapsto w(x,t) \in C^1((0,T);\partial\Omega)$ *any tangent vector vector field on the boundary , the following facts are equivalents:*

(2.5) $\forall w(x,t)\,, with\,, w\cdot\vec{n} = 0\,, \lim_{\nu\to 0}\nu\int_0^T\int_{\partial\Omega}(\frac{\partial u_\nu}{\partial\vec{n}}(\sigma,t))_\tau w(\sigma,t)d\sigma dt = 0$

(2.6) $\lim_{\nu\to 0}\nu\int_0^T\int_{\partial\Omega}((\frac{\partial u_\nu}{\partial\vec{n}}(\sigma,t))_\tau u_\tau(\sigma,t))_- d\sigma dt = 0$

(2.7) $u_\nu(t) \to u(t)$ *in* $L^2(\Omega)$ *uniformly in* $t \in [0,T]\,,$

(2.8) $u_\nu(t) \to u(t)$ *weakly in* $L^2(\Omega)$ *for each* $t \in [0,T]\,,$

(2.9) $\lim_{\nu\to 0}\nu\int_0^T\int_\Omega |\nabla u_\nu(x,t)|^2 dxdt = 0\,,$

(2.10) $\lim_{\nu\to 0}\nu\int_0^T\int_{\Omega\cap\{0<d(x,\partial\Omega)<\frac{\nu}{2}\}} |\nabla u_\nu(x,t)|^2 dxdt = 0\,.$

(2.11) $\lim_{\nu\to 0}\frac{1}{\nu}\int_0^T\int_{\Omega\cap\{\frac{\nu}{4}<d(x,\partial\Omega)<\frac{\nu}{2}\}} |u_\nu(x,t)|^2 dxdt = 0\,.$

3. Remaks and Conclusion

About the theorem 1. The hypothesis (2.2) is genuinely local. A similar hypothesis may be done concerning the pressure. However in the absence of boundary no such hypothesis is needed . Here something very weak (cf (2.1)) seems to be compulsory. Then the convergence to a solution which when restricted to any subdomain $\tilde{\Omega} \subset\subset \Omega$ is $C^{0,\alpha}$ follows. Next with the construction given in [1] the condition (2.3) seems compulsory. Observe such condition involves the Bernouilli pressure in any small neighborhood of the boundary, hence it is automatically satisfied if the sequence u_ν is uniformly (with respect to ν) continuous near the boundary. Extension to such observation to general conservation laws with an entropy leads to some conditions which in general (cf [2]) correspond to hypothesis making the system well posed for short time with smooth initial data.

About the theorem 2. The point (2.6) is direct consequence of (2.5) . From (2.6) one deduces by a Gronwall estimate (2.7) from which (2.8) , (2.9) and (2.10) follow while (2.11) is deduced from (2.10) by the Poincaré inequality. Eventually (2.5) is deduced from (2.11) with the scalar product of the Navier-Stokes equations by an extension of $\hat{w}$ of w inspired by [8] .

When valid these criteria concern situations where the vanishing viscosity limit solution is smooth and where no turbulent wake should escape from the boundary into the bulk of the fluid. They are obviously valid if the Prandlt boundary layer would have up to the time T a smooth solution describing the behavior of these vanishing viscosity solution. In particular the criteria (2.6) says that this is the case when the recirculation of the fluid (due to the no slip boundary condition is not too big with respect to the Reynolds number. In the same way the criteria (2.11) says that the issue of the convergence to a smooth limit and in the same way the absence of anomalous energy dissipation is governed by the amount of energy produced in a sublayer of size of the order of ν detached from the boundary . This is in full agreement with consideration proposed in [6], with numerical simulations in [10] and of course with the theory of the Von Karman Prandlt boundary layer cf. [9] sections : 42-44.

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On the Linear Stability of Vortex Columns

Thierry Gallay
(joint work with Didier Smets)

We investigate the linear stability of *columnar vortices*, which are axisymmetric stationary solutions of the three-dimensional Euler equations

$$\partial_t u + (u \cdot \nabla) u \;=\; -\nabla p\,, \qquad \operatorname{div} u \;=\; 0\,, \qquad x \in \mathbb{R}^3\,, \quad t \in \mathbb{R}\,, \tag{1.1}$$

where $u = u(x,t) \in \mathbb{R}^3$ is the velocity of the fluid and $p = p(x,t) \in \mathbb{R}$ the associated pressure. Introducing cylindrical coordinates (r, θ, z), we decompose the velocity field as $u = u_r e_r + u_\theta e_\theta + u_z e_z$, where e_r, e_θ, e_z are unit vectors in the radial, azimuthal, and vertical directions, respectively. Columnar vortices are stationary solutions of (1.1) of the form

$$u \;=\; V(r)\, e_\theta\,, \qquad p \;=\; P(r)\,, \qquad r > 0\,, \tag{1.2}$$

where $V : \mathbb{R}_+ \to \mathbb{R}$ is the velocity profile, and $P : \mathbb{R}_+ \to \mathbb{R}$ is the associated pressure determined by the centrifugal balance $rP'(r) = V(r)^2$. Other physically relevant quantities are the *angular velocity* Ω and the *vorticity* W:

$$\Omega(r) \;=\; \frac{V(r)}{r}\,, \qquad W(r) \;=\; \frac{1}{r}\,\frac{\mathrm{d}}{\mathrm{d}r}\big(rV(r)\big) \;=\; r\Omega'(r) + 2\Omega(r)\,. \tag{1.3}$$

We always assume that W is a smooth function satisfying $W(0) > 0$, $W'(0) = 0$, and $W(r) \to 0$ (sufficiently fast) as $r \to \infty$.

Stability of columnar vortices was first investigated by Lord Kelvin in 1880 [6]. In this pioneering work, Kelvin reduces the analysis to a family of one-dimensional problems by considering perturbations of the form

$$u \;=\; V(r) e_\theta + u_{m,k}(r,t)\, e^{im\theta}\, e^{ikz}\,, \qquad p \;=\; P(r) + p_{m,k}(r,t)\, e^{im\theta}\, e^{ikz}\,, \tag{1.4}$$

where $m \in \mathbb{Z}$ is the angular Fourier mode and $k \in \mathbb{R}$ is the vertical wave number. In [6], the stability analysis is only performed in the particular case of Rankine's vortex, where W is piecewise constant. A very important contribution was made in 1917 by Lord Rayleigh [5], who established a sufficient condition for spectral stability with respect to *axisymmetric* perturbations ($m = 0$):

$$\Phi(r) \;\geq 0 \quad \text{for all } r > 0\,, \qquad \text{where} \quad \Phi(r) \;=\; 2\Omega(r) W(r)\,. \tag{1.5}$$

That condition is also necessary, as was proved by J. L. Synge in 1933. In a different direction, if we restrict our attention to *two-dimensional* perturbations

($k = 0$), the techniques introduced by Rayleigh for the study of shear flows [4] allow us to prove spectral stability provided the vorticity profile is monotone:

$$W'(r) \leq 0\,, \qquad \text{for all } r > 0\,. \tag{1.6}$$

This criterion, however, is not known to be sharp.

It is widely believed, and sometimes rashly stated in the physical literature, that (1.5), (1.6) provide sufficient conditions for spectral stability with respect to arbitrary perturbations, with no particular symmetry. This conjecture is in agreement with experiments and numerical simulations, but we did not find it supported by any serious argument. The most sophisticated result in the aftermath of Rayleigh's work is due to Howard and Gupta [3], who proved in 1962 that a columnar vortex is spectrally stable with respect to perturbations of the form (1.4) provided

$$k^2 \frac{\Phi(r)}{\Omega'(r)^2} \geq \frac{m^2}{4}\,, \qquad \text{for all } r > 0\,. \tag{1.7}$$

Unfortunately, it is clear that no vortex profile satisfies condition (1.7) for all values of m and k, so that Howard and Gupta's approach does not provide any unconditional stability result.

In a joint work with D. Smets [1,2], we consider a columnar vortex of the form (1.2) where the velocity profile V satisfies the following assumptions:

Assumption H1: *The vorticity profile $W : [0,\infty) \to (0,\infty)$ associated to V is a $\mathcal{C}^2$ function satisfying $W'(0) = 0$, $W'(r) < 0$ for all $r > 0$, $r^3 W'(r) \to 0$ as $r \to \infty$, and $\int_0^\infty W(r) r \,\mathrm{d}r < \infty$.*

Assumption H2: *The $\mathcal{C}^1$ function $J : (0,\infty) \to (0,\infty)$ defined by*

$$J(r) = \frac{\Phi(r)}{\Omega'(r)^2}\,, \qquad r > 0\,, \tag{1.8}$$

satisfies $J'(r) < 0$ for all $r > 0$ and $rJ'(r) \to 0$ as $r \to \infty$.

Roughly speaking, assumption H1 means that we consider a localized vortex satisfying both the monotonicity condition (1.6) and Rayleigh's criterion (1.5). The more technical assumption H2 appears to be satisfied in all classical examples, including the Lamb-Oseen vortex, but we do not know if it is essential.

The linearized equations at the columnar vortex (1.2) take the following form:

$$\begin{aligned} \partial_t u_r + \Omega \partial_\theta u_r - 2\Omega u_\theta &= -\partial_r p\,, \\ \partial_t u_\theta + \Omega \partial_\theta u_\theta + W u_r &= -\tfrac{1}{r}\partial_\theta p\,, \\ \partial_t u_z + \Omega \partial_\theta u_z &= -\partial_z p\,, \end{aligned} \tag{1.9}$$

where the pressure $p = P[u]$ is determined (up to a constant) by the elliptic equation

$$-\frac{1}{r}\partial_r\big(r\partial_r p\big) - \frac{1}{r^2}\partial_\theta^2 p - \partial_z^2 p = \frac{2}{r}\partial_r\big(r\Omega\big)\partial_\theta u_r - \frac{2}{r}\partial_r\big(r\Omega\, u_\theta\big)\,. \tag{1.10}$$

We want to solve the evolution equation (1.9) in the Hilbert space

$$X = \left\{ u = (u_r, u_\theta, u_z) \in L^2(\mathbb{R}^3)^3 \,\middle|\, \frac{1}{r}\partial_r(ru_r) + \frac{1}{r}\partial_\theta u_\theta + \partial_z u_z = 0 \right\}, \tag{1.11}$$

which incorporates the incompressibility condition. We rewrite (1.9) as $\partial_t u = Lu$, where L is the integro-differential operator in X defined by

$$Lu = \begin{pmatrix} -\Omega\partial_\theta u_r + 2\Omega u_\theta - \partial_r P[u] \\ -\Omega\partial_\theta u_\theta - W u_r - \frac{1}{r}\partial_\theta P[u] \\ -\Omega\partial_\theta u_z - \partial_z P[u] \end{pmatrix}. \tag{1.12}$$

Our main result can be stated as follows:

Theorem. *Assume that the vorticity profile W satisfies assumptions H1, H2 above. Then the linear operator L defined in* (1.12) *is the generator of a strongly continuous group $(e^{tL})_{t\in\mathbb{R}}$ of bounded linear operators in X. Moreover, for any $\epsilon > 0$, there exists a constant $C_\epsilon \geq 1$ such that*

$$\|e^{tL}\|_{X\to X} \leq C_\epsilon\, e^{\epsilon|t|}, \qquad \text{for all } t \in \mathbb{R}. \tag{1.13}$$

In technical terms, estimate (1.13) means that *growth bound* of the group e^{tL} is equal to zero. Equivalently, the spectrum of e^{tL} is contained in the unit circle of the complex plane for any $t \in \mathbb{R}$. In view of the Hille-Yosida theorem, this implies that the spectrum of the generator L is entirely contained in the imaginary axis, and that the following resolvent bound holds for any $a > 0$:

$$\sup\left\{ \|(z-L)^{-1}\|_{X\to X} \,\middle|\, z \in \mathbb{C},\ |\Re(z)| \geq a \right\} < \infty. \tag{1.14}$$

In fact, since X is a Hilbert space, the celebrated Gearhart-Prüss theorem asserts that the resolvent bound (1.14) is *equivalent* to the group estimate (1.13). From this point of view, the theorem above can be seen as a (strong) spectral stability result for columnar vortices under assumptions H1, H2.

There are reasons to believe that the group norm $\|e^{tL}\|_{X\to X}$ grows at least linearly as $|t| \to \infty$, which would mean that columnar vortices are not linearly stable solutions of the Euler equations. Improving the subexponential estimate (1.13) into a polynomial bound would already be a substantial progress. Of course, the most important question is nonlinear stability, but in the absence of useful variational characterizations of columnar vortices this is a widely open problem.

As for the above theorem, the most difficult part of the proof consists in ruling out the existence of unstable *eigenvalues* of the linear operator L in the subspace defined by the Fourier parameters $m \in \mathbb{Z}$ and $k \in \mathbb{R}$. This can be done by a *homotopy* argument, namely a continuous deformation of the vortex profile to a reference vortex satisfying Howard and Gupta's criterion (1.7). In this way, it is sufficient to perform a spectral analysis in a small neighborhood of the imaginary axis, where critical layers occur; see [1] for details.

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Recent progress on nonlinear inviscid damping for two dimensional Euler equation

Hao Jia

Two dimensional Euler equation is globally well-posed for smooth initial data, by the classical results in [28]. The long time behavior is however not well understood. Statistical theories were developed by Onsager [22] and Kraichnan [21] to explain the emergence of large scale structures and inverse energy cascade observed in numerical simulations and physical experiments [3, 23]. However, rigorously justifying these theories from the two dimensional Euler equation seems to be out of reach of current PDE techniques, similar to the situation in statistical mechanics. Mathematically, it is more promising to firstly study dynamics near physically relevant steady state solutions, such as vortices and shear flows.

Stability analysis is a classical problem in hydrodynamics. Early pioneers include Kelvin [10], Orr [13], Rayleigh [14], Taylor [24], among many others. Recently, there has been significant development in the analysis of linearized flow near shear flows and vortices, see [2, 7, 16, 17, 19, 20]. In particular, the works [2, 16] proved optimal decay rate of the stream function in Sobolev spaces, from which linear stability follows.

Nonlinear asymptotic stability is on the other hand much harder. Bedrossian and Masmoudi [1] made a breakthrough and proved the first nonlinear asymptotic stability result for the Couette flow which is the linear shear flow in $\mathbb{T} \times \mathbb{R}$. The result was subsequently extended by Ionescu and the speaker [8] to the domain $\mathbb{T} \times [0, 1]$. The main motivation was to consider finite energy solutions and study the possible boundary effects. Asymptotic stability near general shear flows is still open.

In [9] Ionescu and the speaker initiated the nonlinear asymptotic analysis of vortices, in the case of point vortices. Point vortices are physically relevant as approximation to the common circumstances where vorticity concentrates sharply in small regions. The case of general vortices remains open.

There are several major difficulties in passing from the nonlinear analysis of Couette flow to general monotone shear flows. The **first** key difficulty is that the main equations are no longer constant coefficient equations. This causes serious issues since the norms we use to control the nonlinearity are defined using Fourier transforms in a rather delicate way and have no known physical space characterization. The **second** key difficulty is to understand the linear inviscid damping in Gevrey spaces. For some time only Sobolev spaces results were known and the methods could not be used in nonlinear problem.

The first difficulty has recently been resolved by Ionescu and the speaker in the analysis of point vortices [9]. The main idea is to establish finer smoothness properties of the weights and use commutator type argument. Although using commutator argument in proving bounds in variable coefficient setting is not new, the implementation in our case is quite subtle and it turns out that the weight has just enough regularity for such an argument to work.

The second difficulty was recently resolved by the speaker in [19], which proved linear inviscid damping in Gevrey spaces. A crucial point is that the argument in [19] is based on commutator type estimates which can be adapted to the nonlinear problem.

Based on these progresses, the **main remaining difficulty** in proving asymptotic stability and nonlinear inviscid damping near general shear flows is to find methods that effectively combine the tools developed in the linear and nonlinear analysis. It is still a nontrivial task, since the nonlinear analysis is based on controlling a time dependent energy functional, while the analysis of the linearized equation depends heavily on spectral analysis and precise regularity analysis of generalized eigenfunctions associated with continuous spectrum.

Another goal is to prove nonlinear inviscid damping and axi-symmetrization near general vortices. There are many similarities in axi-symmetrization near vortices and inviscid damping near general shear flows. The techniques developed in the context of general shear flows and point vortices [8, 9, 19] will be useful in the general vortices problem as well. However, we expect that the problem of axi-symmtrization on the nonlinear level includes at least two additional important difficulties.

The **first** key difficulty is the degeneracy of rate of mixing at $r = 0$. The **second** key difficulty has to do with $r = 0$ acting somewhat like a boundary and could have a nontrivial boundary effect.

We note that there is another mechanism on the linearized level for the vortices, called "vortex depletion", first oberved in [4] and then rigorously proved in [2, 17], which might help with the above mentioned difficulties. The extension of the mechanism to nonlinear level is however not straightforward.

In the talk, we will discuss these progresses and the difficulties moving to nonlinear inviscid damping near general coherent structures in details.

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Dissipation Enhancement, Mixing and Blowup Suppression

Gautam Iyer
(joint work with Yuanyuan Feng, Xiaoqian Xu, Andrej Zlatoš)

Consider a passive scalar advected by an incompressible, time dependent, velocity field governed by the evolution equation

$$\partial_t\theta + u\cdot\nabla\theta - \kappa\Delta\theta = 0\,. \tag{1.1}$$

For simplicity, we consider equation (1.1) with periodic boundary conditions and mean zero initial data. Our interest is in studying the time decay of $\|\theta_t\|_{L^2}$ as $t\to\infty$. An old result of Poon [11] (see also [9]) shows the double exponential lower bound

$$\|\theta_t\|_{L^2}^2 \geqslant \|\theta_0\|_{L^2}^2 \exp\Big(\frac{-2\kappa\|\nabla\theta_0\|_{L^2}^2}{\|\theta_0\|_{L^2}^2}\Big(\frac{e^{2dUt}-1}{2dU}\Big)\Big)\,, \quad \text{where} \quad U = \sup_{t\geqslant 0}\|\nabla u_t\|_{L^\infty}\,.$$

To the best of our knowledge, we do not know if this is sharp on the torus. In discrete time, however, we show in [4] that the above bound *is* sharp on the torus.

To explain further, let $\varphi\colon \mathbb{T}^d\to\mathbb{T}^d$ be a volume preserving diffeomorphism, and consider the pulsed diffusion

$$\theta_{n+1} = e^{\kappa\Delta}(\theta_n\circ\varphi)\,. \tag{1.2}$$

If φ is the flow map of u after time 1, then (1.2) is equivalent to solving (1.1) for unit time without the diffusion term, and then solving (1.1) for unit time without the convection term. Our result on the L^2 decay can now be stated as follows.

Theorem 1. *For the pulsed diffusion* (1.2) *we have the double exponential lower bound*

$$\|\theta_n\|_{L^2} \geqslant \|\theta_0\|_{L^2}\exp\Big(-\frac{\kappa\|\theta_0\|_{H^1}^2}{\|\theta_0\|_{L^2}^2}\Big(\frac{\|\nabla\varphi\|_{L^\infty}^{2(n+1)} - \|\nabla\varphi\|_{L^\infty}^2}{\|\nabla\varphi\|_{L^\infty}^2 - 1}\Big)\Big)\,,$$

where $\|\nabla\varphi\|_{L^\infty}$ is the supremum of the matrix norm of $\nabla\varphi$. Moreover, there exists a smooth, volume preserving diffeomorphism on the torus for which the above bound is achieved. Explicitly, if φ is any toral automorphism which has no proper invariant rational subspaces, and has no eigenvalues that are roots of unity, then there exists finite constants C, $\gamma>0$ such that

$$\|\theta_n\|_{L^2}^2 \leqslant \|\theta_0\|_{L^2}^2\exp\Big(-\frac{\kappa(1+\gamma)^n}{C}\Big)\,, \tag{1.3}$$

for all $\theta_0\in\dot L^2$.

We now turn to studying upper bounds for solutions to (1.1). Note first that the incompressibility of u and the Poincaré inequality immediately imply that $\|\theta_t\|_{L^2}$ is decreasing as a function of t, and

$$\|\theta_{s+t}\|_{L^2} \leqslant e^{-4\pi^2\kappa t}\|\theta_s\|_{L^2}\,.$$

Thus, we are guaranteed

$$\|\theta_{s+t}\|_{L^2} \leqslant \frac{1}{e}\|\theta_s\|_{L^2}\,, \quad \text{for every} \quad t \geq \frac{1}{4\pi^2\kappa}\,, \tag{1.4}$$

and every $s \geqslant 0$. However, u generates gradients through filamentation, which causes solutions to dissipate energy faster. This may result in the lower bound in (1.4) being attained at much smaller times, and the smallest time t at which this happens is known as the *dissipation time*.

Definition 1.1 (Dissipation time)**.** Let $\mathcal{S}^u_{s,t}$ be the solution operator to (1.1) on $\mathbb{T}^d \times (0,\infty)$. That is, for any $f \in L^2(\mathbb{T}^d)$, the function $\theta_t = \mathcal{S}^u_{s,t} f$ solves (1.1) with initial data $\theta_s = f$, and periodic boundary conditions. Define the *dissipation time* of u by

$$\tau^*(u,\kappa) \stackrel{\text{def}}{=} \inf\Big\{t \geq 0 \,\Big|\, \|\mathcal{S}_{s,s+t}\|_{\dot{L}^2\to\dot{L}^2} \leqslant \frac{1}{e} \text{ for all } s \geqslant 0\Big\}\,. \tag{1.5}$$

Here $\dot{L}^2$ is the space of all mean zero, square integrable functions on the torus $\mathbb{T}^d$.

Note that (1.4) implies $\tau^*(u,\kappa) \leqslant O(1/\kappa)$ as $\kappa \to 0$. If, however, u is mixing, then this can be dramatically improved (see for instance [1–4, 12]). Recall, u is (strongly 1, 1) mixing with rate function h if all H^1 solutions to the transport equation

$$\partial_t\phi + u\cdot\nabla\phi = 0$$

satisfy the bound

$$\|\phi_{s+t}\|_{H^{-1}} \leqslant h(t)\|\phi_s\|_{H^1}\,, \quad \text{for all } s,t \geqslant 0\,.$$

Theorem 2. *If u is strongly mixing with rate function h, then*

$$\tau^*(u,\kappa) \leqslant \frac{1}{\kappa H(\kappa)}\,, \qquad \textit{where} \qquad H(\kappa) \stackrel{\text{def}}{=} \sup\Big\{\lambda \,\Big|\, h\Big(\frac{1}{2\sqrt{\lambda\kappa}}\Big) \leqslant \frac{1}{2\lambda}\Big\}\,.$$

If, for instance, u is exponentially mixing (i.e. $h(t) \leqslant c_1 e^{-c_2 t}$), then the above shows

$$\tau^*(u,\kappa) \leqslant C|\ln\kappa|^2\,,$$

for some constant C, which is much better than the elementary bound obtained from (1.4).

Finally, we study how the above ideas can be used in the study of nonlinear PDEs. The first model problem we consider is the Keller-Segel equation, a simplified model of chemotaxis [7, 10]:

$$\partial_t\rho = \Delta\rho + \chi\nabla\cdot(\rho\nabla^{-1}(\rho-\bar{\rho}))\,. \tag{1.6}$$

Here ρ represents the population density of the bacteria, and the equation (1.6) models the evolution of ρ by diffusion, with a bias directed by the concentration

gradient of a chemoattractant emitted by the bacteria. The quantity $\chi > 0$ is a sensitivity parameter, $\bar{\rho} = \int_{\mathbb{T}^d} \rho_0$, and $\nabla^{-1} = \nabla\Delta^{-1}$. It is well known that if ρ_0 is small solutions to equation (1.6) remain smooth for all time, where as if ρ_0 is large solutions to (1.6) blow up in finite time (see for instance [6]).

We study equation (1.6) with an additional convection term and show in [5] that the blow up can always be avoided if the dissipation time of the imposed drift is small enough. This is motivated by a similar result in [8].

Theorem 3. *Consider the advective Keller-Segel system*

$$\partial_t \rho + u \cdot \nabla \rho = \Delta \rho + \chi \nabla \cdot (\rho \nabla^{-1}(\rho - \bar{\rho}))\,. \tag{1.7}$$

with smooth nonnegative initial data ρ_0. There exists $\tau_0 = \tau_0(\|\rho_0\|_{L^2}, \chi)$ such that if $\tau^*(u, 1) \leqslant \tau_0$, then the solution to (1.7) *remains globally smooth.*

In order to apply this theorem, we need to construct velocity fields with arbitrarily small dissipation time. If u generates a mixing flow (or even a weakly mixing flow), then Theorem 2 (or the results in [1–4]) show that if $u_A(x,t) = Au(x, At)$, then $\tau^*(u_A, \kappa) \to 0$ as $A \to \infty$. However, examples of such flows are not easy to construct. We remedy this by showing that one can always find a cellular flow with an arbitrarily small dissipation time.

Theorem 4. *In two or three dimensions, given any κ, $\tau_0 > 0$ there exists a smooth cellular flow v such that $\tau^*(v, 1) < \tau_0$.*

We prove this by using a result in [13] to bound the dissipation time by the inverse of the effective diffusivity of the rescaled flow.

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Separation of time-scales in fluid mechanics

Michele Coti Zelati
(joint work with Jacob Bedrossian and Theodore D. Drivas)

Consider a two-dimensional periodic domain $\mathbb{T}^2$ and a passive scalar $f : [0,\infty) \times \mathbb{T}^2 \to \mathbb{R}$ that is advected by a smooth divergence-free (i.e. incompressible) velocity field $\boldsymbol{u} : \mathbb{T}^2 \to \mathbb{R}^2$, and therefore satisfies the initial-value problem

$$\begin{cases} \partial_t f + \boldsymbol{u} \cdot \nabla f = \nu \Delta f, \\ f(0) = f^{in}, \end{cases} \tag{1.1}$$

for a *mean-free* initial datum $f^{in} \in L^2$. A simple L^2 estimate that uses the incompressibility of $\boldsymbol{u}$ and the Poincaré inequality reads

$$\|f(t)\|_{L^2} \leqslant \|f^{in}\|_{L^2} \mathrm{e}^{-\nu t}. \tag{1.2}$$

In fact, not much information is used about the flow $\boldsymbol{u}$ other than incompressibility, and the decay rate is that of the heat equation, from which the natural *diffusive* time-scale $O(\nu^{-1})$ appears. However, for small ν, we expect the inviscid mixing to be the leading order dynamics (at least for some time) and hence we can predict a faster decay rate than the one prescribed by the heat equation. The important behavior to detect consists of a cascading mechanism due to the inviscid mixing, and its interaction with a small diffusion of order ν.

A rigorous mathematical framework for so-called relaxation enhancing flows $\boldsymbol{u}$ has been developed in [1]. Roughly speaking, a velocity field $\boldsymbol{u}$ is relaxation enhancing if by the diffusive time-scale $O(\nu^{-1})$, arbitrarily much energy is already dissipated. The main result of [1] characterizes relaxation enhancing flows in terms of the spectral properties of the operator $\boldsymbol{u} \cdot \nabla$. Precisely, $\boldsymbol{u}$ is relaxation enhancing if and only if the operator $\boldsymbol{u} \cdot \nabla$ has no nontrivial eigenfunctions in $\dot{H}^1$. In particular, weakly mixing flows (i.e., those with only continuous spectrum) fall in this class.

A sufficient condition for $\boldsymbol{u}$ to be relaxation enhancing is that the corresponding passive scalar f^ν obeys an estimate of the type

$$\|f(t)\|_{L^2} \leqslant C_0 \mathrm{e}^{-\varepsilon_0 \nu^q t} \|f^{in}\|_{L^2}, \qquad \forall t > 0, \tag{1.3}$$

for some $q \in (0,1)$ depending on $\boldsymbol{u}$ and some positive constants C_0, ε_0 independent on ν. In this case, it is apparent the $\boldsymbol{u}$ induces a faster time-scale $O(\nu^{-q})$. In general, the dependence of q on $\boldsymbol{u}$ is very hard to detect. The case of general shear $\boldsymbol{u} = (u(y), 0)$ with a finite number of critical points was treated in [2]: the enhanced dissipation time-scale was proved to be $O(\nu^{-\frac{n}{n+2}})$, where $n \in \mathbb{N}$ denotes the maximal order of vanishing of u' at the critical points.

The presentation proposed a stochastic interpretation of the enhanced dissipation time-scales appearing for shear flows, in order to prove that the time-scales of [2] are sharp. Consider the following approximate version of (1.1)

$$\begin{cases} \partial_t f + y^n \partial_x f = \nu \partial_{yy} f, \\ f(0) = f^{in}. \end{cases} \tag{1.4}$$

The corresponding characteristics are given by The characteristics are given by

$$\begin{aligned} \mathrm{d}X_x &= Y_y^n \mathrm{d}t, \\ \mathrm{d}Y_y &= \sqrt{2\nu}\mathrm{d}W, \end{aligned}$$

where W is standard one-dimensional Brownian motion and $(x, y) \in \mathbb{T}^2$ are the initial conditions. Assuming $x = y = 0$, it is not hard to see that

$$\mathbb{E}|X_x(t)|^2 = c_n \nu^n t^{n+2}, \qquad \mathbb{E}|Y_y(t)|^2 = 2\nu t, \tag{1.5}$$

for some $c_n > 0$. Now using the fluctuation dissipation relation

$$2\nu \int_0^t \|\nabla f(s)\|_{L^2}^2 \mathrm{d}s = \int_{\mathbb{T}^2} \mathrm{Var}\left[f^{in}(X_x(t), Y_y(t))\right] \mathrm{d}x\mathrm{d}y,$$

for a well-designed (Lipschitz, localized) datum, one can show that

$$\int_{\mathbb{T}^2} \mathrm{Var}\left[f^{in}(X_x(t), Y_y(t))\right] \mathrm{d}x\mathrm{d}y \leqslant C\|f^{in}\|_{L^2}^2 \nu^n t^{n+2}.$$

This provides an upper bound on the energy dissipation, and therefore a lower bound on the mixing rates, which correspond to time-scales $O(\nu^{-\frac{n}{n+2}})$. This work is contained in [3].

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Instabilities and non-uniqueness in ideal fluids

László Székelyhidi Jr.

We discuss some examples of 'wild' weak solutions, and their significance, in various models, in particular with respect to the existence and multiplicity of weak solutions arising in classical fluid interface problems related to instabilities. We address two models: the unstable Muskat problem and the vortex sheet problem. These are related to the classical Rayleigh-Taylor and Kelvin-Helmholtz instabilities respectively. The sharp interface problem in both models is strongly ill-posed. We show that with a suitable 'mixing' ansatz weak solutions can be constructed via a relaxation technique and convex integration. These are not solutions of the

sharp interface problem, but rather are weak solutions of the underlying 'Eulerian' model: the incompressible porous medium equation, and the 2D incompressible Euler equations, respectively. The emphasis here is on the connection between the presence of instabilities on the one hand and the loss of compactness in the solution spaces and existence of multiple weak solutions on the other hand. This is based on joint work with C. Förster (Leipzig), F. Noisette (Paris) and F. Mengual (Madrid).

Wall Bounded Turbulence and Polymer Drag Reduction

Theodore D. Drivas
(joint work with Huy Q. Nguyen and Joonhyun La)

In this talk, we discuss two aspects of wall-bounded turbulence: anomalous dissipation/wall-friction drag due to bulk turbulence and shedding of thin viscous boundary layers, as well as how these effects can be mitigated by the addition of polymer on the boundary.

Regarding anomalous dissipation, we present an extension of Onsager's theory of ideal turbulence to accommodate the presence of solid boundaries. Specifically, I will state two sets of sufficient conditions for energy dissipation to vanish in the infinite Reynolds number limit along sequences of Leray–Hopf weak solutions of the Navier-Stokes equations. Both sets of conditions require that the third-order velocity structure function have inertial-range scaling exponent satisfying $\zeta_3 > 1$ in the bulk (i.e. that the family $\{u^\nu\}_{\nu>0}$ is uniformly bounded in $L^3(0,T;B^\sigma_{3,\infty}(\Omega))$ for some $\sigma > 1/3$ away from solid walls). One set of conditions requires furthermore that, in a ν–dependent thin boundary layer, the velocity remains continuous and the pressure bounded. This shows that discontinuities must emerge at the wall in order for shedding boundary layers to be a viable source of anomalous dissipation (as suggested by numerical study of [R. Nguyen van yen, M. Farge, and K. Schneider. Phys. Rev. Letts. 106.18 (2011)]). The other set of conditions requires instead

$$\lim_{\nu\to 0} \nu \int_0^T \int_{\{x:\mathrm{dist}(x,\Omega)\leqslant \nu^\beta\}} |\nabla u^\nu(x,t)|^2 dxdt = o(\nu^{1-\beta}), \quad \beta = \min\{1, \frac{1}{2(1-\sigma)}\}.$$

Note that if $\sigma = 1/3$ (the K41 value) in the bulk, then $\beta = 3/4$ and $1-\beta = 1/4$. Thus the boundary layer in the above condition becomes of width $\nu^{3/4}$ (Kolmogorov–length), and the assumption is that must decay no faster than the Blasius prediction $\nu^{1/4}$. Interesting the French-washing machine experiments of [Cadot, O., Couder, Y., Daerr, A., Douady, S., and Tsinober, A. Phys. Rev. E, 56(1), 427, (1997)] indicates that there is anomalous dissipation in the bulk (thereby requiring at least one of our conditions to be violated) but the near-wall dissipation in the boundary layer decays with a rate of nearly $\nu^{1/4}$. This suggests that turbulence is as rough/singular as needed to support anomalous dissipation but not (much) rougher. The work presented is joint with Huy Q. Nguyen and can be found in [1, 2].

The next part of the talk concerns polymer drag reduction. The problem of minimizing energy dissipation and wall drag in turbulent pipe and channel flows is a classical one which is of great importance in practical engineering applications. Remarkably, the addition of trace amounts of polymer into a turbulent flow has a pronounced effect on reducing friction drag. To study this mathematically, we introduce a new boundary condition for Navier-Stokes equations which models the situation where polymers are grafted to the wall. Our boundary condition - derived from a fluid-polymer stress balance - becomes an evolution equation for the stresses along the solid walls. In the simplest setting, the polymers are described by a bead-spring model with a Hookean dumbell spring potential and our system closes in the macroscopic fluid variables. It reads

$$\begin{aligned}
\partial_t u^\nu + u^\nu \cdot \nabla u^\nu &= -\nabla p^\nu + \frac{1}{\mathrm{Re}}\Delta u^\nu + f && \text{in} \quad \Omega \times (0,T) \\
u^\nu|_{t=0} &= u_0 && \text{on} \quad \Omega \times \{t=0\} \\
\nabla \cdot u^\nu &= 0 && \text{in} \quad \Omega \times [0,T) \\
u^\nu \cdot \hat{n} &= 0 && \text{on} \quad \partial\Omega \times [0,T) \\
\left(\partial_t + \frac{1}{\mathrm{Wi}}\right)\left(2(D(u^\nu)\hat{n}) \cdot \hat{\tau}_i + \frac{\alpha}{2} u^\nu \cdot \hat{\tau}_i\right) &= -\frac{\alpha \mathrm{Re}}{\mathrm{St}} u^\nu \cdot \hat{\tau}_i && \text{on} \quad \partial\Omega \times (0,T).
\end{aligned}$$

where $D(v) = 1/2(\nabla v + (\nabla v)^t)$, and the Reynolds number Re, Weissenberg number Wi, the relative stress strength St and the ratio of polymer to domain size α are appropriately defined non-dimensional numbers. In two spatial dimensions, we show that the above system admits a global classical solutions. Furthermore, we show that the energy dissipation (and wall-friction momentum drag) vanishes inversely proportional to the Reynolds number, in qualitative agreement with observations of drag in laminar flow. The work presented is joint with Joonhyun La and can be found in [3].

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On the vanishing viscosity problem and the limit $\alpha \to 0$ of the Euler-α equations with Dirichlet boundary conditions

Helena J. Nussenzveig Lopes
(joint work with A. V. Busuioc, D. Iftimie, M. C. Lopes Filho)

The following system of equations are a model for incompressible fluid flow:

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nu \Delta u, \\ \operatorname{div} u = 0, \end{cases} \tag{1.1}$$

$\nu \geqslant 0$. If $\nu = 0$ then this system comprises the Euler equations while if $\nu > 0$ then these are the Navier-Stokes equations; the parameter ν is the viscosity of the flow.

If the fluid domain has boundaries then we need to supplement these equations with boundary conditions. The natural boundary conditions are $u \cdot n = 0$ for the Euler system, called *non-penetration boundary conditions*, and $u = 0$ for the Navier-Stokes system, called *no-slip* or *Dirichlet boundary conditions*.

The vanishing viscosity problem consists in studying the limit, or limits, of solutions of the Navier-Stokes equations with $\nu \to 0$. Formally setting $\nu = 0$ reduces the Navier-Stokes system to the Euler equations, however, in the presence of boundaries, boundary layers arise in the small viscosity regime due to the discrepancy in boundary conditions. The vanishing viscosity problem is yet to be fully understood and it is recognized as a classical open problem in fluid dynamics.

Consider a smooth fluid domain $\Omega \subset \mathbb{R}^3$, with boundary, and smooth initial data u_0. Denote the Euler solution by u^E and the Navier-Stokes solution, with the same initial data, by u^ν; assume that $(0, T)$ is a common interval of existence. Using energy methods it is easy to show that, if $u^E(t, \cdot) \equiv 0$ at the boundary, for all $0 < t < T$, and if u^E is Lipschitz continuous, then the vanishing viscosity limit holds, i.e.

$$u^\nu \to u^E \quad \text{in } L^\infty(0, T; L^2(\Omega)).$$

The Euler evolution does not, in general, preserve the no-slip boundary condition. Examples of Euler flows for which no-slip *is preserved* include all circularly symmetric planar flows with compactly supported vorticty with integral zero, see [2].

The Kato criterion, see [11], provides a condition for a Lipschitz Euler solution to be a vanishing viscosity limit; its proof also uses the energy method. More precisely, if u^E is a Lipschitz solution of the Euler equations with initial data u_0 then u^E is the limit, as $\nu \to 0$, in $L^\infty(0, T; L^2(\Omega))$, of solutions of Navier-Stokes with initial data u_0, if and only if

$$\nu \int_0^T \int_{\Gamma_\nu} |\nabla u^\nu|^2 \, dx ds \xrightarrow[\nu \to 0]{} = 0,$$

where $\Gamma_\nu \subset \Omega$ is a region near $\partial\Omega$ with thickness proportional to ν. The term on the left-hand-side is the dissipation of the (Navier-Stokes) flow near the boundary.

Results of experiments suggest it may be unreasonable to expect vanishing viscosity limits to be Lipschitz solutions of the Euler equations. In fact we expect

large gradients to form near the boundary, detach and affect the bulk of the fluid flow, giving rise to possibly non-smooth vanishing viscosity limits.

Let us consider vanishing viscosity limits which are merely *weak solutions* of the Euler equations. The vanishing viscosity problem is well understood if we impose additional symmetry. This is the case for circularly symmetric flows in 2D, see [1,2,12,13,15], for parallel-plane and parallel-pipe flows in 3D, see [8,10,16,17], and also for the Oseen problem, see [9]. For all these problems it is shown that the vanishing viscosity limit holds in the strong topology of $L^\infty(0,T;L^2)$ and the limit is a weak solution of Euler. The proofs are obtained by semigroup methods.

The following result is a criterion for the vanishing viscosity *weak* limit to be a *weak solution* of the 2D Euler equations. We note that conditions are placed only in the *interior* of the fluid domain.

Let $\Omega \subset \mathbb{R}^2$ bounded, smooth, simply connected. Let $u^\nu \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega))$ be Leray-Hopf solutions of the Navier-Stokes equations, with viscosity ν, such that $u^\nu \rightharpoonup U$ weak-$*$-$L^\infty(0,T;L^2(\Omega))$. Let $\omega^\nu = \operatorname{curl} u^\nu$ denote vorticity.

Theorem 1 (see [4])**.** *Suppose:*

(1) $\{\omega^\nu\} \subset L^\infty((0,T);L^1_{loc}(\Omega))$ *and, for each* $\mathcal{K} \subset\subset \Omega$*, there exists* $C_\mathcal{K} > 0$ *such that*

$$\sup_\nu \sup_{t\in(0,T)} \|\omega^\nu(t,\cdot)\|_{L^1(\mathcal{K})} \leqslant C_\mathcal{K} < \infty;$$

(2) *For any* $\mathcal{K} \subset\subset \Omega$ *we have*

$$\sup_\nu \int_0^T \left(\sup_{x\in\mathcal{K}} \int_{\{|x-y|<r\}\cap\Omega} |\omega^\nu(t,y)|\, dy \right) dt \xrightarrow[r\to 0]{} 0.$$

Then U *is a weak solution of Euler.*

In other words, if there is a uniform L^1_{loc}-estimate on vorticity, and if vorticity does not *concentrate* in the interior of the fluid domain, then any weak limit is a weak Euler solution. In particular this criterion allows weak solutions of Euler with no point vortices to be vanishing viscosity limits. The proof was inspired by Delort/Schochet proof of existence of weak Euler vortex sheet solutions.

Theorem 1 follows-up on prior work of Constantin and Vicol, see [5], who placed assumptions on the local enstrophy instead. In the same paper they have results in 3D as well, under a local structure function assumption. See also [7] for related work.

The vanishing viscosity problem is a difficult, open problem in fluid dynamics. We propose to investigate a similar problem, for which solutions also develop boundary layers, namely the limit $\alpha \to 0$ of the Euler-α equations.

The Euler-α equations on a domain Ω are given by

$$\begin{cases} \partial_t v + (u \cdot \nabla) v + \sum_j v_j \nabla u_j = -\nabla p, \\ \operatorname{div} u = 0, \\ v = u - \alpha \Delta u. \end{cases}$$

Above, p is the (scalar) pressure and v is the (unfiltered) velocity (and $u = (I - \alpha\Delta)^{-1} v$ is the filtered velocity). The constant $\alpha > 0$ has units of length^2 and represents the truncation scale of oscillations in the flow.

We note that formally setting $\alpha = 0$ reduces to the incompressible Euler equations, as then the term $\sum v_j \nabla u_j$ becomes a gradient.

If we impose Dirichlet boundary conditions, $u = 0$ on $\partial\Omega$, then, as in the vanishing viscosity problem, boundary layers form as $\alpha \to 0$.

The vanishing α limit, under Dirichlet boundary conditions, has been studied for planar flows, assuming various degrees of regularity of initial data. More precisely, it has been established that,

(1) if $u(t = 0) = u_0^\alpha \in H^3$ and $\alpha^{1/2} \|\nabla u_0^\alpha\|_{L^2} \to 0$, $\alpha^{3/2} \|u_0^\alpha\|_{H^3}$ uniformly bounded for small α, and $u_0^\alpha \to \bar{u_0} \in H^3$, s-$L^2$, then the vanishing α limit is a solution of the Euler equations, see [14]. Note that the initial data of the Euler equations is $u_0 \in H^3$. The proof is by *energy estimates*, inspired on the proof of the Kato criterion.

(2) if $u(t = 0) = u_0^\alpha \in W^{3,p}$, $1 < p < \infty$ and $\alpha^{1/2} \|\nabla u_0^\alpha\|_{L^2}$, $\|\operatorname{curl}(u_0^\alpha - \alpha \Delta u_0^\alpha)\|_{L^p}$ are bounded for small α, and $u_0^\alpha \to \bar{u_0} \in W^{1,p}$, strongly L^2, then again the limit as $\alpha \to 0$ is a solution of the Euler equations, see [3]. The proof is based on *compactness in* L^2 of u^α, which comes from estimates for u^α and $q^\alpha \equiv \operatorname{curl}(u^\alpha - \alpha\Delta u^\alpha)$. In fact, one gets u^α bounded in $L^\infty(H^s)$, $s = s(p)$, from the analyticity of the Stokes semigroup.

Neither of these results contains a description of the boundary layer, or of the flow near the boundary. There is no clear understanding of the role of the boundary in the vanishing α limit with Dirichlet conditions.

We propose to study, instead, the case of the half-plane, $\Omega = \mathbb{H}$ and we discuss initial data for which $q_0 = q_0^\alpha \equiv \operatorname{curl}(u_0^\alpha - \alpha\Delta u_0^\alpha) \in \mathcal{BM}_+(\mathbb{H}) \cap H^{-1}(\mathbb{H})$, i.e. the potential vorticity is initially independent of α and it is a nonnegative bounded Radon measure in the Sobolev space H^{-1}, called *vortex sheet initial data.* What follows is joint work with A.V. Busuioc, D. Iftimie and M.C. Lopes Filho. Our main result is:

Theorem 2. *Let q_0 be vortex sheet initial data in $\mathbb{H}$. Then*

(1) *there exists a global solution u^α, q^α of the α-Euler equations, $u^\alpha = 0$ on $\partial\mathbb{H}$.*

(2) *If the singular part of q_0 has* distinguished sign *($q_0 \in (\mathcal{BM}_+ + L^1) \cap H^{-1}$) and if*

$$u^\alpha \rightharpoonup u, \text{ as } \alpha \to 0, \text{ weak-} * L^\infty_{loc}(\mathbb{R}_+; L^2(\mathbb{H}))$$

and

$$q^\alpha \rightharpoonup \omega, \ \text{as } \alpha \to 0, \ \text{weak-}* \, L^\infty_{loc}(\mathbb{R}_+; \mathcal{BM}(\mathbb{H})),$$

then (u, ω) *is a weak (vortex sheet) solution of the incompressible 2D Euler equations.*

To prove this result we begin with the vorticity formulation of the α-Euler equations on $\mathbb{H}$, which is

$$\begin{cases} \partial_t q^\alpha + u^\alpha \cdot \nabla q^\alpha = 0, \\ \operatorname{div} u^\alpha = 0, \\ \operatorname{curl} v^\alpha = q^\alpha \\ v^\alpha = u^\alpha - \alpha \Delta u^\alpha \equiv (I - \alpha\Delta) u^\alpha \\ u^\alpha\big|_{\partial\mathbb{H}} = 0 \end{cases}$$

A crucial step is to invert the elliptic system for u^α, given q^α. This is done in two different ways. First, we recall the Biot-Savart kernel $K_{\mathbb{H}}$. Next we note that, since $\operatorname{curl} K_{\mathbb{H}}[q^\alpha] = q^\alpha$ we get that $(I - \alpha\Delta)u^\alpha$ and $K_{\mathbb{H}}[q^\alpha]$ differ by a gradient field. Therefore, if $\mathbb{P}$ is the Leray projection, onto div-free vector fields tangent to $\partial\mathbb{H}$, and $\mathbb{A} = -\mathbb{P}\Delta$ is the Stokes operator, then, because $\mathbb{P}K_{\mathbb{H}}[q^\alpha] = K_{\mathbb{H}}[q^\alpha]$ and $\mathbb{P}u^\alpha = u^\alpha$, we find

$$u^\alpha = (I + \alpha\mathbb{A})^{-1} K_{\mathbb{H}} q^\alpha,$$

where the inverse has zero boundary conditions. But this is not useful enough.

Let $G_\alpha = G_\alpha(x, y)$ denote the Green's function for $I - \alpha\Delta$ in *all of* $\mathbb{R}^2$.

The method of images for $-\Delta$ gives

$$K_{\mathbb{H}}[q] = (K * q_{odd})\big|_{\mathbb{H}},$$

where q_{odd} is the odd extension of q to the full plane.

Denote

$$u^\alpha_{int} \equiv (G_\alpha * K * q^\alpha_{odd})\big|_{\mathbb{H}}$$

Then $\operatorname{curl}(I - \alpha\Delta)u^\alpha_{int} = q^\alpha$ in $\mathbb{H}$, but u^α_{int} does not vanish on $\partial\mathbb{H}$, it is only tangent.

Let $u^\alpha_{bdry} = u^\alpha - u^\alpha_{int}$. Of course $\operatorname{curl}(I - \alpha\Delta)u^\alpha_{bdry} = 0$, and hence $w \equiv u^\alpha_{bdry}$ satisfies the following Stokes problem:

$$\begin{cases} \operatorname{div} w = 0, \\ w - \alpha\Delta w + \nabla p = 0 \\ (w_1)\big|_{\partial\mathbb{H}} = -[(u^\alpha_{int})_1]\big|_{\partial\mathbb{H}} \equiv g \\ (w_2)\big|_{\partial\mathbb{H}} = 0. \end{cases}$$

Using a technique due to Solonnikov based on Fourier methods we find:

$$w_1 = -2\alpha g *_1 \partial_2 G_\alpha + \frac{2\alpha}{\pi} g *_1 \partial_2 G_\alpha *_H \frac{x_2^2 - x_1^2}{|x|^4}$$

$$w_2 = -\frac{4\alpha}{\pi} g *_1 \partial_2 G_\alpha *_H \frac{x_1 x_2}{|x|^4},$$

where $*_1$ is convolution in the first variable and $*_H$ is an adapted convolution on $\mathbb{H}$. We still need to identify g. We obtain:

$$g(x_1) = -2 \int_{\mathbb{H}} G_\alpha(x_1 - y_1, y_2)[K * q^\alpha_{odd}]_1(y_1, y_2)\, dy_1 dy_2.$$

The key estimate we show is:

Lemma 1.1.

$$\|g\|_{L^1(\mathbb{R})} \leqslant \|q^\alpha\|_{\mathcal{BM}(\mathbb{H})}.$$

This Lemma together with easy estimates for $\partial_2 G_\alpha$ and the fact that $\dfrac{x_2^2 - x_1^2}{|x|^4}$ and $\dfrac{x_1 x_2}{|x|^4}$ are homogeneous of degree -2 give:

Proposition 1.1. For all $x \in \mathbb{H}$

$$|u^\alpha_{bdry}(x)| \leqslant C\|q^\alpha\|_{\mathcal{BM}(\mathbb{H})} \left(\frac{\alpha^{1/4}}{x_2^{3/2}} + \frac{\exp(-x_2/2\sqrt{\alpha})}{x_2} \right).$$

It follows from the Proposition 1.1 that the boundary correction is vanishingly small, uniformly, as long as $x_2 > \varepsilon > 0$.

The remainder of the proof of Theorem 2 is an adaptation of the Delort/Schochet proof, see [6,18], using the explicit expression for u^α_{int}. More precisely, we find easily that

$$\|q^\alpha\|_{\mathcal{BM}} \leqslant C \quad \text{and } \|u^\alpha\|_{L^2} \leqslant C,$$

so, passing to subsequences as needed,

$$q^\alpha \rightharpoonup \omega \text{ weak} * L^\infty \mathcal{BM} \quad \text{and } u^\alpha \rightharpoonup u \text{ weak} * L^\infty L^2.$$

We then write a weak formulation for the transport equation satisfied by q^α and we proceed to pass to the limit in each term. The linear terms are easy to handle by weak convergence. The nonlinear term is handled in much the same way as in the Delort/Schochet proof. There is one significant difference, namely the proof that weak limits of q^α do not contain Diracs. Instead of the usual bound in H^{-1}, which is not valid for q^α, we use instead that $q^\alpha \rightharpoonup \omega$, and we prove that $\omega \in H^{-1}$. This in turn implies that ω does not contain Diracs, which is enough to pass to the limit in the nonlinear term.

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Isentropic Euler system: Some good and bad news

Eduard Feireisl
(joint work with A. Abbatiello, E. Chiodaroli, F. Flandoli, M. Hofmanová)

We consider the Euler system describing the evolution of the mass density $\varrho = \varrho(t,x)$ and the momentum $\mathbf{m} = \mathbf{m}(t,x)$ of a gas in the isentropic regime occupying a bounded domain $\Omega \subset R^d, d = 2,3$:

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = 0,\ p(\varrho) = a\varrho^\gamma,\ a > 0,\ \gamma > 1$$

with the impermeability condition

$$\mathbf{m} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

or, alternatively, the space periodic boundary conditions

$$\Omega = \mathbb{T}^d.$$

The initial conditions are

$$\varrho(0,\cdot) = \varrho_0,\ \mathbf{m}(0,\cdot) = \mathbf{m}_0.$$

We consider the weak (distributional) solutions that are *admissible*, meaning they satisfy the energy inequality:

$$\partial_t E \leqslant 0,\ \text{where } E = \int_\Omega \mathcal{E} \mathrm{d}x$$

with

$$\mathcal{E} = \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho),\ P'(\varrho)\varrho - P(\varrho) = p(\varrho),\ P(\varrho) = \frac{a}{\gamma-1}\varrho^\gamma$$

The first result concerns ill posedness for Riemann integrable initial data:

Theorem 1 (A.Abbatiello, EF 2019, [1])**.** *Let $\mathcal{R}$ be the set of bounded, Riemann integrable functions in $\Omega \subset R^d$, $d = 2, 3$. Let ϱ_0, $\mathbf{m}_0$ be given such that*

$$\varrho_0 \in \mathcal{R},\ 0 \leqslant \underline{\varrho} \leqslant \varrho_0 \leqslant \overline{\varrho},$$

$$\mathbf{m}_0 \in \mathcal{R},\ \mathrm{div}_x \mathbf{m}_0 \in \mathcal{R},\ \mathbf{m}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

Let $\{\tau_i\}_{i=1}^\infty \subset (0,T)$ be an arbitrary (countable dense) set of times.

Then the Euler problem admits infinitely many weak solutions ϱ, $\mathbf{m}$ with a strictly decreasing total energy profile such that

$$\varrho \in C_{\mathrm{weak}}([0,T]; L^\gamma(\Omega)),\ \mathbf{m} \in C_{\mathrm{weak}}([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d))$$

but

$$t \mapsto [\varrho(t,\cdot), \mathbf{m}(t,\cdot)]\ \boxed{\textit{is not}}\ \textit{strongly continuous at any } \tau_i,\ i = 1, 2, \dots.$$

The next result shows that adding a random perturbation does not save well posedness for the Euler system. We consider the complete Euler system with the energy as a new variable driven by a random force:

$$\mathrm{d}\varrho + \mathrm{div}_x \mathbf{m}\ \mathrm{d}t = 0$$

$$\mathrm{d}\mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}\right)\ \mathrm{d}t + \nabla_x p\ \mathrm{d}t = -\frac{1}{2}\mathbf{m} \circ \mathrm{d}W$$

$$\mathrm{d}E + \mathrm{div}_x \left((E+p)\frac{\mathbf{m}}{\varrho}\right) \mathrm{d}t = -E \circ \mathrm{d}W,$$

supplemented with the entropy inequality

$$\mathrm{d}(\varrho s) + \mathrm{div}_x(s\mathbf{m})\mathrm{d}t \geqslant -c_v \varrho \circ \mathrm{d}W,$$

and the impermeability boundary conditions. Here W denotes the standard Wiener process and $\circ$ stands for the Stratonowich stochastic integral.

Theorem 2 (E.Chiodaroli, EF, F. Flandoli [2])**.** *There is a set of initial data $\{\varrho_0, \vartheta_0\} \in Y$ dense in $[L^1(\Omega)]^2$ such that for any $[\varrho_0, \vartheta_0] \in Y$ there exists $\mathbf{m}_0 \in L^\infty(\Omega; R^d)$ such that the stochastically driven Euler system admits infinitely many global in time admissible weak solutions. The solutions are strong in the stochastic sense and weak in the PDE sense.*

Finally, we present a result concerning strong convergence of a sequence of consistent approximate solutions. These are quantities $[\varrho_n, \mathbf{m}_n]$ satisfying

$$\int_0^T \int_{R^d} [\varrho_n \partial_t \varphi + \mathbf{m}_n \cdot \nabla_x \varphi] \, \mathrm{d}x \, \mathrm{d}t = e_{1,n}[\varphi]$$

$$\int_0^T \int_{R^d} \left[\mathbf{m}_n \cdot \partial_t \boldsymbol{\varphi} + 1_{\varrho_n > 0} \frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} : \nabla_x \boldsymbol{\varphi} + p(\varrho_n) \mathrm{div}_x \boldsymbol{\varphi} \right] \mathrm{d}x \, \mathrm{d}t = e_{2,n}[\boldsymbol{\varphi}]$$

for any smooth test function φ,

$$\mathcal{E}(\varrho_n, \mathbf{m}_n) \equiv \int_{R^d} \frac{1}{2} \frac{|\mathbf{m}_n - \varrho_n \mathbf{u}_\infty|^2}{\varrho_n} + P(\varrho_n) - P'(\varrho_\infty)(\varrho_n - \varrho_\infty) - P(\varrho_\infty) \leqslant c.$$

$[\varrho_n, \mathbf{m}_n]$ represent a sequence of approximate solutions to the isentropic Euler system on the whole space R^d with the far field conditions

$$\varrho \to \varrho_\infty, \ \mathbf{u} \equiv \frac{\mathbf{m}}{\varrho} \to \mathbf{u}_\infty \text{ as } |x| \to \infty.$$

We say that the approximation is *consistent*, if

$$e_{1,n}[\varphi] \to 0, \ e_{2,n}[\boldsymbol{\varphi}] \to 0 \text{ as } n \to \infty \text{ for any fixed test function.}$$

Theorem 3 (EF, M.Hofmanová [3])**.** *Let $[\varrho_n, \mathbf{m}_n]$ be a consistent approximation of the isentropic Euler system such that*

$$\varrho_n \to \varrho, \ \mathbf{m}_n \to \mathbf{m}$$

in the sense of distributions (weakly). Suppose that $[\varrho, \mathbf{m}]$ is a weak solution to the Euler system.

Then the convergence is strong (a.a. pointwise).

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Formation of shocks for 2d Euler

Steve Shkoller
(joint work with Tristan Buckmaster, Vlad Vicol)

We consider the Cauchy problem for the two-dimensional isentropic compressible Euler equations

$$\partial_t(\rho u) + \operatorname{div}(\rho\, u \otimes u) + \nabla p(\rho) = 0\,, \tag{1.1}$$

$$\partial_t \rho + \operatorname{div}(\rho u) = 0\,, \tag{1.2}$$

where $u : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ denotes the velocity vector field, $\rho : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}_+$ denotes the strictly positive density, and the pressure $p : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}_+$ is defined by the ideal gas law

$$p(\rho) = \tfrac{1}{\gamma}\rho^\gamma\,, \qquad \gamma > 1\,.$$

The sound speed $c(\rho) = \sqrt{\frac{\partial p}{\partial \rho}}$ is then given by $c = \rho^\alpha$ where $\alpha = \frac{\gamma-1}{2}$. The Euler equations (1.1)–(1.2) are a system of conservation laws: (1.1) is the conservation of momentum, which can be equivalently written as $\partial_t u + u \cdot \nabla u + \rho^{\gamma-2}\nabla\rho = 0$, and (1.2) is the conservation of mass.

This paper is devoted to the construction of solutions to (1.1)–(1.2) which form a shock in finite time: specifically, starting from smooth initial data with $O(1)$ amplitude and a minimum slope of $-1/\varepsilon$ with $\varepsilon > 0$ sufficiently small, we construct solutions to the 2D Euler equations (1.1)–(1.2) on a time interval $t_0 \le t \le T_*$, $t_0 = -\varepsilon$ and $T_* = O(\varepsilon^{5/4})$, for which $\rho(\cdot, t)$ and $u(\cdot, t)$ remain bounded, while $|\nabla\rho(\cdot, t)| \to \infty$ and $|\nabla u(\cdot, t)| \to \infty$ as $t \to T_*$; moreover, no other type of singularity can form prior to $t = T_*$, and detailed information on the singularity formation at $t = T_*$ is provided, including blowup time, location, and profile regularity.

We are particularly interested in devising solutions to (1.1)–(1.2) which have vorticity at the shock, by which we mean solutions which are not small perturbations of irrotational flows. As such, our strategy will be to construct solutions that are perturbations of purely azimuthal wave motion whose simplest (constant) profiles are of the $x^\perp$-type with $O(1)$ vorticity at this most basic level. As we shall describe in great detail below, this is in contrast to those solutions which are small perturbations of irrotational simple plane waves.

We are thus motivated to develop a framework of analysis for solutions which are perturbations of purely azimuthal waves. Obviously, polar coordinates provide a natural setting for describing such perturbative solutions, but more fundamentally, we have discovered that the use of homogeneous solutions to (1.1)–(1.2) leads to a remarkable reduction of the Euler dynamics precisely to this nearly-azimuthal wave regime, in which bounded azimuthal waves steepen and then shock, while radial waves (and their slopes) remain bounded. Owing to the inherent vorticity in the most basic wave motion, the solutions are fundamentally two-dimensional in their evolution. We provide a precise description of the shock formation for such Euler solutions, including the *blowup time and location*, by a transformation to self-similar variables that contain dynamically evolving modulation functions

that keep track of the location, time, and amplitude of the blowup. At the blowup time $t = T_*$, the wave profile is of Hölder-class $C^{1/3}$. In the special case that the adiabatic exponent γ is equal to 3 and for purely azimuthal initial velocity fields, a series of surprising cancellations reduces the 2D Euler dynamics to an elementary study of the Burgers equation. The solution for the special case that $\gamma = 3$ can be viewed as the purely azimuthal wave motion, and its shock formation is completely characterized for all time.

Main Theorem. *For an open set of smooth initial data with $O(1)$ amplitude and with minimum initial slope given at initial time t_0 to equal $-1/\varepsilon$, for $\varepsilon > 0$ taken sufficiently small, there exist smooth solutions of the Euler equations with $O(1)$ vorticity, which form an asymptotically self-similar shock in finite time T_*, such that $T_* - t_0 = O(\varepsilon)$. The solutions have $O(1)$ vorticity at the shock, are dominated by azimuthal wave motion, and the location and time of the first singularity can be explicitly computed. The blowup profile at the first singularity is shown to be a cusp with $C^{1/3}$ regularity.*

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On the vanishing viscosity limit of the Navier-Stokes equations and the Triple Deck

Vlad Vicol

(joint work with I. Kukavica & F. Wang (first half) and S. Iyer (second half))

In this talk we discuss two recent results: [2] and [3], concerning the vanishing viscosity limit of the Navier-Stokes equations on the half space, with Dirichlet boundary conditions.

The first half of the talk discusses the joint work with Kukavica and Wang [3]. We consider initial datum which is real-analytic analytic (with respect to ∂_x and $y\partial_y$) in an $O(1)$ strip in the vicinity of the domain, and has finite Sobolev regularity on the complement this strip. We prove that for such data the solution of the Navier-Stokes equations converges in the vanishing viscosity limit to the solution of the Euler equation, on a constant time interval. In particular, our main result implies both the result of Sammartino-Caflisch [5] (which assumes analyticity on the entire half-plane), and also the more recent result of Maekawa [4] (which assumes that the initial vorticity vanishes identically in an $O(1)$ strip next to the boundary).

The second half of the talk discusses the joint work with Iyer [2]. In the immediate vicinity of a separation point, the breakdown of the assumptions on which Prandtl equations are derived signals the limitations of the classical Prandtl boundary layer theory, and in practice one appeals to higher order boundary layer theories. The work [1] shows that the well-known higher order models: *Prescribed*

Displacement Thickness and *Interactive Boundary Layer*; are linearly ill-posed, even for real-analytic initial datum. In the work [2] we consider another classical higher order boundary layer model, *Stewardson's Triple Deck*. We first present the derivation of the model, which appears to lose two derivatives through the pressure-displacement relation which links pressure to the tangential slip. By splitting the Triple Deck system into two coupled equations: a Prandtl type system on in the lower deck and a Benjamin-Ono type equation at the top of the lower deck, we extract a crucial leading order cancellation, which enables us to prove the local well-posedness of the model in tangentially real analytic spaces.

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On boundary value problem for steady Navier–Stokes system in 2D exterior domains

Mikhail Korobkov

(joint work with K. Pileckas, R. Russo)

We study solutions to stationary Navier–Stokes system in two dimensional exterior domains, namely, existence of these solutions and their asymptotical behaviour. The talk is based on the recent joint papers with K.Pileckas and R.Russo where the uniform boundedness and uniform convergence at infinity for arbitrary solution with finite Dirichlet integral were established. Here no restrictions on smallness of fluxes are assumed, etc. In the proofs we develop the ideas of the classical paper of Amick (Acta Math. 1988).

Why it is so hard to get numerical evidences of possible finite time blow-ups for Euler

Alexis F. Vasseur
(joint work with Misha Vishik)

Consider the incompressible Euler equation in a domain $\Omega \subset \mathbb{R}^3$:

$$\begin{aligned} &\mathrm{d}tu + (u\cdot\nabla)u + \nabla P = 0, \qquad x\in\Omega, \quad t\in(0,T^*),\\ &\operatorname{div} u = 0, \qquad x\in\Omega, \quad t\in(0,T^*), \end{aligned} \tag{1.1}$$

endowed with a smooth initial value $u^0 \in H^s(\Omega)$, for a $s > 9/2$. The domain Ω can be $\mathbb{R}^3$, $\mathbb{T}^3$ or any smooth bounded domain Ω where we add the impermeability condition:

$$u\cdot\mathbf{n} = 0, \qquad \text{on } \partial\Omega, \tag{1.2}$$

where $\mathbf{n}$ is the normal of $\partial\Omega$. It is well known that there exists a solution of this equation on (at least) a small timespan $(0,T^*)$ such that for every $T < T^*$, $u \in C^0(0,T;H^s(\Omega)) \cap C^1(0,T;H^{s-1}(\Omega))$. From the assumption $s > 9/2$, this implies that on this lifespan u, ∇u and $\nabla\nabla u$ are $C^1([0,T]\times\Omega)$ for all $T < T^*$. In two dimensions of space, due to the absence of vorticity stretching, the solution can always be extended as a global smooth solution for all time. Whether it is still the case in dimension 3, for all smooth initial values, remains one of the fundamental questions both for the Euler equation, and its viscous counterpart the Navier-Stokes equation. This talk is dedicated to the study of the link between the linear stability of the solutions, and the propagation of their regularity. Let $T^* > 0$ be the biggest time (possibly infinite) such that for every $T < T^*$ the solution of the Euler equation u exists and lies in $C^0(0,T;H^s(\Omega)) \cap C^1(0,T;H^{s-1}(\Omega))$. Let $1 < p < \infty$. For any $T < T^*$, we consider the semigroup generated by the linearization of the Euler equation about the solution u:

$$\begin{aligned} &\mathrm{d}tv + (u\cdot\nabla)v + (v\cdot\nabla)u + \nabla P' = 0, \qquad x\in\Omega, \quad t\in(0,T),\\ &\operatorname{div} v = 0, \qquad x\in\Omega, \quad t\in(0,T),\\ &v\cdot\mathbf{n} = 0, \qquad \text{on } \partial\Omega. \end{aligned} \tag{1.3}$$

The solution v is uniquely defined for any initial value in $H^1(\Omega)$. We denote $\gamma_p(T)$ the growth in $L^p(\Omega)$ norm of the semi-group:

$$\gamma_p(T) = \sup_{v^0\in H^1(\Omega), \|v^0\|_{L^p(\Omega)}\leqslant 1} \|v(T)\|_{L^p(\Omega)}.$$

It is easy to show that the regularity on $[0,T]$ of the solution u implies the boundedness of $\gamma_p(t)$ on $[0,T]$. Indeed, there exists a constant depending only on Ω and p such that

$$\gamma_p(T) \leqslant e^{C_p \int_0^T \|\nabla u\|_{L^\infty(\Omega)}\, dt}.$$

Therefore regularity controls linear stability. This paper is dedicated to the proof of the other causality. We denote the vorticity $\omega = \text{curl } u$. Our main theorem is the following:

Theorem 1. *Let $\Omega \subset \mathbb{R}^3$ be either $\mathbb{R}^3$, $\mathbb{T}^3$, or a bounded smooth domain. Consider a smooth initial value $u^0 \in H^s(\Omega)$, $s > 9/2$, with $\omega^0 = \text{curl } u^0$, and denote T^* the biggest time (possibly infinite) such that the solution u of the Euler equation (1.1) (1.2) exists and lies in $C^0(0,T;H^s(\Omega)) \cap C^1(0,T;H^{s-1}(\Omega))$ for all $T < T^*$. Then, for every $1 < p < \infty$,and every $T < T^*$,*

$$\gamma_p^2(T) \geqslant \frac{\|\omega(T)\|_{L^\infty(\Omega)}}{\|\omega^0\|_{L^\infty(\Omega)}}.$$

Beale Kato and Majda showed that the supremum norm of the vorticity controls the regularity of the Euler solution. More precisely, they showed that, as long as

$$\int_0^T \|\omega(t)\|_{L^\infty(\Omega)}\, dt$$

is bounded, there exists $\varepsilon > 0$ such that u can be extended to a solution to the Euler equation (1.1) (1.2) on $[0, T+\varepsilon]$ with $u \in C^0(0,T+\varepsilon;H^s(\Omega)) \cap C^1(0,T+\varepsilon;H^{s-1}(\Omega))$. Therefore, Theorem 1 implies the following result.

Corollary 1. *Let $\Omega \subset \mathbb{R}^3$ be either $\mathbb{R}^3$, $\mathbb{T}^3$, or a bounded smooth domain. Consider a smooth initial value $u^0 \in H^s(\Omega)$ for $s > 9/2$ with $\omega^0 = \text{curl } u^0$, and assume that the corresponding solution u to the Euler equation (1.1) (1.2) lies in $C^0(0,T;H^s(\Omega)) \cap C^1(0,T;H^{s-1}(\Omega))$ for all $T < T^*$. Assume that*

$$\sup_{0<T<T^*} \|u(T)\|_{H^s(\Omega)} = \infty.$$

Then, for any $1 < p < \infty$:

$$\int_0^{T^*} \gamma_p^2(t)\, dt = \infty,$$

and especially

$$\lim \sup_{T \to T^*} \gamma_p(T) = \infty.$$

This is equivalent to the contrapositive which states that stability controls the regularity. This result shows that if the solution u blows up at $t = T^*$, then small perturbations on the initial value induce huge discrepancies on the solution when time approaches T^*. Numerical experiments involves unavoidable numerical inaccuracies. Therefore, due to the growing instabilities of the exact solution close to the blow-up time, we cannot expect any predictability of the numerical experiment about the blow-up. This explains why, even with the current computational power, it is so difficult to obtain numerical scenarios for a possible blow-up. The difficulty to predict finite time blow-ups is well documented. Note that the result of Corollary 1 covers the case of blow-ups at the boundary.

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Partial Dissipation and Stability

Jiahong Wu

In collaboration with Peter Constantin, Charlie Doering and a group of young colleagues, we have been attempting to understand how partial or fractional dissipation affects the regularity, stability and large-time behavior of solutions. This talk presents three examples in which we successfully established the global stability and obtained the large-time behavior. They include the 2D Boussinesq equations near the hydrostatic equilibrium, several partially dissipated magnetohydrodynamic systems near a background magnetic field, and partially dissipated Oldroyd-B models near the trivial solution.

In a joint work with Charlie Doering, Kun Zhao and Xiaoming Zheng [1], we studied the stability and large-time behavior of the two-dimensional Boussinesq equations without buoyancy diffusion near the hydrostatic equilibrium. Hydrostatic equilibrium or hydrostatic balance in fluid dynamics refers to the status of a fluid when it is at rest. This occurs when the gravity is balanced out by the pressure-gradient force. Our atmosphere is mostly in the hydrostatic equilibrium. The pressure-gradient force prevents gravity from collapsing Earth's atmosphere into a thin shell, whereas gravity prevents the pressure gradient force from diffusing the atmosphere into space. [1] obtains the global stability and the large-time asymptotics of the full nonlinear perturbation. In particular, it shows that the kinetic energy and the first order spatial and temporal derivatives of the velocity field converge to zero as time goes to infinity, regardless of the magnitude of the initial data, and the flow stratifies in the vertical direction in a weak topology. Strikingly, the second-order spatial derivatives of the velocity are shown to be bounded uniformly in time. In a followup work [2], we investigate the evolution of the temperature and provide a precise description of the final buoyancy distribution in case of general initial conditions. In addition, [2] also obtained analytical results on the explicit decay rate of the velocity field.

In a joint work with Hongxia Lin, Ruihong Ji and Li Yan [3] and another joint work with Yi Zhu [4], we successfully established the global stability on perturbations near a background magnetic field to several incompressible magnetohydrodynamic (MHD) equations with only partial dissipation. These stability results provide significant examples for the stabilizing effects of the magnetic field on electrically conducting fluids. The vorticity gradient of the 2D incompressible Euler equation can grow double exponentially in time while the same quantity to the 2D Navier-Stokes equation decays algebraically in time. However, the stability and large-time behavior of the vorticity gradients of the 2D Navier-Stokes equation with only vertical or horizontal dissipation appears to be unknown. When the partially dissipated Navier-Stokes is coupled with the equations of the magnetic field, we do have stability.

In a joint work with Peter Constantin, Jiefeng Zhao and Yi Zhu [5], we establish a new small data global well-posedness result on the incompressible Oldroyd-B model with only dissipation in the equation of stress tensor (without stress tensor damping or velocity dissipation). The dissipation is not necessarily given by the

standard Laplacian operator and any fractional dissipation with fractional power equal to or greater than 1/2 suffices. The previous best result in this direction requires the full Laplacian dissipation and also a damping term in the stress tensor.

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The Navier-Stokes Equations in Domains with Moving Boundaries

Reinhard Farwig

(joint work with Hideo Kozono, Kazuyuki Tsuda, David Wegmann)

Consider the (Navier-)Stokes system on a bounded or exterior domain $\Omega(t) \subset \mathbb{R}^n$ with moving boundary and Dirichlet boundary conditions. This problem goes back to E. Hopf [7] and O.A. Ladyzhenskaya [10] who proved the existence of weak and strong solutions in L^2, respectively. Periodic solutions in the L^2–setting for bounded domains were constructed by T. Miyakawa & Y. Teramoto [11] and Y. Teramoto [14]. J. Neustupa [12] analyzed the problem for a very general class of bounded and unbounded domains without smoothness of the boundary in L^2 proving also an adequate energy estimate of weak solutions. Finally, the existence of suitable weak solutions satisfying a localized energy inequality was proved by H.J. Choe, Y. Jang & M. Yang [1].

The first results on L^q-solutions involving the concept of maximal regularity are due to J. Saal [13] in 2006. He proved that the Stokes operator in a domain with moving boundary has the property of maximal regularity provided that the Stokes operator is invertible. This result can be applied to get global-in-time solutions if the domain is bounded or - by a spectral shift - if the domain is an exterior one leading to exponentially increasing terms.

For $t \in (0, T]$ let $\Omega(t) \subset \mathbb{R}^n$ $(n \geqslant 2)$ be a bounded domain with $\partial\Omega(t) \in C^3$, set $Q := \bigcup_{t\in(0,T]} \Omega(t) \times \{t\}$ and let $\Omega_0 := \Omega(t_0)$ denote a reference domain. The

Navier-Stokes equations on the non-cylindrical space-time domain is described by

$$\begin{aligned} v_t - \Delta v + v \cdot \nabla v + \nabla p &= f \quad \text{in} \quad Q, \\ \operatorname{div} v &= 0 \quad \text{in} \quad Q, \\ v &= 0 \quad \text{on} \quad \bigcup_{t\in(0,T]} \partial\Omega(t) \times \{t\}, \\ v(0) &= v_0. \end{aligned} \tag{1.1}$$

Here $v = v(x,t)$ denotes the unknown velocity of an incompressible, viscous fluid (with viscosity $\nu = 1$) and $p = p(x,t)$ denotes the pressure, respectively, at time $t \in (0,T]$ and position $x \in \Omega(t)$. Moreover, let f be a given external force. For simplicity, we assume that $v = 0$ on $\partial\Omega(t)$; the correct kinetic boundary condition will lead by classical trace arguments to a Navier-Stokes system with perturbation terms which are of order zero and one in v.

In the following, let $\Omega(\cdot)$ be a family of either bounded or exterior domains.

Assumption 0. There exists a family of domains $\Omega(t) \subset \mathbb{R}^n$ with $\partial\Omega(t) \in C^3$, $t \in \mathbb{R}$, and a map

$$\phi : \overline{\Omega_0 \times (0,\infty)} \to \mathbb{R}^n,\ (\xi,t) \mapsto \phi(\xi,t),$$

where $\Omega_0 \subset \mathbb{R}^n$ is a reference domain with $\partial\Omega_0 \in C^3$. Furthermore, the function ϕ has the following properties:

(1) For $t \in \mathbb{R}_+$ the map $\phi(\cdot,t)\colon \overline{\Omega_0} \to \overline{\Omega(t)}$ is a C^3-diffeomorphism with $\phi(\cdot,0) = \mathrm{id}$. Let $\phi(\cdot,t)^{-1}$ denote the inverse for fixed t.
(2) Concerning the function ϕ as a map on $\Omega_0 \times \mathbb{R}_+$ we assume that

$$\phi \in C_b^{3,1} := \big\{ f \in C^0(\Omega_0 \times \mathbb{R}_+) \mid \partial_t^k \partial_\xi^\alpha f \in C_b^0,\ 0 \leqslant 2k + |\alpha| \leqslant 3,\ k \in \mathbb{N}_0,\ \alpha \in \mathbb{N}_0^3 \big\}.$$

(3) The map ϕ is volume preserving, that is, $\det \nabla_\xi \phi(\cdot,t) = 1$ for all t.
(4) For global-in-time solutions: $\phi(\cdot,t) \to \phi(\cdot,\infty)$ as $t \to \infty$ in $C^3(\Omega_0)$ and $\partial_t \phi(\cdot,t) \to 0$ as $t \to \infty$ in $C^1(\Omega_0)$. In this case, $\Omega(\infty) = \phi(\Omega_0,\infty)$ is a domain of class C^3.
(5) For exterior domains: The map ϕ acts locally in space, *i.e.*, there exists an $R > 0$ such that $\phi(\xi,t) = \xi$ for all $|\xi| > R$.
(6) For t-periodic solutions: The map ϕ is μ-Hölder continuous in time in the semi-norm of $C_b^{3.1}$, *i.e.*, there exist constants $\delta_0 > 0$ and $\mu \in (0,1]$ such that $|\phi(t) - \phi(s)|_{C_b^{3,1}} \leqslant \delta_0 |t-s|^\mu$ for $t,s \in \mathbb{R}$.

System (1.1) on Q will be reformulated as a problem in the cylindrical space-time domain $\Omega_0 \times (0,T]$ as follows, *cf.* [2,13]:

Let $t \in (0,T]$ and $v : \Omega(t) \to \mathbb{R}^n$ be a function with parameter t. Given ϕ as in Assumption 0 let us define the map

$$\Phi(t) : L^q(\Omega(t)) \to L^q(\Omega_0), \quad v(\cdot,t) \mapsto u(\cdot,t) := ((\nabla\phi)^{-1}(\cdot,t))v(\phi(\cdot,t),t)$$

which maps solenoidal vector fields to solenoidal ones. Based on the Helmholtz projection $P_{\Omega(t)}$ for the domain $\Omega(t)$, the operators

$$P(t) = \Phi(t) P_{\Omega(t)} \Phi(t)^{-1} : L^q(\Omega_0) \to L^q(\Omega_0)$$

are projections onto $L^q_\sigma(\Omega_0)$ since $\Phi(t) : L^q_\sigma(\Omega(t)) \to L^q_\sigma(\Omega_0)$ is an isomorphism. The same holds for the spaces $W^{k,q}$, $k = 1, 2$ and $W^{1,q}_0$. However, $P(t)$ does not coincide with the Helmholtz projection $P_{\Omega_0} = P(0)$ because its kernel given by $G_q(t) = \{\nabla^\phi(t)(\pi \circ \psi) : \pi \in \dot{W}^{1,q}(\Omega(t))\}$ may depend on t; here

$$\nabla^\phi(t)g := (\nabla\phi(t))^{-1}\big((\nabla\phi(t))^{-1}\big)^\top \nabla g.$$

Concerning the differential operators Δ and ∂_t we calculate

$$\Phi(t)\Delta_x\Phi(t)^{-1} = \Delta_\xi + \sum_{|\alpha|\leqslant 2} a_\alpha(\xi, t)\partial^\alpha,$$

$$\Phi(t)\partial_t\Phi(t)^{-1} = \partial_t + \sum_{|\beta|\leqslant 1} b_\beta(\cdot, t)\partial^\beta,$$

where $a_\alpha(\cdot, t)$, $b_\beta(\cdot, t)$ are bounded compactly supported matrix-valued functions. Finally, we define the modified Stokes operators

$$\begin{aligned} A(t) &= P(t)\Big(-\Delta - \sum_{|\alpha|\leqslant 2} a_\alpha\partial^\alpha + \sum_{|\beta|\leqslant 1} b_\beta\partial^\beta\Big) \\ &= \Phi(t)A_{\Omega(t)}\Phi(t)^{-1} + P(t)\sum\nolimits_\beta b_\beta\partial^\beta, \end{aligned}$$

with domain $\mathcal{D}(A(t)) = W^{2,q}(\Omega_0) \cap W^{1,q}_0(\Omega_0) \cap L^q_\sigma(\Omega_0)$. Thus we get the non-autonomous Cauchy problem

$$\partial_t u(t) + A(t)u(t) = \tilde{f}(t), \quad u(0) = u_0. \tag{1.2}$$

The aim of the talk is threefold. First, in the case of exterior domains, we extend the known maximal regularity result in $L^s(0, T; L^q)$ for the Stokes and Navier-Stokes problem to global-in-time solutions under suitable smallness assumptions on the prescribed motion of $\partial\Omega(t)$ for large t. The approach is based on results by M. Giga, Y. Giga & H. Sohr ([6]) on abstract non-autonomous systems for large t and on local-in-time results of [13]. The main result on the nonlinear problem reads as follows:

Theorem 1. *Let $1 < q < \frac{3}{2}$ and $2 < p < \infty$ such that $\frac{2}{p} + \frac{3}{q} = 3$. Let $f \in L^p(0, \infty; L^q_\sigma(\Omega_0))$, and let the initial value u_0 be an element of the real interpolation space $(L^q_\sigma, \mathcal{D}(A_{\Omega_0}))_{1-\frac{1}{p},p}$. Then there exists an $\varepsilon > 0$ with the following property: If*

$$\sup_t |\phi - \mathrm{id}|_{C^{3,1}_b} + \|f\|_{q,p} + \|u_0\| < \varepsilon,$$

then the Navier-Stokes system (1.1) possesses a unique global-in-time solution u with $u_t, A(\cdot)u \in L^p(0, \infty; L^q_\sigma(\Omega_0))$.

The second result concerns the existence of time-periodic solutions in L^q-spaces for bounded domains with a periodically moving boundary.

Theorem 2. *Let $n \geqslant 2$, $P(\cdot)f \in C(\mathbb{R}; L^q(\Omega_0))$ be T-periodic with μ as in Assumption 0 satisfying $\frac{n}{2q} < \mu \leqslant 1$. If*

$$|\!|\!| f, \phi |\!|\!| := \sup\nolimits_{t\in[0,T]} \|P(t)f(t)\|_{L^q(\Omega_0)} + \sup\nolimits_{t\in[0,T]} |\phi(t) - \phi(0)|_{C^{3,1}_b}$$

is sufficiently small, then there exists a unique T-periodic solution u in the space $C(\mathbb{R}; W^{1,q}_{0,\sigma}(\Omega_0))$ such that

$$\sup_t \|u(t)\|_{W^{1,q}_{0,\sigma}(\Omega_0)} \leqslant C|||f,\phi|||,$$

where $C > 0$ is a constant independent of f and ϕ.

To prove Theorem 2, we construct by Banach's fixed point theorem mild T-periodic solutions in the class $C^0(\mathbb{R}; W^{1,q}_{0,\sigma}(\Omega_0))$. For lack of initial values, the classical variation-of-constants formula is replaced by the representation

$$u(t) = \int_{-\infty}^{t} S(t,\tau)\big\{P(\tau)f(\tau) - P(\tau)u\cdot\nabla^{\phi}(\tau)u\big\}\,\mathrm{d}\tau$$

using the evolution operator $S(t,s)$; this formula was introduced by Kozono-Nakao ([9]) to get existence of time-periodic solutions for the Navier-Stokes system in exterior domains. To apply this formula, we need that $S(t,s)$ decays exponentially fast as $t-s\to\infty$. This property is natural if the family of domains $(\Omega(t))$ is bounded. It is an open problem how this approach can be generalized to the case of exterior domains.

To show as a third result ([4]) that the T-periodic solution of Theorem 2 is smooth with $\partial_t u, A(t)u$ in L^q we generalize the Fujita-Kato iteration ([5,8]) to the non-autonomous setting following the work of Yagi [15]. In this context, estimates of fractional powers of the modified Stokes operators $A(t)$ play a prominent role. We note that the domains $\mathcal{D}(A(t))$ are t-independent, but that this property is not proved for fractional powers $A(t)^{\alpha}$, $0<\alpha<1$, due to the structure of the coefficients a_α and b_β.

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Newtonian gravitational collapse beyond dust dynamics

Juhi Jang
(joint work with Yan Guo and Mahir Hadžić)

The simplest hydrodynamic model to describe the motion of gaseous stars is given by the Euler-Poisson system:

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) &= 0 \\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P &= -\rho \nabla \Phi \\ \Delta \Phi &= 4\pi \rho \end{aligned}$$

where ρ is the density, $\mathbf{u}$ is the velocity, Φ is the gravitational potential, and $P = \rho^\gamma$, $1 < \gamma < 2$. The system admits static equilibria, called polytopes or Lane-Emden stars, having exact balance between the gravity and the pressure. The exponent γ plays an important role in the existence, uniqueness and stability of Lane-Emden solutions. In particular, if $\frac{6}{5} < \gamma < 2$, at least one compactly supported solution exists and the enthalpy behaves like a distance function near the boundary: $\rho^{\gamma-1} \sim (R - r)$. This boundary behavior causes the standard theory of hyperbolic system inaccessible even for local-in-time existence theory. A suitable framework is to consider the free boundary problem:

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) &= 0 && \text{in } \Omega(t) \\ \rho \left(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) + \nabla \rho^\gamma &= -\rho \nabla \Phi && \text{in } \Omega(t) \\ \Delta \Phi &= 4\pi \rho, && \text{in } \mathbb{R}^3 \\ \lim_{|x| \to \infty} \Phi(t, x) &= 0 \\ \rho &= 0 && \text{on } \partial\Omega(t) \\ \mathcal{V}(\partial\Omega(t)) &= \mathbf{u} \cdot \mathbf{n}(t) && \text{on } \partial\Omega(t) \\ (\rho(0, \cdot), \mathbf{u}(0, \cdot)) &= (\rho_0, \mathbf{u}_0)\,, \ \Omega(0) = \Omega_0 \end{aligned}$$

where $\Omega(t)$ is the support of the fluid to be solved dynamically. The free boundary Euler or Euler-Poisson system is well-posed under the physical vacuum boundary condition [6]:

$$-\infty < \nabla \left(\frac{dP}{d\rho} \right) \cdot \mathbf{n} < 0, \quad P = \rho^\gamma$$

which is satisfied by Lane-Emden stars. A recent development [4, 7, 8] shows that the vacuum free boundary Euler problem admits global-in-time solutions which demonstrate stable expansion phenomena. Even in the presence of gravity, such expansion solutions exist for $\gamma = \frac{4}{3}$ (the mass-critical case) [3] or when perturbed by expanding Euler solutions [5].

On the other hand, collapsing phenomena driven by self-gravitation is still poorly understood. For $\gamma > \frac{4}{3}$ (sub-critical) it is known that the collapse by density concentration for the Euler-Poisson system cannot occur [1]. For $\gamma = \frac{4}{3}$, due to the special symmetries, self-similar collapsing solutions exist since 1980 due to Goldreich and Weber. Despite its strong belief, no collapsing solutions have been available for $1 < \gamma < \frac{4}{3}$ (super-critical) until recently [2] where we provided an affirmative answer by constructing an infinite dimensional family of collapsing solutions.

A key starting point is to introduce a small scaling parameter representing the size of the domain into the system and identify the regime where the gravitational force dominates the pressure gradient and the pressureless blowup triggers a collapse to the full system. To this end, for any $\lambda > 0$, consider the mass preserving rescaling:

$$\rho = \lambda^{-3}\tilde{\rho}(s, y), \quad u = \lambda^{-1/2}\tilde{u}(s, y), \quad \Phi = \lambda^{-1}\tilde{\Phi}(s, y),$$

where $s = \lambda^{-3/2}t, \;\; y = \lambda^{-1}x$. If $(\rho, \mathbf{u}, \Phi)$ solve the Euler-Poisson, then the rescaled quantities $(\tilde{\rho}, \tilde{\mathbf{u}}, \tilde{\Phi})$ solve

$$\begin{aligned} \partial_s \tilde{\rho} + \nabla \cdot (\tilde{\rho}\tilde{\mathbf{u}}) &= 0 \\ \tilde{\rho}\left(\partial_s \tilde{\mathbf{u}} + (\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{u}}\right) + \varepsilon \nabla(\tilde{\rho}^\gamma) + \tilde{\rho}\nabla\tilde{\Phi} &= 0 \\ \Delta\tilde{\Phi} = 4\pi\,\tilde{\rho}, \quad \lim_{|y|\to\infty} \tilde{\Phi}(s, y) &= 0 \end{aligned}$$

where $\varepsilon := \lambda^{4-3\gamma}$. For $\lambda \ll 1$ the factor ε in front of the pressure is small precisely in the supercritical range $1 < \gamma < \frac{4}{3}$. For sufficiently small ε, one might expect the leading order singular behavior to be driven by the pressure-less dynamics. However, from the PDE point of view, the pressure enters to the system as the highest order term, and in fact, controlling the pressure is the most challenging task in the construction of collapsing solutions.

To analyze the rescaled system, we use the Lagrangian coordinates and study the flow map:

$$\partial_s \eta(s, y) = \tilde{\mathbf{u}}(s, \eta(s, y)), \quad \eta(0, y) = \eta_0(y)$$

Furthermore, we restrict ourselves to radial flows and consider

$$\eta(s, y) = \chi(s, r)y, \quad r = |y|, \quad r \in [0, 1],$$

and denote $\chi(0,r)$ by $\chi_0(r)$. The Jacobian determinant of $D\eta$ expressed in terms of χ takes the form $\mathcal{J}[\chi] := \chi^2(\chi + r\partial_r\chi)$. The momentum equation reduces to a nonlinear second order degenerate hyperbolic equation for χ:

$$\chi_{ss} + \frac{G(r)}{\chi^2} + \varepsilon P[\chi] = 0$$

where $G(r) = \frac{1}{r^3}\int_0^r 4\pi w^\alpha s^2\, ds$ (mean density), $w(r)^\alpha := \tilde{\rho}_0(\chi_0(r)r)\mathcal{J}[\chi_0](r)$, and nonlinear pressure operator P

$$P[\chi] := \frac{\chi^2}{w^\alpha r^2}(r\partial_r)\left(w^{1+\alpha}\mathcal{J}[\chi]^{-\gamma}\right).$$

The density is recovered by $\tilde{\rho}(s,\chi(s,r)y) = w^\alpha(r)\mathcal{J}[\chi]^{-1}$. The goal is then to show that there exists a choice of initial conditions $\chi(0) = \chi_0$ and $\partial_s\chi(0) = \chi_1$ with a choice of the enthalpy w such that $\mathcal{J}[\chi]$ becomes zero in finite time! This will give the existence of initial density and velocity in Eulerian coordinates.

The construction will be done via the truncated expansion: $\chi = \sum_{I=0}^M \varepsilon^i\chi_i + \mathcal{R}$. Plugging this ansatz and comparing the coefficients in ε, we have $\chi_0 = (1-g(r)s)^{\frac{2}{3}}$, $g(r) = \sqrt{\frac{9G(r)}{2}}$ solving the pressureless Euler-Poisson system: $\chi_{ss} + \frac{G(r)}{\chi^2} = 0$, χ_i solving the inhomogeneous ODEs, and the remainder $\mathcal{R}$ satisfies the PDE with the source generated by expansion coefficients. It turns out to be possible to choose χ_i smaller than χ_0 due to the super-criticality $\gamma < \frac{4}{3}$ and its feedback by the pressure with suitable flatness of g. This gain is crucial to control the source term and various coefficients appearing in the equation for $\mathcal{R}$. Desired initial data will be obtained by solving $\mathcal{R}$ equation backward in time.

The χ constructed by the above procedure behaves like the dust solution χ_0. In particular, $|\frac{\chi}{\chi_0}| \sim 1$ and $|\frac{\mathcal{J}[\chi]}{\mathcal{J}[\chi_0]}| \sim 1$. Further, for any $r \in [0,1]$

$$\lim_{s\to\frac{1}{g(r)}} \frac{\chi}{\chi_0} = \lim_{s\to\frac{1}{g(r)}} \frac{\mathcal{J}[\chi]}{\mathcal{J}[\chi_0]} = 1.$$

Consequently, the corresponding density blows up along the space time singularity $s = \frac{1}{g(r)}$, and the support shrinks to a point, and mass is absorbed into the singularity.

Interesting future directions include non-radial collapse, more general pressure laws, dynamics of collapsing solutions.

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A generalization of the entropy identity for Burgers' equation and application to the Kuramoto-Sivashinsky equation

Felix Otto
(joint work with Michael Goldman and Marc Josien)

1. Theme

We are interested here in large-scale and low-regularity estimates on solutions of the forced inviscid Burgers' equation

$$\partial_t u(t,x) + u(t,x)\partial_x u(t,x) = \partial_x g(t,x) \qquad \text{for } t \in \mathbb{R}_+ \text{ and } x \in \mathbb{R}, \tag{1.1}$$

where u and g are assumed to be smooth and L-periodic in x, for a large length $L \gg 1$. More precisely, we show that the following estimate is almost true:

$$\int_0^T \int_0^L \left| |\partial_x|^{\frac{1}{3}} u \right|^3 \mathrm{d}x\mathrm{d}t \lesssim \int_0^L |u(0,x)|^2 \,\mathrm{d}x + \int_0^T \int_0^L \left| |\partial_x|^{\frac{2}{3}} g \right|^{\frac{3}{2}} \mathrm{d}x\mathrm{d}t, \tag{1.2}$$

where the operator $|\partial_x|^\alpha$ is of Fourier symbol $|k|^\alpha$ and roughly corresponds to a derivative of order α. In this perspective, the conservative operator $u\partial_x u$ acts as a coercive term; we could formally read $u\partial_x u$ as $|\partial_x|^{\frac{1}{3}} Q'\big(|\partial_x|^{\frac{1}{3}} u\big)$, for $Q(z) := |z|^3$.

Rigorously, estimate (1.2) holds in Besov spaces (or fractional Sobolev spaces) as

$$\sup_{l>0} \int_0^T \int_0^L \frac{|\delta_l u|^3}{l} \mathrm{d}x\mathrm{d}t \lesssim \int_0^L |u(0,x)|^2 \,\mathrm{d}x + \int_0^\infty \frac{\mathrm{d}l}{l} \int_0^T \int_0^L \frac{|\delta_l g|^{\frac{3}{2}}}{l} \mathrm{d}x\mathrm{d}t,$$

for $\delta_l u := u^l - u$ and $u^l(x) := u(x+l)$. It may be written in a more compact form:

$$\|u\|^3_{\dot{B}^{1/3}_{3,\infty}} \lesssim \|u(0,\cdot)\|^2_{\mathrm{L}^2} + \|g\|^{\frac{3}{2}}_{\dot{B}^{2/3}_{3/2,1}} . \tag{1.3}$$

Estimate (1.3) was first established in [3] by studying fine properties of Besov spaces. This was further simplified in [1], by appealing to the regularizing effect of Burgers' equation found by Golse & Perthame [2] (considering the entropy solution of (1.1) with $g = 0$). These authors were using the kinetic formulation of Lions, Perthame and Tadmor '94, and the div-curl structure investigated by Tartar '08, De Lellis, Otto & Westdickenberg '04.

We show here an elementary proof of (1.3) via a modified Howarth-von Kármán-Monin (HvKM) identity (see [1]).

2. Connection between [2] and the HvKM identity

2.1. **HvKM identity for the forced Euler equation.** We first enunciate the so-called HvKM identity for the forced Euler equation:

$$\partial_t u + u \cdot \nabla u + \nabla p = f \qquad \text{with } \nabla \cdot u = 0. \tag{2.1}$$

It reads:

$$\partial_t \int_{\mathbb{R}^d} \frac{1}{2} |\delta_l u|^2 \, \mathrm{d}x + \nabla_l \cdot \int_{\mathbb{R}^d} \frac{1}{2} |\delta_l u|^2 \, \delta_l u \mathrm{d}x = \int_{\mathbb{R}^d} \delta_l u \cdot \delta_l f \mathrm{d}x, \tag{2.2}$$

or, in its local version (which generalizes the local energy identity, taking $l \uparrow \infty$):

$$\partial_t \Big(\frac{1}{2} |\partial_l u|^2 \Big) + \nabla_l \cdot \Big(\frac{1}{2} |\delta_l u|^2 \, \delta_l u \Big) + \nabla_x \cdot \Big(\frac{1}{2} |\delta_l u|^2 \, u + \delta_l p \delta_l u \Big) = \delta_l u \cdot \delta_l f.$$

2.2. **Forced inviscid Burgers' equation.** Getting back to Burgers' equation (1.1), where we momentarily denote $f := \partial_x g$, the entropy identity reads

$$\partial_t \eta \, (u) + \partial_x q \, (u) = \eta'(u) f.$$

provided $q'(z) = z\eta'(z)$. Defining the additional quantity Q by $Q' \, (z) = z\eta'(z) - \eta(z)$, it deforms into

$$\partial_t \eta \, (\delta_l u) + \partial_l Q \, (\delta_l u) + \partial_x \left(u^l \eta \, (\delta_l u) - q \, (\delta_l u) \right) = \eta' \, (\delta_l u) \, \delta_l \, (\partial_t u + u \partial_x u) \, . \tag{2.3}$$

The latter identity turns into the entropy identity as $l \uparrow \infty$ (for fixed (t, x)).

Integrating (2.3) yields a global formulation

$$\partial_t \int \eta \, (\delta_l u) \, \mathrm{d}x + \partial_l \int Q(\delta_l u) \mathrm{d}x = \int \eta' \, (\delta_l u) \, \delta_l f \mathrm{d}x. \tag{2.4}$$

Three special cases of (2.4) are especially relevant: $\eta(z) = |z| \Rightarrow Q(z) = 0$, in which we obtain the L^1 contraction:

$$\partial_t \int \left| u^l - u \right| \mathrm{d}x = \int \operatorname{sign} \left(u^l - u \right) \delta_l f \mathrm{d}x;$$

$\eta(z) = \frac{1}{2}|z|^2 \Rightarrow Q(z) = \frac{1}{6} z^3$, which gives the HvKM identity for Burgers' equation:

$$\partial_t \int \frac{1}{2} \, (\delta_l u)^2 \, \mathrm{d}x + \partial_l \int \frac{1}{6} \, (\delta_l u)^3 \, \mathrm{d}x = \int \delta_l u \delta_l f \mathrm{d}x;$$

and $\eta(z) = \frac{1}{2} z_+^2 \Rightarrow Q(z) = \frac{1}{6} z_+^3$, which produces the modified HvKM identity (in which the second left-hand integral is coercive):

$$\partial_t \int \frac{1}{2} \, (\delta_l u)_+^2 \, \mathrm{d}x + \partial_l \int \frac{1}{6} \, (\delta_l u)_+^3 \, \mathrm{d}x = \int (\delta_l u)_+ \, \delta_l f \mathrm{d}x. \tag{2.5}$$

2.3. **Coercivity estimate.** By integrating the estimate (2.5) in t and then l, and using the Hölder inequality, we obtain

$$\sup_{l \neq 0} \frac{1}{|l|} \int_0^T \int \frac{1}{6} \, (\delta_l u)_+^3 \, \mathrm{d}x \mathrm{d}t \leqslant \sup_l \left\{ \int \frac{1}{2} \, (\delta_l u(0, \cdot))_+^2 \, \mathrm{d}x + \int_0^T \int |\delta_l u| \, \delta_l f \mathrm{d}x \mathrm{d}t \right\}.$$

Recalling that $f = \partial_x g$, we derive (1.3).

3. Application to the Kuramoto-Shivashinsky equation

We consider L-periodic solutions of the Kuramoto-Sivashinsky equation:

$$\partial_t u + u\partial_x u + \partial_x^2 u + \partial_x^4 u = 0 \qquad \text{for } t \in \mathbb{R}_+, x \in \mathbb{R}, \tag{3.1}$$

where, by the Galilean invariance, we may assume that $\int_0^L u = 0$. In this equation, the second term provides the energy cascade, the third one pumps in the energy on large scales $\gg 1$, and the fourth one takes out the energy on small scales $\gg 1$. It is conjectured in Michelson '86 that the function u remains bounded for long time and large scales:

$$\limsup_{T,L\uparrow\infty} \frac{1}{T}\int_0^T \frac{1}{L}\int_0^L |u|^2 \,\mathrm{d}x\mathrm{d}t \lesssim 1.$$

We prove the weaker result (see Goldman, Josien & Otto [1]):

Theorem. *The following estimate holds:*

$$\limsup_{T\uparrow\infty} \frac{1}{T}\int_0^T \frac{1}{L}\int_0^L \left|\left|\partial_x\right|^{\frac{1}{3}} u\right|^3 \mathrm{d}x\mathrm{d}t \lesssim \ln^{\frac{5}{6}+} L. \tag{3.2}$$

We denote by $\langle h\rangle$ the average $\frac{1}{TL}\int_0^T\int_0^L h\mathrm{d}x\mathrm{d}t$. The proof of (3.2) relies on three ingredients (modulo the logarithmic loss due our use of Besov spaces $\dot{\mathrm{B}}^{1/3}_{3,\infty}$ instead of $\dot{\mathrm{B}}^{1/3}_{3,3}$): the coercivity of Burgers' equation $\partial_t u + u\partial_x u = -\partial_x^4 u$ (ignoring the term $\partial_x^2 u$), which induces a gain in homogeneity (from 3 to 2 in u):

$$\left\langle \left| \left|\partial_x\right|^{\frac{1}{3}} u\right|^3 \right\rangle \lesssim \left\langle \left|\partial_x u\right|^{\frac{5}{2}} \right\rangle^{\frac{2}{5}} \left\langle \left|\partial_x^3 u\right|^{\frac{5}{3}} \right\rangle^{\frac{3}{5}};$$

the energy estimate on $\partial_t u + u\partial_x u + \partial_x^4 u = -\partial_x^2 u$ (the second left-hand term is irrelevant), which is neutral in terms of homogeneity:

$$\left\langle \left|\partial_x^2 u\right|^2 \right\rangle \lesssim \left\langle \left| \left|\partial_x\right|^{\frac{1}{3}} u\right|^3 \right\rangle^{\frac{2}{3}};$$

the "maximal regularity" for the capillary Burgers' equation $\partial_t u + \partial_x^4 u = -u\partial_x u$ (ignoring the term $\partial_x^2 u$), which suffers from a loss in homogeneity (from 1 to 2 in u):

$$\left\langle \left|\partial_x^5 u\right|^{\frac{5}{4}} \right\rangle \lesssim \left\langle \left|\partial_x u\right|^{\frac{5}{2}} \right\rangle.$$

The result (3.2) is then obtained by combining these three ingredients with the two following interpolation inequalities:

$$\left\langle \left|\partial_x^3 u\right|^{\frac{5}{3}} \right\rangle \lesssim \left\langle \left|\partial_x^5 u\right|^{\frac{5}{4}} \right\rangle^{\frac{4}{9}} \left\langle \left|\partial_x^2 u\right|^2 \right\rangle^{\frac{5}{9}} \text{ and } \left\langle \left|\partial_x u\right|^{\frac{5}{2}} \right\rangle \lesssim \left\langle \left| \left|\partial_x\right|^{\frac{1}{3}} u\right|^3 \right\rangle^{\frac{1}{2}} \left\langle \left|\partial_x^2 u\right|^2 \right\rangle^{\frac{1}{2}}.$$

In this perspective, remark that all the norms are in form of $\left\langle \left| \left|\partial_x\right|^{\frac{10}{p}-3} u\right|^p \right\rangle^{\frac{1}{p}}$.

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Asymptotic criticality of the Navier-Stokes regularity problem

Zoran Grujić

(joint work with Liaosha Xu)

3D Navier-Stokes equations (NSE) – describing a flow of 3D incompressible viscous Newtonian fluid – read

$$u_t + (u \cdot \nabla)u = -\nabla p + \triangle u,$$

supplemented with the incompressibility condition $\operatorname{div} u = 0$, where u is the velocity of the fluid and p is the pressure (the viscosity is set to 1).

NS regularity problem has been *super-critical* in the sense that there has been a 'scaling gap' between any presently known regularity criterion and the corresponding *a priori* bound; a classical example is the LPS regularity criterion, $u \in L^p(0,T;L^q)$,

$$\frac{3}{q} + \frac{2}{p} = 1$$

vs. the *a priori* bound $u \in L^p(0,T;L^q)$,

$$\frac{3}{q} + \frac{2}{p} = \frac{3}{2}.$$

Moreover, all the known regularity criteria are (at best) scaling-invariant, while all the *a priori* bounds have been on the scaling level of the energy bound, regardless of the functional framework utilized. This abstract delineates a mathematical set-up, based on a suitably defined 'scale of sparseness' of the super-level sets of the higher order derivatives of the velocity field, in which the scaling gap shrinks to 0 as the order of the derivative goes to infinity (in the vicinity of the possible singular time), demonstrating *asymptotic criticality.*

`local sparseness`

Let S be an open subset of $\mathbb{R}^3$, x_0 a point in $\mathbb{R}^3$, $r \in (0,\infty)$, and $\delta \in (0,1)$ (m^n will denote the n-dimensional Lebesgue measure)

S is *1D δ-sparse around x_0 at scale r* if there exists a unit vector $\mathbf{d}$ in S^2 such that

$$\frac{m^1\big(S \cap (x_0 - r\mathbf{d}, x_0 + r\mathbf{d})\big)}{2r} \leqslant \delta$$

S is *3D δ-sparse around x_0 at scale r* if

$$\frac{m^3\big(S \cap B(x_0,r)\big)}{m^3\big(B(x_0,r)\big)} \leqslant \delta$$

(It is straightforward to check that 3D δ-sparseness at scale r implies 1D $(\delta)^{\frac{1}{3}}$-sparseness at scale ρ, for some ρ in $(0,r]$ (at any given pair (x_0,r).)

In what follows, let us denote the positive and the negative parts of the vectorial components of f by $f_i^{\pm}$, and compute the norm of a vector $v=(a,b,c)$ as $|v| = \max\{|a|,|b|,|c|\}$.

For a positive exponent α, and a selection of parameters λ in $(0,1)$, δ in $(0,1)$ and $c_0>1$, the class of functions $Z_\alpha(\lambda,\delta;c_0)$ consists of bounded, continuous functions $f:\mathbb{R}^3\to\mathbb{R}^3$ subjected to the following uniformly local condition. For x_0 in $\mathbb{R}^3$, select the/a component $f_i^{\pm}$ such that $f_i^{\pm}(x_0)=|f(x_0)|$, and require that the set

$$\left\{x\in\mathbb{R}^3 :\ f_i^{\pm}(x) > \lambda\|f\|_\infty\right\}$$

be 1D δ-sparse around x_0 at scale $c\,\frac{1}{\|f\|_\infty^\alpha}$, for some c, $\frac{1}{c_0}\le c\le c_0$. Enforce this for all x_0 in $\mathbb{R}^3$ (shortly, we require sparseness of the/a *locally maximal* component only).

Consider the higher order spatial fluctuations of the velocity field (more spatial intermittency, an increased chance to deviate from 'the scaling' ..). As an example, consider the sequence of functional classes $Z^{(k)}_{\alpha_k}$ defined by

$$u\in Z^{(k)}_{\alpha_k} \quad \text{if} \quad D^{(k)}u\in Z_{\alpha_k};$$

then we have the following (for $k=k(\|u_0\|_2,\|u_0\|_\infty)$ large enough:

Regularity class	A priori bound	Energy-level class
$u(\tau)\in Z^{(k)}_{\frac{1}{k+1}}$ for a suitable $\tau<T$	$u(\tau)\in Z^{(k)}_{\frac{1}{k+\frac{3}{2}}}$ a.e. $\tau<T$ whenever $\|D^{(k)}u(\tau)\|_\infty$ is sufficiently large	$u(\tau)\in Z^{(k)}_{\frac{1}{\frac{3}{2}(k+1)}}$ a.e. $\tau<T$

The key technical component of the proof is managing the interplay between the higher order and the lower order derivatives; in particular, local-in-time dynamics of the forms

$$\sup_{t_1<t<t_2} \max_{0\le\kappa_1(t)\le i\le\kappa_2(t)\le k} \left\{ c_{i,k} \, \|D^{(i)}u(t)\|_\infty^{\frac{k+1}{i+1}} \right\}$$

plays the central role.

`a remark on what happens in Sobolev (and other 'standard' functional) spaces as the order of the derivative increases (the scaling gap remains the same):`

Regularity class	A priori bound	Energy-level class
$\int_0^T \|u(t)\|_{H^m}^{\frac{4}{2m-1}} \, dt < \infty$	$\int_0^T \|u(t)\|_{H^m}^{\frac{2}{2m-1}} \, dt < \infty$	$\int_0^T \|u(t)\|_{H^m}^{\frac{2}{2m-1}} \, dt < \infty$

Back to the $Z_{\alpha_k}^{(k)}$ framework.

`a remark on 'constants':`

Regularity class-scale	A priori bound-scale	Energy-level class-scale
$\frac{1}{C_1(k)} \frac{1}{\|D^{(k)}u\|_\infty^{\frac{1}{k+1}}}$	$C_2(\|u_0\|_2) \frac{1}{\|D^{(k)}u\|_\infty^{\frac{1}{k+\frac{3}{2}}}}$	$C_3(\|u_0\|_2) \frac{1}{\|D^{(k)}u\|_\infty^{\frac{1}{\frac{3}{2}(k+1)}}}$

`a remark on scaling of the dynamical quantities:`

Regularity class-scale	A priori bound-scale	Energy-level class-scale
$\frac{1}{\|D^{(k)}u\|_\infty^{\frac{1}{k+1}}} \approx r$	$\frac{1}{\|D^{(k)}u\|_\infty^{\frac{1}{k+\frac{3}{2}}}} \approx r^{\frac{k+1}{k+\frac{3}{2}}}$	$\frac{1}{\|D^{(k)}u\|_\infty^{\frac{1}{\frac{3}{2}(k+1)}}} \approx r^{\frac{2}{3}}$

$$r^{\frac{k+1}{k+\frac{3}{2}}} \to r, \quad k \to \infty$$

asymptotic criticality!

Fluid-Squeezing singularities for the incompressible Euler equations

Diego Córdoba
(joint work with Alberto Enciso and Nastasia Grubic)

I will discuss a new result on the existence of a stationary solution with a fluid-squeezing singularity for the two-fluid incompressible Euler equations. The proof is based on a new set of estimates that permits us to analyze a fluid squeezed by a self-intersecting interface. We will exploit these estimates in the dynamical case and show a local existence result starting from a fluid-squeezing singularity.

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Global existence of Navier-Stokes equations for non-decaying initial data

Tai-Peng Tsai
(joint work with Hyunju Kwon, Zachary Bradshaw)

Consider the Cauchy problem of incompressible Navier-Stokes equations in $\mathbb{R}^3$ with uniformly locally square integrable initial data. If the square integral of the initial datum on a ball vanishes as the ball goes to infinity, the existence of a time-global weak solution was known. However, such data do not include constants, and the only known global solutions for non-decaying data are either for perturbations of constants, for data in critical Morrey space $M^{2,1}$, or when the velocity gradients are in L^q with finite q. In this talk, I will outline how to construct global weak solutions for non-decaying initial data, first for those whose local oscillations decay, no matter how slowly, and second for those whose uniform local square integral grows in scale under a certain rate. These are joint work with Kwon and Bradshaw, respectively.

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A uniqueness/weak-strong uniqueness result for 3D incompressible/compressible fluid-rigid body interaction problem

Šárka Nečasová
(joint work with Ondřej Kreml, Boris Muha, Tomasz Piasecki, Ana Radošević)

We study a 3D nonlinear moving boundary fluid-structure interaction problem describing the interaction of the fluid flow with a rigid body. The fluid flow is governed by 3D incompressible/compressible Navier-Stokes equations, while the motion of the rigid body is described by a system of ordinary differential equations called Euler equations for the rigid body. The equations are fully coupled via dynamical and kinematic coupling conditions. We consider two different kinds of kinematic coupling conditions: no-slip and slip. In both cases we prove a generalization of the well-known weak-strong uniqueness result for the Navier-Stokes equations to the fluid-rigid body system. More precisely, in incompressible case we prove that weak solutions that additionally satisfy Prodi-Serrin $\mathrm{L}^r - \mathrm{L}^s$ condition are unique in the class of Leray-Hopf weak solutions, see [4].

In the compressible case we show that the strong solution, which is known to exist under certain smallness assumptions, is unique in the class of weak solutions to the problem, see [3].

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain which represents a container containing a fluid and a rigid body, and let $S_0 \subset \Omega$ be a connected open set representing the rigid body at the initial time $t = 0$ with the center of mass denoted by $\mathbf{q}_0 \in \Omega$. The motion of the rigid body is fully described by two functions $\mathbf{q} : [0, T] \to \mathbb{R}^3$ and $\mathbb{Q} : [0, T] \to SO(3)$, where $SO(3)$ is the 3D rotation group, representing the position of the center of mass and the rotation around the center of mass at the time moment t, respectively. More precisely, the trajectories of all points of the body are described by an orientation preserving isometry

$$\mathbf{B}(t, \mathbf{y}) = \mathbf{q}(t) + \mathbb{Q}(t)(\mathbf{y} - \mathbf{q}(0)), \qquad \mathbf{y} \in S_0,\ t \in [0, T], \tag{1.1}$$

and at time t the body occupies the set $S(t) = \{\mathbf{x} \in \mathbb{R}^3 : \ \mathbf{x} = \mathbf{B}(t, \mathbf{y}), \mathbf{y} \in S_0\} = \mathbf{B}(t, S_0), t \in [0, T]$, The fluid domain at time t is defined by $\Omega_F(t) = \Omega \setminus \overline{S(t)}$. Since the domain changes in time, we introduce the following notation: $(0, T) \times \Omega_F(t) = \bigcup_{t\in(0,T)} \{t\} \times \Omega_F(t) = Q_T$. The fluid flow is described by the incompressible/compressible Naiver-Stokes equations. with $\mathbf{u}$ is the fluid velocity, ϱ_F is the fluid density, $\mathbb{T} = -p\mathbb{I} + 2\mu\, \mathbb{D}\mathbf{u}$ is the fluid Cauchy stress tensor, $\mathbb{D}\mathbf{u} = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)$ is the deformation-rate tensor, p is the fluid pressure and $\mu > 0$ is the fluid velocity.

The Eulerian velocity of the rigid body is given by:

$$\mathbf{u}_S(t, \mathbf{x}) := \partial_t \mathbf{B}(t, \mathbf{B}^{-1}(t, \mathbf{x})) = \mathbf{a}(t) + \mathbb{P}(t)(\mathbf{x} - \mathbf{q}(t)) \qquad \text{for all} \quad \mathbf{x} \in S(t), \tag{1.2}$$

where $\mathbf{a}(t) = \mathbf{q}'(t)$ is the translation velocity and $\mathbb{P}(t) = \mathbb{Q}'(t)\mathbb{Q}^T$ is the angular velocity. The angular velocity $\mathbb{P}$ is a skew–symmetric matrix and therefore there exists a vector $\boldsymbol{\omega} = \boldsymbol{\omega}(t) \in \mathbb{R}^3$ such that $\mathbb{P}(t)\mathbf{x} = \boldsymbol{\omega}(t) \times \mathbf{x}, \forall \mathbf{x} \in \mathbb{R}^3$.

The coupling conditions: The fluid and the rigid body are coupled via dynamic and kinematic coupling condition. *The dynamic boundary condition* is the balance of forces and torques. *The kinematic condition*: we can consider *the no-slip condition* which says that the fluid and the rigid body velocities are equal at the rigid body boundary, or *the Navier's slip boundary condition* which allows for the discontinuity of the tangential component of the velocity along the interface.

Incompressible case
We consider the following problem: Find $(\mathbf{u}, p, \mathbf{q}, \boldsymbol{\omega})$ such that

$$
\begin{aligned}
&\left.\begin{array}{l}\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \operatorname{div}(\mathbb{T}(\mathbf{u},p)),\\ \operatorname{div}\mathbf{u} = 0\end{array}\right\} \text{ in } \textstyle\bigcup_{t\in(0,T)}\{t\}\times\Omega_F(t),\\
&\left.\begin{array}{l}\frac{d^2}{dt^2}\mathbf{q} = -\int_{\partial S(t)}\mathbb{T}(\mathbf{u},p)\mathbf{n}\,\mathrm{d}S(\mathbf{x}),\\ \frac{d}{dt}(\mathbb{J}\boldsymbol{\omega}) = -\int_{\partial S(t)}(\mathbf{x}-\mathbf{q}(t))\times\mathbb{T}(\mathbf{u},p)\mathbf{n}\,\mathrm{d}S(\mathbf{x})\end{array}\right\} \text{ in } (0,T),\\
&\mathbf{u} = \mathbf{q}' + \boldsymbol{\omega}\times(\mathbf{x}-\mathbf{q}), \quad \text{on } \textstyle\bigcup_{t\in(0,T)}\{t\}\times\partial S(t),\\
&\mathbf{u} = 0 \quad \text{on } \partial\Omega,\\
&\mathbf{u}(0,.) = \mathbf{u}_0 \qquad \text{in } \Omega, \quad \mathbf{q}(0) = \mathbf{q}_0, \quad \mathbf{q}'(0) = \mathbf{a}_0, \quad \boldsymbol{\omega}(0) = \boldsymbol{\omega}_0.
\end{aligned}
\tag{1.3}
$$

We define a function space: $V(t) = \{\mathbf{v} \in H_0^1(\Omega) : \operatorname{div}\mathbf{v} = 0,\ \mathbb{D}\mathbf{v} = 0 \text{ in } S(t)\}$.

Definition 1.1. The couple $(\mathbf{u}, \mathbf{B})$ is a weak solution to the system (1.3) if the following conditions are satisfied: 1) The function $\mathbf{B}(t,\cdot) : \mathbb{R}^3 \to \mathbb{R}^3$ is an orientation preserving isometry given by the formula (1.1), which defines a time-dependent set $S(t) = \mathbf{B}(t, S)$, and the corresponding Eulerian velocity $\mathbf{u}_S$ given by (1.2) is compatible with $\mathbf{B}(t,\cdot)$.
2) The function $\mathbf{u} \in L^2(0,T;V(t)) \cap L^\infty(0,T;L^2(\Omega))$ satisfies the integral equality

$$
\int_0^T \int_{\Omega\setminus\partial S(t)} \{\mathbf{u}\cdot\partial_t\boldsymbol{\psi} + (\mathbf{u}\otimes\mathbf{u}) : \mathbb{D}\boldsymbol{\psi} - 2\,\mathbb{D}\mathbf{u} : \mathbb{D}\boldsymbol{\psi}\,\}\,\mathrm{d}\mathbf{x}\mathrm{d}t = -\int_\Omega \mathbf{u}_0\boldsymbol{\psi}(0)\,\mathrm{d}\mathbf{x},
$$

which holds for any test function $\boldsymbol{\psi} \in H^1(0,T;V(t))$, $\boldsymbol{\psi}(T,.) = 0$.

Now we can state the main result of incompressible section, for details see [4].

Theorem 1.1. [4] Let $(\mathbf{u}_1, \mathbf{B}_1)$ and $(\mathbf{u}_2, \mathbf{B}_2)$ be two weak solutions corresponding to the same data. Assume that $d(S_i(t), \partial\Omega) > \delta_i$, $i = 1, 2$, for some constants $\delta_i > 0$. If $\mathbf{u}_2$ satisfies the following condition:

$$
\mathbf{u}_2 \in L^r(0,T;L^s(\Omega)) \quad \text{for some } s, r \text{ such that} \quad \frac{3}{s} + \frac{2}{r} = 1,\ s \in (3, +\infty]
\tag{1.4}
$$

then $(\mathbf{u_1}, \mathbf{B_1}) = (\mathbf{u_2}, \mathbf{B_2})$.

Compressible case

We consider the following problem: find $(\varrho, \mathbf{u}, \mathbf{q}, \boldsymbol{\omega})$

$$
(1.5)\quad
\begin{aligned}
&\left.\begin{aligned}
&\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0,\\
&\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) - \operatorname{div}\, \mathbb{T}(\nabla_x \mathbf{u}) = 0,
\end{aligned}\right\} \text{ in } Q_T,\\
&\left.\begin{aligned}
&\tfrac{d^2}{dt^2}\mathbf{q} = -\textstyle\int_{\partial S(t)} (\mathbb{T}(\nabla_x \mathbf{u}) - p(\varrho)\mathcal{I})\mathbf{n}\,\mathrm{d}S(\mathbf{x}),\\
&\tfrac{d}{dt}(\mathbb{J}\boldsymbol{\omega}) = -\textstyle\int_{\partial S(t)} (\mathbf{x} - \mathbf{q}(t)) \times (\mathbb{T}(\nabla_x \mathbf{u}) - p(\varrho)\mathcal{I})\mathbf{n}\,\mathrm{d}S(\mathbf{x})
\end{aligned}\right\} \text{ in } (0,T),
\end{aligned}
$$

$$
(1.6)\quad
\begin{aligned}
&\mathbf{u} = \mathbf{q}' + \boldsymbol{\omega} \times (\mathbf{x} - \mathbf{q}), \quad \text{on } \textstyle\bigcup_{t\in(0,T)} \{t\} \times \partial S(t),\\
&\mathbf{u} = 0 \quad \text{on } \partial\Omega,\\
&\varrho \mathbf{u}(0) = (\varrho \mathbf{u})_0,\ \varrho(0) = \varrho_0 \text{ in } \Omega,\\
&\mathbf{q}(0) = \mathbf{q}_0, \quad \mathbf{q}'(0) = \mathbf{a}_0, \quad \boldsymbol{\omega}(0) = \boldsymbol{\omega}_0.
\end{aligned}
$$

We recall that the rate of the strain tensor of the fluid and its stress tensor are defined by $\mathbb{D}(\mathbf{u}) = \frac{1}{2}(\nabla_x \mathbf{u} + (\nabla_x \mathbf{u})^T)$ and $\mathbb{T}(\nabla_x \mathbf{u}) = 2\mu\, \mathbb{D}(\mathbf{u}) + \lambda \operatorname{div}_x \mathbf{u}\mathcal{I}$. A pressure satisfies $p = a\varrho^\gamma$, with a positive constant, $\gamma > 1$.

Theorem 1.2. [3] Let $\bar{\varrho}$ be the mean value of ϱ_0 in $\Omega_F(0)$. Assume that $d(\mathcal{B}_0, \partial\Omega) > 0$ and the initial conditions $(\varrho_0, \mathbf{u}_0, a_0, \boldsymbol{\omega}_0)$ satisfy appropriate compatibility conditions, see [1, formulas (15), (16)]. Let moreover $\|\varrho_0 - \bar{\varrho}\|_{H^3(\mathcal{F}_0)} + \|\mathbf{u}_0\|_{H^3(\mathcal{F}_0)} + |a_0| + |\boldsymbol{\omega}_0| < \delta$, be satisfied, $\gamma > 3/2$ and let $(\varrho_2, \mathbf{u}_2, a_2, \boldsymbol{\omega}_2)$ be the strong solution to (1.5) on $(0,T)$ given by [1] satisfying $d(\mathcal{B}(t), \partial\Omega) \geqslant \kappa > 0\, \forall t \in [0,T]$ for some $d(\mathcal{B}_0, \partial\Omega) > \kappa > 0$. Let $(\varrho_1, \mathbf{u}_1, B_1)$ be a weak solution to (1.5) on $(0,T)$ emanating from the same initial data given by [2] . Then $B_1(t) = B_2(t)$, and $(\varrho_{1\mathcal{F}}, \mathbf{u}_{1\mathcal{F}}) = (\varrho_{2\mathcal{F}}, \mathbf{u}_{2\mathcal{F}})$ for all $(t, \mathbf{x}) \in Q_{\mathcal{F}1} = Q_{\mathcal{F}2}$.

Concluding remarks:

Incompressible case:

(1) Weak-strong uniqueness holds in the case of Navier type of boundary conditions, see [4].

(2) Work in progress with A. Mazuccato and N. Chemetov, global existence of weak solution in case of slip boundary conditions

(3) Work in progress with B. Muha and A. Radosevič concerning regularity of solution

Compressible case

(1) Work in progress with A. Schlömerkemper, M. Ramaswamy, A. Roy concerning existence of weak solution with the Navier type of boundary condition

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Endpoint maximal regularity and application to a free boundary problem for the incompressible Navier-Stokes equations

Raphaël Danchin
(joint work with Matthias Hieber, Piotr B. Mucha, Yoshihiro Shibata, Patrick Tolksdorf)

We are concerned with a free boundary problem for viscous incompressible flows with no surface tension and no gravity. The governing equations for the velocity field $v = v(t,x)$ and the pressure $P = P(t,x)$ are the incompressible Navier-Stokes equations in the (unknown) moving domain $\Omega_t \subset \mathbb{R}^3$, namely

$$
\begin{cases}
\partial_t v + (v\cdot\nabla)v - \operatorname{div}\mathbb{T}(v,P) = 0, & t\in\mathbb{R}_+, x\in\Omega_t\\
\operatorname{div}(v) = 0, & t\in\mathbb{R}_+, x\in\Omega_t\\
\mathbb{T}(v,P)\overline{n} = 0, & t\in\mathbb{R}_+, x\in\partial\Omega_t\\
v|_{t=0} = v_0, & x\in\Omega_0,\\
\Omega_t|_{t=0} = \Omega_0, & \\
v\cdot\overline{n} = -(\partial_t\eta)/|\nabla_x\eta|, & t\in\mathbb{R}_+, x\in\partial\Omega_t.
\end{cases}
\tag{1.1}
$$

Above, $\mathbb{T}(v,P)$ denotes the stress tensor of the form

$$\mathbb{T}(v,P) = D(v) - P\mathrm{Id} \quad \text{with} \quad D(v) = \nabla v + [\nabla v]^\top.$$

The boundary $\partial\Omega_t$ is described by the equation $\eta(t,x) = 0$. By $\overline{n}$ we denote the outward unit normal vector to $\partial\Omega_t$, i.e., $\overline{n} = \nabla_x\eta/|\nabla_x\eta|$. Finally, v_0 stands for the velocity field at time $t = 0$ and we assume that Ω_0 is the half-space $\mathbb{R}^3_+$ (hence $-\overline{n}|_{t=0} = e_3 := (0,0,1)$).

For small and smooth enough v_0, the global existence of a unique strong solution has been established by Y. Shibata in [6] in the case where the fluid moves in a bounded container. However, in the half-space case we aim at investigating here only local existence is known (unless surface tension is taken into consideration, see [5]).

Compared to the bounded domain case, the main difficulty in our framework is that both the domain and the boundary are unbounded. Consequently, 0 is in the spectrum of the Stokes operator corresponding to the linearized equations and the classical methods based either on the maximal regularity for the Stokes semi-group in $\mathbb{R}^3_+$ or on energy estimates fail to yield the global existence. A closer analysis reveals that what we lack is a *global-in-time* L^1 integrability estimate of

the solution of the linearized system:

$$
(1.2) \qquad \begin{cases} \partial_t v - \operatorname{div} \mathbb{T}(v,P) = f & \text{in } \mathbb{R}_+ \times \mathbb{R}^3_+, \\ \operatorname{div} v = g & \text{in } \mathbb{R}_+ \times \mathbb{R}^3_+, \\ \mathbb{T}(v,P)e_3 = h & \text{on } \mathbb{R}_+ \times \partial\mathbb{R}^3_+, \\ v|_{t=0} = v_0 & \text{in } \mathbb{R}^3_+. \end{cases}
$$

In the present work, we establish the global existence of a strong solution for (1.1) if some 'critical' norm of v_0 is small enough. After performing a Lagrangian change of coordinates like in [2], we recast the equations as in (1.2), where all the nonlinearities are in f, g and h, and the general strategy is to implement a fixed point in a suitable functional framework, that will be given by endpoint maximal estimates for (1.2). Typically, those estimates involve spaces like $L_1(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+))$. Let us recall that those estimates are obvious if one considers $\mathbb{R}^n$ instead of $\mathbb{R}^n_+$ and that they have been generalized by the first two authors in [1, 3] to the bounded domain, exterior domain or half-space cases for the Stokes system *supplemented with Dirichlet boundary conditions*. The method therein was based on localization and explicit computations of the Stokes semi-group in the half-space, and involved a number of steps with complicated computations.

The approach that is chosen here is borrowed from an abstract interpolation argument by G. Da Prato and P. Grisvard [4] that just requires the semi-group associated to the linear system under consideration to be bounded and analytic, a property that possesses the Stokes operator supplemented with the above Neumann boundary conditions (see [7]). However, in order to obtain in the end estimates in $L_1(\mathbb{R}_+; X)$, the space X has to be 'homogeneous' (so as to respect the scaling invariance of (1.2)) so that one has to revisit Da Prato-Grisvard approach in the context of 'abstract homogeneous spaces' constructed from an homogenous version of the domain of the Stokes operator.

Let us emphasize that this part of our work is rather general (as it just requires the semi-group to be bounded analytic), and is likely to have other applications in fluid mechanics.

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On the regularity criterion for the axisymmetric 3D Euler equations

Dongho Chae
(joint work with J. Wolf)

In this talk we present new local blow-up criterion for smooth axisymmetric solutions to the 3D incompressible Euler equation. If the vorticity satisfies $\int_0^{t_*}(t_* - t)\|\omega(t)\|_{L^\infty(B(x_*,R_0))}dt < +\infty$ for a ball $B(x_*, R_0)$ away from the axis of symmetry, then there exists no singularity at $t = t_*$ in the torus $T(x_*, R)$ generated by rotation of the ball $B(x_*, R_0)$ around the axis. This implies that possible singularity at $t = t_*$ in the torus $T(x_*, R)$ is excluded if the vorticity satisfies the blow-up rate $\|\omega(t)\|_{L^\infty(T(x_*,R))} = O\left(\frac{1}{(t_*-t)^\gamma}\right)$ as $t \to t_*$, where $\gamma < 2$, and the torus $T(x_*, R)$ does not touch the axis. This would be published in [1]. The crucial part of the proof is establishment of a new blow-up criterion for the 2D Boussinesq equations with variable coefficients, which is a localized version of the result in [2].

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Weak solutions of the Euler equations, D'Alembert Principle, and Turbulence

Alexander Shnirelman

Let $M \subset \mathbb{R}^n$ be a bounded domain with a sufficiently regular boundary. We consider the motion of an ideal incompressible fluid in M with the slipping condition on ∂M which is described by the classical Euler equations

$$\begin{aligned} \frac{\partial u}{\partial t} + \nabla u \cdot u + \nabla p &= 0 \\ \nabla \cdot u &= 0 \\ u_n|_{\partial M} &= 0 \end{aligned}$$

A vector field $u(x,t) \in L^2(M \times [0,T], \mathbb{R}^n)$ is called a weak solution of the Euler equations if for any test field $v(x,t) \in \mathbb{C}_0^\infty$, $\nabla \cdot v = 0$, and any scalar function $\varphi(x,t) \in c^\infty$,

$$\int_0^T \int_M \left[\left(u, \frac{\partial v}{\partial t} \right) + (u \otimes u, \nabla v) \right] dxdt = 0$$

$$\int_0^T \int_M (u, \nabla \varphi)\, dxdt = 0$$

Weak solutions (or some sort thereof) are regarded as candidates to describe the turbulent motion of the fluid, especially the phenomenon of anomalous energy dissipation. However, the known nontrivial weak solutions (see, for example, [1], [2], [3]) appear quite nonphysical. For example, there exist weak solutions $u(x,t) \in C^{1/3-\epsilon}$ such that the kinetic energy $E(t)$ is any given continuous function $e(t)$, while for any physically reasonable weak solution the energy should be monotone decreasing. Moreover, a closer look at the construction described in these works shows that the flow is in fact driven by forces with infinitesimally small space scale. This means that in these flows we see the inverse energy cascade, namely the energy is pumped in at a small scale, and then propagates to the larger scales due to the nonlinearity. This is contrary to the direct cascade observed in the real turbulence, and hence the known weak solutions are not good models of turbulent flows. Beside this, the construction is time-reversible (i.e. if $u(x,t)$ is a weak solution constructed by the proposed method with some parameters of construction, then $-u(x,-t)$ is also an acceptable solution constructed by the same method with some different parameters). The only known exception is the solution constructed in [4] whose energy is monotone decreasing. Its construction is explicitly time-irreversible; however, it has many unphysical properties, and is not a good model of turbulence, too.

Our construction of a weak solution is based on the D'Alembert Principle. Remind that if a material point moves without friction along a smooth surface $D \subset X$ where X is an euclidean space, then its trajectory $x(t)$ satisfies the equations

$$x(t) \in D; \qquad \dot{x}(t) \in T_{x(t)}D; \qquad \ddot{x}(t) \perp T_{x(t)}D$$

In other words, the acceleration is orthogonal to the constraint (i.e. the surface D).

In the case of fluid, let Ω be the space of fluid particles, i.e. Ω is a measure space endowed with σ-algebra $\mathcal{F}$ and a probability measure μ. The space $X = L^2(\Omega, \mathbb{R}^n)$, and the "surface"

$$D = \{f \in X \mid \ f(\omega) \in M \text{ for a.a.} \omega \in \Omega, \text{ and} f_*(d\mu) = dx\}$$

This set is far from being smooth. The only available piece of information about its regularity was found in [5]. Let $f, g \in D$. Let $\delta(f,g)$ be the path distance between them along D, i.e. the infimum of the length of curves h_t, $0 \le t \le 1$, $h_t \in D$, $h_0 = f$, $h_1 = g$. Then there exists $C > 0$ such that for any $f, g \in D$,

$$\delta(f,g) \le C \|f - g\|_X^\alpha;$$

for $n = 3$, $\alpha = 2/7$, and this exponent can't be increased (note that if D were smooth then the exponent α should be 1). Hence the D'Alembert Principle should be modified to accomodate such irregular constraint.

To this end, we define at every $f \in D$ a substitute of a tangent space

$$Y_f = \{V \in L^2(\Omega, \mathbb{R}^n) \mid \exists v \in L^2(M, \mathbb{R}^n) \text{ s.t. } \nabla \cdot v = 0, \\ v_n|_{\partial M} = 0, \text{ and } V(\omega) = v(f(\omega)) \text{ a.e.}\}.$$

Let P_f be the orthogonal projection onto Y_f; suppose $f(0) = f_0 \in D$ and $\dot{f}(0) = V_0 \in Y_{f_0}$ be the initial conditions. Then the Lagrangian weak solution $f(t)$ is defined by the following equations:

$$f(t) = f_0 + \int_0^t V(s)ds$$
$$V(s) = \left(\overrightarrow{\prod}_{0 \le \sigma \le s} P_{f(\sigma)}\right) \cdot V_0$$

The operator in the left hand side is the ordered product of projections, i.e.

$$\overrightarrow{\prod}_{0 \le \sigma \le s} P_{f(\sigma)} = \lim P_{f(\sigma_N)} \cdot \ldots \cdot P_{f(\sigma_0)}$$

as $N \to \infty$, and $\max_{0 \le k \le N-1}(\sigma_{k+1} - \sigma_k) \to 0$.

We construct the approximate trajectory using the D'Alembert Principle as follows. Fix some small $\tau > 0$, and define the following sequences:

$$g_1 = f_0 + \tau V_0; \quad f_1 = pr_D g_1; \quad V_1 = P_{f_1} V_0 \tag{1.1}$$
$$\ldots$$
$$g_{k+1} = f_k + \tau V_k; \quad f_{k+1} = pr_D g_{k+1}; \quad V_{k+1} = P_{f_{k+1}} V_k;$$
$$\ldots$$

Here $pr_D g$ is the closest to g point $f \in D$ (if there are many such points, we take arbitrary one of them).

Thus we define a sequence of points $f_k \in D$ and vectors $V_k \in T_{f_k}$. Now define a function $f(t) = f_k$ for $t = k\tau$, interpolate it linearly between $k\tau$ and $(k+1)\tau$, and define $V(t) = V_k$ for $k\tau \le t < (k+1)\tau$. This is the approximate trajectory and velocity. Let us denote them by $f(t, f_0, V_0, \tau)$ and $V(t, f_0, V_0, \tau)$, including explicitly all the parameters these function depend of.

Now we have to pass to limit as $\tau \to 0$. To this end we use the tools of the Nonstandard Analysis (NSA) [6]. We consider a nonstandard model ${}^*U$ of the standard mathematical universe U. In the nonstandard universe we define the functions ${}^*f(t, f_0, V_0, \tau)$ and ${}^*V(t, f_0, V_0, \tau)$ which are the *-images of the standard functions in ${}^*U$; all their arguments are now nonstandard. Let us fix an infinitesimaly small $\tau \in {}^*\mathbb{R}^+ \setminus \mathbb{R}$; define $f(t) = st({}^*f(t, f_0, V_0, \tau))$ and $V(t) = st({}^*V(t, f_0, V_0, \tau))$ where

$st(a)$ denotes the standard part of a nonstandard entity a. Then $V(t) \in Y_{f(t)}$, namely $V(t)(\omega) = v(f(t)(\omega), t)$.

Theorem. *$v(x,t)$ is a weak solution of the Euler equations satisfying the initial condition $v(x,0) = v_0(x)$.*

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