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Many-Body Quantum Systems

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Abstract. The interaction among fundamental particles in nature leads to many interesting effects in quantum statistical mechanics; examples include superconductivity for charged systems and superfluidity in cold gases. It is a huge challenge for mathematical physics to understand the collective behavior of systems containing a large number of particles, emerging from known microscopic interactions. In this workshop we brought together researchers working on different aspects of many-body quantum mechanics to discuss recent developments, exchange ideas and propose new challenges and research directions.

Mathematics Subject Classification (2010): 82B10, 82C10, 81Q05, 81Q10, 35Q40, 35Q20, 35Q41, 35Q55, 35Q56.

Introduction by the Organizers

The workshop *Many-Body Quantum Systems*, organized by Christian Hainzl (LMU Munich), Benjamin Schlein (University of Zurich), Robert Seiringer (IST Austria) and Simone Warzel (TU Munich) was well attended with over 50 participants, coming from Europe, North America and Asia.

During the workshop we had 21 lectures, covering a large variety of subjects in mathematical physics, including equilibrium and non-equilibrium properties of Bose gases at low temperature, the Hartree-Fock theory of weakly interacting fermions, quantum spin systems and their topological phases, as well as the Fröhlich and related models of polarons.

1. Equilibrium properties of Bose gases

In the last years, there has been substantial progress in the mathematical understanding of the low-energy properties of interacting Bose gases. In this line of research, Jan Philip Solovej presented a rigorous lower bound for the ground state energy of a dilute Bose gas in the thermodynamic limit, matching the celebrated Lee-Huang-Yang formula predicted by physicists more than 60 years ago. Chiara Boccato discussed a series of works establishing the validity of the Bogoliubov approximation for the low-energy excitation spectrum of interacting Bose gases in the Gross-Pitaevskii regime, relevant for the description of recent experiment on cold atomic gases. In his talk, Andreas Deuchert showed a mathematical derivation of the free energy of trapped Bose gases (in the Gross-Pitaevskii regime) at positive temperatures, and presented a proof of Bose-Einstein condensation for this system. Horst Knörrer reported on recent progress in a long-term program aiming at a proof of Bose-Einstein condensation for dilute Bose gases in the thermodynamic limit.

2. Time-evolution of Bose gases

Also the time-evolution of interacting Bose gases and its approximation through effective equations have been closely investigated by the mathematical physics community in recent years. Along this line of research we had two talks in Oberwolfach. Lea Bossmann discussed her recent works showing that the time-evolution of 3-dimensional Bose Einstein condensates that are confined in 2- and 1-dimensional traps can be described through nonlinear one-particle (2- or 1-dimensional) Schrödinger equations. Michele Correggi talked about dynamics of many-boson systems in the semiclassical limit.

3. Quantum spin systems

While analytically easier to handle, quantum spin systems capture many of the essential properties of many-body theories. For this reason, they attracted a lot of attention in the last years. Yoshiko Ogata discussed her recent results about classification of topological phases in quantum spin chains. Amanda Young, on the other hand, displayed a family of 2-dimensional quantum spin systems for which the existence of a gap at the bottom of their spectrum and be proved. Martin Fraas discussed states in quantum Hall systems exhibiting anyonic properties. The free energy of the ferromagnetic Heisenberg spin chain was the topic of the talk by Marcin Napiorkowski, establishing that the low temperature asymptotics matches to leading order the one of an ideal Bose gas, which was known previously only in dimensions three and higher.

4. Polarons and open quantum systems

Polaron systems model electrons coupled to the quantized oscillations of the lattice where they move, or, more generally, impurities immersed in a quantum bath. They have been used in physics as toy models to explain the existence of an effective

attraction between electrons leading to the emergence of superconductivity. In the last years, the mathematical physics community has made a significant effort to reach a rigorous understanding of the properties of polarons. In this workshop, Simone Rademacher reported on recent progress in the mathematical analysis of the dynamics of a Fröhlich polaron and on its description through the Landau-Pekar equations in the strong coupling limit. Jonas Lampart explained his recent results on self-adjointness of Hamiltonian operators describing the Bogoliubov-Fröhlich polaron of an impurity immersed an a quantum gas.

Polarons are example of open quantum system where a small system (the electron) is coupled with an infinite system (the quantized oscillation field, in the case of the polaron). Different types of open quantum systems have been discussed by Alain Joye, who presented his research on adiabatic transitions in two-level systems that are coupled to a Bose field.

5. Effective theories

Complex many-body quantum systems are often described in the physics literature through simplified effective theories. As discussed above, one of the main goals of mathematical statistical mechanics is the rigorous justification of these theories, starting from fundamental, microscopic descriptions of many-body systems. Another important goal for mathematical physicists is the study of the mathematical properties of effective theories.

Along this line of research, Edwin Langmann discussed mathematical questions related with the Bardeen-Cooper-Schrieffer model of superconductivity. David Gontier presented recent results about minimizers of Hartree-Fock functional describing systems of atoms and molecules and their symmetry with respect to translations.

6. Other

Systems of many-particles whose evolution follows classical laws of physics or stochastic processes are also an important subject in mathematical physics. Recent results in this direction were presented by Chiara Saffirio, who discussed a rigorous derivation of a linear Boltzmann equation with a memory term approximating the classical time-evolution of a Lorentz gas in an external magnetic field, and by Michael Loss, whose talk was devoted to the proof of the so-called Kac conjecture regarding the relaxation time in the stochastic Kac model. Jeremy Sok gave a lecture about his work on the dynamics of classical spin systems and their continuum limit.

Jan Derezinski presented a detailed analysis of an analytic family of Schrödinger operators, which appear naturally in various places in mathematical physics. Finally, disordered systems have been used in physics for 60 years to understand transport properties of matter. In the workshop, two talks focused on systems with randomness. Peter Müller discussed how much delocalization is needed to guarantee the validity of an area law for the entanglement entropy. Antti Knowles,

on the other hand, presented his recent results on the localization and delocalization properties of eigenvectors of the adjacency matrix of random graphs.

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Workshop: Many-Body Quantum Systems

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Abstracts

On the Lee-Huang-Yang universal formula for the asymptotics of the ground state energy of a Bose gas in the dilute limit

Jan Philip Solovej
(joint work with Søren Fournais)

We study N bosons governed by the Hamiltonian

$$H_N = \sum_{i=1}^{N} -\Delta_i + \sum_{1 \leqslant i < j \leqslant N} v(x_i - x_j)$$

acting on $\bigotimes_{\mathrm{sym}}^{N} L^2(\Lambda)$. Here the N bosons are confined to a box $\Lambda = [0, L]^3$ and we will assume that the 2-body interaction $v \in L^1(\mathbb{R}^3)$ is a non-negative, spherically symmetric function of compact support.

The object of interest is the ground state energy $E(N) = \inf \mathrm{spec} H_N$. Or more precisely the *thermodynamic limit* of the ground state energy density

$$e(\rho) = \lim_{\substack{L \to \infty \\ N/L^3 \to \rho}} L^{-3} E(N).$$

Our aim is to understand the asymptotic behavior of $e(\rho)$ as $\rho \to 0$. A two term asymptotic formula was suggested by Lee-Huang-Yang [7] in 1957. The remarkable property of this formula is that it depends on the potential v through only one parameter, *the scattering length*. The scattering length can be defined as the finite number a given by

$$a = \lim_{x \to \infty} |x| \omega(x), \text{ where } \left(-\Delta + \frac{1}{2} v \right) (1 - \omega) = 0.$$

Formally the scattering length can be approximated through the Born series, where the first two terms are

$$a = a_0 + a_1 + \ldots, \quad a_0 = \frac{1}{8\pi} \int v = \frac{1}{8\pi} \hat{v}(0) \text{ and } a_1 = -\frac{1}{128\pi^3} \int \frac{\hat{v}(p)^2}{p^2} dp.$$

The universal Lee-Huang-Yang formula is then

$$e(\rho) = 4\pi\rho^2 a \left(1 + \frac{128}{15\sqrt{\pi}} \sqrt{\rho a^3} + o(\sqrt{\rho a^3}) \right), \tag{1}$$

in the limit when $\rho a^3 \to 0$.

This formula has a long history. The leading term was suggested by Lenz [6]. Bogolubov in his celebrated 1947 paper [2] on the theory of superfluidity introduced the approximation that now carries his name and arrived at the following expression for the energy density

$$e_{\mathrm{Bog}}(\rho) = \frac{1}{2} \rho^2 \int v - \frac{1}{16\pi^3} \int \left[p^2 + \rho \hat{v}(p) - \sqrt{p^4 + 2\rho p^2 \hat{v}(p)} \right] dp.$$

Bogolubov realized however that this result had to be wrong for very strong potentials, e.g. the hard core, where even the first term would be infinite. In a footnote in his paper he thanks Landau for suggesting that it would be reasonable to expect that $\int v$ should be replaced by $8\pi a$.

We can rewrite Bogolubov's expression as follows

$$\begin{aligned} e_{\mathrm{Bog}}(\rho) &= 4\pi\rho^2(a_0+a_1)+\frac{1}{16\pi^3}\int\left[\sqrt{p^4+2\rho p^2\widehat{v}(p)}-p^2-\rho\widehat{v}(p)+\rho^2\frac{\widehat{v}(p)^2}{2p^2}\right]dp \\ &= 4\pi\rho^2(a_0+a_1)+\frac{1}{16\pi^3}(\rho\widehat{v}(0))^{5/2}\int\left[\sqrt{p^4+2p^2}-p^2-1+\frac{1}{2p^2}\right]dp \\ &\quad +o(\rho\widehat{v}(0))^{5/2} \\ &= 4\pi\left(\rho^2(a_0+a_1)+\frac{128}{15\sqrt{\pi}}(\rho a_0)^{5/2}+o(\rho a_0)^{5/2})\right) \end{aligned}$$

Note that it was by adding and subtracting the term $4\pi\rho^2 a_1$ that it was possible to get a convergent integral. Note also that if we replace a_0+a_1 by a in the first term and a_0 by a in the second term we arrive at the Lee-Huang-Yang formula.

Bogolubov did not notice this rewriting in his original paper. Lee, Huang, and Yang did not do the rewriting explicitly as above, but used the pseudopotential method to arrive at the convergent integral. The pseudopotential is a zero range potential, which is actually known to be unstable for many bosons. It is therefore very difficult to make the argument of Lee, Huang, and Yang rigorous.

The rigorous work on establishing the Lee-Huang-Yang formula started with Dyson [3] who already in 1957 gave an upper bound which had the correct leading order, but an error term of order $O(\rho^2 a(\rho a^3)^{1/3})$. For the hard core potential this is still the best known upper bound. Lieb and Yngvason [5] gave a lower bound agreeing with the first term, but with an error worse than the upper bound of Dyson.

Erdős, Schlein, and Yau [4] made the interesting observation that for sufficiently regular potentials an upper bound agreeing with the first term could be reached using quasi-free trial states. This is interesting as it is exactly quasi-free states that minimize Bogolubov's approximating Hamiltonian. Shortly after Yau and Yin [8] generalized this approach to prove an upper bound with the correct second order Lee-Huang-Yang term. The trial state is however much more complicated than a quasi-free state.

In [1] Boccato et. al. establish a two term formula correspoding to the Lee-Huang-Yang formula for dilute confined Bose gases.

The purpose of this talk was to present recent joint work with Fournais establishing the Lee-Huang-Formula as a lower bound for all L^1 interaction potentials. This complements the upper bound of Yau and Yin and thus finally proves the Lee-Huang-Yang formula for a large class of interaction potentials.

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The excitation spectrum of the Bose gas in the Gross–Pitaevskii regime

Chiara Boccato

(joint work with Christian Brennecke, Serena Cenatiempo, Benjamin Schlein)

We consider a gas of N bosons trapped in a box $\Lambda = [-1/2, 1/2]^3$ with periodic boundary conditions. The Hamiltonian acts on permutation symmetric functions in $L^2_s(\Lambda^N)$, and we represent it in second quantized form as

$$H_N = \sum_{p \in \Lambda^*} p^2 a^*_p a_p + \frac{1}{2N} \sum_{p,q,r \in \Lambda^*} \widehat{V}(r/N) a^*_{p+r} a^*_q a_p a_{q+r}, \tag{1}$$

where for every $p \in \Lambda^* = 2\pi\mathbb{Z}^3$, a^*_p and a_p are the usual creation and annihilation operators. Here $\widehat{V}$ is the Fourier transform of the interaction potential $V \in L^3(\mathbb{R}^3)$, which we take non-negative, spherically symmetric and compactly supported. The scaling (in position space $N^2V(N\cdot)$) models a very dilute system for large N: the range of interactions is of order N^{-1}, while the mean interparticle distance is much larger, of order $N^{-1/3}$; this is called the Gross–Pitaevskii regime.

Bose–Einstein condensation has been proven for this model for the reduced densities associated to the ground state [10] and the reduced densities associated to sequences of approximated ground states [11, 15]. In [14, 12, 11, 15] the ground state energy has been computed to be $E_N = 4\pi\mathfrak{a}_0 N + o(N)$. With $\mathfrak{a}_0$ we denote the scattering length of V.

We present now our works [2, 3] in the theorems below. Theorem 1 improves the previous results on Bose-Einstein condensation, determining its convergence rate; Theorem 2 improves the results on the ground state energy showing the next order in N. In Theorem 2 we also determine the excitation spectrum of (1).

Theorem 1 (Optimal rate for Bose–Einstein condensation)**.** *Let $V \in L^3(\mathbb{R}^3)$ be non-negative, spherically symmetric, compactly supported. Let $\psi_N \in L^2_s(\Lambda^N)$ be a sequence with $\|\psi_N\| = 1$ and such that*

$$\langle \psi_N, H_N \psi_N \rangle \leqslant 4\pi \mathfrak{a}_0 N + \zeta$$

for a $\zeta > 0$. Then the reduced density matrix $\gamma_N = Tr_{2,\dots,N} |\psi_N\rangle\langle\psi_N|$ associated with ψ_N is such that

$$1 - \langle \varphi_0, \gamma_N \varphi_0 \rangle \leqslant \frac{C(\zeta+1)}{N} \tag{2}$$

for all $N \in \mathbb{N}$ large enough, where $\varphi_0(x) = 1$ for all $x \in \Lambda$ is the one–particle zero–momentum mode.

Theorem 2 (Ground state energy and excitation spectrum)**.** *Let V be as in Theorem 1. Then, for $N \to \infty$, the ground state energy is given by*

$$\begin{aligned} E_N &= 4\pi(N-1)\mathfrak{a}_0 + e_\Lambda \mathfrak{a}_0^2 \\ &\quad - \frac{1}{2} \sum_{p \in \Lambda_+^*} \left[p^2 + 8\pi\mathfrak{a}_0 - \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2} - \frac{(8\pi\mathfrak{a}_0)^2}{2p^2} \right] + \mathcal{O}(N^{-1/4}) \,. \end{aligned} \tag{3}$$

Here we introduced the notation $\Lambda_+^* = 2\pi\mathbb{Z}^3 \backslash \{0\}$ and we defined

$$e_\Lambda = 2 - \lim_{M \to \infty} \sum_{\substack{p \in \mathbb{Z}^3 \backslash \{0\}: \\ |p_1|, |p_2|, |p_3| \leqslant M}} \frac{\cos(|p|)}{p^2} .$$

Moreover, the spectrum of $H_N - E_N$ below a threshold ζ consists of eigenvalues given, in the limit $N \to \infty$, by

$$\sum_{p \in \Lambda_+^*} n_p \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2} + \mathcal{O}(N^{-1/4}(1+\zeta^3)) \,. \tag{4}$$

Here $n_p \in \mathbb{N}$ for all $p \in \Lambda_+^*$ and $n_p \neq 0$ for finitely many $p \in \Lambda_+^*$ only.

Remarks:

1. In (2) we establish a bound, uniform in N, for the number of excited particles over the condensate. The techniques we use to prove Theorem 1 also allow us to prove stronger bounds for the energy of excited particles and powers of the number of excitations. This is crucial for the proof of Theorem 2.
2. We find in the ground state energy a contribution $e_\Lambda \mathfrak{a}_0^2$ of order one originating from the fact that we work in a finite volume. The expression in (3) is the analogue of the Lee–Huang–Yang formula for the thermodynamic limit of the Bose gas (see [18, 7] for recent rigorous results).
3. In the spectrum (4) we see the dispersion relation of excitations $\mathcal{E}(p) = \sqrt{|p|^4 + 16\pi\mathfrak{a}_0 p^2} = \sqrt{16\pi\mathfrak{a}_0}|p|\big(1 + \mathcal{O}(p^2)\big)$. $\mathcal{E}(p)$ is linear for small momenta, as a consequence of interactions. Linearity of $\mathcal{E}(p)$ is believed to be at the basis of the phenomenon of superfluidity [4].

Results analogous to Theorem 1 and 2 have been proven in the mean field limit [17, 8, 9, 6]. The Gross Pitaevskii is mathematically more involved, because of the emergence of the scattering length in (3) and (4).

Theorem 2 confirms Bogoliubov's prediction of 1947. In [4] Bogoliubov proposed a heuristic method to compute the excitation spectrum of a dilute Bose gas in the thermodynamic limit. He considers an effective Hamiltonian, where the condensate contribution has been computed by a c-number substitution and contributions that are cubic and quartic in creation and annihilation operators have been neglected. The resulting Hamiltonian is quadratic and can be easily diagonalized. In the resulting formula for the excitation spectrum however only the first Born approximation of the scattering length appears. In his final result he substitutes it with the full scattering length, obtaining expressions analogous to (3) and (4).

In our works we develop a rigorous implementation of Bogoliubov theory. To prove Theorem 1, we introduce an excitation Hamiltonian $\mathcal{L}_N$, factoring out the Bose-Einstein condensate. The idea comes from [9] and implements rigorously Bogoliubov's c-number substition. We define then generalized Bogoliubov transformations (used also in [1, 5]) to model correlations among particles and to define a renormalized excitation Hamiltonian $\mathcal{G}_N$. A second renormalization, this time through the exponential of an operator which is cubic in creation and annihilation operators leads to a new Hamiltonian $\mathcal{R}_N$. We apply then localization techniques from [13] to conclude the proof of (2).

To prove Theorem 2 we observe that $\mathcal{G}_N$ contains cubic and quartic operators of order one, which therefore should contribute to the second order of the ground state energy and to the excitation spectrum. These terms were neglected in Bogoliubov's method. With a new cubic unitary map e^A we extract these contributions and define a new excitation Hamiltonian $\mathcal{J}_N = e^{-A}\mathcal{G}_N e^A$. This is now approximately quadratic in creation and annihilation operators. We control the errors using our bounds for the number of excitations obtained before (see Remark 2). In our final excitation Hamiltonian the potential energy contributions are now renormalized and only the scattering length appears. Finally (3) and (4) follow from diagonalization.

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Dynamics of the strongly coupled polaron

Simone Rademacher
(joint work with Nikolai Leopold, Benjamin Schlein, Robert Seiringer)

The polaron is a model of an electron together with its self-induced polarization field in an ionic crystal. A polaron, whose extension is large compared to the lattice spacing of the crystal, is described through the Fröhlich Hamiltonian

$$H_\alpha = -\Delta + \int_{\mathbb{R}^3} \frac{dk}{|k|} \left[b_k^* e^{-ik\cdot x} + b_k e^{ik\cdot x} \right] + \int_{\mathbb{R}^3} dk\, b_k^* b_k \tag{1}$$

acting on the Hilbert space $L^2\left(\mathbb{R}^3\right) \otimes \mathcal{F}$. Here, the polarization field is considered to be quantized through so called phonons on the bosonic Fock space $\mathcal{F}$. The Fröhlich Hamiltonian H_α is written in strong coupling units. It depends on the coupling constant $\alpha > 0$ of the electron to the polarization through the creation and annihilation operators b_k^* resp. b_k. They satisfy the rescaled commutation relations

$$[b_k^*, b_{k'}] = \alpha^{-2} \delta\left(k - k'\right), \quad [b_k, b_{k'}] = [b_k^*, b_{k'}^*] = 0 \quad \forall k, k' \in \mathbb{R}^3. \tag{2}$$

We are interested in the dynamics of the polaron for large coupling constant $\alpha \to \infty$. The time evolution of the polaron $\Psi_t \in L^2\left(\mathbb{R}^3\right) \otimes \mathcal{F}$ is given through the

Schrödinger equation

$$i\partial_t \Psi_t = H_\alpha \Psi_t. \tag{3}$$

We consider initial data of product form $\Psi_0 = \psi_0 \otimes W\left(\alpha^2 \varphi_0\right) \Omega$ where the phonon field is a coherent state. Pekar used this product ansatz to derive the first order contribution of the ground state energy of the strongly coupled polaron. We address the question whether the time evolved Pekar product state $\psi_t \otimes W\left(\alpha^2 \varphi_t\right) \Omega$ with respect to the Landau-Pekar equations

$$\begin{cases} i\partial_t \psi_t & = \left(-\Delta + V_{\varphi_t}\right) \psi_t \\ i\alpha^2 \partial_t \varphi_t(k) & = \varphi_t(k) + |k|^{-1} \widehat{|\psi_t|^2}(k) \end{cases} \tag{4}$$

with $V_\varphi(x) = 2\mathrm{Re} \int_{\mathbb{R}^3} dk\ |k|^{-1} \varphi(k) e^{ik\cdot x}$ provides an approximation of the solution of the Schrödinger equation (3) in the strong coupling limit $\alpha \to \infty$. Note that in the Landau-Pekar equations, the electron wave function ψ_t satisfies a Schrödinger equation, while the phonon field φ_t is described through a classical field equation. We are interested in initial data satisfying

Assumption 1. *Let $\varphi_0 \in L^2\left(\mathbb{R}^3\right)$ such that*

$$e\left(\varphi_0\right) = \inf\left\{\langle \psi, \left(-\Delta + V_{\varphi_0}\right) \psi\rangle : \psi \in H^1\left(\mathbb{R}^3\right),\ \|\psi\|_2 = 1\right\} < 0.$$

This assumption ensures the existence of a unique positive ground state ψ_{φ_0} of $h_{\varphi_0} = -\Delta + V_{\varphi_0}$ with corresponding eigenvalue separated from the rest of the spectrum by a gap of size $\Lambda_0 > 0$. We show that the Landau-Pekar equations (4) approximate the time evolution (3) of an Pekar product state with φ_0 satisfying Assumption 1 and $\psi_0 = \psi_{\varphi_0}$ being the ground state of h_{φ_0} :

Theorem 1. *Let φ_0 satisfy Assumption 1 and $\alpha_0 > 0$. Let $(\psi_t, \varphi_t) \in H^1\left(\mathbb{R}^3\right) \times L^2\left(\mathbb{R}^3\right)$ denote the solution of the Landau-Pekar equations with initial data $(\psi_{\varphi_0}, \varphi_0) \in H^1\left(\mathbb{R}^3\right) \times L^2\left(\mathbb{R}^3\right)$ and $\omega(t) = \alpha^2 \mathrm{Im}\langle \varphi_t, \partial_t \varphi_t\rangle + \|\varphi_t\|_2^2$. Then, there exists a constant $C > 0$ such that*

$$\left\|e^{-iH_\alpha t} \psi_{\varphi_0} \otimes W\left(\alpha^2 \varphi_0\right) \Omega - e^{-i\int_0^t ds\ \omega(s)} \psi_t \otimes W\left(\alpha^2 \varphi_t\right) \Omega\right\| \leqslant C\alpha^{-1} |t|^{1/2} \tag{5}$$

for all $\alpha \geqslant \alpha_0$.

It follows from Theorem 1 that the Pekar ansatz is a good approximation for times $|t| \ll \alpha^2$. First results in this direction valid on smaller times scales resp. for a smaller class of initial data have been obtained in [1, 3, 4]. The proof of Theorem 1 is based on a non-linear adiabatic theorem for the Landau-Pekar equations:

Theorem 2. *Let φ_0 satisfy Assumption 1. Let $(\psi_t, \varphi_t) \in H^1\left(\mathbb{R}^3\right) \times L^2\left(\mathbb{R}^3\right)$ denote the solution of the Landau-Pekar equations with initial data $(\psi_{\varphi_0}, \varphi_0) \in H^1\left(\mathbb{R}^3\right) \times L^2\left(\mathbb{R}^3\right)$. Then, there exist $C, C_{\Lambda_0} > 0$ such that*

$$\left\|\psi_t - e^{-i\int_0^t ds\ e(\varphi_s)} \psi_{\varphi_t}\right\|_2^2 \leqslant C\alpha^{-2}, \quad \forall |t| \leqslant C_{\Lambda_0} \alpha^2. \tag{6}$$

The restriction to times $|t| \leqslant C_{\Lambda_0} \alpha^2$ ensures the persistence of the spectral gap of h_{φ_t}. For a similar result as stated in Theorem 2 in one dimension, we refer to [2]. It remains an open problem to prove an approximation of the dynamics of the strongly coupled polaron valid on larger time scales.

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Proof of Kac's conjecture for the hard sphere gas

Michael Loss

(joint work with Eric Carlen, Maria Carvalho)

In 1956 Mark Kac [1] published his Master Equation approach to kinetic theory. This consists of a linear evolution equation describing a spatially homogeneous gas of N particles that undergo binary collisions at exponentially distributed times. He assumed for simplicity that the particles move in one dimension and more importantly he also assumed that the scattering cross sections did not depend on the velocities of the particles. Such kind of colliding particles are called Maxwellian molecules. In this paper he introduced the notion of a chaotic distribution and proved that his Master Equation propagates chaos which led him to a rigorous derivation of a model Boltzmann equation, the Kac-Boltzmann equation. The Master Equation turns out to be ergodic, i.e., any initial distribution tends towards the unique equilibrium state. This led him to the question of speed of convergence. Kac conjectured that the gap of the generator of the evolution should be bounded below by a constant independent of the number of particles and this was finally proved in 2000 by Elise Jeanvresse [2]and shortly thereafter the gap was explicitly computed in [3]. The Kac program was revisited in an important paper by Stephane Mischler and Clement Mouhot [4]. The authors considered momentum preserving collisions and, more importantly, allowed for the dependence of the scattering cross sections on the velocities, in particular hard spheres were included in their work. They proved propagation of chaos, which is considerably more difficult than the case of Maxwellian moelcules, and they also proved approach to equilibrium in the Wasserstein distance with a polynomial rate independent of the number of particles. What was left open is the analogous gap conjecture for this realistic model and this is the topic of this report.

We denote by $\vec{v} = (v_1, \dots, v_N) \in \mathbb{R}^{3N}$ where v_i is the velocity associated with particle i. The generator for the master equation that describes the collisions of N hard spheres (with suitable normalizations) is given by

$$\mathcal{L}_N f(\vec{v}) = N \begin{pmatrix} N \\ 2 \end{pmatrix}^{-1} \sum_{i<j} |v_i - v_j| \int_{S^2} [f(\vec{v}) - f(R_{i,j,\sigma}\vec{v})] d\sigma \tag{1}$$

where

$$(R_{i,j,\sigma}\vec{v})_k = \begin{cases} v_i^*(\sigma) & k = i \\ v_j^*(\sigma) & k = j \\ v_k & k \neq i,j \end{cases} , \tag{2}$$

and

$$\begin{aligned} v_i^*(\sigma) &= \frac{v_i + v_j}{2} + \frac{|v_i - v_j|}{2}\sigma \\ v_j^*(\sigma) &= \frac{v_i + v_j}{2} - \frac{|v_i - v_j|}{2}\sigma \ . \end{aligned} \tag{3}$$

The velocities v_i, v_j are the velocities before the collision and v_i^*, v_j^* are the velocities after the collision. The parameter $\sigma \in \mathbb{S}^2$ parametrizes the collision. One can easily check that the total energy as well as the total momentum is conserved and we set

$$E = \frac{1}{N}\sum_{i=1}^N |v_i|^2 \ , \ p = \frac{1}{N}\sum_{i=1}^N v_i \ .$$

We denote the space of velocity vectors satisfying the energy and momentum constraint by $\mathcal{S}_{N,E,p}$ and endow it with the uniform normalized measure σ_N induced by the Lebesgue measure on $\mathbb{R}^{3N}$. The quadratic form associated with (1) is

$$\mathcal{E}(f,f) = \frac{N}{2}\begin{pmatrix} N \\ 2 \end{pmatrix}^{-1} \sum_{i<j} \int_{S_{N,E,p}} \int_{S^2} |v_i - v_j| \left[f(\vec{v}) - f(R_{i,j,\sigma}\vec{v})\right]^2 d\sigma d\sigma_N \ .$$

and we define the gap by

$$\Delta_N(E,p) = \inf\left\{ \mathcal{E}(f,f) \ : \ f \in L^2(\mathcal{S}_{N,E,p},\sigma_N) \ , \ \langle f, 1\rangle_{L^2(\sigma_N)} = 0 \quad \text{and} \quad \|f\|^2_{L^2(\sigma_N)} = 1 \right\}.$$

Our main result is

THEOREM 0.1. *There exists a positive constant K, independent of N such that*

$$\Delta_N(E,p) \geqslant (E - |p|^2)^{1/2} K \ .$$

By scaling it is enough to consider the case where $E = 1$ and $p = 0$ and we abbreviate $\mathcal{S}_{N,1,0}$ by $\mathcal{S}_N$. The first step is to consider an auxiliary quadratic form

$$\mathcal{D}(f,f) = \frac{1}{N}\sum_{k=1}^N \int_{\mathcal{S}_N} \left[\frac{N^2 - (1 + |v_k|^2)N}{(N-1)^2}\right]^{1/2} \left[f^2 - fP_k f\right] d\sigma_N \ . \tag{4}$$

where $P_k f$ is a marginal, i.e., the unique function $f^{(1)}(v_k)$ such that for any $g(v_k)$, $\int_{\mathcal{S}_N} f(\vec{v}) g(v_k) d\sigma_N = \int_{\mathcal{S}_N} f^{(1)}(v_k) g(v_k) d\sigma_N$. This quadratic form defines a linear operator that is the generator of another stochastic process, which we call the conjugate process.

The gap associated with the quadratic form $\mathcal{D}$ is defined by

(5)
$$\widehat{\Delta}_N = \inf \left\{ \mathcal{D}_N(f,f) \ : \ f \in L^2(\mathcal{S}_N) \ , \ \langle f, 1 \rangle_{L^2(\mathcal{S}_N)} = 0 \quad \text{and} \quad \|f\|^2_{L^2(\mathcal{S}_N)} = 1 \right\} .$$

The following theorem provides the link between the Kac process and the conjugate process:

THEOREM 0.2.

(6)
$$\Delta_N \geqslant \frac{N}{N-1} \Delta_{N-1} \widehat{\Delta}_N \ .$$

A detailed analysis of $\mathcal{D}$ shows that

$$\widehat{\Delta}_N \geqslant (1 - \frac{1}{N} - \frac{C}{N^{3/2}}) \ ,$$

where C is a constant independent of N. If we pick N_0 such that $(1 - \frac{1}{N_0} - \frac{C}{N_0^{3/2}}) > 0$, we get that

$$\Delta_N \geqslant \Pi_{j=N_0}^{\infty} (1 - \frac{C}{j^{3/2}}) \Delta_{N_0}$$

where the infinite product converges. In a separate analysis one establishes that for any N, $\Delta_N > 0$. The detailed proofs can be found in https://arxiv.org/pdf/1812.03874.pdf.

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Adiabatic transitions of a two-level system coupled to a Bose field

Alain Joye

(joint work with Marco Merkli & Dominique Spehner)

This work is devoted to the transition probability between the energy eigenstates of a two-level time-dependent system in contact with a reservoir which plays the role of an environment the two-level system is weakly coupled to. We model the reservoir by a Bose field minimally coupled to the two-level system in an appropriate way specified below; therefore, we are in the framework of time-dependent open quantum systems. The quantum process we study is a basic step in the understanding of energy transfers within driven small quantum systems in contact with an environment. As the properties of genuinely time-dependent quantum processes are often unreachable without resorting to specific asymptotic regimes, we consider the problem above in the adiabatic and small coupling limits.

The adiabatic regime, characterized by a slow time variation of the Hamiltonian governing the quantum system under scrutiny, yields rather detailed approximations of the true quantum dynamics in a variety of physically relevant situations. The adiabatic theorem was first stated for time dependent Hamiltonians with isolated eigenvalues in [BF, K1], and then extended to accommodate isolated parts of spectrum, see [N1, ASY]. This version applies to the two-level system we consider in absence of coupling. Adiabatic approximations for Hamiltonians with eigenvalues embedded in the continuous spectrum were then provided by [AE]. This is the spectral situation for the total Hamiltonian of the two-level system coupled to the Bose field. Later, adiabatic theorems were formulated in [A-SF, J3, AFGG1] for non-self-adjoint generators, leading to extensions to gapless, non self-adjoint generators provided in [S]. Such results apply to effective Markovian descriptions of open quantum system by means of time dependent Lindblad generators.

In our setup, the total Hamiltonian given by the time-dependent two-level system minimally coupled to the Bose field is varying on a slow time scale $1/\varepsilon$, under the supplementary assumption that the coupling of intensity $\lambda > 0$ commutes at all times with the Hamiltonian of two-level system. This means that the system and reservoir do not exchange energy instantaneously.

More precisely, the Hilbert space of the total system is $\mathcal{H}_{\rm tot} = \mathbb{C}^2 \otimes \mathcal{F}_+(L^2(\mathbb{R}^3))$, whith $\mathcal{F}_+(L^2(\mathbb{R}^3))$ the bosonic Fock space on $L^2(\mathbb{R}^3)$, the one-particle momentum Hilbert space. The Hamiltonian of the reservoir reads $H_R = \int \omega(k) a^*(k) a(k) dk^3$, where $a^*(k)/a(k)$ are the bosonic creation/annihilation operators with momentum k and $\omega(k) = |k|$ is the dispersion relation. The smooth time-dependent Hamiltonian of the two-level system writes $H_S(t) = \sum_{j=1}^2 e_j(t) P_j(t)$, for $t \in [0,1]$, with corresponding spectral projectors $P_j(t)$ and instantaneous eigenvalues $e_j(t)$ separated by a gap: $\inf_{t\in[0,1]} |e_1(t) - e_2(t)| > 0$. The coupling to the reservoir is linear in the field operator $\phi(g) = \int_{\mathbb{R}^3} \big(\overline{g(k)} a(k) + g(k) a^*(k)\big) dk^3/\sqrt{2}$, where $g \in L^2(\mathbb{R}^3)$ is the form factor: the interaction Hamiltonian is given by a coupling constant $\lambda > 0$ times $H_{\rm int}(t) = B(t) \otimes \phi(g)$, where $B(t)$ is a slowly-varying self-adjoint operator on $\mathbb{C}^2$ with eigenvalues $b_1(t), b_2(t)$. Our main assumption is that $[H_S(t), B(t)] \equiv 0$ for all $t \in [0,1]$, an expression of the instantaneous nondemolition character of the coupling. The evolution operator $U_{\lambda,\varepsilon}(t)$ on $\mathcal{H}_{\rm tot}$ is thus given by

$$i\varepsilon \partial_t U_{\lambda,\varepsilon}(t) = (H_S(t) \otimes \mathbb{I} + \lambda H_{\rm int}(t) + \mathbb{I} \otimes H_R) U_{\lambda,\varepsilon}(t), \quad U_{\lambda,\varepsilon}(0) = \mathbb{I},$$

where the factor ε in the LHS expresses the adiabatic scaling.

We further suppose that all derivatives of the total Hamiltonian vanish at the initial time. Starting initially in an eigenstate of the two-level system tensored with Ω the vacuum vector of the Bose field, we compute the transition probability

$$p_{1\to 2}^{(\lambda,\varepsilon)}(t) = \mathrm{Tr}\Big((P_2(t) \otimes \mathbb{I})\, U_{\lambda,\varepsilon}(t) (P_1(0) \otimes |\Omega\rangle\langle\Omega|)\, U_{\lambda,\varepsilon}(t)^* \Big) \tag{1}$$

of the state of the full system at some later rescaled time t to the other instantaneous eigenstate of the system, regardless of the state of the field, in the adiabatic and small coupling regime $(\varepsilon, \lambda) \to (0,0)$. The adiabatic theorem applied to the uncoupled two-level system tells us that the transition probability between its

levels equals $p_{1\to 2}^{(0,\varepsilon)}(t) = \varepsilon^2 q_{1\to 2}(t) + \mathcal{O}(\varepsilon^3)$, where $q_{1\to 2}(t) \geqslant 0$ is explicit and generically non zero. By contrast, the gapless adiabatic theorem applied to the total Hamiltonian only tells us that the transition probability we are interested in tends to zero, if applicable at all. Our choice of coupling together with the smallness of λ allows us to determine the leading order of the transition probability $p_{1\to 2}^{(\lambda,\varepsilon)}(t)$ in appropriate regimes of $(\varepsilon, \lambda) \to (0, 0)$, and to compare with $p_{1\to 2}^{(0,\varepsilon)}(t)$. Introducing the reservoir autocorrelation function $\gamma(t) = \langle e^{it\omega} g, g\rangle_{L^2}$ and $\widehat{\gamma}$ its Fourier transform, we prove that for g regular enough

$$p_{1\to 2}^{(\lambda,\varepsilon)}(t) = p_{1\to 2}^{(0,\varepsilon)}(t) + \frac{\lambda^2}{2\varepsilon} \int_0^t p_{1\to 2}^{(0,\varepsilon)}(s) b_{12}^2(s) \widehat{\gamma}\big(e_{12}(s)\big)\, ds + \mathrm{err}(\varepsilon, \lambda).$$

Here $b_{12}(t) = b_1(t) - b_2(t)$, $e_{12}(t) = e_1(t) - e_2(t)$, and, in the asymptotic regime $\varepsilon \ll \lambda \ll \varepsilon^{1/3}$, the error $\mathrm{err}(\varepsilon, \lambda) \ll \min(\varepsilon^2, \varepsilon\lambda^2)$ is smaller than both explicit terms above. This shows in particular that the leading order correction due to the presence of the reservoir is $\mathcal{O}(\varepsilon\lambda^2)$, and that the critical regime making both terms of the same order is $\lambda = \sqrt{\varepsilon}$. Moreover, $\widehat{\gamma}$ being positive definite, the integral correction vanishes if $e_1(t)$ is the ground state of the two-level system.

This question was addressed in the literature by means of an effective description of the presence of the reservoir encoded in a time dependent Lindblad operator [AFGG1, AFGG2, FH]. When the dissipator part of the Lindbladian is chosen to be of dephasing type, the authors compute the transition probability between the levels of the small system in the adiabatic limit. While it is difficult to bluntly compare the two approaches, the coupling we choose induces similar features as those induced by a dephasing Lindbladian. In particular, the leading order of the transition probability obtained within the effective approach is of order ε in the adiabatic regime, all other parameters being kept fixed.

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Classification of SPT phases in quantum spin chains

Yoshiko Ogata

A quantum spin chain is given by $\mathcal{A} := \bigotimes_{\mathbb{Z}} M_d$, where M_d is the algebra of $d \times d$-matrices. Physical models on $\mathcal{A}$ is given by interactions Φ:

$$\Phi : \text{finite subsets of } \mathbb{Z} \ni X \mapsto \Phi(X) = \Phi(X)^* \in \mathcal{A}_X. \tag{1}$$

Here $\mathcal{A}_X := \bigotimes_X M_d$, regarded as a subalgebra of $\mathcal{A}$. An interaction Φ is uniformly bounded if $\sup_X \|\Phi(X)\| < \infty$. It is of finite range if there is $R > 0$ such that $\Phi(X) = 0$ if the diameter of X is larger than R. For a uniformly bounded finite range interaction Φ and a finite subset Λ of $\mathbb{Z}$, set $H_\Phi(\Lambda) := \sum_{X \subset \Lambda} \Phi(X)$. Then the following limit exists for each $A \in \mathcal{A}$

$$\tau^t_\Phi(A) := \lim_{\Lambda \nearrow \mathbb{Z}} Ad(e^{itH_\Phi(\Lambda)})(A), \tag{2}$$

and defines a C^*-dynamics τ_Φ on $\mathcal{A}$. Let δ_Φ be a generator of τ_Φ with a domain $\mathcal{D}(\delta_\Phi)$. A state ω is called a τ_Φ-ground state if the following inequalities hold:

$$-i\omega\left(A^*\delta_\Phi(A)\right) \geqslant 0, \quad A \in \mathcal{D}(\delta_\Phi). \tag{3}$$

We say that Φ has a unique gapped ground state if the τ_Φ-ground state ω is unique and there is a $\gamma > 0$ such that

$$-i\omega\left(A^*\delta_\Phi(A)\right) \geqslant \gamma\omega\left(A^*A\right), \quad A \in \mathcal{D}(\delta_\Phi), \text{ with } \omega(A) = 0. \tag{4}$$

We denote the set of all uniformly bounded finite range interactions with unique gapped ground state by $\mathcal{P}$. The classification of the unique gapped ground state phases is the classification of $\mathcal{P}$ with respect to the following criterion: $\Phi_0 \sim \Phi_1$ if there is a smooth path in $\mathcal{P}$ connecting them. Now we introduce a symmetry to the game. In this talk, we considered the reflection symmetry. (Analogous result holds for on-site symmetry.) The reflection on $\mathcal{A}$ is an automorphism Γ given by

$$\Gamma(Q^{(j)}) := Q^{(-j-1)}, \quad Q \in M_d,\ j \in \mathbb{Z}, \tag{5}$$

where $Q^{(j)}$ is an element of $\mathcal{A}$ acting as Q on the j-th site and as 1 on the rest. Let $R : \mathbb{Z} \to \mathbb{Z}$ be a map $R(j) := -j-1$. Then an interaction Φ is reflection invariant if $\Gamma(\Phi(X)) = \Phi(R(X))$ for all X. We denote the set of all uniformly bounded finite range *reflection invariant* interactions with unique gapped ground state by $\mathcal{P}_\Gamma$. The criterion for the classification of $\mathcal{P}_\Gamma$ is $\Phi_0 \sim_\Gamma \Phi_1$ if there is a smooth path in $\mathcal{P}_\Gamma$ connecting them. It may happen that two elements $\Phi_0, \Phi_1 \in \mathcal{P}_\Gamma$ are equivalent with respect to $\sim$ but not with $\sim_\Gamma$. This is the concept of symmetry protected topological (SPT) phases introduced by Gu and Wen [GW]. In this talk, I asked a question how to detect that $\Phi_0, \Phi_1 \in \mathcal{P}_\Gamma$ are not equivalent with respect to $\sim_\Gamma$. In order to think about this problem, we introduced an invariant of the classification

$\sim_\Gamma$. It is defined as follows. Let ω be a unique gapped ground state for $\Phi \in \mathcal{P}_\Gamma$. Thanks to a Theorem by Matsui [M], we know that a unique gapped ground state satisfies a property called the split property. This split property combined with the reflection symmetry, we see that there is an irreducible representation $(\mathcal{H}, \pi)$ of $\mathcal{A}_R$ such that

$$(\hat{\mathcal{H}} := \mathcal{H} \otimes \mathcal{H}, \hat{\pi} := \pi \circ \Gamma|_{\mathcal{A}_L} \otimes \pi, \Omega) \tag{6}$$

is a GNS representation of ω for some $\Omega \in \hat{\mathcal{H}}$. By the reflection symmetry, we have a unitary operator U on $\hat{\mathcal{H}}$ such that

$$U\hat{\pi}(A)\Omega = \hat{\pi} \circ \Gamma(A)\Omega, \quad A \in \mathcal{A}. \tag{7}$$

Because of the symmetric structure of the GNS representation, we see that there is a $\sigma_\Phi \in \pm 1$ such that

$$\Gamma(\xi \otimes \eta) = \sigma_\Phi(\eta \otimes \xi), \quad \xi, \eta \in \mathcal{H}. \tag{8}$$

This σ_Φ is the index we define. Our main result is the following [O]: This index σ_Φ is an invariant of $\sim_\Gamma$. The index is a generalization of the index considered in [PTBO1], [PTBO2].

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How much delocalisation is needed for an enhanced area law of the entanglement entropy?

Peter Müller

(joint work with Leonid Pastur, Ruth Schulte)

The talk addresses the results of the paper [MPS19].

The entanglement entropy of a pure state Ψ of a quantum system with respect to its bipartition into a subsystem and its "environment" is given by the von Neumann entropy $S_\Psi := -\mathrm{Tr}(\varrho_\Psi \log_2 \varrho_\Psi)$, where ϱ_Ψ is the reduced density operator obtained by partially tracing out the environment Hilbert subspace from the density operator $|\Psi\rangle\langle\Psi|$ of the whole system. Here, the trace $\mathrm{Tr}(\cdot)$ is over the Hilbert subspace of the remaining subsystem, and we have used the Dirac notation for rank-1 projections. A concrete realisation of the situation is given by a multiparticle or spin system in d space dimensions, which is divided into two spatial parts, say, a cube Λ_L for the subsystem and its complement for the environment. If

the ground state Ψ is energetically separated by a spectral gap or a (suitable) mobility gap from the excited states, then the corresponding entanglement entropy is typically not an extensive quantity, but obeys an area law, $S_\Psi \sim L^{d-1}$. Here, L is the side length of Λ_L. We refer to the overview [ECP10] from a physics perspective and to, e.g., [ARS15, ARNSS17, EPS17, BW18, FS18, Sto16] for mathematically rigorous results.

On the other hand, if there is not even a mobility gap separating the ground state then one expects a growth rate faster than the area but slower than the volume, see e.g. [MS16]. A typical and interesting situation is that of a system probed at a quantum critical point. In this case, a logarithmic enhancement to the area law, $S_\Psi \sim L^{d-1} \ln L$, is observed [JK04, RM09, CC09]. We aim to understand the properties of a mobility gap above the ground state leading to the presence or absence of a logarithmic enhancement to the area law. Clearly, such finer questions require a more detailed understanding, and it is natural to study first sufficiently simple situations in order to make progress.

Quasi-free fermion gases provide an excellent testing ground to achieve this goal. Their ground states $\Psi = \Psi_{E_F}$ are parametrised by the Fermi energy $E_F \in \mathbb{R}$, and the corresponding entanglement entropy can be entirely expressed in terms of single-particle quantities $S_{E_F}(L) := S_{\Psi_{E_F}} = \mathrm{Tr}\{h(1_{\Lambda_L} 1_{]-\infty,E_F]}(H) 1_{\Lambda_L})\}$. Here, H is the single-particle Hamiltonian, 1_{Λ_L} the multiplication operator by the indicator function of Λ_L, the trace is over the single-particle Hilbert space and $h(\lambda) := -\lambda \log_2 \lambda - (1-\lambda) \log_2 (1-\lambda)$ for $\lambda \in [0,1]$ (with the convention $0 \log_2 0 := 0$). The following cases are mathematically understood by now: (i) In case the single-particle Hamiltonian is a multi-dimensional random Schrödinger operator and the Fermi energy falls into the region of complete localisation, the validity of an area law is established in [PS14, EPS17, PS18a]. The proofs rely on the exponential decay in space of the Fermi projection for E_F in the region of complete localisation. (ii) In the case of a free Fermi gas in d dimensions or if the single-particle Hamiltonian is a one-dimensional periodic Schrödinger operator, a logarithmically enhanced area law is shown to occur [Wol06, HLS11, LSS14, LSS17, PS18b]. The latter four works even provide the exact asymptotics of S_{E_F} by establishing suitable Szegő-type formulas based on deep work of Sobolev [Sob13, Sob15].

In (i) the Fermi energy lies in a spectral region of dense pure point spectrum with corresponding eigenfunctions that are not only exponentially localised in space but also give rise to dynamical localisation. In contrast, in (ii) the spectrum is absolutely continuous with delocalised generalised eigenfunctions. It is therefore only natural to ask the following questions: *Is a logarithmic enhancement to an area law possible without absolutely continuous spectrum? In other words, is a weaker breakdown of localisation already sufficient? Is it possible for disordered fermions to violate the area law at all?*

We will answer these questions affirmatively by considering the random dimer model in one dimension with Bernoulli disorder. Its random Hamiltonian on $\ell^2(\mathbb{Z})$

is given by

$$H := -\sum_{x\in\mathbb{Z}} \big(|\delta_x\rangle\langle\delta_{x+1}| + |\delta_{x+1}\rangle\langle\delta_x|\big) + v\sum_{x\in\mathbb{Z}} V(x)|\delta_x\rangle\langle\delta_x|.$$

Here $v > 0$ is the disorder strength and $\{\delta_x\}_{x\in\mathbb{Z}}$ the canonical basis of $\ell^2(\mathbb{Z})$. The family of 0-1-Bernoulli random variables $\big(V(2x)\big)_{x\in\mathbb{Z}}$ is independently and identically distributed and every other pair of consecutive sites shares the same value of the potential, that is, $V(2x) = V(2x+1)$ for all $x \in \mathbb{Z}$.

The almost-sure spectrum of H consists only of a dense pure point part with exponentially decaying eigenfunctions, but for $0 < v < 2$ the two spectral values 0 and v are critical in the sense that the localisation length diverges there. Thus, whenever $E_F \neq 0$ and $E_F \neq v$, the analysis of [PS14, EPS17] applies and provides the existence of the limit $\lim_{L\to\infty} \mathbb{E}[S_{E_F}(L)]$ of the disorder-averaged entanglement entropy, i.e., the validity of the area law. We complement the picture in [MPS19] by showing the following

Theorem. *There exists a maximal disorder strength $v_{max} \in]0, 2[$ such that for every $v \in]0, v_{max}]$ and for a critical Fermi energy $E_F \in \{0, v\}$, we have*

$$\liminf_{L\to\infty} \frac{\mathbb{E}[S_{E_F}(L)]}{\ln L} > 0.$$

Thus, for weak disorder and a critical Fermi energy, there is at least a logarithmic enhancement to the area law.

Since self-averaging is known to fail for entanglement entropies of one-dimensional random systems [PS18a], it is desirable to obtain an almost-sure statement as well. We have also achieved this in [MPS19], but for a slightly modified version of the entanglement entropy.

An important ingredient in our proof of the theorem are the delocalisation properties – approximate clock spacing of eigenvalues and flatness of eigenfunctions – for the finite-volume random dimer model in a critical energy window proved by Jitomirskaya, Schulz-Baldes and Stolz [JSBS03].

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Abelian Anyons in Fractional Quantum Hall Effect

Martin Fraas

(joint work with Sven Bachmann, Alex Bols, Wojciech de Roeck)

A key signature of fractional quantum Hall effect (FQHE) is that the low energy excitations behave as anyons. In some cases, for example 1/3-FQHE, the anyons are Abelian, in others, for example 2/5-FQHE, the anyons are non-abelian. The connection between fractional quantum Hall conductance and the properties of the low energy excitations is well understood in topological quantum field theories [1, 2]. The low energy theory is described by a 'loop-soup' model and the anyonic nature of the excitations is determined by commutation properties of the loops.

This picture is expected to also apply to microscopic models but it was not known how to construct these loop operators. Our work fills this gap and provides a construction of these unitary operators, under certain natural conditions, for Abelian anyons. Extension to the non-abelian case is an important open problem.

In an Abelian 'loop soup' model, the ground state space P is invariant under application of unitaries U_α associated to closed loops α in the underlying space. The loop operators associated to loops α, β have a commutation relation

$$U_\alpha^* U_\beta^* U_\alpha U_\beta P = e^{i2\pi\phi} P, \quad 0 \leqslant \phi < 1. \tag{1}$$

A unitary associated to an open curve γ, obtained by restricting a closed loop unitary to the part γ, creates an excitation ϵ at the end point of the loop, see figure 1.

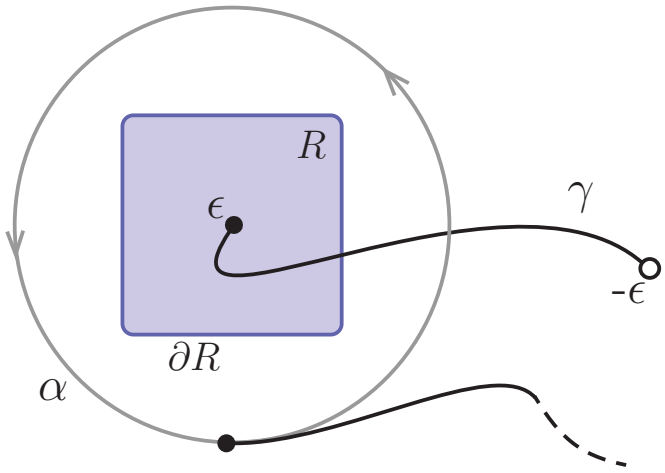

FIGURE 1. The unitary U_γ creates an excitation ϵ at its endpoint. The excitations always come in pairs, we imagine that the corresponding excitation $-\epsilon$ at the other end point is far away.

The commutation relation (1) then immediately implies that taking the second excitation around the loop α around ϵ leads to a statistical factor ϕ. If $\phi \neq 0$ we get a non-trivial statistics of excitations. We will see that ϕ is related to the Hall conductance of the model and also to the charge of the excitation ϵ. In a lattice system, this phenomena was demonstrated in excatly solvable models like the toric code [3] and the Levin-Wen model [4].

Our goal is to construct the loop operators U_α starting from a generic microscopic model. We consider a quantum lattice system on an $L \times L$ discrete torus described by a local Hamiltonian H. We assume the following properties of the model,

(1) H is gapped,
(2) The ground state space is p-dimensional and topologically ordered [5],
(3) The Hamiltonian conserves charge.

The latter assumption means that to each region Ω on the torus we can associate a charge operator Q_Ω and the commutator $[H, Q_\Omega]$ is supported on the boundary

$\partial\Omega$ of Ω. Assumptions (1) and (3) then imply [6, 7] that, for a particular function $\tilde{W}$ decaying at infinity, the operator

$$\bar{Q}_\Omega = \int_{-\infty}^{\infty} \tilde{W}(t) e^{itH} Q_\Omega e^{-itH} dt,$$

commutes with P. For a loop $\alpha = \partial\Omega$, the loop operator that we aimed to construct is given by

$$U_\alpha = \exp(2\pi i \bar{Q}_\Omega).$$

Since charge has an integer spectrum the operator is indeed supported on $\partial\Omega$ with exponentially decaying tails.

On torus there are two classes of closed loops, with representatives α_1, α_2, that are not boundaries of a region. They can be obtained as one part of a boundary. Let $\Gamma = \{0 \leqslant x_1 \leqslant L/2\}$ be the half torus, then $\partial\Gamma$ consists of two parts $\partial_\pm$ each of which is a non-contractible closed loop, we can take $\partial_- = \alpha_1$. Let Q be the charge in Γ. Then the operator $\bar{Q} = Q - K_- - K_+$ with $K_\pm$ located at $\partial_\pm$ and the integrality of charge implies that

$$U_{\partial\Gamma} = e^{2\pi i(Q-K_-)} e^{2\pi i(Q-K_+)}.$$

Exponential decay of correlations then imply that also

$$U_{\alpha_1} = e^{2\pi i(Q-K_-)}$$

commutes with P (up to finite size corrections decaying in L) and is a loop operator associated with a non-contractible loop $\partial_- = \alpha_1$.

Let U_{α_2} be the unitary corresponding to the second non-contractible loop α_2. In [8], we establish the commutation relation (1) and connect the angle to the Hall conductance.

Theorem. There exists an integer n such that

$$U^*_{\alpha_1} U^*_{\alpha_2} U_{\alpha_1} U_{\alpha_2} P = e^{2\pi i \frac{n}{p}}.$$

Moreover for any ground state $\psi \in P$,

$$(\psi, (U^*_{\alpha_2} Q U_{\alpha_2} - Q)_- \psi) = \frac{n}{p}, \tag{2}$$

where $(U^*_{\alpha_2} Q U_{\alpha_2} - Q)_-$ is the charge transported through ∂_- by the application of U_{α_2}.

By a standard Laughlin argument, $\frac{e^2}{h}\frac{n}{p}$ has the meaning of the Hall conductance. We end the discussion by noting that (2) can be also interpreted as giving the charge Q of the excitation ϵ in the box R, see figure 1. If γ is an open curve that is obtained by restricting α_2 then the expression (2) is exactly the excess charge of $U_\gamma \psi$ compare to the charge in the ground state ψ. Indeed, the only contribution to $U^*_\gamma Q U_\gamma - Q$ comes from the intersection of γ with the boundary ∂R and equals to the contribution $(U^*_{\alpha_2} Q U_{\alpha_2} - Q)_-$ coming from one of the two intersections of the closed loop α_2 with the boundary ∂R.

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A gapped family of two-dimensional AKLT models

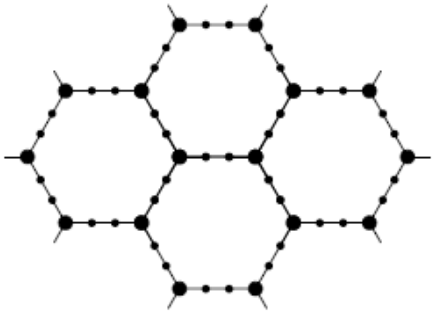
Amanda Young

(joint work with Houssam Abdul-Rahman, Marius Lemm, Angelo Lucia, Bruno Nachtergaele)

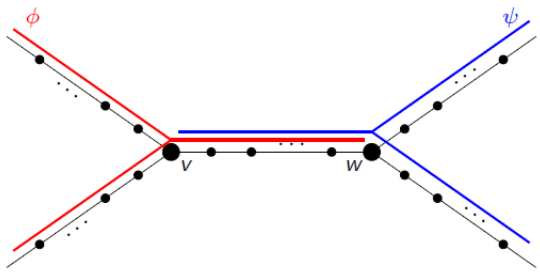
A central question in the study of quantum spin systems is the behavior of the low-lying excitations and, in particular, if there is a spectral gap above the ground state energy. While the importance of the spectral gap is well-known, it is generically undecidable [8, 5], and there are few approaches for determining its existence [9, 10, 11, 12, 15, 16, 19]. Furthermore, few results hold for multi-dimensional systems [4, 6, 7, 17]. In our work [1], we prove that a family of two-dimensional AKLT models are gapped above the ground state.

One of the most famous quantum spin systems is the family of isotropic antiferromagnet models introduced by Affleck, Kennedy, Lieb and Tasaki (AKLT) [2, 3]. In their seminal work, the authors proved the spin-1 AKLT chain satisfies the three conditions of the Haldane phase: a spectral gap above the ground state, exponential decay of correlations, and a unique ground state in the thermodynamic limit. Generalizations of the AKLT model to other regular lattices were introduced and conjectured to be gapped, including the spin-3/2 model on the hexagonal lattice. It has been shown that the spin-3/2 model has a unique ground state with exponential decay of correlations [3, 13]. However, while it was recently shown that the spin-3/2 model on an infinite chain of hexagons is gapped [18], the spectral gap question on the hexagonal lattice still remains open.

In the present work [1], we consider AKLT models defined on a decorated hexagonal lattice and prove they are gapped as long as the decoration parameter is at least three. The lattice, denoted $\Gamma^{(n)}$ where $n \geqslant 1$ is the *decoration parameter*, is defined by replacing each edge of the hexagonal lattice with a chain of n particles, see Figure 1a. Each site $x \in \Gamma^{(n)}$ is either a spin-1 or spin-3/2 particle determined



(A) The decorated hexagonal lattice.



(B) A visualization of the support of the ground states used to calculate ϵ_n.

FIGURE 1. The decorated hexagonal lattice for $n = 2$ and finite-size criterion sub-volume $S_v \cup S_w$ used to estimate γ_n.

by its degree via $s_x = \deg(x)/2$. We define the AKLT Hamiltonian as usual with nearest neighbor interactions given by the orthogonal projection onto the subspace of maximal total spin, i.e.

$$H_\Lambda = \sum_{\substack{\text{edges} \\ \{x,y\} \subset \Lambda}} P_{x,y}^{(s_x+s_y)} \tag{1}$$

for any finite volume $\Lambda \subset \Gamma^{(n)}$. To establish that the system is gapped, we proved that if $n \geqslant 3$, then there exists $\gamma_n > 0$ for which

$$H_\Lambda^2 \geqslant \gamma_n H_\Lambda \tag{2}$$

for any finite volume Λ with periodic boundary conditions. Since AKLT models are frustration-free (implying the ground state energy is zero), the value γ_n is necessarily a lower bound on the spectral gap.

To calculate γ_n we reduce the spectral gap question to estimating the overlap of ground states that are supported on a quasi one-dimensional system that is independent of Λ. Using translation invariance, we show that

$$H_\Lambda^2 \geqslant \frac{\gamma}{2}(1 - 3\epsilon_n) H_\Lambda$$

where, for any two nearest-neighbor spin-3/2 particles v and w, γ is the spectral gap H_{S_v} and

$$\epsilon_n = \sup_{\substack{0 \neq \phi, \psi \in \mathcal{H}_{S_v \cup S_w} \\ \phi \in \mathcal{G}_{S_v},\, \psi \in \mathcal{G}_{S_w} \\ \phi, \psi \perp \mathcal{G}_{S_v \cup S_w}}} \frac{|\langle \phi, \psi \rangle|}{\|\phi\| \|\psi\|}. \tag{3}$$

Here, $S_v \subseteq \Lambda$ is the star consisting of v and the three spin-1 chains connected to v, and $\mathcal{G}_X$ is the ground state space of H_X. See [1] for more details.

In (3), the vector $\phi \in \mathcal{H}_{S_v \cup S_w}$ is a ground state of H_{S_v} but an excited state for the system supported on $S_v \cup S_w$. An analogous statement holds for ψ. As ground states, the support of ϕ and ψ overlap on a one-dimensional AKLT chain of length n (see Figure 1b), which has exponential decay of correlations. This suggests that the excitations of ϕ and ψ are approximately localized to disjoint regions of space and, moreover, that ϕ and ψ become orthogonal as n grows. This indicates that $\epsilon_n < 1/3$ for sufficiently large n.

To make this argument rigorous, we use the valence bond picture for the ground states on $S_v \cup S_w$ to define a tensor network state representation that is injective for $n \geqslant 2$. Here, the ground states have the usual AKLT valence bond description where each site x is identified with the symmetric subspace of $\deg(x)$ spin-1/2 particles, and every pair of connected spin-1/2 particles is projected into a singlet state, see Firgure 2. Similar to the AKLT chain, the valence bond picture can be encoded into tensors $T_x = \{T_x^{(s)} : |s| \leqslant \frac{\deg(x)}{2}\} \subseteq \mathbb{C}^{2n_x^l \times 2n_x^r}$ where $n_x^l, n_x^r \in \{1, 2\}$ are, respectively, the number of edges to the left and right of x. For spin-1 particles, T_x is the usual set of MPS matrices. This construction induces a matrix algebra $\mathcal{K}_X$ and tensor network state $\Gamma_X : \mathcal{K}_X \to \mathcal{H}_X$ such that $\mathcal{G}_X = \operatorname{ran}(\Gamma_X)$ for each $X \in \{S_v, S_w, S_v \cup S_w\}$. Moreover, Γ_X is injective for $n \geqslant 2$.

Due to the injectivity of Γ_X, techniques used for estimating the spectral gap of one-dimensional finitely correlated state models, [9], can be modified to estimate the spectral gap for this system. For each $X \in \{S_v, S_w, S_v \cup S_w\}$ we can define an inner-product $\langle \cdot, \cdot \rangle_X$ on $\mathcal{K}_X$ for which

$$|\langle \Gamma_X(B), \Gamma_X(C) \rangle - \langle B, C \rangle_X| \leqslant C \cdot 3^n \|B\|_X \|C\|_X.$$

For vectors $\phi \in \mathcal{G}_{S_v}$ and $\psi \in \mathcal{G}_{S_w}$ we show there are matrices B_ϕ^n, $C_\psi^n \in \mathcal{K}_{S_v \cup S_w}$ and a constant $A_n = O(3^{-n})$ for which

$$\left| \langle \phi, \psi \rangle - \left\langle B_\phi^n, C_\psi^n \right\rangle_{S_v \cup S_w} \right| \leqslant A_n \|\phi\| \|\psi\|.$$

Under the additional assumption that $\phi, \psi \in \mathcal{G}_{Y_v \cup Y_w}^{\perp}$, it can be further determined that the matrices B_ϕ^n, C_ψ^n become orthogonal as $n \to \infty$, i.e.

$$\left| \left\langle B_\phi^n, C_\psi^n \right\rangle_X \right| \leqslant A_n^2 B_n \|\phi\| \|\psi\| \quad \text{where} \quad B_n = O(1).$$

The specific values of A_n and B_n are such that $\epsilon_n \leqslant A_n + A_n^2 B_n < 3^{-n+2}$ which establishes the result.

We conclude with a few comments on this result and its methods. First, the result that $n \geqslant 3$ is not sharp. While we can show that our method does not hold for $n = 1$, our approach was further extended to AKLT models on the decorated square lattice in [20]. In this subsequent work, numerical results showed that $\epsilon_2 < 1/3$ for both the decorated hexagonal and square lattices. Finally, the tensor network methods we applied in this work are rather generally applicable and are not specific to the AKLT interaction. They should be able to prove spectral gaps for other decorated models with sufficiently injective tensor network states.

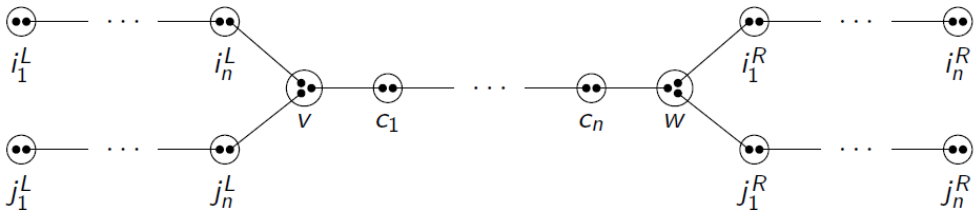


FIGURE 2. The valence bond state representation of the ground states on $S_v \cup S_w$.

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On the continuous limit of classical spin chains with long-range interactions

Jérémy Sok
(joint work with Enno Lenzmann)

We consider the continuum limit of classical spin chains with long range interactions. The spins $S_k \in \mathbb{S}^2$ are placed on the sites of a regular lattice $h\mathbb{Z} \subset \mathbb{R}$ or $\{z_k = e^{i\,2k\pi/N}\} \subset \mathbb{S}^1$ and vary in time according to

$$\partial_t S_k = \sum_{j \neq k} J_{|k-m|}(S_k - S_m) \wedge S_k,$$

where $J_k \geqslant 0$ denotes an interaction potential. Using the canonical Poisson structure on $\prod_k \mathbb{S}^2$, it can be rewritten as an Hamiltonian system with Hamiltonian

$$\mathbb{H} = 4^{-1} \sum_{\substack{(k,j) \\ k \neq j}} J_{|k-m|} |S_k - S_j|^2.$$

This family of equations are used to model the evolution of magnetization within a material. We are especially interested in the $1/r^2$ interaction, *i.e.* $J_{|k-m|} = C_h|k-m|^{-2}$ on $\mathbb{R}$ or $C_N|z_k - z_j|^{-2}$ on $\mathbb{S}^1$, where $C_h \sim h$ or $C_N \sim N^{-1}$. The corresponding equation appears as the large spin limit of the celebrated Haldane-Shastry quantum spin chains.

In the discrete system, it turns out that taking the continuum limit $h \to 0^+$ on $\mathbb{R}$ or $N \to +\infty$ on $\mathbb{S}^1$ yields the so-called *half-wave maps equation*

$$\partial_t S = (\sqrt{-\Delta} S) \wedge S.$$

The goal of this talk is to rigorously derive this continuum limit.

Given a regular initial data $S(x)$, we consider the time-evolution of its sampling according to the discrete equation. At each time, the discrete spin system is seen as the sampling of the (vector-valued) finite frequency band function given by the sampling theorem. In the continuum limit, we obtain an $\mathbb{S}^2$-valued function $S(t;x)$ solution to the half-wave maps equation. Depending on the regularity of the initial data, the limit is known to be a strong solution to the equation (at least up to some time $T_*(S) > 0$) or just a global in time solution to the weak form of the equation.

The generalized linear Boltzmann equation for magnetotransport

Chiara Saffirio
(joint work with A. Nota, S. Simonella)

We consider the dynamics of a test particle in the plane $\mathbb{R}^2$ bouncing among a configuration of N fixed hard disks centred at $(c_1, \dots, c_N) \in \mathbb{R}^{2N}$ and distributed according to the Poisson law of intensity $\mu_\varepsilon = \varepsilon\mu$, where $\mu \in \mathbb{R}_+$ and $\varepsilon > 0$ is the radius of the obstacles. More precisely, the probability of finding N hard disks in a given a bounded measurable set $\mathcal{A} \subset \mathbb{R}^2$ is

$$\mathbb{P}_\varepsilon(dc_1, \dots, dc_N) = e^{-\mu_\varepsilon |\mathcal{A}|} \frac{\mu_\varepsilon}{N!} dc_1 \dots dc_N \,, \tag{1}$$

where we denoted by $|\mathcal{A}|$ the Lebesgue measure of $\mathcal{A}$.

This model is referred to as the Lorentz gas and it was introduced by Lorentz in 1905 (cf. [6]) to describe the motion of electrons in metals. In the late 60s Gallavotti used this model to provide the first rigorous derivation of the linear Boltzmann equation from a Lorentz gas with a random distribution of fixed hard disks in the low-density limit (cf. [4], [5]). He proved that, as $\varepsilon \to 0$, the one-particle correlation function converges in $L^1(\mathbb{R}^2 \times \mathbb{R}^2)$ towards a solution to the linear Boltzmann equation

$$(\partial_t + v \cdot \nabla_x) f(t,x,v) = 2\mu \int_{S^1} [n \cdot v]_+ \left(f(t,x,v - 2(n \cdot v)n) - f(t,x,v) \right) dn \,,$$

where f is a probability density on $\mathbb{R}^2 \times \mathbb{R}^2$ representing the probability of finding the test particle at time t in the position x with velocity v, S^1 denotes the unit sphere and $[n \cdot v]_+$ is the positive part of the scalar product $n \cdot v$.

In our setting, the obstacles are hard disks randomly distributed according to (1), under the action of a uniform, constant, magnetic field orthogonal to the plane. More precisely, the Lorentz force acting on the test particle reads as $F(v) = Bv^\perp$ where B is the magnitude of the magnetic field and $v^\perp = (v_2, -v_1)$. Moreover, the test particle is assumed to have unit mass and unit charge. Starting from the initial position x with initial velocity v, the particle moves under the action of the Lorentz force $Bv^\perp$ up to the first instant of contact with an obstacle. Then it is elastically reflected and so on (cf. Fig. 1).

Bobylev et al, in [1] and later in [2, 3], observed that the presence of a given external field, in the two dimensional Lorentz model, strongly affects the derivation of the linear Boltzmann equation in the Boltzmann-Grad limit. This is easily seen by the following simple computation: the probability of performing a complete cyclotron orbit without hitting any obstacle is given by

$$\mathbb{P}_\varepsilon(\{\mathcal{C}\}) = e^{-\mu_\varepsilon |\mathcal{A}_\varepsilon(R)|} = e^{-4\pi R\mu} \,,$$

where $R > 0$ is the radius of the cyclotron orbit,

$$\mathcal{C} = \{\text{zero obstacles in } \mathcal{A}_\varepsilon(R) \text{ spanned by the tagged particle}\} \,,$$

$\mathcal{A}_\varepsilon(R)$ is the annulus of radius R and width ε, and $|\mathcal{A}_\varepsilon(R)|$ denotes the area of the set $\mathcal{A}_\varepsilon(R)$. It follows that $\mathbb{P}_\varepsilon(\{\mathcal{C}\})$ is strictly positive. These events are not

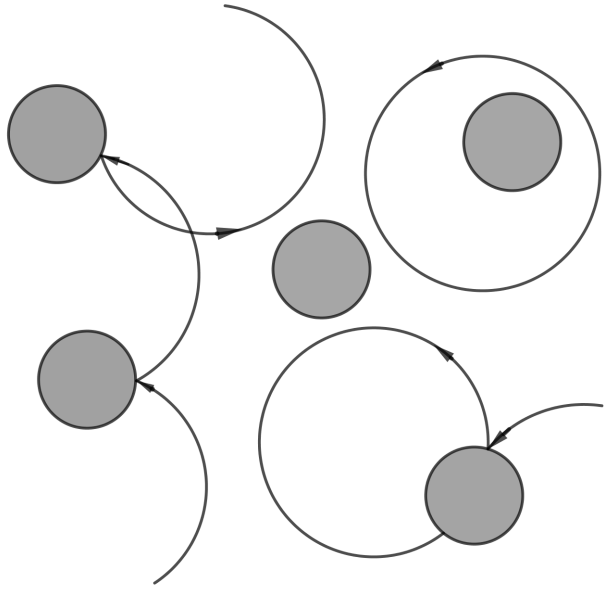

FIGURE 1. Test particle's motion

described by the standard Boltzmann equation, as the Markovianity of the limit system can not be attained. In order to take into account those effects, Bobylev et al. in [1, 2, 3] have introduced the following generalized Boltzmann equation:

$$\frac{D}{Dt}f(t,x,v) = 2\mu \sum_{k=0}^{[t/T_L]} e^{-2\mu k T_L} \int_{S_1} dn\,(v\cdot n) \\ [\chi(\{v\cdot n>0\})\sigma_n + \chi(\{v\cdot n<0\})]f(t-kT_L, x, S_n^{-k}v), \tag{2}$$

where $\chi(A)$ denotes the characteristic function of the set A. More precisely we denote by $\chi(\{v\cdot n>0\})$ the characteristic function of the set of vectors $n\in S^1$ such that $v\cdot n \geqslant 0$. The operator

$$\frac{D}{Dt} = (\partial_t + v\cdot\nabla_x + (v\times B)\cdot\nabla_v)$$

is the generator of the free cyclotron motion with frequency B and $[t/T_L]$ the number of cyclotron periods T_L completed before time t, being $[x]$ the integer part of x. In the right hand side of (2) $T_L = 2\pi/B$ is the cyclotron period. The angular integration over the unit vector n in (2) is over the entire unit sphere S_1 centred at the origin. The operator σ_n is defined by

$$\sigma_n\phi(v) = \phi(v-2(v\cdot n)n)$$

where $\phi(v)$ is an arbitrary function of v. The precollisional velocity $v' = v-2(v\cdot n)n$ becomes v after the elastic collision with the hard disk. Finally, the shift operator S_n^{-k}, when acting on v, rotates the velocity through the angle $-k\theta$, where θ is the scattering angle (from v' to v) determined by n.

Despite a memory term appears in the equation, the low-density limit largely simplifies the picture, as the differential cross section becomes independent of B and, out of a zero measure set of initial data, the test particle is either a circling particle or a wondering particle which collides with infinitely many different obstacles, thus avoiding configurations leading the test particle to be trapped in clusters of a finite number of scatterers. Our result provides a rigorous derivation of the generalised linear Boltzmann Eq. (2) first proposed in [1].

Main result. For a given configuration of hard disks $(c_1, \dots, c_N)$, let $\gamma_\varepsilon^{-t}(x, v)$ be the (backward) Hamiltonian flow solution to the Newton equations for the tagged particle with initial datum (x, v) moving in a Poisson distribution of hard disks of radius ε and subject to a magnetic field perpendicular to the plane. For a given probability distribution $f_0 = f_0(x, v)$, for any $t \geqslant 0$, let

$$f_\varepsilon(t, x, v) = \mathbb{E}_\varepsilon[f_0(\gamma^{-t}_{\mathbf{c}_N,\varepsilon}(x, v))].$$

be the one-particle correlation function, where $\mathbb{E}_\varepsilon$ denotes the expectation with respect to the probability $\mathbb{P}_\varepsilon$ defined in (1).

If $f_0 \in \mathrm{Lip}(\mathbb{R}^2 \times \mathbb{R}^2)$, then for all $T > 0$

$$\lim_{\varepsilon \to 0} \|f_\varepsilon - f\|_{L^\infty([0,T];L^1(\mathbb{R}^2 \times \mathbb{R}^2))} = 0\,,$$

where f is a solution to (2) with initial datum f_0.

The proof is based on an extension of the argument used by Gallavotti in [4, 5], focused on a clever parametrization of the scatterers in terms of impact times and impact vectors. We prove that, in presence of a constant magnetic field perpendicular to the plane, such a parametrization is still well defined through a change of variables up to a negligible set of configurations. Moreover, we introduce two new ideas: a time splitting which allows to consider separately the Markovian and the non-Markovian part of the generalized Boltzmann Eq. (2); we distinguish between internal (with the seme obstacle) and external (with different obstacles) recollisions, being the first ones responsible for the memory term in the equation and the second ones negligible as $\varepsilon \to 0$. The convergence is attained by means of a monotonicity argument.

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Gross–Pitaevskii Limit of a Homogeneous Bose Gas at Positive Temperature

Andreas Deuchert
(joint work with Robert Seiringer)

The experimental realization of Bose–Einstein condensation (BEC) in trapped alkali gases in 1995 [3, 1] triggered numerous experimental, theoretical and mathematical investigations of the properties of dilute quantum gases. While early experiments could only prepare such systems in harmonic traps, modern experimental techniques allow also for the engineering of box-like trapping potentials with approximate Dirichlet boundary conditions at the edges [7].

In a recent work we considered together with Jakob Yngvason the trapped Bose gas in the Gross–Pitaevskii (GP) limit at positive temperature [4]. We were able to prove that the difference between the interacting free energy and the one of the ideal gas is to leading order given by the minimum of a GP energy functional. Additionally, we could show that the one-particle density matrix (1-pdm) of every approximate minimizer of the Gibbs free energy functional is to leading order given by the one of the ideal gas, where the condensate wave function has been replaced by the minimizer of the GP energy functional. The proof of these statements is based on the fact that particles inside the thermal cloud live on a much larger length scale than particles inside the condensate, and hence the interaction can be seen to leading order only in the condensate. The experimentally relevant and mathematically more difficult case of the gas in a box, where the condensate and the non-condensed particles live on the same length scale and all interactions are relevant to leading order, had been left as an open problem. In the remainder of this note we consider this case. More precisely, we consider a system of N bosonic particles confined in a three-dimensional box of side length L, whose Hamiltonian reads

$$H_N = \sum_{i=1}^{N} -\Delta_i + \sum_{1 \leqslant i < j \leqslant N} v_N(x_i - x_j). \tag{1}$$

By Δ_i we denote the Laplacian with periodic boundary conditions acting on the i-th particle. With our methods we can also treat Dirichlet boundary conditions but we do not consider this case here. The interaction potential is of the form

$$v_N(x) = (N/L)^2 \, v(Nx/L), \tag{2}$$

with a nonnegative potential v. The scattering length a_N of v_N scales as $a_N/L \sim N^{-1}$, which assures that the interaction energy per particle is of the same order as the spectral gap of the Laplacian in the box. The scattering length is a combined measure for the range and the strength of a potential and it is the only parameter that enters the low energy scattering cross-section in quantum mechanics. Ground state properties of the dilute Bose gas have been investigated from a mathematical point of view in great detail in the GP limit and in the thermodynamic limit. See [8] for an overview and [2, 6] for two recent developments concerning the excitation

spectrum of the Bose gas in the GP limit and the next-to-leading order correction of the ground state energy in the thermodynamic limit.

In this note we are interested in the Bose gas at positive temperature, and therefore consider the canonical free energy of the system. As the ground state energy, it can be characterized by a variational principle called the Gibbs variational principle. Denote by $S_N = \{\Gamma \in \mathcal{L}(\mathcal{H}_N) \mid 0 \leqslant \Gamma \leqslant 1, \mathrm{Tr}[\Gamma] = 1\}$ the set of states on the bosonic N-particle Hilbert space $\mathcal{H}_N$ consisting of permutation symmetric, square integrable wave functions. Accordingly, Tr denotes the trace over $\mathcal{H}_N$. The free energy is given by

$$F(T,N,L) = \inf_{\Gamma\in\mathcal{S}_N} \underbrace{\{\mathrm{Tr}\left[H_N\Gamma\right] - TS(\Gamma)\}}_{=\mathcal{F}(\Gamma)} \quad \text{with} \quad S(\Gamma) = -\mathrm{Tr}\left[\Gamma\ln(\Gamma)\right]. \tag{3}$$

The quantity S is called the von-Neumann entropy and the unique minimizer of the free energy functional $\mathcal{F}$ is the Gibbs state of the system. By $F_0(\beta,N,L)$ we denote the free energy of the ideal gas ($v = 0$). We are interested in temperatures of the order of the critical temperature of the ideal gas or smaller, that is, in inverse temperatures β such that $\beta \geqslant$ const. $(4\pi)^{-1}((N/L^3)/\zeta(3/2))^{-2/3}$. Here ζ denotes the Riemann zeta function. Apart from the above result for the trapped Bose gas in the GP limit [4], the only other known result concerning the dilute Bose gas at positive temperature concerns the free energy asymptotics in the thermodynamic limit, see [10] for the upper bound and [9] for the lower bound. We note that the second reference is a key ingredient for parts of the proof of Theorem 1 below. The first reference is not relevant for us because we give a much simpler proof for the upper bound for the free energy in the case of the GP limit. This is possible because the gas is much more dilute in this regime compared to the setting in [10]. Our main result in [5] is the following statement:

Theorem 1: *Assume that v is a nonnegative interaction potential with finite scattering length. Denote by $\varrho_0(\beta,N,L)$ the expected condensate density in the canonical Gibbs state of the ideal Bose. In the combined limit $N\to\infty$, $\beta\varrho^{2/3} \geqslant$* const.

$$F(\beta,N,L) = F_0(\beta,N,L) + 4\pi a_N L^3\left(2\varrho^2 - \varrho_0\left(\beta,N,L\right)^2\right)(1+o(1)). \tag{4}$$

Moreover, for any sequence of states $\Gamma_N \in \mathcal{S}_N$ with one-particle density matrices γ_N and

$$\mathcal{F}\left(\Gamma_N\right) = F_0(\beta,N,L) + 4\pi a_N L^3\left(2\varrho^2 - \varrho_0\left(\beta,N,L\right)^2\right)(1+o(1)), \tag{5}$$

we have

$$\|\gamma_N - \gamma_{N,0}\|_1 \leqslant o(N). \tag{6}$$

Here $\gamma_{N,0}$ denotes the one-particle density matrix of the canonical Gibbs state of the ideal gas and $\|\cdot\|_1$ is the trace norm.

We remark that the free energy in (4) has the same form as the one of the dilute Bose gas in the thermodynamic limit. The reason for this are the periodic boundary conditions. They also imply that the condensate wave function equals

a constant. The structure of the interaction term in (4) can be understood if one realizes that the exchange term in the interaction is only relevant for particles sitting in different one-particle orbitals, which is typically the case for particles in the thermal cloud but not for particles in the condensate. This is the reason why the interaction energy between particles inside the thermal cloud is twice as large as the one between particles in the condensate. The quantities related to the canonical ideal gas in Theorem 1, that is, $F_0(\beta, N, L)$, $\varrho_0(\beta, N, L)$ and $\gamma_{N,0}$, can be replaced by their grand canonical equivalents and the statement remains true. This is advantageous because explicit formulas are available for the latter quantities. We also remark that (6) implies BEC in the sense

$$\lim_{N\to\infty} \sup_{\|\phi\|=1} \frac{\langle \phi, \gamma_N \phi \rangle}{N} = \left[1 - \left(\frac{\beta_c}{\beta} \right)^{3/2} \right]_+ , \tag{7}$$

where β_c denotes the inverse critical temperature of the ideal gas.

We conclude our discussion by highlighting two main technical novelties of our proof: The first key ingredient of our approach is a novel use of the Gibbs variational principle that goes hand in hand with the c-number substitution, which is a central technical tool in the proof in [9]. It allows us to work with a general state Γ instead of with a version of the grand canonical Gibbs state. Like this, we can keep the information that Γ has exactly N particles and we are able to quantify the coercivity of the Gibbs free energy functional $\mathcal{F}$. The second main novelty is a lower bound for the bosonic relative entropy of two density matrices (the difference between their free energies) in terms of a function of their trace norm difference. This bound is used in the proof of (6) and allows us to substantially simplify parts of the analysis with respect to a similar part of the proof in [4].

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Symmetry Breaking in a Gas of Bosons

Horst Knörrer
(joint work with T. Balaban, J. Feldman, E. Trubowitz)

The talk was a status report on our program to rigorously demonstrate symmetry breaking in a cold, weakly interacting gas of bosons hopping on a three dimensional lattice. Technically, to show that the correlation functions decay at a non–integrable rate when the chemical potential is sufficiently positive, the non–integrability reflecting the presence of a long range Goldstone boson mediating the interaction between quasiparticles in the superfluid condensate. It is already known [5], [7] that the correlation functions are exponentially decreasing when the chemical potential is sufficiently negative. See also [9].

The first step in the program was to express the positive temperature partition function and thermodynamic correlation functions in a periodic box (a discrete three–dimensional torus) as 'temporal' ultraviolet limits of four–dimensional (coherent state) lattice functional integrals. By a lattice functional integral we mean an integral with one (in this case complex) integration variable for each point of the lattice. By a 'temporal' ultraviolet limit, we mean a limit in which the lattice spacing in the inverse temperature direction (imaginary time direction) is sent to zero while the lattice spacing in the three spatial directions are held fixed. The argument, presented in [1],[2] is a mathematically rigorous interpretation of the standard physics argument (see [8] (2.66)) which writes the correlations in terms of an ill defined infinite dimensional complex Gaussian measure[1].

Then, by a complete large field/small field renormalization group analysis, we expressed the temporal ultraviolet limit for the partition function as a four–dimensional lattice functional integral with the lattice spacing in all four directions being of the order one, preparing the way for an infrared renormalization group analysis of the thermodynamic limit. An overview is given in [3].

Next, we initiated the infrared analysis by tracking the evolution of the effective interaction generated by the iteration of a renormalization group map that is characteristic of a parabolic covariance[2]: in each renormalization group step the spatial lattice directions expand by a factor $L > 1$, the inverse temperature direction expands by a factor L^2 and the running chemical potential grows by a factor of L^2, while the running coupling constant decreases by a factor of L^{-1}. Consequently, the effective potential, initially close to a paraboloid, develops into a Mexican hat with a moderately large radius and a moderately deep circular well of minima. This construction ends after a finite number (depending on the "coupling constant" $\mathfrak{v}_0$, a suitable norm of the two body interaction) of steps once the chemical potential, which initially was of the order of the coupling constant, has grown to a small 'ϵ' power of the coupling constant. The renormalization group iterates have moved away from the trivial noninteracting fixed point and it is no

[1] In slightly different contexts, the fact that such an integral does not make sense, was pointed out in [6] or in [10], section IV

[2] Morally, the $1 + 3$ dimensional heat operator.

longer possible to solve the equations giving the critical point of the effective action simply by perturbing off of the linearized equations. For an overview of the construction see [4].

The technical implementation of the parabolic renormalization group uses $1+3$ dimensional block spin averages, while we had used decimation for the effectively one dimensional problem of evaluating the temporal ultraviolet limit. In both regimes oscillatory integrals occur, which we control by stationary phase calculations. In the treatment of the temporal ultraviolet problem, the critical points of the integrands are easy to determine, while in the "parabolic regime" they are solutions to (weakly) nonlinear systems of parabolic equations. In both cases the critical points are not real, but a Stokes' argument allows to shift the relevant multi dimensional contour to the 'reals'. The evaluation of fluctuation integrals is similar in both cases: it relies on an adaptation and simplification of polymer expansions. In the parabolic step, there is an important new feature: the chemical potential has to be renormalized.

To analyze the output of the block spin convolution (a single renormalization group step), it is de rigueur for the small field/large field style of renormalization group implementations to introduce local small field conditions on the integrand and then decompose the integral into the sum over all partitions of the discrete torus into small and large field regions on which the conditions are satisfied and violated, respectively. Small field contributions are to be controlled by powers of the coupling constant $\mathfrak{v}_0$ uniformly in the volume of the small field region. Large field contributions are to be controlled by a factor $e^{-1/\mathfrak{v}_0^{\varepsilon}}$, $\varepsilon > 0$, raised to the volume of the large field region. Morally, in small field regions, perturbation expansions in the coupling constant converge and exhibit all physical phenomena. Large field regions give multiplicative corrections that are smaller than any power of the coupling constant. So, in the leading terms, every point is small field.

If the actions in our functional integrals were sums of positive terms (as in a Euclidean O(n) model) it would be routine to extract an exponentially small factor per point of a large field region. They are not. There are explicit purely imaginary terms. We analyze the parabolic flow of the leading term, in which all points are small field, as long as it is possible to expand around zero field. Nevertheless, we show that our actions *do* have positivity properties and consequently there is at least one factor $e^{-1/\mathfrak{v}_0^{\varepsilon}}$ whenever there is a large field region. A stronger bound of a factor per point of a large field region has been achieved in the "ultraviolet regime" and shown to be reasonable in the "parabolic regime". Such a stronger bound gives complete control of the full renormalization group flow in this regime.

For the final step of the program (which is not yet completed), we continue the parabolic evolution, but expand around fields concentrated at the bottom of the well rather than about zero (much as is done in the Bogoliubov Ansatz) and track it through an additional finite number of steps until the running chemical potential is sufficiently larger than one, so that the well is sufficiently deep. At that point we turn to an elliptic renormalization group map that expands both the temporal (inverse temperature) and spatial lattice directions by the same factor L.

It is expected that the elliptic evolution can be controlled through infinitely many steps, all the way to the symmetry broken fixed point. The system is superrenormalizable in the entire parabolic regime because the running coupling constant is geometrically decreasing. However in the elliptic regime, the system is only strictly renormalizable.

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Free energy asymptotics of the quantum Heisenberg spin chain

Marcin Napiórkowski

(joint work with Robert Seiringer)

The quantum ferromagnetic Heisenberg model is one the most important and widely studied models of statistical mechanics. The low temperature properties of the model are usually examined using spin wave theory. Briefly, spin wave theory (or spin wave approximation) assumes, that the low-energy behaviour of the system can be described in terms of non-interacting collective excitations of spins called spin waves. The spin wave approximation has been very successful, predicting many features that have been later tested experimentally.

Our goal is to rigorously justify this approximation in the case of the one dimensional chain. The model is defined in terms of the Hamiltonian

$$H_L = \sum_{x=1}^{L-1} \left(S^2 - \vec{S}_x \cdot \vec{S}_{x+1} \right). \tag{1}$$

Here $\vec{S} = (S^1, S^2, S^3)$ denote the three components of the spin operators corresponding to spin S, i.e., they are the generators of the rotations in a $2S + 1$

dimensional representation of $SU(2)$. The Hamiltonian H_L acts on the Hilbert space $\mathfrak{H}_L = \bigotimes_{x=1}^{L} \mathbb{C}^{2S+1}$. We added a constant S^2 for every site in order to normalize the ground state energy of H_L to zero.

We are interested in the free energy of the system. It is given by

$$f(\beta, S) = \lim_{L\to\infty} f_L(\beta, S). \tag{2}$$

where

$$f_L(\beta, S) = -\frac{1}{\beta L} \ln \mathrm{Tr} e^{-\beta H_L}.$$

We are interested in the behavior of $f(S, \beta)$ in the low temperature limit $\beta \to \infty$ for fixed S. In my talk I announced the following result which confirms the predictions of spin wave approximation as far as the free energy is concerned.

Theorem 1. *Consider the Hamiltonian* (1) *and the corresponding free energy* (2)*. For any* $S \geqslant 1/2$*,*

$$\lim_{\beta\to\infty} f(\beta, S) S^{\frac{1}{2}} \beta^{\frac{3}{2}} = C_1 := \frac{1}{2\pi} \int_{\mathbb{R}} \ln\left(1 - e^{-p^2}\right) dp = \frac{-\zeta(\frac{3}{2})}{2\sqrt{\pi}}. \tag{3}$$

The expression on the right hand side of (3) is the free energy of a system of non-interacting bosons. This is what spin wave approximation predicts as, according to Holstein and Primakoff [5], spin waves can be interpreted as free bosonic quasiparticles called magnons.

Recently, this kind of result has been obtained in three and more dimensions by Correggi, Giuliani and Seiringer in [3]. We would like stress that previous results in this direction either provided non exact upper bounds (see [4, 6]) or were obtained in the large S limit [2, 1].

Let me make a few remarks about the proof. We work in the Holstein–Primakoff representation, i.e. we set

$$S_x^+ = \sqrt{2S}\, a_x^\dagger \left[1 - \frac{a_x^\dagger a_x}{2S}\right]_+^{1/2}, \quad S_x^- =: \sqrt{2S} \left[1 - \frac{a_x^\dagger a_x}{2S}\right]_+^{1/2} a_x\,, \quad S_x^3 =: a_x^\dagger a_x - S\,, \tag{4}$$

where $a_x^\dagger, a_x$ are bosonic creation and annihilation operators, $S^\pm = S^1 \pm iS^2$, and $[\,\cdot\,]_+ = \max\{0, \cdot\}$ denotes the positive part. The operators $a^\dagger$ and a act on the space $\ell^2(\mathbb{N}_0)$ via $(a\,f)(n) = \sqrt{n+1} f(n+1)$ and $(a^\dagger f)(n) = \sqrt{n} f(n-1)$, and satisfy the canonical commutation relations $[a, a^\dagger] = 1$. One readily checks that (4) defines a representation of $SU(2)$ of spin S, and the operators S_x leave the space $\bigotimes_{x=1}^{L} \ell^2([0, 2S]) \cong \mathfrak{H}_L = \bigotimes_{x=1}^{L} \mathbb{C}^{2S+1}$, which can be naturally identified with a subspace of the Fock space $\mathcal{F} := \bigotimes_{x=1}^{L} \ell^2(\mathbb{N}_0)$, invariant.

The Hamiltonian H_L in (1) can be expressed in terms of the bosonic creation and annihilation operators as

$$H_L = S\sum_{x=1}^{L-1}\Big(-a_x^\dagger\sqrt{1-\frac{n_x}{2S}}\sqrt{1-\frac{n_{x+1}}{2S}}a_{x+1} - a_{x+1}^\dagger\sqrt{1-\frac{n_{x+1}}{2S}}\sqrt{1-\frac{n_x}{2S}}a_x + n_x + n_{x+1} - \frac{1}{S}n_x n_{x+1}\Big) =: S\sum_{x=1}^{L-1}(a_x^\dagger - a_{x+1}^\dagger)(a_x - a_{x+1}) - K =: T - K\,,$$

where we denote the number of particles at site x by $n_x = a_x^\dagger a_x$. In the equation above we introduced the operator T which describes bosons hoping freely on the lattice. According to spin wave approximation, this part of the Hamiltonian is the dominant one at low temperatures.

This motivates the choice of the trial state for the upper bound. First, however, we use an energy localization into Dirichlet boxes of size $\sqrt{\beta S} \ll \ell \ll \beta S$, just like in [2, 3]. The trial state we then choose is of the form

$$\Gamma = \frac{\mathcal{P}e^{-\beta T}\mathcal{P}}{\mathrm{Tr}_{\mathcal{F}}\mathcal{P}e^{-\beta T}\mathcal{P}}. \tag{5}$$

Here, $\mathcal{P}$ is defined by

$$\mathcal{P} = \prod_{x=1}^{\ell} f(n_x)$$

where

$$f(n_x) = \begin{cases} 1 & \text{if} \quad n_x = 0; \\ \left[\prod_{j=1}^{n_x}\left(1-\frac{j-1}{2S}\right)\right]^{\frac{1}{2}} & \text{if} \quad n_x = 1, 2, \ldots, 2S; \\ 0 & \text{if} \quad n_x > 2S. \end{cases}$$

As mentioned before, the operator T is the Hamiltonian on Fock space describing free bosons on $\Lambda_\ell = [1, \ldots, \ell]$, now with Dirichlet boundary conditions.

As far as the upper bound is concerned, the choice of $\mathcal{P}$ is the main difference between the proof in one and three dimensions. In fact, in three dimensions it was enough to choose $\mathcal{P}$ to be the projection onto $n_x \leqslant 1$ for any x on the lattice. In the three dimensional case, the crucial inequality was

$$\mathcal{P}H\mathcal{P} \leqslant T + (2S-1)\sum_{x\sim y} n_x n_y$$

where $x \sim y$ means x and y are nearest neihgbours. The presence of the last term is related to the hard-core constraint $n_x \leqslant 2S$. In three dimensions Correggi, Giuliain and Seiringer where able to show that this term is small and thus bound the Hamiltonian in terms of free bosons on the lattice. In one dimension one can not expect this term to be small. This is why we had to modify the trial state. In particular, with the new choice of $\mathcal{P}$ one can show that $\mathcal{P}H\mathcal{P} \leqslant T$. The rest of the proof of the upper bound is analogous to the one in three dimensions [3].

The lower bound is more complicated and we refer to the upcoming article for the details.

Finally, let us remark that the trial state (5) can be used also in two dimensions. It leads to the upper bound on the $2d$ free energy f^{2d} of the form

$$f^{\mathrm{2d}}(\beta, S) \leqslant -\frac{\pi}{24} S^{-1} \beta^{-2} \left(1 - \mathcal{O}((\beta S)^{-\kappa})\right) \tag{6}$$

for any $\kappa < \frac{1}{3}$. This yields an upper bound which is in agreement with spin wave approximation. A lower bound matching (6) remains an open problem.

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Derivation of 1d and 2d Gross–Pitaevskii equations for strongly confined 3d bosons

Lea Bossmann

(joint work with Stefan Teufel)

Since two decades, it has been experimentally possible to realise quasi-1d/2d Bose–Einstein condensates (BECs) in cigar- and disc-shaped traps [6], and our aim is to contribute to the mathematically rigorous understanding of such systems. We show that in the joint limit of infinite particle number and infinite trap asymmetry, condensation is preserved by the time evolution, and the dynamics of the condensate are asymptotically described by an effective 1d/2d Gross–Pitaevskii (GP) equation.

We study the dynamics of a BEC of N 3d bosons in an extremely asymmetric trap, where the particles are in one or two directions confined to a region of order $0 < \varepsilon \ll 1$. In the coordinates

$$z = (x, y) \in \mathbb{R}^3\,, \quad x \in \mathbb{R}^d\,, \quad y \in \mathbb{R}^{3-d}\,, \quad d = 1, 2\,,$$

where $x \in \mathbb{R}^d$ denotes the longitudinal direction(s) and $y \in \mathbb{R}^{3-d}$ is the coordinate in the confined direction(s), the Hamiltonian is given as

$$H = \sum_{j=1}^{N} \left(-\Delta_{z_j} + \tfrac{1}{\varepsilon^2} V^{\perp}\left(\tfrac{y_j}{\varepsilon}\right)\right) + \sum_{1 \leqslant i < j \leqslant N} w_\mu(z_i - z_j)\,. \tag{1}$$

The transverse confinement is modelled by the potential $V^{\perp} : \mathbb{R}^{3-d} \to \mathbb{R}$, which is rescaled by ε. We assume $V^{\perp}$ to be such that the ground state χ of the operator

$-\Delta_y + V^\perp$ with eigenvalue E_0 is localised on a length scale of order one. Consequently, the normalised ground state χ^ε of $-\Delta_y + \frac{1}{\varepsilon^2}V^\perp(\frac{y}{\varepsilon})$, corresponding to the eigenvalue $\frac{E_0}{\varepsilon^2}$, is localised on the scale ε and given as

$$\chi^\varepsilon(y) = \varepsilon^{-\frac{3-d}{2}}\chi\left(\tfrac{y}{\varepsilon}\right) . \tag{2}$$

The interaction w_μ between the particles is defined as

$$w_\mu(z) := \mu^{-2}w(\mu^{-1}z) , \qquad \mu := \tfrac{\varepsilon^{3-d}}{N} , \tag{3}$$

where $w : \mathbb{R}^3 \to \mathbb{R}$ is assumed non-negative, spherically symmetric, bounded and compactly supported. The scaling (3) of the interaction w describes a Bose gas in the GP scaling regime at a longitudinal length scale of order one and a transverse length scale of order ε.

To realistically model the physical situation, we take the two limits $N \to \infty$ and $\varepsilon \to 0$ simultaneously. Our analysis does not cover all possible sequences $(N, \varepsilon) \to (\infty, 0)$ but is constrained by the requirements

(ac) *admissibility condition:* $\begin{cases} N\varepsilon^{\frac{2}{5}^-} \to 0 & \text{for } d = 1 , \\ N\varepsilon^2 \to 0 & \text{for } d = 2 , \end{cases}$

(mc) *moderate confinement:* $N\varepsilon^{0^+} \to \infty \qquad \text{for } d = 2 ,$

where we use the notation x^+ and x^- to denote $(x + \sigma)$ and $(x - \sigma)$ for arbitrary but fixed $\sigma > 0$. Condition (ac) ensures that ε shrinks sufficiently fast compared to N^{-1}, while condition (mc), which is required only for disc-shaped traps, states that ε cannot shrink too fast either.

We assume that the initial data exhibit BEC in the joint limit $(N, \varepsilon) \to (\infty, 0)$,

$$\lim_{(N,\varepsilon)\to(\infty,0)} \mathrm{Tr}_{L^2(\mathbb{R}^3)} \left| \gamma_0^{(1)} - |\varphi_0^\varepsilon\rangle\langle\varphi_0^\varepsilon| \right| = 0 , \tag{4}$$

where the limit is taken along a sequence satisfying (ac) and (mc). Here, $\gamma_0^{(1)}$ denotes the one-particle reduced density matrix of the initial state $\psi_0^{N,\varepsilon} \in L^2_+(\mathbb{R}^{3N})$, and $\varphi_0^\varepsilon \in L^2(\mathbb{R}^3)$ is the condensate wave function. Given the strong confinement, the condensate wave function φ_0^ε is assumed to factorise into the transverse ground state χ^ε and a longitudinal part $\Phi_0 \in H^{2d}(\mathbb{R}^d)$,

$$\varphi_0^\varepsilon(z) = \Phi_0(x)\chi^\varepsilon(y) . \tag{5}$$

Moreover, we assume that

$$\lim_{(N,\varepsilon)\to(\infty,0)} \left| E_{w_\mu}^{\psi_0^{N,\varepsilon}}(0) - \mathcal{E}_b^{\Phi_0}(0) \right| = 0 . \tag{6}$$

Here,

$$E_{w_\mu}^{\psi}(t) := \tfrac{1}{N}\langle\psi, H(t)\psi\rangle_{L^2(\mathbb{R}^{3N})} - \tfrac{E_0}{\varepsilon^2} \tag{7}$$

denotes the "renormalised" energy per particle, and

$$\mathcal{E}_b^{\Phi}(t) := \langle\Phi, \left(-\Delta_x + \tfrac{b}{2}|\Phi|^2\right)\Phi\rangle_{L^2(\mathbb{R}^d)} \tag{8}$$

is the effective longitudinal energy per particle, with

$$b := 8\pi a \int_{\mathbb{R}^{3-d}} |\chi(y)|^4 \mathrm{d}y = 8\pi a\, \varepsilon^{3-d} \int_{\mathbb{R}^{3-d}} |\chi^\varepsilon(y)|^4 \mathrm{d}y\,. \tag{9}$$

In [8, 10], it is shown that both assumptions (4) and (6) are satisfied if $\psi_0^{N,\varepsilon}$ is the ground state of H with an additional longitudinal confining potential, and for Φ_0 the minimiser of the corresponding functional (8).

The state $\psi^{N,\varepsilon}(t)$ of the system at time t is determined by

$$\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t}\psi^{N,\varepsilon}(t) = H\psi^{N,\varepsilon}(t)\,, \qquad \psi^{N,\varepsilon}(0) = \psi_0^{N,\varepsilon}\,. \tag{10}$$

Our main result [2, 3] shows that condensation in a factorised state as in (5) is preserved by the N-body time evolution, where the time-evolved longitudinal part $\Phi(t)$ solves the d-dimensional GP equation

$$\mathrm{i}\tfrac{\partial}{\partial t}\Phi(t,x) = \left(-\Delta_x + b|\Phi(t,x)|^2\right)\Phi(t,x)\,, \qquad \Phi(0) = \Phi_0, \tag{11}$$

with b as in (9). This is summarised in the following theorem:

Theorem 1. *Let $d \in \{1,2\}$, $\Phi_0 \in H^{2d}(\mathbb{R}^d)$, and denote by T_{ex} the maximal time of $H^{2d}(\mathbb{R}^d)$-existence of the solution of* (11)*. Let $\psi_0^{N,\varepsilon}$ satisfy* (4) *and* (6)*, and let $\psi^{N,\varepsilon}(t)$ and $\Phi(t)$ denote the solutions of* (10) *and* (11)*, respectively. Then, for any $0 \leqslant T < T_{\mathrm{ex}}$,*

$$\lim_{(N,\varepsilon)\to(\infty,0)} \sup_{t\in[0,T]} \mathrm{Tr}\left|\gamma^{(1)}_{\psi^{N,\varepsilon}(t)} - |\Phi(t)\chi^\varepsilon\rangle\langle\Phi(t)\chi^\varepsilon|\right| = 0\,. \tag{12}$$

We conclude with some remarks:

(1) Our proof [1, 3] includes all interactions of the form

$$w_{\mu,\beta}(z) = \mu^{1-3\beta} w(\mu^{-\beta} z)\,, \quad \beta \in (0,1]\,.$$

Besides, (1) may additionally contain a bounded and sufficiently regular external potential $V(t,z)$, which leads to the additional term $V(t,(x,0))$ in (11). We obtain a non-optimal estimate of the rate of the convergence of the reduced densities, which grows exponentially in time.

(2) To prove Theorem 1, we adapt Pickl's method [9] to the situation with strong confinement. To handle the dimensional reduction, we require the constraints (ac) and (mc), which seem to be of technical nature.

(3) Similar results for interactions $w_{\mu,\beta}$ with β in subsets of $(0,\frac{1}{2})$ have been obtained in [4, 5, 7]. Note that these works require constraints similar to (ac) and (mc).

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On some families of exactly solvable Schrödinger operators

Jan Dereziński
(joint work with Laurent Bruneau, Vladimir Georgescu and Serge Richard)

Schrödinger operators that can be analyzed in terms of special functions play an important role in applications. It is natural to organize them in holomorphic families of closed operators. Clearly, for most values of parameters these operators are non-self-adjoint. Even if we are interested mostly in self-adjoint cases, the existence of an analytic continuation (which is in general non-selfadjoint) plays an important role in applications.

There exists a large literature concerning exactly solvable Schrödinger operators. However, the literature is restricted almost exclusivealy to the self-adjoint case, e.g. [7], or treats these operator formally, without the Hilbert space setting, e.g. [6].

1. Potential $\frac{1}{x^2}$

Consider an operator on $L^2(\mathbb{R}_+)$ formally defined by

$$L_\alpha := -\partial_x^2 + \Big(\alpha - \frac{1}{4}\Big)\frac{1}{x^2}, \tag{1}$$

where $\alpha \in \mathbb{C}$. Let us set $\alpha = m^2$. In [1] the following holomorphic family of closed realizations of (1) was introduced:

$$\begin{aligned} &H_m, \quad -1 < \mathrm{Re}(m), \\ &\text{defined as } L_{m^2} \text{ with boundary conditions } \sim x^{\frac{1}{2}+m}. \end{aligned} \tag{2}$$

The operators H_m are homogeneous of degree -2. H_m is the only closed realization of L_{m^2} for $\mathrm{Re}(m) \geqslant 1$. In the region $-1 < \mathrm{Re}(m) < 1$ this operator function can be obtained as the unique analytic continuation from $\mathrm{Re}(m) \geqslant 1$. Most probably the line $\mathrm{Re}(m) = -1$ is the boundary of its domain of holomorphy.

In $-1 < \mathrm{Re}(m) < 1$ there exist also realizations of L_{m^2} with mixed boundary conditions, which are no longer homogeneous. As described in [4], it is natural to organize them into two holomorphic families:

(3) $H_{m,\kappa}, \quad -1 < \mathrm{Re}(m) < 1, \quad \kappa \in \mathbb{C} \cup \{\infty\},$

defined as L_{m^2} with boundary conditions $\sim x^{\frac{1}{2}+m} + \kappa x^{\frac{1}{2}-m}$;

(4) $H_0^\nu, \quad \nu \in \mathbb{C} \cup \{\infty\},$

defined as L_0 with boundary conditions $\sim x^{\frac{1}{2}}\big(\nu + \ln(x)\big)$.

The operators $H_{m,\kappa}$ have curious point spectrum, situated along a segment of an appropriate logarithmic spiral.

2. Potential $\frac{1}{x}$

Let us now conside the differential operator

$$L_{\beta,\alpha} := -\partial_x^2 + \Big(\alpha - \frac{1}{4}\Big)\frac{1}{x^2} - \frac{\beta}{x} \tag{5}$$

on $L^2(\mathbb{R}_+)$. Again, it is useful to set $\alpha = m^2$. In [5] the family

(6) $H_{\beta,m}, \quad \beta \in \mathbb{C}, \quad -1 < \mathrm{Re}(m),$

defined as L_{β,m^2} with boundary conditions $\sim x^{\frac{1}{2}+m}\Big(1 - \dfrac{\beta}{1+2m}x\Big)$.

is introduced. Similarly as above, for $\mathrm{Re}(m) \geqslant 1$ the operator $H_{\beta,m}$ is the only closed realization of L_{β,m^2}. Inside the strip $-1 < \mathrm{Re}m < 1$ it can be obtained by analytic continuation.

This family is holomorphic except for a singularity at $(\beta, m) = (0, -\frac{1}{2})$. This singularity is not visible if we restrict ourselves to $\beta = 0$. In fact, we have $H_{0,m} = H_m$, and H_m is analytic in $\mathrm{Re}(m) > -1$, which includes $-\frac{1}{2}$. Note that $H_{-\frac{1}{2}}$ is the 1-dimensional Laplacian with the Neumann boundary condition. However, when we include the additional variable β, a singularity appears. More precisely, we have the identity

$$H_{\beta,-\frac{1}{2}} = H_{\beta,\frac{1}{2}}, \quad \beta \neq 0,$$

and $H_{\beta,m}$ is holomorphic around $(0, \frac{1}{2})$. But $H_{-\frac{1}{2}} \neq H_{\frac{1}{2}}$, because $H_{\frac{1}{2}}$ is the 1-dimensional Laplacian with the Dirichlet boundary condition. The singularity of $H_{\beta,m}$ can be explained by the behavior of the point spectrum.

3. Homogeneous rank one perturbations

Let us rename the operators $H_{\beta,m}$ and H_m^ν of Sect. 1 to $\tilde{H}_{\beta,m}$ and $\tilde{H}_m^\nu$. Following [2, 3] we will describe another closely related family of exactly solvable operators.

Consider the multiplication operator

$$Xf(x) := xf(x)$$

on the Hilbert space $L^2[0,\infty[$. Let $m \in \mathbb{C}$, $\lambda \in \mathbb{C} \cup \{\infty\}$. Consider a formal expression

$$X + \lambda |x^{\frac{m}{2}}\rangle\langle x^{\frac{m}{2}}|. \tag{7}$$

Note that X is homogeneous of degree 1 and the perturbation is formally homogeneous of degree $1+m$. The perturbation is not a well defined operator. However, using the theory of singular rank one perturbation (sometimes called the Aronszajn-Donoghue method) one can interpret (7) as a well defined closed operator $H_{m,\lambda}$ or H_0^ρ.

There exists a close link between the operators of Sect. 1 and those of this section. As proven in [2], the pairs of operators $\tilde{H}_m$, $\tilde{H}_{m,\kappa}$ are similar to H_m, $H_{m,\lambda}$, where

$$\lambda \frac{\pi}{\sin(\pi m)} = \kappa \frac{\Gamma(m)}{\Gamma(-m)}, \tag{8}$$

and $\tilde{H}_0$, $\tilde{H}_0^\nu$ are similar to H_0, H_0^ρ, where $\rho = -2\nu$.

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Symmetry breaking in the reduced Hartree-Fock Dirac model for crystals

David Gontier

(joint work with Mathieu Lewin and Faizan Nazar)

In this talk, we study the reduced Hartree-Fock Dirac (rHFD) model for crystals, which is a non-linear mean field approximation which allows to study systems with an infinity of electrons in a $\mathcal{R}_0$-periodic potential (the crystal). It takes the form of a minimisation problem over the set of one-body density matrices γ which commutes with $\mathcal{R}$-translations, where $\mathcal{R}_0 \subset \mathcal{R}$.

In the first part of the talk, we prove that the model breaks symmetry when the Dirac constant is large enough. In other words, *the minimiser with $\mathcal{R} = 2\mathcal{R}_0$ is energetically favourable compared to the one with $\mathcal{R} = \mathcal{R}_0$*. The proof relies on a rescaling, which exhibits a new minimisation problem over self-adjoint operators γ, of the form

$$J(N) := \inf \left\{ \mathrm{Tr}(-\Delta\gamma) - \frac{1}{p}\int_{\mathbb{R}^d} \rho_\gamma^p, \quad \gamma \in \mathcal{S}(L^2(\mathbb{R}^d)),\ 0 \leqslant \gamma \leqslant 1,\ \mathrm{Tr}(\gamma) = N \right\}.$$

In the second part of the talk, we study whether the problem $J(N)$ has minimisers, for all $d \geqslant 1$ and $1 < p < 1 + d/2$. Such a minimiser is solution to the *fermionic non-linear Schrödinger equation*

$$\gamma = \sum_{i=1}^N |u_i\rangle\langle u_i|, \quad (-\Delta - \rho^{p-1})u_i = \mu_i u_i, \quad \text{with} \quad \rho := \sum_{i=1}^N |u_i|^2.$$

Following usual concentration-compactness arguments (see *e.g.* [Lew11]), we see that the existence of minimisers is closely related to the binding inequality

$$\forall 1 \leqslant K \leqslant N-1, \quad J(N) < J(N-K) + J(K).$$

Unfortunately, we were not able to prove binding inequality for all $1 < p < 1 + d/2$. We were only able to prove it for p sufficiently close to 1 and all $N \in \mathbb{N}$, and for all $1 < p < \min(2, 1 + d/2)$ and an infinite sequence $N_1 < N_2 < \cdots$. Our proof relies on the so called *bubble decomposition*, together with the inequality $J(2N) < 2J(N)$ and the pigeonhole principle.

The aforementioned symmetry breaking in rHFD is a consequence of the binding inequality $J(2N) < 2J(N)$.

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The BEC-Polaron and the Bogoliubov-Fröhlich Hamiltonian

Jonas Lampart

The Bogoliubov-Fröhlich Hamiltonian models the interaction of an impurity particle with the excitations of a Bose-Einstein Condensate. For a heuristic derivation of this model, one

- considers a very dilute Bose gas, taking all interactions to be essentially contact interactions;
- uses the Bogoliubov approximation (see[5, 7]) for the BEC, describing excitations by a free quantum field;
- neglects interactions between the impurity and the excitation that are more than linear in the creation/annihilation operators of excitations.

The (formal) Hamiltonian arrived at by this heuristic is (using the Fourier representation for the field)

$$H_{\text{formal}} = -\Delta_x + d\Gamma(\omega) + a(v_x) + a^*(v_x), \tag{1}$$

where x denotes the position of the impurity, $d\Gamma(\omega)$ is the energy of the free field with dispersion

$$\omega(k) = |k|\sqrt{1+k^2}, \tag{2}$$

and the coupling form-factor for given $x \in \mathbb{R}^3$ is

$$v_x(k) = \frac{1}{(2\pi)^{d/2}} \left(\frac{k^2}{1+k^2}\right)^{1/4} e^{ikx}. \tag{3}$$

This Hamiltonian has the same form as other polaron Hamiltonians, such as Fröhlichs optical polaron. Such models have been studied thoroughly for general dispersion relations ω and interactions v. However, in our case there is a strong ultraviolet (UV) singularity, since the interaction does not decay at all as $k \to \infty$. Due to this there was, until recently, no known way of defining a Hamiltonian corresponding to H_{formal} without UV-cutoff. In the article [2] we solved the UV problem for a model with the same type of singularity using the method of interior-boundary conditions (IBCs). We then applied these ideas to the Bogoliubov-Fröhlich Hamiltonian, reformulating them in the language of renormalisation, in [3].

In the IBC-approach [9], one tries to define the domain of a Hamiltonian, corresponding to the formal expression H_{formal} in some distributional sense, in terms of (singular) boundary conditions. These relate the value of the wavefunction with n bosons (excitations) at a collision configuration $y_j = x$, $j \in \{1, \dots, n\}$ and the wave-function with fewer bosons at the point $(x, y_1, \dots, y_{j-1}, y_{j+1}, \dots y_n) \in \mathbb{R}^{3+3(n-1)}$. Thinking of such boundary conditions is particularly natural in the case of contact interactions, but generalised conditions of this type can also describe the domain of the Fröhlichs polaron and the Nelson model [4].

For the Hamiltonian treated in [2] (with $\omega(k) = k^2 + 1$, $v_x(k) = e^{ikx}$), one thereby obtains a self-adjoint operator $(H_{\text{IBC}}, D_{\text{IBC}})$ on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3) \otimes \mathfrak{F}(L^2(\mathbb{R}^3))$, defined on elements satisfying the asymptotic condition

$$\psi^{(n)}(x, y_1, \dots, y_n) \sim \left(\frac{c_{-1}}{|x-y_n|} + c_0 \log|x-y_n|\right) \psi^{(n-1)}(x, y_1, \dots, y_{n-1}), \tag{4}$$

as $y_n \to x$ (and by, symmetry, similar conditions as $y_j \to x$).

The Hamiltonian $(H_{\text{IBC}}, D_{\text{IBC}})$ can also be arrived at using a renormalisation procedure. This reveals the presence of a logarithmically divergent contribution to the ground state energies of the cut-off Hamiltonians H_Λ, in addition to the divergence proportional to Λ that was expected from (e.g.) the Nelson model. Such a divergence had also been observed in the physics literature [1].

Consider the the cut-off Hamiltonian H_Λ, where interactions are set to zero for $|k| \geqslant \Lambda$ ($H_0 = H_{\Lambda=0}$ is the free Hamiltonian). Let also $\sigma > 0$ and set

$$G_\Lambda = -(H_0 + \sigma)^{-1} a^*(v_{x,\Lambda}). \tag{5}$$

Then we have the simple identity

$$H_\Lambda = (1 - G_\Lambda^*)(H_\Lambda + \sigma)(1 - G_\Lambda) - \sigma - a(v_{x,\Lambda})(H_0 + \sigma)^{-1}a^*(v_{x,\Lambda}). \tag{6}$$

It is not difficult to see that the resolvent of H_0 regularises v sufficiently for G_Λ to make sense also for $\Lambda = \infty$.

Lemma 1. [2, Sect.3] *The operator $G := G_\infty$ is bounded from $\mathcal{H}$ to $D(H_0^s)$ for $0 \leqslant s < 1/4$, and for σ large enough $1 - G$ is invertible on $D(H_0^s)$.*

Expressions such as $a(v_{x,\Lambda})(H_0 + \sigma)^{-1}a^*(v_{x,\Lambda})$ are also known from the theory of point interactions with conserved particle number. Spelling out the action of this operator in Fourier space and for fixed Boson number, n, we find

$$\begin{aligned} &- a(v_{x,\Lambda})(H_0 + \sigma)^{-1}a^*(v_{x,\Lambda})\psi^{(n)}(p, K) \\ &= -\sum_{j=1}^{n+1} \int \frac{v_\Lambda(k_{n+1})v_\Lambda(k_j)}{(p - k_{n+1})^2 + \sum_i \omega(k_i) + \sigma}\psi^{(n)}(p + k_j - k_{n+1}, \hat{K}_j)dk_{n+1}, \end{aligned} \tag{7}$$

where $\hat{K}_j$ is $K = (k_1, \ldots, k_{n+1})$ without the j-th entry. The operator appearing in this sum are of different type for $j = n + 1$ and $j \leqslant n$:

- For $j = n + 1$ the variable of integration is not an argument of $\psi^{(n)}$, so the operator acts as multiplication by the function

$$t_{\sigma,\Lambda}(p, \hat{K}_{n+1}) = -\int \frac{|v_\Lambda(k_{n+1})|^2}{(p - k_{n+1})^2 + \sum_i \omega(k_i) + \sigma}dk_{n+1}. \tag{8}$$

  This function diverges as $\Lambda \to \infty$, but $t_{\sigma,\Lambda}(p, \hat{K}_{n+1}) - t_{0,\Lambda}(0, 0)$ has a limit,

$$\lim_{\Lambda\to\infty} \Big(t_{\sigma,\Lambda}(p, \hat{K}_{n+1}) - t_{0,\Lambda}(0, 0)\Big) =: T_{\mathrm{d}}(p, \hat{K}_{n+1}). \tag{9}$$

  Note that $t_{0,\Lambda}(0, 0) \sim e\Lambda$.
- For $j \neq n + 1$, the summand in (7) acts as an integral operator that is well-defined for $\Lambda = \infty$, on a suitable domain. We denote the sum of these operators for $j = 1, \ldots, n$ by T_{od}.

We set $T := T_{\mathrm{d}} + T_{\mathrm{od}}$. This operator is known as the Skornyakov–Ter-Matirosyan operator in the theory of point interactions. The following lemma is based on ideas from this field [6](see [2, Lem.17] for details on the bound, and [8, Lem.3.6] concerning the convergence).

Lemma 2. *The operator T is bounded from $D(T) = D(H_0^{1/2})$ to $\mathcal{H}$, and*

$$\lim_{\Lambda\to\infty} \big(-a(v_{x,\Lambda})(H_0 + \sigma)^{-1}a^*(v_{x,\Lambda}) - t_{0,\Lambda}(0, 0)\big) = T \tag{10}$$

in the norm of operators from $D(H_0^{1/2+\epsilon})$ to $\mathcal{H}$, for any $\epsilon > 0$.

Given these lemmas, one is tempted to define the renormalised Hamiltonian as

$$(1 - G^*)(H_0 + \sigma)(1 - G) + T - \sigma. \tag{11}$$

This is correct for the Nelson model, for example, but does not work for the more singular Bogoliubov-Fröhlich model. The problem is that the expression has no obvious domain, since the natural domain of the first term is

$$\{\psi \in \mathcal{H} : (1-G)\psi \in D(H_0)\} = (1-G)^{-1}D(H_0) \tag{12}$$

(which is dense because $(1-G)^{-1}$) is continuous), but

$$(1-G)^{-1}D(H_0) \cap D(T) = \{0\}, \tag{13}$$

as one easily checks (cf.[4, Sect.4]).

The solution is to perform an additional transformation, similar to this first one. This will reveal the additional logarithmic divergence and allow us to define the renormalised operator.

Set $T_\Lambda := \big(-a(v_{x,\Lambda})(H_0+\sigma)^{-1}a^*(v_{x,\Lambda}) - t_{0,\Lambda}(0,0)\big)$, and (for σ large enough)

$$\widetilde{G}_\Lambda = -(H_0+T_\Lambda+\sigma)^{-1}a^*(v_{x,\Lambda}). \tag{14}$$

Then we have the identity, similar to (6),

$$H_\Lambda = (1-\widetilde{G}_\Lambda^*)(H_0+T_\Lambda+\sigma)(1-\widetilde{G}_\Lambda) - \sigma - a(v_{x,\Lambda})(H_0+T_\Lambda+\sigma)^{-1}a^*(v_{x,\Lambda}) - T_\Lambda. \tag{15}$$

Now by the resolvent formula and the definition of T_Λ

$$\begin{aligned} &-a(v_{x,\Lambda})(H_0+T_\Lambda+\sigma)^{-1}a^*(v_{x,\Lambda}) - T_\Lambda \\ &\quad = a(v_{x,\Lambda})(H_0+\sigma)^{-1}T_\Lambda(H_0+T_\Lambda+\sigma)^{-1}a^*(v_{x,\Lambda}) + t_{0,\Lambda}(0,0) \\ &\quad = a(v_{x,\Lambda})(H_0+\sigma)^{-1}T_\Lambda(H_0+\sigma)^{-1}a^*(v_{x,\Lambda}) + R_\Lambda + t_{0,\Lambda}(0,0). \end{aligned} \tag{16}$$

From lemmas 1, 2, one easily sees that the remainder R_Λ remains bounded as $\Lambda \to \infty$. The principal term can be treated in a similar way to T_Λ: One spells out the expression, and separates it into integral operators and multiplication operators. The multiplication operators are renormalised by subtracting the value of the multiplier with all arguments set equal to zero. This procedure yields the following:

Lemma 3. *There exists a family s_Λ of real numbers such that*

$$\lim_{\Lambda\to\infty} \big(a(v_{x,\Lambda})(H_0+\sigma)^{-1}T_\Lambda(H_0+T_\Lambda+\sigma)^{-1}a^*(v_{x,\Lambda}) - s_\Lambda\big) =: S \tag{17}$$

in the sense of operators from $D(H_0^\epsilon)$ to $\mathcal{H}$, for any $\epsilon > 0$.

By perturbation theory one can also see that $\widetilde{G}$ has the same mapping properties as G. It thus makes sense to set (with $\widetilde{G} := \widetilde{G}_\infty$)

$$H_{\mathrm{ren}} := (1-\widetilde{G}^*)(H_0+T+\sigma)(1-\widetilde{G}) - \sigma + S \tag{18}$$

on the domain

$$D(H_{\mathrm{ren}}) = (1-\widetilde{G})^{-1}D(H_0) \subset D(H_0^s) \tag{19}$$

(for $s < 1/4$).

The main result of [2], [3] follows essentially from these lemmas. It is:

Theorem 1. *The operator H_{ren} is self-adjoint on $D(H_{\mathrm{ren}})$ and bounded from below. For $E_\Lambda := t_{0,\Lambda}(0,0) + s_\Lambda$ we have convergence of $H_\Lambda - E_\Lambda$ to H_{ren} in norm-resolvent sense. Moreover, $H_{\mathrm{ren}} = H_{\mathrm{IBC}} +$ const., where the latter is defined in* [2].

We conclude with the remark that the convergence shows, in particular, that H_{ren} is independent of σ. The numbers $E_\Lambda = e_1\Lambda + e_2 \log \Lambda + O(1)$ are explicit and the expression for H_{ren} obtained by this procedure is also rather explicit. It is easy to check whether a a given $\psi \in \mathcal{H}$ is an element of $D(H_{\mathrm{ren}})$, since the condition $(1-G)\psi \in D(H_0)$ can be checked sector-wise, and on the n-boson sector depends on $\psi^{(n)}$ and $\psi^{(n-1)}$ only (see also [4, Sect.4]).

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Quasi-Classical Dynamics

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(joint work with Marco Falconi and Marco Olivieri)

We study the dynamical properties of a coupled quantum system composed of a *small subsystem* given, *e.g.*, by non-relativistic trapped particles, interacting with a *semiclassical environment*, which may be described by the action of a quantized bosonic field. We name such a regime *quasi-classical* [8, 9, 10] (see also [1, 2, 11, 14] for similar settings) to stress that only the environment is semiclassical, while the small subsystem remains quantum.

A typical realistic setting realizing the quasi-classical regime is the one of a very *intense* field (for instance a laser or a strong magnetic field), *i.e.*, when the state of the field contains a large number of carriers or field's excitations, so that it can be considered *macroscopic*: if we denote by N the number of particles in the

small subsystem, which is assumed to be constant, and by $\mathrm{d}\mathcal{G}(1) = \int \mathrm{d}k\, a^\dagger(k)a(k)$ the field's number operator, the state of the field Ψ gives rise to a macroscopic behavior, if $\langle \Psi \,|\mathrm{d}\mathcal{G}(1)|\, \Psi\rangle \gg N \sim 1$. It is then mathematically convenient to introduce a semiclassical parameter ε, such that $1/\varepsilon$ is proportional to the average number of field's excitations, and therefore, when $\varepsilon \to 0$, the system becomes quasi-classical. We thus set $\langle \Psi_\varepsilon \,|\mathrm{d}\mathcal{G}(1)|\, \Psi_\varepsilon\rangle \sim 1/\varepsilon$, which can be suitably rewritten as

$$\langle \Psi_\varepsilon \,|\varepsilon \mathrm{d}\mathcal{G}(1)|\, \Psi_\varepsilon\rangle = \int \mathrm{d}\mathbf{k}\, \left\langle \Psi_\varepsilon \left| a_\varepsilon^\dagger(\mathbf{k}) a_\varepsilon(\mathbf{k}) \right| \Psi_\varepsilon \right\rangle \leqslant C\,, \tag{1}$$

where the rescaled creation and annihilation operators $a_\varepsilon^\#(\,\cdot\,) := \sqrt{\varepsilon} a^\#(\,\cdot\,)$ satisfy the following canonical commutation relations:

$$\left[a_\varepsilon(\mathbf{k}), a_\varepsilon^*(\mathbf{k}')\right] = \varepsilon\delta(\mathbf{k}-\mathbf{k}')\,. \tag{2}$$

Hence, when $\varepsilon \to 0$, the environment becomes classical while the particle subsystem retains its quantum features. Due to the coupling between the two subsystems, one expects to obtain in the limit a quantum system driven by a classical environment, which is in fact the effective model emerging from next Theorem 1. We anticipate that the proof of such a result relies on an adaptation to the quasi-classical setting of the framework of infinite dimensional semiclassical analysis, originally introduced in [3, 4, 5, 6], and further discussed in [12, 13].

Let us now specify in more detail the microscopic model we plan to study, which is the *Nelson model* with ultraviolet cut-off: the full system is described by the Hilbert space $\mathscr{H}\otimes\mathscr{K}_\varepsilon$, where $\mathscr{H} = L^2(\mathbb{R}^{dN})$ takes into account the particle degrees of freedom, and $\mathscr{K}_\varepsilon = \mathcal{G}_\varepsilon(\mathfrak{h}) = \bigoplus_{n=0}^\infty \mathfrak{h}^{\otimes_{\mathrm{s}} n}$ is the symmetric Fock space of the field, with one-excitation space $\mathfrak{h}$ and with ε-dependent canonical commutations relations $\left[a_\varepsilon(z), a_\varepsilon^\dagger(w)\right] = \varepsilon \langle z|w\rangle_{\mathfrak{h}}$.

The full Hamiltonian H_ε then reads

$$H_\varepsilon = K_0 + \nu(\varepsilon)\mathrm{d}\mathcal{G}_\varepsilon(\omega) + \sum_{j=1}^N \left[a_\varepsilon^\dagger\big(\lambda(\mathbf{x}_j)\big) + a_\varepsilon\big(\lambda(\mathbf{x}_j)\big)\right]\,, \tag{3}$$

where $K_0 := \mathcal{K}_0 \otimes 1$ and $\mathcal{K}_0 = -\Delta + U(\mathbf{x}_1, \ldots, \mathbf{x}_N)$, with U positive and trapping, *i.e.*, such that[1] $(-\Delta + U + \lambda)^{-1} \in \mathscr{L}^\infty(L^2(\mathbb{R}^{Nd}))$, for any $\lambda > 0$. The function $\nu(\varepsilon)$ is ideally varying between 1 and $1/\varepsilon$, so that

$$\lim_{\varepsilon\to 0} \varepsilon\nu(\varepsilon) = \nu \in \{0, 1\}\,, \tag{4}$$

which is going to determine the classical dynamics of the environment in the limit (see (7)). Finally, the coupling factor is $\lambda(\mathbf{x};\mathbf{k}) = \lambda_0(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}$, with $\lambda_0 \in L^2(\mathbb{R}^d)$, independent of ε (ultraviolet cut-off).

[1] We denote by $\mathscr{L}^k(\mathscr{K})$ and $\mathscr{B}(\mathscr{K})$ the Schatten ideal of order k and the space of bounded operators over the Hilbert space $\mathscr{K}$.

The quasi-classical counterpart of H_ε is naively obtained by replacing the operators with the corresponding classical symbols, which leads to the effective potential

$$\mathcal{V}(z) = \sum_{j=1}^{N} 2\Re \langle z|\lambda(\mathbf{x}_j)\rangle_{\mathfrak{h}} \in \mathscr{B}(\mathscr{H}) \ , \tag{5}$$

depending on the classical state of the field $z \in \mathfrak{h}$.

A state of the microscopic system is a normalized density matrix Γ_ε over $\mathscr{H} \otimes \mathscr{K}_\varepsilon$. Its classical counterpart is a *state-valued measure* $\mathfrak{m} \in \mathscr{M}\left(\mathfrak{h}; \mathscr{L}^1_+(\mathscr{H})\right)$, *i.e.* a measure taking values in trace class positive operators $\mathscr{L}^1_+(\mathscr{H})$. Each state-valued measure $\mathfrak{m}$ can be conveniently described by its norm Radon-Nikodým decomposition $\mathrm{d}\mathfrak{m}(z) = \gamma_{\mathfrak{m}}(z)\mathrm{d}\mu_{\mathfrak{m}}(z)$, where $\mu_{\mathfrak{m}}$ is a scalar Borel measure and $\gamma_{\mathfrak{m}}(z) \in \mathscr{L}^1_+(\mathscr{H})$ is a $\mu_{\mathfrak{m}}$-integrable, a.e. defined function taking values in normalized density matrices.

In order to establish a notion of convergence between quantum states and their classical limits, we define the Fourier transform of a state $\Gamma_\varepsilon \in \mathscr{L}^1(\mathscr{H} \otimes \mathscr{K}_\varepsilon)$ as $\widehat{\Gamma}_\varepsilon(\eta) := \mathrm{tr}_{\mathscr{K}_\varepsilon}\left(\Gamma_\varepsilon W_\varepsilon(\eta)\right) \in \mathscr{L}^1(\mathscr{H})$, for any $\eta \in \mathfrak{h}$, where $W_\varepsilon(\eta)$ is the Weyl operator $W_\varepsilon(\eta) := e^{i(a^\dagger_\varepsilon(\eta)+a_\varepsilon(\eta))}$. Analogously, the Fourier transform of a state-valued measure $\mathfrak{m} \in \mathscr{M}(\mathfrak{h}; \mathscr{L}^1_+(\mathscr{H}))$ is

$$\widehat{\mathfrak{m}}(\eta) := \int_{\mathfrak{h}} \mathrm{d}\mu_{\mathfrak{m}}(z)\, \gamma_{\mathfrak{m}}(z) e^{2i\Re\langle \eta|z\rangle_{\mathfrak{h}}} \in \mathscr{L}^1(\mathscr{H}) \ .$$

Definition 1 (Quasi-classical convergence)**.**
Let $\Gamma_\varepsilon \in \mathscr{L}^1_{+,1}(\mathscr{H} \otimes \mathscr{K}_\varepsilon)$ and $\mathfrak{m} \in \mathscr{M}\left(\mathfrak{h}; \mathscr{L}^1_+(\mathscr{H})\right)$. We say that

$$\Gamma_\varepsilon \xrightarrow[\varepsilon\to 0]{} \mathfrak{m}, \tag{6}$$

if and only if $\widehat{\Gamma}_\varepsilon(\eta) \to \widehat{\mathfrak{m}}(\eta)$ pointwise for all $\eta \in \mathfrak{h}$ in weak-$*$ topology in $\mathscr{L}^1(\mathscr{H})$.

The main question addressed in [10] is the quasi-classical convergence of the evolved state $\Gamma_\varepsilon(t) := e^{-iH_\varepsilon t}\Gamma_\varepsilon e^{iH_\varepsilon t}$, or, equivalently, the derivation of an effective dynamics for the small subsystem in the limit $\varepsilon \to 0$. In order to study such a limit, we make the following very general assumption on the initial datum $\Gamma_\varepsilon(0) = \Gamma_\varepsilon$:

$$\exists \delta > 0, \exists C_\delta < +\infty \quad \text{s.t.} \quad \mathrm{Tr}\left(\Gamma_\varepsilon (K_0 + \mathrm{d}\mathcal{G}_\varepsilon(1) + 1)^\delta\right) \leqslant C_\delta \ , \tag{A}$$

which is compatible with the interpretation (1).

The main result in [10] is that, for all $t \in \mathbb{R}$, $\Gamma_{\varepsilon_n}(t)$ also converges to the quasi-classical state $\mathfrak{m}_t$ defined by the norm Radon-Nikodým decomposition

$$\mathrm{d}\mathfrak{m}_t = \mathcal{U}_{t,0}(z)\gamma_{\mathfrak{m}}(z)\mathcal{U}^\dagger_{t,0}(z)\, \mathrm{d}\left(e^{-it\nu\omega} \sharp \mu_{\mathfrak{m}}\right) \ , \tag{7}$$

where $\mathfrak{m}$ is the classical state at time $t = 0$ and $\mathcal{U}_{t,s}(z)$ is the two-parameter unitary group, which is weakly generated by the time-dependent Schrödinger operator $\mathcal{K}_t := \mathcal{K}_0 + \mathcal{V}(e^{-it\nu\omega}z)$. Finally, $e^{-it\nu\omega} \sharp \mu_{\mathfrak{m}}$ is the push-forward of the measure $\mu_{\mathfrak{m}}$ by the flow $e^{-it\nu\omega}$, which yields a free evolution for $\nu = 1$ and no dynamics at all for $\nu = 0$. We recall that $\left(e^{-it\nu\omega} \sharp \mu_{\mathfrak{m}}\right)(S) = \mu_{\mathfrak{m}}\left(e^{it\nu\omega}S\right)$, for any measurable S, and the push-forward does not affect the Radon-Nikodým derivative.

Theorem 1 (Quasi-classical evolution in the Schrödinger picture)**.**
Let $\Gamma_\varepsilon \in \mathscr{L}^1_+(\mathscr{H} \otimes \mathscr{K}_\varepsilon)$ be a normalized state satisfying (A)*. Then, there exist at least one subsequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ and one probability measure $\mathfrak{m} \in \mathscr{M}\left(\mathfrak{h}; \mathscr{L}^1_+(\mathscr{H})\right)$, such that $\|\mathfrak{m}(\mathfrak{h})\|_{\mathscr{L}^1(\mathscr{H})} = 1$ and $\Gamma_{\varepsilon_n} \xrightarrow[n\to+\infty]{} \mathfrak{m}$. Moreover, for all $t \in \mathbb{R}$,*

$$\Gamma_{\varepsilon_n}(t) \xrightarrow[n\to+\infty]{} \mathfrak{m}_t \ , \tag{8}$$

where $\mathfrak{m}_t$ is given by (7)*.*

We can pictorially represent the content of Theorem 1 above in the following commutative diagram

$$\begin{array}{ccc} \Gamma_\varepsilon & \xmapsto{\quad e^{-iH_\varepsilon t} \quad} & \Gamma_\varepsilon(t) \\ \Big\downarrow{\scriptstyle \varepsilon\to 0} & & \Big\downarrow{\scriptstyle \varepsilon\to 0} \\ \gamma_{\mathfrak{m}}(z)\,\mathrm{d}\mu_{\mathfrak{m}}(z) & \xmapsto{\quad \text{effective dynamics} \quad} & \mathcal{U}_{t,0}(z)\gamma_{\mathfrak{m}}(z)\mathcal{U}^\dagger_{t,0}(z)\,\mathrm{d}\left(e^{-it\nu\omega} \sharp\, \mu_{\mathfrak{m}}\right) . \end{array} \tag{9}$$

where we have decomposed the initial state-valued measure as $\mathrm{d}\mathfrak{m}(z) = \gamma_{\mathfrak{m}}(z)\mathrm{d}\mu_{\mathfrak{m}}(z)$ and the convergence is always along a given subsequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$.

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Eigenvalues and eigenvectors of critical Erdős-Reńyi graphs

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(joint work with Johannes Alt, Raphaël Ducatez)

An (undirected simple) graph on N vertices can be characterized by its adjacency matrix $A = A^* = (A_{xy})_{x,y=1}^N \in \{0,1\}^{N\times N}$, where A_{xy} is 1 if x and y are adjacent and 0 otherwise. The eigenvalues and eigenvectors of A play a central role in spectral graph theory. By making A random, we obtain a natural construction of sparse random matrices. Such a random operator can also be interpreted as a random Hamiltonian, describing a particle whose hopping is restricted to edges of a sparse graph.

Arguably the simplest model of a random graph is the *Erdős-Rényi graph* $G(N,p)$, where each edge $\{x,y\}$ of the complete graph is kept with probability p independently of the others. This means that the upper-triangular entries $(A_{xy} : x \leqslant y)$ are independent Bernoulli(p) random variables. Here the parameter $p \equiv p_N$ can depend on N, and the graph is sparse if $p_N \to 0$ as $N \to \infty$. Despite its simple definition, the eigenvalues and eigenvectors of the adjacency matrix of $G(N,p)$ exhibit very complex behaviour, depending on the scale of the parameter p. To discuss this behaviour in more detail, we parametrize $p = d/N$.

Let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_N$ denote the eigenvalues of the (conveniently normalized) adjacency matrix $d^{-1/2}A$. The most basic question about the distribution of the eigenvalues is the convergence of the empirical eigenvalue measure $\frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i}(\mathrm{x})$, which, with high probability, converges in distribution to Wigner's semicircle law $\varrho(x)\mathrm{x}$ with $\varrho(x) = \frac{1}{2\pi}\sqrt{(4-x^2)_+}$ if and only if $d \to \infty$ as $N \to \infty$. Note that this statement says nothing about the fluctuations or even locations of individual eigenvalues.

The expected behaviour of the eigenvalues and eigenvectors changes dramatically around the *critical scale* $d \asymp \log N$. Informally, the critical scale corresponds to the threshold below which the concentration of the degree sequence $D_1, \ldots, D_N$ fails, where $D_x = \sum_y A_{xy}$ is the degree of vertex x. Thus, for[1] $d \gg \log N$ all degrees are with high probability close to d, while for $d \ll \log N$ the degrees differ wildly. A famous manifestation of this phenomenon is the connectivity threshold for $G(N, d/N)$: for $d/\log N > 1$ the graph is with high probability connected, and for $d/\log N$ the graph has with high probability isolated vertices. It is well known that as long as $d > 1$, the graph has a unique giant component (of size of order N), while all other components are of order $O(\log N)$. Throughout the following, we only consider the giant component, discarding the smaller components.

In the recent works [1, 2], we analyse the eigenvalues and eigenvectors of $G(N, d/N)$ on the critical scale. We uncover a phase diagram consisting of localized and delocalized phases, which is analogous to the well-known conjectured phase diagram of the three-dimensional Anderson model at weak disorder [4]. See

[1]Here $a \ll b$ or $b \gg a$ means that $a = o(b)$.

Figure 1 for a depiction of this diagram. It summarizes the content of the following theorem, proved in [1, 2] and [3]. For its statement, we introduce the function $\Lambda : [2, \infty) \to [2, \infty)$ defined by $\Lambda(\alpha) = \frac{\alpha}{\sqrt{\alpha - 1}}$.

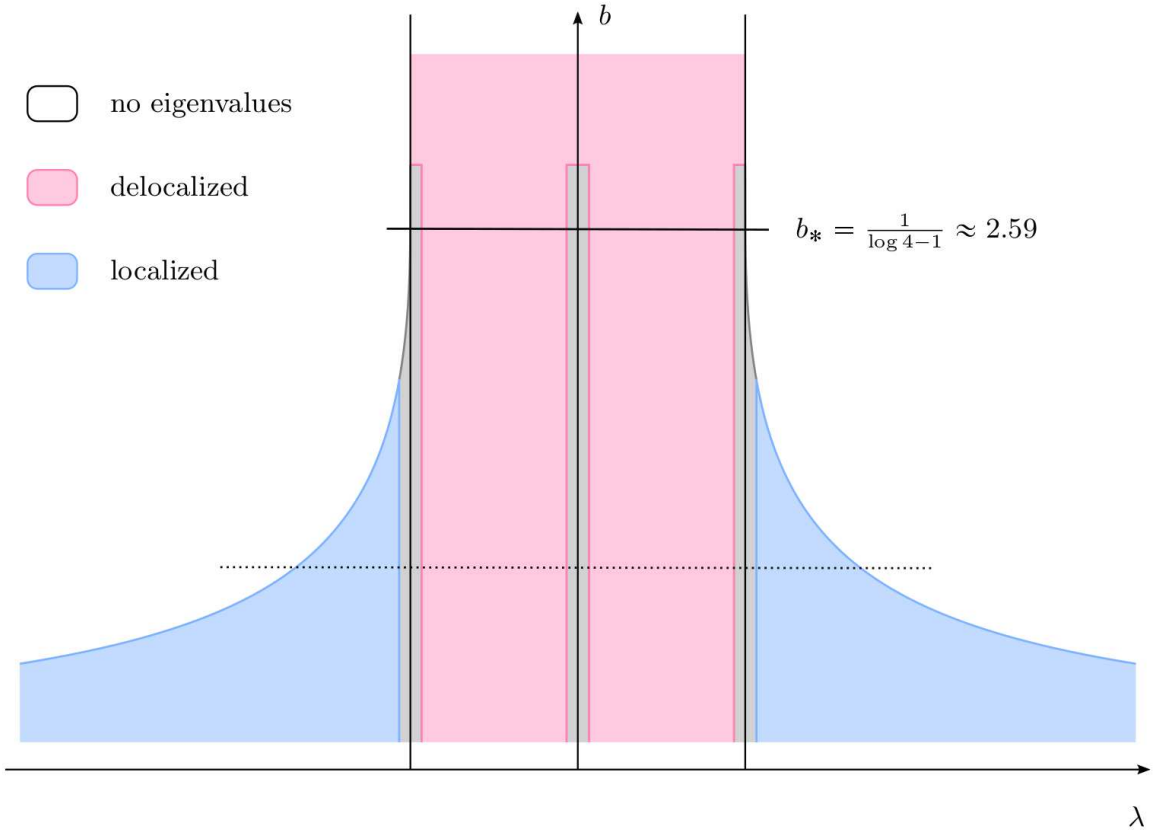

FIGURE 1. The phase diagram of the graph $G(N, d/N)$ on the critical scale $d \asymp \log N$. We parametrize $d = b \log N$, and plot the diagram in the (λ, b)-plane, where λ is the location in the spectrum. The coloured regions are where the spectrum is located with high probability. In the red region, we prove complete eigenvector delocalization. In the blue region, we prove eigenvector localization. The grey regions, which have width $o(1)$ around the points $\{0, \pm 2\}$, have so far not been analysed. For $b \geqslant b_* = \frac{1}{\log 4 - 1}$, the spectrum converges with high probability to $[-2, 2]$. For $b < b_*$, the spectrum converges with high probability to $[-f(b), f(b)]$ where $f(b) > 2$ is a decreasing function of b.

Theorem 1. *Suppose that $\sqrt{\log N} \ll d \leqslant N/2$. Denote by $\alpha_x = D_x/d$ the normalized degree of x. Then for any $\nu > 0$ the following statements hold with probability at least $1 - O_\nu(N^{-\nu})$.*

(1) ***Eigenvalue locations.*** *Let $\sigma \in S_N$ satisfy $\alpha_{\sigma(1)} \geqslant \alpha_{\sigma(2)} \geqslant \cdots \geqslant \alpha_{\sigma(N)}$. Let $L = \max\{l \geqslant 1 : \alpha_{\sigma(l)} \geqslant 2 + o(1)\}$ denote the expected number of*

eigenvalues larger than $2 + o(1)$. *Then for* $1 \leqslant l \leqslant L$ *we have* $|\lambda_{l+1} - \Lambda(\alpha_{\sigma(l)}) \leqslant o(1)$ *and* $|\lambda_{L+2} - 2| = o(1)$.

(2) ***Eigenvector delocalization.*** *The eigenvector* $\underline{u}_i$ *is completely delocalized (i.e.* $\|\underline{u}_i\|_\infty^2 \leqslant N^{-1+o(1)}$*) if* $d \geqslant C \log N$ *for some constant* C *or if the associated eigenvalue* λ_i *satisfies* $o(1) \leqslant |\lambda_i| \leqslant 2 - o(1)$.

(3) ***Eigenvector localization.*** *Let* $\lambda \geqslant 2 + o(1)$ *be an eigenvalue with eigenvector* $\mathbf{u} \in \mathbb{S}^{N-1}$. *Define the set of* vertices in resonance with λ,

$$\mathcal{W}(\lambda) = \left\{x : \alpha_x \geqslant 2, |\Lambda(\alpha_x) - \lambda| = o(1)\right\}.$$

For $r \geqslant 1$ *define the set of resonant balls* $\mathcal{B}_r(\lambda) = \bigcup_{x \in \mathcal{W}(\lambda)} B_r(x)$. *Then for* $r \gg 1$ *we have* $\sum_{x \notin \mathcal{B}_r(\lambda)} u(x)^2 = o(1)$.

(All factors $o(1)$ *are in fact quantitative functions of* d.*)*

This result indicates that at criticality $G(N, d/N)$ provides a model of a random operator that exhibits analogous characteristics to the Anderson model in three dimensions. However, it seems to be more amenable to a rigorous analysis, informally because it tends to be more delocalized. This means not only that we can rigorously establish the existence of a delocalized phase, but also that in the localized phase the eigenvectors are localized around not one localization centre (as in Anderson localization), but around many (the resonant vertices $\mathcal{W}(\lambda)$). Heuristically, this difference may be understood by noting that $G(N, d/N)$ is much better connected than $\mathbb{Z}^3/N\mathbb{Z}^d$, with a diameter logarithmic in N instead of polynomial.

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