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Homology and *K*-Theory of Torsion-Free Ample Groupoids and Smale Spaces

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HOMOLOGY AND K-THEORY OF TORSION-FREE AMPLE GROUPOIDS AND SMALE SPACES

VALERIO PROIETTI AND MAKOTO YAMASHITA

Abstract. Given an ample groupoid, we construct a spectral sequence with groupoid homology with integer coefficients on the second sheet, converging to the K-groups of the groupoid C*-algebra when the groupoid has torsion-free stabilizers and satisfies the strong Baum–Connes conjecture. The construction is based on the triangulated category approach to the Baum–Connes conjecture by Meyer and Nest. For the unstable equivalence relation of a Smale space with totally disconnected stable sets, this spectral sequence shows Putnam's homology groups on the second sheet.

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Introduction

In this paper, we look at the K-theory of ample Hausdorff groupoids, that is, étale groupoids on totally disconnected spaces, and its relation to groupoid homology. Such groupoids are closely related to dynamical systems on Cantor sets, such as (sub)shifts of finite type (also called topological Markov shifts) in symbolic dynamics. While this remains a fundamental example, the second half of the last century saw a rapid development of the theory which resulted in several generalizations involving various geometric, combinatorial, and functional analytic structures.

One prominent example is the framework of Smale spaces introduced by Ruelle [Rue04], who designed them to model the basic sets of Axiom A diffeomorphisms [Sma67]. This turned out to be a particularly nice class of hyperbolic dynamical systems, where Markov partitions provide a symbolic approximation of the dynamics. Examples of Smale spaces include hyperbolic toral automorphisms, and more generally Anosov diffeomorphisms, see [Bow08] and references therein.

Ample groupoids arise from Smale spaces with totally disconnected stable sets. This is especially useful in the study of dynamical systems whose topological dimension is not zero, but whose dynamics is completely captured by restricting to a totally disconnected transversal. Such spaces include generalized solenoids [Tho10b, Wil74] and substitution tiling spaces [AP98, Theorem 3.3], and can be characterized as certain inverse limits [Wie14].

Beyond the theory of dynamical systems, these groupoids also play an important role in the theory of operator algebras, where they provide an invaluable source of examples of C*-algebras. These are obtained by considering the (reduced) groupoid C*-algebras [Ren80], generalizing the crossed product algebras for group actions on the Cantor set. The resulting C*-algebras capture interesting aspects of the homoclinic and heteroclinic structure of expansive dynamics [Mat19, Put96, Tho10a], extending the correspondence between topological Markov shifts and the Cuntz–Krieger algebras.

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Another important class of Cantor systems comes from minimal homeomorphisms of the Cantor set. This study was initiated by Giordano, Putnam and Skau [GPS95], in which they classified minimal homeomorphisms up to orbit equivalence. Actions of $\mathbb{Z}^k$ on the Cantor set, which are higher rank analogues, also naturally appear from tiling spaces. More generally, essentially free ample groupoids appear in the study of actions of $\mathbb{N}^k$ by local homeomorphisms on zero-dimensional spaces, where they are known as Deaconu–Renault groupoids [Dea95, ER07]. This is a convenient framework to understand higher-rank graph C*-algebras. The étale groupoids, and related invariants such as topological full groups, of such systems proved to be a rich source of interesting examples in the structure theory of discrete groups and operator algebras, see for example [JM13, Mat13, Phi05].

The K-groups of groupoid C*-algebras and groupoid cohomology with integer coefficients are known to have close parallels, for example in various cohomological invariants of tiling spaces. In fact, groupoid homology [CM00] has even closer properties to K-groups, and the comparison of these invariants (for topologically free, minimal, and ample Hausdorff groupoids) was recently popularized by Matui [Mat12]. While his conjectural isomorphism in its original form ("HK conjecture") has counterexamples [Sca19], in situations where one expects low homological dimension we do have an isomorphism, see for example [FKPS19, Ort18].

Our main result gives a correspondence between groupoid homology and K-groups for reduced crossed products by torsion-free ample groupoids satisfying the strong Baum–Connes conjecture [Tu99a], as follows.

Theorem A (Theorem 4.6). *Let G be an ample groupoid with torsion-free stabilizers satisfying the strong Baum–Connes conjecture, and A be a separable G-C^*-algebra which is KK^X-nuclear for $X = G^{(0)}$. Then there is a convergent spectral sequence*

$$E^2_{pq} = H_p(G, K_q(A)) \Rightarrow K_{p+q}(G \ltimes A).$$

In particular, for $A = C_0(X)$ we obtain a spectral sequence with $E^2_{pq} = E^3_{pq} = H_p(G, K_q(\mathbb{C}))$ converging to $K_{p+q}(C^*_r G)$. Similarly to discrete groups, amenable groupoids satisfy the (strong) Baum–Connes conjecture, which cover most of our concrete examples in this paper.

Note that, for groupoids with low homological dimension, this spectral sequence degenerates for degree reasons. Moreover the top-degree group in groupoid homology tends to be torsion-free, so that there are no extension problems, thus leading to the positive cases where the HK conjecture holds.

Turning to Smale spaces, there is another homology theory proposed by Putnam [Put14] We show that one of the variants, $H^s_\bullet$, fits into this scheme for the groupoid $R^u(Y, \psi)$ of the unstable equivalence relation on the underlying space, as follows.

Theorem B (Theorem 4.12). *Let (Y, ψ) be an irreducible Smale space with totally disconnected stable sets, and $R^u(Y, \psi)$ be the groupoid of the unstable equivalence relation. Then there is a convergent spectral sequence*

$$E^2_{pq} = E^3_{pq} = H^s_p(Y, \psi) \otimes K_q(\mathbb{C}) \Rightarrow K_{p+q}(C^*(R^u(Y, \psi))).$$

This result gives a partial answer to a question raised by Putnam [Put14, Section 8.4.1]. An immediate consequence is that the K-groups of $C^*(R^u(Y, \psi))$ are of finite rank.

Although we give an independent proof of Theorem B, it can also be obtained from the combination of Theorem A and the result below.

Theorem C (Theorem 4.15). *For any étale groupoid G that is Morita equivalent to $R^u(Y, \psi)$, we have an isomorphism $H^s_p(Y, \psi) \simeq H_p(G, \mathbb{Z})$.*

In order to prove the result above, we turn the definition of Putnam's homology into a resolution of modules which computes groupoid homology. As a corollary we obtain a Künneth formula for $H^s_\bullet$, generalizing a result in [DKW16].

Our proofs of Theorem A and B above are based on the triangulated category approach to the Baum–Connes conjecture by Meyer and Nest [Mey08, MN06, MN10]. Building on their theory of projective resolutions and complementary subcategories from homological ideals, we show that an explicit projective resolution can be obtained from adjoint functors and associated simplicial objects. Applying this to the restriction functor $\mathrm{KK}^G \to \mathrm{KK}^X$ and induction functor $\mathrm{KK}^X \to \mathrm{KK}^G$ for $X = G^{(0)}$ gives the standard bar complex computing the groupoid homology. Then, the spectral sequence in Theorem A appears as a particular case of the "ABC spectral sequence" of [Mey08].

This paper is organized as follows. In Section 1 we lay out the basic notation and definitions for all the background objects of the paper.

In Section 2, we discuss the multiple fibered product of groupoid homomorphisms, generalizing a construction in [CM00], which provides the spatial implementation of the groupoid bar complex in the case of the inclusion map $G^{(0)} \to G$ regarded as a groupoid homomorphism. For Smale spaces, we look at an s-bijective map $f\colon (\Sigma, \sigma) \to (Y, \psi)$ from a shift of finite type, which underlies Putnam's homology through multiple fiber products for f. A key technical result is a transversality result in Proposition 2.9, which allows us to relate the multiple fiber products of f to the multiple groupoid fibered products.

In Section 3, we look at a simplicial object arising from adjoint functors and relate it to the categorical approach to the Baum–Connes conjecture. In a triangulated category, a homological ideal with enough projectives, and a pair of complementary subcategories, appear from adjunction of functors [Mey08]. Our observation is that the canonical comonad construction from homological algebra gives a concrete model of projective resolution. We then use this to show that, when G is an étale groupoid satisfying the strong Baum–Connes conjecture, any G-C*-algebra A which is KK^X-nuclear as a $C_0(X)$-algebra belongs to the triangulated subcategory of KK^G generated by the image of the induction functor $\mathrm{KK}^X \to \mathrm{KK}^G$ for $X = G^{(0)}$.

We then combine these results in Section 4 to obtain our main results mentioned above. Now, let us summarize the ingredients which go into the correspondence between groupoid homology and K-theory. By the adjunction of the functors $\mathrm{Ind}_X^G\colon \mathrm{KK}^X \to \mathrm{KK}^G$ and $\mathrm{Res}_X^G\colon \mathrm{KK}^G \to \mathrm{KK}^X$, for any G-C*-algebra A we have an exact triangle in KK^G,

$$P \to A \to N \to \Sigma P,$$

with $\mathrm{Res}_X^G N \simeq 0$ and P being orthogonal to all such N. If G has torsion-free stabilizers and satisfies the strong Baum–Connes conjecture and A is KK^X-nuclear, we actually have $P \simeq A$ in KK^G. In addition, for any homological functor F, we have a spectral sequence from the Moore complex of the simplicial object $(F(L^{n+1}A))_{n=0}^\infty$ with $L = \mathrm{Ind}_X^G \mathrm{Res}_X^G$, converging to $F(P)$.

For an ample groupoid G, with the functor $F = K_\bullet(G \ltimes \text{-})$, this complex is isomorphic to the bar complex computing the groupoid homology of G with coefficient in $K_\bullet(A)$. For the groupoid of the unstable equivalence relation on a Smale space (Y, ψ) with totally disconnected stable sets, we follow the same scheme, but replace X by the subgroupoid coming from an s-bijective factor map from a shift of finite type. The resulting complex is isomorphic to the one defining Putnam's homology $H^s_\bullet(Y, \psi)$.

Finally, in Section 5 we discuss some examples. We also compare our construction with the counterexample to the HK conjecture from [Sca19].

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1. Preliminaries

In this section we recall the most important objects and notions at the basis of this paper. We will deal with C*-algebras endowed with a groupoid action, and will consider these as objects of the equivariant Kasparov category. In addition, we will introduce a special class of topological dynamical systems, called Smale spaces, which will be a key example to which we apply our results.

1.1. **Groupoids and Morita equivalence.** Let G be a groupoid with base space $X = G^{(0)}$. We let $s, r\colon G \to X$ denote respectively the source and range maps. In addition, we let $G_x = s^{-1}(x)$, $G^x = r^{-1}(x)$, and for a subset $A \subset X$, we write $G_A = \bigcup_{x\in A} G_x$, $G^A = \bigcup_{x\in A} G^x$, and $G|_A = G^A \cap G_A$.

Definition 1.1. A topological groupoid G is *étale* if s and r are local homeomorphisms, and *ample* if it is étale and $G^{(0)}$ is totally disconnected.

If G is étale and $g \in G$, then by definition, for small enough neighborhoods U of $s(g)$ there is a neighborhood U' of g such that $s(U') = U$, and the restriction of s and r to U' are homeomorphisms onto the images. When this is the case, we write $g(U) = r(U')$ and use g as a label for the map $U \to g(U)$ induced by the identification of $U \sim U' \sim g(U)$.

Throughout the paper we assume that a topological groupoid is second countable, locally compact Hausdorff, and admits a Haar system, so that its (reduced) groupoid C*-algebra makes sense. (We always mean continuous Haar systems in this paper.) In particular, G and X are σ-compact and paracompact. Recall that the condition on Haar system is automatic for étale groupoids, as we can take the counting measure on G^x.

A locally compact groupoid is *amenable* if there is a net of probability measures on G^x for $x \in G^{(0)}$ which is approximately invariant, see [ADR00]. In this case, the full and reduced C*-norms are equal, and the completion of the compactly supported functions in the regular representation is $*$-isomorphic to the full groupoid C*-algebra.

The notion of Morita equivalence of groupoids in the sense of [MRW87] plays an important role in this paper. We review it here below for convenience. First, recall a topological groupoid G is *proper* if the map $(r \times s)\colon G \to X \times X$ is proper. Furthermore, if Z is a locally compact, Hausdorff G-space, we say that G *acts properly* on Z if the transformation groupoid $G \ltimes Z$ is proper. The map $Z \to G^{(0)}$ underlying the G-action is called the *anchor map*.

Definition 1.2. The groupoids G and H are *Morita equivalent* if there is a locally compact Hausdorff space Z such that

- Z is a free and proper left G-space with anchor map $\rho\colon Z \to G^{(0)}$;
- Z is a free and proper right H-space with anchor map $\sigma\colon Z \to H^{(0)}$;
- the actions of G and H on Z commute;
- $\rho\colon Z \to G^{(0)}$ induces a homeomorphism $Z/H \to G^{(0)}$;
- $\sigma\colon Z \to H^{(0)}$ induces a homeomorphism $G\backslash Z \to H^{(0)}$.

This can be conveniently packaged by a *bibundle* over G and H: that is, a topological space Z with G and H acting continuously from both sides with surjective and open anchor maps, such that that the maps

$$G \times_{G^{(0)}} Z \to Z \times_{H^{(0)}} Z, \quad (g, z) \mapsto (gz, z), \qquad Z \times_{H^{(0)}} H \to Z \times_{G^{(0)}} Z, \quad (z, h) \mapsto (z, zh)$$

are homeomorphisms.

An important class of Morita equivalences comes from *generalized transversals* [PS99]. For a topological space X and $x \in X$, let us denote the family of the open neighborhoods of x by $\mathcal{O}(x)$.

Definition 1.3. Let G be a topological groupoid. A *generalized transversal* for G is given by a topological space T and an injective continuous map $f\colon T \to G^{(0)}$ such that:

- $f(T)$ meets every orbit of G; and
- the *condition Ar* for neighborhoods of $x \in G$ and $f^{-1}(rx)$, i.e.,

$$\forall x \in G^{f(T)}, U_0 \in \mathcal{O}(x), V_0 \in \mathcal{O}(f^{-1}(rx)) \quad \exists U \in \mathcal{O}(x), V \in \mathcal{O}(f^{-1}(rx))\colon$$
$$U \subset U_0, V \subset V_0, \forall y \in U \quad \exists!\, z \in U, s(y) = s(z), r(z) \in f(V).$$

Under the above setting, there is a (finer) topology on the subgroupoid $H = G|_{f(T)}$ such that H is étale and Morita equivalent to G [PS99, Theorem 3.6]. The equivalence is implemented by the principal bibundle $G^{f(T)}$ with a natural finer topology from that of G and T.

1.2. **Groupoid equivariant C*-algebras.** Let us fix our conventions for G-C*-algebras.

Definition 1.4. A *$C_0(X)$-algebra* is a C*-algebra A endowed with a nondegenerate $*$-homomorphism from $C_0(X)$ to the center of the multiplier algebra $\mathcal{M}(A)$.

Thus, if $a \in A$, we have $a = fb = bf$ for some $f \in C_0(X)$ and $b \in A$, and the second equality holds for all f and b. For an open set $U \subset X$, we put $A_U = AC_0(U)$. For a locally closed subset $Y \subset X$, that is, if $Y = U \setminus V$ for some open sets $U, V \subset X$, we put $A_Y = A_U/A_{U \cap V}$, and we put $A_x = A_{\{x\}} = A/AC_0(X \setminus \{x\})$ for $x \in X$.

A $C_0(X)$-algebra is $C_0(X)$-*nuclear* if it is a continuous field of C*-algebras over X such that every fiber A_x is nuclear. There is another way to define this in terms of completely positive maps factoring through $M_n(C_0(X))$, see [Bau98].

Definition 1.5. Let A and B be $C_0(X)$-algebras which admit faithful $C_0(X)$-equivariant nondegenerate representations on Hilbert C*-$C_0(X)$-modules $\mathcal{E}$ and $\mathcal{E}'$. Then their C*-algebraic relative tensor product $A \otimes_{C_0(X)} B$ is defined as the closure of the image of $A \otimes^{\mathrm{alg}}_{C_0(X)} B$ in the adjointable operators $\mathcal{L}(\mathcal{E} \otimes_{C_0(X)} \mathcal{E}')$.

Although we do not need it, the above definition can be extended to arbitrary $C_0(X)$-algebras [Kas88, Definition 1.6].

Remark 1.6. If A or B is $C_0(X)$-nuclear, we have

$$A \otimes_{C_0(X)} B \simeq (A \otimes_{\max} B)_{\Delta(X)} \simeq (A \otimes_{\min} B)_{\Delta(X)},$$

where $\Delta(X) = \{(x, x) \mid x \in X\} \subset X \times X$, see [Bla96, Section 3.2].

If $f \colon Y \to X$ is a continuous map, $C_0(Y)$ is a $C_0(X)$-algebra. It is a continuous field (hence $C_0(X)$-nuclear) if and only if f is open [BK04]. This induces a functor $f^*A = C_0(Y) \otimes_{C_0(X)} A$ from the category of $C_0(X)$-algebras to that of $C_0(Y)$-algebras. For $Y = G$ and $f = s$, we write $s^*A = C_0(G) \,{}^s{\otimes}_{C_0(X)} A$, and similarly for $f = r$.

Definition 1.7. Let G be a second countable locally compact Hausdorff groupoid, and put $G^{(0)} = X$. A *continuous action* of G on a $C_0(X)$-algebra A is given by an isomorphism of $C_0(G)$-algebras

$$\alpha \colon C_0(G) \,{}^s{\otimes}_{C_0(X)} A \to C_0(G) \,{}^r{\otimes}_{C_0(X)} A$$

such that the induced homomorphisms $\alpha_g \colon A_{s(g)} \to A_{r(g)}$ for $g \in G$ satisfy $\alpha_{gh} = \alpha_g \alpha_h$. In this case, we say that A is a G-*C*-algebra*.

For an étale groupoid G, the above amounts to giving α_g as isomorphisms $A_U \to A_{g(U)}$ for small enough neighborhoods U of $s(g)$, compatible with the natural actions of $C_0(U) \simeq C_0(g(U))$ and multiplicative in g.

In [LG99], Le Gall constructed the equivariant KK-category of separable and trivially graded G-C*-algebras with morphism sets $\mathrm{KK}^G(A, B)$, generalizing Kasparov's construction for transformation groupoids. This will be our main framework to work in.

Remark 1.8. Le Gall uses a different convention for $A \otimes_{C_0(X)} B$, namely $(A \otimes_{\max} B)_{\Delta(X)}$, which is different from ours in general. However these definitions agree in all the relevant cases, such as $B = C_0(Y)$ for a locally compact space Y endowed with an open map $Y \to X$, then B would be $C_0(X)$-nuclear, see Remark 1.6. For example, the range map $G \to X$ is open because there exists a Haar system [Ren80, Proposition 2.4].

The algebraic balanced tensor product $C_c(G) \,{}^s{\otimes}_{C_0(X)} A$ admits an A-valued inner product induced by the measures on the sets G_x from the Haar system, and we denote its closure as a right Hilbert A-module by $E^G_A = L^2(G; A)$. (This can be interpreted as $L^2(G) \otimes_{C_0(X)} A$, where the canonical right $C_0(X)$-Hilbert module $L^2(G) = L^2(G; C_0(X))$.) The *reduced crossed product* $G \ltimes_\alpha A$ is the C*-algebra generated by the "convolution product" representation of $C_c(G) \,{}^s{\otimes}_{C_0(X)} A$ on E^G_A, see [KS04, MW08] for the details. In this paper we always take reduced crossed products, although they will be isomorphic to the full ones in most of our concrete examples as we mostly consider amenable groupoids.

1.3. **Equivariant sheaves over ample groupoids.** The nerve $(G^{(n)})_{n=0}^\infty$ of G form a simplicial space, with the face maps are given by

$$d^n_i \colon G^{(n)} \to G^{(n-1)}, \quad (g_1, \dots, g_n) \mapsto \begin{cases} (g_2, \dots, g_n) & \text{if } i = 0 \\ (g_1, \dots, g_i g_{i+1}, \dots, g_n) & \text{if } 1 \le i \le n-1 \\ (g_1, \dots, g_{n-1}) & \text{if } i = n, \end{cases}$$

with $d^1_1 = r$ and $d^1_0 = s$ as maps $G \to X$, while the degeneracy maps are given by insertion of units. These structure maps are étale maps.

Suppose further that G be an ample groupoid, and C be a commutative group. For a topological space Y, we denote the group of compactly supported continuous functions from Y to C by $C_c(Y, C)$. The *groupoid homology* of G with coefficients in C, denoted $H_\bullet(G, C)$, is the homology of the chain complex $(C_c(G^{(n)}, C))_{n=0}^\infty$ with differential

$$\partial_n = \sum_{i=0}^{n} (-1)^i (d_i^n)_* \colon C_c(G^{(n)}, C) \to C_c(G^{(n-1)}, C), \qquad (d_i^n)_*(f)(x) = \sum_{d_i^n(y)=x} f(y).$$

(This is well defined as d_i^n is étale.)

This is a special case of groupoid homology with coefficients in equivariant sheaves [CM00]. Let us quickly review this more general setting. When G is a topological groupoid with base space X, a *G-equivariant sheaf* (of commutative groups) over X is a sheaf (of commutative groups) F over X, together with a morphism $s^*F \to r^*F$ of sheaves over G, with analogous multiplicativity conditions to the case of G-C*-algebras.

In fact, when G is ample, such G-sheaves are represented by *unitary $C_c(G, \mathbb{Z})$-modules* [Ste14]. Here, we consider the convolution product on $C_c(G, \mathbb{Z})$, and a module M over $C_c(G, \mathbb{Z})$ is said to be unitary if it has the factorization property $C_c(G, \mathbb{Z})M = M$. The correspondence is given by $\Gamma_c(U, F) = C_c(U, \mathbb{Z})M$ for compact open sets $U \subset X$ if F is the sheaf corresponding to such a module M.

A sheaf F on a topological space Y is called *soft* if, for any closed subspace $K \subset Y$ and $s \in \Gamma(K, F)$, there is a global section $s' \in \Gamma(Y, F)$ such that $s'|_K = s$. When Y is second countable locally compact and Hausdorff, this is equivalent to *c-softness*, where in the above K is moreover assumed to be compact.

Proposition 1.9. *Let Y be a totally disconnected, second countable, locally compact Hausdorff space. Then any sheaf of commutative groups on Y is soft.*

This seems to be folklore, but can be obtained as follows. As Y is locally compact Hausdorff and totally disconnected, each point has a base of neighborhood consisting of compact open sets. Thus, fixing a point y and its compact open neighborhood U, any closed subset of U, being compact, also has a base of neighborhoods consisting of compact open subsets of Y. This, combined with the paracompactness of Y, implies the (c-)softness of sheaves [God73, Sections II.3.3 and II.3.4].

Back to equivariant sheaves over (second countable) ample groupoids, with G, F, and M as above, the *homology of G with coefficient in F*, denoted $H_\bullet(G, F)$, is the homology of the chain complex $(C_c(G^{(n)}, \mathbb{Z}) \otimes_{C_c(X,\mathbb{Z})} M)_{n=0}^\infty$ with differentials coming from the simplicial structure as above. Concretely, the differential is given by

$$\partial_n \colon C_c(G^{(n)}, \mathbb{Z}) \otimes_{C_c(X,\mathbb{Z})} M \to C_c(G^{(n-1)}, \mathbb{Z}) \otimes_{C_c(X,\mathbb{Z})} M$$

$$\partial_n(f \otimes m) = \sum_{i=0}^{n-1} (-1)^i (d_i^n)_* f \otimes m + (-1)^n \alpha_n(f \otimes m),$$

where α_n is the concatenation of the last leg of $C_c(G^{(n)}, \mathbb{Z})$ with M induced by the module structure map $C_c(G, \mathbb{Z}) \otimes M \to M$. This definition agrees with the one given in [CM00] as there is no need to take c-soft resolutions of equivariant sheaves by Proposition 1.9.

More generally, if $F_\bullet$ is a chain complex of G-sheaves modeled by a chain complex of unitary $C_c(G, \mathbb{Z})$-modules $M_\bullet$, we define $\mathbb{H}_\bullet(G, F_\bullet)$, the *hyperhomology* with coefficient $F_\bullet$, as the homology of the double complex $(C_c(G^{(p)}, \mathbb{Z}) \otimes_{C_c(X,\mathbb{Z})} M_q)_{p,q}$.

As usual, a chain map of complexes of G-sheaves $f \colon F_\bullet \to F'_\bullet$ is a *quasi-isomorphism* if it induces quasi-isomorphisms on the stalks. When $F_\bullet$ and $F'_\bullet$ are bounded from below, such maps induce an isomorphism of the hyperhomology [CM00, Lemma 3.2].

1.4. **Triangulated categorical structures.** The framework of triangulated categories is ideal for extending basic constructions from homotopy theory to categories of C*-algebras. Much work in this direction has been carried out by Meyer and Nest in [Mey08, MN06, MN10].

We follow their convention which we quickly recall here. The fundamental axiom requires that there is an autoequivalence Σ, and any morphism $f \colon A \to B$ should be part of an exact triangle:

$$A \to B \to C \to \Sigma A.$$

An additive functor F between triangulated categories is said to be exact when it intertwines suspensions and preserves exact triangles.

We say that $\mathcal{T}$ has *countable direct sums* if, given a sequence of objects $(A_n)_{n=1}^\infty$ in $\mathcal{T}$, there is an object $\bigoplus_{n=1}^\infty A_n$ such that

$$\mathcal{T}\left(\bigoplus_{n=1}^\infty A_n, B\right) \simeq \prod_{n=1}^\infty \mathcal{T}(A_n, B)$$

naturally in the A_n and B. An exact functor F is *compatible with direct sums* if it commutes with countable direct sums (see [Mey08, Proposition 3.14]).

As before let G be a second countable locally compact Hausdorff groupoid with a Haar system. Note that triangulated categories involving KK-theory have no more than countable direct sums, because separability assumptions are needed for certain analytical results in the background.

Proposition 1.10 ([Pro18a, Section A.3])**.** *The equivariant Kasparov category* KK^G *is triangulated.*

Here, the suspension functor Σ is given by $\Sigma A = C_0(\mathbb{R}, A)$. Note that Bott periodicity implies $\Sigma^2 \simeq \mathrm{id}$, so that Σ is also a model of Σ^{-1}. The exact triangles are defined as the triangles isomorphic to mapping cone triangles for equivariant $*$-homomorphisms. See Section A.4 for some details.

We also note that functors such as $A \mapsto G \ltimes A$ and $A \mapsto D \otimes A$ preserve mapping cones, hence define triangulated functors into appropriated (equivariant) KK-categories. These are also compatible with countable direct sums.

We call a subcategory *thick* when it is closed under direct summands.

Definition 1.11. We call a pair $(\mathcal{L}, \mathcal{N})$ of thick triangulated subcategories of $\mathcal{T}$ *complementary* if $\mathcal{T}(P, N) = 0$ for all $P \in \mathcal{L}, N \in \mathcal{N}$, and for any $A \in \mathcal{T}$, there is an exact triangle

$$P_A \to A \to N_A \to \Sigma P_A$$

where $P_A \in \mathcal{L}$ and $N_A \in \mathcal{N}$.

Let us list some of the basic properties of a pair of complementary subcategories (see [MN06, Proposition 2.9]).

- We have $N \in \mathcal{N}$ if and only if $\mathcal{T}(P, N) = 0$ for all $P \in \mathcal{L}$. Analogously, we have $P \in \mathcal{L}$ if and only if $\mathcal{T}(P, N) = 0$ for all $N \in \mathcal{N}$.
- The exact triangle as above, with $P_A \in \mathcal{L}$ and $N_A \in \mathcal{N}$, is uniquely determined up to isomorphism and depends functorially on A. In particular, its entries define functors

$$P\colon \mathcal{T} \to \mathcal{L}, \quad A \mapsto P_A, \qquad\qquad N\colon \mathcal{T} \to \mathcal{N}, \quad A \mapsto N_A.$$

  The functors P and N are respectively left adjoint to the embedding functor $\mathcal{P} \to \mathcal{T}$ and right adjoint to the embedding functor $\mathcal{N} \to \mathcal{T}$.
- The *localizations* $\mathcal{T}/\mathcal{N}$ and $\mathcal{T}/\mathcal{L}$ exist and the compositions

$$\mathcal{L} \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{N}, \qquad\qquad \mathcal{N} \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{L}$$

  are equivalences of triangulated categories.

Most concrete examples come from *homological ideals with enough projectives*, as we quickly recall here. Let $\mathcal{T}$ and $\mathcal{S}$ be triangulated categories with countable direct sums, and $F\colon \mathcal{T} \to \mathcal{S}$ be an exact functor compatible with direct sums. The system of morphisms

$$\mathcal{I}(A, B) = \ker(F\colon \mathcal{T}(A, B) \to \mathcal{S}(FA, FB))$$

is an example of *homological ideal* compatible with countable direct sums.

Remark 1.12. We do not lose generality by assuming that $\mathcal{S}$ is a stable abelian category, and that F is a stable functor, see [MN10, Remark 19]. More concretely, we can always replace the target triangulated category $\mathcal{S}$ by the category of *representable* contravariant functors $\mathcal{S} \to \mathrm{Ab}$, which are cokernels of the natural transforms $\mathcal{S}(\text{-}, A) \to \mathcal{S}(\text{-}, B)$ induced by some $f\colon A \to B$.

An object $P \in \mathcal{T}$ is called *$\mathcal{I}$-projective* if $\mathcal{I}(P, A) = 0$ for all objects $A \in \mathcal{T}$. An object $N \in \mathcal{T}$ is called *$\mathcal{I}$-contractible* if id_N belongs to $\mathcal{I}(N, N)$. Let $\mathcal{P}_\mathcal{I}, \mathcal{N}_\mathcal{I} \subseteq \mathcal{T}$ be the full subcategories of projective and contractible objects, respectively. We say that $\mathcal{I}$ *has enough projectives* if for any $A \in \mathcal{T}$, there is an $\mathcal{I}$-projective object P and a morphism $P \to A$ such that, in the associated exact triangle

$$P \to A \to C \to \Sigma P,$$

the morphism $A \to C$ belongs to $\mathcal{I}$. With $\mathcal{I} = \ker F$ as above, the latter condition is equivalent to $FP \to FA$ being a split surjection for all A.

We denote by $\langle P_{\mathcal{I}} \rangle$ the ($\aleph_0$-)*localizing* subcategory generated by the projective objects, i.e., the smallest triangulated subcategory that is closed under countable direct sums and contains $P_{\mathcal{I}}$. In particular, $\langle P_{\mathcal{I}} \rangle$ is closed under isomorphisms, suspensions, and if

$$A \to B \to C \to \Sigma A$$

is an exact triangle in $\mathcal{T}$ where any two of the objects A, B, C are in $\langle P_{\mathcal{I}} \rangle$, so is the third. Note that $N_{\mathcal{I}}$ is localizing, and any localizing subcategory is thick.

Theorem 1.13 ([Mey08, Theorem 3.16]). *Let $\mathcal{T}$ be a triangulated category with countable direct sums, and let $\mathcal{I}$ be a homological ideal with enough projective objects. Suppose that $\mathcal{I}$ is compatible with countable direct sums. Then the pair of localizing subcategories $(\langle \mathcal{P}_{\mathcal{I}} \rangle, \mathcal{N}_{\mathcal{I}})$ in $\mathcal{T}$ is complementary.*

Remark 1.14. Note that if $(\mathcal{L}, \mathcal{N})$ is a complementary pair, then $\ker P$ has enough projectives and we have $\mathcal{L} = \mathcal{P}_{\ker P}$, $\mathcal{N} = \mathcal{N}_{\ker P}$. Thus the above construction is universal, although $\mathcal{I}$ is not uniquely determined from $(\langle \mathcal{P}_{\mathcal{I}} \rangle, \mathcal{N}_{\mathcal{I}})$.

Definition 1.15. Let $F \colon \mathcal{T} \to \mathcal{S}$ be an exact functor compatible with countable direct sums. Given an object $A \in \mathcal{T}$ and a chain complex

$$\cdots \xrightarrow{\delta_{n+1}} P_n \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} A, \tag{1}$$

we say that (1) is an (even) $\mathcal{I}$-*projective resolution* of A if each P_n is $\mathcal{I}$-projective and the chain complex

$$\cdots \xrightarrow{F(\delta_2)} F(P_1) \xrightarrow{F(\delta_1)} F(P_0) \xrightarrow{F(\delta_0)} F(A) \longrightarrow 0$$

is split exact, i.e., is contractible by chain homotopy in $\mathcal{S}$.

There is also an intrinsic formulation of $\mathcal{I}$-*exactness* for chain complexes corresponding to the second condition above, and the above definition does not depend on the choice of F with $\mathcal{I} = \ker F$. Moreover, if $\mathcal{I}$ has enough projectives, any A has an $\mathcal{I}$-projective resolution. In particular, two $\mathcal{I}$-projective resolutions of A are chain homotopy equivalent, and we obtain functor $\mathcal{T} \to \mathrm{Ho}(\mathcal{T})$. Moreover, if $P_\bullet$ is an $\mathcal{I}$-projective resolution of A, the object P_A in Definition 1.11 (an $\mathcal{I}$-*simplicial approximation* of A) belongs to the localizing subcategory generated by the objects P_k.

Definition 1.16. An *odd* $\mathcal{I}$-projective resolution is an $\mathcal{I}$-projective resolution where the boundary maps of positive index have degree one, i.e., the morphism $\delta_n \colon P_n \to P_{n-1}$ gets replaced, for $n \geq 1$, by a morphism $\delta_n \colon P_n \to \Sigma P_{n-1}$.

Evidently, if (P_n, δ_n) is an odd projective resolution, then (P'_n, δ'_n) is an even resolution, where $P'_n = \Sigma^{-n} P_n$ and $\delta'_n = \Sigma^{-n} \delta_n$.

Let $K \colon \mathcal{T} \to \mathcal{C}$ be a covariant homological functor into a stable abelian category. We put $K_n(A) = K(\Sigma_{-n} A)$. Let us recall a few extra constructions on K motivated by homological algebra.

Definition 1.17. Let $(\mathcal{L}, \mathcal{N})$ be a complementary pair, with $P \colon \mathcal{T} \to \mathcal{L}$. The *localization* of K with respect to $\mathcal{N}$ is defined by $\mathbb{L}^{\mathcal{N}} K = K \circ P$.

The defining morphisms $P(A) \to A$ induce a natural transformation $\mathbb{L}^{\mathcal{N}} K \Rightarrow K$.

Definition 1.18. Let $\mathcal{I}$ be a homological ideal with countable direct sums and enough projectives. The p-th *derived functor* of K with respect to $\mathcal{I}$ is defined as

$$\mathbb{L}^{\mathcal{I}}_p K(A) = H_p(K(P_\bullet)),$$

where $P_\bullet$ is any $\mathcal{I}$-projective resolution of A.

This is well-defined because projective resolutions are unique up to chain homotopy. Note that unless K is compatible with $\mathcal{I}$-exact sequences, one cannot expect $\mathbb{L}^{\mathcal{I}}_0 K \simeq K$. When $(\mathcal{L}, \mathcal{N})$ is a complementary pair, we can think of the localization $\mathbb{L}^{\mathcal{N}} K$ as the derived functor $\mathbb{L}^{\ker P}_0 K$ for $P \colon \mathcal{T} \to \mathcal{L}$ up to the embedding of Remark 1.12.

Building on the idea of Christensen [Chr98] to understand the Adams spectral sequence, Meyer constructed the following spectral sequence.

Theorem 1.19 ([Mey08, Theorems 4.3 and 5.1]). *Let $\mathcal{I}$ be a homological ideal with countable direct sums and enough projectives, and let $K\colon \mathcal{T} \to \mathrm{Ab}$ be a homological functor to the category of commutative groups. Then there is a convergent spectral sequence*

$$E^r_{pq} \Rightarrow \mathbb{L}^{\mathcal{N}_\mathcal{I}} K_{p+q}(A),$$

with the E^2-sheet $E^2_{pq} = \mathbb{L}^{\mathcal{I}}_p K_q(A)$.

The E^r-differentials $d^r\colon E^r_{pq} \to E^r_{p-r,q+r-1}$ come from a choice of *phantom tower* for A and the associated *exact couple*, but their precise form will not be important for us.

1.5. **The Baum–Connes conjecture for groupoids.** Because we are particularly interested in spectral sequences which approximate the K-groups of groupoid C*-algebras, the Baum–Connes conjecture naturally plays a fundamental role. The notion of pair of complementary subcategories introduced earlier allows for a general formulation of this conjecture in terms of localization at the subcategory contractible objects.

However, as our main focus is on torsion-free amenable groupoids, we will not need the full machinery for our applications, hence we limit ourselves to simply recalling the main positive result concerning the conjecture for groupoids with the *Haagerup property.* Namely, G is said to have the Haagerup property if it acts properly and isometrically by affine maps on a continuous field of (real) Hilbert spaces, or equivalently, if there is a proper conditionally negative type function on G [BCV95]. Analogously to the case of groups, amenable groupoids have this property.

Suppose G is a second countable, locally compact, Hausdorff groupoid with Haar system. In the following, the crossed product is understood to be *reduced.*

Definition 1.20. A G-algebra A is said to be *proper* if there is a locally compact Hausdorff proper G-space Z such that A is a $G \ltimes Z$-algebra.

Evidently, a commutative G-C*-algebra is proper if and only if its spectrum is a proper G-space.

Remark 1.21. If G is locally compact, σ-compact, and Hausdorff, then there is a locally compact, σ-compact, and Hausdorff model of $\underline{E}G$, the classifying space for proper actions of G; in our case G is second countable hence $\underline{E}G$ is too [Tu99b, Proposition 6.15]. In Definition 1.20 for a proper G-algebra we can always assume Z to be a model of $\underline{E}G$. Indeed if $\phi\colon Z \to \underline{E}G$ is a G-equivariant continuous map, then $\phi^*\colon C_0(\underline{E}G) \to \mathcal{M}(C_0(Z)) = C_b(Z)$ can be precomposed with the structure map $\Phi\colon C_0(Z) \to Z(\mathcal{M}(A))$, making A into an $G \ltimes \underline{E}G$-algebra.

We will need the following result proved by J.-L. Tu.

Theorem 1.22 ([Tu99a]). *Suppose that G has the Haagerup property. Then there exists a proper G-space Z with an open surjective structure morphism $Z \to X$, and a $G \ltimes Z$-C*-algebra P which is a continuous field of nuclear C*-algebras over Z, and such that $P \simeq C_0(X)$ in KK^G.*

As a consequence, if A is a separable G-C*-algebra, then we have that $A \otimes_{C_0(X)} P$ is a proper G-C*-algebra that is KK^G-equivalent to A.

In this paper, for a general topological groupoid G we say that it satisfies the strong Baum–Connes conjecture if the conclusions of the previous theorem hold. This definition implies the standard version of the conjecture. More precisely, if $D\colon P \to C_0(X)$ is the isomorphism from Theorem 1.22, there is a commutative diagram

$$\begin{array}{ccc}
\varinjlim_{Y \subseteq \underline{E}G} \mathrm{KK}^G(C_0(Y), A) & \xrightarrow{\mu^G_A} & K_\bullet(G \ltimes A) \\
\downarrow & \nearrow_{j_G(D \,\widehat{\otimes}\, \mathrm{id}_A)} & \\
K_\bullet(G \ltimes (A \otimes_{C_0(X)} P)) & &
\end{array}$$

where all arrows are isomorphisms and Y runs over G-compact invariant subsets of $\underline{E}G$ ([EM10, Theorem 6.12], see also [MN06]). The functor j_G above is the descent morphism of Kasparov [Kas88] which has been generalized to this context in [LG99, Laf07].

1.6. **Induction and restriction for groupoid KK-theory.** Suppose G is a groupoid as in the previous subsection.

Let $H \subseteq G$ be an open subgroupoid with the same base space $X = G^{(0)} = H^{(0)}$. Note that H has a Haar system automatically by restriction. We have a natural restriction functor $\operatorname{Res}_H^G\colon \mathrm{KK}^G \to \mathrm{KK}^H$. It admits a left adjoint, which is an analogue of induction, as follows. Full details will appear elsewhere in a joint work of the first named author with C. Bönicke.

Let B be an H-C*-algebra, with structure map $\rho\colon C_0(X) \to Z(\mathcal{M}(B))$. As before, take the $C_0(G)$-algebra

$$B' = C_0(G)\; {}^s\!\otimes_{C_0(X)} B.$$

This has a right action of H, by combination of the right translation on $C_0(G)$ and the action on B twisted by the inverse map of H. We then set

$$\operatorname{Ind}_H^G(B) = B' \rtimes H = (C_0(G)\; {}^s\!\otimes_{C_0(X)} B) \rtimes_{\mathrm{diag}} H.$$

This can be regarded as the crossed product of B' by the transformation groupoid $G \rtimes H$ for the right translation action of H on G. Moreover, notice that G also acts on B' by left translation on $C_0(G)$. This induces a continuous action of G on $\operatorname{Ind}_H^G(B)$.

Let A be a G-C*-algebra. Then the Haar system on G induces an A-valued inner product on $C_c(G) \otimes_{C_0(X)} A$, and by completion we obtain a right Hilbert A-module $E_A^G = L^2(G; A)$. We then have the following, see Section A.5 for details.

Proposition 1.23. *Under the above setting, E_A^G implements an equivariant strong Morita equivalence between A and $\operatorname{Ind}_G^G A$.*

Let κ denote the inclusion homomorphism

$$\operatorname{Ind}_H^G \operatorname{Res}_H^G(A) = (C_0(G)\; {}^s\!\otimes_{C_0(X)} A) \rtimes H \to (C_0(G)\; {}^s\!\otimes_{C_0(X)} A) \rtimes G = \operatorname{Ind}_G^G A,$$

induced by $H \subseteq G$ because H is open, and let ι denote the map

$$\operatorname{Ind}_H^H B = (C_0(H)\; {}^s\!\otimes_{C_0(X)} B) \rtimes H \to (C_0(G)\; {}^s\!\otimes_{C_0(X)} B) \rtimes H = \operatorname{Res}_H^G \operatorname{Ind}_H^G(B)$$

induced by the ideal inclusion $C_0(H) \subseteq C_0(G)$.

Theorem 1.24. *The functor Ind_H^G induces a functor $\mathrm{KK}^H \to \mathrm{KK}^G$, and there is a natural isomorphism*

$$\mathrm{KK}^G(\operatorname{Ind}_H^G B, A) \simeq \mathrm{KK}^H(B, \operatorname{Res}_H^G A)$$

defining an adjunction $(\epsilon, \eta)\colon \operatorname{Ind}_H^G \dashv \operatorname{Res}_H^G$ with counit and unit natural morphisms

$$\epsilon_A = [\kappa] \otimes_{\operatorname{Ind}_G^G A} [E_A^G] \in \mathrm{KK}^G(\operatorname{Ind}_H^G \operatorname{Res}_H^G A, A), \quad \eta_B = [\bar{E}_B^H] \otimes_{\operatorname{Ind}_H^H B} [\iota] \in \mathrm{KK}^H(B, \operatorname{Res}_H^G \operatorname{Ind}_H^G B).$$

In fact, our main result Corollary 4.7 only requires this for $H = X$ in ample groupoids G, for which the results [Bön18] are enough.

Example 1.25. If G is the transformation groupoid $\Gamma \ltimes X$ and $H = X$, the previous theorem amounts to

$$\mathrm{KK}^{\Gamma \ltimes X}(C_0(\Gamma) \otimes B, A) \simeq \mathrm{KK}^X(B, A)$$

for any $C_0(X)$-algebra B and G-algebra A, where the Γ-action on $C_0(\Gamma) \otimes B$ is given by translation on the factor $C_0(\Gamma)$.

1.7. **Smale spaces.** A Smale space is given by a self-homeomorphism on a compact metric space which admits contracting and expanding directions. The precise definition requires the definition of a bracket map satisfying certain axioms [Put14, Rue04], as follows.

Definition 1.26. A *Smale space* (X, ϕ) is given by a compact metric space (X, d) and a homeomorphism $\phi\colon X \to X$ such that:

- there exist constant $0 < \epsilon_X$ and a continuous map

  $$\{(x, y) \in X \times X \mid d(x, y) \le \epsilon_X\} \to X, \qquad (x, y) \mapsto [x, y]$$

  satisfying the *bracket axioms*:

  $$[x, x] = x, \qquad [x, [y, z]] = [x, z],$$
  $$[[x, y], z] = [x, z], \qquad \phi([x, y]) = [\phi(x), \phi(y)],$$

  for any x, y, z in X when both sides are defined.

- there exists $0 < \lambda < 1$ satisfying the *contraction axioms*:

$$[x, y] = y \Rightarrow d(\phi(x), \phi(y)) \leq \lambda d(x, y),$$
$$[x, y] = x \Rightarrow d(\phi^{-1}(x), \phi^{-1}(y)) \leq \lambda d(x, y),$$

  whenever the brackets are defined.

Suppose $x \in X$ and $0 < \epsilon \leq \epsilon_X$. We define the *local stable sets* and the *local unstable sets* around x as

$$X^s(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon, [y, x] = x\},$$
$$X^u(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon, [x, y] = x\}.$$

The bracket $[x, y]$ can be characterized as the unique element of $X^s(x, \epsilon) \cap X^u(y, \epsilon)$ when $2d(x, y) < \epsilon < \epsilon_X$. This means that, locally, we can choose coordinates so that

$$[\text{-}, \text{-}]\colon X^u(x, \epsilon) \times X^s(x, \epsilon) \to X$$

is a homeomorphism onto an open neighborhood of $x \in X$ for $0 < \epsilon < \epsilon_X/2$.

A point $x \in X$ is called *non-wandering* if for all open sets $U \subseteq X$ containing x there exists $N \in \mathbb{N}$ with $U \cap \phi^N(U) \neq \emptyset$. Periodic points are dense among the non-wandering points [Put15, Theorem 4.4.1]. We say that X is non-wandering if any point of X is non-wandering. *We will set a blanket assumption that Smale spaces are non-wandering.* This holds in virtually all interesting examples.

It can be shown that any non-wandering Smale space (X, ϕ) can be partitioned in a finite number of ϕ-invariant clopen sets $X_1, \ldots, X_n$, in a unique way, such that $(X_k, \phi|_{X_k})$ is *irreducible* for $k = 1, \ldots, n$ [Put00]. Irreducibility means that for every (ordered) pair U, V of nonempty open sets in X, there exists $N \in \mathbb{N}$ such that $U \cap \phi^n(V) \neq \emptyset$, $n \geq N$.

Example 1.27. The standard definition of a *shift of finite type* is given in [LM95, Definition 2.1.1]. However, an equivalent and more convenient definition is to start out with a finite directed graph G. A *directed graph* $G = (G^0, G^1, i, t)$ consists of finite sets G^0 and G^1, called vertices and edges, such that each edge $e \in G^1$ is given by a directed arrow from $i(e) \in G^0$ to $t(e) \in G^0$. Then a shift of finite type (Σ_G, σ) is defined as the space of bi-infinite sequences of paths

$$\Sigma_G = \{e = (e_k)_{k \in \mathbb{Z}} \in (G^1)^{\mathbb{Z}} \mid t(e_k) = i(e_{k+1})\},$$

together with the left shift map $\sigma(e)_k = e_{k+1}$. The metric is such that $d(e, f) \leq 2^{-n-1}$ if e, f coincide on the interval $[-n, n]$. In particular, $d(e, f) = 2^{-1}$ means that e, f share the central edge, i.e., $e_0 = f_0$. Then we can define

$$[e, f] = (\ldots, f_{-2}, f_{-1}, e_0, e_1, e_2, \ldots).$$

The pair (Σ_G, σ) is a Smale space with constant $\epsilon = 1/2$.

We are particularly interested in groupoids encoding the *unstable equivalence relation* of Smale spaces. Given $x, y \in X$, we say they are

- *stably equivalent*, denoted by $x \sim_s y$, if

$$\lim_{n \to \infty} d(\phi^n(x), \phi^n(y)) = 0;$$

- *unstably equivalent*, $x \sim_u y$, if

$$\lim_{n \to \infty} d(\phi^{-n}(x), \phi^{-n}(y)) = 0.$$

We denote the graphs of these relations as

$$R^s(X, \phi) = \{(x, y) \in X \times X \mid x \sim_s y\}, \tag{2}$$
$$R^u(X, \phi) = \{(x, y) \in X \times X \mid y \sim_u y\},$$

and treat them as groupoids, with source, range, and composition maps given by

$$s(x, y) = y, \qquad r(x, y) = x, \qquad (x, y) \circ (w, z) = (x, z) \quad \text{if } y = w.$$

The orbit of $x \in X$ under the stable (resp. unstable) equivalence relation is called the *global stable* (resp. *unstable*) *set*, and is denoted by $X^s(x)$ (resp. $X^u(x)$). They satisfy the following identities:

$$X^s(x) = \bigcup_{n\geq 0} \phi^{-n}(X^s(\phi^n(x), \epsilon)), \tag{3}$$

$$X^u(x) = \bigcup_{n\geq 0} \phi^{n}(X^s(\phi^{-n}(x), \epsilon)), \tag{4}$$

for any fixed $\epsilon < \epsilon_X$.

This leads to locally compact Hausdorff topologies on the above groupoids [Put96]: consider the induced topology on

$$G_s^n = \{(x,y) \mid y \in \phi^{-n}(X^s(\phi^n(x),\epsilon))\}, \quad G_u^n = \{(x,y) \mid y \in \phi^{n}(X^s(\phi^{-n}(x),\epsilon))\}$$

as subsets of $X \times X$. Then, as $R^s(X,\phi)$ is the union of the increasing sequence, it has the inductive limit topology of these spaces. Since the inclusion $G_s^n \to G_s^{n+1}$ is open, $R^s(X,\phi)$ is a locally compact Hausdorff groupoid. Of course, analogous considerations make $R^u(X,\phi)$ a locally compact Hausdorff groupoid.

To get an étale groupoid, we can take a transversal $T \subset X$ and restrict the base space to T, putting $G|_T = G_T^T$. A canonical choice is to take $T = X^s(x)$, with the inductive limit topology from (3), which is an example of generalized transversal. Slightly generalizing this, for a subset $P \subseteq X$, we write $X^s(P)$ meaning the union of all $X^s(x)$'s for $x \in P$, with the disjoint union topology. Analogously we define $X^u(P) = \bigcup_{x\in P} X^u(x)$. Let us put

$$R^s(X,P) = R^s(X,\phi)|_{X^u(P)}, \qquad R^u(X,P) = R^u(X,\phi)|_{X^s(P)}.$$

As we indicated after Definition 1.3, since we consider finer topologies on the sets $X^s(x)$, $X^u(x)$ than the ones induced by the inclusion into X, we need to endow $R^s(X,P), R^u(X,P)$ with a different topology, following [PS99]. Concretely, this is achieved by taking the "holonomy groupoid" topology for the maps in (5) (see for example [Kil09, Theorem 2.17], see also [Tho10a] under the name "topology of local conjugacies"). For each pair $(x,y) \in G_s^n$, consider maps

$$X^u(y,\delta) \to X^u(x,\delta), \quad z \mapsto \phi^{-n}([\phi^n(z), \phi^n(x)]), \tag{5}$$

defined for any $\delta > 0$ satisfying

$$\phi^n(X^u(x,\delta)) \subseteq X^u\left(\phi^n(x),\epsilon\right), \quad \phi^n(X^u(y,\delta)) \subseteq X^u(\phi^n(y),\epsilon).$$

This way, $R^s(X,P)$ becomes an étale groupoid, which is Morita equivalent to $R^s(X,\phi)$. Here, the equivalence is implemented by the set $R^s(X,\phi)_{X^u(P)}$, together with the topology generated by the sets of the form $U \cap s^{-1}V$ for open sets $U \subset R^s(X,\phi)$ and $V \subset X^u(P)$. Analogous considerations hold for $R^u(X,P)$.

Theorem 1.28 ([PS99, Theorem 1.1])**.** *These groupoids are amenable.*

1.8. **Maps of Smale spaces.** A continuous and surjective map $f\colon (X,\phi) \to (Y,\psi)$ between Smale spaces is called a *factor map* if it intertwines the respective self-maps, i.e.,

$$f \circ \phi = \psi \circ f. \tag{6}$$

Equation (6) is enough to guarantee that f preserves the local product structure. In particular, there is $\epsilon_f > 0$ such that both $[x_1, x_1]$ and $[f(x_1), f(x_2)]$ are defined and $f([x_1,x_2]) = [f(x_1), f(x_2)]$ for all x_1, x_2 with $d(x_1,x_2) < \epsilon_f$.

Proposition 1.29 ([Put15, Lemma 5.2.10])**.** *If $y_0 \in Y$ is a periodic point with $f^{-1}(y_0) = \{x_1,\dots,x_N\}$, given $\epsilon_X > \epsilon > 0$, there exists $\delta > 0$ such that*

$$f^{-1}(Y^u(y_0,\delta)) \subseteq \bigcup_{i=1}^N X^u(x_i,\epsilon).$$

Definition 1.30. A factor map $f\colon (X,\phi) \to (Y,\psi)$ is called *s-resolving* if it induces an injective map from $X^s(x)$ to $Y^s(f(x))$ for each $x \in X$. It is called *s-bijective*, if moreover these induced maps are bijective.

Theorem 1.31 ([Put05, Corollary 3]). *Let (X, ϕ) be an irreducible Smale space such that $X^s(x, \epsilon)$ is totally disconnected for every $x \in X$ and $0 < \epsilon < \epsilon_X$. Then there is an irreducible shift of finite type (Σ, σ) and an s-bijective factor map $f\colon (\Sigma, \sigma) \to (X, \phi)$.*

Theorem 1.32 ([Put15, Theorem 5.2.4]). *Let $f\colon X \to Y$ is an s-resolving map between Smale spaces. There is a constant $N \geq 1$ such that for any $y \in Y$ there exist $x_1, \dots, x_n$ in X, with $n \leq N$, satisfying*

$$f^{-1}(Y^u(y)) = \bigcup_{k=1}^{n} X^u(x_k).$$

For any $y \in Y$ the cardinality of the fiber $f^{-1}(y)$ is less than or equal to N.

Let us list several additional facts about s-resolving maps, which can be found in [Put15]. First, if each point in Y is non-wandering, then f is s-bijective. Second, the induced maps $X^s(x) \to Y^s(f(x))$ and $X^u(x) \to Y^u(f(x))$ are both continuous and proper in the inductive limit topology of the presentation in (3) and (4). If, moreover, f is s-bijective, the map $X^s(x) \to Y^s(f(x))$ is a homeomorphism. Assume that X and Y are irreducible, and P is an at most countable subset of X such that no two points of P are stably equivalent after applying f. Then

$$f \times f\colon R^u(X, P) \to R^u(Y, f(P))$$

is a homeomorphism onto an open subgroupoid of $R^u(Y, f(P))$.

2. Fibered products of groupoids

In this section we consider the groupoids associated to resolutions of Smale spaces and prove several key Morita equivalences.

2.1. Multiple fibered product of groupoids.

We start by defining the appropriate notion of fibered product between groupoids which will be used in the following proofs.

Definition 2.1. Let $\alpha\colon H \to G$ be a homomorphism of groupoids, and $n \geq 2$. We define the n-th fibered product of H with respect to α as the groupoid $H^{\times_G n}$ defined as follows:

- the object space is the set

$$(H^{\times_G n})^{(0)} = \{(y_1, g_1, y_2, \dots, g_{n-1}, y_n) \mid y_k \in H^{(0)}, g_k \in G^{\alpha(y_k)}_{\alpha(y_{k+1})}\}$$

- the arrows from $(y_1, g_1, y_2, \dots, g_{n-1}, y_n)$ to $(y'_1, g'_1, y'_2, \dots, g'_{n-1}, y'_n)$ are given by the n-tuples $(h_1, \dots, h_n) \in H^{y'_1}_{y_1} \times \dots \times H^{y'_n}_{y_n}$ such that the squares in

$$\begin{array}{ccccccc}
\alpha(y'_1) & \xleftarrow{g'_1} & \alpha(y'_2) & \xleftarrow{g'_2} & \cdots & \xleftarrow{g'_{n-1}} & \alpha(y'_n) \\
{\scriptstyle \alpha(h_1)}\big\uparrow & & {\scriptstyle \alpha(h_2)}\big\uparrow & & & & \big\uparrow{\scriptstyle \alpha(h_n)} \\
\alpha(y_1) & \xleftarrow[g_1]{} & \alpha(y_2) & \xleftarrow[g_2]{} & \cdots & \xleftarrow[g_{n-1}]{} & \alpha(y_n)
\end{array}$$

are all commutative.

(Of course, we can put $H^{\times_G 1} = H$). We say that an arrow in $H^{\times_G n}$ is represented by the tuple $(h_1, g'_1, h_2, \dots, g'_{n-1}, h_n)$ in the above situation. This way we can think of $H^{\times_G n}$ as a subset of $H \times G \times \cdots G \times H$, and in the setting of topological groupoids this gives a compatible topology on $H^{\times_G n}$ (for example, local compactness passes to $H^{\times_G n}$).

Remark 2.2. The above definition makes sense for n-tuples of different homomorphisms $\alpha_k\colon H_k \to G$, so that we can define $H_1 \times_G \cdots \times_G H_n$ as a groupoid. The case of $n = 2$ appears in [CM00].

We will need a slight generalization of Definition 2.1 falling under this more general setting, where $H_j = H$ except for one value $j = k$, with $H_k = G$.

Definition 2.3. In the setting of Definition 2.1, define a groupoid $G \times_G H^{\times_G n}$ as follows:

- the object space is the set

$$(G \times_G H^{\times_G n})^{(0)} = \{(g_0, y_1, g_1, y_2, \dots, g_{n-1}, y_n) \mid y_k \in H^{(0)}, g_0 \in G_{\alpha(y_1)}, g_k \in G^{\alpha(y_k)}_{\alpha(y_{k+1})} (k \geq 1)\}$$

- a morphisms from $(g_0, y_1, g_1, y_2, \dots, g_{n-1}, y_n)$ to $(g'_0, y'_1, g'_1, y'_2, \dots, g'_{n-1}, y'_n)$ is given by $k \in G_{rg_0}^{rg'_0}$ and an n-tuple $(h_1, \dots, h_n) \in H_{y_1}^{y'_1} \times \cdots \times H_{y_n}^{y'_n}$ such that the squares in

$$\begin{array}{ccccccccc} rg'_0 & \xleftarrow{g'_0} & \alpha(y'_1) & \xleftarrow{g'_1} & \alpha(y'_2) & \xleftarrow{g'_2} & \cdots & \xleftarrow{g'_{n-1}} & \alpha(y'_n) \\ {\scriptstyle k}\uparrow & & {\scriptstyle \alpha(h_1)}\uparrow & & {\scriptstyle \alpha(h_2)}\uparrow & & & & \uparrow{\scriptstyle \alpha(h_n)} \\ rg_0 & \xleftarrow[g_0]{} & \alpha(y_1) & \xleftarrow[g_1]{} & \alpha(y_2) & \xleftarrow[g_2]{} & \cdots & \xleftarrow[g_{n-1}]{} & \alpha(y_n) \end{array}$$

  are all commutative.

Again we say that an arrow of $G \times_G H^{\times_G n}$ is represented by $(k, g'_0, h_1, \dots, h_n)$ in the above situation. As in the case of $H^{\times_G n}$, this induces a compatible topology in the setting of topological groupoids.

Proposition 2.4. *Let $\alpha\colon H \to G$ be a homomorphism of topological groupoids. Then $H^{\times_G n}$ and $G \times_G H^{\times_G n}$ are Morita equivalent as topological groupoids.*

Proof. Consider the space

$$Z = \{(g_0, h_1, g_1, h_2, \dots, g_{n-1}, h_n) \mid (g_0, \dots, g_{n-1}) \in G^{(n)}, \alpha(rh_k) = sg_{k-1}\}.$$

We define a left action of $G \times_G H^{\times_G n}$ as follows. The anchor map is

$$Z \to (G \times_G H^{\times_G n})^{(0)}, (g_0, h_1, \dots, h_n) \mapsto (g_0, rh_1, g_1, \dots, rh_n),$$

and an arrow of $G \times_G H^{\times_G n}$ with source $(g_0, rh_1, g_1, \dots, rh_n)$ acts by

$$(k, g'_0, h'_1, \dots, h'_n).(g_0, h_1, \dots, h_n) = (g'_0, h'_1 h_1, g'_1, \dots, h'_n h_n).$$

On the other hand, there is a right action of $H^{\times_G n}$ defined as follows. The anchor map is

$$Z \to (H^{\times_G n})^{(0)}, \quad (g_0, h_1, \dots, h_n) \mapsto (sh_1, g'_1, \dots, sh_n), \quad (g'_k = \alpha(h_k)^{-1} g_k \alpha(h_{k+1})).$$

An arrow of $H^{\times_G n}$ with range $(sh_1, g'_1, \dots, sh_n)$ acts by

$$(g_0, h_1, \dots, h_n).(h''_1, g_1, h''_2, \dots, h''_n) = (g_0, h_1 h''_1, g_1, \dots, h_n h''_n).$$

We claim that Z is a bibundle implementing the Morita equivalence (compatibility with topology will be obvious from the concrete "coordinate transform" formulas).

Comparing between $Z \times_{(G \times_G H^{\times_G n})^{(0)}} Z$ and $Z \times_{(H^{\times_G n})^{(0)}} H^{\times_G n}$ amounts to comparison of pairs (h_k, h'_k) with $rh_k = rh'_k$ on the one hand, and the composable pairs $(h_k, h''_k) \in H^{(2)}$ on the other. There is a bijective correspondence between the two sides, given by the coordinate transform $h'_k = h_k h''_k$. Comparing $Z \times_{(H^{\times_G n})^{(0)}} Z$ with $G \times_G H^{\times_G n} \times_{(G \times_G H^{\times_G n})^{(0)}} Z$ amounts to comparing:

- on the side of $Z \times_{(H^{\times_G n})^{(0)}} Z$: $((g_0, h_1), (g'_0, h'_1))$ with $(g_0, \alpha(h_1), (g'_0, \alpha(h'_1)) \in G^{(2)}$ and $sh_1 = sh'_1$, and $(h_k, h'_k) \in H^{(2)}$ with $sh_k = sh'_k$ for $k \geq 2$;
- on the side of $G \times_G H^{\times_G n} \times_{(G \times_G H^{\times_G n})^{(0)}} Z$: $(k, g''_0) \in G^{(2)}$, $(h_1, h''_1) \in H^{(2)}$ with $sh_1 = sg''_0$, and $(h_k, h''_k) \in H^{(2)}$ for $k \geq 2$.

Again we have a bijective correspondence by $h'_k = h''^{-1}_k$, $g_0 = g''_0 \alpha(h_1)^{-1}$, and $g'_0 = k g''_0 \alpha(h_1)^{-1}$. ☐

A slight generalization is obtained by considering the groupoid $H^{\times_G a} \times_G G \times_G H^{\times_G b}$ for $a, b \geq 0$. This is defined as $H^{\times_G (a+b+1)}$ in Definition 2.1, with the difference that h_{a+1} is not in $H_{y_{a+1}}^{y'_{a+1}}$, and instead in $G_{\alpha(y_{a+1})}^{\alpha(y'_{a+1})}$.

Proposition 2.5. *The groupoid $H^{\times_G a} \times_G G \times_G H^{\times_G b}$ is Morita equivalent to $H^{\times_G (a+b)}$.*

Proof. Recall the construction in the proof of Proposition 2.4 for the Morita equivalence between $G \times_G H^{\times_G b}$ and $H^{\times_G b}$: we have the space

$$Z = \{(g_0, h_1, g_1, \dots, h_b) \mid (g_0, \dots, g_{b-1}) \in G^{(b)}, \alpha(rh_k) = rg_k\},$$

which is a bimodule between these groupoids. Based on this, put

$$\tilde{Z} = \{(h_1, g_1, h_2, \dots, g_a, g_{a+1}, h_{a+1}, g_{i+2}, \dots, h_{a+b}) \mid (g_1, \dots, g_{a+b}) \in G^{(a+b)}, \\ \alpha(rh_k) = rg_k \ (k \leq a), \alpha(rh_k) = sg_k \ (k > a)\}.$$

This has obvious "composition" actions of $H^{\times_G a} \times_G G \times_G H^{\times_G b}$ from the left and $H^{\times_G (a+b)}$ from the right. By a similar argument as before, we can see that $\tilde{Z}$ implements a Morita equivalence. ☐

Next let us show the compatibility of fiber products and generalized transversals.

Proposition 2.6. *Let $\alpha\colon H \to G$ be a homomorphism of topological groupoids, and $f\colon T \to H^{(0)}$ be a generalized transversal. Consider the space*

$$\tilde{T} = \{(t_1, g_1, t_2, \dots, t_n) \mid t_k \in T, g_k \in G^{f(t_k)}_{f(t_{k+1})}\}$$

with the induced topology from the natural embedding into $T^n \times G^{n-1}$. The map

$$\tilde{f}\colon \tilde{T} \to (H^{\times_G n})^{(0)}, \quad (t_1, g_1, t_2, \dots, t_n) \mapsto (f(t_1), g_1, f(t_2), \dots, f(t_n))$$

is a generalized transversal for $H^{\times_G n}$.

Proof. Let us check the conditions in Definition 1.3. First, $\tilde{T}$ meets all orbits of $H^{\times_G n}$. Indeed, if we take a point $(y_1, g_1, y_2, \dots, g_{n-1}, y_n) \in (H^{\times_G n})^{(0)}$, we can find $t_k \in T$ and $h_k \in H^{f(t_k)}_{y_k}$ for $k = 1, \dots, n$. Then there are unique g'_k such that $(h_1, \dots, h_n)$ represents an arrow from $(y_1, g_1, y_2, \dots, g_{n-1}, y_n)$ to $(f(t_1), g'_1, \dots, f(t_n))$.

Next, let us check the condition Ar. Thus, take an arrow x represented by $(h_1, g_1, h_2, \dots, g_{n-1}, h_n)$ with range $rx = (f(t_1), g_1, f(t_2), \dots, g_{n-1}, f(t_n))$, open neighborhood U_0 of x, and another V_0 of rx. We may assume that these neighborhoods are of the form

$$U_0 = (U'_1 \times U''_1 \times U'_2 \times \dots \times U'_n) \cap H^{\times_G n}, \qquad (U'_k \in \mathcal{O}(h_k), U''_k \in \mathcal{O}(g_k))$$
$$V_0 = (V'_1 \times V''_1 \times V'_2 \times \dots \times V'_n) \cap \tilde{T}, \qquad (V'_k \in \mathcal{O}(t_k), V''_k \in \mathcal{O}(g_k)).$$

Then, for each k we can find $\tilde{U}_k \in \mathcal{O}(h_k)$ with $\tilde{U}_k \subset U'_k$, $\tilde{V}_k \in \mathcal{O}(t_k)$ with $\tilde{V}_k \subset V'_k$ realizing the condition Ar. We claim that

$$U = (\tilde{U}_1 \times U''_1 \times \dots \times \tilde{U}'_n) \cap H^{\times_G n}, \quad V = (\tilde{V}_1 \times V''_1 \times \dots \times \tilde{V}_n) \cap \tilde{T}$$

do the job. Indeed, if $y = (\tilde{h}_1, \tilde{g}_1, \cdots, \tilde{h}_n) \in U$, another element $z = (\tilde{h}'_1, \tilde{g}'_1, \cdots, \tilde{h}'_n)$ as the same source as y if and only if $s\tilde{h}_k = s\tilde{h}'_k$ and $f(\tilde{h}_k)^{-1}\tilde{g}_k f(\tilde{h}_{k+1}) = f(\tilde{h}'_k)^{-1}\tilde{g}'_k f(\tilde{h}'_{k+1})$ hold for all k. Moreover, $rz \in \tilde{T}$ if and only if $rh'_k \in f(T)$ for all k. The elements $\tilde{g}'_k$ are determined by the $\tilde{h}'_k$, and we can find such $\tilde{h}'_k$ uniquely by condition Ar for U'_k and V'_k. ☐

Suppose $f\colon T \to H^{(0)}$ is a generalized transversal for H such that $\alpha f\colon T \to G^{(0)}$ is also a transversal for G. Then α induces a homomorphism of étale groupoids from $H' = H|_{f(T)}$ to $G' = G|_{\alpha f(T)}$.

Corollary 2.7. *In the setting above, $H^{\times_G n}$ is Morita equivalent to $H'^{\times_{G'} n}$.*

Proof. The construction of Proposition 2.6 gives a generalized transversal for $\tilde{f}\colon \tilde{T} \to (H^{\times_G n})^{(0)}$. The étale groupoid obtained by this is isomorphic to $H'^{\times_{G'} n}$. ☐

2.2. **Transversality for Smale spaces.** Let (Y, ψ) be a non-wandering Smale space with totally disconnected unstable sets, and $f\colon (\Sigma, \sigma) \to (Y, \psi)$ be an s-resolving (hence s-bijective) factor map from a shift of finite type.

Let Σ_n denote the fibered product of $n+1$ copies of Σ with respect to f. Then $\sigma_n = \sigma \times \dots \times \sigma|_{\Sigma_n}$ defines a Smale space, which is again a shift of finite type. If $a = (a^0, \dots, a^n)$ and $b = (b^0, \dots, b^n)$ are points of Σ_n, they are unstably (resp. stably) equivalent if and only if a^k is unstably (resp. stably) equivalent to b^k for all k.

Theorem 2.8. *In the setting above, set $G = R^u(Y, \psi)$, $H = R^u(\Sigma, \sigma)$, and $\alpha = f \times f\colon H \to G$ be the induced groupoid homomorphism. Then $H^{\times_G n+1}$ is Morita equivalent to $R^u(\Sigma_n, \sigma_n)$ as a locally compact groupoid.*

We will apply this to the s-bijective maps from Theorem 1.31. A key step is the following proposition, which is our first technical result.

Proposition 2.9. *Let $f\colon (\Sigma, \sigma) \to (Y, \psi)$ be an s-bijective factor map from a shift of finite type. Suppose $a^0, \dots, a^n$ in Σ are points such that $f(a^0) \sim_u f(a^k)$ for all k. Then there are points $b^0, \dots, b^n$ in X satisfying*

$$a^k \sim_u b^k, \qquad f(b^0) = f(b^k)$$

for $k = 0, \dots, n$.

Lemma 2.10. *Let d be the standard metric of Σ, and $a, b \in \Sigma$ be points such that $d(a, b) < \epsilon_\Sigma$. Then we have $d([a, b], b) = d([b, a], a)$.*

Proof. As we saw in Example 1.27, the brackets are given by $[a, b] = (\dots, b_{-1}, a_0, a_1, \dots)$ and $[b, a] = (\dots, a_{-1}, a_0, b_1, \dots)$. Hence both distances are computed (in the same way) from the minimum $n > 0$ such that $a_n \neq b_n$. □

Proof of Proposition 2.9. A graphical illustration for the case $n = 1$ is provided in Figure 1.

Because the maps ψ^n for $n \in \mathbb{Z}$ preserve the unstable equivalence relation, we can assume $d(f(a^0), f(a^k)) < \epsilon < \epsilon_Y$ and $f(a^k) = [f(a^k), f(a^0)]$ holds for all k. Let $0 < \delta < \epsilon_Y/2$ be such that the maps $f\colon \Sigma^s(a^k, \delta) \to Y^s(f(a^k), \epsilon)$ are homeomorphisms onto their images.

Choose a periodic point $y_0 \in Y$ close to the points $f(a^k)$, so that $y_1 = [y_0, f(a^k)]$ and $z_k = [f(a^k), y_0]$ are well-defined. Note that y_1 does not depend on k. We claim that there are points $b^k \in \Sigma$ such that $a^k \sim_u b^k$ and $f(b^k) = y_1$.

Write $f^{-1}(y_0) = \{c^1, \dots, c^m\}$, with $m \leq N$ as in Theorem 1.32. Replacing ψ and σ by an appropriate power, we may assume that each c^i is fixed by σ.

Since f is s-bijective, there is a unique point $\bar{z}^k \in Y^s(a^k, \delta)$ satisfying $f(\bar{z}^k) = z_k$. As y_0 is fixed by ψ and $z_k \sim_u y_0$, we have the convergence $\psi^{-n}(z_k) \to y_0$. Consider the sequence $(\sigma^{-n}(\bar{z}^k))_{n=0}^{\infty}$. Since Σ is compact, we can take a cluster point w, which should be among the c^i's. Then, as the c^i's are fixed by σ, our sequence can only cluster around one of them. We thus obtain $\sigma^{-n}(\bar{z}^k) \to c^{i_k}$ for some i_k, and we get $\bar{z}^k \sim_u c^{i_k}$. Again using s-bijectivity, there is a unique $b^k \in \Sigma^s(c^{i_k})$ such that $f(b^k) = y_1$. It remains to prove that $b^k \sim_u a^k$. By Proposition 1.29, there is δ such that

$$f^{-1}(Y^u(y_0, \delta)) \subseteq \bigcup_{i=1}^{m} \Sigma^u(c^i, \epsilon'),$$

where $2\epsilon' < \min(\epsilon_f, \epsilon_\Sigma)$. Take $M > 0$ such that $\psi^{-M}(z_k) \in Y^u(y_0, \delta)$, so that we have $\sigma^{-M}(\bar{z}^k) \in \Sigma^u(c^{i_k}, \epsilon')$. Then take points $u^1, \dots, u^n$ from $\Sigma^s(c^{i_k})$ such that

$$d(c^{i_k}, u^1), d(u^1, u^2), \dots, d(u^n, \sigma^{-M}(b^k)) < \epsilon'.$$

Then we can inductively define

$$v^1 = [\sigma^{-M}(\bar{z}^k), u^1], v^2 = [v^1, u^2], \dots, v^{n+1} = [v^n, \sigma^{-M}(b^k)]$$

since $d(v^i, u^i)$ remains equal to $d(\sigma^{-M}(\bar{z}^k), c^{i_k}) < \epsilon'$ by Lemma 2.10.

Mapping down by f, we have the same relation as above for the points $\psi^{-M}(z_k)$, $f(u^i)$, and $f(v^i)$. This shows, for example, $\psi^M(f(v^1)) = [z_k, \psi^M(f(u^1))]$, and by induction, we obtain $\psi^M(f(v^{n+1})) = [z_k, y_1] = f(a^k)$. Again s-bijectivity implies $\sigma^M(v^{n+1}) = a^k$, and we obtain $a^k \sim_u b^k$. □

Remark 2.11. Although we presented a somewhat metric geometrical proof, it is possible to turn part of it into a more direct argument using a symbolic presentation of Σ; as the points c^i are represented by periodic sequences, $\bar{z}^k$ and b^k will be represented by sequences which are periodic in one direction. Combined with the consistency condition for f, it is possible to show $a^k \sim_u b^k$ from this.

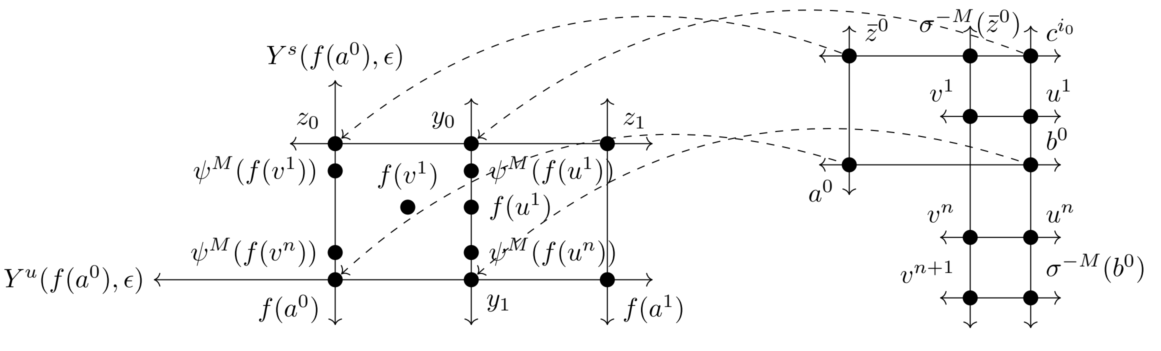

FIGURE 1. The configuration of points in the proof of Proposition 2.9. The vertical direction represents the stable direction, while the horizontal direction represents the unstable direction.

Proof of Theorem 2.8. We have an embedding of the groupoid $R^u(\Sigma_n, \sigma_n)$ into $H^{\times_G n+1}$ by the correspondence

$$(a^0, \dots, a^n) \mapsto (a^0, \mathrm{id}_y, a^1, \dots, \mathrm{id}_y, a^n) \quad (y = f(a^0) = \dots = f(a^n))$$

at the level of objects, and by

$$((a^0, \dots, a^n), (b^0, \dots, b^n)) \mapsto ((a^0, b^0), \dots, (a^n, b^n))$$

at the level of arrows. Proposition 2.9 implies that $\Sigma_n \subset (H^{\times_G n+1})^{(0)}$ meets all orbits of $H^{\times_G n+1}$. Moreover, $a, b \in \Sigma_n$ are connected by an arrow in $(H^{\times_G n+1})^{(0)}$ if and only if they are connected in $R^u(\Sigma_n, \sigma_n)$. Thus, $R^u(\Sigma_n, \sigma_n) \curvearrowright (H^{\times_G n+1})^{\Sigma_n} \curvearrowleft H^{\times_G n+1}$ gives a Morita equivalence between the two groupoids. It is a routine task to see that this is compatible with the topology on the two groupoids. ☐

Remark 2.12. Although we do not need it, we can replace (Σ, σ) in Theorem 2.8 by another Smale space (X, ϕ) with totally disconnected stable sets. The generalization of Proposition 2.9 with X can be reduced to the above one as follows. Fix a s-bijective factor map $f' \colon (\Sigma, \sigma) \to (X, \phi)$ from a shift of finite type. Starting from $a^k \in X$ with unstably equivalent image in Y, taking inverse images $\bar{a}^k \in \Sigma$ of a^k, we can find points $\bar{b}^k \in \Sigma$ satisfying the assertion for the map ff'. Then the points $b^k = f'(\bar{b}^k)$ satisfy the assertion for f.

Combining Proposition 2.4, Corollary 2.7, and Theorem 2.8, we obtain the following.

Theorem 2.13. *In addition to $f \colon (\Sigma, \sigma) \to (Y, \psi)$ as above, let $f' \colon T \to \Sigma$ be a generalized transversal for the locally compact groupoid $R^u(\Sigma, \sigma)$ such that $ff' \colon T \to Y$ defines a generalized transversal for $R^u(Y, \psi)$. Denote the corresponding étale groupoids by*

$$H = R^u(\Sigma, \sigma)|_{f'(T)}, \qquad G = R^u(Y, \psi)|_{ff'(T)}.$$

The groupoid $G \times_G H^{\times_G n+1}$ with respect to the natural inclusion $H \to G$ is Morita equivalent to $R^u(\Sigma_n, \sigma_n)$ as a topological groupoid.

3. Approximation in the equivariant KK-category

In this section we study a special situation in which Theorem 1.13 can be applied. It yields a pair of complementary subcategories which is completely characterized by a pair of adjoint functors on triangulated categories. In the setting of the equivariant Kasparov category, we obtain this pair from the induction and restriction functors, and use it to translate the strong Baum–Connes conjecture to a statement about the localizing subcategory generated the image of the induction functor.

3.1. **Simplicial approximation from adjoint functors.** One powerful way to check that a homological ideal has enough projectives is to relate it to adjoint functors between triangulated categories. More precisely, let $\mathcal{S}$ and $\mathcal{T}$ be triangulated categories with countable direct sums, and $E \colon \mathcal{S} \to \mathcal{T}$ and $F \colon \mathcal{T} \to \mathcal{S}$ be exact functors compatible with countable direct sums, with natural isomorphisms

$$\mathcal{S}(A, FB) \simeq \mathcal{T}(EA, B) \quad (A \in \mathcal{S}, B \in \mathcal{T}). \tag{7}$$

Then $\mathcal{I} = \ker F$ has enough projectives and the $\mathcal{I}$-projective objects are retracts of EA for some $A \in \mathcal{S}$ [MN10, Section 3.6].

Our next goal is to give an explicit projective resolution in this setting. In fact, this situation is quite standard in homological algebra: such adjoint functors give a *comonad* $L = EF$ on $\mathcal{T}$, from which we obtain a simplicial object $(L^{n+1}A)_{n=0}^{\infty}$ giving a “resolution” of A [Wei94, Section 8.6].

Proposition 3.1. *In the above setting, any $A \in \mathcal{T}$ admits an $\mathcal{I}$-projective resolution $P_\bullet$ consisting of $P_n = L^{n+1}A$. The pair of subcategories $(\langle E\mathcal{S} \rangle, \mathcal{N}_{\mathcal{I}})$ is complementary.*

Proof. Let us denote the structure morphisms of the adjunction as

$$\epsilon_B \in \mathcal{T}(LB, B), \qquad \eta_A \in \mathcal{S}(A, FEA),$$

so that the isomorphism (7) is given by

$$\mathcal{S}(A, FB) \to \mathcal{T}(EA, B) \qquad \mathcal{T}(EA, B) \to \mathcal{S}(A, FB)$$
$$f \mapsto \epsilon_B E(f) \qquad g \mapsto F(g)\eta_A.$$

As already observed in [MN10], the objects of the form EA are $\mathcal{I}$-projective. Indeed, if $g \in \mathcal{T}(EA, B)$ is in the kernel of F, the corresponding morphism in $\mathcal{S}(A, FB)$ is zero by the above presentation.

Next, let us recall the comonad structure on L. There are natural transformations $L \to \mathrm{id}_{\mathcal{T}}$ and $L \to L^2$ defining a coalgebra structure on L. The counit is simply given by the morphisms ϵ_B, while the comultiplication is given by $\delta_B = E(\eta_{FB}) \in \mathcal{T}(LB, L^2 B)$. The compatibility condition between these reduces to consistency between ϵ and η.

Now we are ready to define a structure of simplicial object on $(P_n)_{n=0}^\infty$ as in the assertion. The *face* morphisms $d_i^n \colon P_n \to P_{n-1}$ $(0 \le i \le n)$ are

$$d_i^n = L^i(\epsilon_{L^{n-i}A}) \colon L^{n+1}A \to L^n A,$$

while the *degeneracy* morphisms $s_i^n \colon P_n \to P_{n+1}$ $(0 \le i \le n)$ are

$$s_i^n = L^i(\delta_{L^{n-i}A}) \colon L^{n+1}A \to L^{n+2}A,$$

see [Wei94, Paragraph 8.6.4]. The associated Moore complex on $(P_n)_{n=0}^\infty$ is given by

$$\delta_n = \sum_{i=0}^n (-1)^i d_i^n \colon P_n \to P_{n-1}, \tag{8}$$

together with the augmentation morphism $\delta_0 = \epsilon \colon P_0 = LA \to A$.

Let us check the $\mathcal{I}$-exactness of the augmented complex, or as in Definition 1.15, the split exactness of

$$\cdots \to FL^2A \to FLA \to FA \to 0$$

for all A in a natural way. We claim that the the complex

$$\cdots \to FL^2A \to FLA \to FA \to 0$$

in $\mathcal{S}$ is contractible. Again this is a consequence of a standard argument: the contracting homotopy is given by $h_n = \eta_{FL^nA} \colon FL^nA \to FL^{n+1}A$ for $n \ge 0$, see [Wei94, Proposition 8.6.10].

Finally, the assertion that $\langle E\mathcal{S} \rangle$ and $\mathcal{N}_{\mathcal{I}}$ are complementary follows from Theorem 1.13. □

We will apply the previous proposition in the setting of K-theory, more precisely for $\mathcal{T} = \mathrm{KK}^G$, $\mathcal{S} = \mathrm{KK}^H$, $E = \mathrm{Ind}_H^G$, $F = \mathrm{Res}_H^G$.

3.2. **The Baum–Connes conjecture for torsion-free groupoids.** Hereafter it is assumed that G is étale and that it satisfies the conclusion of Theorem 1.22. We are going to use the notion of $\mathcal{R}\mathrm{KK}(X)$-nuclearity as defined by Bauval [Bau98, Definition 5.1] (see also [Ska88]). Here, we call it KK^X-nuclearity. Namely, a $C_0(X)$-algebra A is KK^X-nuclear if $\mathrm{id}_A \in \mathrm{KK}^X(A, A)$ is represented by an X-A-A-Kasparov cycle $(\pi, \mathcal{E}, T)$ such that the left action $\pi \colon A \to \mathcal{L}(\mathcal{E})$ is *strictly* $C_0(X)$*-nuclear* with respect to the identification $\mathcal{L}(\mathcal{E}) = \mathcal{M}(\mathcal{K}(\mathcal{E}))$.

Our next goal is to prove the following result.

Theorem 3.2. *Suppose that G is an étale groupoid with torsion-free stabilizers satisfying the conclusion of Theorem 1.22, and that $H \subseteq G$ is an étale subgroupoid with the same base space X. If A is a G-C*-algebra which is KK^X-nuclear as a $C_0(X)$-algebra, it belongs to the localizing subcategory generated by the image of $\mathrm{Ind}_H^G \colon \mathrm{KK}^H \to \mathrm{KK}^G$.*

Above, H is an open subgroupoid of G because $H^{(0)} = X$ and H is étale. The key step is to prove the special case when $H = X$.

Proposition 3.3. *Under the assumptions of Theorem 3.2, A belongs to the localizing subcategory generated by the objects $\mathrm{Ind}_X^G B$ for $C_0(X)$-algebras B.*

The following lemma clarifies the local picture of proper actions.

Lemma 3.4. *Let G be an étale groupoid with torsion-free stabilizers, and $G \curvearrowright Z$ a proper action on a locally compact Hausdorff space with the anchor map $\rho \colon Z \to X$. Then each $z \in Z$ has an open neighborhood U satisfying:*

- *U has a compact closure in Z;*
- *the saturation GU can be identified as $G \times_X U$ as a G-space.*

Proof. This is essentially contained in Proposition 2.42 of the extended version of [Tu04], but let us give a proof. First, observe that any $w \in Z$ has trivial stabilizer. Indeed, on the one hand it can be identified with the inverse image of (w, w) for the action map $\phi\colon G \ltimes Z \to Z \times_X Z$, hence is a compact set by the properness of the action. On the other hand, it is a subgroup of the stabilizer of $\rho(w)$, which is a torsion-free group, hence it must be trivial.

Next, fix an open neighborhood V of z, and put $C = (G \ltimes Z) \smallsetminus V$, where V is identified with an open subset of $G \ltimes Z$ by taking the identity morphisms of $v \in V$. Since Z is locally compact Hausdorff, ϕ is closed (with compact fibers) and in particular $\phi(C)$ is closed in $Z \times_X Z$, and it does not contain (z, z) by the above observation.

Take an open neighborhood U of z such that $U \times_X U$ does not meet $\phi(C)$. Then the restriction of the action map to $G \times_X U$ is a bijection onto GU. Indeed, if (g, u) and (g', u') had the same image in GU, we would have

$$(u, u') \in U \times_X U \cap \phi(G \ltimes Z) \subset \phi(V),$$

which implies $u = u'$ and then $g = g'$.

Finally, as $G \ltimes Z$ is an étale groupoid, the action map $G \times_X U \to Z$ is an open map. Then we obtain that the bijective continuous map $G \times_X U \to GU$ is a homeomorphism. ☐

For the next proof we use the *equivariant E-theory* of $C_0(Y)$-algebras [PT00]. The equivariant E-groups $\mathrm{E}^Y(A, B)$ (denoted by $\mathcal{R}\mathrm{E}(Y; A, B)$ in [PT00]) define a triangulated category with countable direct sums and a triangulated functor $\mathrm{KK}^Y \to \mathrm{E}^Y$ compatible with countable direct sums.

Lemma 3.5. *Let Y be a second countable locally compact space, and $(V_k)_{k=0}^\infty$ be a countable and locally finite open covering of Y. If A is a KK^Y-nuclear $C_0(Y)$-algebra, and if N is a $C_0(Y)$-algebra such that N_{V_k} is KK^{V_k}-equivalent to 0 for all k, then we have $\mathrm{KK}^Y(A, N) = 0$.*

Proof. By assumption on A, we have $\mathrm{KK}^Y(A, N) \simeq \mathrm{E}^Y(A, N)$ [PT00, Theorem 4.7]. In order to show the latter group vanishes, it is enough to show $\mathrm{E}^Y(N, N) = 0$.

Put $N_k = N_{V_0 \cup \cdots \cup V_k}$. We first claim that $\mathrm{E}^Y(N_k, N) = 0$ for all k. By induction, it is enough to prove this for $k = 1$. We have an extension of $C_0(Y)$-algebras

$$0 \to N_0 \to N_1 \to N_{V_1 \cup V_0 \smallsetminus V_0} \to 0.$$

By assumption N_0 is contractible in KK^Y (hence in E^Y). We also have the contractibility of $N_{V_1 \cup V_0 \smallsetminus V_0}$, as it is a reduction of the KK^{V_1}-contractible object N_{V_1} to $V_1 \cup V_0 \smallsetminus V_0 = V_1 \smallsetminus V_0$. Now, the functor $B \mapsto \mathrm{E}^Y(B, N)$ satisfies excision [PT00, Theorem 4.17], which gives an exact sequence of the form

$$0 = \mathrm{E}^Y(N_{V_1 \cup V_0 \smallsetminus V_0}, N) \to \mathrm{E}^Y(N_1, N) \to \mathrm{E}^Y(N_0, N) = 0,$$

and we obtain $\mathrm{E}^Y(N_1, N) = 0$.

The inclusion maps make $(N_k)_{k=0}^\infty$ an inductive system, and N is its inductive limit as a $C_0(Y)$-algebra. This inductive system is *admissible* in the sense of [MN06, Section 2.4] (this condition is automatic for inductive systems in E^Y, but this example is already admissible in KK^Y). In particular, there is an exact triangle of the form

$$\Sigma N \to \bigoplus_k N_k \to \bigoplus_k N_k \to N.$$

Since we already have $\mathrm{E}^Y(\bigoplus_k N_k, N) \simeq \prod_k \mathrm{E}^Y(N_k, N) = 0$, we obtain $\mathrm{E}^Y(N, N) = 0$. ☐

Lemma 3.6. *Let X, Y be locally compact spaces, and $f\colon Y \to X$ be an open continuous map. Suppose A is a KK^X-nuclear $C_0(X)$-algebra, and B is a $C_0(Y)$-nuclear $C_0(Y)$-algebra. Then $A \otimes_{C_0(X)} B$ is KK^Y-nuclear as a $C_0(Y)$-algebra.*

Proof. Let $(\pi, \mathcal{E}, T)$ be a $C_0(X)$-equivariant Kasparov cycle from A to A representing id_A and such that the left action $A \to \mathcal{M}(\mathcal{K}(\mathcal{E}))$ is strictly $C_0(X)$-nuclear. Similarly, take an analogous one $(\pi, \mathcal{E}', T')$ for B. Then their "cup product" $(\mathcal{E}, T) \otimes_{C_0(X)} (\mathcal{E}', T')$ [Kas88, Proposition 2.21] represents $\mathrm{id}_{A \otimes_{C_0(X)} B}$.

We claim that this cup product has the underlying Hilbert bimodule

$$\mathcal{E} \otimes_{C_0(X)} \mathcal{E}' \simeq (\mathcal{E} \otimes \mathcal{E}')_{\Delta(X)} = (\mathcal{E} \otimes \mathcal{E}')/(\mathcal{E} \otimes \mathcal{E}') C_0(\Delta(X)^{\complement}).$$

By definition, it has the underlying bimodule

$$\mathcal{E} \otimes_A (A \otimes_{C_0(X)} B) \otimes_{A \otimes_{C_0(X)} B} (\mathcal{E}' \otimes_B (A \otimes_{C_0(X)} B)).$$

By the assumption on B and f, B is also $C_0(X)$-nuclear. We thus have the identification $A \otimes_{C_0(X)} B \simeq (A \otimes B)_{\Delta(X)}$, see Remark 1.6. From this we obtain isomorphisms like $\mathcal{E} \otimes_A (A \otimes_{C_0(X)} B) \simeq (\mathcal{E} \otimes B)_{\Delta(X)}$, and consequently, the above bimodule is isomorphic to $(\mathcal{E} \otimes \mathcal{E}')_{\Delta(X)}$.

Thus, it is enough to show that the left action map $\pi \otimes \pi' \colon A \otimes_{C_0(X)} B \to L((\mathcal{E} \otimes \mathcal{E}')_{\Delta(X)})$ is strictly $C_0(Y)$-nuclear. Let $S \colon A \to L(\mathcal{E})$ (resp. $S' \colon B \to L(\mathcal{E})$) be a completely positive $C_0(X)$-linear map factoring through $M_m(C_0(X))$ (resp. $M_n(C_0(Y))$) approximating π (resp. π'). Then $S \otimes S'$ induces a completely positive map $(A \otimes B)_{\Delta(X)} \to L((\mathcal{E} \otimes \mathcal{E}')_{\Delta(X)})$ factoring through $M_m(C_0(X)) \otimes_{C_0(X)} M_n(C_0(Y)) \simeq M_{mn}(C_0(Y))$. This construction is compatible with approximation in the pointwise convergence for the strict topology of adjointable morphisms. ☐

Remark 3.7. If A is moreover $C_0(X)$-nuclear, then $A \otimes_{C_0(X)} B$ is $C_0(Y)$-nuclear with fibers $A_{f(y)} \otimes B_y$.

Proof of Proposition 3.3. Let A be a KK^X-nuclear G-algebra. By Theorem 1.22, there is a paracompact proper G-space Z and a $G \ltimes Z$-C*-algebra P such that A is KK^G-equivalent to $A \otimes_{C_0(X)} P$. By Lemma 3.6, $A \otimes_{C_0(X)} P$ is KK^Z-nuclear. Thus, we may assume that A is a KK^Z-nuclear $G \ltimes Z$-C*-algebra.

Let $U \subset Z$ be an open set satisfying the conditions of Lemma 3.4, and put $V = GU$. Then the G-algebra $A_V = C_0(V)A$ is isomorphic to $\mathrm{Ind}_X^G A_U$. Indeed, the latter is $C_0(G) \otimes_{C_0(X)} A_U$, and the G-equivariant isomorphism $V \simeq GU$ induces $A_V \simeq C_0(G) \otimes_{C_0(X)} A_U$.

Now, take countably many open sets $(U_i)_{i \in I}$ satisfying the conditions of Lemma 3.4, such that the sets $V_i = GU_i$ cover Z and $(V_i/G)_i$ is a countable and locally finite open cover of Z/G (this is possible by second countability). We want to say that the (unreduced) "Čech complex" of objects $A_{V_{i_1} \cap \dots \cap V_{i_k}}$ give a resolution of A in $\mathrm{KK}^{G \ltimes Z}$. Then, combined with the "induction functor" $\mathrm{KK}^{G \ltimes Z} \to \mathrm{KK}^G$ (which is really given by the restriction of $C_0(Z)$-algebras to $C_0(X)$-algebras), we get that A is indeed in $\langle \mathrm{Ind}_X^G \mathrm{KK}^X \rangle$. Suppose U and U' are open sets of Z satisfying the conditions of Lemma 3.4, and put $V = GU$ and $V' = GU'$. Then there is an open set W satisfying the conditions of Lemma 3.4 with $V \cap V' = GW$; indeed, we can take $W = U \cap V'$. This implies that the G-algebras $A_{V_{i_1} \cap \dots \cap V_{i_k}}$ are all of the form $\mathrm{Ind}_X^G B$.

Now, set $\tilde{Z} = \coprod_i V_i$, and regard it as a $G \ltimes Z$-space by the canonical equivariant map $\tilde{Z} \to Z$. The functors $\mathrm{Ind}_{\tilde{Z}}^Z \colon \mathrm{KK}^{G \ltimes \tilde{Z}} \to \mathrm{KK}^{G \ltimes Z}$ and $\mathrm{Res}_{\tilde{Z}}^Z \colon \mathrm{KK}^{G \ltimes Z} \to \mathrm{KK}^{G \ltimes \tilde{Z}}$ make sense. Concretely, if B is a G-equivariant $C_0(Z)$-algebra, we have

$$\mathrm{Res}_{\tilde{Z}}^Z B = \bigoplus_i B_{V_i}$$

endowed with an obvious action of G, while for a G-equivariant $C_0(\tilde{Z})$-algebra B, we set $\mathrm{Ind}_{\tilde{Z}}^Z B$ to be the same C*-algebra as B regarded as a $C_0(Z)$-algebra. Then we have the standard adjunction

$$\mathrm{KK}^{G \ltimes Z}(\mathrm{Ind}_{\tilde{Z}}^Z B, B') \simeq \prod_i \mathrm{KK}^{G \ltimes V_i}(B_{V_i}, B'_{V_i}) \simeq \mathrm{KK}^{G \ltimes \tilde{Z}}(B, \mathrm{Res}_{\tilde{Z}}^Z B').$$

From this, we see that $L = \mathrm{Ind}_{\tilde{Z}}^Z \mathrm{Res}_{\tilde{Z}}^Z$ satisfies

$$L^k A = \bigoplus_{i_1, \dots, i_k} A_{V_{i_1} \cap \dots \cap V_{i_k}}.$$

By Proposition 3.1, we obtain an exact triangle

$$P \to A \to N \to \Sigma P$$

in $\mathrm{KK}^{G \ltimes Z}$, such that P is in the localizing subcategory generated by objects of the form $\mathrm{Ind}_{U_i}^{G \ltimes Z} B$, and $N \in \ker \mathrm{Res}_{G \ltimes \tilde{Z}}^{G \ltimes Z}$.

It remains to prove that $N = 0$ in $\mathrm{KK}^{G \ltimes Z}$. For this it is enough to show that the morphism $A \to N$ in the above triangle is zero. Indeed, P will then be a direct sum of A and ΣN, but there is no nonzero morphism from P to ΣN.

Since the action of G on Z is free and proper, there is an equivalence of categories between $\mathrm{KK}^{G\ltimes Z}$ and $\mathrm{KK}^{Z/G}$, and similar statements hold for the G-invariant open sets V_i. Under this correspondence, A corresponds to a $\mathrm{KK}^{Z/G}$-nuclear algebra. Now, Lemma 3.5 implies that $\mathrm{KK}^{G\ltimes Z}(A, N) = 0$. ☐

We are now ready to prove the main result of the section.

Proof of Theorem 3.2. Consider the functors

$$\mathrm{Res}^G_H\colon \mathrm{KK}^G \to \mathrm{KK}^H, \qquad \mathrm{Ind}^G_H\colon \mathrm{KK}^H \to \mathrm{KK}^G$$

as in Section 1.6. By Proposition 3.1, we have a complementary pair $(\langle \mathcal{P}_\mathcal{I}\rangle, \mathcal{N}_\mathcal{I})$ for $\mathcal{I} = \ker \mathrm{Res}^G_H$, with $\langle \mathcal{P}_\mathcal{I}\rangle$ being generated by the image of Ind^G_H as a localizing subcategory.

Moreover, we have a natural isomorphism of functors $\mathrm{Ind}^G_X \simeq \mathrm{Ind}^G_H \mathrm{Ind}^H_X$. Concretely, if A is a $C_0(X)$-algebra, $\mathrm{Ind}^G_X A = C_0(G)\, {}^s{\otimes}_{C_0(X)} A$ and $\mathrm{Ind}^G_H \mathrm{Ind}^H_X = (C_0(G)\, {}^s{\otimes}^r_{C_0(X)} C_0(H)\, {}^s{\otimes}_{C_0(X)} A) \rtimes H$ are G-equivariantly strongly Morita equivalent via a Hilbert C*-bimodule completion of $C_c(G)\, {}^s{\otimes}^r_{C_0(X)} C_c(H)\, {}^s{\otimes}_{C_0(X)} A$. Combined with Proposition 3.3, we obtain that A belongs to $\langle \mathcal{P}_\mathcal{I}\rangle$. ☐

The following is a direct consequence of Theorem 1.22 and Theorem 3.2.

Corollary 3.8. *Let G, H, and A be as in Theorem* 3.2. *Let $P_H(A) \in \langle \mathrm{Ind}^G_H \mathrm{KK}^H\rangle$ be the algebra appearing in the exact triangle*

$$P_H(A) \to A \to N \to \Sigma P_H(A)$$

that we get by applying Proposition 3.1. *Then we have $P_H(A) \simeq A$ in KK^G.*

Corollary 3.9. *Let G, H, and A be as in Theorem* 3.2. *Then we have a convergent spectral sequence*

$$E^2_{pq} = H_p(K_q(G \ltimes L^{\bullet+1}A)) \Rightarrow K_{p+q}(G \ltimes A), \tag{9}$$

where $L^n A = (\mathrm{Ind}^G_H \mathrm{Res}^G_H)^n(A)$.

Proof. The reduced crossed product functor

$$\mathrm{KK}^G \to \mathrm{KK}, \quad A \mapsto G \ltimes A$$

is exact and compatible with direct sums, while

$$\mathrm{KK} \to \mathrm{Ab}, \quad B \mapsto K_0(B)$$

is a homological functor. Thus, their composition

$$K_0(G \ltimes \text{-})\colon \mathrm{KK}^G \to \mathrm{Ab}$$

is a homological functor, cf. [MN10, Examples 13 and 15]. Now we can apply Theorem 1.19 to get a spectral sequence

$$H_p(K_q(G \ltimes P_\bullet)) \Rightarrow K_{p+q}(G \ltimes P_H(A)),$$

where $P_\bullet$ is a $(\ker \mathrm{Res}^G_H)$-projective resolution of A. The $(\ker \mathrm{Res}^G_H)$-projective resolution from Proposition 3.1 gives the left hand side of (9). Now the claim follows from Corollary 3.8. ☐

Remark 3.10. It would be an interesting question to cast the above constructions to groupoid equivariant E-theory [Pop04], since we mostly use formal properties of KK^G. However, since some parts of our constructions involve reduced crossed products, there are some details to be checked. (Note that H need not be a proper subgroupoid.)

4. Homology and K-theory

In this section we relate the construction of the previous section to groupoid homology for ample groupoids with torsion-free stabilizers. As for the Smale spaces with totally disconnected stable sets, a similar construction will allow us to relate to Putnam's homology.

Suppose G is a second countable locally compact Hausdorff étale groupoid, and H is an open subgroupoid with the same base space. Let us analyze the chain complex in (8) more concretely. Let $s_n\colon G^{(n)} \to X$ be the map $(g_1, \ldots, g_n) \mapsto s g_n$.

Lemma 4.1. *Let A be an H-C^*-algebra. The $C_0(G^{(n)})$-algebra $s_n^* A$ is endowed with a continuous action of the groupoid $G \times_G H^{\times_G n}$.*

Proof. We give a concrete proof for $n = 1$, as the general case can be done following the same idea. We use $(C_0(G) \otimes_{\min} A)_{\Delta(X)}$ as a model of $C_0(G) \otimes_{C_0(X)} A$, and analogous models for other relative C*-algebra tensor products as well. Recall that the arrow set of $G \times_G H$ can be identified with the set of triples (g, g_1, h) where $(g, g_1) \in G^{(2)}$, $h \in H$, and $s(g_1) = s(h)$. Then

$$C_0(G \times_G H)\ {}^s\!\otimes_{C_0(G)} (C_0(G)\ {}^s\!\otimes_{C_0(X)} A) \simeq (C_0(G \times_G H \times G) \otimes A)_Y,$$

where Y is the space of tuples (g, g_1, h, g_1, x) with (g, g_1, h) as above and $x = s(g_1)$. On the other hand,

$$C_0(G^{(2)})\ {}^s\!\otimes^s_{C_0(X)} C_0(H)\ {}^s\!\otimes_{C_0(G)} A \simeq (C_0(G^{(2)} \times H) \otimes A)_Z,$$

where Z is the space of quadruples (g, g_1, h, x) where the components are related as above. Via the obvious homeomorphism between Y and Z, we have the identification of these algebras. The structure map $\alpha\colon C_0(H)\ {}^s\!\otimes_{C_0(G)} A \to C_0(H)\ {}^r\!\otimes_{C_0(G)} A$ of the H-C*-algebra induces an isomorphism onto

$$C_0(G^{(2)})\ {}^s\!\otimes^s_{C_0(X)} C_0(H)\ {}^r\!\otimes_{C_0(G)} A \simeq (C_0(G^{(2)} \times H) \otimes A)_{Z'},$$

where Z' is the space of quadruples (g, g_1, h, y) with (g, g_1, h) as above and $y = r(h)$. Finally, we have

$$C_0(G \times_G H)\ {}^r\!\otimes_{C_0(G)} (C_0(G) \otimes_{C_0(X)} A) = (C_0(G \times_G H \times G) \otimes A)_{Y'},$$

where Y' is the space of tuples (g, g_1, h, g_1', y) where (g, g_1, h) is as above, $g_1' = g g_1 h^{-1}$, and $y = s(g_1') = r(h)$. Again the obvious bijection between Y' and Z' induces an isomorphism of the last two algebras, and combining everything we obtain an isomorphism

$$C_0(G \times_G H)\ {}^s\!\otimes_{C_0(G)} s_1^* A \to C_0(G \times_G H)\ {}^r\!\otimes_{C_0(G)} s_1^* A$$

which is the desired structure morphism of $G \times_G H$-C*-algebra. □

Proposition 4.2. *In the setting above, the functor $L = \operatorname{Ind}_H^G \operatorname{Res}_H^G\colon \mathrm{KK}^G \to \mathrm{KK}^G$ satisfies*

$$G \ltimes L^n A \simeq (G \times_G H^{\times_G n}) \ltimes s_n^* A.$$

Proof. We have $L^n A = H^{\times_G n} \ltimes s_n^* A$ by expanding the definitions. □

Using the Morita equivalence between $G \times_G H^{\times_G n}$ and $H^{\times_G n}$, we can replace the formula above with $H^{\times_G n} \ltimes s_{n-1}^* A$. This enables us to transport the simplicial structure on $(G \ltimes L^{n+1} A)_{n=0}^\infty$ to $(H^{\times_G (n+1)} \ltimes s_n^* A)_{n=0}^\infty$. The proof is again straightforward from definitions.

Proposition 4.3. *The induced simplicial structure on $(H^{\times_G (n+1)} \ltimes s_n^* A)_{n=0}^\infty$ has face maps d_i^n represented by the composition of* KK*-morphisms*

$$C_r^*(H^{\times_G (n+1)}) \to C_r^*(H^{\times_G i} \times_G G \times_G H^{\times_G (n-i)}) \to C_r^*(H^{\times_G n}),$$

where the first morphism is induced by the inclusion $H^{\times_G (n+1)} \to H^{\times_G i} \times_G G \times_G H^{\times_G (n-i)}$ as an open subgroupoid, and the second morphism is given by the Morita equivalence of Proposition 2.5*.*

4.1. **Induction from unit space and groupoid homology.** Let us further assume that G is ample, and consider the case $H = H^{(0)} = X = G^{(0)}$. Proposition 2.4 says that we can replace $G \times_G H^{\times_G (n+1)}$ by the Morita equivalent groupoid $H^{\times_G (n+1)}$. Now, this is just $G^{(n)}$ as a locally compact space with trivial groupoid structure. Here we obtain the complex of groupoid homology which was descrbed in Section 1.3. Looking at the relevant coefficients, for the algebra $C_0(X)$ we have $K_0(C_0(X)) \simeq C_c(X, \mathbb{Z})$, which corresponds to the constant sheaf $\underline{\mathbb{Z}}$ on X. More generally, any G-C*-algebra gives a G-sheaf on X.

Proposition 4.4. *Let A be a G-C*-algebra. Then $K_i(A)$ is a unitary $C_c(G, \mathbb{Z})$-module.*

Proof. We show that $K_i(A)$ is a unitary $C_c(X, \mathbb{Z})$-module, and the associated sheaf is a G-sheaf.

The structure map of $C_0(X)$-algebra induces a $*$-homomorphism $C_0(X) \otimes A \to A$. Combined with the canonical map $K_0(C_0(X)) \otimes K_i(A) \to K_i(C_0(X) \otimes A)$, we obtain a map $K_0(C_0(X)) \otimes K_i(A) \to K_i(A)$, hence a $C_c(X, \mathbb{Z})$-module structure on $K_i(A)$.

Next let us check the unitarity of this module. By total disconnectedness and second countability of X, we can take an increasing sequence of compact open sets $(U_k)_{k=1}^\infty$ in X such that $\phi_k = \chi_{U_k}$ form an approximate unit of $C_0(X)$. Replacing A by its suspension if necessary, it is enough to check that, for any class $c \in K_0(A)$, there is k such that $[\phi_k] \in K_0(C_0(X))$ satisfies $[\phi_k]c = c$.

By definition, c is represented by a formal difference $[e]-[f]$ of projections $e, f \in M_n(A^+)$ such that $\pi([e]) = \pi([f])$ in $K_0(\mathbb{C})$, where n is some integer, A^+ is the unitization of A, and $\pi\colon A^+ \to \mathbb{C}$ is the canonical quotient map. By conjugating by a unitary in $M_n(\mathbb{C})$, we can arrange $\pi(e) = \pi(f)$. Then we can write the components of e as $e_{ij} = \alpha_{ij} + e'_{ij}$ for $\alpha_{ij} \in \mathbb{C}$ and $e'_{ij} \in A$, and those of f as $f_{ij} = \alpha_{ij} + f'_{ij}$.

Now, put $x^{(k)}_{ij} = (1-\phi_k)\alpha_{ij}$, $y^{(k)}_{ij} = \phi_k e_{ij}$, and $z^{(k)}_{ij} = \phi_k f_{ij}$. These form projections $x^{(k)} \in M_n(A^+)$, $y^{(k)}, z^{(k)} \in M_n(A)$, such that $x^{(k)} + y^{(k)}$ and $x^{(k)} + z^{(k)}$ are still projections. If $\phi_k e'_{ij}$ is close enough to e'_{ij} in norm, e is close to $x^{(k)} + y^{(k)}$ in norm, and we obtain $[e] = [x^{(k)} + y^{(k)}]$ in $K_0(A^+)$ for large enough k. Similarly, we obtain $[f] = [x^{(k)} + z^{(k)}]$ for large enough k. Then we have $c = [y^{(k)}] - [z^{(k)}]$, and for this k we indeed have $[\phi_k]c = c$.

It remains to give an action of G on the associated sheaf F. Take $g \in G$, and choose its open compact neighborhood U such that s and r restrict to homeomorphisms on U. Then the action of G induces an isomorphism $A_{s(U)} \to A_{r(U)}$. In turn this induces $\chi_{s(U)}K_i(A) \to \chi_{r(U)}K_i(A)$, which can be interpreted as the action of g from $\Gamma(s(U), F)$ to $\Gamma(r(U), F)$. A routine bookkeeping shows that these maps patch up to give an action morphism $s^*F \to r^*F$ on G. □

Proposition 4.5. *When A is a G-C^*-algebra, there is an isomorphism of chain complexes*

$$(K_i(G \ltimes L^{\bullet+1}A), \delta_\bullet) \simeq (C_c(G^{(\bullet)}, \mathbb{Z}) \otimes_{C_c(X,\mathbb{Z})} K_i(A)), \partial_\bullet)$$

for $L = \operatorname{Ind}_X^G \operatorname{Res}_X^G$.

Proof. From the equivalence of groupoids between $G \ltimes G^{(n+1)}$ and $G^{(n)}$ (where we consider $G^{(k)}$ as spaces), we have $K_i(G \ltimes L^{n+1}A) \simeq K_i(C_0(G^{(n)})\ {}^s\!\otimes_{C_0(X)} A)$.

Since $G^{(n)}$ is totally disconnected, we have

$$K_0(C_0(G^{(n)})) \simeq C_c(G^{(n)}, \mathbb{Z}), \qquad K_1(C_0(G^{(n)})) = 0.$$

Thus, we have an isomorphism of unitary $C_c(G, \mathbb{Z})$-modules

$$K_i(C_0(G^{(n)})\ {}^s\!\otimes_{C_0(X)} A) \simeq C_c(G^{(n)}, \mathbb{Z}) \otimes_{C_c(X,\mathbb{Z})} K_i(A).$$

We have a (semi-)simplicial structure on $K_0(C_0(G^{(n)}))$ from Proposition 4.3. It is a routine calculation to compare this with the one above from the nerve structure. □

Thus, we obtain an isomorphism of homology groups

$$H_p(K_q(G \ltimes L^{\bullet+1}A), \delta_\bullet) \simeq H_p(G, K_q(A)).$$

Theorem 4.6. *Let G be a second countable Hausdorff ample groupoid with torsion-free stabilizers satisfying the strong Baum–Connes conjecture, and A be a KK^X-nuclear separable G-C^*-algebra. Then there is a convergent spectral sequence*

$$E^r_{pq} \Rightarrow K_{p+q}(G \ltimes A), \tag{10}$$

with $E^2_{pq} = H_p(G, K_q(A))$.

Proof. We obtain the convergent spectral sequence by Corollary 3.9, and Proposition 4.5 gives the description of the E^2-sheet. □

Specializing to the case $A = C_0(X)$, we obtain our main result.

Corollary 4.7. *Let G be as above. Then there is a convergent spectral sequence*

$$E^r_{pq} \Rightarrow K_{p+q}(C^*_r(G)),$$

with $E^2_{pq} = E^3_{pq} = H_p(G, K_q(\mathbb{C}))$.

Proof. As $K_q(\mathbb{C}) = 0$ for odd q, by degree reasons the E^2-differential is trivial. This implies $E^2_{pq} = E^3_{pq}$. □

Remark 4.8. Looking at the bidegree of differentials at the E^3-sheet, we see that the above spectral sequence collapses at the E^2-sheet if $H_k(G, \mathbb{Z})$ vanishes for $k \geq 3$. If, in addition, $H_2(G, \mathbb{Z})$ is torsion-free, one has

$$K_0(C^*_r G) \simeq H_0(G, \mathbb{Z}) \oplus H_2(G, \mathbb{Z}), \qquad K_1(C^*_r G) \simeq H_1(G, \mathbb{Z}).$$

This covers the transformation groupoids of minimal $\mathbb{Z}$-actions on the Cantor space considered in [Mat12] and the Deaconu–Renault groupoids of rank 1 and 2 (in particular k-graph groupoids for $k = 1, 2$) in [FKPS19], and groupoids of 1-dimensional generalized solenoids [Yi20]. The Exel–Pardo groupoid model [EP17] for Katsura's realization [Kat08] of Kirchberg algebras also belong to this class [Ort18]. For the groupoid of tiling spaces (see Section 5.2) one can do slightly better; if G is a groupoid associated with some tiling in $\mathbb{R}^d$, one has the vanishing of $H_k(G, \mathbb{Z})$ for $k > d$ and $H_d(G, \mathbb{Z})$ is torsion-free. Comparing the rank of $H_\bullet(G, \mathbb{Z})$ and $K_\bullet(C^*G)$, we see that the higher differentials are always zero on $H_d(G, \mathbb{Z})$, and the spectral sequence collapses if $d \leq 3$.

Remark 4.9. For the transformation groupoids $\Gamma \ltimes X$ where $X = E\Gamma$ is a "nice" manifold (such as carrying an invariant Riemannian metric with nonpositive sectional curvature), [Kas88] gives a spectral sequence analogous to (10).

Remark 4.10. In Theorem 4.6, without assuming that G has torsion-free stabilizers, or that it satisfies the strong Baum–Connes conjecture, we still have a convergent spectral sequence

$$E^2_{pq} = H_p(G, K_q(A)) \Rightarrow K_{p+q}(G \ltimes P_A)$$

where P_A is a $(\ker \operatorname{Res}^G_X)$-simplicial approximation of A, see Section 5.4 for an example.

4.2. **Putnam's homology for Smale spaces.** Let (Y, ψ) be an irreducible Smale space with totally disconnected stable sets. Then there is an irreducible shift of finite type (Σ, σ) and an s-bijective factor map $f\colon (\Sigma, \sigma) \to (Y, \psi)$. Recall that Σ_n stands for the fiber product of $(n+1)$-copies of Σ over Y, with $\Sigma_0 = \Sigma$.

For any shift of finite type (Σ, σ) the group $K_0(C^*(R^u(\Sigma, \sigma)))$ can be described as Krieger's *dimension group* $D^s(\Sigma, \sigma)$ [Kri80]. This group is generated by the elements $[E]$ for compact open sets E in the stable orbits in Σ. We can restrict to a collection of stable orbits which form a generalized transversal, and also assume that E is contained in a *local* stable orbit as well [Pro18b, Lemma 1.3].

The simplicial structure on the groups $D^s(\Sigma_\bullet, \sigma_\bullet)$ is induced by natural face maps $\delta^s_k\colon \Sigma_n \to \Sigma_{n-1}$, which delete the k-th entry of a point in Σ_n. This yields a well defined map between the corresponding dimension groups, via the assignment $[E] \mapsto [\delta^s_k(E)]$. This way the groups $D^s(\Sigma_\bullet, \sigma_\bullet)$ form a simplicial object, and the associated homology $H^s_\bullet(Y, \psi)$, called *stable homology* of (Y, ψ), does not depend on f [Put14, Section 5.5; Pro18b].

For a suitable choice of $P \subseteq \Sigma$, we have an open inclusion of étale groupoid $f \times f\colon R^u(\Sigma, P) \to R^u(Y, f(P))$. We set $G = R^u(Y, f(P))$ and $H = (f \times f)(R^u(\Sigma, P))$. Notice that G is ample and H is AF [Put15, Tho10a].

Proposition 4.11. *There is an isomorphism of chain complexes*

$$(K_0(G \ltimes L^{\bullet+1}C_0(X)), \delta_\bullet) \simeq (D^s(\Sigma_\bullet, \sigma_\bullet), d^s(f)_\bullet), \qquad (K_1(G \ltimes L^{\bullet+1}C_0(X)), \delta_\bullet) \simeq 0$$

Before going into the proof, let us recall the concept of *correspondences* between groupoids. In general, if G and H are topological groupoids, a correspondence from G to H is a topological space Z together with commuting proper actions $G \curvearrowright Z \curvearrowleft H$, such that the anchor map $Z \to H^{(0)}$ is open (surjective) and induces a homeomorphism $G \backslash Z \simeq H^{(0)}$. Of course, one source of such correspondence is Morita equivalence. Another example is provided by continuous homomorphisms $f\colon G \to H$, where one puts $Z = \{[g, h] \mid f(sg) = rh\}$ with the relation $[g_1 g_2, h] = [g_1, f(g_2)h]$.

If G and H are (second countable) locally compact Hausdorff groupoids with Haar systems, a correspondence Z induces a right Hilbert $C^*_r(H)$-module $C^*_r(Z)_{C^*_r(H)}$ with a left action of $C^*_r(G)$ [MSO99]. If the action of $C^*_r(G)$ is in $\mathcal{K}(C^*_r(Z)_{C^*_r(H)})$, we obtain a map $K_\bullet(C^*_r(G)) \to K_\bullet(C^*_r(H))$. While finding a good characterization of this condition in terms of Z seems to be somewhat tricky, in concrete examples as below it is not too difficult.

On the other hand, composition of such Hilbert modules are easy to describe. If H' is another topological groupoid with Haar system, and Z' is a correspondence from H to H', we have the identification

$$C^*_r(Z)_{C^*_r(H)} \otimes_{C^*_r(H)} C^*_r(Z')_{C^*_r(H')} \simeq C^*_r(Z \times_H Z')_{C^*_r(H')}.$$

Proof of Proposition 4.11. By Proposition 4.2 and Theorem 2.13, the C*-algebra $G \ltimes L^{n+1}C_0(X)$ is strongly Morita equivalent to $C^*(R^u(\Sigma_n, \sigma_n))$. From this we have the identification of the underlying modules, and it remains to compare the corresponding simplicial structures. Let us give a concrete comparison of the maps $K_0(C^*_r(H^{\times_G n+1})) \to D^s(\Sigma_{n-1}, \sigma_{n-1})$ corresponding to the 0-th face maps, as the general case is completely parallel.

Let us put $\tilde{G} = R^u(Y, \psi)$, $\tilde{H} = R^u(\Sigma, \sigma)$, and take a (generalized) transversal T' for $R^u(\Sigma_n, \sigma_n)$, and put $K = R^u(\Sigma_n, \sigma_n)|_{T'}$, $K' = R^u(\Sigma_{n-1}, \sigma_{n-1})|_{\delta_0(T')}$ so that we have

$$D^s(\Sigma_n, \sigma_n) \simeq K_0(C^*_r(K)), \qquad D^s(\Sigma_{n-1}, \sigma_{n-1}) \simeq K_0(C^*_r(K')).$$

We denote the generalized transversal of $\tilde{H}^{\times_{\tilde{G}} n}$ induced by P, as in Proposition 2.6, by $\tilde{T}_n$.

The map δ_0 induces a groupoid homomorphism $K \to K'$, and hence a correspondence Z_{δ_0} from K to K'. Composing this with the Morita equivalence bibundle ${}_{\tilde{T}_{n+1}}(\tilde{H}^{\times_{\tilde{G}} n+1})_{T'}$, we obtain a correspondence

$${}_{\tilde{T}_{n+1}}(\tilde{H}^{\times_{\tilde{G}} n+1})_{T'} \times_K Z_{\delta_0} \tag{11}$$

from $H^{\times_G n+1}$ to K' representing the effect of δ_0 on the K-groups.

As for the 0-th face map d_0 of $H^{\times_G \bullet+1}$, let Z be the Morita equivalence bibundle between $G \times_G H^{\times_G n}$ and $H^{\times_G n}$ from Proposition 2.4. Since $H^{\times_G n+1}$ is an open subgroupoid of $G \times_G H^{\times_G n}$, Z becomes a correspondence from $H^{\times_G n+1}$ to $H^{\times_G n}$. Composing this with the Morita equivalence ${}_{\tilde{T}_n}(\tilde{H}^{\times_{\tilde{G}} n})_{\delta_0(T')}$ between $H^{\times_G n}$ and K', we obtain the correspondence

$$Z \times_{H^{\times_G n}} {}_{\tilde{T}_n}(\tilde{H}^{\times_{\tilde{G}} n})_{\delta_0(T')} \tag{12}$$

from $H^{\times_G n+1}$ to K' representing the effect of d_0.

It remains to check that the above correspondences are isomorphic, hence giving isomorphic Hilbert modules. Expanding the ingredients of (12), we obtain the space

$$W = \{(g_0, h_1, g_1, h_2, \dots, g_{n-1}, h_n) \mid (g_0, \dots, g_{n-1}) \in G^{(n)}, h_k \in \tilde{H}^{s g_{k-1}}, (s h_1, \dots, s h_n) \in \delta_0(T')\}.$$

On the other hand, (11) gives $W \times_K K'$ with

$$W' = \{(h_1, g_1, h_2, \dots, g_n, h_{n+1}) \mid (g_1, \dots, g_n) \in G^{(n)},$$
$$h_k \in \tilde{H}^{r g_k}, h_{n+1} \in \tilde{H}^{s g_n}, (s h_1, \dots, s h_{n+1}) \in T'\}.$$

The operation - $\times_K K'$ "kills" the component h_1, and we obtain the identification with W. ☐

Thus, we obtain isomorphisms of homology groups

$$H_p(K_q(G \ltimes L^{\bullet+1} C_0(X)), \delta_\bullet) \simeq H^s_p(Y, \psi) \otimes K_q(\mathbb{C}).$$

Theorem 4.12. *Let (Y, ψ) be an irreducible Smale space with totally disconnected stable sets. Then there is a convergent spectral sequence*

$$E^r_{pq} \Rightarrow K_{p+q}(C^*(R^u(Y, \psi))), \tag{13}$$

with $E^2_{pq} = E^3_{pq} = H^s_p(Y, \psi) \otimes K_q(\mathbb{C})$.

Proof. The proof is parallel to that of Corollary 4.7, but this time we use Corollary 3.9 and Proposition 4.11. ☐

Corollary 4.13. *The K-groups $K_i(C^*(R^u(Y, \psi)))$ have finite rank.*

Proof. By the above theorem, for $i = 0, 1$, the rank of $K_i(C^*(R^u(Y, \psi)))$ is bounded by that of $\bigoplus_k H^s_{i+2k}(Y, \psi)$. The latter is of finite rank by [Put14, Theorem 5.1.12]. ☐

Remark 4.14. By the Pimsner–Voiculescu exact sequence, the same can be said for the unstable Ruelle algebra $\mathbb{Z} \ltimes_\psi C^*(R^u(Y, \psi))$. If the stable relation $R^s(Y, \psi)$ also has finite rank K-groups, the Ruelle algebras will have finitely generated K-groups by [KPW17].

In fact, Putnam's homology is isomorphic to groupoid homology in the above setting, which gives an alternative proof of the previous result.

Theorem 4.15. *We have $H^s_\bullet(Y, \psi) \simeq H_\bullet(G, \mathbb{Z})$.*

Proof. Let us consider $G^{(n+1)}$ as an $H^{\times_G (n+1)}$-space by the anchor map

$$(g_0, \dots, g_n) \mapsto (g_1, \dots, g_n) \in G^{(n)} = (H^{\times_G (n+1)})^{(0)}$$

and the action map

$$(h_1, g_1, h_2, \dots, h_n)(g_0, \dots, g_n) = (g_0 h_1^{-1}, g_1', \dots, g_n')$$

in the notation of Definition 2.1. Then $H_0(H^{\times_G (n+1)}, C_c(G^{(n+1)}, \mathbb{Z}))$ is a unitary $C_c(X, \mathbb{Z})$-module by the action from the left, and the associated sheaf F_n on X is a G-sheaf by the left translation action of G. At $x \in X$, the stalk can be presented as

$$(F_n)_x = H_0(H^{\times_G (n+1)}, C_c((G^{(n+1)})^x, \mathbb{Z})) = C^c((G^{(n+1)})^x, \mathbb{Z})_{H^{\times_G (n+1)}}. \tag{14}$$

Indeed, the sheaf corresponding to the $C_c(X, \mathbb{Z})$-module $C_c(G^{(n+1)}, \mathbb{Z})$ has the stalk $C^c((G^{(n+1)})^x, \mathbb{Z})$ at x, and taking coinvariants by $H^{\times_G (n+1)}$ commutes with taking stalks.

We then have

$$H_0(G, F_n) \simeq H_0(G \times_G H^{\times_G (n+1)}, \mathbb{Z}) \simeq H_0(H^{\times_G (n+1)}, \mathbb{Z}).$$

The simplicial structure on $(G \times_G H^{\times_G (n+1)})_n$ leads to the complex of G-sheaves

$$\cdots \to F_2 \to F_1 \to F_0, \tag{15}$$

and $H^s_\bullet(Y, \psi)$ is the homology of the complex obtained by applying the functor $H_0(G, \text{-})$ to (15).

We first claim that the augmented complex

$$\cdots \to F_2 \to F_1 \to F_0 \to \underline{\mathbb{Z}} \to 0 \tag{16}$$

is exact. It is enough to check the exactness at the level of stalks. In terms of the presentation (14), we have the chain complex

$$\cdots \to C_c((G^{(2)})^x, \mathbb{Z})_{H^{\times_G 2}} \to C_c(G^x, \mathbb{Z})_H \to \mathbb{Z}$$

with differential

$$\begin{aligned} d((g_1, \dots, g_{n+1})) = (g_1 g_2, g_3, \dots, g_{n+1}) - (g_1, g_2 g_3, \dots, g_{n+1}) + \cdots \\ + (-1)^{n-1}(g_1, \dots, g_n g_{n+1}) + (g_1, \dots, g_n), \end{aligned}$$

where $(g_1, \dots, g_n) \in (G^{(n)})^x$ represents the image of $\delta_{(g_1, \dots, g_n)} \in C_c((G^{(n)})^x, \mathbb{Z})$ in the coinvariant space, and the augmentation is given by $d(g) = 1$ at $n = 0$. This has a contracting homotopy given by $\mathbb{Z} \to C_c(G^x, \mathbb{Z})_H$, $a \to a(\mathrm{id}_x)$ and

$$C_c((G^{(n)})^x, \mathbb{Z})_{H^{\times_G n}} \to C_c((G^{(n+1)})^x, \mathbb{Z})_{H^{\times_G n+1}}, \quad (g_1, \dots, g_n) \to (\mathrm{id}_x, g_1, \dots, g_n),$$

hence (16) is indeed exact.

We next claim that $H_k(G, F_n) = 0$ for $k > 0$. Let H_{n+1} be a subgroupoid of G which is Morita equivalent to $H^{\times_G (n+1)}$ (this exists by choosing a good transversal for (Σ_n, σ_n)). Then the module $H_0(H^{\times_G (n+1)}, C_c(G^{(n+1)}, \mathbb{Z}))$ representing F_n is isomorphic to $H_0(H_{n+1}, C_c(G, \mathbb{Z}))$. Thus, it is enough to check the claim when $n = 0$.

Let us write $M = H_0(H, C_c(G, \mathbb{Z}))$, and consider the double complex of modules $C_c(G^{(p+1)} \times_X H^{(q)}, \mathbb{Z})$ for $p, q \geq 0$, with differentials coming both from the simplicial structures on $(G^{(p)})_{p=0}^\infty$ and $(H^{(q)})_{q=0}^\infty$, cf. [CM00, Theorem 4.4]. For fixed p, this is a resolution of $C_c(G^{(p)}, \mathbb{Z}) \otimes_{C_c(X, \mathbb{Z})} M$, hence the double complex computes $H_\bullet(G, F)$. For fixed q, this is a resolution of $H_0(G, C_c(G \times_X H^{(q)}, \mathbb{Z})) \simeq C_c(H^{(q)}, \mathbb{Z})$, and this double complex also computes $H_\bullet(H, \mathbb{Z})$. Since H is Morita equivalent to an AF groupoid, $H_k(H, \mathbb{Z}) = 0$ by [Mat12, Theorem 4.11]. We thus obtain $H_k(G, F_n) = 0$.

Finally, consider the hyperhomology $\mathbb{H}_\bullet(G, F_\bullet)$. On the one hand, by the above vanishing of $H_k(G, F_n)$, this is isomorphic to the homology of the complex $(H_0(G, F_n))_n$, i.e., $H^s_\bullet(Y, \psi)$. On the other hand, since $(F_n)_n$ is quasi-isomorphic to $\underline{\mathbb{Z}}$ concentrated in degree 0, we also have $\mathbb{H}_\bullet(G, F_\bullet) \simeq H_\bullet(G, \mathbb{Z})$. ☐

We then have the following Künneth formula from the corresponding result for groupoid homology [Mat16, Theorem 2.4].

Corollary 4.16. *Let (Y_1, ψ_1) and (Y_2, ψ_2) be Smale spaces with totally disconnected stable sets. Then we have a split exact sequence*

$$0 \to \bigoplus_{a+b=k} H^s_a(Y_1, \psi_1) \otimes H^s_b(Y_2, \psi_2) \to H^s_k(Y_1 \times Y_2, \psi_1 \times \psi_2) \to \bigoplus_{a+b=k-1} \mathrm{Tor}(H^s_a(Y_1, \psi_1), H^s_b(Y_2, \psi_2)) \to 0.$$

Remark 4.17. As usual, the splitting is not canonical. This generalizes [DKW16, Theorem 6.5], in which one of the factors is assumed to be a shift of finite type. Indeed, if (Y_1, ψ_1) is a shift of finite type, the first direct sum reduces to $D^s(Y_1, \psi_1) \otimes H^s_k(Y_2, \psi_2)$, while the second direct sum of torsion groups vanishes as the dimension group $D^s(Y_1, \psi_1)$, being torsion-free, is flat.

Remark 4.18. Theorem 4.15 holds in general without the assumption of total disconnectedness on stable sets. We plan to expand on this direction in a forthcoming work.

5. Examples

5.1. **Solenoid.** One class of motivating example is that of *one-dimensional solenoids* [vD30, Wil67]. Let us first explain the easiest example, the m^∞-solenoid. Consider the space

$$Y = \{(z_0, z_1, \dots) \mid z_k \in S^1, z_k = z_{k+1}^m\},$$

which is the projective limit of

$$S^1 \xleftarrow[z^m \leftarrow\!\shortmid z]{} S^1 \xleftarrow[z^m \leftarrow\!\shortmid z]{} S^1 \xleftarrow[z^m \leftarrow\!\shortmid z]{} \cdots . \tag{17}$$

A compatible metric is given by

$$d((z_k)_k, (z'_k)_k) = \sum_k m^{-k} d_0(z_k, z'_k),$$

where d_0 is any metric on S^1 compatible with its topology; for example, one may take the arc-length metric $d_0(e^{is}, e^{it}) = |s - t|$ when $|s - t| \le \pi$.

There is a natural "shift" self-homeomorphism

$$\phi\colon Y \to Y, \quad (z_0, z_1, \dots) \mapsto (z_0^m, z_1^m = z_0, z_2^m = z_1, \dots),$$

with inverse given by $\phi^{-1}((z_0, z_1, \dots)) = (z_1, z_2, \dots)$. Then (Y, ϕ) is a Smale space.

Denote by π the canonical projection $Y \to S^1$ on the first factor. As each step of (17) is an m-to-1 map, $\pi^{-1}(z_0)$ can be identified with the Cantor set $\Sigma = \prod_{n=1}^\infty \{0, 1, \dots, m-1\}$ for any $z_0 \in S^1$. This allows us to write local stable and unstable sets around $z = (z_k)_k$, as

$$Y^s(z, \epsilon) = \pi^{-1}(z_0) \cong \Sigma, \qquad Y^u(z, \epsilon) = \{(e^{itm^{-k}} z_k)_{k=0}^\infty \mid |t| < \delta_\epsilon\} \tag{18}$$

for small enough $\epsilon > 0$, with $\delta_\epsilon > 0$ depending on ϵ. Note that π defines a fiber bundle with fiber Σ, and $Y^u(z, \epsilon) \times \Sigma \to Y$ corresponding to the bracket map gives local trivializations.

Now, the groupoid $R^u(Y, \phi)$ is the transformation groupoid $\mathbb{R} \ltimes_\alpha Y$ for the flow

$$\alpha_t(z_0, z_1, \dots) = (e^{it} z_0, e^{itm^{-1}} z_1, \dots, e^{itm^{-k}} z_k, \dots) \quad (t \in \mathbb{R}).$$

Restricted to the transversal $\pi^{-1}(1)$, we obtain the "odometer" transformation groupoid $\mathbb{Z} \ltimes_\beta \Sigma$, where Σ is identified with $\varprojlim_k \mathbb{Z}_{m^k}$, and the generator $1 \in \mathbb{Z}$ acts by the $+1$ map on $\mathbb{Z}_{m^k}$.

There is a well-known factor map from the two-sided full shift on m letters onto (Y, ϕ). Namely, writing

$$\Sigma' = \{0, 1, \dots, m-1\}^{\mathbb{Z}} = \{(a_n)_{n=-\infty}^\infty \mid 0 \le a_n < m\},$$

we have a continuous map $f\colon \Sigma' \to Y$ by

$$f((a_n)_n) = (z_k)_{k=0}^\infty, \quad z_k = \exp\biggl(2\pi i \sum_{j=0}^\infty m^{-j-1} a_{-k+j}\biggr).$$

Then we have $f\sigma = \phi f$ for $\sigma\colon \Sigma' \to \Sigma'$ defined by $\sigma((a_n)_n) = (a_{n-1})_n$.

This allows us to compute all relevant invariants separately. As for the K-groups, by Connes's Thom isomorphism,

$$K_0(C^* R^u(Y, \phi)) \simeq K^1(Y) \simeq \mathbb{Z}\left[\frac{1}{m}\right], \qquad K_1(C^* R^u(Y, \phi)) \simeq K^0(Y) \simeq \mathbb{Z}.$$

As for groupoid homology, we have

$$H_\bullet(\mathbb{Z} \ltimes_\beta \Sigma, \mathbb{Z}) \simeq H_\bullet(\mathbb{Z}, C(\Sigma, \mathbb{Z}))$$

where right hand side is the groupoid homology of $\mathbb{Z}$ with coefficient $C(\Sigma, \mathbb{Z})$ endowed with the $\mathbb{Z}$-module structure induced by β. This leads to

$$H_0(\mathbb{Z} \ltimes_\beta \Sigma, \mathbb{Z}) \simeq C(\Sigma, \mathbb{Z})_\beta \simeq \mathbb{Z}\left[\frac{1}{m}\right], \qquad H_1(\mathbb{Z} \ltimes_\beta \Sigma, \mathbb{Z}) \simeq C(\Sigma, \mathbb{Z})^\beta \simeq \mathbb{Z},$$

with coinvariants and invariants of β, while $H_n(\mathbb{Z} \ltimes_\beta \Sigma, \mathbb{Z}) = 0$ for $n > 1$. The computation for $H^s_\bullet(Y, \phi)$ will be more involved, but one finds [Put14, Section 7.3] that

$$H^s_0(Y, \phi) \simeq D^s(\Sigma', \sigma) \simeq \mathbb{Z}\left[\frac{1}{m}\right], \qquad H^s_1(Y, \phi) \simeq \mathbb{Z},$$

and $H^s_n(Y, \phi) = 0$ for $n > 1$. Thus the spectral sequences of Corollary 4.7 and Theorem 4.12 collapse at the E^2-sheet, and there is no extension problem.

5.2. **Substitution tiling.** We follow the convention of [KP00], and consider substitution tilings of finite local complexity. Thus, we are given a finite set P of *prototiles* in $\mathbb{R}^d$ and a substitution rule ω for P. The associated hull Ω admits a self-homeomorphism induced by ω, again denoted by ω, giving a Smale space (Ω, ω). Under reasonable assumptions on ω, the translation action τ of $\mathbb{R}^d$ on Ω is free and minimal. Then, analogously to the case of solenoids, the groupoid of the unstable equivalence relation is the transformation groupoid $\mathbb{R}^d \ltimes_\tau \Omega$. Moreover, by [SW03], there is a transversal $X \subset \Omega$ that is homeomorphic to a Cantor set, such that $(\mathbb{R}^d \ltimes_\tau \Omega)|_X$ is the transformation groupoid $\mathbb{Z}^d \ltimes_\alpha X$ for some action $\alpha\colon \mathbb{Z}^d \curvearrowright X$, see also [KP03, Section 5].

Let us quickly explain how a spectral sequence of more classical nature arises in this setting. By Connes's Thom isomorphism, the right hand side is

$$K_n(C^* R^u(\Omega, \omega)) \simeq K^{n+d}(\Omega).$$

Now, Ω can be identified with a projective limit of a self-map of branched d-dimensional manifold obtained by gluing (collared) prototiles [AP98]. This leads to the Atiyah–Hirzebruch spectral sequence

$$E_2^{p,q} = \check{H}^p(\Omega, K_q(\mathbb{C})) \Rightarrow K^{p+q}(\Omega), \tag{19}$$

that is, $E_2^{p,q}$ is the p-th Čech cohomology of Ω with constant sheaf $\underline{\mathbb{Z}}$ when q is even, and $E_2^{p,q} = 0$ otherwise (for dimension reasons we also have $E_2^{p,q} = 0$ if $p > d$). Since Ω is a compact Hausdorff space, this is also equal to the sheaf cohomology as derived functor. Since the action τ is free and $\mathbb{R}^d$ is contractible, Ω is a model of the classifying space BG and the universal principal bundle EG for the groupoid $G = R^u(\Omega, \omega) = \mathbb{R}^d \ltimes_\tau \Omega$ (up to nonequivariant homotopy). In particular, we can interpret the sheaf cohomology on Ω as groupoid cohomology of G, see [Moe98, Tu06].

Let us relate our construction to this. Using the transversal X, we have

$$H^\bullet(G|_X, \mathbb{Z}) \simeq H^\bullet(\mathbb{Z}^d, C(X, \mathbb{Z})), \qquad H_\bullet(G|_X, \mathbb{Z}) \simeq H_\bullet(\mathbb{Z}^d, C(X, \mathbb{Z})),$$

where we consider $C(X, \mathbb{Z})$ as a module over $\mathbb{Z}^d$ by the action induced by α. Moreover we have $H_k(\mathbb{Z}^d, M) \simeq H^{d-k}(\mathbb{Z}^d, M)$ for any $\mathbb{Z}^d$-module M, see for example [Bro94, Section VIII.10]. This shows that

$$H_k(G|_X, \mathbb{Z}) \simeq H^{d-k}(G|_X, \mathbb{Z}) \simeq H^{d-k}(G, \mathbb{Z})$$

for the étale groupoid $G|_X$, and the spectral sequence of Corollary 4.7 is comparable to (19).

Let us also remark that these observations imply

$$H^s_k(\Omega, \omega) \simeq \check{H}^{d-k}(\Omega),$$

giving a positive answer to [Put14, Question 8.3.2] in the case of tiling spaces.

Remark 5.1. A spectral sequence analogous to (13) is given in [SB09], as an analogue of the Serre spectral sequence for the Anderson–Putnam fibration structure $\Omega \to \Gamma_k$ over the k-collared prototile space. It would be an interesting question to compare these.

5.3. **Semidirect product by torsion-free groups.** Suppose that a group Γ acts on a (second countable locally compact Hausdorff) groupoid G. Then we can form a semidirect product $\Gamma \ltimes G$: its object set is the same as that of G, its arrow set is the direct product $\Gamma \times G$, with structure maps $s(\gamma, g) = s(g)$, $r(\gamma, g) = \gamma r(g)$, and composition rule $(\gamma, g)(\gamma', g') = (\gamma\gamma', \gamma'^{-1}(g)g')$. We then have a following analogue of the permanence property of the strong Baum–Connes conjecture for extension of torsion free discrete groups [OO01].

Proposition 5.2. *Suppose that Γ is torsion-free and satisfies the strong Baum–Connes conjecture, and that G is an ample groupoid with torsion-free stabilizers satisfying the strong Baum–Connes conjecture. Then any KK^X-nuclear separable $\Gamma \ltimes G$-C*-algebra A belongs to the localizing subcategory generated by the image of $\mathrm{Ind}_X^{\Gamma\ltimes G}\colon \mathrm{KK}^X \to \mathrm{KK}^{\Gamma\ltimes G}$.*

Proof. Let us fix A as in the assertion. First consider the functor

$$F\colon \mathrm{KK}^\Gamma \to \mathrm{KK}^{\Gamma\ltimes G}, \quad B \mapsto B \otimes A,$$

where Γ acts on $B \otimes A$ diagonally and G acts on the leg of A. This is a triangulated functor compatible with countable direct sums.

By assumption on Γ, the trivial Γ-C*-algebra $\mathbb{C}$ belongs to the localizing subcategory generated by objects of the form $C_0(\Gamma) \otimes B'$ for separable C*-algebras B'. Thus, $A = F(\mathbb{C})$ belongs to the localizing subcategory generated by the $C_0(\Gamma) \otimes B' \otimes A$.

Now, we claim that the $\Gamma \ltimes G$-C*-algebra $C_0(\Gamma) \otimes A$ is isomorphic to $\mathrm{Ind}_G^{\Gamma\ltimes G} \mathrm{Res}_G^{\Gamma\ltimes G} A$, by an analogue of Fell's absorption principle. Both algebras can be interpreted as the direct sum of copies of A indexed by the elements of Γ. For $C_0(\Gamma) \otimes A$, the action of G becomes component-wise action on this direct sum, while the action of Γ is the combination of translation on indexes and component-wise action. For $\mathrm{Ind}_G^{\Gamma\ltimes G} \mathrm{Res}_G^{\Gamma\ltimes G} A$, the action of G preserves direct summands, but twisted by the effect of γ on G on the γ-th component. The action of Γ simply becomes translation of indexes. We can move from one presentation to another by applying γ or γ^{-1} on the γ-th component.

We thus have A in the localizing subcategory generated by $\mathrm{Ind}_G^{\Gamma\ltimes G} \mathrm{Res}_G^{\Gamma\ltimes G} A \otimes B'$ for separable C*-algebras B'. By Proposition 3.3, $\mathrm{Res}_G^{\Gamma\ltimes G} A \in \mathrm{KK}^G$ belongs to the localizing subcategory generated by the image of $\mathrm{Ind}_X^G\colon \mathrm{KK}^X \to \mathrm{KK}^G$. Combined with natural isomorphism $\mathrm{Ind}_G^{\Gamma\ltimes G} \mathrm{Ind}_X^G \simeq \mathrm{Ind}_X^{\Gamma\ltimes G}$, we obtain the assertion (note that the C*-algebras B' above receive trivial action). ☐

Consequently, if G is moreover ample, the conclusion of Theorem 4.6 holds for $\Gamma \ltimes G$.

Let (Y, ψ) be a Smale space with totally disconnected stable sets. Then the groupoid $\mathbb{Z} \ltimes_\psi R^u(Y, \psi)$ behind the unstable Ruelle algebra R_u fits into the above setting. Indeed, as a generalized transversal of $R^u(Y, \psi)$ take $X = Y^s(P)$ for some set P of periodic points of ψ. Then X is stable under ψ, and $\mathbb{Z} \ltimes_\psi (R^u(Y, \psi)|_X)$ is Morita equivalent to $\mathbb{Z} \ltimes_\psi R^u(Y, \psi)$.

Remark 5.3. Recall that Y is of the form $\varprojlim Y_0$ for a projective system of some compact metric space Y_0 and a suitable self-map $g\colon Y_0 \to Y_0$ as the connecting map at each step [Wie14], analogous to the standard presentation of the m^∞-solenoid above. Suppose further that g is open and the groupoid C*-algebra of stable relation $R^s(Y, \psi)$ has finite rank K-groups. Then, combining [KPW17] and [DGMW18], we see that $K_\bullet(R_u)$ fits in an exact sequence

$$K^{*+1}(C(Y_0)) \xrightarrow{1-[E_g]} K^{*+1}(C(Y_0)) \longrightarrow K_\bullet(R_u) \longrightarrow K^\bullet(C(Y_0)) \xrightarrow{1-[E_g]} K^\bullet(C(Y_0)),$$

where E_g is the $C(Y_0)$-bimodule associated with g. It would be an interesting problem to compare the two ways to compute $K_\bullet(R_u)$.

The setting of this section can also be applied to the study of Deaconu–Renault groupoids (see [FKPS19, Section 6] for proofs). Let X be a locally compact Hausdorff space, and σ be an action of $\mathbb{N}^k$ on X by surjective local homeomorphisms. The associated Deaconu–Renault groupoid $G = G(X, \sigma)$ is defined by

$$G = \{(x, p - q, y) \in X \times \mathbb{Z}^k \times X \mid \sigma^p(x) = \sigma^q(y)\}.$$

There is a natural cocycle $c\colon G(X, \sigma) \to \mathbb{Z}^k$ given by $c(x, n, y) = n$, and the resulting skew-product groupoid $G \times_c \mathbb{Z}^k$ is free and AF. By considering the automorphisms $\alpha_p\colon ((x, m, y), n) \mapsto ((x, m, y), n + p)$, we obtain a semidirect product groupoid $\tilde{G} = G \times_c \mathbb{Z}^k \rtimes_\alpha \mathbb{Z}^k$, which is homologically similar to G. Moreover, $H_\bullet(G)$ is the group homology of $\mathbb{Z}^k$ with coefficients in $H_0(G \times_c \mathbb{Z}^k)$. On the K-theory side, Takai–Takesaki duality implies that $C^*_r G$ is stably isomorphic to $C^*_r \tilde{G}$. Hence, for the purpose of comparing homology and K-theory, we can use $\tilde{G}$ in place of G.

5.4. **A non-example.** Scarparo has found a counterexample to the HK conjecture [Sca19]. In his example G is the transformation groupoid of an action α of the infinite dihedral group $\Gamma = \mathbb{Z}_2 \ltimes \mathbb{Z}$ on the Cantor set X. Thus, it is amenable and in particular satisfies the strong Baum–Connes conjecture. However, α is not free, and the simplicial approximation $P(C(X))$ arising from restriction to the unit space is indeed not KK^G-equivalent to $C(X)$. Let us explain the ingredients in more detail.

Let $(n_i)_{i=0}^\infty$ be a strictly increasing sequence of integers such that, for $i \geq 1$, $n_{i+1}/n_i \in \mathbb{N}$ for all i. We take the model $X = \varprojlim \mathbb{Z}_{n_i}$. Then $\mathbb{Z}$ acts by the odometer action, i.e., $1 \in \mathbb{Z}$ acts by the $+1$ map on each factor $\mathbb{Z}_{n_i}$. There is a consistent action of $\mathbb{Z}_2$, where the nontrivial element $g = [1] \in \mathbb{Z}_2$ acts by multiplication by -1, giving rise to an action α of Γ on X. Note that α is topologically free but not free, nor does it have torsion-free stabilizers.

Put $G = \Gamma \ltimes_\alpha X$, and

$$M = \left\{ \frac{m}{n_i} \mid m \in \mathbb{Z}, i \geq 1 \right\}.$$

The C*-algebra $C^*G = \Gamma \ltimes_\alpha C(X)$ is an AF algebra, with

$$K_0(C^*G) \simeq \begin{cases} M \oplus \mathbb{Z} & \text{if } n_{i+1}/n_i \text{ is even for infinitely many } i, \\ M \oplus \mathbb{Z}^2 & \text{otherwise,} \end{cases}$$

see [BEK93]. On the other hand, the groupoid homology is

$$\begin{aligned} H_0(G, \mathbb{Z}) &\simeq M, \\ H_{2k}(G, \mathbb{Z}) &\simeq 0, \\ H_{2k-1}(G, \mathbb{Z}) &\simeq \begin{cases} \mathbb{Z}_2 & \text{if } n_{i+1}/n_i \text{ is even for infinitely many } i, \\ \mathbb{Z}_2^2 & \text{otherwise,} \end{cases} \end{aligned}$$

for $k > 1$, see [Sca19]. This shows that groupoid homology cannot form a spectral sequence converging to $K_\bullet(C^*G)$, much less being isomorphic to it.

Fortunately, there is a somewhat concrete description of $P(C(X))$ in this case. Consider the antipodal action of $\mathbb{Z}_2$ on S^n, that is, g acts by the restriction of the multiplication by -1 on $\mathbb{R}^{n+1}$. Then the contractible space $S^\infty = \varinjlim S^n$ is a model of the universal bundle $E\mathbb{Z}_2$. We want to make sense of an analogue of Poincaré dual for this.

Let $Y_n = C_0(T^*S^n)$ denote the function algebra of the total space of the cotangent bundle of S^n, and Y'_n denote the $\mathbb{Z}_2$-graded C*-algebra of continuous sections of the C*-algebra bundle $\mathrm{Cl}_\mathbb{C}(T^*S^n)$ over S^n with complex Clifford algebras $\mathrm{Cl}_\mathbb{C}(T^*_x S^n)$ as fibers. These admit naturally induced actions of $\mathbb{Z}_2$, and Y_n is $\mathrm{KK}^{\mathbb{Z}_2}$-equivalent to Y'_n [Kas16, Theorem 2.7].

Let us recall the (equivariant) Poincaré duality between $C(S^n)$ and Y'_n [Kas88, Section 4]. The natural Clifford module structure on the differential forms of S^n, together with $D'_n = d + d^*$, give an unbounded model of a K-homology class $[D'_n] \in K^0_{\mathbb{Z}_2}(Y'_n)$. Composed with the product map $m\colon Y'_n \otimes C(S^n) \to Y'_n$, we obtain the class $[D_n] = m \otimes_{Y'_n} [D'_n] \in K^0_{\mathbb{Z}_2}(Y'_n \otimes C(S^n))$. The dual class $[\Theta_n] \in K_0^{\mathbb{Z}_2}(C(S^n) \otimes Y'_n)$ is defined as a certain class localized around the diagonal.

Let $j\colon S^n \to S^{n+1}$ be the embedding at the equator (which is a $\mathbb{Z}_2$-equivariant continuous map), and let $j'\colon Y'_n \to Y'_{n+1}$ be the $\mathrm{KK}^{\mathbb{Z}_2}$-morphism dual to the restriction map $j^*\colon C(S^{n+1}) \to C(S^n)$. Thus, we have

$$j' = [\Theta_{n+1}] \otimes_{C(S^{n+1}) \otimes Y_{n+1}} (\mathrm{id}_{Y'_n} \otimes j^* \otimes \mathrm{id}_{Y'_n}) \otimes_{Y_n \otimes C(S^n)} [D_n],$$

see [Kas88, Theorem 4.10].

Lemma 5.4. *We have* $j' \otimes_{Y'_{n+1}} [D'_{n+1}] = [D'_n]$ *in* $K^0_{\mathbb{Z}_2}(Y'_n)$.

Proof. As a $\mathrm{KK}^{\mathbb{Z}_2}$-morphism, $[D'_n]$ is the dual of the embedding $\eta_n\colon \mathbb{C} \to C(S^n)$, hence the claim reduces to $\eta_{n+1} = j\eta_n$. ☐

Take the homotopy colimit $Y'_\infty = \varinjlim Y'_n$ in $\mathrm{KK}^{\mathbb{Z}_2}$ (to be precise, we are working in the enlarged category of $\mathbb{Z}_2$-graded C*-algebras). By the above lemma, the morphisms $[D'_n]$ induce a morphism

$[D'_\infty] \in \mathrm{KK}^{\mathbb{Z}_2}(Y'_\infty, \mathbb{C})$. Transporting this by the $\mathrm{KK}^{\mathbb{Z}_2}$-equivalence, we obtain $Y_\infty = \varinjlim Y_n$ and $[D_\infty] \in \mathrm{KK}^{\mathbb{Z}_2}(Y_\infty, \mathbb{C})$.

Lemma 5.5. *The image of $[D_\infty]$ in $\mathrm{KK}(Y_\infty, \mathbb{C})$ is a* KK*-equivalence.*

Proof. In the nonequivariant KK-category, Y_n is equivalent to $\mathbb{C}^2$ or $\mathbb{C} \oplus \Sigma\mathbb{C}$ depending on the parity of n, and there is a distinguished summand which is equivalent to $\mathbb{C}$ (at the even degree) spanned by the K-theoretic fundamental class of T^*S^n. Moreover, the morphism corresponding to $[D'_n]$ is a projection onto this summand.

The KK-morphisms corresponding to j' preserve the fundamental class while killing the other direct summand. Thus, the limit is equivalent to $\mathbb{C}$, spanned by the image of the fundamental classes, and $[D_\infty]$ gives the equivalence. ☐

Since $\mathbb{Z}_2$ acts freely on T^*S^n, each Y_n is orthogonal to the kernel of restriction functor $\mathrm{KK}^{\mathbb{Z}_2} \to \mathrm{KK}$. The discussion so far can be readily adjusted to the groupoid G, as follows. Here, $Y_n \otimes C(X)$ is a G-C*-algebra for which Y_n only sees the action of $\mathbb{Z}_2$.

Proposition 5.6. *The G-C*-algebra $Y_n \otimes C(X)$ belongs to the localizing subcategory generated by the image of $\mathrm{Ind}_X^G\colon \mathrm{KK}^X \to \mathrm{KK}^G$.*

Proof. First, $G \ltimes (T^*S^n \times X)$ is a free groupoid. Indeed, it is the transformation groupoid of the action $\Gamma \curvearrowright T^*S^n \times X$, but any element $\gamma \in \Gamma$ that has a fixed point in X is either conjugate to $(g, 0)$ or $(g, 1)$. (Here, g is the nontrivial element of $\mathbb{Z}_2$ and we identify Γ with $\mathbb{Z}_2 \times \mathbb{Z}$ as a set.) By the freeness of $\mathbb{Z}_2 \curvearrowright T^*S^n$, these elements cannot have fixed points in $T^*S^n \times X$.

We thus obtain that $Y_n \otimes C(X)$ belongs to the localizing subcategory generated by the image of $\mathrm{Ind}_{T^*S^n \times X}^{G \ltimes (T^*S^n \times X)}$, see the proof of Proposition 3.3. Using the triangulated functor $\mathrm{KK}^{G \ltimes (T^*S^n \times X)} \to \mathrm{KK}^G$ given by restricting the scalars of $C_0(T^*S^n \times X)$-algebras to $C(X)$, we obtain the assertion. ☐

Corollary 5.7. *We have $P_{\mathcal{I}} C(X) \simeq Y_\infty \otimes C(X)$ for $\mathcal{I} = \ker \mathrm{Res}_X^G$, with the corresponding KK^G-morphism $Y_\infty \otimes C(X) \to C(X)$ given by $[D_\infty] \otimes \mathrm{id}_{C(X)}$.*

Consequently, the spectral sequence of groupoid homology converges to the K-theory groups of the algebra $G \ltimes (Y_\infty \otimes C(X))$.

Appendix A. Structure of groupoid equivariant KK-theory

As in the other parts of paper, G denotes a locally compact Hausdorff groupoid with Haar system, and we write $X = G^{(0)}$. We denote the category of separable G-C*-algebras by C^*_G. We regard $C_c(G)$ as a $C_0(X)$-module via pullback by s, and denote its completion a right Hilbert $C_0(X)$-module with respect to the inner product by the Haar system by $L^2(G)$.

A.1. **Invariant ideals.**

Let us check that continuous actions of G restrict to kernel of equivariant homomorphisms.

Proposition A.1. *Let $f\colon A \to B$ be an equivariant homomorphism of G-C*-algebras. Then $I = \ker f$ is a G-C*-algebra.*

Proof. Since I is an ideal of A, it inherits a structure of $C_0(X)$-algebra. We need to show that there is an isomorphism of $C_0(G)$-algebras

$$s^*I = C_0(G) \;{}^s\!\otimes_{C_0(X)} I \to r^*I = C_0(G) \;{}^r\!\otimes_{C_0(X)} I$$

defining a continuous action of G. By the nuclearity of $C_0(G)$ as a C*-algebra,

$$0 \to C_0(G) \otimes I \to C_0(G) \otimes A \to C_0(G) \otimes B \to 0$$

is exact.

We first claim that s^*I is the kernel of $s^*A \to s^*B$ induced by f. By the $C_0(X)$-nuclearity of $C_0(G)$, we can write

$$s^*I = (C_0(G) \otimes I)_{\Delta(X)},$$

etc. Then we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I' & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_0(G) \otimes I & \longrightarrow & C_0(G) \otimes A & \longrightarrow & C_0(G) \otimes B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & s^* I & \longrightarrow & s^* A & \longrightarrow & s^* B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with $I' = C_0((G \times X) \smallsetminus (G \times_X X))(C_0(G) \otimes I)$, etc., and we know the exactness of the vertical sequences and top and middle horizontal sequences. Then the bottom sequence is also exact, which establishes the claim.

Then looking at the action map

$$s^* A \to r^* A,$$

we see that $s^* I$ is mapped onto $r^* I = \ker(r^* A \to r^* B)$ bijectively. □

A.2. **Stabilization.** When $\mathcal{E}$ is a Hilbert A-module, we denote the Hilbert A-module direct sum of countable copies of $\mathcal{E}$ by $\mathcal{E}^\infty$.

Lemma A.2. *Let A be a separable G-C*-algebra, and $\mathcal{E}$ be a countably generated Hilbert G-A-module. If E is full as an right Hilbert A-module, we have*

$$L^2(G)^\infty \otimes_{C_0(X)} \mathcal{E} \simeq L^2(G)^\infty \otimes_{C_0(X)} A$$

as Hilbert G-A-modules.

Proof. By fullness, we have $\mathcal{E}^\infty \simeq A^\infty$ as Hilbert A-modules [Lan95, Proposition 7.4]. Then the assertion follows from [Pop04, Lemma 3.6]. □

Proposition A.3. *Let F be a functor from C^*_G to an additive category. Then the following conditions are equivalent:*

1. *if $\mathcal{E}$ is a Hilbert G-A-module which is full over A, the natural maps*
$$F(A) \to F(\mathcal{K}(A \oplus \mathcal{E})), \qquad F(\mathcal{K}(\mathcal{E})) \to F(\mathcal{K}(A \oplus \mathcal{E}))$$
are isomorphisms;
2. *same as above, but just for $\mathcal{E} = \mathcal{E}' \otimes_{C_0(X)} A$ where $\mathcal{E}'$ is a Hilbert G-$C_0(X)$-module which is full over $C_0(X)$;*
3. *same as above, but just for $\mathcal{E}' = L^2(G)^\infty$.*

Proof. The only nontrivial implication is from (3) to (1). Since $\mathcal{K}(L^2(G)^\infty \otimes_{C_0(X)} A)$ is isomorphic to $\mathcal{K}(L^2(G)^\infty) \otimes_{C_0(X)} A$, (3) implies that $F(A) \simeq F(\mathcal{K}(L^2(G)^\infty) \otimes_{C_0(X)} A)$. Suppose $\mathcal{E}$ is as in (1). Then $\mathcal{K}(L^2(G)^\infty) \otimes_{C_0(X)} \mathcal{K}(\mathcal{E})$ is isomorphic to $\mathcal{K}(L^2(G)^\infty \otimes_{C_0(X)} A)$ by this observation and Lemma A.2. We thus obtain $F(A) \simeq F(\mathcal{K}(\mathcal{E}))$, and a routine bookkeeping gives that this can be indeed induced by maps as in (1). □

If the conditions in the above proposition is satisfied, we say that F is *stable*.

A.3. **Universal property.** Again suppose F is a functor from C^*_G to an additive category. As usual, F is *homotopy invariant* if the evaluation maps $A \otimes C([0,1]) \to A$ at $0 \le t \le 1$ induce isomorphisms $F(A \otimes C([0,1])) \simeq F(A)$, and is *split exact* if an extension $I \to A \to B$ with splitting $B \to A$ by an equivariant $*$-homomorphism induces an isomorphism $F(A) \simeq F(I) \oplus F(B)$.

Proposition A.4. *The canonical functor $\mathrm{C}^*_G \to \mathrm{KK}^G$ is a universal functor satisfying stability, homotopy invariance, and split exactness.*

Before getting into the proof, recall that an element in $\mathrm{KK}^G(A,B)$ is by definition represented by a G-A-B-Kasparov cycle $(\pi,\mathcal{E},T)$, where $\mathcal{E}$ is a $\mathbb{Z}_2$-graded right Hilbert B-module, π is a $*$-homomorphism from A to $\mathcal{L}(\mathcal{E})$, and T is a certain odd endomorphism of $\mathcal{E}$. Note that T is only assumed to be G-equivariant up to compact errors. A key ingredient is the following result of Oyono-Oyono, which allows us to replace such cycles by strictly equivariant ones. (To be precise, his result is for odd cycles, but his construction is compatible with grading on the underlying Hilbert module, otherwise we can work with suspensions.)

Proposition A.5 ([Laf07, Section A.4])**.** *Under the above setting, there is an odd G-equivariant endomorphism T' on $\tilde{\mathcal{E}} = L^2(G)^\infty \otimes_{C_0(X)} \mathcal{E}$ such that $(\iota\otimes\pi,\tilde{\mathcal{E}},T')$ is a G-$(\mathcal{K}(L^2(G)^\infty)\otimes_{C_0(X)} A)$-$B$-Kasparov cycle, and such that $(S\otimes\pi(a))(1\otimes T-T')$ is a compact endomorphism for all $S\in\mathcal{K}(L^2(G)^\infty)$ and $a\in A$.*

Another important ingredient is the "Cuntz picture" of $\mathrm{KK}^G(A,B)$. To simplify the notation, put $\tilde{A}=\mathcal{K}(L^2(G)^\infty)\otimes_{C_0(X)} A$. A *$G$-equivariant quasi-homomorphism* from $\tilde{A}$ to $\tilde{B}$ is given by pair of G-equivariant $*$-homomorphisms ϕ^+,ϕ^- from $\tilde{A}$ to $\mathcal{M}(\tilde{B})$ such that $\phi^+(a)-\phi^-(a)\in\tilde{B}$ for all $a\in\tilde{A}$. This induces a Kasparov G-$\tilde{A}$-B-cycle

$$\left(\phi^+\oplus\phi^-,(L^2(G)^\infty\otimes_{C_0(X)} B)^{\oplus 2},T=\begin{bmatrix}0&1\\1&0\end{bmatrix}\right). \tag{20}$$

Proof of Proposition A.4. Let us explain how to present $\mathrm{KK}^G(A,B)$ in terms of equivariant quasi-homomorphisms using Proposition A.5. Let us start with a G-A-B-Kasparov cycle $(\pi,\mathcal{E},T)$. Adding a degenerate direct summand, we may assume that E is full as a Hilbert B-module. Take a G-equivariant endomorphism T' on $\tilde{\mathcal{E}}$ as above. $1\otimes T$ and T' define homotopic Kasparov cycles, so any class in $\mathrm{KK}^G(A,B)$ has a G-equivariant representative by replacing A with $\tilde{A}$.

Doing the same for G-A-$(B\otimes C([0,1]))$-Kasparov cycles, we see that, if $(\pi_0,\mathcal{E}_0,T_0)$ and $(\pi_1,\mathcal{E}_1,T_1)$ are homotopic cycles, then T_0' are T_1' are homotopic through a G-equivariant path. Consequently $\mathrm{KK}^G(A,B)$ is the quotient set of the G-$\tilde{A}$-B-Kasparov cycles $(\pi,\mathcal{E},T)$, with G-equivariant T, up to the equivalence relation generated by G-equivalent homotopy, G-equivariant unitary equivalence, and ignoring the difference of direct sum with degenerate cycles.

Moreover, we can replace T by a G-equivariant endomorphism satisfying $T=T^*=T^{-1}$ without breaking the equivalence relation, see [Bla86, Chapter 17]. By Lemma A.2, we may assume that T is represented on $L^2(G)^\infty\otimes_{C_0(X)} B$. Then we can write T in the form of (20), and the left action of $\tilde{A}$ is given by a G-equivariant $*$-homomorphism $\pi\colon\tilde{A}\to\mathcal{M}(\tilde{B})$. Finally, the commutation relation with T implies that π is of the form $\phi^+\oplus\phi^-$ for an equivariant quasi-homomorphism (ϕ^+,ϕ^-) from $\tilde{A}$ to $\tilde{B}$. Consequently, $\mathrm{KK}^G(A,B)$ is isomorphic to the set of equivalence classes of equivariant quasi-homomorphisms $(\phi^+,\phi^-)\colon\tilde{A}\to\tilde{B}$ up to equivariant homotopy, equivariant unitary equivalence, and ignoring the difference of direct sum with degenerate ones.

Next let us relate quasi-homomorphisms to split extensions, cf. [Bla86, PT00]. Let (ϕ^+,ϕ^-) be an equivariant quasi-homomorphism from $\tilde{A}$ to $\tilde{B}$. Put

$$D=\{(a,\phi^+(a)+b)\mid a\in\tilde{A},b\in\tilde{B}\}\subset\tilde{A}\oplus\mathcal{M}(\tilde{B}),$$

which is a G-C*-algebra by Lemma A.6. Moreover, this fits into a split extension

$$\tilde{B}\xrightarrow{\ j\ }D\underset{s}{\overset{q}{\rightleftarrows}}\tilde{A},$$

with $j(b)=(0,b)$, $q(a,\phi^+(a)+b))=a$, and $s(a)=(a,\phi^-(a))$.

Suppose that $F\colon\mathrm{C}^*_G\to\mathcal{C}$ is a functor into an additive category satisfying stability, homotopy invariance, and split exactness. We want to show that there is a uniquely determined functor $\tilde{F}\colon\mathrm{KK}^G\to\mathcal{C}$ factoring F up to natural isomorphisms. Given an equivariant quasi-homomorphism (ϕ^+,ϕ^-) from $\tilde{A}$ to $\tilde{B}$, construct D as above. Then we have an identification $F(D)\simeq F(\tilde{B})\oplus F(\tilde{A})$, so the projection onto the first summand combined with stability gives a morphism $\phi_*\colon F(D)\to F(B)$. Moreover, there is another equivariant $*$-homomorphism $f\colon\tilde{A}\to D$ defined by $f(a)=(a,\phi^+(a))$. We then obtain $\tilde{F}(\phi^+,\phi^-)\colon F(A)\to F(B)$ by combining $\phi_*\circ F(f)$ with stability for A. This construction is compatible with the equivalence relation on quasi-homomorphisms, and we obtain a well-defined functor $\tilde{F}\colon\mathrm{KK}^G\to\mathcal{C}$ extending F.

Uniqueness follows from functoriality and the following observation: if $B \to D \to A$ is a split extension, D is a model for the direct sum $B \oplus A$ in KK^G. More concretely, the ideal inclusion $j\colon B \to D$ defines a homomorphism $\tilde{\jmath}\colon D \to \mathcal{M}(B)$, and $(\tilde{\jmath}, \tilde{\jmath} s q)$ in the above notation defines a quasi-homomorphism from D to B. This is a projection of D to B in KK^G, and together with the other maps in the extension these KK^G-morphisms give the structure morphisms of the direct sum. □

Lemma A.6. *The algebra D in the above proof is a G-C^*-algebra.*

Proof. First let us check that D is a $C_0(X)$-algebra. If $f \in C_0(X)$, $a \in \tilde{A}$ and $b \in \tilde{B}$, we obviously put $f.(a, \phi^+(a) + b) = (fa, \phi^+(fa) + fb)$. Since fa and fb can approximate a and b, we see that this defines a nondegenerate homomorphism $C_0(X) \to \mathcal{M}(D)$.

Next, the maps

$$D \to \tilde{A}, \quad (a, \phi^+(a) + b) \mapsto a, \qquad\qquad D \to \tilde{B}, \quad (a, \phi^+(a) + b) \mapsto b$$

are $C_0(X)$-linear and completely bounded. From this we see that $C_0(G)\, {}^s\otimes_{C_0(X)} D$ is a direct sum of $C_0(G)\, {}^s\otimes_{C_0(X)} \tilde{A}$ and $C_0(G)\, {}^s\otimes_{C_0(X)} \tilde{B}$ as an operator space, and similar decomposition holds for $C_0(G)\, {}^r\otimes_{C_0(X)} D$. Then the action maps on $\tilde{A}$ and $\tilde{B}$ given an action map on D. □

A.4. **Triangulated structure.** Let $f\colon A \to B$ be an equivariant $*$-homomorphism of G-C*-algebras. As usual, its mapping cone is given by

$$\mathrm{Con}(f) = \{(a, b_*) \in A \oplus C_0((0,1], B) \mid f(a) = b_1\},$$

which inherits a structure of G-C*-algebra from A and B.

An exact triangle in KK^G is a diagram of the form

$$A \to B \to C \to \Sigma A$$

such that there exists a homomorphism $f\colon A' \to B'$ of G-C*-algebras and a commutative diagram

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & \Sigma A \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Sigma B' & \longrightarrow & \mathrm{Con}(f) & \longrightarrow & A' & \longrightarrow & B', \end{array}$$

in KK^G, where vertical arrows are equivalences and the rightmost downward arrow is equal to the leftmost downward arrow up to applying Σ and Bott periodicity $\Sigma^2 B' \simeq B'$ in KK^G.

Thus, we are really defining a triangulated category structure on the opposite category of KK^G. Generally the opposite category of a triangulated category is again triangulated with "the same" exact triangles with suspension and desuspension exchanged, but for KK^G we have $\Sigma^2 \simeq \mathrm{id}$ and we can ignore that issue.

The crucial step is to check the axiom (TR1), in particular that any KK^G-morphism is represented by a G-equivariant $*$-homomorphism up to KK^G-equivalence, see [Laf07, Lemma A.3.2]. Having established that, the rest is quite standard; one can follow [MN06, Appendix A] to check that the triangles of the form

$$\Sigma B \to \mathrm{Con}(f) \to A \to B$$

satisfy the axioms (TR2), (TR3), and (TR4) for the opposite category of KK^G.

Finally, suppose that an equivariant $*$-homomorphism $f\colon A \to B$ is surjective with a $C_0(X)$-linear completely positive section $B \to A$. Then the G-C*-algebra $I = \ker f$ is isomorphic to $\mathrm{Con}(f)$ in KK^G, by the embedding homomorphism

$$I \to \mathrm{Con}(f), \quad a \mapsto (a, 0).$$

It follows that there is an exact triangle of the form

$$I \longrightarrow A \xrightarrow{f} B \longrightarrow \Sigma I.$$

A.5. **Induction functor for subgroupoids.** Suppose that G acts freely and properly from right on a second countable, locally compact, Hausdorff space Y. Then the transformation groupoid $Y \rtimes G$ is Morita equivalent to the quotient space Y/G as a groupoid, with Y being the bibundle implementing the equivalence. This induces the strong Morita equivalence between $G \ltimes C_0(Y) \simeq C^*(Y \rtimes G)$ and $C_0(Y/G)$. In particular, for the case $Y = G$ and action given by right translation, we get the isomorphism between $G \ltimes C_0(G)$ and $\mathcal{K}(L^2_r(G))$, where $L^2_r(G)$ is the right Hilbert $C_0(X)$-module completion of $C_c(G)$ with $C_0(X)$-module structure from $r^*\colon C_0(X) \to C_b(G)$ and inner product from the Haar system.

Proof of Proposition 1.23. As in the assertion, let A be a G-C*-algebra. We have two actions of G: on the one hand, it acts on s^*A by the combination of right translation on G and the original action on A, while on the other hand it acts on r^*A by the right translation on G and trivially on A. Then, the structure morphism $\alpha\colon s^*A \to r^*A$ of the action intertwines these two actions. Morally s^*A can be thought of as a space of sections $f(g) \in A_{sg}$ for $g \in G$, with the action of G given by $f^{g'}(g) = g'^{-1} f(gg')$ for $(g, g') \in G^{(2)}$, while r^*A as a space of sections $f(g) \in A_{rg}$ with G acting by $f^{g'}(g) = f(gg')$ for $(g, g') \in G^{(2)}$. We have $(\alpha f)(g) = gf(g)$ for the sections of the first kind, and these formulas give $(\alpha f^{g'})(g) = gf(gg') = (\alpha f)^{g'}(g)$.

Now, $\operatorname{Ind}_G^G \operatorname{Res}_G^G(A)$ is the crossed product of s^*A by G, while $\mathcal{K}(L^2_r(G)) \otimes_{C_0(X)} A$ is the crossed product of r^*A by G. Consequently we get an isomorphism between these algebras. The extra action of G on $\operatorname{Ind}_G^G \operatorname{Res}_G^G(A)$ comes from the action of G on s^*A given by the combination of the left translation on G and the trivial action on A. Under the above isomorphism, this corresponds to the action on r^*A given by the combination of left translation on G and the original action on A. Thus, it corresponds to the diagonal action of G on $\mathcal{K}(L^2_r(G)) \otimes_{C_0(X)} A$. ☐

More generally, the same argument gives an isomorphism

$$\phi\colon \operatorname{Ind}_H^G \operatorname{Res}_H^G A \simeq (C_0(G) \rtimes H)\ {}^r\!\otimes_{C_0(X)} A,$$

where G acts diagonally on the algebra on the right.

The functor $\operatorname{Ind}_H^G\colon \mathrm{C}^*_H \to \mathrm{C}^*_G$ preserves split extensions, respects equivariant Morita equivalence, and is compatible with homotopy. Then, by Proposition A.4 it extends to a functor $\operatorname{Ind}_H^G\colon \mathrm{KK}^H \to \mathrm{KK}^G$. Let us give a more concrete description at the level of Kasparov cycles.

Consider an H-equivariant right Hilbert module $\mathcal{E}$ over B. By using an approximate unit in B, we can equip $\mathcal{E}$ with a compatible $C_0(X)$-action. We can form the Hilbert module $C_0(G) \otimes \mathcal{E}$ over $G_0(G) \otimes B$, and restrict on the diagonal to get $s^*\mathcal{E} = (C_0(G) \otimes \mathcal{E})_{\Delta(X)}$ over $s^*B \simeq (C_0(G) \otimes B)_{\Delta(X)}$. This still has an action of H, analogous to the right action of H on s^*B.

Assume moreover $(\pi, \mathcal{E}, T)$ is an equivariant Kasparov module between H-C*-algebras. So $\mathcal{E}$ is a graded right Hilbert module over B, T is an odd adjointable (or self adjoint) endomorphism, and $\pi\colon C \to \mathcal{L}(\mathcal{E})$ is a $*$-representation, with commutation relations as in [LG99]. Then $s^*\mathcal{E}$ as a right Hilbert module over s^*B, with a left module structure over s^*C. Moreover we can extend T to s^*T on $s^*\mathcal{E}$ as the restriction of $1_{C_0(G)} \otimes T$, with the right commutation properties (they hold before restriction to $\Delta(X)$). Finally, we take the crossed product by the right action of H,

$$\operatorname{Ind}_H^G(\pi, \mathcal{E}, T) = j_H(s^*\pi, s^*\mathcal{E}, s^*T).$$

This way, we obtain a map $\operatorname{Ind}_H^G\colon \mathrm{KK}^H(C, B) \to \mathrm{KK}^G(\operatorname{Ind}_H^G C, \operatorname{Ind}_H^G B)$, realizing the extension of Ind_H^G to KK^H.

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