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Discrete Geometry

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Abstract. A number of important recent developments in various branches of discrete geometry were presented at the workshop, which took place in hybrid format due to a pandemic situation. The presentations illustrated both the diversity of the area and its strong connections to other fields of mathematics such as topology, combinatorics, algebraic geometry or functional analysis. The open questions abound and many of the results presented were obtained by young researchers, confirming the great vitality of discrete geometry.

Mathematics Subject Classification (2010): 52Bxx, 52Cxx.

Introduction by the Organizers

Discrete Geometry deals with the structure and complexity of discrete geometric objects, from finite point sets in the plane to intersection patterns of convex sets in high dimensional spaces. It goes back to classical problems such as Kepler's conjecture on the density of packings of balls in space, and Hilbert's third problem on decomposing polyhedra, as well as works by Minkowski, Steinitz, Hadwiger, and Erdős form the heritage of this area. Over the past years, several outstanding problems were solved, for example: (1) Natan Rubin improved the 30-year old upper bound on the size of weak epsilon-nets, (2) Karim Adiprasito proved the *g-conjecture*, from 1970, on the characterization of face vectors of triangulated spheres, etc.

The workshop gathered 17 participants on-site and 33 remote participants. The outstanding contributions by young scholars include the lecture by Avvakumov of his construction (with Adiprasito and Karasev) of a triangulation of $\mathbb{RP}^n$ of subexponential size. Another one was Kühne and Yashfe's answer to a question of Björner concerning configuration spaces of matroids as multilinear arrangements.

There were 21 other, mostly short, lectures presenting new connections to classical topics such as combinatorics (Keszegh) or stochastic geometry (Akopyan), as well as new developments in classical topics such as arrangements and matroids (Kühne, Steiner, Yashve), polytopes and triangulations (Cano, de Loera, Padrol, Welzl), euclidean geometry (Pak, Tóth), combinatorial convexity and topology (Frick, Paták, Tancer, Yuditski, Zerbib), and convex geometry (Montejano, Patáková). Altogether, 13 of the talks were given by remote participants.

In order to encourage collaboration, the workshop started with an opening session in which the junior participants presented themselves and outlined their research focus. An open problem session took place on Tuesday evening; the collection of open problems resulting from this session can be found in this report. The program left ample time for research and discussions. The small size of the group and larger fraction of junior researchers among the on-site participants made the atmosphere of the Oberwolfach Institute even more stimulating than usually. There were several small informal sessions on specific topics of common interest. On Tuesday afternoon, most participants joined the traditional outing to St. Roman with the black forest cherry cake, enjoying the beautiful autumn air under somewhat adverse weather conditions.

Subject classification. The topics of the conference belong mainly to classes 52C and 52B in the AMS-classification scheme. They fall into category 4 (Geometry) of the International Mathematical Union (1995) classification. There is only a minor overlap with other Oberwolfach meetings like "Convex Geometry and its Applications" or "Topological and Geometric Combinatorics".

Workshop: Discrete Geometry

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Abstracts

The Beauty of Random Polytopes Inscribed in the 2-sphere

Arseniy Akopyan
(joint work with Herbert Edelsbrunner, Anton Nikitenko)

We study properties of random polytopes X_n defined as the convex hull of n points chosen uniformly at random on the unit sphere in $\mathbb{R}^3$.

First, we prove the following results, the asymptotic version of which was obtained in [4, 2].

Theorem 1 (Random Triangle)**.** *A uniformly chosen random facet of X_n is an acute triangle with probability $\frac{1}{2}$.*

We also give an elementary proof of the following results from [1] and [3]

Theorem 2.

$$\begin{aligned}\mathbf{E}[\mathrm{W}(X_n)] &= \mathrm{W}(\mathbb{B}^3) \cdot \tfrac{n-1}{n+1},\\ \mathbf{E}[\mathrm{A}(X_n)] &= \mathrm{A}(\mathbb{B}^3) \cdot \tfrac{n-1}{n+1}\tfrac{n-2}{n+2},\\ \mathbf{E}[\mathrm{V}(X_n)] &= \mathrm{V}(\mathbb{B}^3) \cdot \tfrac{n-1}{n+1}\tfrac{n-2}{n+2}\tfrac{n-3}{n+3},\end{aligned}$$

in which $\mathrm{W}, \mathrm{A}, \mathrm{V}$ *map a* 3*-dimensional convex body to its mean width, surface area, and volume.*

Here we give the proof of the area part. We need the following lemma.

Lemma (non-Bertrand paradox in 3d)**.** *The probability distribution on lines intersecting $\mathbb{S}^2$, defined by choosing two points uniformly and independently on $\mathbb{S}^2$, coincides with the Crofton measure, i.e., the isometry-invariant measure on lines in $\mathbb{R}^3$ normalized to have the total measure* 1 *of lines intersecting the unit ball.*

Proof of the area part of Theorem 2. The $\mathbf{E}[\frac{\mathrm{A}(X_n)}{\mathrm{A}(\mathbb{B}^3)}]$ is the probability that a random chord—which, by the Lemma, has the distribution of X_2—intersects X_n. Joining all points together, it is the probability that the *extra* two points span a diagonal of X_{n+2}. There are $\frac{1}{2}(n+2)(n+1)$ pairs of vertices and (by Euler's formula) $3n$ edges, so this probability is

$$\frac{\frac{1}{2}(n+2)(n+1) - 3n}{\frac{1}{2}(n+1)(n+2)} = \frac{(n-1)(n-2)}{(n+1)(n+2)}.$$

Multiplying by the area of $\mathbb{S}^2$, we get the claimed identity. □

Similar results hold for X_ϱ, the convex hull of the stationary Poisson point process with intensity $\varrho > 0$ on the unit sphere in $\mathbb{R}^3$.

Theorem 3.

$$\begin{aligned}\mathbf{E}[\mathrm{W}(X_\varrho)] &= \mathrm{W}(\mathbb{B}^3)\cdot 2\pi\varrho^{0.5}e^{-2\pi\varrho}\,\mathrm{I}_{1.5}(2\pi\varrho),\\ \mathbf{E}[\mathrm{A}(X_\varrho)] &= \mathrm{A}(\mathbb{B}^3)\cdot 2\pi\varrho^{0.5}e^{-2\pi\varrho}\,\mathrm{I}_{2.5}(2\pi\varrho),\\ \mathbf{E}[\mathrm{V}(X_\varrho)] &= \mathrm{V}(\mathbb{B}^3)\cdot 2\pi\varrho^{0.5}e^{-2\pi\varrho}\,\mathrm{I}_{3.5}(2\pi\varrho),\end{aligned}$$

in which $\mathrm{I}_\alpha(x)$ *is the modified Bessel function of the first kind.*

Theorem 4. *The sums of lengths of the edges on the two inscribed random polytopes satisfy*

$$\begin{aligned}\mathbf{E}[\mathrm{L}(X_n)] &= \binom{n}{3}\frac{512}{3\pi}\cdot B\left(n-\tfrac{1}{2},\tfrac{5}{2}\right) && \left[=\tfrac{64}{3\sqrt{\pi}}\sqrt{n}\cdot(1+o(1))\right],\\ \mathbf{E}[\mathrm{L}(X_\varrho)] &= \tfrac{128}{3}\varrho^{0.5}\cdot 2\pi\varrho^{0.5}e^{-2\pi\varrho}\,\mathrm{I}_2(2\pi\varrho) && \left[=\tfrac{64}{3\sqrt{\pi}}\sqrt{4\pi\varrho}\cdot(1+o(1))\right].\end{aligned}$$

We finish the report with the following “conjecture” motivated by the figure below.

Conjecture. *Knowing the total mean curvature of a random polytope X, we can find its volume and surface area.*

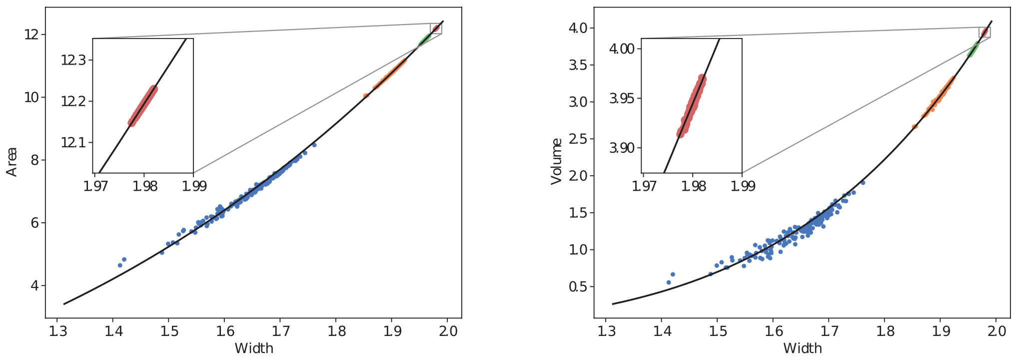

FIGURE.
Projections of the graph of $\left(\mathrm{W}(\mathbb{B}^3)\,\frac{t-1}{t+1};\ \mathrm{A}(\mathbb{B}^3)\,\frac{t-1}{t+1}\frac{t-2}{t+2};\ \mathrm{V}(\mathbb{B}^3)\,\frac{t-1}{t+1}\frac{t-2}{t+2}\frac{t-3}{t+3}\right)$, and the triplets of expected intrinsic volumes into the width-area plane on the *left* and the width-volume plane on the *right*. *Top:* the 150 *blue*, *orange*, *green*, and *red* points belong to polytopes with 10, 40, 100, and 200 vertices each.

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A subexponential size $\mathbb{RP}^n$

Sergey Avvakumov
(joint work with Karim Adiprasito, Roman Karasev)

1. Introduction

While in general, every smooth manifold allows for a triangulation, it is a notoriously hard problem to construct small triangulations of manifolds, and usually poses a difficult challenge. And so, outside of special cases, there are few cases known where the upper bounds and lower bounds come even close to each other.

Let us focus on minimality in terms of the number of vertices that a triangulation of a given manifold would have, a study Banchoff and Kühnel initiated [Küh95]. The best lower bounds in this area are usually either homological, or homotopic in nature. Indeed,

- it is clear that the number of vertices cannot be lower than the ball-category, or the more studied Lusternik-Schnirelmann category [CLOT03]. In particular
- it is bounded from below in terms of the cup length of the space in question. In fact, it is easy to show and observed by Arnoux and Marin that for a space of cup length n, one needs $\binom{n+2}{2}$ vertices [AM91].Finally
- Murai gave a lower bound in terms of the Betti numbers of (closed and orientable) manifolds [Mur15], which was simplified and generalized to general manifolds by Adiprasito and Yashfe [Adi18, AY20]. This bound in general is not so good for interesting manifolds, as it seems insensitive to any interesting multiplicative structure in the cohomology ring, let alone homotopy.

On the example of $\mathbb{RP}^n$, the bound by Arnoux and Marin is best. Still, the best construction so far is essentially still Kühnel's observation that the barycentric subdivision of the n-simplex yields a triangulation of the $(n-1)$-sphere on $2^{n+1}-2$ vertices, with a $\mathbb{Z}_2$-action, such that antipodal vertices are at distance at least 3 from each other. This yields a triangulation of $\mathbb{RP}^{n-1}$ of size $2^n - 1$.

Since then, no substantial improvement has been made for the general problem, and focus has shifted to experimental study of low-dimensional cases (see [Lut99] for an excellent survey) and improvements of the base of the exponential [BS15, VZ19].

Surprisingly perhaps, and at least counter to the prevailing expectation of experts, we are not only constructing a triangulated sphere, but a polytope, counter to intuition coming from concentration inequalities on the sphere, see for instance [Bar13].

2. Main result

Theorem 2.1. For all positive integers n, there exists a convex centrally symmetric n-dimensional polytope P such that:

- All the vertices of P lie on the unit sphere.

- For any vertex $A \in P$, if $F, F' \subset P$ are faces with $A \in F$ and $-A \in F'$, then $F \cap F' = \emptyset$.
- The number of vertices of P is less than $e^{(\frac{1}{2}+o(1))\sqrt{n}\log n}$.

Corollary 2.2. For all positive integers n, there exists a triangulation of $\mathbb{R}P^{n-1}$ with at most $e^{(\frac{1}{2}+o(1))\sqrt{n}\log n}$ vertices.

The corollary is proved by symmetrically triangulating ∂P and then taking the quotient of the $\mathbb{Z}_2$ action. The result is a simplicial complex by the second property of P.

3. Proof of Theorem 2.1

Let V be a subset of the set of non-empty subsets of $\{1, \dots, n\}$. We identify any $A \in V$ with a unit vector in $\mathbb{R}^n$ whose endpoint has its ith coordinate equal to $1/\sqrt{|A|}$ if $i \in A$ and 0 otherwise.

A centrally symmetric polytope $P(V)$ is the convex hull of the endpoints of the vectors $V \sqcup -V$.

Our proof of Theorem 2.1 consists of the following three claims. By $\langle \cdot, \cdot \rangle$ we denote the inner product.

Claim 3.1. Suppose that V satisfies the following properties:

(1) $\{i\} \in V$ for all $i \in \{1, \dots, n\}$.
(2) If $A \in V$ and $|A| > 1$, then $A \setminus i \in V$ for any $i \in A$.
(3) Let $A, B \in V$ be vertices and $X \in S^{n-1}$ be a unit vector with non-negative coordinates such that $\langle A, B \rangle = 0$ and $\langle A, X \rangle = \langle B, X \rangle$. Then there is $C \in V$ such that $\langle C, X \rangle > \langle A, X \rangle = \langle B, X \rangle$.

Then $P(V)$ satisfies the conditions of Theorem 2.1, except maybe the condition on the number of vertices.

Claim 3.2. Suppose that V satisfies the following properties:

(1) $\{i\} \in V$ for all $i \in \{1, \dots, n\}$.
(2) If $A \in V$ and $|A| > 1$, then $A \setminus i \in V$ for any $i \in A$.
(3) For every $A, B \in V$ with $A \cap B = \emptyset$, there are $i \in A$ and $j \in B$ such that either
    (3a) $B \sqcup i \in V$ and $A \sqcup j \setminus i \in V$,
    or
    (3b) $A \sqcup j \in V$ and $B \sqcup i \setminus j \in V$.

Then V satisfies the requirements of Claim 3.1.

Claim 3.3. There is a set V of size at most $e^{(\frac{1}{2}+o(1))\sqrt{n}\log n}$ satisfying the requirements of Claim 3.2.

Remark 3.4. The set of combinatorial conditions on V in the statement of Claim 3.2 is not the only one which makes the claim work. Although, other conditions we found give a worse bound on the number of vertices. In general, we don't know what conditions on V are necessary for our proof of Theorem 2.1 to go through.

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Cells in the box and a hyperplane

Imre Bárány

(joint work with Péter Frankl)

1. Main results

It is well-known that a line can intersect the interior of at most $2n-1$ cells of the $n \times n$ chessboard. What happens in high dimensions? This is the question addressed here.

Write $Q_n = Q_n^d = [0,n]^d$, $Q^d = Q_1^d$ so $Q_n^d = nQ^d$. Let $e_1, \dots, e_d$ be the standard basis vectors of $\mathbb{R}^d$ and of the integer lattice $\mathbb{Z}^d$. For $z = (z_1, \dots, z_d) \in \mathbb{Z}^d$ define the unit cube

$$C(z) = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d : z_i \le x_i \le z_i + 1, i \in [d]\}$$

that we are going to call a *cell*. Here $[d]$ stands for the set $\{1, 2, \dots, d\}$. For $v \in \mathbb{R}^d$, $(v \neq 0)$ let $A(v,t)$ denote the hyperplane $\{x \in \mathbb{R}^d : vx = t\}$ where vx is the scalar product of the two vectors. Define $N^d(n)$ as the maximal number of cells in Q_n^d that a hyperplane $A(v,t)$ can intersect properly, meaning that $A(v,t)$ intersects the interior of $C(z)$.

As we mentioned earlier $N^2(n) = 2n-1$. Variants of this result have appeared as olympiad problems in several countries. József Beck used a slightly stronger version of this fact to answer questions of Dirac, Motzkin, and Erdős in a seminal paper [6]. Here we show that $N^3(n) = \frac{9}{4}n^2 + O(n)$ and determine the asymptotic behaviour of $N^d(n)$ for $d > 3$. Our main result is

Theorem 1. *$N^d(n) = V_d n^{d-1}(1+o(1))$, where $V_d > 0$ is a constant depending only on d.*

For $d = 3$ more precise result is given in

Theorem 2. *$N^3(n) \leq \frac{9}{4}n^2+2n+1$. Moreover, for n large enough, $N^3(n) \geq \frac{9}{4}n^2 + n-5$ for even n and $N^3(n) \geq \frac{9}{4}n^2+n-\frac{17}{4}$ for odd n, and $N^3(2) = 7$, $N^3(3) = 19$, and $N^3(4) = 35$.*

For the defintion of the constant V_d in Theorem 1 we let $|v|$ resp. $|v|_1$ denote the ℓ_2 and ℓ_1 norm of the vector $v \in \mathbb{R}^d$. Set

$$V_d(v) = \frac{|v|_1}{|v|} \max_{t \in \mathbb{R}} \mathrm{vol}_{d-1}(A(v,t) \cap Q^d),$$

and

$$V_d = \max\{V_d(v) : v \in \mathbb{R}^d, v \neq 0, t \in \mathbb{R}\}.$$

It is a consequence of the Brunn-Minkowski theorem cf [7] that for fixed v the quantity $\mathrm{vol}_{d-1}(A(v,t) \cap Q^d)$ is maximal when $(A(v,t) \cap Q^d)$ is the central section of Q^d, that is $A(v,t)$ contains the centre of Q^d which is the point $e/2$ where $e = e_1 + \ldots + e_d$. In this case of course $t = ev/2$. It is known that

$$1 \leq \mathrm{vol}_{d-1}(A(v, ev/2) \cap Q^d \leq \sqrt{2},$$

the upper bound is a famous result of Keith Ball [2], the lower bound is trivial. This implies that

$$\sqrt{d} \leq V_d \leq \sqrt{2d}.$$

It is known that the sequence $V_2, V_3, \ldots$ is increasing, $V_2 = 2, V_3 = \frac{9}{4}, V_4 = \frac{8}{3}$ etc and its limit is $\sqrt{\frac{6d}{\pi}}$. We conjectured that the vector $v = e$ gives the maximum in the definition of V_d. This has been recently proved by Iskander Aliev [1].

2. Tools

Write $S(v,t)$ for the strip $\{x \in \mathbb{R}^d : vx \leq t \leq v(x+e)\}$ where $v \in \mathbb{R}^d$ is a Euclidean unit vector and $t \in \mathbb{R}$. It is clear that $N^d(n)$ is equal to the maximum number of lattice points in $Q^d_n \cap S(v,t)$ with the maximum taken over all Euclidean unit vectors v and reals t.

Let $K \subset \mathbb{R}^d$ be a convex body. A cell $C(z)$, $z \in \mathbb{Z}^d$ called *inside* if $C(z) \subset K$, *outside* if $C(z) \cap K = \emptyset$, and *boundary* otherwise. The following inequality is fairly simple and probably well known.

$$\big|\mathrm{vol} K - |K \cap \mathbb{Z}^d|\big| \leq |\text{boundary cells of } K|. \tag{1}$$

This estimate useless for the convex body $C = Q_n^d \cap S(v,t)$ since it has no inside cells and vol$C \approx |$boundary cells of $C|$. In the proof we choose a basis F of $\mathbb{Z}^d$ more suitable for C.

Given a basis $F = \{f_1, \dots, f_d\}$ of $\mathbb{Z}^d$ we define the F-*box* with parameters $\alpha, \beta \in \mathbb{R}^d$ as

$$B(\alpha,\beta,F) = \{x = \sum_1^d x_i f_i \in \mathbb{R}^d : \alpha_i \le x_i \le \beta_i, i \in [d]\}.$$

This is a parallelotope. We of course assume that $\alpha_i \le \beta_i$ for all i. The minimal box containing a convex body K is denoted by $B(K,F)$. This is the F-box $B(\alpha,\beta,F)$ with all α_i maximal and β_i minimal under the condition that $K \subset B(\alpha,\beta,F)$. We will make use of the following theorem of Bárány and Vershik from [5] and [8] as well.

Theorem 3. *For every convex body K in $\mathbb{R}^d$ there is a basis F such that*

$$\mathrm{vol}B(K,F) \ll \mathrm{vol}K.$$

The notation $\ll$ means, as usual, that the quantity on the LHS is smaller than the one on the RHS times a positive constant that only depends on d. Of course one can use F-cells (i.e. basic parallelotopes in the basis F) and call them inside, outside, and boundary F-cells with respect to K. Then inequality (1) becomes

$$\left|\mathrm{vol}K - |K \cap \mathbb{Z}^d|\right| \le |\text{boundary } F\text{-cells of } K|. \tag{2}$$

The following result may be useful in other cases as well. It is similar to the well-known fact that the surface area of a convex subset of a convex set K is smaller that the surface area of K itself. To our surprise we couldn't find it anywhere in the literature.

Theorem 4. *Assume K, L are convex bodies in $\mathbb{R}^d$ and $K \subset L$. Then*

$$|\text{boundary cells of } K| \le |\text{boundary cells of } L|.$$

We need a non-degeneracy condition on K:

$$K \cap \mathbb{Z}^d \text{ contains } d+1 \text{ affinely independent vectors.} \tag{3}$$

Under this condition and with minimal box $B(K,F) = B(\alpha,\beta,F)$ we have $\alpha_i \le \lceil \alpha_i \rceil < \lfloor \beta_i \rfloor \le \beta_i$ for all $i \in [d]$. Setting $\gamma_i = \beta_i - \alpha_i$, vol$B(K,F) = \prod_1^d \gamma_i$. The number of boundary cells of $B(K,F)$ is easy to estimate: it is at most

$$2\sum_{i=1}^d \prod_{j \ne i} (\gamma_j + 2) \ll \sum_{i=1}^d \prod_{j\ne i} \gamma_j = \mathrm{vol}B(K,F)\left(\frac{1}{\gamma_1} + \dots \frac{1}{\gamma_d}\right).$$

Combining the previous theorems we have

Theorem 5. *Let K be a convex body in $\mathbb{R}^d$ satisfying (3), and let F be the basis from Theorem 3. Then*

$$\left|\mathrm{vol}K - |K \cap \mathbb{Z}^d|\right| \ll \mathrm{vol}K\left(\frac{1}{\gamma_1} + \dots + \frac{1}{\gamma_d}\right).$$

This is the main tool for proving Theorem 1.

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Hamiltonicity for generalized Delaunay graphs

Pilar Cano

(joint work with Prosenjit Bose, Maria Saumell, Rodrigo I. Silveira)

The study of the combinatorial properties of geometric graphs has played an important role in the area of Discrete and Computational Geometry. One of the fundamental structures that has been studied intensely is the *Delaunay triangulation* of a planar point set (see [12] for an encyclopedic treatment of this structure). It was conjectured by Shamos [14] that the Delaunay triangulation contains a Hamiltonian cycle. This was disproved by Dillencourt [9]. However, Dillencourt [10] showed that Delaunay triangulations are *almost* Hamiltonian, in the sense that they are 1-tough.[1]

Focus then shifted on determining how much to loosen the definition of the Delaunay triangulation to achieve Hamiltonicity. One such direction is to relax the *empty disk* requirement. Given a planar point set S and two points $p, q \in S$, the k-Delaunay graph (k-DG) with vertex set S has an edge pq provided that there exists a closed disk with p and q on its boundary containing at most k points of S different from p and q.[2] If the disk with p and q on its boundary is restricted to disks with pq as diameter, then the graph is called the k-Gabriel graph (k-GG). For the k-Relative Neighborhood graph (k-RNG), pq is an edge provided that there are at most k points of S whose distance to both p and q is less than $|pq|$. Note that $k\text{-}RNG \subseteq k\text{-}GG \subseteq k\text{-}DG$. Chang et al. [8] showed that

[1] A graph is *1-tough* if removing any k vertices from it results in at most k connected components.

[2] Note that this implies that the standard Delaunay triangulation is the 0-DG.

19-RNG is Hamiltonian.[3] Abellanas et al. [1] proved that 15-GG is Hamiltonian. Currently, the lowest known upper bound is by Kaiser et al. [11] who showed that 10-GG is Hamiltonian. All of these results are obtained by studying properties of *bottleneck Hamiltonian cycles.* Given a planar point set, a bottleneck Hamiltonian cycle is a Hamiltonian cycle whose maximum edge length is minimum among all Hamiltonian cycles of the point set. Biniaz et al. [3] showed that there exist point sets such that its 7-GG does not contain a bottleneck Hamiltonian cycle, implying that this approach cannot yield an upper bound lower than 8. Despite this, it is conjectured that 1-DG is Hamiltonian [1].

Another avenue that has been explored is the relaxation of the shape defining the Delaunay triangulation. Delaunay graphs where the disks have been replaced by various convex shapes have been studied in the literature. As for Hamiltonicity in convex shape Delaunay graphs, not much is known. Bonichon et al. [5] proved that every plane triangulation is Delaunay-realizable where homothets of a triangle act as the empty convex shape. This implies that there exist $DG_\triangle$ graphs that do not contain Hamiltonian paths or cycles. Biniaz et al. [4] showed that 7-$DG_\triangle$ contains a bottleneck Hamiltonian cycle and that there exist points sets where 5-$DG_\triangle$ does not contain a bottleneck Hamiltonian cycle. Ábrego et al. [2] showed that the $DG_\square$ admits a Hamiltonian path, while Saumell [13] showed that the $DG_\square$ is not necessarily 1-tough, and therefore does not necessarily contain a Hamiltonian cycle.

We generalize the above results by replacing the disk with an arbitrary convex shape $\mathcal{C}$. We show that the k-Gabriel graph, and hence also the k-Delaunay graph, is Hamiltonian for any convex shape $\mathcal{C}$ when $k \geq 24$. Furthermore, we give improved bounds for point-symmetric shapes, as well as for even-sided regular polygons. Table 1 summarizes the bounds obtained. Finally, we provide some lower bounds on the existence of a Hamiltonian cycle for an infinite family of regular polygons, and bottleneck Hamiltonian cycles for the particular cases of hexagons and squares. In this talk we give a sketch of the proof for the case of general convex shape $\mathcal{C}$. For detailed proofs we refer to [6].

Our results rely on the use of normed metrics and packing lemmas. In fact, in contrast to previous work on Hamiltonicity for generalized Delaunay graphs, our results are the first to use properties of normed metrics to obtain simple proofs for various convex shape Delaunay graphs.

For future research, we point out that our results are based on bottleneck Hamiltonian cycles, in the same way as all previously obtained bounds [1, 8, 11]. However, in several cases, this technique is reaching its limit. Therefore a major challenge to effectively close the existing gaps will be to devise a different approach to prove Hamiltonicity of Delaunay graphs.

[3]According to the definition of k-RNG in [8], they showed Hamiltonicity for 20-RNG.

Type of shape $\mathcal{C}$	$k \leq$	$k \geq$	Bottleneck-$k \geq$
Squares	7	1 [13]	3
Regular hexagons	11	1	6
Regular octagons	12	1	-
Regular t-gons (t even, $t \geq 10$)	11	-	-
Regular t-gons ($t = 3m$ with m odd, $m \geq 3$)	24	1	-
Point-symmetric convex	15	-	-
Arbitrary convex	24	-	-

Table 1. Bounds on the minimum k for which k-$DG_{\mathcal{C}}(S)$ is Hamiltonian and for which k-$GG_{\mathcal{C}}(S)$ contains a $d_{\mathcal{C}}-$bottleneck Hamiltonian cycle.

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The space of monotone paths of a linear program

Jesús A. De Loera

Given a polytope P and linear function f (this pair makes a linear program). We investigate the possible monotone paths inside the oriented graph of P (oriented by the objective function f). As we look at all monotone paths put together we see a rich topological CW-space structure which was first studied by Billera and Sturmfels in their theory of Fiber polytopes and can be used to count how many monotone path are there or to generate them randomly. Our main enumerative results include bounds on the number of monotone paths, and on the the diameter of the CW-complex of monotone paths (how far are two monotone paths from each other?). The picture is fairly complete in dimension three, but plenty of open problems remain for high dimensional polytopes.

Theorem 1. *Let $\mu(P, f)$ be the number of monotone paths on polytope P with objective function f.*

- *For all 3-dimensional polytopes P with n vertices,*
$$\left\lceil \frac{n}{2} \right\rceil + 2 \leq \mu(P, f) \leq T_{n-1},$$
where T_n is the Tribonacci numbers defined by the recurrence $T_0 = T_1 = 1$, $T_2 = 2$ and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 3$.
- *For all d-dimensional ($d \geq 4$) polytopes P on n vertices,*
$$\left\lceil \frac{dn}{2} \right\rceil + 2 - n \leq \mu(P, f) \leq 2^{n-2}.$$

Theorem 2. *Let $G(P, f)$ be the flip graph of polytope P on objective function f. For any 3-dimensional polytope P on n vertices.*
$$\lceil \frac{(n-2)^2}{4} \rceil \leq diam\, G(P, f) \leq (n-2) \lfloor \frac{n-1}{2} \rfloor.$$

These new theorems presented in my talk come from joint work with Christos Athanasiadis (U. Athens) and Zhenyang Zhang (UC Davis) available at the Arxiv.

Continuous dependence of curvature flow on initial conditions in the sphere.

Michael Gene Dobbins

The space of all curves in the sphere is a metric space with Fréchet distance. We prove that a weak form of curvature flow restricted to curves that bisect the sphere depends continuously on initial conditions. As a consequence, this gives an O_3-equivariant strong deformation retraction from the space of all bisectors of the sphere to the space of great circles. Some motivations for this work are to construct a SO_3-equivariant strong deformation retraction from the homeomorphism group of the projective plane to SO_3, and to show that earlier work by the author on spaces of pseudocircle arrangements also holds analogously for pseudoline arrangements [2].

Level-set flow, the *weak* form of curvature flow above, is defined for any simple closed curve γ_0 on the sphere as the compliment of the union of the evolution of all smooth curves that are initially disjoint from γ_0. In set notation, the level-set flow starting from $\gamma(0) = \gamma_0$ is

$$\gamma(t) = \mathrm{S}^2 \setminus \{\alpha(t) : \alpha \text{ evolves by curvature flow}, \alpha(0) \cap \gamma_0 = \emptyset\}.$$

Joseph Lauer showed that as long as the initial curve γ_0 has Lebesgue area 0, $\gamma(t)$ is a smooth solution to curvature on some interval $t \in (0, T)$[3]. By *bisector*, we mean a simple closed curve in S^2 that divides the sphere into 2 regions that each have area 2π. Importantly, this means a bisector must itself have Lebesgue area 0. Lauer further showed that the level-set flow starting from a bisector is a solution to curvature flow for all positive time and converges to a great sphere. The intuition behind level-set flow and Lauer's work build on work of Sigurd Angenent who showed that for a pair of smooth closed curves that are initially distinct, the number of intersection points is finite and decreasing for positive time [1].

The present work shows that, given a sequence of bisectors $\gamma_k(0) \to \gamma_\infty(0)$ converging in Fréchet distance and $t_k \to t_\infty \in [0, \infty]$, we have $\gamma_k(t_k) \to \gamma_\infty(t_\infty)$ in Fréchet distance where $\gamma_k(t)$ is the level-set flow at time t starting from $\gamma_k(0)$.

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The topological Tverberg problem beyond prime powers

Florian Frick

(joint work with Pablo Soberón)

A 1959 result of Bryan Birch [3] asserts that for any straight-line drawing of the complete graph K_{3q} with $3q$ vertices in the plane, there is a partition into q vertex-disjoint 3-cycles that all surround a common point in the plane. It is natural to wonder whether this results holds more generally if the edges are not assumed to be straight-line segments, but only continuous arcs. This question and its natural generalizations to higher dimensions have turned out to be surprisingly resistant.

The natural generalization of Birch's result to dimension d holds: Any $q(d+1)$ points in $\mathbb{R}^d$ may be partitioned into q sets $X_1, \dots, X_q$ of size $d+1$ such that the simplices spanned by the X_i all intersect in a common point. In fact, generically this intersection of simplices will be full-dimensional, and one can save d points; Helge Tverberg [11] proved that any $(q-1)(d+1)+1$ points in $\mathbb{R}^d$ can be partitioned into q sets whose convex hulls all share a common point.

The continuous generalization of Birch's result, and more generally Tverberg's result, has been proved for q a power of a prime [4, 10, 12]. More precisely, any continuous map $f\colon \Delta_{(q-1)(d+1)} \to \mathbb{R}^d$ from the $(q-1)(d+1)$-dimensional simplex to $\mathbb{R}^d$ identifies q points from pairwise disjoint faces, provided that q is a power of a prime. For a linear map f this is precisely Tverberg's theorem. Perhaps surprisingly, the condition that q be a prime power is indeed crucial for this continuous generalization: For any q with at least two distinct prime divisors and d sufficiently large there is a continuous map $f\colon \Delta_{(q-1)(d+1)} \to \mathbb{R}^d$ that never maps q points from pairwise disjoint faces to the same point; see [2, 6, 8, 9]. In fact, Avvakumov, Karasev, and Skopenkov [1] showed that there is such a map $f\colon \Delta_n \to \mathbb{R}^d$ for $n = q(d+1) - q\lceil \frac{d+2}{q+1} \rceil - 2$, provided that q is not a power of prime and $d \geq 2q$.

However, this leaves open the question whether there is a continuous generalization of Birch's original result. Here we prove this generalization and its higher-dimensional versions beyond prime powers; see [7] for details:

Theorem 1. *Let $q \geq 2$ and $d \geq 1$ be integers. Let $n = q(d+1) - 1$. For any continuous map $f\colon \Delta_n \to \mathbb{R}^d$ there are points $x_1, \dots, x_q$ in q pairwise disjoint faces of Δ_n with $f(x_1) = f(x_2) = \cdots = f(x_q)$.*

As a simple consequence of this we obtain a continuous generalization of Birch's theorem:

Corollary 2. *For any continuous drawing of K_{3q} in the plane, where each 3-cycle is embedded, there is a partition of the vertex set into q triples such that the induced 3-cycles all surround a common point.*

Here we require 3-cycles to be embedded since then, by the Jordan curve theorem, each 3-cycle surrounds a well-defined interior region.

Let p be a prime. The p-fold join of a continuous map $f\colon \Delta_n \to \mathbb{R}^d$ is a $\mathbb{Z}/p$-equivariant map $F\colon (\Delta_n)^{*p} \to (\mathbb{R}^{d+1})^p$. Let

$$D = \{(y_1, \dots, y_p) \in (\mathbb{R}^{d+1})^p \ : \ y_1 = y_2 = \cdots = y_p\}$$

denote the diagonal in $(\mathbb{R}^{d+1})^p$. The preimage $F^{-1}(D)$ consists of all ordered p-tuples of (not necessarily distinct) points that f maps to the same point, that is, $F(\lambda_1 x_1 + \cdots + \lambda_p x_p) \in D$ if and only if $\lambda_i = \frac{1}{p}$ for all i and $f(x_1) = f(x_2) = \cdots = f(x_p)$. Since for p a prime the $\mathbb{Z}/p$-action shifting coordinates of $(\mathbb{R}^{d+1})^p$ is free away from the diagonal D, a result of Dold [5] now implies that $F^{-1}(D)$ intersects any $\mathbb{Z}/p$-invariant subcomplex $\Sigma \subset (\Delta_n)^{*p}$ that is homotopically $[(p-1)(d+1)-1]$-connected. The subcomplex $\Sigma \subset (\Delta_n)^{*p}$ that consists only of p-fold joins of pairwise disjoint faces is $(n-1)$-connected, so for $n = (p-1)(d+1)$ this proves the continuous generalization of Tverberg's theorem, provided that $q = p$ is a prime.

The key idea for the proof of Theorem 1 now is to construct for a given integer $q \geq 2$ and a large prime of the form $p = kq + 1$ a $\mathbb{Z}/p$-invariant subcomplex $\Sigma \subset (\Delta_{q(d+1)})^{*p}$ that is $[(p-1)(d+1)-1]$-connected and such that the $\mathbb{Z}/p$-orbit of any vertex contains q consecutive vertices that are pairwise not adjacent. Since Σ is highly connected it follows as before that there are $x_1, \dots, x_p \in \Delta_{q(d+1)}$ with $f(x_1) = f(x_2) = \cdots = f(x_p)$ and such that $\frac{1}{p}x_1 + \cdots + \frac{1}{p}x_p \in \Sigma$. By construction of Σ the points $x_1, x_2, \dots, x_q$ are in pairwise disjoint faces.

This can be used to prove a weaker variant of Theorem 1 for $n = q(d+1)$. To prove the stronger version for $n = q(d+1) - 1$, one can add a dummy vertex to instead argue for $\Delta_{q(d+1)}$ as above. Then observe that for any set $I \subset \mathbb{Z}/p$ of q consecutive numbers modulo p, the points $x_i, i \in I$, are in pairwise disjoint faces of $\Delta_{q(d+1)}$, and the dummy vertex cannot obstruct all of these collections of points, since otherwise q would divide p.

The technical core of the proof of Theorem 1 consists of the construction of suitable complexes Σ, which are highly connected (thus dense) while having large independent sets in each $\mathbb{Z}/p$-orbit (and thus are locally sparse). The construction given in [7] is optimal in the sense that in any $\mathbb{Z}/p$-symmetric $[(p-1)(d+1)-1]$-connected subcomplex of $(\Delta_{q(d+1)})^{*p}$ the largest independent set in some $\mathbb{Z}/p$-orbit has size at most q.

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Coloring hypergraphs defined on ordered vertex sets

Balázs Keszegh

(joint work with Eyal Ackerman, Dömötör Pálvölgyi)

Given a family of hypergraphs $\mathcal{H}$ and a positive integer c, let $m(\mathcal{H}, c)$ denote the least integer such that the vertices of every hypergraph $H \in \mathcal{H}$ can be colored with c colors such that every hyperedge of size at least $m(\mathcal{H}, c)$ is non-monochromatic (i.e., contains two vertices with different colors). In other words, for every hypergraph $H \in \mathcal{H}$ the sub-hypergraph of H that consists of all the hyperedges of size at least $m(\mathcal{H}, c)$ is *c-colorable.* We denote by $\chi_m(\mathcal{H})$ the least integer c for which such a finite $m(\mathcal{H}, c)$ exists (otherwise, define $\chi_m(\mathcal{H}) = \infty$).

A family of geometric (or topological) regions $\mathcal{F}$ and a set of points S naturally define a hypergraph $H(S, \mathcal{F})$ whose vertices are the points in S and whose hyperedge set consists of every subset $S' \subseteq S$ for which there is a region $F' \in \mathcal{F}$ such that $S' = F' \cap S$. The family of (finite) hypergraphs $\mathcal{H}(\mathcal{F})$ defined by a family of geometric regions $\mathcal{F}$ consists of all the hypergraphs $H(S, \mathcal{F})$ for some (finite) point set S.

Typically, one is interested in determining whether it holds that $\chi_m(\mathcal{F}) = 2$ or at least $\chi_m(\mathcal{F}) < \infty$ for a given family of geometric regions $\mathcal{F}$.

When $\chi_m(\mathcal{F}) = 2$ then we are in addition interested in *polychromatic k-colorings*, which are k-colorings such that every hyperedge (region) contains points of all k colors. The following such result of Smorodinsky and Yuditsky about halfplanes is the starting point of our work:

Theorem 0.1. [3] We can color any set of points in the plane with k colors such that every halfplane containing at least $2k - 1$ points contains all colors.

We denote by $(AB)^l$ the alternating sequence of letters A and B of length $2l$. For example, $(AB)^{1.5} = ABA$ and $(AB)^2 = ABAB$.

Definition 1 ($(AB)^l$-free hypergraphs)**.**

*(1) Two subsets A, B of an ordered set of elements form an $(AB)^l$-*sequence *if there are $2l$ elements $a_1 < b_1 < a_2 < b_2 < \ldots < a_l < b_l$ such that $\{a_1, a_2, \ldots, a_l\} \subset A \setminus B$ and $\{b_1, b_2, \ldots, b_l\} \subset B \setminus A$.*

*(2) A hypergraph with an ordered vertex set is $(AB)^l$-*free *if it does not contain two hyperedges A and B that form an $(AB)^l$-sequence.*

*(3) A hypergraph with an unordered vertex set is $(AB)^l$-*free *if there is an order of its vertices such that the hypergraph with this ordered vertex set is $(AB)^l$-free.*

*(4) The family of all $(AB)^l$-free hypergraphs is denoted by $(\mathcal{AB})^l$-*free.

$(AB)^l$-free hypergraphs were introduced in [2].

A *pseudoline arrangement* is a finite collection of x-monotone bi-infinite curves such that any two of the curves intersect at most once. Such curves cut the plane into a top and a bottom component. A family of *pseudo-halfplanes* is the family of top and bottom components defined by a pseudoline arrangement. A family of *upwards pseudo-halfplanes* is the family of top components defined by a pseudoline arrangement.

In [2] it was shown that ABA-free hypergraphs are equivalent to hypergraphs defined by upwards pseudo-halfplanes. E.g., a finite set of translates of an unbounded convex set on a finite point set defines an ABA-free hypergraph. Using this connection, we could generalize Theorem 0.1 to pseudo-halfplanes:

Theorem 0.2. [2] Given a finite family of pseudo-halfplanes and a set of points in the plane, we can color the points with k colors such that every halfplane containing at least $2k-1$ points contains all colors.

For $k = 2$ this implies that $\chi_m(\mathcal{ABA}\text{-}\mathit{free}) = 2$. The first main component of the proof is a generalization of the notion of convex hull vertices to ABA-free hypergraphs. The second is the notion of shallow hitting sets:

Definition 2. *A set R is a c-*shallow hitting set *of the hypergraph H if for every hyperedge $h \in H$ we have $1 \leq |R \cap h| \leq c$.*

In particular, it was proved that ABA-free hypergraphs admit 2-shallow hitting sets using only vertices that are on the 'convex hull' of the hypergraph.

It turns out that $ABAB$-free hypergraphs also have a nice geometric meaning. First, in [1] it was shown that these are the hypergraphs that can be defined on points by upwards pseudo-parabolas, i.e., the top components defined by a family of x-monotone bi-infinite curves that intersect pairwise at most twice. In addition, these are also the hypergraphs that can be defined on points by stabbed pseudo-disks. A family of Jordan-regions is a family of pseudo-disks if the boundaries of every pair of regions intersect at most twice. We say that a family of regions is *stabbed* if their intersection is non-empty.

It was also proved in [2] that $\chi_m(\mathcal{ABAB}\text{-}\mathit{free}) > 2$:

Theorem 0.3 ([2]). For every $m \geq 2$ there exists an $ABAB$-free m-uniform hypergraph which is not 2-colorable.

In [1] we have shown that in fact $\chi_m(\mathcal{ABAB}\text{-}\mathit{free}) = 3$:

Theorem 0.4. Every $ABAB$-free hypergraph is proper 3-colorable.

Going one step further, we have shown that $\chi_m(\mathcal{ABABA}\text{-}\mathit{free}) = \infty$:

Theorem 0.5. For every $c \geq 2$ and $m \geq 2$ there exists an $ABABA$-free m-uniform hypergraph which is not c-colorable.

One can also regard the dual question, when instead of the vertices of the hypergraph we color the hyperedges. Our aim is to find an m such that we can always color the hyperedges in such a way that every vertex that is contained in at least m hyperedges is contained in hyperedges of different colors. The polychromatic coloring problem can be phrased similarly. Note that hyperedge coloring problems are equivalent to the vertex coloring problems on the dual of the hypergraph.

In this direction our results are the following:

The dual of an ABA-free hypergraph is also ABA-free and thus the same is true about them. About pseudo-halfplane hypergraphs the following holds, again generalizing the respective result of Smorodinsky and Yuditsky [3] about halfplanes:

Theorem 0.6. [2] Given a finite family of pseudo-halfplanes and a set of points in the plane, we can color the pseudo-halfplanes with k colors such that every point contained by at least $2k-1$ pseudo-halfplanes is contained by pseudo-halfplanes of all colors.

The main tool for this result is a common generalization of hypergraphs defined by pseudo-halfplanes and their dual, which we call pseudo-hemisphere hypergraphs. In particular, we show that pseudo-hemisphere hypergraphs admit 4-shallow hitting sets while dual pseudo-halfplane hypergraphs admit 3-shallow hitting sets.

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The Universality of the Resonance Arrangement

Lukas Kühne

1. Introduction

The main object considered in this talk is the resonance arrangement:

Definition 1. For a fixed integer $n \geq 1$ we define the hyperplane arrangement $\mathcal{A}_n$ as the *resonance arrangement* in $\mathbb{R}^n$ by setting $\mathcal{A}_n := \{H_I \mid \emptyset \neq I \subseteq [n]\}$, where the hyperplanes H_I are defined by $H_I := \left\{\sum_{i \in I} x_i = 0\right\}$.

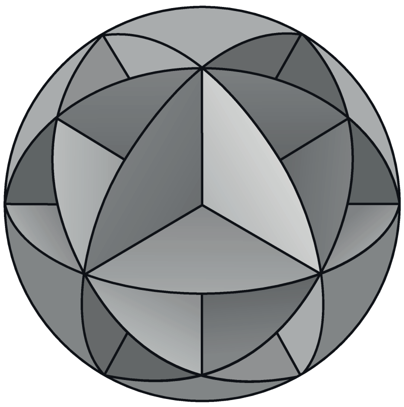

Figure 1. The resonance arrangement $\mathcal{A}_3$ projected onto the hyperplane $H_{\{1,2,3\}}$. There are 16 chambers visible and another 16 antipodal chambers hidden. Thus, $\mathcal{A}_3$ has 32 chambers in total.

The term resonance arrangement was coined by Shadrin, Shapiro, and Vainshtein in their study of double Hurwitz numbers stemming from algebraic geometry [9]. Billera, Billey, Rhoades, and Tewari proved that the product of the defining linear equations of $\mathcal{A}_n$ is Schur positive via a so-called Chern phletysm from representation theory [2]. Recently, Gutekunst, Mészáros, and Petersen established a connection between the resonance arrangement and the type A root polytope [5].

The arrangement $\mathcal{A}_n$ is also the *adjoint of the braid arrangement.* It was studied under this name by Liu, Norledge, and Ocneanu in its relation to mathematical physics [8]. The relevance of the resonance arrangement in physics was also demonstrated by Early in his work on so-called *plates*, cf. [4].

In earlier work, the arrangement $\mathcal{A}_n$ was called *(restricted) all-subsets arrangement* by Kamiya, Takemura, and Terao who established its relevance for applications in psychometrics and economics [6].

2. Universality of $\mathcal{A}_n$

A first contribution presented in this talk is a universality result of the resonance arrangement for rational hyperplane arrangements:

Theorem 2. *Let $\mathcal{B}$ be any hyperplane arrangement defined over $\mathbb{Q}$. Then $\mathcal{B}$ is a minor of $\mathcal{A}_n$ for some large enough n, that is $\mathcal{B}$ arises from $\mathcal{A}_n$ after a suitable sequence of restriction and contraction steps. Equivalently, any matroid that is representable over $\mathbb{Q}$ is a minor of the matroid underlying $\mathcal{A}_n$ for some large enough n.*

The proof is constructive. We gave an example of this construction in the talk.

3. Chambers of $\mathcal{A}_n$

The *chambers of $\mathcal{A}_n$* are the connected components of the complement of the hyperplanes in $\mathcal{A}_n$ within $\mathbb{R}^n$. We denote by R_n the number of chambers of the arrangement $\mathcal{A}_n$. The arrangement $\mathcal{A}_3$ for instance has 32 chambers as shown in 1.

Billera, Tatch Moore, Dufort Moraites, Wang, and Williams observed that the chambers of $\mathcal{A}_n$ are also in bijection with *maximal unbalanced families* of order $n+1$. These are systems of subsets of $[n+1]$ that are maximal under inclusion such that no convex combination of their characteristic functions is constant [1].

The values of R_n are only known for $n \leq 8$, cf. also the entry A034997 in the OEIS. There is no exact formula known for R_n. The recent work of Gutekunst, Mészáros, and Petersen [5] yields $\log_2(R_n) \sim n^2$.

Due to a theorem of Zaslavsky the number of chambers of any arrangement over $\mathbb{R}$ equals the sum of all Betti numbers of the arrangement [10]. The Betti numbers are the absolute values of the coefficients of the characteristic polynomial $\chi(\mathcal{A}_n; t)$ of an arrangement $\mathcal{A}$. The polynomial $\chi(\mathcal{A}_n; t)$ is only known for $n \leq 7$ as computed in [6].

The second result presented in this talk proves that the Betti numbers $b_i(\mathcal{A}_n)$ for any fixed $i > 0$ can be computed for all $n > 0$ from a fixed finite combination of *Stirling numbers of the second kind* $S(n, k)$ which count the number of partitions of n labeled objects into k non-empty blocks. The proof is based on Brylawski's broken circuit complex [3].

Theorem 3. [7] *There exist some positive integers $c_{i,k}$ for all $i \geq 0$ and $i+1 \leq k \leq 2^i$ such that for all $n \geq 1$,*

$$b_i(\mathcal{A}_n) = \sum_{k=1}^{2^i} c_{i,k} S(n+1,k),$$

The first two trivial cases of this theorem are $b_0(A_n) = S(n+1,1)$, and $b_1(\mathcal{A}_n) = S(n+1,2)$. The next two cases can be determined via the broken circuit complex:

$$\begin{aligned} (i) \quad b_2(\mathcal{A}_n) =& 2S(n+1,3) + 3S(n+1,4) \text{ and} \\ (ii) \quad b_3(\mathcal{A}_n) =& 9S(n+1,4) + 80S(n+1,5) + 345S(n+1,6) + \\ & 840S(n+1,7) + 840S(n+1,8). \end{aligned}$$

Example 4. Using 3 we can compute $\chi(\mathcal{A}_3; t)$ as

$$\chi(\mathcal{A}_3; t) = t^3 - 7t^2 + 15t - 9.$$

Thus, the above mentioned result by Zaslavsky again yields $R_3 = 1+7+15+9 = 32$.

Remark 5. The formula for $b_2(\mathcal{A}_n)$ in 3 was also found earlier by Billera (personal communication).

Determining the coefficients $c_{i,k}$ from 3 for $i > 3$ could help developing a better understanding of the exact number R_n for larger n. This however could be quite difficult. The author was not able to discover any pattern in the coefficients $c_{i,k}$ for $i = 1, 2, 3$ yet.

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Extremal question for unit distance graphs

Andrey Kupavskii
(joint work with Nora Frankl)

Determining the maximum possible number of pairs $u_d(n)$ at distance 1 apart in a set of n points in $\mathbb{R}^d$ for $d = 2$ is one of the central questions in combinatorial geometry, known as the Erdös Unit Distance problem. The question dates back to 1946, and despite much effort, the best known upper and lower bounds are still very far apart. For some constants $C, c > 0$, we have

$$n^{1+c/\log\log n} \leq u_2(n) \leq Cn^{4/3},$$

where the lower bound is due to Erdös [3] and the upper bound is due to Spencer, Szemerédi and Trotter [7]. Recently, there has been great progress in a closely related problem of determining the minimum number of distinct distances between n points on the plane due to Guth and Katz, but the powerful algebraic machinery they used has not yet given any improvement for the unit distance question.

The contents of the first part of this extended abstract are based on [5]. This research can be seen as an effort to find generalisations of the Unit Distance problem that are within the reach of our current methods. In what follows, we describe the generalisation that we work with.

Palsson, Senger and Sheffer [6] suggested the following question. Let $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_k)$ be a fixed sequence of k positive reals. A $(k+1)$-tuple $(p_1, \ldots, p_{k+1})$ of distinct points in $\mathbb{R}^d$ is called a *k-chain* if $\|p_i - p_{i+1}\| = \delta_i$ for all $i = 1, \ldots, k$. For every fixed k determine $C_k^d(n)$, the maximum number of k-chains that can be spanned by a set of n points in $\mathbb{R}^d$. We do not include $\boldsymbol{\delta}$ in the notation, since our results do not depend on $\boldsymbol{\delta}$ up to the order of magnitude. The authors of [6] give the following lower bound on $C_k^2(n)$:

$$C_k^2(n) = \Omega\left(n^{\lfloor (k+1)/3 \rfloor + 1}\right).$$

They also provided upper bounds in terms of the maximum number of unit distances.

Proposition 1 (Palsson, Senger, and Sheffer [6])**.**

$$C_k^2(n) = \begin{cases} O\left(n \cdot u_2(n)^{k/3}\right) & \text{if } k \equiv 0 \pmod 3, \\ O\left(u_2(n)^{(k+2)/3}\right) & \text{if } k \equiv 1 \pmod 3, \\ O\left(n^2 \cdot u_2(n)^{(k-2)/3}\right) & \text{if } k \equiv 2 \pmod 3. \end{cases}$$

If $u_2(n) = O(n^{1+\varepsilon})$ for any $\varepsilon > 0$, which is conjectured to hold, then the upper bounds in the proposition above almost match the lower bound given above. However, as we have already mentioned, determining the order of magnitude of $u_2(n)$ has proved to be a very hard problem and is very far from its resolution. Thus, it is interesting to obtain "unconditional" bounds, that depend on the value of $u_2(n)$ as little as possible. In [6], the following "unconditional" upper bounds were proved in the planar case.

Theorem 2 (Palsson, Senger, and Sheffer [6]). *$C_2^2(n) = \Theta(n^2)$, and for every $k \geq 3$ we have*

$$C_k^2(n) = O\left(n^{2k/5+1+\gamma_k}\right),$$

where $\gamma_k \leq \frac{1}{12}$, and $\gamma_k \to \frac{4}{75}$ as $k \to \infty$.

In our main result, in two-third of the cases we almost determine the value of $C_k^2(n)$, no matter what the value of $u_2(n)$ is, by matching the lower bounds given in Theorem 2. Further, we show that in the remaining cases determining $C_k^2(n)$ essentially reduces to determining the maximum number of unit distances.

Theorem 3. *For every integer $k \geq 1$ we have*[1]

$$C_k^2(n) = \tilde{\Theta}\left(n^{\lfloor (k+1)/3 \rfloor + 1}\right) \text{ if } k \equiv 0, 2 \pmod 3,$$

and for any $\varepsilon > 0$ we have

$$C_k^2(n) = \Omega\left(n^{(k-1)/3} u_2(n)\right) \text{ and } C_k^2(n) = O\left(n^{(k-1)/3+\varepsilon} u_2(n)\right) \text{ if } k \equiv 1 \pmod 3.$$

We also obtain similar results in the three-dimensional case.

Finally, we note that for $d \geq 4$ we have $C_k^d(n) = \Theta(n^{k+1})$. Indeed, we clearly have $C_k^d(n) = O(n^{k+1})$. To see that $C_k^d(n) = \Omega(n^{k+1})$, take two orthogonal circles of radius $1/\sqrt{2}$ centred at the origin and choose $n/2$ points on each of them. Then any sequence of $k+1$ points that alternate between the two circles forms a path in which all edges have unit length. The exact value of $u_d(n)$ for large n and even $d \geq 4$ was determined by Brass ($d = 4$) and Swanepoel ($d \geq 6$), by using stability results form extremal graph theory.

The second part of this extended abstract is based on a yet unpublished work with Nora Frankl and is devoted to the following question. Denote by $US_{k,d}(n)$ the maximum number of unit k-vertex simplices that can be determined by a set of n points in $\mathbb{R}^d$. Erdős and Purdy [4] conjectured that $US_{k,d}(n) = \Theta(\min\{n^k, n^{d/2}\}$ for even $d \geq 4$, and later Agarwal and Sharir [2] extended this conjecture to the case of odd $d \geq 5$, suggesting that $US_{k,d}(n) = \Theta(\min\{n^k, n^{d/2-1/6}\}$ in that case. They also proved it for $d \leq 7$ and $k \leq d-1$.

Later, Agarwal, Apfelbaum, Purdy, and Sharir [1] proved that $US_{k,d}(n) = O(n^{d-c})$ for some small constant c, and $k = d, d-1$. We have managed to obtain the first significant improvement of the exponent.

Theorem 4. *$US_{k,d}(n) = \tilde{O}(n^{3d/4}$ for any k, d.*

Actually, the works [4, 2] addressed congruent simplices, and [1] studied similar simplices, but our result is extendable to these settings as well.

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[1] $f(n) = \tilde{\Theta}(g(n))$ means that there exist positive constants c, C such that $C^{-1} \log^{-c} n \leq f(n)/g(n) \leq C \log^{c} n$.

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On the geometric hypothesis of Banach

Luis Montejano

The following is known as the geometric hypothesis of Banach: let V be an m-dimensional Banach space (over the real or the complex numbers) with unit ball B and suppose all n-dimensional subspaces of V are isometric (all the n-sections of B are affinely equivalent). In 1932, Banach conjectured that under this hypothesis V is a Hilbert space (the boundary of B is an ellipsoid). Gromow proved in 1967 that the conjecture is true for $n =$ even and Dvoretzky and V. Milman derived the same conclusion under the hypothesis $n = \infty$. We prove this conjecture for $n = 4k + 1$, with the possible exception of V a real Banach space and $n = 133$. [1] for the real case and [2] for the complex case.

The ingredients of the proof are classical homotopic theory, irreducible representations of the orthogonal group and convex geometry. For the real case, suppose B is an $(n + 1)$-dimensional convex body with the property that all its n-sections through the origin are affinity equivalent to a fixed n-dimensional body K. Using the characteristic map of the tangent vector bundle to the n-sphere, it is possible to prove that if $n =$even, then K must be a ball and using homotopical properties of the irreducible representations we prove that if $n = 4 + 1$ then K must be a body of revolution. Finally, we prove, using convex geometry and topology that, if this is the case, then there must be a section of B which is an ellipsoid and consequently B must be also an ellipsoid. The strategy for the complex case is similar but but taking into account the technical complexities of the case.

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Shard polytopes

Arnau Padrol
(joint work with Vincent Pilaud and Julian Ritter)

Any lattice congruence $\equiv$ of the weak order on the symmetric group $\mathfrak{S}_n$ is completely determined by the permutations with a single descent it contracts [13]. These are naturally encoded by *arcs*, which are quadruples $\alpha := (a, b, A, B)$ with $a < b$ and $A \sqcup B =]a, b[$. If we say that α forces α' if every congruence that contracts α also contracts α', then each lattice congruence $\equiv$ of the weak order corresponds to an upper ideal $\mathcal{A}_\equiv$ of the forcing order among arcs.

Geometrically, the arcs correspond to pieces of hyperplanes, called *shards*, that partition the braid arrangement. Namely, the arc $\alpha := (a, b, A, B)$ corresponds to the shard $\mathsf{S}(\alpha)$ defined as the piece of the hyperplane $\boldsymbol{x}_a = \boldsymbol{x}_b$ given by the inequalities $\boldsymbol{x}_{a'} \le \boldsymbol{x}_a = \boldsymbol{x}_b \le \boldsymbol{x}_{b'}$ for all $a' \in A$ and $b' \in B$. In [11], N. Reading proved that each lattice congruence $\equiv$ of the weak order defines a complete fan $\mathcal{F}_\equiv$, called *quotient fan*, whose dual graph is the Hasse diagram of the lattice quotient $\mathfrak{S}_n/\equiv$. The chambers of $\mathcal{F}_\equiv$ can be seen either by glueing together the chambers of the braid fan that belong to the same congruence class, or as the connected components of the complement of the union of the shards $\mathsf{S}(\alpha)$ for all arcs α in the ideal $\mathcal{A}_\equiv$.

In [9], V. Pilaud and F. Santos showed that this quotient fan is the normal fan of a polytope, called *quotientope*. These realizations were obtained by a careful but slightly obscur choice of right-hand sides defining the inequalities normal to the rays of the braid fan.

We propose an alternative approach to construct polytopal realizations of this quotient fan, using Minkowski sums of elementary polytopes called *shard polytopes*, which have remarkable combinatorial and geometric properties.

To illustrate the idea, let us start with a simple construction. For any arc α, denote by $\mathcal{A}_\alpha$ the arc ideal generated by α. The corresponding congruence $\equiv_\alpha$ is a Cambrian congruence [12] and the corresponding quotient fan $\mathcal{F}_\alpha$ is a Cambrian fan of [16]. It is the normal fan of the α-associahedron Asso_α of [4]. Our motivating observation is the following statement.

Theorem 1. *Let $\equiv$ be a lattice congruence of the weak order, and let $\alpha_1, \dots, \alpha_p$ denote the arcs generating the ideal $\mathcal{A}_\equiv$. Then the quotient fan $\mathcal{F}_\equiv$ is*

- *the common refinement of the Cambrian fans $\mathcal{F}_{\alpha_1}, \dots, \mathcal{F}_{\alpha_p}$, and*
- *the normal fan of the Minkowski sum of the associahedra $\mathsf{Asso}_{\alpha_1}, \dots, \mathsf{Asso}_{\alpha_p}$.*

Note that this approach has the advantage of transfering all the geometric difficulty into the construction of the α-associahedra, which was already done in [4]. We push this idea further by decomposing the α-associahedra into Minkowski sums of more elementary (indecomposable) pieces, called *shard polytopes*.

Theorem 2. *For any arc α, there is a Minkowski indecomposable polytope $\mathsf{SP}(\alpha)$, called the shard polytope of α, such that the union of the walls of the normal fan* *of* $\mathsf{SP}(\alpha)$ *contains the shard* $\mathsf{S}(\alpha)$ *and is contained in the union of the shards* $\mathsf{S}(\alpha')$ *for all arcs* α' *forcing* α*.*

This property enables to construct quotientopes as Minkowski sums of shard polytopes. The idea now is that each shard polytope $\mathsf{SP}(\alpha)$ will be responsible for the shard $\mathsf{S}(\alpha)$ to appear in the normal fan of the Minkowski sum, without introducing unwanted walls.

Corollary 3. *For any lattice congruence* $\equiv$ *of the weak order and any positive coefficients* $\boldsymbol{s}_\alpha > 0$ *for* $\alpha \in \mathcal{A}_\equiv$*, the quotient fan* $\mathcal{F}_\equiv$ *is the normal fan of the Minkowski sum* $\mathsf{SP}(\mathcal{A}_\equiv) := \sum_{\alpha \in \mathcal{A}} \boldsymbol{s}_\alpha \mathsf{SP}(\alpha)$ *of the shard polytopes* $\mathsf{SP}(\alpha)$ *of all arcs* $\alpha \in \mathcal{A}_\equiv$*.*

This construction recovers relevant realizations of specific quotient fans. For example, for the sylvester congruence, we obtain the classical associahedron of [6]. More generally, for the α-Cambrian congruence, we get the α-associahedron of [4]. Moreover, all the quotientopes constructed by V. Pilaud and F. Santos [9] can be obtained this way.

The Minkowski indecomposability of shard polytopes can be rephrased in the realization spaces of the quotient fans. The space of all polytopes whose normal fan coarsens a given fan $\mathcal{F}$ is a cone under Minkowski addition, called (closed) *type cone* by P. McMullen [7] or *deformation cone* by A. Postnikov [10]. Theorem 1 affirms that for each arc $\alpha \in \mathcal{A}_\equiv$, the shard polytope $\mathsf{SP}(\alpha)$ is a representative of a ray of the type cone of the quotient fan $\mathcal{F}_\equiv$. Type cones of Cambrian fans have recently received particular attention with the works of [2, 3, 5, 8]. Their results imply that shard polytopes can be interpreted as Newton polytopes of F-polynomials of cluster variables of acyclic type A cluster algebras [2], and brick polytope summands of certain sorting networks [5]. Furthermore, each shard polytope $\mathsf{SP}(\alpha)$ is (up to a translation) a series-parallel matroid polytope.

As their normal fans coarsen the braid fan, shard polytopes belong to the class of *deformed permutahedra* [10]. It thus follows from [1] that they decompose uniquely as a signed Minkowski sum of faces of the standard simplex. We prove that, conversely, any deformed permutahedron has a unique decomposition as a Minkowski sum and difference of dilated shard polytopes (up to translation).

Besides containing and explaining the construction of [9], the motivation for this new construction is the possibility to extend it to lattice quotients of the poset of regions of hyperplane arrangements beyond the braid arrangement. Indeed, there is a natural geometric realization of lattice quotients of tight arrangements via polyhedral fans [14, 15]. However, no general polytopal realization is known. We achieve the first step in this perspective by constructing quotientopes for any lattice quotient of the weak order of the type B Coxeter group. In contrast to type A, no systematic construction of type B quotientopes was known so far.

The join-irreducible elements of B_n are in bijection with the so-called B*-arcs* and B*-shards*. Lattice congruences $\equiv^{\mathrm{B}}$ of the type B are then in correspondence with upper ideals $\mathcal{A}^{\mathrm{B}}_{\equiv^{\mathrm{B}}}$ of the forcing order among B-arcs. Each congurence defines a quotient fan $\mathcal{F}^{\mathrm{B}}_{\equiv^{\mathrm{B}}}$, whose chambers are the connected components of the complement of the union of the B-shards for all B-arcs in $\mathcal{A}^{\mathrm{B}}_{\equiv^{\mathrm{B}}}$ [11].

Proposition 4. *For any B-arc β, there exists a* shard polytope $\mathsf{SP}(\beta)$ *such that the union of the walls of its normal fan contains the shard $\mathsf{S}(\beta)$ and is contained in the union of the shards $\mathsf{S}(\beta')$ for $\beta \prec \beta'$.*

This property provides the first proof that all type B quotient fans are polytopal, recovering some known realizations, such as the cyclohedra of [4].

Corollary 5. *For any lattice congruence $\equiv^{\mathrm{B}}$ of the type B weak order and any positive coefficients $\boldsymbol{s}_\beta > 0$ for $\beta \in \mathcal{A}^{\mathrm{B}}_{\equiv^{\mathrm{B}}}$, the quotient fan $\mathcal{F}^{\mathrm{B}}_{\equiv^{\mathrm{B}}}$ is the normal fan of the Minkowski sum $\mathsf{SP}(\mathcal{A}^{\mathrm{B}}_{\equiv^{\mathrm{B}}}) := \sum_{\beta \in \mathcal{A}^{\mathrm{B}}} \boldsymbol{s}_\beta \mathsf{SP}(\beta)$ of the shard polytopes $\mathsf{SP}(\beta)$ of all B-arcs $\beta \in \mathcal{A}^{\mathrm{B}}_{\equiv^{\mathrm{B}}}$.*

The existence of shard polytopes for arbitrary hyperplane arrangements with a lattice of regions is still wide open.

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Domes over curves

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(joint work with Alexey Glazyrin)

The study of polyhedra with regular polygonal faces is a classical subject going back to ancient times. It was revived periodically when new tools and ideas have developed, most recently in connection to algebraic tools in rigidity theory. In this paper we study one of most basic problems in the subject – polyhedral surfaces in $\mathbb{R}^3$ whose faces are congruent equilateral triangles. We prove both positive and negative results on the types of boundaries these surfaces can have, suggesting a rich theory extending far beyond the current state of the art.

Formally, let $\gamma \subset \mathbb{R}^3$ be a closed piecewise linear (PL-) curve. We say that γ is *integral* if it is comprised of intervals of integer length. Now, let $S \subset \mathbb{R}^3$ be a PL-surface realized in $\mathbb{R}^3$ with the boundary $\partial S = \gamma$, and with all facets comprised of unit equilateral triangles. In this case we say that S is a *unit triangulation* or *dome over* γ, that γ is *spanned* by S, and that γ *can be domed.*

Question 1 (Kenyon [5]). Is every integral closed curve $\gamma \subset \mathbb{R}^3$ spanned by a unit triangulation? In other words, can every such γ be domed?

For example, the unit square and the (unit sided) regular pentagon can be domed by a regular pyramid with triangular faces. Of course, there is no such simple construction for a regular heptagon. Perhaps, surprisingly, the answer to Kenyon's question is negative in general.

A 3-dimensional *unit rhombus* is a closed curve $\rho \subset \mathbb{R}^3$ with four edges of unit length. This is a 2-parameter family of space quadrilaterals $\rho(a, b)$ parameterized by the *diagonals* a and b, defined as distances between pairs of opposite vertices.

Theorem 2. *Let $\rho(a, b) \subset \mathbb{R}^3$ be a unit rhombus with diagonals $a, b > 0$. Suppose $\rho(a, b)$ can be domed. Then there is a nonzero polynomial $P \in \mathbb{Q}[x, y]$, such that $P(a^2, b^2) = 0$.*

In other words, for $a, b > 0$ algebraically independent over $\overline{\mathbb{Q}}$, the corresponding unit rhombus cannot be domed, giving a negative answer to Kenyon's question. The proof is based on the *theory of places* [3] (see also [6, §41]). In fact, our tools give further examples of a unit rhombi which cannot be domed, such as $\rho\big(\frac{1}{\pi}, \frac{1}{\pi}\big)$.

Our next result is a positive counterpart to the theorem. We show that the set of integral curves spanned by a unit triangulation is everywhere dense within the set of all integral curves.

Let $\gamma, \gamma' \subset \mathbb{R}^3$ be two integral closed curves of equal length. We assume the vertices of γ, γ' are similarly labeled $\big[v_1, \ldots, v_n\big]$ and $\big[v'_1, \ldots, v'_n\big]$, giving a parameterizations of the curves. The *Fréchet distance* $|\gamma, \gamma'|_F$ in this case is given by

$$|\gamma, \gamma'|_F \; = \; \max_{1 \le i \le n} \; |v_i, v'_i|.$$

Theorem 3. *For every integral curve $\gamma \subset \mathbb{R}^3$ and $\varepsilon > 0$, there is an integral curve $\gamma' \subset \mathbb{R}^3$ of equal length, such that $|\gamma, \gamma'|_F < \varepsilon$ and γ' can be domed.*

The proof is an involved explicit argument in part based on the *Steinitz Lemma* (1913), see [1, 2] We conclude with one interesting special case:

Theorem 4. *Every regular integral n-gon in the plane can be domed.*

This gives a new infinite class of *regular polygon surfaces*, comprised of one regular n-gon and many unit triangles.

In [4], Gaifullin and Gaifullin studied the case of doubly periodic surfaces homeomorphic to the plane. In this case they proved the following result:

Theorem 5 ([4, Thm 1.4])**.** *Every embedded doubly periodic triangular surface homeomorphic to a plane has at most one-dimensional doubly periodic flex.*

By a *doubly periodic flex* of the triangular surface S we mean a continuous rigid deformation $\{S_t,\, t \in [0, \delta)\}$ for some $\delta > 0$, which preserves double periodicity, i.e. invariant under the action of $\mathrm{G} = \mathbb{Z} \oplus \mathbb{Z}$ (the action of G can also depend on t). The continuity of S is meant with respect to all dihedral angles. Here we identify deformations modulo changes of parameter t and ask for the dimension of the space of flexing at $t = 0$, i.e. when $S_0 = S$.

In [4, Question 1.5], the authors asked if Theorem 5 can be extended to surfaces which are not homeomorphic to a plane. We give a negative answer to this question by an explicit construction.

Theorem 6. *There is a doubly periodic triangular surface whose doubly periodic flex is three-dimensional.*

The proof is also based on an explicit construction which arises in the analysis of the inductive topological argument in the proof of Theorem 2.

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Helly numbers of disconnected sets

Pavel Paták

Given a finite family $\mathcal{F}$ of sets with empty intersection, it is natural to search for the smallest subfamily $\mathcal{G}$ whose intersection is still empty. The smallest size of such subfamily is called the Helly number of $\mathcal{F}$ and denoted $h(\mathcal{F})$. If $\bigcap \mathcal{F} \neq \emptyset$, we define $h(\mathcal{F}) := 0$.

Any upper bound on $h(\mathcal{F})$ which is valid for a whole class of families $\mathcal{F}$ has significant algorithmic applications, as exemplified by the original Helly's theorem [3]: For a family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$, $h(\mathcal{F}) \leq d+1$.

What to do if the sets are not convex? Or more generally, if the sets do not live in $\mathbb{R}^d$, but instead in some topological space X that has no natural notion of convexity? Such considerations naturally occur for example when considering line transversals, since they live in the Grassmanian.

To partially answer these questions, we show that if $\mathcal{F}$ is a set system in a d-dimensional manifold such that $\bigcap \mathcal{G}$ has at most b path-connected components for every $\mathcal{G} \subseteq \mathcal{F}$, and each of these components is $\lceil \frac{d}{2} \rceil$-connected, the Helly number of $\mathcal{F}$ is $b^{O(d)}$. This is a vast improvement over the previous result of Jiří Matoušek [7].

Let us now establish the essential ingredients of the proof. Given a family $\mathcal{F}$ of sets in a topological space X, we can turn $\mathcal{F}$ into a closure operator cl on X as follows. For a set $S \subseteq X$ we set

$$\mathrm{cl}(S) := \bigcap_{\substack{F \in \mathcal{F} \\ S \subseteq F}} F,$$

where we define $\mathrm{cl}\, S = X$, if no $F \in \mathcal{F}$ contains S.

The Radon number $r(\mathrm{cl})$ of a closure operator cl is defined as the smallest number r such that for every set $S \subseteq X$ with r elements, there are two disjoint subsets $A, B \subseteq S$ with $\mathrm{cl}(A) \cap \mathrm{cl}(B) \neq \emptyset$. If the closure operator cl is obtained from a finite set system $\mathcal{F}$, we have $h(\mathcal{F}) + 1 \leq r(\mathrm{cl})$, see e.g. the proof of Helly's theorem by Radon [11], or its abstract form [6]. As it turns out, Radon's number is one of the most important parameters of the closure operator, since it implies almost all other "convexity theorems": It provides (linear) bounds for Tverberg's numbers [5],[8], gives fractional Helly theorem [4] and establishes existence of weak ε-nets and (p,q)-theorems [1] for the considered families.

Probably the most general theorem in this direction is the following result obtained by Patáková [10]: *For every b and d, there is a number $r(b,d)$ such that the following holds. Let* cl *be a closure operator on $\mathbb{R}^d$ such that the Betti numbers $\beta_k(\mathrm{cl}(S); \mathbb{Z}/2\mathbb{Z})$ are upper bounded by b for all $k = 0, 1, \ldots, \lceil \frac{d}{2} \rceil - 1$ and all $S \subseteq \mathbb{R}^d$. Then $r(\mathrm{cl}) \leq r(b,d)$.*

The proof uses hypergraph Ramsey theorem and hence the values of $r(b,d)$ are usually too huge to be of practical importance.

Let us now sketch how certain non-embeddability results establish bounds for Radon numbers and the ideas that lead to the improvement of Matoušek's bound.

Let cl be a closure operator on $\mathbb{R}^2$, for which $\mathrm{cl}(S)$ is simply-connected for every $S \subseteq \mathbb{R}^2$. Then $r(\mathrm{cl}) \le 4$. Indeed, consider any set S of 4 points, say $p_1, p_2, \ldots, p_4$. Since $\mathrm{cl}\{p_1, p_2\}$ is path-connected, there is a path p_{12} that connects p_1 to p_2 and lies entirely in $\mathrm{cl}\{p_1, p_2\}$. Continuing this way we obtain a drawing of K_4. Now we may look at the triangle $p_{12}p_{23}p_{31}$. Since $\mathrm{cl}\{p_1, p_2, p_3\}$ is simply connected, this triangle can be filled. Continuing this way, we obtain a continuous map from the 2-skeleton of 3-simplex into $\mathbb{R}^2$. It is well known that in any such map there will be two disjoint faces, whose images intersect. These faces then correspond to the desired sets A and B.

In general, assuming that the set S is sufficiently large, we want to find a drawing of K_n and assignments $e \mapsto S_e$ for each edge $e \in E(K_n)$ such that

- each vertex of K_n is a point in S,
- $S_e \subseteq S \setminus V_e$, where V_e are the endpoints of e,
- each edge e is drawn inside $\mathrm{cl}(S_e \cup V_e)$,
- for disjoint edges e and f, the sets S_e and S_f are disjoint, and
- each set S_e is disjoint with $V(K_n)$.

We call such drawing *constrained.* If we suceed we may use the connectivity assumptions and continue by filling triangles, tetrahedra, and so on, until we arrive at some complex that is not “embeddable” into our target space.

Let us recall that we assume that $\mathrm{cl}(S)$ has at most b path-connected components. The new technique that allows us to improve the bounds relies on the following facts:

(1) The definition of constrained drawing carries over to arbitrary graphs.
(2) If we want to find a constrained drawing of the star $K_{1,k}$, it suffices if S contains $\binom{b+1}{2}(k-1)+b+1$ points.
(3) Complete graphs can be built inductively: If t points suffice to find K_{n-1} and x points suffice to find $K_{1,t}$, then x vertices are enough for K_n.

This naturally opens new questions.

(1) How many points are actually needed to find $K_{1,k}$? (Conjecture: $kb+1$)
(2) Can one obtain better bounds for K_n?
(3) Can the approach be adapted to deal with non-trivial homology/homotopy in higher dimensions?

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Barycentric cuts through a convex body

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(joint work with Martin Tancer, Uli Wagner)

Let K be a convex body in $\mathbb{R}^n$ (i.e., a compact convex set with nonempty interior). Given a point p in the interior of K, a hyperplane h passing through p is called *barycentric* if p is the barycenter of $K \cap h$. In 1961, Branko Grünbaum [4] raised the following questions:

Q1: Does there always exist an interior point $p \in K \subseteq \mathbb{R}^n$, $n \geq 3$, which admits at least $n+1$ distinct barycentric hyperplanes?

Q2: If so, is this true for the barycenter of K?

Seemingly, Question 1 was answered affirmatively by Grünbaum himself [5, §6.2] two years later, by using a variant of Helly's theorem to show that there are at least $n+1$ barycentric cuts through the point of K of maximal *depth*. However, when working on Question 2, which remains open, we noticed that one of the auxiliary claims in Grünbaum's proof is incorrect. We provide a counterexample to this claim and hence reopen Question 1.

In order to describe the problematic part, we first need several definitions. For a unit vector v in the unit sphere $S^{n-1} \subseteq \mathbb{R}^n$, let $h_v = h_v^p := \{x \in \mathbb{R}^n : \langle v, x-p \rangle = 0\}$ be the hyperplane orthogonal to v and passing through p, and let $H_v = H_v^p := \{x \in \mathbb{R}^n : \langle v, x-p \rangle \geq 0\}$ be the half-space bounded by h_v in the direction of v. Given p, we define the *depth function* $\delta^p \colon S^{n-1} \to [0,1]$ via $\delta^p(v) = \lambda(H_v \cap K)/\lambda(K)$, where λ is the Lebesgue measure in $\mathbb{R}^n$. The *depth* of a point p in K is defined as $\operatorname{depth}(p,K) := \inf_{v \in S^{n-1}} \delta^p(v)$. It is easy to see that δ^p is a continuous function, therefore the infimum in the definition is attained at some $v \in S^{n-1}$. Any hyperplane h_v through p such that $\operatorname{depth}(p,K) = \delta^p(v)$ is said to *realize the depth* of p. Finally, a *point of maximal depth in* K is a point p_0 in the interior of K such that $\operatorname{depth}(p_0,K) := \max \operatorname{depth}(p,K)$ where the maximum is taken over all points in the interior of K. We note that the point of maximal depth always exists and it is unique.

We remark that the depth function is a special case of the (Tukey) depth of a probability measure in $\mathbb{R}^d$, a well-known notion in statistics [7, 2].

The incorrect claim in the affirmative answer to Question 1 was that through the point of maximal depth, there are always at least $n+1$ distinct hyperplanes realizing the depth. However, we construct a convex body in $\mathbb{R}^3$ with only three such hyperplanes. Our construction is quite simple, it is $\Delta \times [0,1]$, where Δ is an equilateral triangle.

It follows from known results that for $n \geq 2$, there are always at least three distinct barycentric cuts through the point $p_0 \in K$ of maximal depth. Using tools related to Morse theory we are able to improve this bound: four distinct barycentric cuts through the point p_0 are guaranteed if $n \geq 3$. For more details we refer to [6].

Let us finish with a stronger version of Question 2 attributed to Karel Löwner [1, A8], [3, Problem 28]:

Q3: Are there always at least $2^n - 1$ distinct barycentric hyperplanes with respect to the barycenter of a convex body $K \subset \mathbb{R}^n$?

The simplex shows that this is the best possible bound one can hope for.

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Enclosing depth and other depth measures

Patrick Schnider

Medians are an important tool in the statistical analysis and visualization of data. Due to the fact that medians only depend on the order of the data points, and not their exact positions, they are very robust against outliers. However, in many applications, data sets are multidimensional, and there is no clear order of the data set. For this reason, various generalizations of medians to higher dimensions have been introduced and studied. Many of these generalized medians rely on a notion of *depth* of a query point within a data set, a median then being a query point with the highest depth among all possible query points. Several such depth measures have been introduced over time, most famously Tukey depth [4] (also called halfspace depth), simplicial depth, or convex hull peeling depth (see, e.g., [1]). In particular, all of these depth measures are *combinatorial*, i.e., they do not depend on the coordinates of the data points but only on their relative positions (their order type). In this abstract, we consider general classes of combinatorial depth measures, defined by a small set of axioms, and prove relations between them and concrete depth measures, such as Tukey depth (TD)[1] and Tverberg depth (TvD)[2]. We further introduce a new depth measure, called *enclosing depth*, which gives a lower bound for all considered combinatorial depth measures, and we prove that there is always a point whose enclosing depth is linear in the size of the data set.

Let $S^{\mathbb{R}^d}$ denote the family of all finite sets of points in $\mathbb{R}^d$. A depth measure is a function $\varrho : (S^{\mathbb{R}^d}, \mathbb{R}^d) \to \mathbb{R}_{\geq 0}$ which assigns to each pair (S, q) consisting of a finite set of data points S and a query point q a value, which describes how deep the query point q lies within the data set S. A depth measure ϱ is called *combintorial* if it depends only on the order type of $S \cup \{q\}$. We further want that it defines the standard depth in $\mathbb{R}^1$ (and thus gives a correct median)[3]

The first set of depth measures that we consider are *additive* depth measures. A combinatorial depth measure $\varrho : (S^{\mathbb{R}^d}, \mathbb{R}^d) \to \mathbb{R}_{\geq 0}$ is called additive if it satisfies the following conditions:

(i) for all $S \in S^{\mathbb{R}^d}$ and $q, p \in \mathbb{R}^d$ we have $|\varrho(S, q) - \varrho(S \cup \{p\}, q)| \leq 1$ (sensitivity),
(ii) for all $S \in S^{\mathbb{R}^d}$ and $q \in \mathbb{R}^d$ we have $\varrho(S, q) = 0$ for $q \notin \operatorname{conv}(S)$ (locality),
(iii) for all $S \in S^{\mathbb{R}^d}$ there is a $q \in \mathbb{R}^d$ for which $\varrho(S, q) > 0$ (non-triviality),
(iv) for any disjoint subsets $S_1, S_2 \subseteq S$ and $q \in \mathbb{R}^d$ we have $\varrho(S, q) \geq \varrho(S_1, q) + \varrho(S_2, q)$ (additivity).

It is not hard to show that a one-dimensional depth measure which satisfies these conditions has to be the standard depth measure and that no three conditions suffice for this. Further, we can show the following bounds for any additive depth measure ϱ in $\mathbb{R}^d$:

Observation 1.

$$TD(S, q) \geq \varrho(S, q) \geq TvD(S, q) \geq \frac{1}{d} TD(S, q).$$

Here, the first inequality follows from sensitivity and locality, while the second follows from non-triviality and additivity. The last inequality follows from the fact that removing a simplex containing q can decrease the Tukey depth of q by at most d. In $\mathbb{R}^2$, it can be shown that $\mathrm{TvD}(S, q) = \min\{\mathrm{TD}(S, q), |S|/3\}$, and we conjecture that in higher dimensions we have $\mathrm{TvD}(S, q) \geq \frac{1}{d-1} \mathrm{TD}(S, q)$, which would be tight.

[1]The Tukey depth of q is defined as the minimum number of points that can be removed from S such that q is not in the convex hull of the remaining point set

[2]The Tverberg depth of q is defined as the maximum number of pairwise vertex-disjoint simplices spanned by points in S that contain q.

[3]The standard depth in $\mathbb{R}^1$ is the one which counts the number of points of S to the left and to the right of q and then returns the minimum of the two numbers.

However, there are depth measures that give the standard depth in $\mathbb{R}^1$ but which are not additive. One example is enclosing depth, which we will now define. We say that a point set S of size $(d+1)k$ in $\mathbb{R}^d$ *k-encloses* a point q if S can be partitioned into $d+1$ subsets $S_1, \ldots, S_{d+1}$, each of size k, in such a way that for every transversal $p_1 \in S_1, \ldots, p_{d+1} \in S_{d+1}$, the point q is in the convex hull of $p_1, \ldots, p_{d+1}$. Intuitively, the points of S are centered around the vertices of a simplex with q in its interior. The enclosing depth of a point q with respect to a point set S, denoted by $\mathrm{ED}(S,q)$, is now defined as the maximal k for which there exists a subset of S which k-encloses q.

The fact that enclosing depth is not additive but still gives the standard measure in $\mathbb{R}^1$ indicates that the above conditions are too restrictive. We thus give a new set of conditions, defining *central* depth measures. A combinatorial depth measure $\varrho : (S^{\mathbb{R}^d}, \mathbb{R}^d) \to \mathbb{R}_{\geq 0}$ is called *central* if it satisfies sensitivity, locality and the following two conditions:

(iii') for all $S \in S^{\mathbb{R}^d}$ and $q, p \in \mathbb{R}^d$ we have $\varrho(S \cup \{p\}, q) \geq \varrho(S, q)$ (monotonicity),

(iv') for every $S \in S^{\mathbb{R}^d}$ there is a $q \in \mathbb{R}^d$ for which $\varrho(S,q) \geq \frac{1}{d+1}|S|$ (centrality).

It follows from Tverbergs theorem and Observation 1 that every additive depth measure is also central. Also, it can again be shown that a one-dimensional central depth measure has to be the standard depth measure and that no three of the above conditions suffice for this. On the other hand, enclosing depth is also not central, so while central depth measures are a superset of additive depth measures, they still do not include all depth measures that give the standard depth in $\mathbb{R}^1$. It would be interesting to find a set of conditions which define exactly those depth measures.

Similar to above, we can show bounds for central depth measures:

Theorem 2. *Let ϱ be a central depth measure in $\mathbb{R}^d$. Then there exists a constant $c = c(d)$, which depends only on the dimension d, such that*

$$TD(S,q) \geq \varrho(S,q) \geq ED(S,q) - (d+1) \geq c \cdot TD(S,q) - (d+1).$$

Here the first inequality follows again from sensitivity and locality. As for the second inequality, we would like to argue that if S k-encloses q then $\varrho(S,q) = k$. By centrality, there must indeed be a point q' with $\varrho(S,q') = k$ (note that $|S| = k(d+1)$ by definition of k-enclosing), but this point can lie anywhere in the centerpoint region of S and not every point in the centerpoint region is k-enclosed by S. However, by adding $d+1$ points very close to q, we can ensure that q is the only possible centerpoint in the new point set, and the second inequality then follows from sensitivity and monotonicity after removing these points again.

The most involved part of the result is the last inequality:

Theorem 3 ($E(d)$)**.** *There is a constant $c = c(d)$ such that for all $S \in S^{\mathbb{R}^d}$ and $q \in \mathbb{R}^d$ we have $ED \leq c \cdot TD(S,q)$.*

We will denote this statement in dimension d by $E(d)$. Note that $E(1)$ is true and $c(1) = 1$. The statement $E(d)$ turns out to be intimately related to a positive fraction Radon theorem on certain bichromatic point sets. Let $P = R \cup B$ be a bichromatic point set with color classes R (red) and B (blue). We say that B *surrounds* R if for every halfspace h we have $|B \cap h| \geq |R \cap h|$. Note that this in particular implies $|B| \geq |R|$. The positive fraction Radon theorem is now the following:

Theorem 4 ($R(d)$)**.** *Let $P = R \cup B$ be a bichromatic point set where B surrounds R. Then there is a constant $c_1 = c_1(d)$ such that there are integers a and b and pairwise disjoint subsets $R_1, \ldots, R_a \subseteq R$ and $B_1, \ldots, B_b \subseteq B$ with*

(1) $a + b = d + 2$,
(2) $|R_i| = c_1 \cdot |R|$ *for all* $1 \leq i \leq a$,
(3) $|B_i| = c_1 \cdot |R|$ *for all* $1 \leq i \leq b$,
(4) for every transversal $r_1 \in R_1, \ldots, r_a \in R_a, b_1 \in B_1, \ldots, b_b \in B_b$, *we have* $conv(r_1, \ldots, r_a) \cap conv(b_1, \ldots, b_b) \neq \emptyset$.

In other words, the Radon partition respects the color classes. We will denote the above statement in dimension d by $R(d)$. It can be seen that $R(1)$ can be satisfied choosing $a = 1$, $b = 2$ and $c_1(1) = \frac{1}{3}$. To conclude, we will sketch how $R(d-1) \Rightarrow E(d)$. Using the center transversal theorem [3, 5] and the Same Type Lemma [2] it can be shown that $E(d-1) \Rightarrow R(d)$. For space reasons, this proof has to be postponed to a forthcoming full version. By induction, these two claims then imply the above theorems.

Lemma 5. $R(d-1) \Rightarrow E(d)$.

Sketch of proof. Assume without loss of generality that q is the origin an that the halfspace $h : x_d \leq 0$ witnesses $\mathrm{TD}(S, q) = k$. Consider the point set S' derived from S by central projection through q to the hyperplane $x_d = 1$, and color all points from h red and the points from h^c blue. Then S' is a $(d-1)$-dimensional point set where B surrounds R. Further, every Radon partition in S' which respects the color classes corresponds to a simplex in S which contains q. □

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On the Average Complexity of the k-Level

Raphael Steiner
(joint work with M.-K. Chiu, S. Felsner, M. Scheucher, P. Schnider, P. Valtr)

Let $\mathcal{L}$ be an arrangement of n lines in the Euclidean plane. The *vertices* of $\mathcal{L}$ are the intersection points of lines of $\mathcal{L}$. Throughout this article we consider arrangements to be *simple*, i.e., no three lines intersect in a common vertex. Moreover, we assume that no line is vertical. The k*-level* of $\mathcal{L}$ consists of all vertices v which have exactly k lines of $\mathcal{L}$ below v. The $(\leq k)$*-level* of $\mathcal{L}$ consists of all vertices v which have at most k lines of $\mathcal{L}$ below v. We denote the k-level by $V_k(\mathcal{L})$ and its size by $f_k(\mathcal{L})$. Moreover, by $f_k(n)$ we denote the maximum of $f_k(\mathcal{L})$ over all arrangements $\mathcal{L}$ of n lines, and by $f(n) = f_{\lfloor (n-2)/2 \rfloor}(n)$ the maximum size of the *middle level*.

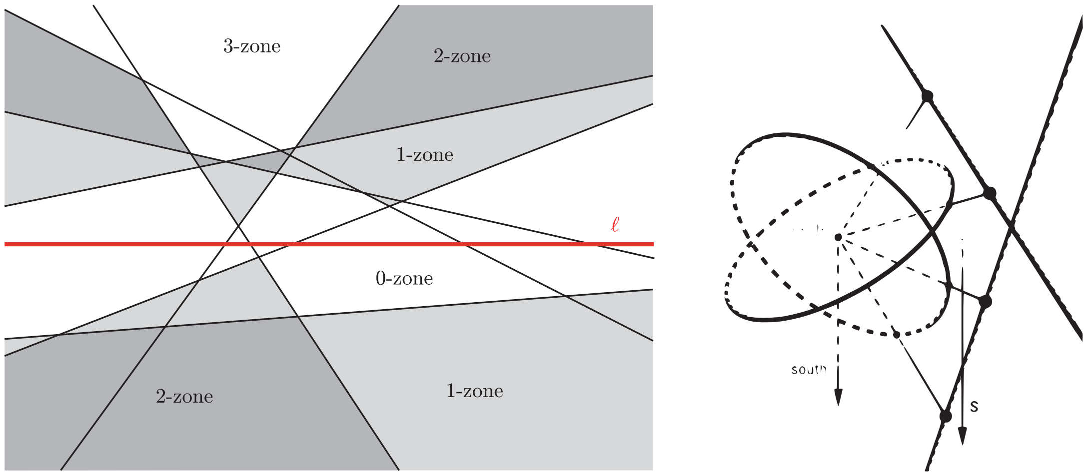
A k*-set* of a finite point set P in the Euclidean plane is a subset K of k elements of P that can be separated from $P \setminus K$ by a line. Paraboloid duality is a bijection $P \leftrightarrow \mathcal{L}_P$ between point sets and line arrangements. The number of k-sets of P equals $|V_{k-1}(\mathcal{L}_P) \cup V_{n-1-k}(\mathcal{L}_P)|$.

In discrete and computational geometry bounds on the number of k-sets of a planar point set, or equivalently on the size of k-levels of a planar line arrangement have important applications. The complexity of k-levels was first studied by Lovász [6] and Erdős et al. [5]. They bound the size of the k-level by $O(n \cdot (k+1)^{1/2})$. Dey [3] used the crossing lemma to improve the bound to $O(n \cdot (k+1)^{1/3})$. In particular, the maximum size $f(n)$ of the middle level is $O(n^{4/3})$. Concerning the lower bound on the complexity, Erdős et al. [5] gave a construction showing that $f(2n) \geq 2f(n) + cn = \Omega(n \log n)$ and conjectured that $f(n) \geq \Omega(n^{1+\varepsilon})$. An alternative $\Omega(n \log n)$-construction was given by Edelsbrunner and Welzl [4]. The current best lower bound $f_k(n) \geq n \cdot e^{\Omega(\sqrt{\log k})}$ was obtained by Nivasch [8]. The complexity of the $(\leq k)$-level in arrangements of lines is better understood. Alon and Györi [1] prove a tight upper bound of $(k+1)(n-k/2-1)$ for its size. For further information, we recommend the survey by Wagner [11].

Generalized Zone Theorem. In order to define “zones”, let us introduce the notion of “distances”. For x and x' being a vertex, edge, line, or cell of an arrangement $\mathcal{L}$ of lines in $\mathbb{R}^2$ we let their *distance* $\operatorname{dist}_{\mathcal{L}}(x, x')$ be the minimum number of lines of $\mathcal{L}$ intersected by the interior of a curve connecting a point of x with a point of x'. Pause to note that the k-level of $\mathcal{L}$ is precisely the set of vertices which are at distance k to the bottom cell.

The $(\leq j)$-zone $Z_{\leq j}(\ell, \mathcal{L})$ of a line ℓ in an arrangement $\mathcal{L}$ is defined as the set of vertices, edges, and cells from $\mathcal{L}$ which have distance at most j from ℓ. See Figure 1a for an illustration.

For arrangements of hyperplanes in $\mathbb{R}^d$ the $(\leq j)$-zone is defined similarly. The classical zone theorem provides bounds for the complexity of the zone ((≤ 0)-zone) of a hyperplane (cf. [7, Chapter 6.4]). A generalization with bounds for the complexity of the $(\leq j)$-zone appears as an exercise in Matoušek’s book [7, Exercise 6.4.2]. In the proof of Theorem 2 we use a variant of the 2-dimensional case (Proposition 1).



(A) (B)

FIGURE 1. a The higher order zones of a line ℓ. b The correspondence between great-circles on the unit sphere and lines in a plane. Using the center of the sphere as the center of projection points on the sphere are projected to the points in the plane.

Proposition 1. *Let $\mathcal{L}$ be a simple arrangement of n lines in $\mathbb{R}^2$ and $\ell \in \mathcal{L}$. The $(\leq j)$-zone of ℓ contains at most $2e \cdot (j+1)n$ vertices strictly above ℓ.*

Arrangements of Great-Circles. Let Π be a plane in 3-space which does not contain the origin and let $\mathbb{S}^2$ be a sphere in 3-space centered at the origin. The central projection Ψ_Π yields a bijection between arrangements of great circles on $\mathbb{S}^2$ and arrangements of lines in Π. Figure 1b gives an illustration.

The correspondence Ψ_Π preserves interesting properties, e.g. simplicity of the arrangements. If $\Psi_\Pi(\mathcal{C}) = \mathcal{L}$ and $\mathcal{L}$ has no parallel lines, then Ψ_Π induces a bijection between pairs of antipodal vertices of $\mathcal{C}$ and vertices of $\mathcal{L}$.

As in the planar case, we define the *distance* between points x, y of $\mathbb{S}^2$ with respect to a great-circle arrangement $\mathcal{C}$ as the minimum number of circles of $\mathcal{C}$ intersected by the interior of a curve connecting x with y. The *k-level* (*$(\leq k)$-level* resp.) of $\mathcal{C}$ is the set of all the vertices of $\mathcal{C}$ at distance k (distance at most k resp.) from the south pole. The $(\leq j)$-zone of a great-circle in $\mathbb{S}^2$ is defined similar to the $(\leq j)$-zone of a line in $\mathbb{R}^2$.

Let Π_1 and Π_2 be two parallel planes in 3-space with the origin between them and let Ψ_1 and Ψ_2 be the respective central projections. For a great-circle arrangement $\mathcal{C}$ we consider $\mathcal{L}_1 = \Psi_1(\mathcal{C})$ and $\mathcal{L}_2 = \Psi_2(\mathcal{C})$. A vertex v from the k-level of $\mathcal{C}$ maps to a vertex of the k-level in one of $\mathcal{L}_1$, $\mathcal{L}_2$ and to a vertex of the $(n-k-2)$-level in the other. Hence, bounds for the maximum size of the k-level of line arrangements carry over to the k-level of great-circle arrangements except for a multiplicative factor of 2.

The $(\leq j)$-zone of a great-circle C in $\mathcal{C}$ projects to a $(\leq j)$-zone of a line in each of $\mathcal{L}_1$ and $\mathcal{L}_2$. Hence, the complexity of a $(\leq j)$-zone in $\mathcal{C}$ is upper bounded by two times the maximum complexity of a $(\leq j)$-zone in a line arrangement. Proposition 1 implies that the $(\leq j)$-zone of a great-circle C in an arrangement of n great-circles contains at most $4e \cdot (j+1)n$ vertices in each of the two open hemispheres bounded by C.

Higher Dimensions. The problem of determining the complexity of the k-level admits a natural extension to higher dimensions. We consider arrangements in $\mathbb{R}^d$ of hyperplanes to be *simple*, meaning that no $d+1$ hyperplanes intersect in a common point. Moreover, we assume that no hyperplane is parallel to the x_d-axis. The *k-level* of $\mathcal{A}$ consists of all vertices (i.e. intersection points of d hyperplanes) which have exactly k hyperplanes of $\mathcal{A}$ below them (with respect to the d-th coordinate). We denote the k-level by $V_k(\mathcal{A})$ and its size by $f_k(\mathcal{A})$. Moreover, by $f_k^{(d)}(n)$ we denote the maximum of $f_k(\mathcal{A})$ among all arrangements $\mathcal{A}$ of n hyperplanes in $\mathbb{R}^d$.

As in the planar case, there remains a gap between lower and upper bounds;

$$\Omega(n^{\lfloor d/2 \rfloor} k^{\lceil d/2 \rceil - 1}) \leq f_k^{(d)}(n) \leq O(n^{\lfloor d/2 \rfloor} k^{\lceil d/2 \rceil - c_d}),$$

here $c_d > 0$ is a small positive constant only depending on d. Details and references can be found in Chapter 11 of Matoušek's book [7]. In dimensions 3 and 4 improved bounds have been established. For example, for $d = 3$, it is known that $f_k^{(3)}(n) \leq O(n(k+1)^{3/2})$ (see [9]). For the middle level in dimension $d \geq 2$ an improved lower bound $f^{(d)}(n) \geq n^{d-1} \cdot e^{\Omega(\sqrt{\log n})}$ is known (see [10] and [8]).

We call the intersection of $\mathbb{S}^d$ with a central hyperplane in $\mathbb{R}^{d+1}$ a *great-$(d-1)$-sphere* of $\mathbb{S}^d$. Similar to the planar case, arrangements of hyperplanes in $\mathbb{R}^d$ are in correspondence with arrangements of great-$(d-1)$-spheres on the unit sphere $\mathbb{S}^d$ (embedded in $\mathbb{R}^{d+1}$). The terms "distance" and "k-level" generalize in a natural way.

1. Our Results

Our first result concerns the average complexity of the k-level in arrangements of great-circles on $\mathbb{S}^2$ when the southpole is chosen uniformly at random among the cells. This question was raised by Barba, Pilz, and Schnider while sharing a pizza [2, Question 4.2].

We prove the following bound on the average complexity.

Theorem 2. *Let $\mathcal{C}$ be a simple arrangement of great-circles. The expected size of the $(\leq k)$-level is at most $16e \cdot (k+2)^2$ when the southpole is chosen uniformly at random among the cells of $\mathcal{C}$.*

Remarkably the bound is independent of the number n of great-circles in the arrangement.

Secondly, we investigate arrangements of randomly chosen great-circles. Here we propose the following model of randomness. On $\mathbb{S}^2$ we have the duality between points and great-circles (each antipodal pair of points defines the normal vector of the plane containing a great-circle). Since we can choose points uniformly at random from $\mathbb{S}^2$, we get random arrangements of great-circles. The duality generalizes to higher dimensions so that we can talk about random arrangements on $\mathbb{S}^d$ for a fixed dimension $d \geq 2$. Using the duality between antipodal pairs of points on $\mathbb{S}^d$ and great-$(d-1)$-spheres, we determine the exact asymptotics of the expected size of the k-level in this random model. Again the bound does not depend on the size of the arrangement.

Theorem 3. *Let $d \geq 2$ be fixed. In an arrangement of n great-$(d-1)$-spheres chosen uniformly at random on the unit sphere $\mathbb{S}^d$ (embedded in $\mathbb{R}^{d+1}$), the expected size of the k-level is of order $\Theta((k+1)^{d-1})$ for all $k \leq n/2$.*

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Optimal bounds for the colorful fractional Helly theorem

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(joint work with Denys Bulavka, Afshin Goodarzi)

Our starting point is the Helly theorem:

Theorem 1 (Helly's theorem [Hel23])**.** *Let $\mathcal{F}$ be a finite family of at least $d+1$ convex sets in $\mathbb{R}^d$. Assume that every subfamily of $\mathcal{F}$ with exactly $d+1$ members has a nonempty intersection. Then all sets in $\mathcal{F}$ have a nonempty intersection.*

Helly's theorem admits numerous extensions and two of them, important in our context, are the fractional Helly theorem and the colorful Helly theorem. The fractional Helly theorem of Katchalski and Liu covers the case when only some fraction of the $d+1$ tuples in $\mathcal{F}$ has nonempty intersection.

Theorem 2 (The fractional Helly theorem [KL79])**.** *For every $\alpha \in (0,1]$ and every non-negative integer d, there is $\beta = \beta(\alpha, d) \in (0,1]$ with the following property. Given a finite family $\mathcal{F}$ of $n \geq d+1$ convex sets in $\mathbb{R}^d$ such that at least $\alpha\binom{n}{d+1}$ of the subfamilies of $\mathcal{F}$ with exactly $d+1$ members have a nonempty intersection. Then there is a subfamily of $\mathcal{F}$ with at least βn members with a nonempty intersection.*

An interesting aspect of the fractional Helly theorem is not only to show the existence of $\beta(\alpha, d)$ but also to provide the largest value of $\beta(\alpha, d)$ with which the theorem is valid. This has been resolved independently by Kalai [Kal84] and by Eckhoff [Eck85] showing that the fractional Helly theorem holds with $\beta(\alpha, d) = 1-(1-\alpha)^{1/(d+1)}$. (There is a simple construction showing that $\beta(\alpha, d)$ cannot be improved beyond this bound.)

The colorful Helly theorem of Lovász covers the case where the sets are colored by $d+1$ colors and only the 'colorful' $(d+1)$-tuples of sets in $\mathcal{F}$ are considered. Given families $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ of sets in $\mathbb{R}^d$ a family of sets $\{F_1, \ldots, F_{d+1}\}$ is a *colorful $(d+1)$-tuple* if $F_i \in \mathcal{F}_i$ for $i \in [d+1]$, where for a non-negative integer $n \geq 1$ we use the notation $[n] := \{1, \ldots, n\}$. (The reader may think of $\mathcal{F}$ from preceding theorems decomposed into color classes $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$.)

Theorem 3 (The colorful Helly theorem [Lov74, Bár82])**.** *Let $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ be finite nonempty families of convex sets in $\mathbb{R}^d$. Let us assume that every colorful $(d+1)$-tuple has a nonempty intersection. Then one of the families $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ has a nonempty intersection.*

Both the colorful Helly theorem and the fractional Helly theorem with optimal bounds imply the Helly theorem. The colorful one by setting $\mathcal{F}_1 = \cdots = \mathcal{F}_{d+1} = \mathcal{F}$ and the fractional one by setting $\alpha = 1$ giving $\beta(1, d) = 1$.

The preceding two theorems can be merged into the following colorful fractional Helly theorem due to Bárány, Fodor, Montejano, Oliveros and Pór:

Theorem 4 (The colorful fractional Helly theorem [BFM$^+$14])**.** *For every $\alpha \in (0,1]$ and every non-negative integer d, there is $\beta_{\mathrm{col}} = \beta_{\mathrm{col}}(\alpha, d) \in (0,1]$ with the following property. Let $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ be finite nonempty families of convex sets in $\mathbb{R}^d$ of sizes $n_1, \ldots, n_{d+1}$ respectively. If at least $\alpha n_1 \cdots n_{d+1}$ of the colorful $(d+1)$-tuples have a nonempty intersection, then there is $i \in [d+1]$ such that $\mathcal{F}_i$ contains a subfamily of size at least $\beta_{\mathrm{col}} n_i$.*

Bárány et al. proved the colorful fractional Helly theorem with the value $\beta_{\mathrm{col}}(\alpha, d) = \frac{\alpha}{d+1}$ and they used it as a lemma [BFM$^+$14, Lemma 3] in a proof of a colorful variant of a (p,q)-theorem. Despite this, the correct bound for β_{col} seems to be of independent interest. In particular, the bound on β_{col} has been subsequently improved by Kim [Kim17] who showed that the colorful fractional Helly theorem is true with $\beta_{\mathrm{col}}(\alpha, d) = \max\{\frac{\alpha}{d+1}, 1-(d+1)(1-\alpha)^{1/(d+1)}\}$. On the other hand, the value of $\beta_{\mathrm{col}}(\alpha, d)$ cannot go beyond $1-(1-\alpha)^{1/(d+1)}$ because essentially the same example as for the standard fractional Helly theorem applies in this setting as well. (Kim [Kim17] provides a slightly different upper bound example showing the same bound.)

Coming back to the lower bound on $\beta_{\mathrm{col}}(\alpha, d)$, Kim explicitly conjectured that $1-(1-\alpha)^{1/(d+1)}$ is also a lower bound, thereby an optimal bound for the colorful fractional Helly theorem. He also provides a more refined conjecture [Kim17, Conjecture 4.2] which implies this lower bound. We are able to prove the refined conjecture which therefore indeed gives the optimal bounds for the colorful fractional Helly theorem.

Theorem 5 (The optimal colorful fractional Helly theorem). *Let $\mathcal{F}_1, \ldots, \mathcal{F}_{d+1}$ be finite nonempty families of convex sets in $\mathbb{R}^d$ of sizes $n_1, \ldots, n_{d+1}$ respectively. If at least $\alpha n_1 \cdots n_{d+1}$ of the colorful $(d+1)$-tuples have a nonempty intersection, for $\alpha \in (0,1]$, then there is $i \in [d+1]$ such that $\mathcal{F}_i$ contains a subfamily of size at least $(1-(1-\alpha)^{1/(d+1)})n_i$.*

In the proof we follow the exterior algebra approach which has been used by Kalai [Kal84] in order to provide optimal bounds for the standard fractional Helly theorem. We have to upgrade Kalai's proof to the colorful setting. This requires guessing the right generalization of several steps in Kalai's proof. However, we honestly admit that after making these 'guesses' we follow Kalai's proof quite straightforwardly.

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Polygons with Prescribed Angles in 2D and 3D

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(joint work with Alon Efrat, Radoslav Fulek, and Stephen Kobourov)

We consider the construction of a polygon P with n vertices whose turning angles at the vertices are given by a sequence $A = (\alpha_0, \ldots, \alpha_{n-1})$. Straight-line realizations of graphs with given metric properties have been one of the earliest applications of graph theory. We extend research on the so-called *angle graphs*, introduced by Vijayan [4] in the 1980s, which are geometric graphs with prescribed angles between adjacent edges.

In the plane, an *angle sequence* A is a sequence $(\alpha_0, \ldots, \alpha_{n-1})$ of real numbers such that $\alpha_i \in (-\pi, \pi)$ for all $i \in \{0, \ldots, n-1\}$. Let $P \subset \mathbb{R}^2$ be an oriented polygon with n vertices $v_0, \ldots, v_{n-1}$ that appear in the given order along P, which is consistent with the given orientation of P. The *turning angle* of P at v_i is the angle in $(-\pi, \pi)$ between the vector $v_i - v_{i-1}$ and $v_{i+1} - v_i$.

The oriented polygon P *realizes* the angle sequence A if the turning angle of P at v_i is equal to α_i, for $i = 0, \ldots, n-1$. A polygon P is *generic* if all its self-intersections are transversal (that is, proper crossings), vertices of P are distinct points, and no vertex of P is contained in a relative interior of an edge of P. Following the terminology of Viyajan [4], an *angle sequence* is *consistent* if there exists a generic closed polygon P with n vertices realizing A. For a polygon P that realizes an angle sequence $A = (\alpha_0, \ldots, \alpha_{n-1})$ in the plane, the *total curvature* of P is $\mathrm{TC}(P) = \sum_{i=0}^{n-1} \alpha_i$, and the *turning number* (also known as *rotation number*) of P is $\mathrm{tn}(P) = \mathrm{TC}(P)/(2\pi)$; it is known that $\mathrm{tn}(P) \in \mathbb{Z}$ in the plane [3].

The *crossing number*, denoted by $\mathrm{cr}(P)$, of a generic polygon is the number of self-crossings of P. The *crossing number* of a consistent angle sequence A is the minimum integer c, denoted by $\mathrm{cr}(A)$, such that there exists a generic polygon $P \in \mathbb{R}^2$ realizing A with $\mathrm{cr}(P) = c$. Our first main results is the following theorem.

Theorem 1. *For a consistent angle sequence $A = (\alpha_0, \ldots, \alpha_{n-1})$ in the plane,*

$$\mathrm{cr}(A) = \begin{cases} 1 & \text{if } \sum_{i=0}^{n-1} \alpha_i = 0, \\ |k| - 1 & \text{if } \sum_{i=0}^{n-1} \alpha_i = 2k\pi \text{ and } k \neq 0. \end{cases}$$

The lower bound follows from a result by Grünbaum and Shepard [2, Theorem 6], using a decomposition due to Wiener [5].

In d-space, $d \geq 3$, the sign of a turning angle no longer plays a role: The *turning angle* of an oriented polygon P at v_i is in $(0, \pi)$, and an angle sequence $A = (\alpha_0, \ldots, \alpha_{n-1})$ is in $(0, \pi)^n$. By the discrete version of Fenchel's theorem [3, Theorem 2.4], we have $\sum_{i=0}^{n-1} \alpha_i \geq 2\pi$ if A is realizable in $\mathbb{R}^d$ for any $d \geq 2$.

The unit-length direction vectors of the edges of P determine a spherical polygon P' in $\mathbb{S}^{d-1}$. Note that the turning angles of P correspond to the spherical lengths of the segments of P'. It is not hard to see that this observation reduces the problem of realizability of A by a polygon in $\mathbb{R}^d$ to the problem of realizability of A by a spherical polygon in $\mathbb{S}^{d-1}$, in the sense defined below, that additionally contains the origin $\mathbf{0}$ in the interior of its convex hull.

Let $\mathbb{S}^2 \subset \mathbb{R}^3$ denote the unit 2-sphere. A *spherical polygon* $P \subset \mathbb{S}^2$ is a closed curve consisting of finitely many spherical segments; and a spherical polygon $P = (\mathbf{u}_0, \ldots, \mathbf{u}_{n-1})$, $\mathbf{u}_i \in \mathbb{S}^2$, realizes an angle sequence $A = (\alpha_0, \ldots, \alpha_{n-1})$ if the spherical segment $(\mathbf{u}_{i-1}, \mathbf{u}_i)$ has (spherical) length α_i, for every i. The *turning angle* of P at $\mathbf{u}_i$ is the angle in $[0, \pi]$ between the tangents to $\mathbb{S}^2$ at $\mathbf{u}_i$ that are co-planar with the great circles containing $(\mathbf{u}_i, \mathbf{u}_{i+1})$ and $(\mathbf{u}_i, \mathbf{u}_{i-1})$. Unlike for polygons in $\mathbb{R}^2$ and $\mathbb{R}^3$, we do not put any constraints on turning angles of spherical polygons (i.e., angles 0 and π are allowed).

Regarding realizations of A by spherical polygons, we prove the following.

Theorem 2. *Let $A = (\alpha_0, \ldots, \alpha_{n-1})$, $n \geq 3$, be an angle sequence. There exists a generic polygon $P \subset \mathbb{R}^3$ realizing A if and only if $\sum_{i=0}^{n-1} \alpha_i \geq 2\pi$ and there exists a spherical polygon $P' \subset \mathbb{S}^2$ realizing A. Furthermore, P can be constructed efficiently if P' is given.*

Theorem 3. *There exists a constructive weakly polynomial-time algorithm to test whether a given angle sequence $A = (\alpha_0, \ldots, \alpha_{n-1})$ can be realized by a spherical polygon $P' \subset \mathbb{S}^2$.*

A simple exponential-time algorithm for realizability of angle sequences by spherical polygons follows from a known characterization [1, Theorem 2.5], which also implies that the order of angles in A does not matter for the spherical realizability. The combination of Theorems 2 and 3 yields our second main result.

Theorem 4. *There exists a constructive weakly polynomial-time algorithm to test whether a given angle sequence $A = (\alpha_0, \ldots, \alpha_{n-1})$ can be realized by a polygon $P \subset \mathbb{R}^3$.*

It remains an open problem to find an efficient algorithms that computes the minimum number of crossings in generic realizations. The evidence that we have points to the following conjecture, whose "only if" part we can prove.

Conjecture 5. *Let $A = (\alpha_0, \ldots, \alpha_{n-1})$, $n \geq 3$, be an angle sequence that can be realized by a polygon in $\mathbb{R}^3$. The sequence A can be realized by a polygon in $\mathbb{R}^3$ without self-intersections if and only if n is even or $\sum_{i=0}^{n-1}(\pi - \alpha_i) \neq \pi$.*

It can be seen that Conjecture 5 is equivalent to the claim that every realization A in $\mathbb{R}^3$ has a self-intersection if and only if A can be realized in $\mathbb{R}^2$ as a *thrackle*, that is, a polygon where every pair of nonadjacent edges cross each other.

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Connectivity of triangulation flip graphs in the plane

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(joint work with Uli Wagner)

Given a finite point set P in *general position* in the plane, a *full triangulation* of P is a maximal straight-line embedded plane graph on P. A *partial triangulation* of P is a full triangulation of some subset P' of P containing all extreme points in P. A *bistellar flip* on a partial triangulation either flips an edge (called *edge flip*), removes a non-extreme point of degree 3, or adds a point in $P \setminus P'$ as vertex of degree 3. The *bistellar flip graph* has all partial triangulations as vertices, and a pair of partial triangulations is adjacent if they can be obtained from one another by a bistellar flip. The *edge flip graph* is defined with full triangulations as vertices, and edge flips determining the adjacencies. Lawson [2] showed in the early seventies that these graphs are connected. Our goal is to investigate the structure of these graphs, with emphasis on their vertex connectivity.

For sets P of n points in the plane in general position, we show that the edge flip graph is $\lceil \frac{n}{2} - 2 \rceil$-vertex connected [5], and the bistellar flip graph is $(n-3)$-vertex connected [6]; both results are tight and resolve, for sets in general position, a question asked in [3]. The latter bound matches the situation for the subfamily of regular triangulations (i.e., partial triangulations obtained by lifting the points to 3-space and projecting back the lower convex hull), where $(n-3)$-vertex connectivity has been known since the late eighties through the secondary polytope due to Gelfand, Kapranov & Zelevinsky [1], and Balinski's Theorem. For the edge flip-graph, we additionally show that the vertex connectivity is at least as large as (and hence equal to) the minimum degree (i.e., the minimum number of flippable edges in any full triangulation), provided that n is large enough.

Our methods also yield several other results: (i) The edge flip graph can be covered by graphs of polytopes of dimension $\lceil \frac{n}{2} - 2 \rceil$ (products of associahedra) and the bistellar flip graph can be covered by graphs of polytopes of dimension $n-3$ (products of secondary polytopes). (ii) A partial triangulation is regular, if it has distance $n-3$ in the Hasse diagram of the partial order of partial subdivisions from the trivial subdivision. (iii) All partial triangulations of a point set are regular iff the partial order of partial subdivisions has height $n-3$. (iv) There are arbitrarily large sets P with non-regular partial triangulations and such that every proper subset has only regular triangulations, i.e. there are no small certificates for the existence of non-regular triangulations.

A natural next question in the plane is to show expansion properties of the flip graphs, ideally yielding rapid mixing of the process of flipping random edges.

The question of whether flip graphs in higher dimensions are connected remained a mystery until Santos [4] showed in 2000, that in dimension 5 and higher, there exist point sets for which the graph (for bistellar flips) is not connected. The question is open in dimensions 3 and 4.

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Representability of c-arrangements

Geva Yashfe

(joint work with Lukas Kühne)

A matroid M with rank function r is said to be representable as a c-arrangement if the polymatroid $c \cdot r$ is representable. Previously, in [2], we studied the representability of matroids as c-arrangements, and showed the following problem is undecidable: Let $\mathbb{F}$ be a field. Given a matroid M, decide whether there exists $c \in \mathbb{N}$ such that M is representable as a c-arrangement over $\mathbb{F}$. This talk presented a strengthening of this result ([3]) to limits of polymatroid rank functions, proved using refinements of the same techniques. We showed the following problem is undecidable: Fix a field $\mathbb{F}$. The input is a finite set E and a matroidal rank function $r : 2^E \to \mathbb{N}$. The problem is to decide whether there exist representable polymatroidal rank functions $\{r_i\}_{i=1}^{\infty}$, where $r_i : 2^E \to \mathbb{N}$, together with a sequence of integers $\{c_i\}_{i=1}^{\infty}$, such that

$$\forall S \subseteq E : \quad \lim_{i\to\infty} \frac{r_i(S)}{c_i} = r(S).$$

One motivation for studying this limit problem is an application to rank inequalities. These are linear inequalities on the values of the rank function which any representable matroidal (or polymatroidal) rank function satisfies. For example, Ingleton's inequalities state that if r is representable, then for any $A_1, \ldots, A_4 \subseteq E$:

$$r(A_1) + r(A_2) + r(A_1 \cup A_2 \cup A_3) + r(A_1 \cup A_2 \cup A_4) + r(A_3 \cup A_4) \leq$$
$$r(A_1 \cup A_2) + r(A_1 \cup A_3) + r(A_1 \cup A_4) + r(A_2 \cup A_3) + r(A_2 \cup A_4).$$

Kinser [1] asked several general questions about representable polymatroids. We answer some of these, showing:

- Not every rational polymatroid which satisfies all rank inequalities has a representable multiple.

- The set of all rank inequalities cannot be classified into a finite number of families, each of which is finite when restricted to the ground set $[n]$ and computable in a suitable sense.

The proof encodes word problems for sofic groups in limit representation problems for matroids (the decision problems described above). The main construction used is similar to the one in [2].

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The ϵ-t-Net Problem

Yelena Yuditsky

(joint work with Noga Alon, Bruno Jartoux, Chaya Keller and Shakhar Smorodinsky)

We study a natural generalization of the classical ϵ-net problem (Haussler–Welzl 1987 [1]), which we call the *ϵ-t-net problem*: Given a hypergraph on n vertices and parameters t and $\epsilon \geq \frac{t}{n}$, find a minimum-sized family S of t-element subsets of vertices such that each hyperedge of size at least ϵn contains a set in S. When $t = 1$, this corresponds to the ϵ-net problem.

We prove that any sufficiently large hypergraph with VC-dimension d admits an ϵ-t-net of size $O(\frac{(1+\log t)d}{\epsilon} \log \frac{1}{\epsilon})$. For some families of geometrically-defined hypergraphs, we prove the existence of $O(\frac{1}{\epsilon})$-sized ϵ-t-nets. For example the dual hypergraph defined with respect to points and pseudo-disks in the plane, and more generally, the dual hypergraph of regions with linear union complexity.

We also present an explicit construction of ϵ-t-nets (including ϵ-nets) for hypergraphs with bounded VC-dimension. In comparison to previous constructions for the special case of ϵ-nets (i.e., for $t = 1$), it does not rely on advanced derandomization techniques. To this end we introduce a variant of the notion of VC-dimension which is of independent interest.

Finally, we use our techniques to generalize the notion of ϵ-approximation and to prove the existence of small-sized ϵ-t-approximations for sufficiently large hypergraphs with a bounded VC-dimension.

We present a few applications for the ϵ-t-nets. For example, for the Turán problem in hypergraphs, edge colorings of hypergraphs and secret sharing.

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Cutting cakes with topological Hall

Shira Zerbib

(joint work with Ron Aharoni, Eli Berger, Joseph Briggs, Erel Segal-Halevi, Shira Zerbib)

An instance of the *cake division* problem consists of some resource (called *cake*), identified with the unit interval, that is to be partitioned into interval parts (called *pieces*), with the aim of distributing the parts among *agents* (sometimes also called *players*). A partition of the cake divides it into intervals $I_1, \ldots, I_n$ of respective lengths $x_1, \ldots, x_n$, from left to right. Since $\sum_{j=1}^{n} x_j = 1$, the partition can be identified with the element $\vec{x} = (x_1, \ldots, x_n)$ in the $(n-1)$-dimensional simplex Δ_{n-1}. The interval $I_j = I_j(\vec{x})$ is called the j-th piece.

Each agent i assigns to each partition $\vec{x}$ a nonempty set $L^i(\vec{x})$ of indices of pieces she *prefers* (that is, finds at least as good as the other pieces). An allocation of pieces from $\vec{x}$ among the agents is called *fair*, or *envy-free*, if every agent i receives a piece I_j with $j \in L^i(\vec{x})$, and the pieces the agents receive are distinct.

Let $A^i_j = \{\vec{x} \mid j \in L^i(\vec{x})\}$. Two conditions are assumed for every fixed i:

(a) A^i_j is a closed set for every j.

(b) ("hungry players") For every partition $\vec{x}$ the set $L^i(\vec{x})$ contains at least one index j with $x_j \neq 0$.

In this talk we are interested in the *multiple cake division* problem, in which the resource consists of multiple copies $C_i,\ i \leq k$ of the unit interval, that are each to be partitioned into a number a_i of pieces. Each agent is to be given a k-tuple of pieces, one piece per C_i. Each agent assigns to each k-tuple of partitions a nonempty set of preferred k-tuples of pieces, at least one of which consists entirely of nonempty pieces. As we shall see, in the general case it is not always possible in this case to satisfy all players. But we shall ask how many players can be made happy. That is — what is the maximum, over all k-tuples of partitions, of the number of players that can receive a k-tuple of pieces they prefer.

The classical fair division theorem due to Stromquist [7] and Woodall [8], states that for $k = 1$ envy-free division into n pieces between n agents is always possible. This theorem can be proved using topological methods, and Meunier and Su [4] recently gave a new topological proof of this result. We use their approach to reduce the fair division problem with k cakes to a problem on matchings in $(k+1)$-uniform hypergraphs, where the reduction is one-directional: the existence of a matching in a hypergraph satisfying certain conditions implies the possibility of satisfying a large number of agents. We shall use a topological version of Hall's theorem Due to Aharoni and Haxell [1], to obtain results on the matchings problem, and thereby also on fair division of multiple cakes.

Below, we summarize some of the known results, and our contribution. We write $(n; a_1, a_2, \ldots, a_k) \rightsquigarrow m$, for $m \leq n$, if for every instance of the fair division problem with n agents and k cakes, where cake j is partitioned into a_j parts, there exists such a partition of the cakes and an envy-free division to a set of agents of size m.

The Stromquist-Woodall theorem is that

Theorem. *For all $n \geq 1$: $(n; n) \rightsquigarrow n$,*

For $k > 1$ the analogous result is false. An example with $k = 2$ (two cakes), $n = 2$ (two agents), and partitions of each cake into 2 pieces, in which there is no envy-free division satisfying both players, was given in [2]. In our notation, this result says: $(2; 2, 2) \not\rightsquigarrow 2$. Some other known results are:

Theorem.

- $(2; 2, 3) \rightsquigarrow 2$ *and* $(3; 2, 2) \rightsquigarrow 2$ [2].
- $(3; 5, 5) \rightsquigarrow 3$ [3].
- $(p; \underbrace{n, \ldots, n}_{k\ times}) \rightsquigarrow \left\lceil \frac{p}{2k(k-1)} \right\rceil$ *whenever* $p \leq k(n-1)+1$, *and*
- $(p; \underbrace{n, \ldots, n}_{k\ times}) \rightsquigarrow \left\lceil \frac{p}{k(k-1)} \right\rceil$ *if* p *divides* $k(n-1)+1$ [5].
- *In particular,* $(2n-1; n, n) \rightsquigarrow n$.

The simplest and possibly most attractive case of the $\rightsquigarrow$ relation is that of $(n; n, *)$, namely when there are n players, and one of the two cakes is partitioned into n pieces. One question is then into how many pieces should the second cake be partitioned in order to make all n agents envy free — what $p > n$ guarantees $(n; n, p) \rightsquigarrow n$. We show that $p = n^2 - n/2$ suffices, and $p = 2n - 2$ does not. We also prove results of the form $(n; n, p) \rightsquigarrow m$, for various values of p. Additionally, we show that the result of [5], $(2n-1; n, n) \rightsquigarrow n$, is sharp in the sense that $(2n-2; n, n) \not\rightsquigarrow n$. Our results are:

- $(n; n, rn) \rightsquigarrow \left\lceil \frac{\lfloor 2r \rfloor n}{\lfloor 2r \rfloor + 2} \right\rceil$ and $(n; n, rn) \rightsquigarrow \left\lceil \frac{2rn}{\lceil 2r \rceil + 2} \right\rceil$ for every $r \geq 1$ such that rn is an integer. In particular:
  - $(n; n, n^2 - n/2) \rightsquigarrow n$ (generalizing the result $(2, 2, 3) \rightsquigarrow 2$ from [2]).
  - $(n; n, \binom{n}{2}) \rightsquigarrow n - 1$.
  - $(n; n, 2n-1) \rightsquigarrow \max\left(\left\lceil \frac{2n-1}{3} \right\rceil, \left\lceil \frac{3n}{5} \right\rceil\right)$.
- $(n; n, n+1) \rightsquigarrow \left\lceil \frac{n+1}{2} \right\rceil$.
- $(n; 2n-1, 2n-1) \rightsquigarrow n$ (generalizing the result $(3; 5, 5) \rightsquigarrow 3$ from [3]).
- $(2n-2; n, n) \not\rightsquigarrow n$ (showing sharpness of the result $(2n-1; n, n) \rightsquigarrow n$ from [5]).
- $(n; n, 2n-2) \not\rightsquigarrow n$ (generalizing the result $(2; 2, 2) \not\rightsquigarrow 2$ from [2]).

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Open Problems in Discrete Geometry

Collected by Günter Rote

PROBLEM 1 (Karim Adiprasito). Repeated halving of simplices

Start with a d-dimensional simplex. Subdivide the longest edge and cut the simplex into two simplices of half the area. Repeat the process recursively with both halves ad infinitum. (The result is in general not a face-to-face decomposition.)

We assume that the starting simplex is sufficiently generic, so that there is never a tie in choosing the longest edge.

In dimensions 2 and 3, this process stabilizes after finitely many iterations in the sense that every new simplex is homothetic to one of the simplices that have already been seen. In dimension 4, this is true provided that one starts with a suitable starting simplex: a generically perturbed orthoscheme.

Question. Does this stabilizing behavior occur for any starting simplex, in any dimension?

Or, on the contrary, can this process lead to arbitrarily badly shaped simplices, where the ratio between the inradius and the circumradius approaches 0?

The question has applications in scientific computing.

PROBLEM 2 (Gil Kalai). Complicated intersections

Consider a fixed real algebraic variety N in $\mathbb{R}^d$, and real algebraic varieties M of a certain type.

We are looking for general statement of the form: If all intersections between M and affine transformations of N are "complicated" then the dimension of the affine hull of M is "small".

The model statement for this is the Sylvester–Gallai Theorem: Here, M is a set of points and N is a line. If all nontrivial intersections have more then 2 points, then the affine hull of M has dimension 1.

PROBLEM 3 (Tomasz Szemberg). Absolute linear Harbourne constants

Harbourne constants have been introduced in connection with the Bounded Negativity Conjecture in algebraic geometry [1]. However, they can be considered purely combinatorially.

Let $\mathbb{K}$ be a field and $\mathcal{L}$ be a set of d lines in the projective plane $\mathbb{P}^2(\mathbb{K})$. Let $\{P_1, \dots, P_s\}$ be the set of points where at least two lines of $\mathcal{L}$ intersect, and let m_i be the number of lines passing through P_i.

The rational number

$$H(\mathbb{K}, \mathcal{L}) = \frac{d^2 - \sum_{i=1}^{s} m_i^2}{s}$$

is the Herbourne constant of the arrangement $\mathcal{L}$.

Taking the minimum over all arrangements of d lines in $\mathbb{P}^2(\mathbb{K})$, we obtain the linear Harbourne constant of d lines over $\mathbb{K}$

$$H(\mathbb{K}, d) = \min_{|\mathcal{L}|=d} H(\mathbb{K}, \mathcal{L}).$$

Finally, taking the minimum over all fields $\mathbb{K}$ we arrive at the absolute linear Harbourne constant of d lines

$$H(d) = \min_{\mathbb{K}} H(\mathbb{K}, d).$$

Question. Compute the numbers $H(d)$.

This has been done for $1 \leq d \leq 31$ and for d of the form $d = q^2 + q + 1$ [2]. The article contains also a conjectural formula for these numbers.

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PROBLEM 4 (Tomasz Szemberg). Projective plane of order 10

A computer proof of the non-existence of a projective plane of order 10 was announced in 1989 [1]. Has this proof ever been independently verified?

Note. (Konrad Swanepoel) It seems that the non-existence of a projective plane of order 10 was independently checked as described in a M.Sc. thesis from 2010 [3]. There are other verifications of parts of the search [2].

Curtis Bright and coworkers are busy using SAT solvers to produce more rigorous computer-based proofs, see `https://cs.uwaterloo.ca/~cbright/#writings`. There is no complete formal proof yet, but it looks as if this is not very far away. The most enlightening of his papers is [4].

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PROBLEM 5 (Luis Montejano). Sections that are bodies of revolution

If all hyperplane sections of a convex body of dimension at least 4 are either single points or bodies of revolution, prove that the body is itself a body of revolution.

PROBLEM 6 (Gil Kalai). 4-POLYTOPES WITH DENSE GRAPHS

Suppose a simplicial 4-polytope P with n vertices has the property that, among every three vertices, at least two of them are joined by an edge. Does it follow that the graph of P contains a "large" complete subgraph, say, of size $n/10$?

PROBLEM 7 (Arseniy Akopyan).
UNBALANCED HAM-SANDWICH CUTS FOR SPHERICAL CAPS

We are given two continuous probability measures on the 2-dimensional sphere and a parameter $0 < \alpha < 1/2$. Is there always a spherical cap (intersection with a half-space) that has measure α for both measures?

PROBLEM 8 (Raphael Steiner).
BICHROMATIC TRIANGLES IN ARRANGEMENTS OF PSEUDOLINES

A pseudoline is a non-self-intersecting infinite curve in $\mathbb{R}^2$ dividing the plane into two connected components. A *simple* arrangement of pseudolines is a set of pseudolines such that any two distinct pseudolines intersect in a point, and no point is contained in three or more pseudolines.

Question. In a simple arrangement of red and blue pseudolines, is there always a bounded triangular face that is incident to a red and a blue pseudoline?

It is easy to see that this is true for planar *line* arrangements. It holds also for the more general class of *approaching* pseudoline arrangements [1].

[1] Stefan Felsner, Alexander Pilz, and Patrick Schnider, *Arrangements of approaching pseudolines.* arXiv:2001.08419, (2020).

PROBLEM 9 (Balázs Keszegh). HEREDITARY POLYCHROMATIC k-COLORINGS

For a hypergraph $\mathcal{H}$ denote by m_k the smallest number for which we can k-color the vertices such that on every hyperedge of size at least m_k, all k colors appear. Denote by m_k^* the maximum of m_k over every induced subhypergraph of $\mathcal{H}$.

Berge showed that if for a hypergraph $m_2^* = 2$ then $m_k^* = k$ for all k. What about larger m_2^*? Does $m_2^* = 3$ imply that m_3^* is finite?

PROBLEM 10 (Emo Welzl). MINIMUM NUMBER OF PARTIAL TRIANGULATIONS

A partial triangulation of a set of n points in the plane is a triangulation of the convex hull that may use the interior points as vertices, but does not have to use all of them.

Question. What is the smallest number of partial triangulations that a set of n points in general position can have? Is it the Catalan number $C_{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2}$?

For full triangulations, where all interior points have to be used, smallest known number of full triangulations, roughly $\sqrt{12}^n$, is obtained by the so-called *double circle*, which is constructed by putting an interior point near the midpoint of every edge of a regular $\frac{n}{2}$-gon. By contrast, n points in convex position have $C_{n-2} \sim 4^n$ full (or equivalently, partial) triangulations. Interestingly, the double circle has exactly the same number C_{n-2} of partial triangulations.

PROBLEM 11 (Karim Adiprasito). The compact part of a polyhedral subdivision

Take a polyhedral subdivision of $\mathbb{R}^3$ into finitely many parts, none of which contains a line, and look at the union of all bounded faces. This set is contractible. Is it collapsible?

PROBLEM 12 (Stefan Langerman). The centerpoint constant for complete intersections

For every set of n lines in the plane, there is a point p such that for every halfspace H containing p there is a subset of at least $\sqrt{n/3}$ of lines all of whose intersections lie in H. There are examples that show that the bound cannot be improved to more than $\sqrt{n}$. What is the right constant?

PROBLEM 13 (Gil Kalai). Sets consisting of two convex pieces

Suppose there is a family of sets in d dimensions, each of which is the disjoint union of two nonempty closed convex sets. Moreover, the intersection of any $2, 3, \dots$ or $d+1$ sets from the family has the same property of consisting of exactly two convex pieces. Does it follow that the whole family has a nonempty intersection?

Micha Perles constructed an example that shows that the statement is not true in the plane if the number "two" of convex pieces is replaced by 48.

PROBLEM 14 (Michael Dobbins). Extending piecewise-linear maps from the boundary to the interior in a continuous manner

Take a fixed reference triangle ABC, and consider a piecewise-linear (PL) one-to-one map from the boundary of the triangle ABC into the plane (in other words, a PL parameterization of a simple polygon).

Such a map can be easily extended to a PL one-to-one map from whole triangular area ABC into the plane.

Can this be done in a way that depends continuously on the boundary map? In other words, is there a continuous function that assigns to every PL one-to-one map $\partial ABC \to \mathbb{R}^2$ a PL one-to-one map $ABC \to \mathbb{R}^2$ extending it?

Reporter: Johanna Kristina Steinmeyer

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