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# Mini-Workshop: Rank One Groups and Exceptional Algebraic Groups

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Abstract. Rank one groups are a class of doubly transitive groups that are natural generalizations of the groups $\mathrm{SL}_2(k)$. The most interesting examples arise from exceptional algebraic groups of relative rank one. This class of groups is, in turn, intimately related to structurable algebras. The goal of the mini-workshop was to bring together experts on these topics in order to make progress towards a better understanding of the structure of rank one groups.

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## Introduction by the Organizers

The aim of the workshop was to bring together researchers from different areas in order to discuss various topics concerning rank one groups. A rank one group is a pair $(G, X)$ consisting of a group $G$ and a set $X$ such that each point stabilizer $G_x$ contains a normal subgroup $U_x$ (called a root group) acting regularly on $X \setminus \{x\}$ and such that all $U_x$ are conjugate in $G$. Thus, in particular, $G$ acts 2-transitively on $X$. Rank one groups were introduced by Jacques Tits in the early 1990s under the name Moufang sets, although they have been studied long before that, and the idea already plays a crucial role in Tits' Bourbaki notes on the Suzuki and Ree groups from 1960/1961. Since 2000, they also appear in the literature under the name abstract rank one groups, which is due to Franz Timmesfeld.

There is a relatively concise list of known examples of rank one groups which are all of algebraic origin (i.e., related to an algebraic group over a field, a classical group, or some variations of these in small characteristics) and it is a prominent open question whether this list is complete. It has been known since a long time that exceptional algebraic groups and non-associative algebras play an important role in the structure theory of rank one groups, but it was only over the last few years that these interactions have been studied thoroughly from an algebraic and geometric point of view. Due to these recent developments, we decided to organize this workshop.

As we had participants with quite different mathematical backgrounds, we were grateful that some of them agreed to give introductory talks on topics which were central to the theme of the conference. Vladimir Chernousov talked on cohomological invariants, Victor Petrov on the classification of simple algebraic groups, Oleg Smirnov on structurable algebras, Yoav Segev on Moufang sets and Richard Weiss on Moufang buildings and polygons. These lectures were given during the first two days of the conference.

In addition to these introductory talks we had 11 research talks, 4 of which were delivered by the younger participants. Most of them concerned geometric and algebraic structures that are directly linked to the structure theory of rank one groups.

Oleg Smirnov talked about Kantor pairs which are used in constructions of rank one groups of nilpotence class $\leq 2$. Victor Petrov talked about a specific construction of Lie algebras that is closely related to one family of exceptional rank one groups of type $E_7$.

The contributions of Jeroen Meulewaeter and Simon Rigby concerned a class of non-associative algebras called structurable algebras. These algebras are closely related to Kantor pairs and produce exceptional rank one groups of nilpotence class $\leq 2$ in a systematic way, as explained in the recent work of Lien Boelaert, Tom De Medts and Anastasia Stavrova.

Richard Weiss presented a characteristic free approach to exceptional rank one groups that is based on descent in Moufang buildings and Tits polygons. Paulien Jansen's talk dealt with a geometric reinterpretation of those Tits polygons.

Matthias Grüninger presented a classification result for boundary Moufang sets, and Hendrik Van Maldeghem provided some new results about Tits webs which are geometries introduced by Tits for rank one groups with non abelian root groups.

Vladimir Chernousov presented a solution of the Kneser-Tits problem for algebraic rank one groups of type $E_{7,1}^{78}$. Philippe Gille discussed several versions of the notion of compactness for smooth affine algebraic groups, which is important for understanding anisotropic kernels of algebraic rank one groups. Torben Wiedemann talked about $C_3$-graded groups.

Apart from the talks we had two further sessions. The first was a problem session where each participant could present an open research problem. The second session was devoted to discuss specifically the classification problem for rank one groups with abelian root groups. There were lively discussions during both sessions; especially during the second session, novel approaches towards the classification problem were revealed.

Our main motivation for organizing this workshop was to stimulate the scientific exchange between researchers from different mathematical areas in order to make progress on various questions on rank one groups. The introductory talks given during the first days of the conference gave a fairly complete up to date account of our current knowledge on rank one groups and laid the ground for further scientific interaction. The intenseness of the discussions during the talks and problem sessions was beyond our expectations and therefore we consider the meeting as most successful. We are confident that it provided a base for further scientific interactions between the participants and it is our hope that it will lead to fruitful new collaborations.

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## Mini-Workshop: Rank One Groups and Exceptional Algebraic Groups

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# Abstracts

## Introduction to Moufang sets

Yoav Segev

I gave two 50 minutes talks in which I introduced some basic notions and results concerning Moufang sets. I also tried to show an application (see [7]), namely, a sketch of the proof that in a special Moufang set $\mathbb{M} := (X, (U_x)_{x\in X})$, given two distinct points $0, \infty \in X$, the root group $U_\infty$ contains no proper, nontrivial subgroup which is invariant (via conjugation) under the two point stabilizer $G_{0,\infty}$ (the *Hua* subgroup), where $G$ is the rank one group associated with $\mathbb{M}$, namely $G := \langle U_x \mid x \in X\rangle$. This result could turn out useful in showing *two major open problems* concerning special Moufang sets: (1) *Show that all special Moufang sets have abelian root groups* (see [2], [1] and [6]) and (2) *Show that all Moufang sets with abelian root groups (hence special, see* [5]*) come from quadratic Jordan division algebras* (see [3], [2] and [4]).

In my talks I defined what a Moufang set is: It consists of a set $X$, with $|X| \geq 3$; A group $G \leq \mathrm{Sym}(X)$ (nowadays called a rank one group); A subgroup $U_x \leq G_x$ ($G_x$ is the stabilizer of $x$ in $G$), for each $x \in X$. *Such that*: ($\mathbb{M}$1) $U_x$ fixes $x$ it is normal in $G_x$ and it acts regularly on $X \setminus \{x\}$, for each $x \in X$. ($\mathbb{M}$2) $\{U_x \mid x \in X\}$ is a conjugacy class of subgroups of $G$. ($\mathbb{M}$3) $G = \langle U_x \mid x \in X\rangle$.

I discussed the notation $\mathbb{M}(U, \tau)$ due to De Medts and Weiss ([3]). In particular I discussed in detail the *$\mu$-maps*, the *hua maps*, I proved some of their properties, and a variety of small technical details that help one to get a grip on the above simple definition of a Moufang set and start classifying those from scratch.

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## On Simple Kantor Pairs

Oleg N. Smirnov
(joint work with Bruce N. Allison, John R. Faulkner)

The talk contains the current version of our classification of simple Kantor pairs. A Kantor pair is a polarized Kantor triple system. These triple systems were introduced by Isai Kantor in 1972 [1]. He classified finite dimensional simple nonpolarized triple systems over algebraically closed fields of characteristic 0 using a classification of 5-gradings on simple Lie algebras.

These triple systems constitute one of the largest classes of nonassociative objects for which such a classification result has been obtained. The class includes Jordan triple systems as well as triple systems constructed from associative algebras, alternative algebras, Jordan algebras, and many other interesting exceptional objects.

Currently we focus on simple Kantor pairs that correspond to nonexceptional simple Lie algebras. Every such pair is either Jordan, or a reflection of a Jordan pair (as in [2]), or constructed from a right-polarized associative triple system. Simplicity criteria for these constructions are also obtained.

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## On the group of $R$-equivalence classes of strongly inner forms of type $E_6$

Vladimir Chernousov
(joint work with Seidon Alsaody and Arturo Pianzola)

In the talk we discuss a proof of the long standing Tits–Weiss conjecture on $U$-operators in Albert algebras and the Kneser–Tits conjecture for algebraic groups of type $\mathrm{E}_{7,1}^{78}$ and $\mathrm{E}_{8,2}^{78}$.

The Tits–Weiss conjecture asserts that the structure group $\mathrm{Str}(A)$ of an arbitrary Albert algebra $A$ is generated by the inner structure group, formed by the so-called $U$-operators, and the central homotheties. This problem was raised by Tits and Weiss in their 2002 book [5], where they studied spherical buildings and the corresponding generalized polygons attached to isotropic groups of relative rank 2. Despite many efforts, this problem has remained out of reach.

If $G$ is an isotropic simple simply connected group over $K$ of relative rank $\geq 2$ then by [4] the group $G(K)$ is generated by $K$-points of isotropic subgroups of $G$ of relative rank 1. This result allows to reduce many problems for $G(K)$ to groups of relative rank 1. For instance, this is the case for the Kneser-Tits problem (see below). Note also that isotropic groups of relative rank 1 give rise to important examples of more general groups of rank one. The latter were introduced by Tits in the early 1990s, who called them Moufang sets. They have proved to be important in the classification of simple groups, incidence geometry, the theory of buildings, and other areas. Further still, rank one groups are useful in studying isotropic groups of exceptional types, where algebraic groups and their associated root subgroups are typically parametrized by a nonassociative structure, and, as emphasized in [2], a rich interplay emerges between rank one groups, nonassociative algebras, and linear algebraic groups.

The Kneser–Tits conjecture for a simple simply connected isotropic group $\mathbf{G}$ over a field $K$ asserts that the abstract group $\mathbf{G}(K)$ of $K$-points of $\mathbf{G}$ coincides with its normal subgroup $\mathbf{G}(K)^+$ generated by the unipotent radicals of the minimal parabolic $K$-subgroups of $\mathbf{G}$. We refer to [1] for a survey of the history and recent results on this conjecture. Its importance comes from the fact that the group $\mathbf{G}(K)^+$ has a natural $BN$-pair structure and hence is projectively simple (i.e. simple modulo its centre), by a celebrated theorem of Tits. So if $\mathbf{G}(K) = \mathbf{G}(K)^+$ we would have many more new examples of projectively simple abstract groups given by $K$-points of isotropic simple simply connected algebraic groups. In this way, we would obtain analogues of finite simple groups of Lie type in the case of infinite fields. It is also worth mentioning that the information about the normal subgroup structure of $\mathbf{G}(K)$ is crucial in the arithmetic of algebraic groups for studying, among other things, congruence subgroups, discrete subgroups, lattices, and locally symmetric spaces. In general, the Kneser–Tits conjecture does not hold, and the first counterexample was constructed by V. Platonov in 1975 [3]. However, it is believed by specialists that the conjecture holds for many isotropic groups of exceptional type, including those of type $\mathrm{E}_{7,1}^{78}$ and $\mathrm{E}_{8,2}^{78}$.

The bridge connecting the Tits–Weiss conjecture and the Kneser–Tits conjecture for the abovementioned forms of type $\mathrm{E}_7$ and $\mathrm{E}_8$ is provided by a theorem of Tits and Weiss, which states that the two conjectures are equivalent. Furthermore, using P. Gille's results in [1] on Whitehead groups, one can easily see that

(1) the Kneser–Tits conjecture for the abovementioned groups reduces to the $R$-triviality of structure groups of Albert algebras, and
(2) the conjecture holds in arbitrary characteristic once it is established in characteristic zero.

Our main result is the following.

**Theorem.** *Let $A$ be an Albert algebra over a field $K$. Then the structure group $\mathbf{Str}(A)$ of $A$ is $R$-trivial, i.e. for any field extension $F/K$ the group of $R$-equivalence classes $\mathbf{Str}(A)(F)/R$ is trivial.*

As explained above, this implies that the Tits–Weiss conjecture on $U$-operators holds for Albert algebras over any field, and that the same is true for the Kneser–Tits conjecture for groups of type $\mathrm{E}_{7,1}^{78}$ and $\mathrm{E}_{8,2}^{78}$. Our proof is of a geometric nature. We carefully analyze the properties of the natural action of the structure group $\mathbf{Str}(A)$ on the corresponding Albert algebra $A$. The information that we need is encoded in the Galois cohomology of the stabilizers of subalgebras of $A$. We compute the Galois cohomology of all these stabilizers and using this information, we explicitly construct a system of generators of $\mathbf{Str}(A)(K)$, which we prove is $R$-trivial.

**Applications.** To a reductive algebraic group $\mathbf{G}$ over a field $K$ one can attach the functor of $R$-equivalence classes

$$\mathcal{G}/R : Fields/K \longrightarrow Groups, \quad F/K \to \mathbf{G}(F)/R$$

where $Fields/K$ is the category of field extensions of $K$ and *Groups* is the category of abstract groups. The experts expect that this functor factors through the subcategory *Abelian* $\subset$ *Groups* of abelian groups and that the group $G(F)/R$ is finite if $F$ is finitely generated over its prime subfield. Furthermore, we expect that the functor $\mathcal{G}/R$ has transfers. The following consequences of our main result provides an evidence that the above mentioned properties might be true in general case.

**Corollary 1.** *Let* $\mathbf{G}$ *be a simple simply connected strongly inner form of type* $\mathrm{E}_6$ *over a field* $K$ *of arbitrary characteristic. Then* $\mathbf{G}(K)/R$ *is an abelian group.*

**Corollary 2.** *Let* $\mathbf{G}$ *be a simple simply connected strongly inner form of type* $\mathrm{E}_6$ *over a field* $K$ *of arbitrary characteristic. Then the functor* $\mathcal{G}/R$ *has transfers.*

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## Moufang polygons and buildings

Richard M. Weiss

A generalized polygon is the same thing as an irreducible spherical building of rank 2. Tits observed that the generalized polygons associated with absolutely simple groups of relative rank 2 as well as the generalized polygons that appear as residues in irreducible spherical buildings of higher rank all have a property he called the Moufang condition. In [1], the classification of Moufang polygons (i.e. generalized polygons satisying the Moufang condition) in terms of various algebraic structures was given. The algebraic structures that arise in this context include anisotropic cubic norm structures, anisotropic quadrangular algebras and composition division algebras. In this talk, we gave a brief overview of this classification and indicated out it can be used to classify irreducible spherical buildings of rank at least 3.

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## $D_6 + A_1$-construction of $E_7$

Victor Petrov

We describe a construction of a Lie algebra of type $E_7$ out of an algebraic structure called "gift" by Skip Garibaldi. "Gift" stands for "generalized Freudenthal triple system" and means that the underlying representation is allowed to be a vector space over quaternions and not over the base field. The automorphism group of the structure is $(HSpin_{12} \times SL_2)/\mu_2$, so the construction corresponds to the map of Galois cohomology

$$H^1(F, D_6 + A_1) \to H^1(F, E_7),$$

and by a simple argument of Steinberg produces all Lie algebras of type $E_7$ up to an odd degree extension. Using basic theory of symmetric space we give necessary and sufficient conditions for isotropy of the resulting Lie algebra.

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## Exceptional Moufang sets

Richard M. Weiss
(joint work with Bernhard Mühlherr)

Let $M$ be a spherical Coxeter diagram with vertex set $S$. Suppose that $|S| > 1$ and let $\Delta$ be a Moufang building of type $M$. Let $\varphi$ be the standard homomorphism from $G := \mathrm{Aut}(\Delta)$ to $\mathrm{Aut}(M)$ and let $\psi$ be the standard homomorphism from $G$ to $\mathrm{Aut}(k)$, where $k$ is the field of definition of $\Delta$. Let $\Gamma$ be a subgroup of $G$. The group $\Gamma$ is called *Galois* if the restriction of $\psi$ to $\Gamma$ is injective. A *polarity* of $\Delta$ is a non-type-preserving automorphism of order 2.

In light of [2, Thm. 12.2(ii)], the following is a special case of [1, Thm. 22.20].

**Theorem 1.** *Suppose that either* $\Gamma$ *is Galois and acts with finite orbits on* $\Delta$ *or that* $\Gamma$ *is generated by a polarity of* $\Delta$ *and* $M = B_2$, $F_4$ *or* $G_2$. *Let* $\Omega$ *be the set of proper residues of* $\Delta$ *fixed by* $\Gamma$. *Suppose that* $\Omega \neq \emptyset$ *and that the residues in* $\Omega$ *are pairwise disjoint. Then the subgroup of* $\mathrm{Sym}(\Omega)$ *induced by the centralizer* $C_G(\Gamma)$ *is a Moufang set.*

We say that a Moufang set arises by descent, Galois or polar, if it arises from a pair $(\Delta, \Gamma)$ as in Theorem 1; we say that it is exceptional if, in addition, the building $\Delta$ is exceptional. Aside from $\mathrm{PSL}_2(F)$ for some field $F$ (or a minor variation of this Moufang set in characteristic 2), every known Moufang set arises by descent.

A Tits $n$-gon is a pair $(\Gamma, \{\equiv_v\}_{v\in V})$, where $\Gamma$ is a bipartite graph with vertex set $V$ and for each $v \in V$, $\equiv_v$ is a symmetric non-reflexive relation on $\Gamma_v$ satisfying axioms given in [3, 1.1.8]. A Moufang $n$-gon is the same thing as a Tits $n$-gon in which the relations $\equiv_v$ are all trivial.

Let $\Delta$ be a Moufang spherical building of type $M$ and let $T = (M, A, \Theta)$ be a Tits index of absolute type $M$ and of relative rank 2. Thus, in particular, $\Theta$ is a subgroup of $\mathrm{Aut}(M)$, $A$ is a $\Theta$-invariant subset of $S$ and there are exactly two $\Theta$-invariant subsets $J_1$ and $J_2$ of $S$ containing $A$. The $J_1$- and $J_2$-residues of $\Delta$ form a bipartite graph which has the natural structure of a Tits $n$-gon, where $2n$ is the order of the relative Coxeter group of $T$. We denote this Tits polygon by $X_{\Delta,T}$. There is a canonical isomorphism from the group of automorphisms of $\Delta$ stabilizing $T$ to $X_{\Delta,T}$, so it makes sense to talk of Galois groups of $X_{\Delta,T}$. An exceptional Tits polygon is a Tits polygon of the form $X_{\Delta,T}$ for $\Delta$ exceptional. The following is proved in [3] and [4]:

**Theorem 2.** *Every exceptional Moufang set with non-abelian root groups arises by descent from a pair $(X_{\Delta,T}, \Gamma)$, where $X_{\Delta,T}$ is an exceptional Tits quadrangle or hexagon and $\Gamma$ is a Galois group of $X_{\Delta,T}$ of order 2.*

The exceptional Tits hexagons are classified by isotopy classes of reduced cubic norm structures and the exceptional Tits quadrangles are classified by isotopy classes of reduced quadrangular algebras. In [3] and [4], we applied Theorem 2 to obtain explicit formulas in terms of these reduced algebras for the structure equation of an arbitrary exceptional Moufang set with non-abelian root groups in arbitrary characteristic.

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## The tensor product of two octonion algebras and its structure group

Simon W. Rigby

Structurable algebras are a class of nonassociative algebras with involution that Allison defined in [2] with the express purpose of rationally constructing all isotropic simple Lie algebras over a field of characteristic zero. Given a simple structurable algebra $(A, ^-)$ over a field $k$ of characteristic neither two nor three, the output of the *Tits-Kantor-Koecher (TKK) construction* is a 5-graded simple Lie algebra denoted by $K(A, ^-)$. Allison [3] together with Hein [5] wrote germane definitions of inverses, structurable division algebras, and isotopes, and proved the following statements:

(1) $K(A, ^-) \cong K(A', ^-)$ as $\mathbb{Z}$-graded algebras if and only if $(A, ^-)$ and $(A', ^-)$ are isotopic.

(2) If $\mathrm{char}(k) = 0$, then $K(A, ^-)$ has relative rank one if and only if $(A, ^-)$ is a structurable division algebra.

The *structure group* of $(A, ^-)$ is an algebraic subgroup $\mathbf{Str}(A, ^-)$ of $\mathbf{GL}(A)$ defined as the set of all isotopies from $(A, ^-)$ to itself. This group is isomorphic to the group of all grade-preserving automorphisms of $K(A, ^-)$, and its connected component $\mathbf{Str}(A, ^-)^\circ$ is isomorphic to a Levi subgroup of $\mathbf{Aut}(K(A, ^-))^\circ$.

There are many reasons to be interested in the structure groups of structurable algebras. For instance, the Kneser-Tits problem for some groups of type $E_8$ (different from those that featured in this talk) has several equivalent formulations in terms of $\mathbf{Str}(J, \mathrm{id})$ for $J$ an exceptional simple Jordan algebra. In this talk, we presented some results on $\mathbf{Str}(A, ^-)$ and $\mathbf{Aut}(K(A, ^-))^\circ$ for the interesting case when $(A, ^-)$ or some scalar extension of it is a tensor product of two octonion algebras.

If $(C_1, \sigma_1)$ and $(C_2, \sigma_2)$ are composition algebras with their canonical involutions, then they are simple structurable algebras and so is their tensor product $(A, ^-) = (C_1 \otimes C_2, \sigma_1 \otimes \sigma_2)$. The *Albert form* of $(A, ^-)$ is a quadratic form on $S = \mathrm{Skew}(A, ^-) \cong \mathrm{Skew}(C_1, \sigma_1) \oplus \mathrm{Skew}(C_2, \sigma_2)$ defined as $Q = n_1' \perp \langle -1 \rangle n_2'$ where $n_i'$ is the pure norm of $C_i$. There is a particularly important isometry $\natural \in O(Q)$, namely the map $(s_1, s_2)^\natural = (s_1, -s_2)$ for all $s_i \in \mathrm{Skew}(C_i, \sigma_i)$. If $\dim C_1 = \dim C_2 = 4$ or $8$, then we call $A$ a *biquaternion* or a *bioctonion* algebra and we have $Q \in \mathbf{I}_6^2(k)$ or $\mathbf{I}_{14}^3(k)$, respectively, where $\mathbf{I}_d^n(k)$ is the set of $d$-dimensional forms represented in the $n$-th power of the fundamental ideal in the Witt ring.

Allison [4] showed that any algebra with involution $(A, ^-)$ that becomes isomorphic to a bioctonion algebra over some field extension $K/k$, does so over a quadratic extension $K = k(\sqrt{d})$. In that case, either $(A, ^-)$ was already a bioctonion algebra over $k$ or there exists an octonion algebra $(C, \sigma)$ over $K$ such that $A$ is the corestriction $\mathrm{cor}_{K/k}(C, \sigma)$. The natural way to define the Albert form of this twisted bioctonion algebra is the trace transfer $Q = \mathrm{tr}_*(\langle \sqrt{d} \rangle n') = \mathrm{cor}_{K/k}(n')$ where $n'$ is the pure norm of $C$. More details on these twisted forms are exposed in [4].

The main results that featured in the talk are summarised below, and they apply to any form of a bioctonion algebra $(A, ^-)$ over a field $k$ of characteristic neither two nor three.

**Theorem 1.** *There is a short exact sequence of algebraic groups*

$$1 \longrightarrow \mathbf{G}_m \longrightarrow \mathbf{\Omega}(Q) \xrightarrow{\theta} \mathbf{Str}(A, ^-)^\circ \longrightarrow 1,$$

*where $\mathbf{\Omega}(Q)$ is the extended Clifford group of the Albert form and $\theta$ is the restriction of the unique homomorphism $\theta : C^+(Q) \to \mathrm{End}(A)$ such that $\theta(st) = -L_s L_{t^\natural}$ for all $s, t \in S$. Further restricting $\theta$ yields an isomorphism of the derived subgroups:*

$$\mathcal{D}(\mathbf{\Omega}(Q)) = \mathbf{Spin}(Q) \xrightarrow{\cong} \mathcal{D}(\mathbf{Str}(A, ^-)^\circ).$$

Some of the ideas behind Theorem 1 are implicit in the 1988 paper of Allison [4]. Using classification results of Tits and Selbach [9, 8] and Rost [7, Theorem 21.3], we obtain:

**Theorem 2.** *Every simple algebraic group of type $E_8$ in the image of the Galois cohomology map $H^1(k, \mathbf{Spin}_{14}) \to H^1(k, E_8)$ is isomorphic to $G = \mathbf{Aut}(K(A, ^-))^\circ$ for some (possibly twisted) bioctonion algebra $(A, ^-)$. The group $G$ has Tits index $E^{91}_{8,1}$, $E^{66}_{8,2}$, $E^{28}_{8,4}$, or $E^{0}_{8,8}$ according as $Q$ has Witt index 0, 1, 3, or 7. The semisimple anisotropic kernel of $G$ is isomorphic to $\mathbf{Spin}(Q_{\mathrm{an}})$.*

According to [6, Theorem 4.3.1], $(A, ^-)$ is a structurable division algebra if and only if $\mathbf{Aut}(K(A, ^-))^\circ$ has $k$-rank one. Combining this statement with Theorem 2, we obtain an analogue for bioctonions of Albert's Theorem on biquaternions [1]:

**Theorem 3.** *The following are equivalent:*

(1) *$Q$ is anisotropic.*
(2) *$(A, ^-)$ is a structurable division algebra.*

Theorem 3 was proved in a different way by Allison [4, Theorem 3.14] with the assumption that $k$ has characteristic zero.

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## Boundary Moufang sets

Matthias Grüninger
(joint work with Pierre-Emmanuel Caprace)

Let $T$ be a thick tree, $X$ a subset of the set of ends of $T$ satisfying certain properties, $G$ a subgroup of $\mathrm{Aut}(T,X)$ and $(U_x)_{x\in X}$ a family of subgroups of $G$ such that $U_x \leq G_x$ for all $x \in X$. Then $\big(T, X, G, (U_x)_{x\in X}\big)$ is called a *boundary Moufang tree* if $\big(X, (U_x)_{x\in X}\big)$ is a Moufang set and $G = \langle U_x \rangle$. The Bruhat-Tits tree of a simple algebraic group $G$ over a field $k$ with a discrete valuation provides an example for a boundary Moufang tree.

The case that is best understood is the following: $T$ is locally finite, $X = T^\infty$ and $G$ and the groups $U_x$ are closed in $\mathrm{Aut}T$. In this case the group $G$ and the root groups $U_x$ are locally compact groups and each group $U_x$ has a contracting automorphism induced by a translation of an apartment containing $x$. Using the theory of locally compact contraction groups, P.-E. Caprace and T. De Medts proved in [1] that if $U_x$ is torsion-free, then $G$ is essentially an algebraic group over a $p$-adic field for some prime $p$ and $T$ is essentially the Bruhat-Tits tree for $G$.

There is analogue of this result in case that the root groups $U_x$ are abelian torsion groups. In this case P.-E. Caprace and I proved in [2] that $T$ is one half of a twin tree satisfying the boundary Moufang condition for twin trees. Using the classification result for boundary Moufang twin trees in [3] one can now prove that $G$ is a simple algebraic group over a field of Laurent series with finite constant field and $T$ is isomorphic to the Bruhat-Tits of $G$.

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## Geometries from structurable algebras and inner ideals

Jeroen Meulewaeter
(joint work with Hans Cuypers, Tom De Medts)

Structurable algebras are a class of non-associative algebras introduced by Allison in 1978, see [1], which includes the class of Jordan algebras. Any structurable algebra $\mathcal{A}$ has an involution and hence a subspace $\mathcal{S}$ of skew elements. Using a generalized Tits-Kantor-Koecher-construction, one can associate a 5-graded Lie algebra $K(\mathcal{A})$ to any structurable algebra $\mathcal{A}$. As a vector space, we have

$$K(\mathcal{A}) = \mathcal{S}_- \oplus \mathcal{A}_- \oplus \ \mathrm{Instrl}(\mathcal{A}) \oplus \mathcal{A}_+ \oplus \mathcal{S}_+,$$

with $\mathcal{S}_-$ and $\mathcal{S}_+$ two copies of $\mathcal{S}$, $\mathcal{A}_-$ and $\mathcal{A}_+$ two copies of $\mathcal{A}$ and $\mathrm{Instrl}(\mathcal{A})$ a certain subspace of $\mathrm{End}(\mathcal{A})$. The $(-2)$-component of the 5-grading is $\mathcal{S}_-$, the $(-1)$-component $\mathcal{A}_-$ and so forth.

When we consider structurable algebras the characteristic is always assumed to be different from 2 and 3 and if we consider Lie algebras the characteristic is always assumed to be different from 2. All algebras are assumed to be finite-dimensional $k$-algebras, with $k$ a field.

There exist well known close connections between simple linear algebraic groups, (classical) Lie algebras, and (algebraic) spherical buildings. As noted before, structurable algebras and Lie algebras are also closely related. In earlier study of low rank geometries related to exceptional groups [2, 3], it became clear that structurable algebras play an important role. The natural question arose to what extent it would be possible to recover those geometries directly from the structurable algebras and their associated Tits-Kantor-Koecher Lie algebra. It turns out that the notion of an inner ideal is essential. We have been able to recover many geometries of rank one and two directly from the algebras in a surprisingly direct fashion. This is related to the extremal geometries studied intensively by Arjeh Cohen and his collaborators, but our approach allows for more geometries. We also discuss a generalization of extremal geometries, which allows one to obtain polar spaces as well.

Inner ideals are defined in a few (related) algebras:

**Definition 1.** An inner ideal in a Jordan algebra $J$ is a subspace $I$ satisfying $U_i(J) \leq I$ for all $i \in I$.

**Example 2.** If $J$ is the Jordan algebra associated with a non-degenerate quadratic form $Q$ with basepoint, then $I \leq J$ is an inner ideal if and only if $Q(I) = 0$ or $I = J$, by [8]. Geometrically, these inner ideals form a polar space.

**Definition 3** ([7])**.** An inner ideal of a skew-dimension one structurable algebra $\mathcal{A}$ is a subspace $I$ satisfying $U_i(\mathcal{A}) \leq I$, for all $i \in I$.

**Example 4** ([7])**.** If $\mathcal{A}$ is a Brown algebra, a 56-dimensional skew-dimension one structurable algebra, then Garibaldi constructed a building of type $E_7$ using the inner ideals of $\mathcal{A}$.

**Definition 5.** An inner ideal of a Lie algebra $L$ is a subspace $I$ which satisfies $[I, [I, L]] \leq I$.

**Example 6.** Using the 5-grading of the Lie algebra $K(\mathcal{A})$, one sees that $\mathcal{S}_-$ is always an inner ideal.

Let $L$ be a Lie algebra. We call $0 \neq x \in L$ extremal if $[x, [x, L]] \leq \langle x \rangle$. In particular $\langle x \rangle$ is a 1-dimensional inner ideal. If moreover $[x, [x, L]] = 0$, then $x$ is called an absolute zero divisor. Any extremal element which is not an absolute zero divisor is called pure and a Lie algebra is called non-degenerate if it does not contain absolute zero divisors.

We call the 1-dimensional inner ideals the extremal points of the Lie algebra, and denote the set containing these by $\mathcal{E}(L)$. We denote the set of extremal elements by $E(L)$. In 2006, Cohen and Ivanyos [4] introduced a point-line geometry in $L$, called the extremal geometry. Its points set is $\mathcal{E}(L)$, its line set is

$$\mathcal{F}(L) = \{\langle x, y\rangle \mid \lambda x + \mu y \in E(L), \text{ for all } (\lambda, \mu) \in k^2 \backslash \{(0,0)\}\},$$

and incidence is just containment. They showed:

**Theorem 7** ([4]). *Let $L$ be a non-degenerate simple Lie algebra generated by its set of extremal elements. If $\mathcal{F}(L) \neq \emptyset$, the extremal geometry $(\mathcal{E}(L), \mathcal{F}(L))$ is isomorphic to a root shadow space of type* $\mathrm{A}_{n,\{1,n\}}$ $(n \geq 2)$, $\mathrm{BC}_{n,2}$ $(n \geq 3)$, $\mathrm{D}_{n,2}$ $(n \geq 4)$, $\mathrm{E}_{6,2}$, $\mathrm{E}_{7,1}$, $\mathrm{E}_{8,8}$, $\mathrm{F}_{4,1}$ *or* $\mathrm{G}_{2,2}$.

However, one class of root shadow spaces is missing in the above list, namely root shadow spaces of type $\mathrm{BC}_{n,1}$. These are the polar spaces. In order to recover the polar spaces one needs a new definition of lines.

We define an *inner line ideal* to be a proper inner ideal containing two distinct extremal points which is minimal with these properties. Then denote by $\mathcal{F}'(L)$ the set of all inner line ideals and call the point-line geometry $(\mathcal{E}(L), \mathcal{F}'(L))$, with containment as incidence, the inner line ideal geometry of $L$.

**Theorem 8** ([5]). *Suppose $L$ is a simple Lie algebra generated by pure extremal elements over a field of characteristic not 2. Then we have one of the following:*

- *$\mathcal{F}(L) \neq \emptyset$ and in this case $\mathcal{F}(L) = \mathcal{F}'(L)$. Hence the inner line ideal geometry coincides with the extremal geometry.*
- *$\mathcal{F}(L) = \emptyset$, but $L$ contains two commuting, linearly independent extremal elements; the inner line ideal geometry is a non-degenerate polar space of rank at least 2.*
- *$L$ does not contain a pair of commuting, linearly independent extremal elements, and the inner line ideal geometry has no lines.*

So by using this more general definition of lines we recover both the extremal geometries and polar spaces.

Using geometric arguments we made a connection with structurable algebras:

**Theorem 9** ([5]). *Let $L$ be a simple non-symplectic Lie algebra generated by pure extremal elements over a field of characteristic different from 2 and 3. Then there exists a skew-dimension one structurable algebra $\mathcal{A}$ such that $L \cong K(\mathcal{A})$.*

Then we discussed the case when the extremal geometry is a generalized hexagon, i.e. is of type $\mathrm{G}_{2,2}$. For any cubic Jordan algebra $J$ there exists a skew-dimension one structurable algebra associated to $J$, denote it by $M(J)$. Now assume $J$ to be anisotropic. It turns out that the only proper non-trivial inner ideals of $K(M(J))$ are 1- and 2-dimensional and form a Moufang hexagon, which coincides with the extremal geometry. One can show that we get an embedding of the proper non-trivial inner ideals of $M(J)$, which form a Moufang set, into the geometry of the proper non-trivial inner ideals of $K(M(J))$, which form a Moufang hexagon.

If there are no inner line ideals in the Lie algebra $L$, then, using [3], the extremal points of $L$ actually form a Moufang set. More generally, we have the following:

**Theorem 10** ([6])**.** *If $\mathcal{A}$ is a central simple structurable division algebra, then the proper non-trivial inner ideals of $K(\mathcal{A})$ form a Moufang set. Moreover any proper non-trivial inner ideal can be mapped onto $\mathcal{S}_-$ if $\mathcal{S} \neq 0$ or onto $\mathcal{A}_-$ otherwise, using a Lie algebra automorphism.*

Note that if the dimension of the inner ideals in the previous theorem is not 1, the inner line ideal geometry has an empty point and line set. This is also the case in this last result relating structurable algebras with generalized polygons:

**Theorem 11** ([6])**.** *Let $F$ be an alternative division algebra with $Z(F) = k$, and set $n = \dim(F)$. Consider the structurable algebra $F \oplus F^{opp}$ with involution mapping $(x, y)$ onto $(y, x)$. Then the proper non-trivial inner ideals of $K(F \oplus F^{opp})$ have dimension $n$ or $2n$ and form a thin generalized hexagon, associated with a Moufang triangle.*

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## Root Graded Groups and the Blueprint Technique

Torben Wiedemann

Let $\phi$ be a finite irreducible root system. A *$\phi$-grading* of a group $G$ is a family of non-trivial subgroups $(U_\alpha)_{\alpha \in \phi}$ generating $G$, called the *root groups of $G$*, such that some commutator relations and a non-degeneracy condition are satisfied and such that so-called Weyl elements exist for each root. They appear in Shi's paper [14] as an analogue of the corresponding notion of root graded Lie algebras. For these Lie algebras, which were introduced by Berman and Moody in [5], there exists a complete classification which is due to Allison, Benkart, Berman, Gao, Moody, Neher, Smirnov and Zelmanov (see [1, 2, 4, 3, 5, 12]). Further, several concepts which are very similar to Shi's notion of a $\phi$-graded group had been studied before. Examples include Faulkner's *groups of Steinberg type* or *groups with Steinberg relations* (see [6, 7]), the notion of *Données radicielles* introduced by Bruhat and Tits and Tits' definition of *RGD-systems*. Our definition of a $\phi$-graded group is very general in the sense that it encompasses all previously mentioned concepts as special cases.

The following theorem was proven by Shi in [14] for the higher rank case. The study of the lower rank case (that is, $\phi = A_2$) has older roots: It dates back to works of Moufang on Moufang planes, see [9, 10]. In the framework of root graded groups, an explicit proof of Theorem 1 for the case $\phi = A_2$ can be obtained by a slight modification of Faulkner's arguments in [7]. (For a modern reference, see [8].)

**Theorem 1.** *Let $\phi$ be a simply laced irreducible root system of rank at least 2. Then every $\phi$-graded group is parametrized (in a suitable sense) by a nonassociative ring $\mathcal{R}$. If the rank of $\phi$ is at least 3, then $\mathcal{R}$ must be associative. If $\phi$ is of type $D$ or $E$, then $\mathcal{R}$ must be commutative.*

In [6], Faulkner constructs an $A_2$-graded group which is parametrized by $\mathcal{R}$ for any alternative ring $\mathcal{R}$. In fact, all known examples of such rings are alternative.

**Open Question.** *Is every ring which parametrizes an $A_2$-graded group alternative?*

For this open problem, there exist some partial results: In [6, (A.14)], Faulkner showed that the assertion is true for $A_2$-graded groups satisfying some additional conditions and in [11, 3.2, 3.9], Mühlherr and Weiss showed that it holds for any $A_2$-graded group which comes from a 5-plump Tits polygon.

If $\phi$ is not simply laced, the situation is more difficult. For $C_n$, serious difficulties arise in the case of characteristic 2. If we exclude this case (and make some additional assumptions), then we have the following result which was proven by Zhang in his PhD thesis [15].

**Theorem 2.** *Let $n \geq 3$ and let $G$ be a $C_n$-graded group which satisfies some additional suitable assumptions. (For example, "characteristic 2" is not allowed.) Then $G$ is parametrized by a nonassociative ring $\mathcal{R}$ with a nuclear involution $r \mapsto r^*$.*

For $C_3$, the situation is similar to the situation for $A_2$: Every known example of a ring which parametrizes a $C_3$-graded group is alternative and for every (finitely generated) alternative ring $\mathcal{R}$, there exists a $C_3$-graded group which is parametrized by $\mathcal{R}$. (For the latter assertion, see [15].) We were able to prove the following theorem.

**Theorem 3.** *Let $\mathcal{R}$ be ring which parametrizes a $C_3$-graded group and assume that $2_{\mathcal{R}}$ is not a zero divisor. Then $\mathcal{R}$ is alternative.*

Theorem 3 follows from certain computations involving a self-homotopy of the longest word in Weyl($C_3$) and some rewriting rules. (A self-homotopy is a sequence of elementary homotopies which transforms a word into itself.) This method, which we call the *blueprint technique*, is inspired by Ronan-Tits' construction of a building from a blueprint in [13]. The idea to apply this technique in the context of root graded groups is due to Mühlherr and Weiss.

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## Tits polygons: a geometric point of view

Paulien Jansen

Recently, Mühlherr and Weiss introduced the notion of a Tits polygon [2]. Essentially, a Tits polygon is a bipartite graph, equipped with an opposition relation in every vertex neighbourhood and with a group which acts very transitively on the graph while preserving all these opposition relations. Under some extra assumptions, the authors proved that these polygons are paremetrized by certain algebraic structures. Most prominent examples of Tits polygons were already studied before, from a geometric point of view, using shadow spaces of buildings. The aim of this talk was to present this geometric framework, while showing its connection with certain Tits polygons of index type. We also present a classification theorem of parapolar spaces inspired by this connection.

**Shadow spaces** [1]**.** Let $\Delta$ be a thick, spherical, irreducible building of type $X_n$, with associated Coxeter system $(W, S)$, $|S| \geq 3$. For any element $j$ of $S$, there is a canonical way to associate a point-line geometry $(\mathcal{P}, \mathcal{L})$ to the pair $(\Delta, j)$: the points $\mathcal{P}$ are defined to be the $j$-vertices of $\Delta$, the lines $\mathcal{L}$ are defined to be the $L$-simplices of $\Delta$, with $L \subseteq S$ consisting of those elements in $S$ that do not commute with $j$. A point is incident with a line if the corresponding simplices in $\Delta$ are incident. The point-line geometry we obtain in this way is called the shadow space of $\Delta$ of type $X_{n,j}$.

A shadow space of type $A_{n,1}$ is a projective space, one of type $C_{n,1}$ or $D_{n,1}$ is a polar space. In all other cases, we obtain a space which has a lot of substructures that form projective and polar spaces.

**Theorem 1.** *Let $(\mathcal{P}, \mathcal{L})$ be a shadow space of type $X_{n,j}$ ($n \geq 3$). Then $X_{n,j}$ is a projective space (in case of $A_{n,1}$), a polar space (in case of $D_{n,1}$ or $C_{n,1}$) or a parapolar space (in all other cases); in particular, it satisfies the following axioms:*

(1) *For every point $p$ and every line $L$, $p$ is collinear with $0, 1$ or all points of $L$.*
(2) *Let $x$ and $y$ be non-collinear points. If there exist at least two points collinear to both, then the convex closure of $x$ and $y$ forms a polar space, substructures obtained in this way are called symps.*

**Tits polygons of index type** [2]**.** For a given spherical Tits index $\mathbf{T}$ (of relative rank 2 and absolute type $X_n$), and a thick building $\Delta$ (of type $X_n$), there exists a canonical construction that results in a Tits polygon. This construction is closely related to the construction of shadow spaces above. In fact, when we take the type $X_n$ to be exceptional, all Tits hexagons and quadrangles obtained this way have as underlying graph the point-line (in case of Tits hexagons) or point-symp (in case of Tits quadrangles) incidence graph of the corresponding root shadow spaces, these are shadow spaces of types:

$$F_{4,1}, E_{6,2}, E_{7,1} \text{ or } E_{8,8}.$$

Moreover, the opposition relation in each vertex neighbourhood can be deduced from the geometry. In the point-line incidence graphs for example:

- Two points $p$, $q$ on a line $L$ are opposite in $L$ if they are distinct.
- Two lines $L$, $M$ through a point $p$ are opposite in $p$ as soon as:
  - We can find a sequence $(L_0 = L, L_1, L_2, L_3 = M)$ of length 3 of lines $L_i$ through $p$ such that every two consecutive lines are contained in a common plane.
  - We cannot find such a sequence of length $< 3$.

**A classification theorem.** We used the observations above to find a new classification theorem for parapolar spaces. We call two lines through a point $p$ opposite in $p$ if the opposition criteria from above holds.

**Theorem 2.** *Let $\Omega = (\mathcal{P}, \mathcal{L})$ be a parapolar space, with $G = Aut(\Omega)$ such that:*

(1) *For every point $p$ and two lines $L_1, L_2$ through $p$, there exists a third line through $p$ opposite both $L_1$ and $L_2$ in $p$.*

(2) *Let $L$ be any line, containing points $p \neq q$. Then the group:*

$$G^{(1)}_{L,p} := \{\sigma \in G \mid \sigma \text{ fixes } L \text{ pointwise and fixes all lines through } p\}$$

*acts transitively on the set*

$$\{\text{lines } M \text{ through } q \mid L \text{ opposite } M \text{ in } q\}.$$

*Then $\Omega$ is the shadow space of a building. Moreover, it is of type*

$$B_{n,2}\,(n \geq 3), D_{n,2}\,(n \geq 4), F_{4,1}, E_{6,2}, E_{7,1} \text{ or } E_{8,8}.$$

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## Residues on Affine Grassmannians


Philippe Gille

(joint work with Mathieu Florence)


The compact Lie groups play an essential role in the theory of Lie groups and it makes sense to generalize the notion of compacity for a smooth affine group $G$ over a base field $k$, that is a closed $k$-subgroup of some $\mathrm{G}L_n$ (e.g. the orthogonal group of a quadratic form). We consider the fourth following candidates.

(I) (rank one subgroups) $G$ does not carry any $k$–subgroup isomorphic to the additive group $\mathbb{G}_a$ nor the multiplicative group $\mathbb{G}_m$;

(II) (Boundedness property) $G\big(k((t))\big)$ is bounded for the valuation topology.

(III) $G\big(k[[t]]\big) = G\big(k((t))\big)$;

(IV) (No point at infinity) There exists a (projective) compactification $X$ of $G$ such that $G(k) = X(k)$.

We have the easy implications (IV) $\Longrightarrow$ (III) $\Longrightarrow$ (II) $\Longrightarrow$ (I). If $k$ is a perfect field and $G$ is smooth, Borel and Tits have shown in 1965 the implication (I) $\Longrightarrow$ (IV) so that all conditions agree [1, th. 8.2]. Furthermore in the case of the real numbers (and for $p$-adic fields), this is equivalent to say that the group $G(k)$ of points is compact (*ibid*, 9.3). For unipotent subgroups over imperfect fields, the equivalence (I) $\Longleftrightarrow$ (III) is due to J. Tits, see [3, Appendice B.2].

For $k$ imperfect and $G$ reductive, we have that (I) $\Longrightarrow$ (II) according to a result of Bruhat-Tits-Rousseau, we refer to Prasad's elementary proof [7]; actually (IV) holds as well by using nice compactifications of $G$ starting with the wonderful compactification in the adjoint case.

The next step is Gabber's talk [6] in Oberwolfach in 2012. Using the theory of pseudo-reductive groups, Gabber proved (among other things) that the four conditions are equivalent in the general case. The main result of today generalizes (partly) Gabber's statement over rings in a quite elementary manner.

**Theorem 1** ([5]). *Let $A$ be a ring (commutative, unital) and let $G$ be a closed $A$–subgroup scheme of $\mathrm{SL}_{N,A}$ for some $N$. Then the following are equivalent:*

*(I)* $\mathrm{Hom}_{A-gp}(\mathbb{G}_a, G) = 1$ *and* $\mathrm{Hom}_{A-gp}(\mathbb{G}_m, G) = 1$*;*

*(III)* $G\big(A[[t]]\big) = G\big(A((t))\big)$ *where* $A((t)) = A[[t]][\frac{1}{t}]$.

We call that property *wound* (ployé in French). The proof goes by associating to an element $g \in G\big(A((t))\big) \setminus G\big(A[[t]]\big)$ its residue $\mathrm{res}(g) : \mathbb{G}_a \to G$ or $\mathbb{G}_m \to G$ which is a non-trivial group homomorphism. The techniques involved apply also to $G$-torsors. The second main result is the following.

**Theorem 2** ([5]). *Let $G$ be an affine algebraic $k$–group over a field $k$. Let $X$ be a $G$–torsor. If $X\big(k((t))\big) \neq \emptyset$, then $X(k) \neq \emptyset$.*

For reductive groups, this statement is due to Bruhat-Tits. The generalization of that statement over a ring is known for $\mathrm{GL}_n$ and for tori according to recent results by Bouthier-Česnavičius [2, 2.1.17, 3.1.7]; we generalize it as well for wound closed subgroup schemes of $\mathrm{SL}_N$ and for $G$ commutative under further assumptions [5, 4.2,4.3]. It is an open question beyond those cases.

Already over a field it is an open question whether the statement does generalize to homogeneous spaces; this is the case in characteristic 0 according to results by M. Florence [4].

Finally, if $G$ is split reductive, the coset $G(k((t)))/G(k[[t]])$ is described by the $k$–points of the affine grassmannian $\mathcal{Q}_G$ [8]. This permits to show that an element $g \in G(k((t))) \setminus G(k[[t]])$ is of rank zero iff $g$ is of the shape $g = g_1\mu(t)g_2$ for $g_1, g_2 \in G(k[[t]])$ and $\mu : \mathbb{G}_m \to G$ a homomorphism.

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## Tits webs

Hendrik Van Maldeghem

Let $\mathcal{M} = (X, (U_x)_{x' in X})$ be a Moufang set with nonabelian root groups. Suppose moreover that $\mathcal{M}$ is constructed from an algebraic group of relative rank 1, or from a Frobenius twist of a group of mixed type $\mathsf{B}_2$, $\mathsf{G}_2$, $\mathsf{F}^*_{4,2}$, where the latter corresponds to the class of exceptional Moufang quadrangles of type $\mathsf{F}_4$. Then, for each $x \in X$, $U_x$ has nilpotency class 2 or 3. Set $V_x \in \{U'_x, Z(U_x)\}$ (so $V_x$ is either the derived group or the center of $U_x$). We define a point-line geometry $\Gamma = (X, \mathcal{L})$, with point set $X$ and line set $\mathcal{L}$ as follows. For $x, y \in X$, let $y^{V_x}$ denote the orbit of $y$ under the acton of $V_x$. Then we define

$$\mathcal{L} = \left\{ y^{V_x} \cup \{x\} \mid x, y \in X, x \neq y \right\}.$$

The members of $\mathcal{L}$ are also called *threads* and the geometry $\Gamma$ a *Tits web*. This definition is due to Jacques Tits (unpublished).

A *linear space* is a point-line geometry with the properties that each line contains at least two points and every pair of distinct points is contained in exactly one line.

**Observation.** *The Tits webs associated to algebraic groups of relative rank* 1 *are linear spaces.*

**Conjecture.** *The automorphism group of a Tits web is precisely the automorphism group of the corresponding Moufang set.*

A slightly weaker form of this conjecture is the following.

**Conjecture′.** *Two Tits webs are isomorphic if and only if the corresponding Moufang sets are isomorphic.*

For several classes of Moufang sets, the conjecture is already proved. These include algebraic groups of type ${}^2\mathsf{A}^{(1)}_{2,1}$ (by Jacques Tits [5]), $\mathsf{F}^{21}_{4,1}$ (by Tom De Medts and the author [1]) and one of the two relative rank 1 forms of type $\mathsf{E}_8$ (by Jacques Tits, unpublished). Furthermore, all cases of Frobenius twists are done by combined work of the author, Fabienne Haot and Koen Struyve [3, 4, 6].

**Example.** We now present an example of how the conjecture can be approached by providing a new proof for the case ${}^2\mathsf{A}^{(1)}_{2,1}$, i.e., the case of a unitary Moufang set in dimension 4 (a so-called *unital* in a pappian projective plane). It is based on ideas from [2].

Let $\Gamma = (X, \mathcal{L})$ be the Tits web. Fix a point $x \in X$ and set $\mathcal{L}_x = \{L \in \mathcal{L} \mid x \in L\}$. Let $L, M \in \mathcal{L}_x$, $L \neq M$, and define

$$\langle L, M \rangle = \left\{ K \in \mathcal{L}_x \;\middle|\; \begin{array}{c} K, L, M \text{ never meet the} \\ \text{same line not containing } x \end{array} \right\} \cup \{L, M\}.$$

Then one shows that $\langle L, M \rangle = \langle L', M' \rangle$, for all $L', M' \in \langle L, M \rangle$, $L' \neq M'$. Moreover, if we set

$$\mathfrak{T}_x = \{\langle L, M \rangle \mid L, M \in \mathcal{L}_x, L \neq M\},$$

then the point-line geometry $\Omega_x = (\mathcal{L}_x, \mathfrak{T}_x)$ is an affine plane, defined over some field $k$ (the same field over which the Moufang set is defined).

Now the lines through $x$ meeting a fixed line not through $x$ turns out to be a conic in $\Omega_x$. Also, all the conics thus obtained have the same two points at infinity over the algebraic closure of $k$. Therefore, the quadratic extension of $k$ related to the group of type ${}^2\mathsf{A}_{2,1}^{(1)}$ is determined and Conjecture$'$ is proved. But one also sees that the automorphism group of the affine plane stabilizing the said set of conics is precisely the point stabilizer in the Moufang set. Hence the full conjecture follows.

In this example, the threads can be identified with the traces of the projective lines when viewing the set $X$ as a Hermitian curve in a projective plane. This is a general phenomenon: the threads are always some geometric objects occurring in "nature", despite their group-theoretic definition.

**Remark for the Suzuki groups.** In the case of a Frobenius twist of type ${}^2\mathsf{B}_2$, the Tits webs occurring are so-called *inversive planes*, except when the ambient group is not defined over a field, but only over a vector space. In that case, the threads are still plane section of the set $X$ in some 3-dimensional projective space, just like what happens in the field case, but not all section occur. This is precisely the hard case in the proof of the conjecture for these Moufang sets. The conjecture can also be proved if we enrich the family of threads with all possible nontrivial plane sections, see [6].

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