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Mini-Workshop: Seshadri Constants

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Abstract. Seshadri constants were defined by Demailly around 30 years ago using the ampleness criterion of Seshadri. Demailly was interested in studying problems related to separation of jets of line bundles on projective varieties, specifically in the context of the well-known Fujita Conjecture. However, Seshadri constants turned out to be objects of fundamental importance in the study of positivity of linear series and many other areas. Consequently, in the past three decades, they have become a central object of study in numerous directions in algebraic geometry and commutative algebra. In this mini-workshop, we studied some of the most interesting current research problems concerning Seshadri constants. We expect that this exploration will help focus research on some of the most important questions in this area in the years to come.

Mathematics Subject Classification (2010): primary: 14C20, secondary: 14G17, 13A35.

Introduction by the Organizers

The mini-workshop *Seshadri constants*, organized by Thomas Bauer (Marburg), Łucja Farnik (Kraków), Krishna Hanumanthu (Kelambakkam) and Jack Huizenga, (University Park) was attended by 17 participants from Europe, North America, and Asia. The participants came from a wide variety of levels, ranging from early postdocs to fully established professors with international stature. Additionally, four of the researchers were women. Through this diversity we made substantial progress on questions in this area, forged new collaborations which will continue for years to come, and provided excellent training opportunities for early career researchers.

Workshop activities consisted of fifteen half hour talks in the mornings, with every participant being given the opportunity to speak if they desired. On the first day there were additionally two problem sessions where suggestions of problems for groups to work on were made. We divided ourselves into three groups which worked together on research problems in the afternoon sessions. We now discuss the problems studied and progress made by each of these working groups.

Line arrangements and Seshadri constants. Let (X, L) be a smooth polarized surface. The *multi-point Seshadri constant* of L at $x_1, \ldots, x_r$ is defined to be

$$\varepsilon := \varepsilon(L; x_1, \ldots, x_r) = \inf_C \frac{L \cdot C}{\sum_i \mathrm{mult}_{x_i}(C)},$$

where the infimum is taken over reduced and irreducible curves C which pass through at least one of the points x_i.

The study of multi-point Seshadri constants is already interesting for points in the projective plane $\mathbb{P}^2$. A famous conjecture of Nagata [10] asserts that if $r \geq 10$ and $x_1, \ldots, x_r \in \mathbb{P}^2$ are sufficiently general, then $\varepsilon = 1/\sqrt{r}$. On the other hand, when the points $x_1, \ldots, x_r$ lie in special position, the constant ε is frequently rational, and it is a very interesting problem to compute it. This problem often requires a great deal of understanding of the position of the points.

In this group we focused on the case where the collection of points is the set of singular points in a line configuration in the plane $\mathbb{P}^2$. In this case there is a natural guess as to what the Seshadri constant ε should be: one of the lines in the configuration should compute the constant. This was recently conjectured by Pokora.

Conjecture ([11]). *Let $x_1, \ldots, x_r \subset \mathbb{P}^2$ be the set of singular points in a configuration of lines. If k is the maximum number of points which lie on a line, then*

$$\varepsilon(L; x_1, \ldots, x_r) = \frac{1}{k}.$$

We discovered a method to study this conjecture via a linear programming problem; this allowed us to solve the conjecture so long as the number of lines in the configuration is at most 12. However, in the case of 12 lines, the very special line configuration known as the dual Hesse configuration requires an essentially different method to solve. Work on this problem continues.

Global generation and nef divisors on self-products of curves. This group started by thinking about the following problem:

Problem. *Does there exist a smooth complex projective surface S and a sequence of ample line bundles (L_n) on S such that setting*

$$\nu(L_n) = \min\{m \mid L_n^{\otimes m} \textit{ is globally generated}\},$$

we have $\nu(L_n) \to \infty$ as $n \to \infty$?

The corresponding problem for "globally generated" replaced by "very ample" was answered in the affirmative by Kollár [5, Ex. 3.7]. After Bauer's talk, we also realized that the problem has *conditionally* been answered affirmatively under the additional assumption that the SHGH conjecture, a generalization of Nagata's conjecture, holds [1, Prop. 4].

During the workshop, most of our time was spent studying the self-product $C \times C$ of a very general smooth complex projective curve, towards answering the problem unconditionally. The reasoning was that a sequence (L_n) as in the statement of Problem is more likely to exist on a surface with a non-polyhedral nef cone. Recall that if X is a projective variety, then the *nef cone* is the convex cone

$$\mathrm{Nef}(X) = \left\{ \xi \in N^1(X)_{\mathbf{R}} \;\middle|\; (\xi \cdot C) \geq 0 \text{ for all curves } C \subseteq X \right\}$$

in the Néron–Severi space $N^1(X)_{\mathbf{R}}$ of X. The non-polyhedrality of the nef cone of $C \times C$ was proved by Rabindranath [12, Thm. 1].

In order to start thinking about the problem, we decided we should first understand the nef cone of $C \times C$. Kouvidakis [9, Thm. 2] showed a certain class symmetric in permuting the factors of $C \times C$ is nef. Following ideas of Vojta [13, Prop. 1.5], Rabindranath showed there is an infinite family of nef classes that force the non-polyhedrality of the nef cone [12, Prop. 3.2]. Using different techniques, Fulger and Murayama [6, Thm. 4.7(ii)] showed the existence of more nef classes.

While checking if some of these known classes could be used to give an answer to problem, we realized that there is room for improvement in the approach of Vojta and Rabindranath. We spent the rest of the time attempting to work out the largest subset of the conjectured nef cone which can be proved to be nef by their approach.

Symbolic defect. This project focuses on the explicit determination of the *symbolic defect* of ideals. The particular case of interest is the ideal of a number of points in the projective space. Let $X = \{P_1, \ldots, P_s\}$ be a finite set of points in the projective space $\mathbb{P}^d$ with defining ideal $I = \bigcap_{i=1}^{s} I_{P_i} \subseteq R = k[x_0, \ldots, x_d]$.

Aside from the usual powers I^n, there is also a more geometric set of powers of I to consider: the span of the forms that vanish with order at least n at every point of X. At least in characteristic zero, this is the n-th symbolic power $I^{(n)}$ of I. It is also given by $I^{(n)} = \bigcap_{i=1}^{s} I_{P_i}^n$, and it is the saturation of I^n with respect to the irrelevant ideal.

In the following we denote by $\mu(-)$ the minimal number of generators of a graded R-module and by m the irrelevant maximal ideal of R. The *symbolic defect* of I, introduced by Galetto, Geramita, Shin and Van Tuyl in [7], is the numerical function

$$\mathrm{sdef}_I : \mathbb{N} \to \mathbb{N}, \quad \mathrm{sdef}_I(n) = \mu\left(\frac{I^{(n)}}{I^n}\right) = \dim_k \left(\frac{I^{(n)}}{I^n + mI^{(n)}}\right).$$

It is a measure of the algebraic complexity of the difference between the usual powers and the symbolic powers of I. Our project focuses on determining the order of growth of the symbolic defect function, defined by

$$\begin{aligned}\operatorname{ord}(\operatorname{sdef}_I) &= \min\left\{t \in \mathbb{N} \mid \operatorname{sdef}_I(n) = O(n^t)\right\}\\ &= \min\left\{t \in \mathbb{N} \mid \limsup_{n\to\infty} \frac{\operatorname{sdef}_I(n)}{n^t} < \infty\right\}.\end{aligned}$$

The literature on the topic of the symbolic defect is quite limited so far, with the notable exceptions of [7] and [4], which presents the opportunity to tackle the following open questions:

(1) Which (homogeneous) ideals I of R have finite $\operatorname{ord}(\operatorname{sdef}_I)$?
(2) Compute $\operatorname{ord}(\operatorname{sdef}_I)$ or find upper bounds on it for ideals I satisfying the property in (1).
(3) For an ideal I with $\operatorname{ord}(\operatorname{sdef}_I) = t$ determine the value of $\limsup_{n\to\infty} \frac{\operatorname{sdef}_I(n)}{n^t}$ and decide whether $\lim_{n\to\infty} \frac{\operatorname{sdef}_I(n)}{n^t}$ exists.

In regards to question (1) we can provide an affirmative answer for ideals of points in $\mathbb{P}^d$. In this same case, question (2) can be answered with the inequality $\operatorname{ord}(\operatorname{sdef}_I) \leq \dim(R) = d + 1$. This is based on the existence of the asymptotic regularity $\lim_{n\to\infty} \frac{\operatorname{reg}(I^{(n)})}{n}$ according to [2, Theorem 3.2], [8, Corollary 2.5]. We note that the asymptotic regularity is the reciprocal of a *Seshadri constant* as explained in [2, Remark 1.3].

Turning to question (3), we were able to make progress in the case of defining ideals for sets of $s \in \{5, 6, 7, 8\}$ general points in $\mathbb{P}^2$, as well as for the defining ideals of start configurations of planar points, i.e. the pairwise intersection points of an arrangement of ℓ general lines in $\mathbb{P}^2$. These ideals enjoy remarkable properties regarding the structure of their blow up algebras. In particular, we note that the symbolic Rees algebra $\mathcal{R}_s(I) = \bigoplus_{n\in\mathbb{N}} I^{(n)}$ is Noetherian, which implies in view of [3, Proposition 2.3] that $\operatorname{ord}(\operatorname{sdef}_I) \leq 1$. In fact, in all the above mentioned cases we show that $\operatorname{ord}(\operatorname{sdef}_I) = 1$ and we can compute the exact value for the limit of a certain subsequence of the sequence $\left\{\frac{\operatorname{sdef}_I(n)}{n}\right\}_{n\in\mathbb{N}}$. Computational evidence suggests that in all these cases $\lim_{n\to\infty} \frac{\operatorname{sdef}_I(n)}{n}$ exists.

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The abstracts on the following pages are presented in the order in which the talks were given.

Mini-Workshop: Seshadri Constants

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Abstracts

Measures of local positivity

Alex Küronya
(joint work with Catriona Maclean and Joaquim Roé)

Our purpose is to study local positivity of line bundles on projective varieties. Let X be a projective variety of dimension n over the complex numbers, $x \in X$ a smooth point, L a big line bundle on X. We say that L is ample at x if there exists a Zariski open neighbourhood $x \in U \subseteq X$ such that the Kodaira map $\phi_{mL}|_U$ is an embedding for all $m \gg 0$. It was proven by Boucksom–Cacciola–Lopez [3] that L is ample at x if and only if $x \notin \mathbb{B}_+(L)$. In what follows we will primarily care about the above situation, and we will assume that we deal with L and x such that L is ample at x.

Once L is ample at x we can try to measure the extent of its positivity there, two well-studied examples of such measures are the Seshadri constant

$$\varepsilon(L;x) = \sup\{t > 0 \mid \pi^*L - tE \text{ is nef }\},$$

and the pseudo-effective threshold

$$\mu(L;x) = \sup\{t > 0 \mid \pi^*L - tE \text{ is pseudo-effective}\}.$$

In both cases $\pi\colon Y \to X$ denotes the blowing-up of X at x, and E is the exceptional divisor of π.

Both invariants have good formal properties: they are left unchanged under numerical equivalence of line bundles, they are super-additive, homogeneous, and can only increase upon restriction to a subvariety containing x.

Most of the above facts are explained by the nature of the respective definitions of $\varepsilon(L;x)$ and $\mu(L;x)$ in that they are defined in terms of the convex geometric structure of the Néron–Severi space.

For this talk the guiding question regarding these invariants is the question whether $\varepsilon(L;x)$ or $\mu(L;x)$ are rational numbers. This is wide open even in dimension two. Note that there is some evidence in both directions: for one, all Seshadri constants that have been computed so far are rational, at the same time by [6] the rationality of Seshadri constants on blow-ups of the projective plane would disprove the Segre–Harbourne–Gimigliano–Hirschowitz (SHGH) conjecture.

We proceed by presenting a dual convex geometric interpretation of $\varepsilon(L;x)$ and $\mu(L;x)$ via Newton–Okounkov theory. For definitions and the basic theory we refer the reader to [2, 5, 8, 13, 14, 15].

In addition to X,L, and x as defined above, we choose an admissible flag $Y_\bullet$ on or over X (the distinction is not too relevant since one ends up using the corresponding flag valuation of the function field of X). The choice of $Y_\bullet$ leads to a compact convex body $\Delta_{Y_\bullet}(L) \subseteq \mathbb{R}^n$.

Let again $\pi\colon Y \to X$ be the blow-up of X at x, and for simplicity let us assume that X is a smooth surface. Then the Newton–Okounkov body $\Delta_{(C,x)}(L)$ is in fact a polygon [12], and its combinatorics is determined by the Zariski chamber structure (cf. [1]) of the surface X.

In this language it was shown in [9, 11] (cf. [10] that the following three statements are equivalent:

(1) the line bundle L is ample at x;
(2) for every $y \in E$ there exists $\delta > 0$ such that $\Delta_\delta^{-1} \subseteq \Delta_{(E,y)}(\pi^* L)$;
(3) there exists $y \in E$ and $\delta > 0$ such that $\Delta_\delta^{-1} \subseteq \Delta_{(E,y)}(\pi^* L)$.

Here $\Delta_\delta^{-1} \subseteq \mathbb{R}^2$ is the simple spanned by the points $(0,0)$,$(\delta,0)$, and (δ,δ).

As local positivity of L at x can be read off from infinitesimal Newton–Okounkov bodies, we can try the same with $\varepsilon(L;x)$ and $\mu(L;x)$. We showed in [9, 11] that this can indeed be done:

(1) $\mu(L;x)$ is the width of an arbitrary Newton–Okounkov body $\Delta_{(E,y)}(L;x)$;
(2) $\varepsilon(L;x) = \sup\{t > 0 \mid \Delta_t^{-1} \subseteq \Delta_{(E,y)}(L)\}$.

This state of affairs is satisfactory, but it turns out that we can obtain new measures of local positivity by considering interesting functions on Newton–Okounkov bodies. Building on ideas from complex analysis it was observed by Boucksom and Chen that multiplicative filtrations on the section ring

$$R(X,L) \;=\; \bigoplus_{m=0}^{\infty} H^0(X, mL)$$

yield concave functions on Newton–Okounkov bodies. We point out that the shape of a Newton–Okounkov body $\Delta_{Y_\bullet}(L)$ changes by changing the flag, nevertheless, certain invariants remain constant. Not unexpectedly, the same holds for functions on Newton–Okounkov bodies.

Keeping our previous notation, we will focus on the decreasing multiplicative filtration on the section ring $R(X,L)$ induced by order of vanishing at a smooth point $p \in X$ (the same results hold for order of vanishing along a smooth subvariety $Z \subseteq X$). We write $\mathrm{ord}_p \colon \Delta_{Y_\bullet}(L) \to \mathbb{R}_{\geq 0}$ for the induced concave function.

It was shown in [7] that

$$\max_{\Delta_{Y_\bullet}(L)} \mathrm{ord}_p \;=\; \mu(L;p),$$

hence the left-hand side is independent of the choice of $Y_\bullet$. Perhaps more importantly, Boucksom–Chen [4] verified that

$$\beta(L;x) \stackrel{\text{def}}{=} \frac{1}{\mathrm{vol}_X(L)} \cdot \int_{\Delta_{Y_\bullet}} \mathrm{ord}_p$$

is also independent of the choice of the flag $Y_\bullet$. The invariant $\beta(L;x)$ plays a crucial role in the diophantine arguments of [16], and in recent results on K-stability in Kähler geometry.

It turns out the $\beta(L;x)$ contributes to our understanding of Seshadri constants as well, because it provides a bridge between $\varepsilon(L;x)$, and volumes of line bundles, whose arithmetic properties are somewhat better understood. Our results in this direction go as follows [14]. With notation as above, let us assume that X is a surface. then

(1) if $\beta(L;x)$ is rational then so is $\varepsilon(L;x)$;
(2) there exists a three-dimensional projective variety $\hat{X}$ and a big line bundle $\hat{L}$ on $\hat{X}$ such that

$$\beta(L;x) \ = \ \mathrm{vol}_{\hat{X}}(\hat{L}).$$

This relates the study of $\varepsilon(L;x)$ to the study of the birational geometry of the threefold $\hat{X}$.

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Newton–Okounkov polygons

Joaquim Roé
(joint work with Julio Moyano-Fernández, Matthias Nickel, and Tomasz Szemberg)

Newton polygons. In a famous letter[1] written in October 1676, Isaac Newton described his method to obtain fractional power series solutions $y = \sum c_i x^{i/n}$ to polynomial equations in two variables $0 = p(x, y) = \sum a_{ij} x^i y^j$. The first step in Newton's algorithm is to draw the convex hull $\Delta(p)$ in $\mathbb{R}^2$ of the set of pairs (j, i) such that $a_{ij} \neq 0$. Then for each lower side Γ of $\Delta(p)$, from the *slope* of Γ and the polynomial $\sum_{(j,i)\in\Gamma} a_{ij} x^i y^j$, the initial term of one of the series above can be determined. A suitable change of variables allows then to iterate the process and, asymptotically, parametrize the branches of the curve $p(x, y) = 0$ near the origin of coordinates.

To appropriately interpret the information contained in the upper sides of $\Delta(p)$, one applies coordinate changes $(x, y) \mapsto (1/x, y/x)$ or $(x, y) \mapsto (x/y, 1/y)$ to conclude that they correspond to branches of the curve at infinity.

The lower sides of the Newton polygon $\Delta(p)$ describe the curve $p(x, y) = 0$ locally near 0. The upper sides of the Newton polygon $\Delta(p)$ describe the curve $p(x, y) = 0$ at infinity. (1)

Polygons vs. valuations. As long as one is interested in the local behavior near p only, the same method can be used for polynomials $p(y)$ whose coefficients are *series* in x (in this case, only the lower part of the Newton polygon is meaningful, because the exponents of x can go to infinity), allowing to study curves on arbitrary smooth surfaces, locally near an arbitrary point. Ostrowski [8] was the first to use Newton polygons over valued fields $K^* \xrightarrow{v} \mathbb{R}$ (motivated mainly by the p-adic case) generalising the construction from the case of power series $K = \mathbb{C}((x))$ and assigning the point $(j, v(a_j))$ to the monomial $a_j y^j$. The connection of Newton's original construction with the theory of valuations becomes even nicer by observing that the map

$$\sum a_{ij} x^i y^j \longmapsto \min_{\text{lex}}\{(j, i) \mid a_{ij} \neq 0\}$$

is a rank 2 valuation $\mathbb{C}[[x, y]] \setminus \{0\} \to \mathbb{Z}^2_{\text{lex}}$, and for every smooth point 0 on a surface S, with local coordinates x, y, the isomorphism $\hat{\mathcal{O}}_{S,0} \cong \mathbb{C}[[x, y]]$ allows to carry this valuation to the field of rational functions on S.

[1]The so-called *epistola posterior* is part of an exchange of letters between Newton and Leibniz through Henry Oldenburg, secretary of the Royal Society, relevant in the controversy over the invention of calculus, and including the description of the 'Newton method' for finding zeros of differentiable functions. The letter is preserved in the Cambridge University Library, see [7].

Toric surfaces. Considering the linear family of all polynomials with the same Newton polygon Δ, one sees that general members are equisingular at the coordinate points of $\mathbb{P}^2$ and Enriques [1] described their embedded resolution in terms of the polygon. The corresponding blowups lead to a toric surface X_Δ where the pullback of the linear family is a *complete linear system* $|D_\Delta|$. The correspondence between polygons Δ and divisors D_Δ is nowadays a standard topic in toric geometry (see [2]). The shape of Δ nicely reflects properties of D_Δ:

The area of Δ equals half of the volume of D_Δ (which equals the selfintersection if and only if D_Δ is nef). (2)

Each side Γ_i of Δ corresponds to a torus-invariant curve C_i in X_Δ (the correspondence is bijective if and only if D_Δ is ample). (3)

The (lattice) length of each side Γ_i equals the intersection number $D_\Delta \cdot C_i$. (4)

The angles between sides (equivalently, the slope of each side) are determined by the selfintersections of the torus-invariant curves. (5)

Until the advent of Newton–Okounkov bodies, the local picture (1) for curve singularities provided by the lower part of Newton polygons could be carried over to curves on arbitrary surfaces via valuation of power series, but properties (2)—(5) only made sense over toric surfaces.

Newton–Okounkov bodies. If S is a surface, 0 is a smooth point on S and x, y are local coordinates determining a rank 2 valuation v as above, then the Newton–Okounkov body of any big divisor D with respect to v is defined (see Lazarsfeld–Mustaţă [6], Kaveh–Khovanskii [3]) as the convex body

$$\Delta_v(D) = \overline{\left\{ \frac{v(s)}{k} \,\middle|\, s \in \mathcal{L}(kD) \right\}}.$$

It is known by the work of Küronya–Lozovanu–Maclean [5, 4] (based on the description of [6]) that $\Delta_v(D)$ is a polygon. If S is a toric surface and the local coordinates are chosen so that both $y = 0$ and $x = 0$ are torus-invariant curves, then $\Delta_v(D)$ is (up to the action of $\mathrm{GL}_n(\mathbf{Z})$) just the Newton polygon associated to D in toric geometry, so it satisfies properties (1)–(5) above, and (2) is well-known to hold in all generality in the following form:

The area of $\Delta_v(D)$ equals half the volume of D for every v. (2')

However, the meaning of the *shape* of Newton–Okounkov polygons in general, and more precisely the existence of generalizations of (1), (3)–(5), and their dependence on v, is still an intriguing subject. In joint work with Tomasz Szemberg [9] we pursue an in-depth analysis of the construction in [6] and [5], to the conclusion that the shape of Newton–Okounkov polygons does reflect the geometry of the pair (D, v) much like Newton polygons do in the toric case.

Associated to each pair (D, v), there is a configuration N of irreducible curves on the surface S such that each side of $\Delta_v(D)$ corresponds to one or more of these irreducible curves. (3')

The lengths of the sides of Δ_v are determined by the intersection numbers of D with the curves in N. (4')

Their slopes are rational and determined by the intersection matrix of N. (5')

And there is even some analogy to property (1) of Newton's classical polygons:

The lower sides of $\Delta_v(D)$ correspond to connected components of N passing through 0 whereas the upper sides are related to connected components of N intersecting $y = 0$ at other points. (1')

The number of sides. It was observed in [4] that the number of sides of $\Delta_v(D)$ is bounded above by $2\rho(S) + 2$, where $\rho(S)$ denotes the Picard number. We show that the slightly stronger bound $2\rho(S) + 1$ holds and is sharp on some surfaces. We also determine, in terms of configurations of negative curves on the surface S, the numbers k for which there is a k-gon $\Delta_v(D)$; somewhat surprisingly, these do not depend on the ample divisor D.

In a further work in progress with Julio Moyano-Fernández and Matthias Nickel we generalize the study to rank 2 valuations coming from local coordinates in birational models of S, to conclude that a similar description holds, with the bound on the number of sides being in fact dependent only on the Picard number of the image surface of S in projective space under the map $|kD|$ for $k \gg 0$.

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Seshadri constants and Fujita's conjecture

Takumi Murayama

Throughout, all varieties will be over an algebraically closed field k.

If L is an ample line bundle on a projective variety X, then by definition, the ℓth tensor power $L^{\otimes \ell}$ of L is globally generated or even very ample for ℓ sufficiently large. This prompts the following:

Question 1. *Let L be an ample line bundle on a projective variety X. What tensor power $L^{\otimes \ell}$ of L is very ample or globally generated?*

The ideal situation is when this power ℓ depends solely on geometric invariants of X. For curves, $L^{\otimes \ell}$ is globally generated for all $\ell \geq 2g$, and is very ample for all $\ell \geq 2g+1$. An example due to Kollár [5, Ex. 3.7], however, shows that there is not such a simple answer for surfaces: different ample line bundles on the same surface may need to be raised to different powers to become very ample.

Instead, a suggestion of Mukai was to consider line bundles of the form $\omega_X \otimes L^{\otimes \ell}$, where $\omega_X = \det \Omega_X$ is the canonical bundle of a smooth variety X. In this direction, Fujita formulated the following:

Conjecture 2 (Fujita [7])**.** *Let L be an ample line bundle on a smooth projective variety X of dimension n. Then, the line bundle $\omega_X \otimes L^{\otimes \ell}$ is globally generated for all $\ell \geq n+1$, and is very ample for all $\ell \geq n+2$.*

In characteristic zero, Fujita's conjecture for global generation is known in dimensions at most five and Fujita's conjecture for very ampleness is known in dimensions at most two (see [12] and the references therein). In positive characteristic, Fujita's conjecture is known for curves, and for surfaces, Fujita's conjecture for global generation is known for surfaces not of general type (see [2] and the references therein), with some weaker results known in the general type case [3].

To study Fujita's conjecture, Demailly introduced the following notion.

Definition 3 (Demailly [4])**.** Let $x \in X$ be a closed point on a projective variety X, and let L be a nef line bundle on X. Denote by $\mu\colon \widetilde{X} \to X$ the blowup of X at x with exceptional divisor E. The *Seshadri constant* of L at x is

$$\varepsilon(L;x) := \sup\bigl\{t \in \mathbf{R}_{\geq 0} \bigm| \mu^*L(-tE) \text{ is nef}\bigr\}.$$

The name comes from Seshadri's ampleness criterion, which states that L is ample if and only if $\inf_{x\in X} \varepsilon(L;x) > 0$. The relationship between Fujita's conjecture and Seshadri constants is given by the following:

Theorem 4 (— [11, Thm. 7.3.1])**.** *Suppose L is a big and nef line bundle on a projective variety X of dimension n, and let s be a non-negative integer. Suppose $\varepsilon(L;x) > n+s$ for a closed point $x \in X$ with singularities of dense F-injective type in characteristic zero, or with F-injective singularities in positive characteristic. Then, the following restriction homomorphism is surjective:*

$$H^0(X, \omega_X \otimes L) \longrightarrow H^0(X, \omega_X \otimes L \otimes \mathcal{O}_X/\mathfrak{m}_x^{s+1}).$$

The $s = 0$ case gives a criterion for when $\omega_X \otimes L$ is globally generated at x, and the $s = 1$ case gives a criterion for when $\omega_X \otimes L$ separates tangent directions at x. When X is smooth of characteristic zero, Theorem 4 is due to Demailly [4]. In positive characteristic, the case $s = 0$ for smooth varieties is due to Mustaţă and Schwede [9]. The general case is due to the author [10, 11].

By Theorem 4, to prove Fujita's conjecture for global generation, it would suffice to show that for ample line bundles, the lower bound $\varepsilon(L; x) \geq 1$ always holds. There is no hope that this approach alone can prove Fujita's conjecture, since there are examples of surfaces with arbitrarily small Seshadri constants:

Example 5 (Miranda [5, Ex. 3.1])**.** Let $\Gamma \subseteq \mathbf{P}^2$ be a curve of degree d and multiplicity m at a point ξ. Let $\Gamma' \subseteq \mathbf{P}^2$ be another curve of degree d meeting Γ transversely. Taking $d \gg 0$ and Γ' sufficiently general, Γ and Γ' span a pencil whose members are all irreducible. Blowing up the base locus of this pencil, we obtain a surface X which maps to $\mathbf{P}^1$.

On this surface, the divisor $L = aC + S$ is ample for $a \geq 2$, where $C \simeq \Gamma$ is a fiber and S is a section, but denoting by x the preimage of ξ in X, we have

$$\varepsilon(L; x) \leq \frac{(L \cdot C)}{\operatorname{mult}_x C} = \frac{1}{m}.$$

Despite Miranda's example, our main results show that one can still use Seshadri constants to prove results toward Fujita's conjecture. The first result gives global generation of adjoint-type line bundles, assuming a weaker lower bound on the Seshadri constant as in Theorem 4.

Theorem 6 (— [11, Thms. C and 8.1.1])**.** *Suppose X is normal projective variety over k of characteristic zero. Let Δ be an effective $\mathbf{Q}$-Weil divisor such that $K_X+\Delta$ is $\mathbf{Q}$-Cartier. Consider a closed point $x \in X$ such that (X, Δ) is log canonical at x. Suppose that D is a Cartier divisor on X such that $H = D-(K_X+\Delta)$ satisfies*

$$\varepsilon\big(\|H\|; x\big) > \operatorname{lct}_x\big((X, \Delta); \mathfrak{m}_x\big).$$

Then, $\mathcal{O}_X(D)$ is globally generated at x.

Here, $\operatorname{lct}_x((X, \Delta); \mathfrak{m}_x)$ is the *log canonical threshold* of the pair (X, Δ) with respect to the ideal sheaf $\mathfrak{m}_x$, and $\varepsilon(\|H\|; x)$ is the *moving Seshadri constant* of H at x as in [6] or [11, Ch. 7]. Theorem 6 holds in positive characteristic as well, if one replaces "log canonical" with "F-pure" throughout.

Theorem 6 implies the following local version of the Angehrn–Siu theorem [1], and gives effective bounds for global generation towards Fujita's conjecture.

Theorem 7 (— [11, Thm. D])**.** *Let X be a normal projective variety of dimension n over k of characteristic zero. Let Δ be an effective $\mathbf{Q}$-Weil divisor such that $K_X + \Delta$ is $\mathbf{Q}$-Cartier. Consider a closed point $x \in X$ such that (X, Δ) is klt at x. Suppose that D is a Cartier divisor on X such that $H = D - (K_X + \Delta)$ satisfies*

$$\operatorname{vol}_{X|Z}(H) > \binom{n+1}{2}^{\dim Z}$$

for every positive-dimensional subvariety $Z \subseteq X$ containing x. Then, $\mathcal{O}_X(D)$ has a global section not vanishing at x.

In particular, if X as above has at worst klt singularities, and if D is a Cartier divisor such that $D \sim_{\mathbf{Q}} K_X + \ell L$ for an ample Cartier divisor L and some $\ell > \binom{n+1}{2}$, then $\mathcal{O}_X(D)$ is globally generated.

Here, $\mathrm{vol}_{X|Z}(H)$ is the *restricted volume* of H along Z [6]. A version of Theorem 7 for smooth varieties is due to Ein, Lazarsfeld, Mustaţă, Nakamaye, and Popa [6, Thm. 2.20]. The non-local statement is due to Angehrn and Siu for smooth varieties [1], and to Kollár for varieties with klt singularities [8].

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Seshadri constants for vector bundles

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(joint work with Takumi Murayama)

Let X be a projective variety over an algebraically closed field, and let $x \in X$. Given L a nef line bundle on X, or just a nef class in the real Néron–Severi space $N^1(X)$, the Seshadri constant $\varepsilon(L; x)$ is a classical measure of the positivity of L at x. Among its applications we mention the following:

(1) (Seshadri ampleness criterion) L is ample iff $\inf_{x\in X}\varepsilon(L;x)>0$.
(2) (Asymptotic jet separation) If L is an ample line bundle, then $\varepsilon(L;x) = \lim_{m\to\infty}\frac{s(L^{\otimes m};x)}{m}$, where $s(L;x)$ is the largest s such that the evaluation map $H^0(X,L)\to H^0\big(X, L\otimes\mathcal{O}_X/\mathfrak{m}_x^{s+1}\big)$ is surjective.
(3) (Characterization of the augmented base locus) If L is a nef class in $N^1(X)$, we have $\{x\in X \mid \varepsilon(L;x)=0\} = \mathbf{B}_+(L)$, where $\mathbf{B}_+(L)=\bigcap_E \operatorname{Supp} E$, as E ranges through the effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors such that $L-E$ has ample class in $N^1(X)$.
(4) (Bounds on jet separation) If X is smooth projective over $\mathbb{C}$ and L is a big and nef line bundle such that $\varepsilon(L;x)>\frac{n+s}{p}$, then $\omega_X\otimes L^{\otimes p}$ separates s-jets at x in the sense of the surjectivity of the evaluation map mentioned above.

See [4, Chapter 5] for details. In our work we extend these results to bundles of arbitrary rank. If $\rho: Y\to X$ is a morphism of projective varieties and if ξ is a ρ-ample class in $N^1(Y)$, we consider

$$\varepsilon(\xi;x) := \inf_C \frac{\xi\cdot C}{\operatorname{mult}_x \rho_* C},$$

the infimum ranging over irreducible curves $C\subset Y$ that meet the fiber Y_x without being contained in it. These are precisely the curves on Y such that $\operatorname{mult}_x\rho_*C>0$, where $\rho_*C = (\deg\rho|_C)\cdot(\rho(C))$ as cycles, with the convention that $\deg(\rho|_C)=0$ when ρ contracts C.

If $\mathcal{V}$ is a coherent sheaf on X, put $Y=\mathbb{P}(\mathcal{V}) = \operatorname{Proj}_{\mathcal{O}_X}\operatorname{Sym}^\bullet\mathcal{V}$, let ρ be the natural map to X, and let ξ be the class of the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$. The class ξ is ρ-ample. Set

$$\varepsilon(\mathcal{V};x) := \varepsilon(\xi;x).$$

The definitions also make sense when ξ and $\mathcal{V}$ are not globally positive, in which case $\varepsilon(\xi;x)$ could be negative. For ample vector bundles, the notion was also studied by Hacon in [3].

As a trivial example, we note that when $\mathcal{V}$ is locally free of rank 1, then $\rho=\mathrm{id}_X$ and $\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)=\mathcal{V}$, hence the new definition of Seshadri constant for the line bundle $\mathcal{V}$ agrees with the classical definition.

A new example is the case of vector bundles on curves. If C is a smooth projective curve over $\mathbb{C}$, and $\mathcal{V}$ is a vector bundle on C, then $\varepsilon(\mathcal{V};x)$ is the smallest slope in the Harder–Narasimhan filtration of $\mathcal{V}$. It is the smallest slope $\frac{\deg Q}{\operatorname{rk} Q}$ of any positive rank quotient Q of $\mathcal{V}$. This was also observed by Hacon in [3]. The result can be extended in positive characteristic at the expense of involving the Frobenius morphism.

Our main results from [2] are the extension of the known results for line bundles. Recall that a coherent sheaf $\mathcal{V}$ is nef/ample if $\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$ is nef/ample.

(1) The verbatim analogue of the Seshadri ampleness criterion holds for ρ-ample classes ξ on projective Y with a morphism $\rho: Y\to X$.
(2) Asymptotic jet separation holds after replacing $L^{\otimes m}$ with $\operatorname{Sym}^m\mathcal{V}$.

(3) If $\mathcal{V}$ is a nef vector bundle, then the vanishing locus of Seshadri constants on X is $\mathbf{B}_+(\mathcal{V}) := \rho(\mathbf{B}_+(\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)))$.
(4) If X is a complex projective manifolds, and $\mathcal{V}$ is ample of rank r such that $\varepsilon(\mathcal{V};x) > \frac{n+s}{p+r}$, then $\omega_X \otimes \mathrm{Sym}^p\mathcal{V} \otimes \det\mathcal{V}$ separates s-jets at x.

As a consequence, in any characteristic, we deduce that the ampleness of coherent sheaves is preserved under tensor products.

We conjecture that $\mathbb{P}^n$ is the only n-dimensional projective manifold X such that $\varepsilon(TX;x_0) > 0$ for some $x_0 \in X$. This is in the spirit of Mori's characterization of $\mathbb{P}^n$ via the ampleness of its tangent bundle, and of characterizations of $\mathbb{P}^n$ via bounds of form $\varepsilon(-K_X;x_0) \geq n+1$ as initiated by [1], and continued more recently by Y. Liu, Z. Zhuang, and T. Murayama. We have proved our conjecture in the cases when $n \leq 2$; when X is homogeneous; when it is Fano; or when $\varepsilon(TX;x) > 0$ for not just one point, or for a very general point, but for Zariski general $x \in X$.

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Surfaces with close to irrational Seshadri constants

Sönke Rollenske
(joint work with Alex Küronya)

Let X be a smooth complex projective surface and L an ample line bundle on X. Classically it is known that for any point $x \in X$ the Seshadri constant satisfies

$$1 \leq \varepsilon(L;x) \leq \sqrt{L^2},$$

but it is a longstanding open problem if the upper bound can be attained if L^2 is not a perfect square (compare [1] for some context).

We study this problem on simple cyclic multiple planes, defined in the following way: fix integers $d \geq 2$ and $m \geq 3$ and let $f \in \mathbb{C}[x, yz]$ be a homogenous polynomial of degree $dm \geq 6$. Then we consider

$$\pi\colon X = X_{d,m} \to \mathbb{P}^2$$

the simple cyclic cover branched over $B = V(f)$; alternatively we can define X as a hypersurface in weighted projective space,

$$X = V(w^d - f(x,y,z)) \subset \mathbb{P}(1,1,1,m) \dashrightarrow \mathbb{P}^2,$$

where the additional variable w has degree m.

We then prove

Theorem. Let f be very general, x a very general point on $X = X_{d,m}$ and $L = \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$. Then

(1) X is a smooth surface with Picard number $\rho(X) = 1$;
(2) the Seshadri constant satisfies

$$\sqrt{d} - \frac{d}{m} \leq \varepsilon(L;x) \leq \sqrt{d} = \sqrt{L^2}.$$

In particular, for large m the Seshadri constant is arbitrarily close to $\sqrt{d}$, which is irrational if d is not a perfect square.

The proof of the first part follows the proof of Cox [3] for the full family of hypersurfaces.

The second part is obtained by first observing that all curves of low degree are pullback of curves from $\mathbb{P}^2$, which can never be submaximal at a general point. On the other hand, T. Bauer proved in [2, Thm. 4.1], that curves resulting in a relatively low Seshadri constant have bounded degree. The result follows from these inequalities after observing that for every curve C on X we have $dC \in |kL|$ for some k.

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Seshadri Constants on hyperelliptic Surfaces and on Surfaces of general type

Praveen Kumar Roy
(joint work with Krishna Hanumanthu)

Seshadri constants quantify the positivity of an ample line bundle on a smooth projective variety. Computing and bounding them is an active area of research. In this talk, we talk about some new results obtained on hyperelliptic surfaces and on surfaces of general type. The primary motivation for the results on hyperelliptic surfaces is [3].

1. Hyperelliptic Surfaces

Hyperelliptic surfaces are minimal smooth surface X of Kodaira dimension $k_X = 0$ satisfying $h^0(\mathcal{O}_X) = 1$ and $h^2(\mathcal{O}_X) = 0$.

Hyperelliptic surfaces are also know as bielliptic surfaces (cf. [7]). More details can be found in [7]. There is an alternate characterization of hyperelliptic surfaces. A smooth surface X is Hyperelliptic if and only if $X \cong (A \times B)/G$, where A and

B are elliptic curves and G is a finite group of translation of A acting on B in such a way that $B/G \cong \mathbb{P}^1$. We have the following two projections.

$$\begin{array}{ccc} X \cong (A \times B)/G & \xrightarrow{\Phi} & A/G \\ \downarrow{\scriptstyle \Psi} & & \\ B/G \cong \mathbb{P}^1 & & \end{array}$$

Every hyperelliptic surface has Picard rank 2. Serrano described a basis for the free group $\mathrm{Num}(X)$ of divisors modulo numerical equivalence for each of the seven types of hyperelliptic surfaces. For each type, Serrano also lists the multiplicities $m_1, \ldots, m_s$ of the singular fibres of Ψ, where s is the number of singular fibres.

Theorem 1.1. [7, Theorem 1.4]. *Let $X \cong (A \times B)/G$ be a hyperelliptic surface. A basis for the group Num(X) of divisors modulo numerical equivalence and the multiplicities of the singular fibres of $\Psi : X \to B/G$ in each type are given in the following table.*

Type of X	G	$m_1, m_2, \ldots, m_s$	*Basis of Num(X)*
1	$\mathbb{Z}_2$	$2, 2, 2, 2$	$A/2,\ B$
2	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$2, 2, 2, 2$	$A/2,\ B/2$
3	$\mathbb{Z}_4$	$2, 4, 4$	$A/4,\ B$
4	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$2, 4, 4$	$A/4, B/2$
5	$\mathbb{Z}_3$	$3, 3, 3$	$A/3, B$
6	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$3, 3, 3$	$A/3, B/3$
7	$\mathbb{Z}_6$	$2, 3, 6$	$A/6, B$

Let X be a hyperelliptic surface. Let $\mu = lcm(m_1, m_2, \ldots, m_s)$ and let $\gamma = |G|$. By Serrano's theorem, a basis of $\mathrm{Num}(X)$is given by A/μ, $(\mu/\gamma)B$.

Notation: We say that L *is a line bundle of type* (a, b) on X if L is numerically equivalent to $a.A/\mu + b.(\mu/\gamma)B$. If L is of type (a, b), we write $L \equiv (a, b)$.

We note the following properties of line bundles on X.

(1) $A^2 = 0, B^2 = 0, A \cdot B = \gamma$.
(2) A divisor $b.(\mu/\gamma)B \equiv (0, b)$ is effective if and only if $b(\mu/\gamma) \in \mathbb{N}$ ([1, Proposition 5.2]).
(3) A line bundle of type (a, b) is ample if and only if $a > 0$ and $b > 0$ ([7, Lemma 1.3]).
(4) If C is an irreducible and reduced curve on X and $x \in C$ is a point of multiplicity m, then $C^2 \geq m^2 - m$.

1.1. **Results about $\varepsilon(L)$.** We define $\varepsilon(L)$ as the minimum of $\varepsilon(L, x)$ as x varies over X and $\varepsilon(L, 1)$ to be the maximum of $\varepsilon(L, x)$.

Theorem 1.2. [4, Theorem 3.1] *Let X be a hyperelliptic surface of odd type (i.e., of type 1, 3, 5, or 7). Let $L \equiv (a, b)$ be an ample line bundle on X. Then* $\varepsilon(L) = \min\{a, b\}$.

Theorem 1.3. [4, Theorem 3.3] *Let X be a hyperelliptic surface of type different from 6 and let L be an ample line bundle on X. Then $\varepsilon(L)$ is rational.*

1.2. **Results about** $\varepsilon(L,1)$**.**

Theorem 1.4. [4, Theorem 3.11] *Let X be a hyperelliptic surface and let L be an ample line bundle on X. If $\varepsilon(L,1) < (0.93)\sqrt{L^2}$, then $\varepsilon(L,1) = \min(L\cdot A, L\cdot B)$.*

2. Surface of general type

A smooth complex algebraic surface X is said to be of general type if the Kodaira dimension $\kappa(X) = 2$ (see [5]).

Motivated by [2, Theorem 1], we prove the following:

Theorem 2.1. [6, Theorem 2.4] *Let X be a surface of general type and K_X be the canonical line bundle on X. If K_X is big and nef and $x_1, x_2, \ldots, x_r \in X$ are $r \geq 2$ points, then we have the following.*

(1) $\varepsilon(X, K_X, x_1, x_2, \ldots, x_r) = 0 \Leftrightarrow$ *at least one of x_i lies on one of the finitely many (-2)-curves on X.*
(2) *If* $0 < \varepsilon(X, K_X, x_1, x_2, \ldots, x_r) < \frac{1}{r}$, *then*

$$\varepsilon(X, K_X, x_1, x_2, \ldots, x_r) = \begin{cases} \frac{1}{r+1} \text{ or } \frac{2}{5} & \text{if } r = 2, \\ \frac{1}{r+1} \text{ or } \frac{1}{r+2} & \text{if } 3 \leq r < 10, \\ \frac{1}{r+1} \text{ or } \frac{1}{r+2} \text{ or } \frac{1}{r+3} & \text{if } r \geq 10. \end{cases}$$

Let C be a smooth complex projective curve of genus $g \geq 2$ and consider a surface $X = C \times C$. Let F_1 and F_2 be fibres corresponding to the two projections from $C \times C \longrightarrow C$ and let δ be the diagonal. Assume that C is a general member of the moduli of smooth curves of genus g, where $g \geq 2$. Then, it is known that the Néron-Severi group $NS(X)$ is spanned by F_1, F_2 and δ [5, 1.5B].

Now we partially answer the question about the rationality of $\varepsilon(X, L)$ [8, Question 1.6]. In other words, under some conditions on a_1, a_2 and a_3 we address the question of rationality in affirmative. Following is our main theorem.

Theorem 2.2. [6, Theorem 3.1] *Let $X = C \times C$, where C is a general member of moduli of smooth curves of genus $g \geq 2$. Let $L \equiv_{num} a_1F_1 + a_2F_2 + a_3\delta$ be an ample line bundle satisfying any of the following conditions on a_1, a_2 and a_3.*

(1) $a_3 = 0$,
(2) $a_3 > 0$, $a_1 \leq a_2$ *and* $a_1^2 + a_3^2 < 2a_1a_2$,
(3) $a_3 > 0$, $a_2 \leq a_1$ *and* $a_2^2 + a_3^2 < 2a_1a_2$,
(4) $a_3 < 0$ *and* $a_2 \geq \left(\frac{2gk^2+2k+1}{2(k+1)}\right) \cdot a_1$, *where* $k = \lceil \frac{|a_3|/a_1}{1-|a_3|/a_1} \rceil$ *or*
(5) $a_3 < 0$ *and* $a_1 \geq \left(\frac{2gl^2+2l+1}{2(l+1)}\right) \cdot a_2$, *where* $l = \lceil \frac{|a_3|/a_2}{1-|a_3|/a_2} \rceil$.

Then $\varepsilon(X, L) \in \mathbb{Q}$.

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Seshadri constants on surfaces with Picard number 1

Justyna Szpond

Seshadri constants constitute an area of intense study since the ground-breaking work of Demailly [2]. They are interesting invariants considered to measure local positivity of a line bundle. In the simplest form, Seshadri constant is defined as follows.

Definition 1 (Seshadri constant)**.** Let X be a smooth projective variety, L an ample line bundle on X and $p \in X$ a point. The real number number

$$\varepsilon(X, L; p) = \inf_{C \ni p} \frac{L.C}{\mathrm{mult}_p C}$$

is the Seshadri constant of L at the point p. If the point p is very general, then we write simply $\varepsilon(X, L; 1)$.

It is well known that there is an upper bound $\varepsilon(X, L; p) \leq \sqrt[n]{L^n}$, where $n = \dim X$. It is also well known, due to examples of Miranda and Viehweg, that for any $\varepsilon > 0$ there exists a triple consisting of a smooth variety X, an ample line bundle L and a point $p \in X$ such that $\varepsilon(X, L; p) < \varepsilon$. It is however not known if such a phenomena can happen for an arbitrary positive ε on a fixed variety X (letting L and p vary). It is therefor desirable to look for lower bounds on Seshadri constants under additional assumption on X. Here we focus on varieties X with Picard number $\varrho(X) = 1$ and very general points on X. An interesting theorem along these lines has been proved by Ein, Küchle and Lazarsfeld in [3].

Theorem 2 (Ein, Küchle, Lazarsfeld)**.** Let X be a smooth projective variety of dimension n and let L be an ample line bundle on X, then

$$\varepsilon(X, L; 1) \geq \frac{1}{n}.$$

It is expected, but not known in general, that the actual bound in Theorem 2 is 1 rather than $1/n$. It is known on surfaces, due to an earlier work of Ein and Lazarsfeld.

Theorem 3 (Ein, Lazarsfeld)**.** *Let X be a smooth projective surface and let L be an ample line bundle on X. Then*

$$\varepsilon(X, L; 1) \geq 1.$$

Assuming additionally that the Picard number of X is 1 we can do considerably better. The first result in this setting has been obtained by Steffens [7].

Theorem 4 (Steffens)**.** *Let X be a smooth projective surface with Picard number 1 and let L be the ample generator. Then*

$$\varepsilon(X, L; 1) \geq \lfloor\sqrt{L^2}\rfloor.$$

Thus is L^2 is a perfect square, we obtain the precise value of Seshadri constant in a very general point. If L^2 is not a perfect square, this result has been improved by Szemberg in [8].

Theorem 5 (Szemberg)**.** *Let X be a smooth projective surface with Picard number 1 and let L be the ample generator whose self-intersection $d = L^2$ is not a perfect square. Let $\beta = \sqrt{d} - \lfloor\sqrt{d}\rfloor$. For $p_0 = \lceil\frac{1}{2\beta}\rceil$ and $m_0 = p_0\lfloor\sqrt{d}\rfloor + 1$ we have*

$$\varepsilon(X, L; 1) \geq \frac{p_0}{m_0} d. \tag{1}$$

The bound in 1 is just the first step in the development of d into a continued fraction. Motivated by results by Bauer and himself in [1], Szemberg conjectured that one can improve the bound in 1 considerably.

Conjecture 6 (Szemberg)**.** *Let X be a smooth projective surface with $\varrho(X) = 1$ and let L be the ample generator of the Picard group with $d = L^2$ not a perfect square. Let (p_0, q_0) be the primitive solution of Pell's equation*

$$q^2 - dp^2 = 1. \tag{2}$$

Then

$$\varepsilon(X, L; 1) \geq \frac{p_0}{q_0} d.$$

Building upon results of Küronya and Lozovanu on Okounkov bodies of polarized surfaces obtained in [6], in the joint paper with Farnik, Szemberg and Tutaj-Gasińska [5] we obtained the following result valid on arbitrary surfaces.

Theorem 7 (Farnik, Szemberg, Szpond, Tutaj-Gasińska)**.** *Let X be a smooth projective surface and L an ample line bundle with $d = L^2$ not a perfect square and (p, q) satisfy (2). Then*

$$\varepsilon(X, L; 1) \geq \frac{p}{q} d \quad \text{or} \quad \varepsilon(X, L; 1) \in \mathrm{Exc}(d; p, q),$$

where

$$\begin{aligned}\text{Exc}(d; p, q) =& \{1, 2, 3, \ldots, \lfloor\sqrt{d}\rfloor\} \cup \\ & \left\{\frac{a}{b} \text{ such that } 1 \leq \frac{a}{b} \leq \frac{p}{q} \cdot d \text{ and } 2 \leq b \leq q^2\right\}.\end{aligned}$$

Assuming additionally that the Picard number of X is 1 and that L is the ample generator, Theorem 7 provides some new evidence towards Conjecture 6, and in fact proves it in certain cases, for example when $p_0 = 1$ or $p_0 = 2$.

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Multi-point Seshadri constants and special configurations of points

Tomasz Szemberg

1. The containment problem

Huneke and Harbourne raised a series of important questions about the containment between symbolic and ordinary powers of homogeneous ideals. In the most elementary version, it asks if the third symbolic power of a radical ideal of points in the complex projective plane is always contained in the second ordinary power of the ideal. It has been answered to the negative by Dumnicki, Tutaj-Gasińska and myself in [6]. The non-containment example is provided by the ideal of singular points of the dual Hesse arrangement of lines.

Soon after finding the first non-containment example, a plethora of additional non-containment examples has been discovered. Notably all of them are directly or indirectly related to arrangements of lines and their singular points. The best known non-containment examples are those provided by Klein and Wiman arrangements, Fermat arrangements (see [12]) and Böröczky examples (see [4] and [13]). Recently Grifo proposed in [8] an asymptotic approach to the package of containment problems raised by Harbourne and Huneke in [10]. This has been further deepened in her joint work with Huneke and Mukundan [9]. A number of significant new results in this circle of problems can be expected soon.

2. Unexpected hypersurfaces

Building upon an example due to Di Gennaro, Illardi and Vallès [5], Cook II, Harbourne, Migliore and Nadel introduced in [3] the notion of unexpected curves, which in the subsequent article by the last three authors joined by Teitler [11] has been extended to hypersurfaces of arbitrary dimension. Roughly speaking, a set Z in a projective space admits an unexpected hypersurface of degree d if the naive count of conditions imposed by a fat point of multiplicity $m \geq 2$ on the linear system of hypersurfaces of degree d fails, i.e., if there are more such hypersurfaces than expected (see [15] for a version with multiple fat points). It is worth to point out that it never happens if Z is an empty set or if Z consists of general points. It was in fact quite surprising to realize that there are indeed sets Z admitting unexpected hypersurfaces.

Interestingly, in the aforementioned example of Di Gennaro, Illardi and Vallès the set Z is taken as the dual points of what is known as the B_3 line arrangement (or directly, as the B_3 root system in $\mathbb{P}^2$). Fermat arrangements of lines in $\mathbb{P}^2$, resp. planes in $\mathbb{P}^3$ have been shown to give rise to unexpected curves, resp. surfaces in [2]. Also Klein and Wiman arrangements of lines are reported to give rise to unexpected curves. As of writing of this abstract, this is known only to computer experiments (verified independently by various groups of researchers).

3. Asymptotic invariants

As hyperplane arrangements have appeared, rather unexpectedly, in the two contexts mentioned above, it is natural to wonder which of them lead to interesting phenomena (non-containment, unexpected hypersurface) and which fail to do so and why. One way to distinguish among various classes of arrangements and also to seek for what interesting classes have in common is to study their asymptotic invariants. In the context of the containment problem, Bocci and Harbourne introduced the real number $\rho(I)$ called the resurgence of I and defined as follows

$$\rho(I) = \sup\left\{\frac{m}{r} : \ I^{(m)} \not\subset I^r\right\}.$$

This is an invariant, quite interesting in its own, which has been computed for several classes of arrangements, e.g., in [7] and [1]. Another invariant, a lot more classical, at least in the realms of complex analysis is the Waldschmidt constant of a set of points. This invariant is quite more delicate and its exact value for some arrangements, e.g., the Klein arrangement is not known. It is subject to a conjecture in [1]. In the present workshop, yet another invariants, the Seshadri constants and their multi-point cousins have been of the main interest. For singular points of all arrangements mentioned above they have been computed recently by Pokora [14]. Rather than repeating his results in detail, I conclude with an interesting question raised by Pokora, which attracted a lot of attention during the workshop.

Conjecture 1 (Pokora)**.** Let Z be the set of all singular points of an arrangement of lines. Then for the multi-point Seshadri constant

$$\varepsilon(\mathbb{P}^2, \mathcal{O}(1); Z) = \frac{1}{\mathrm{mpl}(Z)},$$

where $\mathrm{mpl}(Z)$ is the maximal number of collinear points in Z.

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Rationality of Seshadri constants on general blow ups of $\mathbb{P}^2$

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(joint work with Łucja Farnik, Jack Huizenga, David Schmitz, Tomasz Szemberg)

Let X be a smooth complex projective surface and let L be a nef line bundle on X. The *Seshadri constant* of L at $x \in X$ is defined as the real number

$$\varepsilon(X, L, x) := \inf_{x \in C} \frac{L \cdot C}{\mathrm{mult}_x C},$$

where the infimum is taken over all irreducible and reduced curves C passing through x. It is interesting to study the behaviour of $\varepsilon(X, L, x)$ as the point $x \in X$ varies.

With this in mind, one has the following definition:

$$\varepsilon(X, L) := \inf_{x \in X} \varepsilon(X, L, x).$$

It is well-known that $0 < \varepsilon(X, L) \leq \sqrt{L^2}$. We can ask if $\varepsilon(X, L)$ is always a rational number for every pair (X, L) of a surface X and an ample line bundle L on X; see [5, Question 1.6]. This question has an affirmative answer for several classes of surfaces, such as abelian surfaces [1], Enriques surfaces [4], and most hyperelliptic surfaces [3].

In this talk, we consider the case of blow ups X of $\mathbb{P}^2$ at $r \geq 0$ very general points. We show in [2] that $\varepsilon(L) = \varepsilon(X, L)$ can in fact be irrational, provided a strengthening of SHGH Conjecture is true. Let H denote the pull-back of $\mathcal{O}_{\mathbb{P}^2}(1)$ and let $E_1, \ldots, E_r$ denote the exceptional divisors. Set $E = E_1 + \ldots + E_r$.

We are interested in ample line bundles L on X of the form $dH - mE$. After normalizing, we consider $\mathbb{R}$-divisors of the form $L = \mu H - E$. Since L is ample, we have $\mu^2 > r$. If the Nagata Conjecture is true, then the converse also holds. A curve C satisfying

$$\frac{L \cdot C}{\text{mult}_x C} \leq \sqrt{L^2}$$

is called a *weakly submaximal curve* for L with respect to x (note that if equality holds, then $\sqrt{L^2}$ is rational). If L is ample then we have $\varepsilon(L) \in \mathbb{Q}$ if and only if either $\sqrt{L^2} \in \mathbb{Q}$ or there is a weakly submaximal curve.

Let $L(\mu) := \mu H - E$. Then a real number $\mu_0 \geq \sqrt{r}$ is called the *submaximality threshold* for r if

(1) $L(\mu)$ does not admit a weakly submaximal curve for $\mu < \mu_0$, and
(2) $L(\mu)$ does admit a weakly submaximal curve for $\mu \geq \mu_0$.

Let $r \leq 9$. Suppose that $\mu \in \mathbb{Q}$ and $L(\mu)$ is ample. Then it is well-known that $\varepsilon(L(\mu)) \in \mathbb{Q}$. So the submaximality threshold for $r \leq 9$ is given by $\sqrt{r}$. Our first theorem gives an upper bound for the submaximality threshold μ_0 for an arbitrary r.

Theorem. Let $r \geq 1$ and let $\mu \in \mathbb{R}$. Then we have the following.

(1) For any r, $L(\mu)$ admits a weakly submaximal curve for all $\mu \geq \sqrt{r+1}$. In particular, we have $\mu_0 \leq \sqrt{r+1}$.
(2) If $r = 10$, then $L(\mu)$ admits a weakly submaximal curve for all $\mu \geq 77/24 \approx 3.208$.
(3) If $r = 11$, then $L(\mu)$ admits a weakly submaximal curve for all $\mu \geq 4 - \frac{\sqrt{3}}{3} \approx 3.422$.
(4) If $r = 13$, then $L(\mu)$ admits a weakly submaximal curve for all $\mu \geq \frac{1}{6}(26 - \sqrt{13}) \approx 3.732$.

The famous *SHGH Conjecture* classifies special linear systems on general blow ups X of $\mathbb{P}^2$. Suppose that we have integers $d \geq 0$ and $m_1, \ldots, m_r \geq 0$. Consider the linear series $\mathcal{L} = |dH - m_1E_1 - \cdots - m_rE_r|$ on X. Recall that $\mathcal{L}$ is said to be *special* if $\dim \mathcal{L} > \max\left\{\binom{d+2}{2} - \sum_i \binom{m_i+1}{2} - 1, -1\right\}$. Then SHGH Conjecture is the following statement.

SHGH Conjecture. If $\mathcal{L}$ is special, then every divisor in $\mathcal{L}$ is nonreduced.

We state a strengthening of the SHGH Conjecture and assuming this is true, we give precise value of the submaximality threshold for $r \geq 10$.

Conjecture. Let X be a blow up of $\mathbb{P}^2$ at $r \geq 0$ very general points. Suppose $d \geq 1$, $t \geq 1$, and $m_1, \ldots, m_r \geq 0$ are integers such that

$$\binom{d+2}{2} - \sum_{i=1}^{r} \binom{m_i+1}{2} \leq \max\left\{\binom{t+1}{2} - 2, 0\right\}.$$

Then any curve $C \in |dH - m_1E_1 - \cdots - m_rE_r|$ which has a point of multiplicity t is non-reduced.

The following is our main theorem.

Theorem. Suppose that the above conjecture is true and let $r \geq 10$. Then the submaximality threshold μ_0 for r exists, and

$$\mu_0 = \begin{cases} \frac{77}{24} & \text{if } r = 10 \\ 4 - \frac{\sqrt{3}}{3} & \text{if } r = 11 \\ \frac{1}{6}(26 - \sqrt{13}) & \text{if } r = 13 \\ \sqrt{r+1} & \text{if } r = 12 \text{ or } r \geq 14. \end{cases}$$

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Seshadri constants on fake projective planes

Halszka Tutaj-Gasińska

(joint work with Piotr Pokora)

This is a report on a joint work with Piotr Pokora, [4]. We study the existence of certain submaximal curves in the context of the Seshadri constants for ample line bundles on fake projective planes. Let us recall that by a fake projective plane we understand a smooth complex projective surface of general type having the same Betti numbers as the complex projective plane. The existence of such fake projective planes was proved by Mumford [3], and now we know that there are exactly 50 pairs of fake projective planes. We would like to focus on the case of multipoint Seshadri constants for fake projective planes. In the case of the single point Seshadri constants, L. Di Cerbo [1] proved that these constants coincide with the single point Seshadri constants of the complex projective plane, namely $\varepsilon(X, L_1; P) = 1$, for any point P in fake projective plane and for L_1, any ample generator of Pic$X$. The key advantage of this results is that it provides probably the first sharp result on single point Seshadri constants for surfaces of general type. Here we want to follow this path and look at the multipoint Seshadri constants for ample line bundles on fake projective planes.

Roé in [5] proved a bound on multipoint Seshadri constants on a surface, namely if X is a surface (e.g. fake projective plane with ample line bundle L_1), $p \in X$ any point, and $r \geq 1$, then we have

$$\varepsilon(X, L_1; r) \geq \varepsilon(X, L_1; p) \cdot \varepsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), r).$$

By the result of L. Di Cerbo [1] we know that $\varepsilon(X, L_1; p) = 1$, so we obtain the following inequality

$$\varepsilon(X, L_1; r) \geq \varepsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), r).$$

The main result of the paper, gives us bounds for Seshadri constants on fake projective planes. It tells us, for instance, that for $r \in \{2, 3, 5, 6, 7\}$ we have strict inequality above. This stands to the opposite to the case of single point Seshadri constants where $\varepsilon(X, kL_1; p) = \varepsilon(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k); p')$. Our strategy to show the strict inequality is based on the fact that we are able to exclude the existence of certain *submaximal* curves. In the proof we use Xu-type lemma from [2] and also a result of M. Roth, [6].

The theorem is as follows:

Let X be a fake projective plane and denote by L_1 an ample generator of the Néron-Severi group and assume that r is not a square. Then

$$\varepsilon(X, L_1; r) \geq \frac{1}{\sqrt{r} + \delta(r)},$$

where

$$\delta(2) = 0.031, \quad \delta(3) = 0.018, \quad \delta(5) = 0.014,$$
$$\delta(6) = 0.022, \quad \delta(7) = 0.011, \quad \delta(8) = 0.012,$$

and

$$\delta(r) = 0.013 \text{ for } r \geq 10.$$

Moreover, if $r = s^2$ for $s \in \mathbb{Z}_{>0}$, then

$$\varepsilon(X, L_1; r) = \frac{1}{s}.$$

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Bounded volume denominators and bounded primitive negativity

Thomas Bauer
(joint work with Brian Harbourne, Alex Küronya, Matthias Nickel)

Boundedness conditions of various kinds have generated vivid interest in algebraic geometry. One such that has received recent attention is the Bounded Negativity Problem: Does every smooth projective surface X have *bounded negativity*, in the sense that there is a constant that bounds the self-intersections C^2 of irreducible curves $C \subset X$ from below? In previous work with Pokora and Schmitz [3] we established that bounded negativity is equivalent to the boundedness of Zariski denominators. Here we report on recent joint work with Harbourne, Küronya and Nickel, where we relate these conditions to another natural boundedness question: Are the denominators that appear in the volumes $\mathrm{vol}_X(L)$ of big line bundles L on a smooth surface X bounded by a constant that depends only on X?

Turning to a more detailed description, we start by recalling the Bounded Negativity Conjecture:

Bounded Negativity Conjecture. (see [1]) *For every smooth projective surface X over $\mathbb{C}$ there is a constant $b(X) \in \mathbb{N}$ such that*

$$C^2 \geq -b(X)$$

for every reduced irreducible curve $C \subset X$.

This intriguing conjecture has a quite long history in Algebraic Geometry. The conjectured boundedness becomes wrong in positive characteristic – but even there the counter-examples are very special (see [1] and [6, Exercise V.1.10]). We say that a smooth projective surface X has *bounded negativity*, if the boundedness statement expressed in the conjecture holds for X. In work with Pokora and Schmitz [3] we showed that this condition is equivalent to a statement about Zariski decompositions:

Theorem. ([3]) *For a smooth projective surface X over an algebraically closed field the following are equivalent:*

(i) *X has bounded negativity.*
(ii) *X has bounded Zariski denominators.*

Here the condition of *bounded Zariski denominators* means by definition that there exists an integer $d(X) \geq 1$ such that for every pseudo-effective integral divisor D the denominators appearing in the Zariski decomposition of D are bounded from above by $d(X)$. If such a bound $d(X)$ exists, then by taking the factorial $d(X)!$ one obtains a uniform number that clears denominators in all Zariski decompositions on X.

The purpose of the work reported here is to consider another geometric concept in which denominators occur: volumes of big line bundles. Recall that for a big line bundle L on a smooth projective surface X, the *volume* is by definition the number

$$\mathrm{vol}(L) = \limsup_k \frac{h^0(X, kL)}{k^2/2}\,.$$

One clearly has

$$\mathrm{vol}(L) \in \mathbb{Q}\,,$$

since, using the Zariski decomposition $L = P + N$ with $\mathbb{Q}$-divisors P and N, the volume can be computed as

$$\mathrm{vol}(L) = P^2 \in \mathbb{Q}\,.$$

(Note that rationality of volumes is a feature that is specific to surfaces. In higher dimensions, irrational volumes occur, see [7, 5].) So the theorem above tells us that volumes on a surface X have bounded denominators, if X has bounded negativity. It is natural so ask whether or to what extent the converse of this might hold. Our first result shows that boundedness of volume denominators is equivalent to a variant of bounded negativity that we call *bounded primitive negativity*:

Theorem 1. (see [2]) *For a smooth projective surface X, the following conditions are equivalent:*

(i) *X has bounded volume denominators, i.e., there exists an integer $d_{\mathrm{vol}}(X)$ such that for every (integral) big line bundle L, the volume* $\mathrm{vol}(L)$ *is a rational number with denominator at most $d_{\mathrm{vol}}(X)$.*

(ii) *X has bounded primitive negativity, i.e., there exists an integer $b_{\mathrm{prim}}(X)$ such that for every primitive class $F \in \mathrm{NS}(X)$, whose ray $\mathbb{R}^+ \cdot F$ contains a reduced irreducible curve, one has*

$$F^2 \geq -b_{\mathrm{prim}}(X)\,.$$

Clearly, bounded negativity implies bounded primitive negativity. The other implication, however, is not clear at all: It is conceivable that on some surface one might have a sequence of primitive classes (F_n), whose self-intersections F_n^2 are bounded, and irreducible curves $C_n \equiv k_n F_n$, whose self-intersections C^2 are *not* bounded – this would happen if their "primitivity coefficients" k_n were unbounded. We do not know whether such a situation can arise on any smooth surface – in fact it even seems to be unknown if a prime divisor of negative self-intersection is ever numerically equivalent to kF for a primitive class F with $k > 2$. (Examples with $k = 2$ have long been known.)

Even dropping the assumption $F^2 < 0$ leaves an open question: On a smooth projective surface X consider a primitive class F such that some positive multiple kF is numerically equivalent to an effective divisor. We call the least such integer k

the *semi-effective order* of F. Is there a surface X that carries a sequence of primitive classes F_n with semi-effective orders $k_n \to \infty$? Work by Ciliberto et al. [4] shows that the answer is yes, if the SHGH Conjecture is true. The effective divisors $D_n \equiv k_n F_n$ appearing (conjecturally) in this way, are however reducible. The corresponding question about *irreducible* curves is open even when assuming the SHGH Conjecture.

Our best result in this direction is given by the following theorem. It shows that arbitrarily high semi-effective orders occur with negative definite divisors – without assuming the SHGH Conjecture – on a sequence of surfaces:

Theorem 2. (see [2]) *Over any algebraically closed field K, there exists a sequence of smooth projective surfaces X_n and primitive classes F_n of semi-effective orders k_n with*

$$\lim_{n\to\infty} k_n = \infty$$

such that $k_n F_n$ is numerically equivalent to an effective divisor D_n on X_n having a negative definite intersection matrix.

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Asymptotic Hilbert Polynomial and a bound for Waldschmidt constant

Marcin Dumnicki

(joint work with Łucja Farnik, Justyna Szpond, Halszka Tutaj-Gasińska)

The talk is based on results published in [1, 2].

Let $I = I(Z)$ be the homogeneous ideal of a set Z in projective space $\mathbb{P}^n$ over the field of complex numbers. By an m-th symbolic power we define

$$I^{(m)} = \{F : F \text{ vanish at least to order } m \text{ at every point of } Z\}.$$

Let $\alpha(I)$ denote the least degree of a non-zero form in I. The Waldschmidt constant of I may be defined as

$$\widetilde{\alpha}(I) = \lim_{m\to\infty} \frac{\alpha(I^{(m)})}{m}.$$

As an example, for Z equal to three points in $\mathbb{P}^2$, $\widetilde{\alpha}(I) = 3/2$ when points are not collinear and $\widetilde{\alpha} = 1$ otherwise.

A famous Nagata Conjecture states that if Z is a set of s general points in $\mathbb{P}^2$, $s \geq 9$ then

$$\widetilde{\alpha}(I) = \sqrt{s}.$$

The conjecture is still open, but the upper bound

$$\widetilde{\alpha}(I) \leq \sqrt{s}$$

is known.

Similarly, for any set of s points in $\mathbb{P}^n$, the following upper bound is valid:

$$\widetilde{\alpha}(I) \leq \sqrt[n]{s}.$$

As another example, for the set Z of s skew lines in $\mathbb{P}^n$, the Waldschmidt constant is bounded from above by the largest real root of

$$t^n - nst + (n-1)s.$$

More generally, for the set Z of s pairwise non-intersecting r-dimensional linear subspaces of $\mathbb{P}^n$, the Waldschmidt constant is bounded from above by the largest real root of

$$t^n - s\left(\sum_{j=1}^{r} \binom{n}{j}(t-1)^j\right).$$

To generalize the above for any homogeneous ideals I, we define the asymptotic Hilbert Polynomial

$$\mathrm{aHP}_I(t) = \lim_{m\to\infty} \frac{\mathrm{HP}_{I^{(m)}}(mt)}{m^n},$$

where HP is the usual Hilbert polynomial of a module $\mathbb{C}[x_0,\ldots,x_n]/I$.

It can be shown that the limit exists provided that the Castelnuovo-Mumford regularity of a sequence $I^{(m)}$ is linearly bounded with respect to m.

An example that aHP is a subtle invariant can be made. Consider two sets in $\mathbb{P}^3$. Let Z_1 consists of two intersecting lines and a point, let Z_2 consists of two skew lines. Then

$$\mathrm{HP}_{I_1}(t) = \mathrm{HP}_{I_2}(t) = 2t + 2,$$

while

$$\mathrm{aHP}_{I_1}(t) = t - \frac{5}{6}, \qquad \mathrm{aHP}_{I_2}(t) = t - \frac{2}{3}.$$

Define

$$\Lambda_I(t) = \frac{t^n}{n!} - \mathrm{aHP}_I(t).$$

It is not true that $\widetilde{\alpha}(I)$ would be bounded by a largest real root of Λ_I. The counterexample is for the set Z of a star configuration of lines given by s hypersurfaces in $\mathbb{P}^4$. In this case $\Lambda_I(t)$ has no real roots for $s \geq 4$.

The main theorem can be stated as follows. Let I be the radical homogeneous ideal with linearly bounded regularity of symbolic powers. Assume that in the sequence $\{\mathrm{depth}(I^{(m)})\}$ there exists a constant subsequence of value $n-c$. Then

$$\Lambda_I^{(c)}(\widetilde{\alpha}(I)) \leq 0,$$

where by (c) we mean taking a derivative of Λ c times.

Going back to the example with a star configuration of lines, in this case the first derivative of $\Lambda_I(t)$ has a root, approximately equal to $\frac{s}{\sqrt[3]{6}}$, while $\widetilde{\alpha}(I) = \frac{s}{3}$.

We show that an approach by Λ_I may be better that computer experiments. For a set Z consisting of 5 crosses (i.e. pairs of intersecting lines) in $\mathbb{P}^3$, we get

$$\Lambda_I(t) = \frac{t^3}{6} - 5(t-1),$$

which gives

$$\widetilde{\alpha}(I) \leq 4.88448,$$

while computer experiments suggest that $\alpha(I^{(m)}) = 5m$ (this holds for $m = 1, \ldots, 10$).

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The role of line arrangements in some open problems in algebraic geometry

Brian Harbourne

We discuss a number of topics where line arrangements have played a significant role recently and state some open questions for each.

Topic 1: Computability and rationality of multipoint Seshadri Constants.

Let $Z \subset \mathbb{P}^2$ be a finite set of points. Let $\pi_Z : X_Z \to \mathbb{P}^2$ be the blow up of Z, with E_Z the exceptional locus. Let $L' = \pi_Z^* L$ for a general line $L \subset \mathbb{P}^2$. Then the Seshadri constant for Z is

$$\varepsilon(Z) = \sup\{t : L' - tE_Z \text{ is nef}\}.$$

Question (see [8]): Let $C \subset \mathbb{P}^2$ be a reduced curve, $Z = Z_C = \mathrm{Sing}(C)$, and $D_C \subset X$ the proper transform of C. If $D_C^2 < -1$, must $\varepsilon(Z)$ be rational? Can we compute it? (As a test case, take $C_{\mathcal{L}}$ to be the union of the lines of a supersolvable line arrangement $\mathcal{L}$. Recall that a *modular* point of a line arrangement C is a singular point of C which can see all other singular points by looking down lines of the arrangement. A *supersolvable* line arrangement is a line arrangement with at least one modular point.)

Given a line arrangement $\mathcal{L}$, let $\ell_{\mathcal{L}}$ be the number of singular points on the line of $\mathcal{L}$ with the largest number of singular points. Clearly $\varepsilon(Z_{C_{\mathcal{L}}}) \leq 1/\ell_{\mathcal{L}}$. Pokora [8] observes that equality often holds and asks if it holds for all complex line arrangements.

Topic 2: Computability and rationality of Waldschmidt constants.

Let Z be a finite set of points in $\mathbb{P}^2$. Recall that the Waldschmidt constant for Z is

$$\widehat{\alpha}(Z) = \inf\{t/m : tL' - mE_Z \text{ is effective}\}$$

and we have $\widehat{\alpha}(Z) \geq |Z|\varepsilon(Z)$.

Question: Let $C \subset \mathbb{P}^2$ be a reduced curve, $Z = Z_C = \operatorname{Sing}(C)$. Must $\widehat{\alpha}(Z)$ be rational? Can we compute it? (As a sample open case, take C to be Klein's arrangement of 21 lines [2].)

Topic 3: Bounded Negativity.

Question: Let C be a reduced singular plane curve. How negative can $H(C) = D_C^2/|Z_C|$ be? We have the following facts [3]:

(a) $\inf_C H(C) \leq -2$ (the inf is taken over all reduced, irreducible C; the characteristic is arbitrary);
(b) $\inf_C H(C) = -3$ (the inf is taken over all real line arrangements C);
(c) $\inf_C H(C) \geq -4$ (the inf is taken over all complex line arrangements C).

Question: If C is irreducible (for arbitrary characteristic) must we have $H(C) > -2$? Taking C to be a general image of $\mathbb{P}^1$ in $\mathbb{P}^2$ of degree d gives

$$H(C) = -2 + \frac{6d-4}{(d-1)(d-2)}.$$

Question: If C is a complex line arrangement must we have $H(C) \geq H(C_W) = -\frac{225}{67} \approx -3.36$, where C_W is Wiman's arrangement of 45 lines? Note C_W has 120 triple, 45 quadruple and 36 quintuple points (see [2]).

Question: What is $\inf_C H(C)$ for rational line arrangements C? There is a rational line arrangement C having 37 lines with $H(C) = \frac{-503}{181} \approx -2.779$.

Topic 4: A Containment Problem.

Let $Z \subset \mathbb{P}^2$ be a finite set, and let $I_Z^{(m)} = \cap_{p \in Z} I_p^m$. This is called the mth symbolic power of I_Z.

Question: Is it possible to characterize those Z with $I_Z^{(3)} \not\subseteq I_Z^2$? All known complex examples have been found by taking $Z \subset \operatorname{Sing}(C)$ for certain line arrangements C. The first case found was for the lines to be the components of the curve defined by $(x^3 - y^3)(x^3 - y^3)(y^3 - z^3) = 0$ [5].

Other questions related to symbolic powers arise for line arrangements. For example, can we compute the resurgence $\rho(I_{Z_{C_{\mathcal{L}}}})$ for any line arrangement $\mathcal{L}$ (or at least for supersolvable line arrangements)? Here $\rho(I_Z) = \sup\{m/r : I_Z^{(m)} \not\subseteq I_Z^r\}$.

At least for some supersolvable line arrangements $\mathcal{L}$ it is the case that

$$I_{Z_{C_{\mathcal{L}}}}^{(m\ell_{\mathcal{L}})} = (I_{Z_{C_{\mathcal{L}}}}^{(\ell_{\mathcal{L}})})^m$$

holds for all $m > 0$. We can ask if this holds for all supersolvable line arrangements $\mathcal{L}$ or perhaps even for all line arrangements. This behavior would be a consequence of the symbolic Rees algebra $\oplus I_{Z_{C_{\mathcal{L}}}}^{(m)}$ of $Z_{C_{\mathcal{L}}}$ being Noetherian, so another question is whether $\oplus I_{Z_{C_{\mathcal{L}}}}^{(m)}$ is always Noetherian, or at least if $\mathcal{L}$ is supersolvable.

Topic 5: Unexpected curves.

Let $Z \subset \mathbb{P}^2$ be a finite set, $p \in \mathbb{P}^2$ a general point, $Z' = Z \cup \{p\}$.

Question: Can we classify all (Z, m) with

$$h^0(X_{Z'}, (m+1)L' - mE_p - E_Z) >$$

$$\max\left(0, h^0(X_{Z'}, (m+1)L' - E_Z) - \binom{m+1}{2}\right)?$$

The main technique currently uses properties of the line arrangement dual to Z.

Example: The least m for which there is a Z is $m = 3$, and this Z is unique (up to projective equivalence), coming from the B_3 arrangement of 9 lines dual to the roots of the B_3 root system [6].

There is a nice criterion for unexpectedness when Z is dual to a supersolvable line arrangement.

Theorem ([4]): Let $\mathcal{L} = \{L_1, \dots, L_r\}$ be supersolvable, $m_{\mathcal{L}}$ the maximum multiplicity among the singular points, and $d_{\mathcal{L}} = r$ the number of lines. Then the following are equivalent:

(a) $\mathcal{P}_{\mathcal{L}}$ has an unexpected curve of degree $d = m + 1$ for some m;
(b) $\mathcal{P}_{\mathcal{L}}$ has an unexpected curve of degree d for $d = m_{\mathcal{L}}$; and
(c) $2m_{\mathcal{L}} < d_{\mathcal{L}}$.

Question: Which supersolvable $\mathcal{L}$ have $2m_{\mathcal{L}} < d_{\mathcal{L}}$? Can we classify supersolvable $\mathcal{L}$?

Topic 6: Classifying complex supersolvable $\mathcal{L}$.

Definition: Let $\mathcal{L}$ be supersolvable. We say $\mathcal{L}$ is homogeneous if every modular point has the same multiplicity.

The results of [7] give a complete classification of complex supersolvable nonhomogeneous $\mathcal{L}$. The paper [7] also shows that a complex supersolvable homogeneous $\mathcal{L}$ has at most 4 modular points and completely classifies the case of 3 and 4 modular points. Then [1] classifies (up to incidence structure) complex supersolvable homogeneous $\mathcal{L}$ with 2 modular points.

Question: Can supersolvable complex line arrangements with only one modular point be classified?

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Negative curves on special rational surfaces

Łucja Farnik

(joint work with Marcin Dumnicki, Krishna Hanumanthu, Grzegorz Malara, Tomasz Szemberg, Justyna Szpond, Halszka Tutaj-Gasińska)

Negative curves on algebraic surfaces are a classical object of interest. One of the central and open problems concerning negative curves is the Bounded Negativity Conjecture which asks whether on a fixed surface negativity is bounded. This is not the case in positive characteristic. In characteristic zero, it is easy to bound negativity in some cases but the problem is open in general.

I report on the results from [2]. We study negative curves on surfaces obtained by blowing up special configurations of points in $\mathbb{P}^2$. Our results concern the following configurations: very general points on a cubic, 3–torsion points on an elliptic curve and nine Fermat points. As a consequence, we also show that the Bounded Negativity Conjecture holds for these surfaces.

Turning into details, we say that a reduced and irreducible curve C on a smooth projective surface is *negative*, if its self-intersection number C^2 is less than zero.

The famous Bounded Negativity Conjecture (BNC for short) may be stated as follows.

Bounded Negativity Conjecture. Let X be a smooth projective surface. Then there exists a number τ such that $C^2 \geq \tau$ for any reduced and irreducible curve $C \subset X$.

If the BNC holds on a surface X, then we denote by $b(X)$ the largest number τ such that the Conjecture holds. See [1] for an extended introduction to this problem.

Now we state the first result, see also [3, Remark III.13].

Theorem (Very general points on a cubic)**.** Let D be an irreducible and reduced plane cubic and let $P_1, \dots, P_s$ be very general points on D. Let $f : X \longrightarrow \mathbb{P}^2$ be the blow up at $P_1, \dots, P_s$. If $C \subset X$ is any reduced and irreducible curve such that $C^2 < 0$, then

(a) C is the proper transform of D, or
(b) C is a (-1)-curve.

As a consequence we have the following.

Corollary. Let X be a surface as in the theorem above with $s > 0$. Then the BNC holds for X and we have

$$b(X) = \min\{-1,\ 9 - s\}.$$

Now let us consider blow ups of $\mathbb{P}^2$ at 3-torsion points of an elliptic curve as well as the points of intersection of the Fermat arrangement. In order to study these two cases, we first state the following numerical lemma which seems to be quite interesting in its own right.

Lemma. Let $m_1, \dots, m_9$ be nonnegative real numbers satisfying the following 12 inequalities:

$$\begin{array}{llll} m_1 + m_2 + m_3 \le 1, & m_1 + m_4 + m_7 \le 1, & m_1 + m_5 + m_9 \le 1, & m_1 + m_6 + m_8 \le 1, \\ m_4 + m_5 + m_6 \le 1, & m_2 + m_5 + m_8 \le 1, & m_2 + m_6 + m_7 \le 1, & m_2 + m_4 + m_9 \le 1, \\ m_7 + m_8 + m_9 \le 1, & m_3 + m_6 + m_9 \le 1, & m_3 + m_4 + m_8 \le 1, & m_3 + m_5 + m_7 \le 1. \end{array}$$

Then

$$m_1^2 + \cdots + m_9^2 \le 1.$$

The inequalities are obtained from suitable triples of m_i's from the matrix

$$\begin{pmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{pmatrix}.$$

Using the lemma above we can prove the following theorem.

Theorem (3–torsion points on an elliptic curve)**.** Let D be a smooth plane cubic and let $P_1, \dots, P_9$ be the flexes of D. Let $f : X \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at $P_1, \dots, P_9$. If C is a negative curve on X, then

(a) C is the proper transform of a line passing through two (hence three) of the points $P_1, \dots, P_9$, or
(b) C is an exceptional divisor of f.

Corollary. Let X be a surface as in the theorem above. Then the BNC holds for X and we have

$$b(X) = -2.$$

For the configuration of nine Fermat points the result is as follows.

Theorem (Fermat points)**.** Let $f : X \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at $Q_1, \ldots, Q_9$, where

$$\begin{array}{lll} Q_1 = (\varepsilon : \varepsilon : 1), & Q_2 = (1 : \varepsilon : 1), & Q_3 = (\varepsilon^2 : \varepsilon : 1), \\ Q_4 = (\varepsilon : 1 : 1), & Q_5 = (1 : 1 : 1), & Q_6 = (\varepsilon^2 : 1 : 1), \\ Q_7 = (\varepsilon : \varepsilon^2 : 1), & Q_8 = (1 : \varepsilon^2 : 1), & Q_9 = (\varepsilon^2 : \varepsilon^2 : 1). \end{array}$$

If C is a negative curve on X, then

(a) C is the proper transform of a line passing through two or three of the points $Q_1, \ldots, Q_9$, or
(b) C is a (-1)-curve.

We finish this note with an interesting problem.

Problem. For a positive integer m, let $Z(m)$ be the set of all points of the form $(1 : \varepsilon^\alpha : \varepsilon^\beta)$, where ε is a primitive root of unity of order m and $1 \le \alpha, \beta \le m$. Let $f_m : X(m) \to \mathbb{P}^2$ be the blow up of $\mathbb{P}$ at all the points of $Z(m)$. Is the negativity bounded on $X(m)$? If yes, what is the value of $b(X(m))$?

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Brill-Noether theory for vector bundles and higher rank SHGH conjectures

Jack Huizenga

(joint work with Izzet Coskun)

The SHGH conjecture describes the expected cohomology of line bundles on general blowups of the projective plane. When there is unexpected cohomology, there is a (-1)-curve on the surface contributing to the cohomology. The problem of computing the cohomology of a general vector bundle is a natural higher-rank generalization of this problem.

In joint work with Izzet Coskun [1], we determined the cohomology of the general sheaf in a moduli space of sheaves on a Hirzebruch surface. On the other hand, the problem is more challenging on del Pezzo surfaces, largely because the Picard rank is higher. I will discuss the following theorem of Daniel Levine and Shizhuo Zhang, which solves the problem for del Pezzo surfaces which are the blowup of the projective plane at up to 5 points. Recall that a sheaf is *nonspecial* if it only has nonzero cohomology in at most one degree.

Theorem ([2])**.** Let X be a del Pezzo surface of degree at least 4, and let H be the pullback of a line under the blowdown map $X \to \mathbb{P}^2$. Suppose $\mathcal{V}$ is a general $(-K_X)$-semistable sheaf of numerical invariants (r, c_1, χ), and let $\nu = c_1/r$. If $r \geq 2$, $\chi \geq 0$, and $\nu \cdot H \geq -2$, then $\mathcal{V}$ is nonspecial if and only if $\nu \cdot C \geq -1$ for every (-1)-curve C on X.

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