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Heat Kernels, Stochastic Processes and Functional Inequalities

Organized by
Masha Gordina, Storrs
Takashi Kumagai, Kyoto
Laurent Saloff-Coste, Ithaca
Karl-Theodor Sturm, Bonn

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Abstract. The aims of the 2019 workshop *Heat Kernels, Stochastic Processes and Functional Inequalities* were: (a) to provide a forum to review recent progresses in a wide array of areas of analysis (elliptic, subelliptic and parabolic differential equations, transport, functional inequalities), geometry (Riemannian and sub-Riemannian geometries, metric measure spaces, geometric analysis and curvature), and probability (Brownian motion, Dirichlet spaces, stochastic calculus and random media) that have natural common interests, and (b) to foster, encourage and develop further interactions and cross-fertilization between these different directions of research.

Mathematics Subject Classification (2010): 58J65, 58J35, 60J45, 60K37, 60F17, 53C23.

Introduction by the Organizers

The workshop, organised by Masha Gordina (University of Connecticut), Takashi Kumagai (RIMS, Kyoto University), Laurent Saloff-Coste (Cornell University), and Karl-Theodor Sturm (University of Bonn), was attended by over 50 participants from Australia, Austria, Canada, China, France, Germany, Israel, Italy, Japan, Luxembourg, United Kingdom, and USA. The program consisted of 29 talks and 7 short contributions, leaving sufficient time for informal discussions. The general topic of the workshop was the study of linear and non-linear diffusions in geometric environments: metric measure spaces, Riemannian and sub-Riemannian manifolds, fractals and graphs, and in random environments. The workshop brought together leading experts in three different major fields of mathematics: analysis, stochastics and geometry. The unifying theme was analytic, geometric, and stochastic calculus on degenerate, singular or randomized spaces, based on functional inequalities and heat kernel analysis. The workshop also provided a unique opportunity for interaction between established and early career scientists from these different areas. One after-dinner session was devoted to short communications by junior participants.

The list of the talks provided below illustrates the wide variety of the topics treated during the workshop. Even so no particular pressure was put on the speakers to stress connections across fields, such connections were overwhelmingly present, loud and clear. The questions during and following the talks demonstrated both the high interest of the problems and results that were presented from the point of view of the experts in the field and the curiosity of many participants for concepts and ideas that were unfamiliar to them. The highlights of the conference include (but are not limited to):

(a) New results on the link between elliptic and parabolic Harnack inequalities in the context of Dirichlet forms on metric measure spaces involving quasi-symmetric changes of metric (M. Barlow, M. Murugan and N. Kajino).

(b) Recent progress in the construction of Liouville quantum gravity, namely, the construction of these objects as random metric measure spaces (E. Gwynne and J. Miller), and also the construction of the Brownian sphere based on Brownian trees and tree-indexed Brownian motion (J-F. Le Gall and A. Riera).

(c) The beautiful result by E. Bruè and D. Semola that the dimensional strata-structure of a metric measure space satisfying the Riemannian-curvature-dimension property RCD (K, N) reduces essentially to a unique strata. The proof uses ideas from mass transport and a new regularity result for Lagrangian flows in metric measure spaces.

Another important aspect of the workshop is to encourage interactions between people working on discrete and continuous models. Indeed, viewing continuous models as a limit of discrete models is one of the major recurring theme. The talks by M. Murugan, P. Alonso Ruiz and C. Li involved interactions between analysis on graphs and on manifolds and other continuous spaces. The talks by S. Andres, J. Norris, D. Shiraishi and A. Winter involved sophisticated limit theorems for certain discrete models. In the talk of E. Kopfer, the stability of optimal transports under convergence to continuous limits was analyzed. The conjectural connections between various discrete models and the continuous models of Liouville Quantum gravity remains a vast open area of research. L. Dello Schiavo introduced a random geometry based on Dirichlet form techniques on the space of probability measures. J. Kigami presented analytic characterization of the Ahlfors regular conformal dimension by analysing the Gromov hyperbolic graph associated with a compact metric space.

Functional inequalities appearing in the analysis of random media are key ingredients of the workshop as illustrated in the talks by S. Andres, N. Berger, M. Erbar, E. Gwynne, and C. Smart; and also in the talks by M. Biskup and A. Faggionato. Some recent results concerning heat kernel estimates for jump processes were discussed by J. Wang. The Riesz transforms on non-doubling spaces were addressed in the talk of A. Sikora. Functional inequalities played a central role in several talks on sub-Riemannian or variable metric spaces given by N. Eldredge, E. Milman and A. Thalmaier as well as in the talk of F. Cipriani on spectral invariants and conformal equivalences.

To give a more vivid impression of the mathematical content of the workshop, we now provide a non-technical description of a sample of the topics discussed by the speakers.

In the opening talk, **Mathav Murugan** (UBC, Canada) described in a beautiful way significant new results (obtained jointly with M. Barlow and N. Kajino) which establish a precise bridge between elliptic and parabolic Harnack inequalities, two of the most consequential inequalities in geometric analysis and PDEs. This bridge involves the time scaling exponents that appear in parabolic Harnack inequalities and the notion of "conformal walk-dimension" associated with quasi-symmetric changes of metric. These results provide new insights on heat kernel analysis and the geometry of rough metric measure spaces including fractals.

In his talk, **Robert Haslhofer** (Toronto, Canada) reported on an ongoing joint work with E. Kopfer and A. Naber on differential Harnack inequality on path space. In the previous work by Naber and Haslhofer-Naber on the Bochner inequality on path space, already deep new insights have been gained. In particular, it allowed to characterize Einstein manifolds in terms of functional inequalities on the path space. Now a first version of a Harnack inequality of Hamilton type could be proven on path space.

Liouville quantum gravity is the name given to certain illusive random geometries on the two-sphere. Here, one views triplets (M, μ, d) made of the sphere $M = \mathbb{S}^2$, a measure μ, and a distance function d, as elements in the space of marked metric measure spaces equipped with the Gromov-Hausdorff-Prokhorov topology. The goal is to construct random objects in that space that have universal properties reminiscent to that of Brownian motion. The suggestion that such objects might exit and be useful goes back to the physicist A. Polyakov (1981). Recent progress in this area is the subject of an article in Quanta Magazine (July 2019, *Random Surfaces Hide an Intricate Order*). Clay Research Fellow **Ewain Gwynne** (Cambridge, UK) is an expert on this subject. He gave a very informative and dazzling talk which highlighted recent progress concerning the construction of the Liouville gravity distance, one of the most subtle and important aspect of the theory. In the context of the workshop, the area surveyed by E. Gwynne establish a remarkable and exciting link between the general idea of *random environment* and the study of *metric measure spaces*.

In heat kernel analysis, sharp gradient estimates are both important and difficult to prove as they typically require rather strong geometric assumptions such as a curvature lower bound. In her talk, **Li Chen** (University of Connecticut) reviewed recent results concerning integrated gradient estimates in L^p-norm, $p \in (1, 2]$, whose virtue is to hold under very minimal assumptions on the underlying space, and thus are widely applicable in the context of Dirichlet spaces on metric measure spaces. She also described the connections between such estimates, Riesz transforms, and Besov-space embedding theorems, connecting her talk to the long-term project presented at the workshop in talks by Fabrice Baudoin and Patricia Alonzo Ruiz.

Two workshop talks presented results from a long-term project on the heat semigroup-based Besov spaces and functions of bounded variation (BV) on Dirichlet spaces. **Fabrice Baudoin** (University of Connecticut) described the setting including one of the main assumptions, namely, a weak Bakry-Émery curvature type condition. The aim is to be able to apply this approach to non-smooth spaces such as fractals. A number of results have been presented such as isoperimetric and Sobolev inequalities including a conjecture about L^p-Besov critical exponents, that is, the exponent for which the Besov space is non-trivial. The next talk by **Patricia Alonso Ruiz** (Texas A& M) applied these tools to fractional Laplacians on metric measure spaces. Namely, she considered a class of non-local Dirichlet forms corresponding to a semigroup obtained by subordination with the generator being a fractional Laplacian. This simplifies the presentation though some of their results can be extended to a somewhat more general setting.

Anita Winter (University of Duisburg-Essen) presented a novel notion of continuum trees – called algebraic trees – which generalizes countable graph-theoretic trees to potentially uncountable structures. Under an order-separability condition, algebraic trees can be considered as tree structure equivalence classes of metric trees. In many applications binary trees are of particular interest and the space of binary algebraic tree is shown to be compact.

Jean-François Le Gall (Université Paris-Sud Orsay) gave a brilliant description of a particular construction of the *Brownian sphere*. Roughly speaking, the Brownian sphere is one of the key random objects underlying the metric measure spaces of the Liouville quantum gravity discussed earlier in the week by E. Gwynne. Le Gall explained how the Brownian sphere can be constructed using Brownian trees (a variation on Aldous' continuum random tree) and tree-indexed Brownian motion in the plane. This construction leads to analogies with excursion theory and intriguing new growth-fragmentation random processes associated the geometry of the Brownian sphere, namely, the structure of the complement of a ball on the Brownian sphere.

The study of metric measure spaces with synthetic lower Ricci bounds attracted lot of attention in recent years and many deep results and insights could be obtained. Of particular interest from the perspective of geometric analysis are structural results for infinitesimally Hilbertian metric measure spaces which satisfy a curvature-dimension condition in the sense of Lott-Sturm-Villani. A first breakthrough in this direction was the seminal work of Mondino and Naber leading to dimensional decompositions for such spaces. The open challenge for several years then was to prove constancy of the dimension. Even for Ricci limits, this was proven only few years ago by Colding and Naber. In his talk on joint work with

Daniele Semola, **Elia Brué** (SNS Pisa, Italy) presented their beautiful and technically demanding argument for proving constancy of dimension for RCD(K,N)-spaces.

Jan Maas (IST Austria) reported on the joint work with Eric Carlen on approaches via optimal transport to challenging problems in non-commutative geometry, in particular, to the study of ergodic quantum Markov semigroups on finite-dimensional unital C^*-algebras. They show that the evolution on the set of states that is given by such a quantum Markov semigroup is the gradient flow for the relative entropy with respect to a unique stationary state in a particular Riemannian metric on the set of states. As a consequence, a number of new inequalities for the decay of the relative entropy for ergodic quantum Markov semigroups can be deduced.

Anderson localization is a well-known trapping phenomena in the field of random Schrödinger operators. Let $H = -\Delta + \delta V$ be a Schrödinger operator on $\mathbb{Z}^d$, where Δ is the discrete Laplacian and V is a random potential given by iid 0-1 Bernoulli random variables. **Charles Smart** gave a wonderful talk about his recent work with Jian Ding that shows that for any $\delta > 0$, H has Anderson localization almost surely for $d = 2$, which resolves a long time open problem in this area. While the proof follows the program of Bourgain and Kenig that proves the corresponding results for $\mathbb{R}^d$ (continuum model), it requires completely new ideas and significant efforts, as is often the case for discrete models in random media. One of the key ideas of the proof is to apply a Liouville-type theorem recently obtained by Buhovsky, Logunov, Malinnikova and Sodin.

The last talk of the workshop was given by **Sebastian Andres** (Cambridge, UK). In the first part of his talk, he summarized briefly the history of the quenched invariance principle and heat kernel estimates random conductance models, which helped non-expert to have an overview of the area. He then discussed his recent work with P. Taylor about the quenched and annealed local limit theorem for time dependent random conductance models. He then explained beautiful application of the results to the Ginzburg-Landau $\nabla\varphi$ interface model. He also explained some other possible applications of the theory of random conductance models to various other models motivated by statistical physics.

The workshop participants came from different mathematical areas such as analysis, geometry, probability with a strong presence of mathematical physics. Many of them benefited from this opportunity to interact with the mathematicians who have expertise in a related area of research. The extensive discussions by participants at different stages of their career have produced a number of new ideas and connections. In addition to stimulating the existing joint projects, the workshop helped to initiate new collaborations, in particular for junior researchers.

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Workshop: Heat Kernels, Stochastic Processes and Functional Inequalities

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Abstracts

Heat kernel approach to BV functions in non-local Dirichlet spaces

Patricia Alonso Ruiz
(joint work with F. Baudoin, L. Chen, L. Rogers, N. Shanmugalingam, A. Teplyaev)

Functions of bounded variation (BV) appear naturally in the study of variational problems and are intrinsically related to isoperimetric sets and sets of finite perimeter. In this talk we propose a notion of BV functions in metric measure spaces equipped with a non-local Dirichlet form. Two main reasons that motivate the study of the non-local setting are, first, the extension of results previously obtained by the authors in the local setting [2] and second, to create stronger connections with the PDE research community, where non-local operators and fractional Sobolev spaces appear naturally in many models under investigation.

The prototype of operator we will consider here is the fractional Laplacian, which we see as the infinitesimal generator of a semigroup defined via subordination. The starting point of our discussion is thus a locally compact Ahlfors d_H-regular space (X, μ, d) equipped with a strongly local and regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ whose associated heat semigroup $\{P_t\}_{t>0}$ is assumed to be conservative and admit a jointly continuous heat kernel satisfying two-sided sub-Gaussian estimates

$$p_t(x,y) \asymp t^{-\frac{d_H}{d_W}} \exp\biggl(-c\Bigl(\frac{d(x,y)^{d_W}}{t}\Bigr)^{\frac{1}{d_W-1}}\biggr).$$

From a purely metric-measure theoretic approach that goes back to the seminal paper by Korevaar and Schoen [6], one defines the spaces

$$KS^{\lambda,1}(X) := \Bigl\{f \in L^1(X,\mu) : \ \|f\|_{KS^{\lambda,1}(X)} < \infty\Bigr\},$$

where

$$\|f\|_{KS^{\lambda,1}(X)} := \limsup_{r\to 0+} \int_X \int_{B(x,r)} \frac{|f(x)-f(y)|}{r^\lambda \mu(B(x,r))} d\mu(y)\, d\mu(x)$$

and the critical exponent

$$\lambda_1^\# := \sup\{\lambda > 0 : \ KS^{\lambda,1}(X) \text{ contains non-constant functions.}\}.$$

It was proved in [6] that $KS^{1,1}(\mathbb{R}^d) = BV(\mathbb{R}^d)$ and $\lambda_1^\# = 1$, which motivates the following general definition of the space of BV functions and the variation of a function

$$BV(X) := KS^{\lambda_1^\#,1}(X)$$

$$\mathbf{Var}_{\mathcal{E}}(f) := \liminf_{r\to 0+} \int_X \int_{B(x,r)} \frac{|f(x)-f(y)|}{r^{\lambda_1^\#} \mu(B(x,r))} d\mu(y)\, d\mu(x).$$

As a combination of several ideas due to de Giorgi [4] and Ledoux [7], Miranda et al. provided in [8] a semigroup approach to BV functions on $\mathbb{R}^d$. Based on them, we introduced in [1] the Besov-type space

$$\mathbf{B}^{1,\alpha}(X) := \{f \in L^1(X,\mu) : \|f\|_{1,\alpha} < \infty\},$$

where

$$\|f\|_{1,\alpha} := \sup_{t>0} \frac{1}{t^\alpha} \int_X P_t(|f - f(x)|)(x)\, d\mu(x)$$

as well as the critical exponent

$$\alpha_1^\# := \sup\{\alpha > 0 : \mathbf{B}^{\lambda,1}(X) \text{ contains non-constant functions.}\}.$$

It was proved in [8] that $\mathbf{B}^{1,1/2}(\mathbb{R}^d) = BV(\mathbb{R}^d)$ and $\alpha_1^\# = 1/2$. Notice that, in particular, $\alpha_1^\# = \lambda_1^\#/2$.

The correspondence between the metric-measure theoretic and the heat semigroup approach in the framework of local Dirichlet spaces was obtained in [2] under the following *weak Bakry-Émery non-negative curvature condition*

$$|P_t f(x) - P_t f(y)| \leq C \frac{d(x,y)^\kappa}{t^{\kappa/d_W}} \|f\|_{L^\infty(X,\mu)} \tag{1}$$

for all $t > 0$, $f \in \mathcal{F} \cap L^\infty(X,\mu)$ and $\kappa = d_W - \lambda_1^\#$. In that case, it was possible to prove that $BV(X) = \mathbf{B}^{1,\alpha_1^\#}(X)$. In particular, $\alpha_1^\# = \lambda_1^\#/d_W$ and

$$\mathbf{Var}_\mathcal{E}(f) \simeq \|f\|_{1,\alpha_1^\#} \simeq \liminf_{t\to 0^+} t^{-\alpha_1^\#} \int_X P_t(|f - f(x)|)(x)\, d\mu(x). \tag{2}$$

With these results in hand, we are now ready to move towards the non-local case. In the light of the classical definition of fractional Sobolev spaces due to Gagliardo [5], we may define their analogue in our setting by

$$W^{s,1}(X) := \Big\{ f \in L^1(X,\mu) : \int_X \int_X \frac{|f(x) - f(y)|}{d(x,y)^{d_H + s}} d\mu(y)\, d\mu(x) < \infty \Big\}.$$

Let now $(X, d, \mu, \mathcal{E}, \mathcal{F})$ be the local Dirichlet space we started with and fix $\delta \in (0,1)$. In the sequel we will be dealing with the corresponding subordinated semigroup $\{P_t^{(\delta)}\}_{t>0}$ whose associated heat kernel satisfies the two-sided estimate

$$p_t^{(\delta)}(x,y) \asymp t \cdot (t^{\frac{1}{\delta d_W}} + c d(x,y))^{-d_H - \delta d_W}.$$

A crucial observation concerns the effect of the subordination in the weak Bakry-Émery condition (1). Namely, the subordinated space now satisfies

$$|P_t f(x) - P_t f(y)| \leq C \frac{d(x,y)^\kappa}{t^{\kappa/(\delta d_W)}} \|f\|_{L^\infty(X,\mu)}.$$

Under these assumptions, we can finally characterize the Besov-type space as follows:

$$\mathbf{B}^{1,\alpha}(X) = \begin{cases} KS^{\alpha\delta d_W,1}(X) & \text{if } \alpha \in (0,1), \\ W^{\delta d_W,1}(X) & \text{if } \alpha = 1, \\ \{\text{constant function } 0\} & \text{if } \alpha > 1. \end{cases}$$

In addition, the critical exponent reads $\alpha_1^{\#} = \min\left\{1, \frac{1}{\delta}\left(1 - \frac{\kappa}{d_W}\right)\right\}$, whence

$$\mathbf{B}^{1,\alpha_1^{\#}}(X) = \begin{cases} BV(X) & \text{if } \delta > 1 - \frac{\kappa}{d_W}, \\ W^{\delta d_W, 1}(X) & \text{if } \delta < 1 - \frac{\kappa}{d_W}, \\ \{\text{constant function } 0\} & \text{if } \delta = 1 - \frac{\kappa}{d_W}, \end{cases}$$

and the variation of a function will satisfy (2) when $\delta > 1 - \frac{\kappa}{d_W}$, and

$$\mathbf{Var}_{\mathcal{E}}(f) \simeq \int_X \int_X \frac{|f(x) - f(y)|}{d(x,y)^{d_H + \delta d_W}} d\mu(y)\, d\mu(x)$$

when $\delta < 1 - \frac{\kappa}{d_W}$.

We finish our discussion with a particular application of this notion of BV functions, that concerns the isoperimetric inequality available in this framework. As in the classical theory, we may say that a Borel measurable set $E \subseteq X$ has finite perimeter if the indicator function of the set has bounded variation, i.e. $\mathbf{1}_E \in BV(X)$. In the present non-local setting, the inequality reads

$$\mu(E)^{\frac{d_H - \delta d_W}{d_H}} \leq C \int_E \int_{X \setminus E} \frac{1}{d(x,y)^{d_H + \delta d_W}} d\mu(y)\, d\mu(x). \tag{3}$$

We note that the latter is only meaningful in the range $0 < \delta < 1 - \frac{\kappa}{d_W}$; otherwise the right hand side need not converge. In the case of $X = \mathbb{R}^d$, the parameters are $d_H = d$, $d_W = 2$ and $\kappa = 1$, so that (3) holds for $0 < \delta < 1/2$. The question about existence of sets of bounded perimeter is directly related to that of non-local minimal surfaces, studied for instance in [3].

References

[1] P. Alonso Ruiz, F. Baudoin, L. Chen, L. Rogers, N. Shanmugalingam, A. Teplyaev, *Besov class via heat semigroup on Dirichlet spaces I: Sobolev type inequalities*, arXiv:1811.04267 (2018)

[2] P. Alonso Ruiz, F. Baudoin, L. Chen, L. Rogers, N. Shanmugalingam, A. Teplyaev, *Besov class via heat semigroup on Dirichlet spaces III: BV functions and sub-Gaussian heat kernel estimates*, arXiv:1903.10078 (2019)

[3] L. Caffarelli, J. M. Roquejoffre, O. Savin, *Nonlocal minimal surfaces*, Comm. Pure Appl. Math. (9) **63** (2010), 1111–1144.

[4] E. De Giorgi, *Su una teoria generale della misura $(r-1)$-dimensionale in uno spazio ad r dimensioni*, Ann. Mat. Pura Appl. (4) **36** (1954), 191–213.

[5] E. Gagliardo, *Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili*, Rend. Sem. Mat. Univ. Padova. **27** (1957), 284–305.

[6] N. J. Korevaar, R. M. Schoen, *Sobolev spaces and harmonic maps for metric space targets*, Comm. Anal. Geom. (3-4) **1** (1993), 561–659.

[7] M. Ledoux, *Semigroup proofs of the isoperimetric inequality in Euclidean and Gauss space*, Bull. Sci. Math. (6) **118** (1994), 485–510.

[8] Jr. M. Miranda, , D. Pallara, F. Paronetto, M. Preunkert, *Heat semigroup and functions of bounded variation on Riemannian manifolds*, J. Reine Angew. Math. **613** (2007), 99–119.

Local Limit Theorems for the Random Conductance Model

Sebastian Andres
(joint work with Peter Taylor)

We consider the *random conductance model (RCM)*, which is a well established model for a random walk in random environment. For $d \geq 2$ let $(\mathbb{Z}^d, E_d)$ be the Euclidean lattice equipped with non-oriented nearest neighbour bonds. We endow the graph with positive random weights $\{\omega(e), e \in E_d\}$, where we refer to $\omega(e)$ as the *conductance* of an edge $e \in E_d$. Let $\theta^\omega : \mathbb{Z}^d \to (0, \infty)$ be a positive function which may depend upon the environment $\omega \in \Omega$. For any fixed realisation of conductances consider the continuous time Markov chain $X = \{X_t : t \geq 0\}$ on $\mathbb{Z}^d$ with generator

$$\mathcal{L}_\theta^\omega f(x) := \frac{1}{\theta^\omega(x)} \sum_{y \sim x} \omega(x,y) \left(f(y) - f(x)\right),$$

acting on bounded functions $f : \mathbb{Z}^d \to \mathbb{R}$. This random walk is reversible with respect to the *speed measure* θ^ω. We denote by P_x^ω the law of this process started at $x \in \mathbb{Z}^d$. There are two natural laws on the path space - the quenched law $P_x^\omega(\cdot)$ which concerns $\mathbb{P}$-almost sure phenomena, and the annealed law $\mathbb{E}P_x^\omega(\cdot)$.

The random walk X chooses its next position with probability $\omega(x,y)/\mu^\omega(x)$, after waiting an exponential time with mean $\theta^\omega(x)/\mu^\omega(x)$ at the vertex x where $\mu^\omega(x) := \sum_{y \sim x} \omega(x,y)$. Our main results are statements about the heat kernel of X, which is defined as

$$p_\theta^\omega(t,x,y) \;:=\; \frac{P_x^\omega\left(X_t = y\right)}{\theta^\omega(y)} \qquad \text{for } t \geq 0 \text{ and } x, y \in \mathbb{Z}^d.$$

Perhaps the most natural choice for the speed measure is $\theta^\omega \equiv \mu^\omega$, for which we obtain the constant speed random walk (CSRW) that spends i.i.d. Exp(1)-distributed waiting times at all vertices it visits. Another well-studied process, the variable speed random walk (VSRW) is recovered by setting $\theta^\omega \equiv 1$, so called because as opposed to the CSRW, the waiting time at a vertex x does indeed depend on the location; it is an $\mathrm{Exp}(\mu^\omega(x))$-distributed random variable. In what follows we will consider a general θ^ω assuming it is stationary and satisfies $\mathbb{E}[\theta^\omega(0)] < \infty$ and $\mathbb{E}[\theta^\omega(0)/\mu^\omega(0)] \in (0, \infty)$.

In the study of the random conductance model the question whether a *functional central limit theorem (FCLT)* holds has been object of very active research, see the surveys [5, 7] and references therein. One recent result for general ergodic environments is the following.

Theorem 1 (Quenched FCLT [2])**.** *Suppose $d \geq 2$. Let $(\omega_e)_{e \in E_d}$ be stationary ergodic and $p, q \in (1, \infty]$ be such that $1/p + 1/q < 2/d$ and assume that $\mathbb{E}\big[(\omega_e)^p\big] < \infty$ and $\mathbb{E}\big[(\omega_e)^{-q}\big] < \infty$ for any $e \in E_d$. Then, for $\mathbb{P}$-a.e. ω, the rescaled process $X_t^{(n)} := \frac{1}{n} X_{n^2 t}$ converges (under P_0^ω) in law to a Brownian motion on $\mathbb{R}^d$ with a deterministic non-degenerate covariance matrix Σ^2.*

We are now interested in a deriving a local limit theorem which roughly describes how the transition probabilities of the random walk X can be rescaled in order to get the Gaussian transition density of the Brownian motion with covariance matrix Σ^2 appearing as the limit process in the FCLT in Theorem 1. The Gaussian heat kernel associated with that process will be denoted

$$k_t(x) \equiv k_t^{\Sigma}(x) := \frac{1}{\sqrt{(2\pi t)^d \det \Sigma^2}} \exp\left(-x \cdot (\Sigma^2)^{-1} x/2t\right).$$

In general, a local limit theorem is a stronger statement than an FCLT. Define the two measures $\mu^\omega(x) := \sum_{y\sim x} \omega(x,y)$ and $\nu^\omega(x) := \sum_{y\sim x} \frac{1}{\omega(x,y)}$ on $\mathbb{Z}^d$. Further, for $x \in \mathbb{R}^d$ write $\lfloor x \rfloor = (\lfloor x_1 \rfloor, ..., \lfloor x_d \rfloor) \in \mathbb{Z}^d$.

Theorem 2 (Quenched local limit theorem [4])**.** *Suppose $d \geq 2$. Let $(\omega_e)_{e\in E_d}$ be stationary ergodic and $p, q, r \in (1, \infty]$ satisfying*

$$\frac{1}{r} + \frac{1}{p}\frac{r-1}{r} + \frac{1}{q} < \frac{2}{d}.$$

Assume that

$$\mathbb{E}\left[\left(\frac{\mu^\omega(0)}{\theta^\omega(0)}\right)^p \theta^\omega(0)\right] + \mathbb{E}\left[\nu^\omega(0)^q\right] + \mathbb{E}\left[\theta^\omega(0)^{-1}\right] + \mathbb{E}\left[\theta^\omega(0)^r\right] \;<\; \infty.$$

Then, for any $T_2 > T_1 > 0$ and $K > 0$,

$$\lim_{n\to\infty} \sup_{|x|\leq K} \sup_{t\in[T_1,T_2]} \left| n^d p_\theta^\omega(n^2 t, 0, \lfloor nx \rfloor) - a k_t(x) \right| = 0, \quad \textit{for } \mathbb{P}\textit{-a.e. } \omega,$$

with $a := \mathbb{E}\left[\theta^\omega(0)\right]^{-1}$.

This result extends the local limit theorem in [3, Theorem 1.11] for the CSRW to the case of a general speed measure. Note that in the case of the CSRW or VSRW the moment condition in Theorem 2 coincides with the one in Theorem 1. For the proof of Theorem 2 we adapt Di Giorgi iteration techniques from [8] to derive Hölder regularity of solutions to parabolic PDEs in continuum. This, along with the FCLT, is precisely what is required to prove a local limit theorem. We stress that this approach to show Hölder regularity directly circumvents the need for a parabolic Harnack inequality, in contrast to the proof in [3], which makes it significantly simpler.

Next we consider the *dynamic random conductance model*. We define the dynamic variable speed random walk starting in $x \in \mathbb{Z}^d$ at $s \in \mathbb{R}$ to be the continuous-time Markov chain $(X_t : t \geq s)$ with time-dependent generator

$$\left(\mathcal{L}_t^\omega f\right)(x) := \sum_{y\sim x} \omega_t(x,y)\left(f(y) - f(x)\right),$$

acting on bounded functions $f : \mathbb{Z}^d \to \mathbb{R}$. Note that the counting measure, which is time-independent, is an invariant measure for X. We denote $P_{s,x}^\omega$ the law of this process started at $x \in \mathbb{Z}^d$ at time s. For $x, y \in \mathbb{Z}^d$ and $t \geq s$, we denote by $p^\omega(s,t,x,y) := P_{s,x}^\omega\left[X_t = y\right]$ the heat kernel of $(X_t)_{t\geq s}$. We establish an annealed local limit theorem for the dynamic RCM under a stronger, non-optimal but polynomial moment condition.

Theorem 3 (Annealed local limit theorem [4]). *Suppose $d \geq 2$ and let $(\omega_t(e)$ be space-time ergodic. There exist exponents $p, q \in (1, \infty)$ such that if*

$$\mathbb{E}\big[\omega_0(e)^p\big] \ < \ \infty \quad \text{and} \quad \mathbb{E}\big[\omega_0(e)^{-q}\big] \ < \ \infty$$

for any $e \in E_d$, the following holds. For all $K > 0$ and $0 < T_1 \leq T_2$,

$$\mathbb{E}\bigg[\sup_{|x| \leq K} \sup_{t \in [T_1, T_2]} \Big| n^d p^\omega(0, n^2 t, 0, \lfloor nx \rfloor) - k_t(x) \Big| \bigg] = 0.$$

In general, a quenched FCLT does imply an annealed FCLT. However, the same does not apply to the local limit theorem. In fact, as mentioned above, the proofs of the quenched local limit theorems in [3] and Theorem 2 rely on Hölder regularity estimates on the heat kernel, which involve some random constants depending on the exponential of the conductances. Those constants can be controlled almost surely, but naively taking expectations would require exponential moment conditions. To derive the annealed local limit theorem given the corresponding quenched result, one might hope to employ the dominated convergence theorem, which requires that the integrand above can be dominated uniformly in n by a function of finite expectation. We achieve this using a quenched maximal inequality from [1]. It is precisely the form of the random constants in this inequality that allows us to anneal the result using only polynomial moments, together with a simple probabilistic bound.

A somewhat unexpected context in which one encounters (dynamic) RCMs is the Ginzburg-Landau $\nabla\phi$-interface model, see [6] for a survey. Mathematically, the interface is described by a field of height variables $\{\phi_t(x) : x \in \mathbb{Z}^d, t \geq 0\}$ with dynamics given by the following infinite system of SDEs:

$$\phi_t(x) = \phi_0(x) - \int_0^t \sum_{y:|x-y|=1} V'(\phi_t(x) - \phi_t(y))\, dt + \sqrt{2}\, w_t(x), \qquad x \in \mathbb{Z}^d,$$

where $\{w(x) : x \in \mathbb{Z}^d\}$ is a collection of independent Brownian motions and the potential $V \in C^2(\mathbb{R}, \mathbb{R}_+)$ is even and convex. The formal equilibrium measure for the dynamic is given by the Gibbs measure $Z^{-1} \exp(-H(\phi)) \prod_x d\phi(x)$ on $\mathbb{R}^{\mathbb{Z}^d}$ with formal Hamiltonian $H(\phi) = \frac{1}{2} \sum_{x \sim y} V(\phi(x) - \phi(y))$. We are interested in the decay of the space-time covariances of height variables under an equilibrium Gibbs measure. By the *Helffer-Sjöstrand representation* such covariances can be written in terms of the *annealed* heat kernel of a random walk among dynamic random conductances. More precisely,

$$\operatorname{cov}_\mu\big(\phi_0(0), \phi_t(y)\big) = \int_0^\infty \mathbb{E}_\mu \left[p^\omega_{0,t+s}(0, y)\right] ds,$$

where the covariance and expectation are taken with respect to an ergodic Gibbs measure μ and p^ω denotes the heat kernel of the dynamic RCM with time-dependent conductances given by

$$\omega_t(x,y) := V''\big(\phi_t(y) - \phi_t(x)\big), \qquad \{x,y\} \in E_d,\, t \geq 0.$$

Thus far, all applications of the aforementioned Helffer-Sjöstrand relation have been restricted to gradient models with strictly convex potential function, which corresponds to uniformly elliptic conductances in the random walk picture. However, recent developments in the degenerate setting will also allow some potentials that are convex but *not strictly convex*. As an example in this direction, we use the annealed local limit theorem in Theorem 3 to derive a scaling limit for the space-time covariances of the ϕ-field for a wider class of potentials.

Theorem 4 ([4])**.** *Let $d \geq 3$ and $V'' \geq c_-$. There exists $p \in (1,\infty)$ such that if $\mathbb{E}_\mu[V''(\nabla\phi_t(e))^p] < \infty$ under any stationary, shift-invariant, ergodic ϕ-Gibbs measure μ, then for all $t > 0$ and $x \in \mathbb{R}^d$,*

$$\lim_{n\to\infty} n^{d-2}\mathrm{cov}_\mu\left(\, \phi_0(0),\, \phi_{n^2 t}(\lfloor nx \rfloor)\,\right) = \int_0^\infty k_{t+s}(x)\, ds,$$

where k_t is the heat kernel of a Brownian motion on $\mathbb{R}^d$ with a deterministic non-degenerate covariance matrix.

The moment condition in Theorem 4 on the potential V is satisfied for any V with V'' having polynomial growth. Hence, Theorem 4 applies, for instance, to the anharmonic crystal potential $V(x) = x^2 + \lambda x^4$ for $\lambda > 0$.

References

[1] S. Andres, A. Chiarini, J.-D. Deuschel, and M. Slowik. Quenched invariance principle for random walks with time-dependent ergodic degenerate weights. *Ann. Probab.*, 46(1):302–336, 2018.

[2] S. Andres, J.-D. Deuschel, and M. Slowik. Invariance principle for the random conductance model in a degenerate ergodic environment. *Ann. Probab.* **43**, no. 4, 1866–1891 (2015).

[3] S. Andres, J.-D. Deuschel, and M. Slowik. Harnack inequalities on weighted graphs and some applications to the random conductance model. *Probability Theory and Related Fields*, 164(3-4):931–977, 2016.

[4] S. Andres and P. Taylor. Local limit theorem for the random conductance model and applications to the Ginzburg-Landau $\nabla\phi$ interface model. *Preprint, available at arXiv:1907.05311*, 2019.

[5] M. Biskup. Recent progress on the random conductance model. *Probab. Surv.* **8**, 294–373 (2011).

[6] T. Funaki. Stochastic interface models. In *Lectures on probability theory and statistics*, volume 1869 of *Lecture Notes in Math.*, pages 103–274. Springer, Berlin, 2005.

[7] T. Kumagai. *Random walks on disordered media and their scaling limits*, volume 2101 of *Lecture Notes in Mathematics*. Springer, Cham, 2014.

[8] Z. Wu, J. Yin, and C. Wang. *Elliptic & parabolic equations*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006.

Isoperimetric inequalities in Dirichlet spaces

Fabrice Baudoin

(joint work with P. Alonso Ruiz, L. Chen, L. Rogers, N. Shanmugalingam, A. Teplyaev)

A basic motivating geometric question for this work is: What is the good mathematical structure on a space that allows to define an intuitively reasonable notion of perimeter for "good" sets? In [1, 2, 3, 4] we argue that Dirichlet spaces provide a good framework in which we can define sets of finite perimeter and prove theorems generalizing in an elegant way classical results from the Euclidean space, like the classical isoperimetric inequality.

Let X be a good measurable space (like a Polish space) equipped with a σ-finite measure μ. Let $(\mathcal{E}, \mathcal{F} = \mathbf{dom}(\mathcal{E}))$ be a densely defined closed symmetric form on $L^2(X,\mu)$. A function v on X is called a normal contraction of the function u if for almost every $x, y \in X$ $|v(x) - v(y)| \le |u(x) - u(y)|$ and $|v(x)| \le |u(x)|$. The form $\mathcal{E}$ is called a Dirichlet form if it is Markovian, that is, has the property that if $u \in \mathcal{F}$ and v is a normal contraction of u then $v \in \mathcal{F}$ and $\mathcal{E}(v,v) \le \mathcal{E}(u,u)$. Let $\{P_t\}_{t\in[0,\infty)}$ denote the self-adjoint heat semigroup on $L^2(X,\mu)$ associated with the Dirichlet space $(X, \mu, \mathcal{E}, \mathcal{F})$: $\mathcal{E}(f,f) = \lim_{t\to 0^+} \frac{1}{t}\langle (I - P_t)f, f\rangle$. As is well-known, $P_t : L^2(X,\mu) \cap L^p(X,\mu) \to L^p(X,\mu)$, $1 \le p \le \infty$, can be extended into a contraction semigroup $P_t : L^p(X,\mu) \to L^p(X,\mu)$. We assume $P_t 1 = 1$.

For $\alpha > 0$, consider the L^1 Besov type space

$$\mathbf{B}^{1,\alpha}(\mathcal{E}) = \left\{ f \in L^1(X,\mu), \limsup_{t\to 0} \frac{1}{t^\alpha} \int_X P_t(|f - f(y)|) d\mu(y) < +\infty \right\}$$

and

$$\alpha_1^\#(\mathcal{E}) = \sup\{\alpha > 0 : \mathbf{B}^{1,\alpha}(\mathcal{E}) \text{ contains non a.e. constant functions}\}.$$

The space of bounded variation functions associated to the Dirichlet form $\mathcal{E}$ is defined as $BV(\mathcal{E}) = \mathbf{B}^{1,\alpha^\#}(\mathcal{E})$ and for $f \in BV(\mathcal{E})$, one defines its variation as

$$\mathbf{Var}_\mathcal{E}(f) = \liminf_{t\to 0} \frac{1}{t^\alpha} \int_X P_t(|f - f(y)|) d\mu(y).$$

A set $E \subset X$ is called a $\mathcal{E}$-Caccioppoli set if $1_E \in BV(\mathcal{E})$. In that case, its $\mathcal{E}$-perimeter is defined as $P_\mathcal{E}(E) = \mathbf{Var}_\mathcal{E}(1_E)$.

- **Euclidean space:** The following can be deduced from M. Miranda Jr, D. Pallara, F. Paronetto, M. Preunkert, 2007. Assume that $\mathcal{E}$ is the standard Dirichlet form on $\mathbb{R}^n$, $\mathcal{E}(f,f) = \int_{\mathbb{R}^n} \|\nabla f\|^2 dx$, $\quad f \in W^{1,2}(\mathbb{R}^n)$, then $\alpha_1^\#(\mathcal{E}) = \frac{1}{2}$, $BV(\mathcal{E}) = \mathbf{BV}(\mathbb{R}^n)$ and for $f \in BV(\mathcal{E})$, $\mathbf{Var}_\mathcal{E}(f) = \frac{2}{\sqrt{\pi}} \|Df\|(\mathbb{R}^n)$.
- **Sierpinski triangle:** Consider on the Sierpinski triangle SG the Dirichlet form

$$\mathcal{E}(f) \simeq \limsup_{r\to 0^+} \frac{1}{r^{d_W}} \int_{SG} \int_{B(x,r)} \frac{|f(y) - f(x)|^2}{\mu(B(x,r))} d\mu(y)\, d\mu(x)$$

where d_W is the walk dimension of the Sierpinski triangle. Then $\alpha_1^{\#}(\mathcal{E}) = \frac{d_H}{d_W}$, where d_H is the Hausdorff dimension of the Sierpinski triangle and

$$\mathbf{Var}_{\mathcal{E}}(f) \simeq \liminf_{r\to 0^+} \int_{SG}\int_{B(x,r)} \frac{|f(y)-f(x)|}{r^{d_H}\mu(B(x,r))}\, d\mu(y)\, d\mu(x)$$

A set $E \subset SG$ is a $\mathcal{E}$-Caccioppoli set if its boundary is finite.

- **Riemannian manifolds:** Assume that $\mathcal{E}$ is the standard Dirichlet form on a complete Riemannian manifold $\mathbb{M}$ with Ricci curvature bounded from below: $\mathcal{E}(f,f) = \int_{\mathbb{M}} \|\nabla f\|^2\, dx, \quad f \in W^{1,2}(\mathbb{M})$, then $\alpha_1^{\#}(\mathcal{E}) = \frac{1}{2}$, $BV(\mathcal{E}) = \mathbf{BV}(\mathbb{M})$ and for $f \in BV(\mathcal{E})$,

$$\mathbf{Var}_{\mathcal{E}}(f) \simeq \|Df\|(\mathbb{M}).$$

In the case of Riemannian manifolds, the space $\mathbf{BV}(\mathbb{M})$ and the associated notion of variation $\|Df\|(\mathbb{M})$ we are using are for instance presented in the paper: Heat semigroup and functions of bounded variation on Riemannian manifolds by M. Miranda Jr, D. Pallara, F. Paronetto & M. Preunkert.

Let now $(X,\mu,\mathcal{E},\mathcal{F})$ be a Dirichlet space. We consider the following property:

$$\mathcal{P}_\infty : \qquad \mathbf{Var}_{\mathcal{E}}(f) \simeq \sup_{t>0} \frac{1}{t^{\alpha_1^{\#}(\mathcal{E})}} \int_X P_t(|f - f(y)|) d\mu(y).$$

Theorem: (Weak Bakry-Emery estimates I) *Let $(X,\mu,\mathcal{E},\mathcal{F})$ be a strictly local metric Dirichlet space that is locally doubling and that locally supports a 2-Poincaré inequality on balls. If there exists a constant $C>0$ such that*

$$\| |\nabla P_t f| \|_{L^\infty(X,\mu)} \le \frac{C}{\sqrt{t}} \|f\|_{L^\infty(X,\mu)}, \quad t>0.$$

Then, $\alpha_1^{\#}(\mathcal{E}) = \frac{1}{2}$ and $\mathcal{P}_\infty$ is satisfied.

The theorem applies to $RCD(0,\infty)$ spaces, Carnot groups and large classes of sub-Riemannian manifolds with non-negative Ricci curvature in the sense of Baudoin-Garofalo.

Theorem: (Weak Bakry-Emery estimates II) *Let $(X,\mu,\mathcal{E},\mathcal{F})$ be a metric Dirichlet space with a heat kernel admitting sub-Gaussian estimates. If there exists a constant $C>0$ such that $|P_t f(x) - P_t f(y)| \le C\frac{d(x,y)^\kappa}{t^{\kappa/d_W}} \|f\|_{L^\infty(X,\mu)}, \quad t>0.$ where $\kappa = d_W(1-\alpha_1^{\#}(\mathcal{E}))$ then $\mathcal{P}_\infty$ is satisfied.*

This applies to the unbounded Sierpinski triangle and their products and large classes of fractals or products of fractals. This is however a conjecture on the Sierpinski carpet.

Theorem: *Assume $\mathcal{P}_\infty$ is satisfied and that P_t admits a measurable heat kernel $p_t(x,y)$ satisfying, for some $C>0$ and $\beta>0$,*

$$p_t(x,y) \le Ct^{-\beta}, \quad t>0.$$

Then, if $0 < \alpha_1^{\#}(\mathcal{E}) < \beta$*, there exists a constant* $C > 0$ *such that for every* $f \in BV(\mathcal{E})$*,*

$$\|f\|_{L^q(X,\mu)} \leq C\mathbf{Var}_{\mathcal{E}}(f),$$

where $q = \frac{\beta}{\beta - \alpha_1^{\#}(\mathcal{E})}$.

Under the assumptions of this theorem, one therefore obtains the following general isoperimetric inequality for Caccioppoli sets in Dirichlet spaces

$$\mu(E)^{\frac{\beta-\alpha_1^{\#}(\mathcal{E})}{\beta}} \leq CP_{\mathcal{E}}(E).$$

It generalizes the isoperimetric inequality which was known in Riemannian manifolds or Carnot groups (due to N. Varopoulos) but also applies to new situations like fractals.

References

[1] Patricia Alonso Ruiz, Fabrice Baudoin, Li Chen, Luke Rogers, Nageswari Shanmugalingam, Alexander Teplyaev: *Besov class via heat semigroup on Dirichlet spaces I: Sobolev type inequalities*, 2018, in revision for JFA,

[2] Patricia Alonso Ruiz, Fabrice Baudoin, Li Chen, Luke Rogers, Nageswari Shanmugalingam, Alexander Teplyaev: *Besov class via heat semigroup on Dirichlet spaces II: BV functions and Gaussian heat kernel estimates*, 2018

[3] Patricia Alonso Ruiz, Fabrice Baudoin, Li Chen, Luke Rogers, Nageswari Shanmugalingam, Alexander Teplyaev: *Besov class via heat semigroup on Dirichlet spaces III: BV functions and sub-Gaussian heat kernel estimates*, 2018

[4] Patricia Alonso Ruiz, Fabrice Baudoin, Li Chen, Luke Rogers, Nageswari Shanmugalingam, Alexander Teplyaev: *BV functions and fractional Laplacians on Dirichlet spaces*, 2019

Harnack inequalities for balanced environments

Noam Berger

(joint work with Moran Cohen, David Criens, Jean-Dominique Deuschel and Xiaoqin Guo)

In this talk consider difference equations in balanced random environments. To set a precise model we let E be the set of (positive and negative) unit vectors in the lattice $\mathbb{Z}^d$ and let Ω be the set of all functions (later referred to as 'environments') ω from $\mathbb{Z}^d \times E$ to $\mathbb{R}$, satisfying

(1) (ω is an environment for a random walk)

$$\forall_{x\in\mathbb{Z}^d, e\in E}\omega(x,e) \geq 0 \quad ; \quad \forall_{x\in\mathbb{Z}^d}\sum_{e\in E}\omega x, e = 1.$$

(2) (ω is balanced)

$$\forall_{x\in\mathbb{Z}^d, e\in E}\omega(x,e) = \omega(x,-e).$$

In particular, we wish to study ω-harmonic functions, i.e. functions $f : A\bigcup\partial A \to \mathbb{R}$, $A \subseteq \mathbb{Z}^d$ satisfying

$$\forall_{x\in A} \sum_{e\in E} \omega x, e\big(f(x+e) - f(x)\big) = 0$$

and ω-caloric functions, i.e. functions $f : \big(A\bigcup\partial A\big) \times \big([0,T]\bigcap\mathbb{Z}\big) \to \mathbb{R}$, satisfying

$$\forall_{x\in A,0<t<T} \sum_{e\in E} \omega x, e\big(f(x+e,t) - f(x,t-1)\big) = 0$$

These were first considered by Lawler [5] and by Kuo and Trudinger [3] who, independently and simultaneously, proved Harnack inequalities in uniform elliptic cases. Here uniform ellipticity means that the values of ω are bounded away from zero.

closely related to the behaviour of harmonic and caloric functions is the behavior of the random walk on the environment ω This has been studied by Lawler [4], by Guo and Zeitouni [2] and by Berger and Deuschel [1].

Lawler proved an invariance principle for uniformly elliptic environments under the very general condition of stationarity and ergodicity. The methods followed those of Papanicolaou and Varadhan [6] who proved similar results in continuous settings. However, once one relaxes enough the condition of uniform ellipticity, stationarity and ergodicity would no longer suffice to prove invariance principles or Harnack inequalities, as many counter examples show. Thus, Guo and Zeitouni showed an invariance principle for iid environments that are merely elliptic, and Berger and Deuschel for iid environments that are non-elliptic. In the latter result, the assumption of iid cannot be relaxed.

Given the results on invariance principles, we set to prove the (harder) Harnack inequalities. We started by proving an elliptic Harnack inequality, and could prove that, in the iid case, there exists a Harnack constant C s.t. a.s. there exists a (random) finite R_0 s.t. for all $R > R_0$ and all non-negative ω-harmonic function $f : B_{2R} \to \mathbb{R}$, we have

$$\sup_{x\in B_R} f(x) \leq C \inf_{x\in B_R} f(x),$$

and our Harnack constant C is optimal.

We then proved a parabolic Harnack inequality. Here the situation is more complex: In the same setting, there exists a Harnack constant C s.t. a.s. there exists a (random) finite R_0 s.t. for all $R > R_0$ and all non-negative ω-caloric function $f : B_{2R} \times [0,R^2] \to \mathbb{R}$, satisfying the mild growth condition

$$\sup_{(x,t)\in B_{2R}\times[0,R^2]} f(x,t) \leq e^{R^{2-\epsilon}} \inf_{(x,t)\in B_{2R}\times[0,R^2]} f(x,t), \tag{1}$$

we have

$$\sup_{(x,t)\in B_R\times[R^2/4,R^2/2]} f(x,t) \leq C \inf_{(x,t)\in B_R\times[3R^2/4,R^2]} f(x,t),$$

and our Harnack constant C is optimal.

It turns out that the growth condition (1) is necessary (and optimal). The growth condition stems from percolation theoretical properties of non-elliptic environments.

References

[1] Noam Berger and Jean-Dominique Deuschel. A quenched invariance principle for non-elliptic random walk in iid balanced random environment. *Probability Theory and Related Fields*, 158(1-2):91–126, 2014.

[2] Xiaoqin Guo and Ofer Zeitouni. Quenched invariance principle for random walks in balanced random environment. *Probability Theory and Related Fields*, 152(1-2):207–230, 2012.

[3] Hung Ju Kuo and Neil S Trudinger. Linear elliptic difference inequalities with random coefficients. *Mathematics of computation*, 55(191):37–53, 1990.

[4] Gregory F Lawler. Weak convergence of a random walk in a random environment. *Communications in Mathematical Physics*, 87(1):81–87, 1982.

[5] Gregory F Lawler. Estimates for differences and harnack inequality for difference operators coming from random walks with symmetric, spatially inhomogeneous, increments. *Proceedings of the London Mathematical Society*, 3(3):552–568, 1991.

[6] George C Papanicolaou and Srinivasa RS Varadhan. Diffusions with random coefficients. *Statistics and probability: essays in honor of CR Rao*, pages 547–552, 1982.

Exceptional points of random walks in planar domains

Marek Biskup

(joint work with Yoshihiro Abe, Sangchul Lee)

Motivated by the seminal work of Erdös and Taylor [7] and Dembo, Peres, Rosen and Zeitouni [5] on the most visited and thick points of planar random walks, we study various exceptional sets associated with the local time of random walks in planar domains at times of order of the cover time. More specifically, for lattice domains $D_N \subset \mathbb{Z}^2$ approximating, in the sense of vague convergence of the scaled harmonic measure, a nice continuum domain $D \subset \mathbb{R}^2$, we study discrete-time simple symmetric random walk $X = \{X_k : k \geq 0\}$ on a graph with vertex set $D_N \cup \{\varrho\}$, where ϱ is a boundary vertex obtained by collapsing $\mathbb{Z}^2 \setminus D_N$ while keeping all edges incident with D_N.

Denoting by

$$\ell_n(x) := \sum_{k=0}^{n} 1_{\{X_k = x\}} \tag{1}$$

the total time the walk spends at x in the first n steps, we are interested in the (time-reparametrized) local time

$$L_t(x) := \frac{1}{\deg(x)} \ell_{\lfloor t \deg(\overline{D}_N) \rfloor}(x), \tag{2}$$

where $\deg(x)$ is the degree of x in the resulting graph and $\deg(\overline{D}_N)$ is the sum of all degrees, including the one at ϱ. We will observe the walk at times that are proportional, in the limit as $N \to \infty$ and with the constant of proportionality denoted by θ, to the cover time of the whole graph. This corresponds to taking t to infinity along a sequence $\{t_N\}_{N\geq 1}$ such that

$$\lim_{N\to\infty} \frac{t_N}{(\log N)^2} = \frac{1}{\pi}\theta. \tag{3}$$

The value $\theta := 1$ then corresponds to the leading-order scaling of the cover time.

In our parametrization, a typical value of L_{t_N} is $t_N + O(\sqrt{t_N})$, while the maximal value is $(\sqrt{\theta}+1)^2/\theta$-multiple and the smallest value is $((\sqrt{\theta}-1)\vee 0)^2/\theta$-multiple thereof, with probability tending to one. Motivated by this, we designate $x \in D_N$ to be a λ-*thick point* if

$$L_{t_N}(x) \geq \frac{1}{\pi}(\sqrt{\theta}+\lambda)^2(\log N)^2, \quad \lambda \in (0,1], \tag{4}$$

and a λ-*thin point* if

$$L_{t_N}(x) \leq \frac{1}{\pi}\big((\sqrt{\theta}-\lambda)\vee 0\big)^2(\log N)^2, \quad \lambda \in (0,\sqrt{\theta}\,]. \tag{5}$$

We are also able to address the r-*light points*, which are those where $L_{t_N}(x) \leq r$, and the *avoided points* which are those not visited at all, i.e., $L_{t_N}(x) = 0$.

A convenient way to study these exceptional points is via vague convergence of random measures. For the thick and thin points, this is achieved by

$$\zeta_N^D := \frac{1}{W_N}\sum_{x\in D_N} \delta_{x/N} \otimes \delta_{(L_{t_N}^{D_N}(x)-a_N)/\sqrt{2a_N}}, \tag{6}$$

where $\{a_N\}_{N\geq 1}$ is a centering sequence with leading-order asymptotic as on the right-hand sides of (4) or (5) depending on whether λ-thick or λ-thin points are of concern. As it turns out, the right choice of the normalization constant is

$$W_N := \frac{N^2}{\sqrt{\log N}} \mathrm{e}^{-\pi\frac{(\sqrt{2t_N}-\sqrt{2a_N})^2}{\log N}}. \tag{7}$$

In order to formulate a convergence result, let $\mathfrak{d}\colon D \to [0,\infty)$ be the unique continuous function vanishing outside D whose Laplacian is (a positive) constant on D and such that $\mathfrak{d}(\cdot)-1$ has vanishing Lebesgue integral over D. Denote by σ_D^2 the integral of the continuum Green function in D with respect to both variables. Our main result concerning the thick points is then:

Theorem 1 (Thick points). *Given $\lambda \in (0,1)$ and $\theta > 0$, let $\{t_N\}_{N\geq 1}$ and $\{a_N\}_{N\geq 1}$ be positive sequences with leading-order growth as in* (3) *and* (4)*, respectively. Then for X sampled from P^{x_N}, for any $x_N \in D_N$,*

$$\zeta_N^D(\mathrm{d}x\,\mathrm{d}h) \ \underset{N\to\infty}{\overset{\text{law}}{\longrightarrow}}\ c(\theta,\lambda)\,\mathrm{e}^{\alpha\lambda(\mathfrak{d}(x)-1)Y} Z_\lambda^{D,0}(\mathrm{d}x) \otimes \mathrm{e}^{-\alpha\lambda h}\mathrm{d}h, \tag{8}$$

where $\alpha := \sqrt{8\pi}$, $c(\theta,\lambda)$ is an explicit constant, $Y = \mathcal{N}(0,\sigma_D^2)$ and $Z_\lambda^{D,0}$ is the zero-average Liouville Quantum Gravity measure in D independent of Y, at parameter λ-times critical.

We remark that, as shown in [4], the Liouville Quantum Gravity measure Z_λ^D appears as the distribution of the λ-thick points of the Discrete Gaussian Free Field (DGFF); see [3] for a review. Its zero-average counterpart $Z_\lambda^{D,0}$ is obtained by conditioning the DGFF to have zero average over D_N. The above theorem hinges on the fact (proved in [2]) that

$$Z_\lambda^D(\mathrm{d}x) \overset{\text{law}}{=} \mathrm{e}^{\alpha\lambda\mathfrak{d}(x)Y} Z_\lambda^{D,0}(\mathrm{d}x) \tag{9}$$

for $Y = \mathcal{N}(0,\sigma_D^2)$ independent of $Z_\lambda^{D,0}$. With the normalization of Z_λ^D fixed by prescribing $\mathbb{E}Z_\lambda^D(D)$, the constant $c(\theta,\lambda)$ is completely explicit.

Our main result for the thin points is completely analogous to Theorem 1. As mentioned earlier, we are also able to control the scaling limit of the light and avoided points. Focusing on the latter, these are encoded via the measure

$$\kappa_N^D := \frac{1}{\widehat{W}_N} \sum_{x\in D_N} 1_{\{L_{t_N}^{D_N}(x)=0\}} \delta_{x/N} \tag{10}$$

on D, where the normalization is now given by

$$\widehat{W}_N := N^2 \mathrm{e}^{-2\pi \frac{t_N}{\log N}}. \tag{11}$$

Looking back at (5), the avoided points should correspond to the $\sqrt{\theta}$-thin points and so we expect the measure $Z_{\sqrt{\theta}}^{D,0}$ play a role in their description. This is indeed confirmed in:

Theorem 2 (Avoided points). *Let $\{t_N\}_{N\geq 1}$ be a positive sequence satisfying* (3) *for some $\theta \in (0,1)$. Then for X sampled from P^{x_N}, for any $x_N \in D_N$,*

$$\kappa_N^D(\mathrm{d}x) \underset{N\to\infty}{\overset{\text{law}}{\longrightarrow}} \widehat{c}(\theta)\, \mathrm{e}^{\alpha\sqrt{\theta}(\mathfrak{d}(x)-1)Y} Z_{\sqrt{\theta}}^{D,0}(\mathrm{d}x), \tag{12}$$

where $\widehat{c}(\theta)$ is an explicit constant and Y and $Z_{\sqrt{\theta}}^{D,0}$ are as in Theorem 1.

Theorem 2 implies that the number of avoided points scaled by $\widehat{W}_N$ tends to the total mass of the measure on the right of (12). For the scaling of their overall number we get $\widehat{W}_N = N^{2(1-\theta)+o(1)}$. Similarly, by (8) and a simple integral, the number of λ-thick points normalized by W_N tends to a non-degenerate random variable. For their overall number we thus get $W_N = N^{2(1-\lambda^2)+o(1)}$.

The above results are proved in full detail in joint papers with Y. Abe [1] and with Y. Abe and S. Lee [2]. The conclusions can be augmented to include some control of the law of the local time near the exceptional points. The proofs are based on the Second Ray-Knight Theorem (a.k.a. Dynkin isomorphism) from [6] that gives a strong connection of the local time to the DGFF and, in particular, removes the need to peform second moment calculations for the local time. The limit conclusions draw on earlier joint work with O. Louidor [4] where similar limit results were derived for the thick points of the two-dimensional DGFF. The results affirm the *universality* of Gaussian Free Field for these extremal problems.

References

[1] Y. Abe and M. Biskup (2019). Exceptional points of two-dimensional random walks at multiples of the cover time. arXiv:1903.04045
[2] Y. Abe, M. Biskup and S. Lee (2019). Exceptional points of discrete-time random walks in planar domains. arXiv:1911.11810
[3] M. Biskup (2019). Extrema of the two-dimensional Discrete Gaussian Free Field. In: M.T. Barlow and G. Slade (eds.): *Random Graphs, Phase Transitions, and the Gaussian Free Field*, Springer Proceedings in Mathematics & Statistics, vol. 304, pp 161–405.
[4] M. Biskup and O. Louidor (2019). On intermediate level sets of two-dimensional discrete Gaussian free field. *Ann. Inst. Henri Poincaré* **55**, no. 4, 1948–1987.
[5] A. Dembo, Y. Peres, J. Rosen, and O. Zeitouni (2001). Thick points for planar Brownian motion and the Erdős-Taylor conjecture on random walk. *Acta Math.* **186** 239-270.
[6] N. Eisenbaum, H. Kaspi, M. B. Marcus, J. Rosen, and Z. Shi (2000). A Ray-Knight theorem for symmetric Markov processes. *Ann. Probab.* **28**, 1781-1796.
[7] P. Erdős and S.J. Taylor (1960). Some problems concerning the structure of random walk paths. *Acta Math. Acad. Sci. Hungar.* **11**, 137–162.

Long range, ergodic random conductance model

Filip Bosnić

Let $\alpha \in (0,2)$, $d \in \mathbb{N}$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We consider a random conductance $c(\omega, x, y)$ ($\omega \in \Omega, x, y \in \mathbb{Z}^d$) on discrete lattice $\mathbb{Z}^d$ which is assumed to be strictly positive and symmetric, $c(x,y) = c(y,x) > 0$. For $n \in \mathbb{N}$ we associate to c a random form

$$\mathcal{E}_\omega^{(n)}(f,g) = \frac{1}{n^{2d}} \sum_{x,y \in Z_m} \frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{d+\alpha}} c\left(\omega, nx, ny\right)$$

on the rescaled lattice $Z_n := (\mathbb{Z}/n)^d$ and shorten $\mathcal{E}_\omega := \mathcal{E}_\omega^{(1)}$. Suppose that, for $\mathbb{P}$-a.e. ω, $c(\omega)$ satisfies the following ergodic-like sumability condition: for every $p \in \mathbb{R}$ and every sequence $A_1 \subset A_2 \subset \ldots \subset \mathbb{Z}^d \times \mathbb{Z}^d$ of convex sets

$$\frac{1}{\# A_k} \sum_{(x,y) \in A_k} c(\omega, x, y)^p \xrightarrow{k \to \infty} \mathbb{E}[c^p]$$

where the claim is that the limit exists in $[0, \infty]$ and we name it $\mathbb{E}[c^p]$ as it is, by ergodic theorem, equal to $\mathbb{E}[c(x,y)^p]$ if the conductance is truly ergodic. Then for $p, q \geq 1$, c such that $\mathbb{E}[c^p] + \mathbb{E}[c^q] < \infty$ and $q^{-1} + (p-1)^{-1} \leq \alpha/d$ the following large scale parabolic Hölder regularity estimate holds. There exist $R_0 : \Omega \to (0, \infty)$ and $\theta > 0$ such that $\mathbb{P}$-a.s. in ω, for every weak solution u of parabolic equation $(\partial_t u, \varphi) = \mathcal{E}_\omega(u, \varphi)$ $\forall \varphi$ in the cylinder $C(R) := [0, R^\alpha] \times B(0,R)$ and every $r > 0$,

$$\sup_{C(r)} u - \inf_{C(r)} u \leq 6 \left(\frac{r \vee R_0(\omega)}{R} \right)^\theta .$$

Let $X_t^{(n)}$ be a sequence of (rescaled) random walks associated to Dirichlet forms $\mathcal{E}_\omega^{(n)}$ starting from $0 \in Z_m$, let Y_t be the rotationally symmetric α-stable Lévy process on $\mathbb{R}^d$ and call its Dirichlet form $\mathcal{E}^{(\infty)}$. If, for $\mathbb{P}$-a.e. ω, $\mathcal{E}_\omega^{(n)} \xrightarrow{n \to \infty} \mathcal{E}^{(\infty)}$ in generalized Mosco sense (as is the case if c is ergodic in the appropriate sense) then Hölder regularity presented before implies that $X_t^{(n)} \xrightarrow{n\to\infty} Y_t$ in finite dimensional distributions for almost every realization of c.

Constancy of the dimension for RCD spaces via regularity of Lagrangian flows

Elia Brué
(joint work with Daniele Semola)

In this talk we present the constancy of the dimension theorem for $\mathsf{RCD}(\mathsf{K},\mathsf{N})$ spaces. After the introduction of the so-called *curvature-dimension* condition $\mathsf{CD}(\mathsf{K},\mathsf{N})$ in the seminal and independent works [19, 20] and [17], the notion of $\mathsf{RCD}(\mathsf{K},\mathsf{N})$ space was proposed in [13] after the study of its infinite-dimensional counterpart $\mathsf{RCD}(\mathsf{K},\infty)$ in [1]. This class, includes smooth and weighted Riemannian manifolds with Ricci bounded below by K and dimension smaller than N, Ricci limits, and Alexandrov spaces. Despite their generality, RCD spaces enjoy important analytic and geometric properties typical of smooth manifolds with Ricci bounded below. Let us mention the characterization via Bochner-inequality [2, 5, 12, 3, 8], the Bishop-Gromov inequality, the Laplacian comparison, the splitting theorem and many others.

Structure theory: Nowadays $\mathsf{RCD}(\mathsf{K},\mathsf{N})$ spaces have a well-developed structure theory, analogous to the one obtained by Cheeger, Colding, Naber and collaborators for Ricci limit spaces.

Given an $\mathsf{RCD}(\mathsf{K},\mathsf{N})$ space $(X,\mathsf{d},\mathfrak{m})$ and an integer $k\in(0,N]$ we define $\mathcal{R}_k$, the set of *k-regular points*, as the collection of $x\in \operatorname{supp}\mathfrak{m}$ satisfying

$$\left(X, r^{-1}\mathsf{d}, \frac{\mathfrak{m}}{\mathfrak{m}(B_r(x))}, x\right) \to (\mathbb{R}^k, \mathsf{d}_{\mathrm{eucl}}, c_k\mathcal{L}^k, 0^k) \quad \text{as r} \to 0^+$$

in the pmGH topology. Informally, $\mathcal{R}_k$ is the set of points where the intrinsically defined tangent space is the Euclidean space of dimension k.

The following structure theorem for RCD spaces collects results from [18, 15, 16, 11] and provides a complete generalization of the structure theory for Ricci limits obtained by Cheeger and Colding.

Theorem 1. *Let* $(X,\mathsf{d},\mathfrak{m})$ *be an* $\mathsf{RCD}(\mathsf{K},\mathsf{N})$ *metric measure space. Then*

$$\mathfrak{m}\left(X \setminus \bigcup_{k=1}^{[N]} \mathcal{R}_k\right) = 0.$$

Moreover, $\mathcal{R}_k$ *is* $(\mathfrak{m},k)$*-rectifiable (i.e., it can be covered, up to a* $\mathfrak{m}$*-negligible set, with a countable family of bi-Lipschitz images of Borel subset of* $\mathbb{R}^k$*), and* $\mathfrak{m}|_{\mathcal{R}^k} \ll \mathcal{H}^k$*.*

Since the work of Mondino and Naber [18] in 2014, the following question has been considered of central importance:

"Is there an integer $n \in (0, N]$ such that $\mathfrak{m}(X \setminus \mathcal{R}_n) = 0$?".

The analogous question for Ricci limits has been positively answered after the celebrated work [9] by Colding and Naber.

The main content of my result in [6] is a new quantitative estimate for flows associated to Sobolev vector fields over RCD spaces and, as an application, we answer positively the question above.

Theorem 2 ([6])**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(\mathsf{K}, \mathsf{N})$ *space. Then there exists* $n \in (0, N]$*, called essential dimension of* X*, such that* $\mathfrak{m}(X \setminus \mathcal{R}_n) = 0$.

Let me start by explaining the link between quantitative flow estimates and the *constancy of the dimension* result.

It is known since the works [4, 14] that flows associated to Sobolev vector fields exist and act "almost transitively", in a suitable measure theoretic sense, on X. In particular, assuming by contradiction the existence of $m < n$ such that $\mathfrak{m}(\mathcal{R}_n) > 0$ and $\mathfrak{m}(\mathcal{R}_m) > 0$, we can bring a portion of positive $\mathfrak{m}$-measure of $\mathcal{R}_n$ to $\mathcal{R}_m$ by means of a flow map associated to a Sobolev vector field. Therefore, if we had at our disposal a strong enough regularity result for these flow maps (for instance an approximate bi-Lipschitz regularity would suffice), then we would have found a contradiction.

It is worth remarking that the approximate bi-Lipschitz regularity for flow maps associated to Sobolev vector fields is a reasonable property, and in the Euclidean setting holds true as a consequence of Crippa-DeLellis' estimates [10].

In the joint work with D. Semola [7], we obtain a first result in this regard proving the analogue of Crippa-DeLellis' estimates in the setting of *Ahlfors regular* $\mathsf{RCD}(\mathsf{K}, \mathsf{N})$ spaces.

Even though this class of spaces is quite wide (it includes non-collapsed Ricci limits and Alexandrov spaces), it is still too restrictive for applications to the constancy of the dimension. In order to cover *collapsed* spaces it is of fundamental importance to get rid of the Ahlfors regularity assumption in the regularity statement.

Unfortunately, whether a statement of this kind holds true or not is still an open problem. What we have obtained in [6] is a completely new estimate in terms of the *Green function* of the Laplacian. For the sake of simplicity I am going to state it in a simplified, but still meaningful, setting.

Theorem 3. *Let* X *be a Riemannian manifold of dimension* N *with non-negative Ricci curvature, satisfying*

$$\int_1^\infty \frac{r}{Vol(B_r(p))} \,\mathrm{d}r < \infty, \quad \text{for some (and thus all) } p \in X.$$

Let us denote by $G(x, y) = G_x(y)$ *the positive Green-function of the Laplacian, i.e. the solution to* $-\Delta G_x = \delta_x$.

There exists then a constant $C = C(N)$ such that, for any smooth time dependent vector field $\{b\}_{t\in[0,T]}$ satisfying $||\div b_t||_{L^\infty} \leq L$, the flow map X_t enjoys the following point-wise bound:

$$\frac{1}{C}\exp\{-\Phi(x)-\Phi(y)\} \leq \frac{G(x,y)}{G(X_t(x),X_t(y))} \leq C\exp\{\Phi(x)+\Phi(y)\} \quad \forall x,y\in X, \tag{1}$$

where $\Phi : X \to [0,\infty)$ fulfills

$$||\Phi||_{L^2} \leq Ce^L \int_0^T ||\nabla b_t||_{L^2}\,\mathrm{d}t.$$

The just stated result holds for abstract $\mathsf{RCD}(0,\mathsf{N})$ spaces and, up to changing the Green function with the one associated to the operator $\lambda I - \Delta$ for some $\lambda \in \mathbb{R}$, we can extend the result to $\mathsf{RCD}(\mathsf{K},\mathsf{N})$ spaces. It is worth remarking that even the smooth statement is new and potentially interesting, for instance it can be extend to collapsed Ricci limits using that (1) is stable under mGH convergence.

Note that in the Euclidean space the Green function is a power of the distance, thus in this case 3 coincides with a usual approximate Lipschitz estimate. This is not the case in *collapsed* and *weighted* spaces where the Green function is not necessarily comparable with a power of the distance. In particular, for the application to the constancy of the dimension, one has to find a counterpart for the "preservation of the Hausdorff dimension via bi-Lipschitz maps" formulated just in terms of Green functions. This is indeed possible, the argument builds upon the study of the asymptotic behavior of the Green function, along with a measure theoretic result in the spirit of Sard's lemma.

References

[1] L. Ambrosio, N. Gigli, G. Savaré: *Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below,* Invent. Math., **195** (2014), 289–391.

[2] L. Ambrosio, N. Gigli, G. Savaré: *Bakry-Émery curvature-dimension condition and Riemannian Ricci curvature bounds.* Ann. Probab., **43** (2015), 339–404.

[3] L. Ambrosio, A. Mondino, G. Savaré: *Nonlinear diffusion equations and curvature conditions in metric measure spaces.* ArXiv preprint: 1509.07273 (2015), to appear in Memoirs Amer. Math. Soc.

[4] L. Ambrosio, D. Trevisan: *Well posedness of Lagrangian flows and continuity equations in metric measure spaces,* Analysis and PDE, **7** (2014), 1179–1234.

[5] K. Bacher, K.-T. Sturm: *Localization and tensorization properties of the curvature-dimension condition for metric measure spaces,* J. Funct. Anal., **259** (2010), no. 1, 28–56.

[6] E. Brué, D. Semola: *Constancy of the dimension for $RCD(K,N)$ spaces via regularity of Lagrangian flows.* Comm. Pure and Applied Math. DOI: 10.1002/cpa.21849.

[7] E. Brué, D. Semola: Regularity of Lagrangian flows over $\mathrm{RCD}^*(K,N)$ spaces, accepted paper: J. Reine Angew. Math.

[8] F. Cavalletti, E. Milman: *The Globalization Theorem for the Curvature Dimension Condition,* preprint on arXiv: 1612.07623.

[9] T. H. Colding, A. Naber: *Sharp Hólder continuity of tangent cones for spaces with a lower Ricci curvature bound and applications,* Ann. of Math. (2), **176** (2012), 1173–1229.

[10] G. Crippa, C. De Lellis: *Estimates and regularity results for the Di Perna-Lions flow*, J. Reine Angew. Math., **616** (2008), 15–46.

[11] G. De Philippis, A. Marchese, F. Rindler: *On a conjecture of Cheeger,* Measure theory in non-smooth spaces, De Gruyter Open, Warsaw, (2017), 145–155.
[12] M. Erbar, K. Kuwada, K.-T. Sturm: *On the equivalence of the entropic curvature-dimension condition and Bochner's inequality on metric measure spaces.* Invent. Math., **201** (2015), 993–1071.
[13] N. Gigli: *On the differential structure of metric measure spaces and applications,* Mem. Amer. Math. Soc., **236** (2015), vi–91.
[14] N. Gigli, S. Mosconi: *The abstract Lewy-Stampacchia inequality and applications,* J. Math. Pures Appl. (9), 2015 **104**, 258–275.
[15] N. Gigli, E. Pasqualetto: *Behaviour of the reference measure on* RCD *spaces under charts,* preprint on ArXiv: 1607.05188v2.
[16] M. Kell, A. Mondino: *On the volume measure of non-smooth spaces with Ricci curvature bounded below,* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), **18** (2018), 593–610.
[17] J. Lott, C. Villani: *Ricci curvature for metric-measure spaces via optimal transport,* Ann. of Math. (2), **169** (2009), 903–991.
[18] A. Mondino, A. Naber: *Structure theory of metric measure spaces with lower Ricci curvature bounds,* J. Eur. Math. Soc. **21** (2019), no. 6, 1809–1854.
[19] K.-T. Sturm: *On the geometry of metric measure spaces I,* Acta Math., **196** (2006), 65–131.
[20] K.-T. Sturm: *On the geometry of metric measure spaces II*, Acta Math., **196** (2006), 133–177.

A regularization property of heat semigroups and its applications

Li Chen

Let (X, μ) be a measurable space (e.g. Riemannian manifolds, graphs or Dirichlet spaces) equipped with a self-adjoint operator L. Let $\{P_t\}_{t>0}$ be the associated heat semigroup. We are interested in the following regularization property of the heat semigroup: for $1 < p \leq \infty$,

$$(G_p) \qquad \| \, |\nabla P_t f| \, \|_p \leq \frac{C}{\sqrt{t}} \|f\|_p,$$

where ∇ denotes the gradient on Riemannian manifolds or its proper substitutes in other settings like "carré du champ".

To fix idea we consider complete Riemannian manifolds.

- (G_2) is always true by spectral theory.
- (G_∞) is the so-called weak Bakry-Émery curvature condition.
- When $2 < p \leq \infty$, (G_p) is linked to the geometry of the underlying space, Riesz transforms, harmonic functions, Sobolev and isoperimetric inequalities, and regularity problems of some PDEs.
- When $1 < p < 2$, (G_p) is of different nature. Surprisingly it is always true.

In this talk we focus on the study of (G_p) for $1 < p \leq 2$ and its applications. We start with the setting of graphs. Let (V, E) be an infinite connected graph with symmetric weight μ and let d be the graph distance. Then μ induces a weight on vertices and a measure on the graph. Denote by $p(x, y)$ the transition probability and by P the associated Markov operator. The discrete Laplacian is the operator $I - P$. The (length of) discrete gradient is the "carré du champ" defined as $|\nabla f(x)|^2 = \frac{1}{2} \sum_{y \sim x} p(x, y) |f(x) - f(y)|^2$. Nick Dungey in [6] proved that if (V, E, μ) satisfies the "local doubling" property: $\mu(B(x,1)) \leq C\mu_x$, then for any $1 < p \leq 2$,

$$\| |\nabla e^{-t(I-P)} f| \|_p \leq \frac{C}{\sqrt{t}} \|f\|_p. \tag{1}$$

A deep original idea introduced in the proof is the use of "pseudo-gradient":

$$\Gamma_p(f) = pf(I-P)f - f^{2-p}(I-P)f^p,$$

where $1 < p \leq 2$. This notion mimics the chain rule for the Laplace-Beltrami operator on Rieammnian manifolds and is comparable to the carré du champ in certain sense. Hence one can use the analyticity of heat semigroup and Hölder's inequality to deduce the desired gradient estimate.

Motivated by Dungey's proof, we can show that (G_p), $1 < p \leq 2$, always holds on any complete connected Riemannian manifold without any geometry or volume assumption. A significant difference from the discrete case is that the local doubling volume property is not needed. The proof in [2] relies on the chain rule for the Laplace-Beltrami operator on appropriate function u:

$$\Delta u^p(x,t) = -p(p-1)u^{p-2}(x,t)|\nabla u(x,t)|^2 + pu^{p-1}\Delta u,$$

as well as a delicate cut-off argument.

Coming back to the setting of graphs, a natural question to ask is whether or not one can remove the "local doubling" assumption in Dungey's result. Together with T. Coulhon and B. Hua [4], we work on locally finite connected graph (V, E) endowed with a symmetric weight μ on edges and a weight ν on vertices such that $\sup_{x\in V} \frac{\sum_{y\sim x} \mu_{xy}}{\nu_x} < \infty$. The associated bounded Laplacian is defined by

$$\Delta_{\mu,\nu} f = \frac{1}{\nu_x} \sum_{y\sim x} \mu_{xy}(f(x) - f(y)).$$

Considering the gradient on edges $|Df|(\{x,y\}) = |f(y) - f(x)|$, we prove that

$$\| |De^{-t\Delta_{\mu,\nu}} f| \|_{\ell^p(E,\mu)} \leq \frac{C}{\sqrt{t}} \|f\|_{\ell^p(V,\nu)}. \tag{2}$$

Our proof adopts a symmetrization argument for the "pseudo-gradient". That is, one writes for $1 < p \leq 2$

$$\Gamma_p(f)(x) = pf\Delta_{\mu,\nu} f - f^{2-p}\Delta_{\mu,\nu}(f^p) = \sum_y \frac{\mu_{xy}}{\nu_x} \gamma_p(f(x), f(y)),$$

where $\gamma_p(\alpha, \beta) = p\alpha(\alpha - \beta) - \alpha^{2-p}(\alpha^p - \beta^p), \forall \alpha, \beta \geq 0$. A crucial observation is that $\gamma_p(\alpha,\beta) + \gamma_p(\beta,\alpha) \simeq (\alpha-\beta)^2$. Hence one can run the gradient estimate by using the analyticity of heat semigroup and Hölder's inequality.

In [1], we further carry (G_p) to Dirichlet spaces. Let X be a good measurable space equipped with a σ-finite measure μ. Let $(\mathcal{E}, \mathcal{F} = \mathrm{Dom}(\mathcal{E}))$ be a Dirichlet form on $L^2(X,\mu)$ and $\{P_t\}_{t>0}$ be the associated heat semigroup. Assume that P_t

is conservative, i.e. $P_t 1 = 1$. Then the analogue of (G_p), $1 < p \leq 2$, has the form

$$\|P_t f\|_{p,1/2} \leq \frac{C}{\sqrt{t}} \|f\|_p, \tag{3}$$

where $\|\cdot\|_{p,1/2}$ is the seminorm of the heat semigroup-based Besov space introduced in [1]:

$$\mathbf{B}^{p,\alpha}(X) = \left\{ f \in L^p(X), \|f\|_{p,\alpha} := \sup_{t>0} \frac{1}{t^\alpha} \left(\int_X P_t(|f - f(y)|^p)(y) d\mu(y) \right)^{1/p} < \infty \right\}.$$

The symmetrization argument in [4] can also be applied in this setting.

The property (G_p), $1 < p \leq 2$, on Riemannian manifolds or its substitutes (1), (2) on graphs and (3) on Dirichlet spaces describe the regularity of heat semigroups in different settings. These properties turn to be very powerful tools dealing with problems arising in analysis. We describe two applications here.

- One application is on the L^p boundedness of Riesz transform $\nabla\Delta^{-1/2}$ on Riemannian manifolds or graphs. Assuming volume doubling property and Gaussian heat kernel upper bound, Coulhon and Duong [5] proved that $\nabla\Delta^{-1/2} : L^p \to L^p$ for $1 < p \leq 2$. The key ingredient is a weighted version of (G_2) which was proved by Grigor'yan in [7] using integration by parts. Replacing the Gaussian upper bound by a sub-Gaussian one (which is satisfied by some fractal-like manifolds or graphs), we prove in [3] the same results. In this case Grigor'yan's approach does not work anymore. As a natural substitute, we use a weighted version of (G_p) for $1 < p < 2$, which follows from Dungey's idea on the use of chain rule.
- The other application is on the study of critical exponents of Besov spaces, i.e., $\alpha^*(p) = \sup\{\alpha : \mathbf{B}^{p,\alpha} \text{ is nontrivial}\}$. The property (3) leads to a stronger version of pseudo-Poincaré inequality for $p \geq 2$. As a consequence, one can deduce $\alpha^*(p) \leq 1/2$ for $p \geq 2$.

References

[1] P. Alonso Ruiz, F. Baudoin, L. Chen, L. Rogers, N. Shanmugalingam, and A. Teplyaev, *Besov class via heat semigroup on Dirichlet spaces I: Sobolev inequalities*, arXiv:1811.04267, 2018.

[2] L. Chen, *Sub-Gaussian heat kernel estimates and quasi Riesz transforms for* $1 \leq p \leq 2$, Publ. Mat. **59(2)** (2015), 313–338.

[3] L. Chen, T. Coulhon, J. Feneuil, and E. Russ, *Riesz transform for* $1 \leq p \leq 2$ *without Gaussian heat kernel bound*, J. Geom. Anal., **27(2)** (2017), 1489–1514.

[4] L. Chen, T. Coulhon, and B. Hua, *Riesz transform for bounded Laplacians on graphs*, Math. Z., to appear.

[5] T. Coulhon and X. T. Duong, *Riesz transforms for* $1 \leq p \leq 2$, Trans. Amer. Math. Soc., **351(3)** (1999),1151–1169.

[6] N. Dungey, *A Littlewood-Paley-Stein estimate on graphs and groups*, Studia Math., **189(2)**(2008), 113–129.

[7] A. Grigor'yan, *Upper bounds of derivatives of the heat kernel on an arbitrary complete manifold*, J. Funct. Anal., **127(2)** (1995), 363–389.

Listening to the shape of a drum

Fabio E.G. Cipriani
(joint work with J.-L. Sauvageot, C.N.R.S. Paris France)

In a famous paper H. Weyl [7] shown that the volume and the dimension of a Euclidean domain can be captured from the asymptotics of the eigenvalues of the Laplace operator. In a as much famous paper [4] M. Kac analyzed this and other connections between the geometry of a domain or Riemannian manifold and the spectrum of the Laplace operator. However the question posed by the title of his conference relied on a vain hope as J. Milnor had just showed in [6] the existence of two non isometric 16-dimensional tori sharing the same spectrum. Since then an enormous amount of efforts have been dedicated to the elucidation of questions of spectral geometry, often involving other disciplines such as Probability or Topology as in the works of S. Bochner, K. Yano, H.P. Jr. McKean and I.M. Singer.

The starting point of our work was to reconsider Kac's question from a new point of view: instead to remain in the isometric category to compare metric spaces we moved to the conformal category. This is after all natural as the original latin acceptation of the adjective *conformal*, i.e. *conformalis*, is *sharing the same shape*. Hence in this work we judge two Euclidean domains as equivalent if they are transformed one into the other by a conformal transformation, i.e. a transformation preserving orthogonality (and more in general, preserving angles between curves). By the rigidity theorem of J. Liouville, these are just restrictions of Möbious transformations, i.e. compositions of a finite number of translations, rotations, dilations and symmetry with respect to the unit sphere (in fact reflections with respect to spheres of arbitrary centers and radii generate the whole Möbious group).

A first novelty of our work concerns the tools we introduce. They rely on potential theory and in particular on the notion of multiplier of the Sobolev space $H^1(\Omega)$, the latter viewed as the form domain of the Dirichlet integral $\mathcal{D}$.

A second novelty is to disregard the overtones $0 < \mu_1(\Omega, dx) < \cdots < \mu_n(\Omega, dx) < \cdots$ of the spectrum of the Dirichlet form $\mathcal{D}$ with respect to $L^2(\Omega, dx)$ but rather to take into account only the first non zero eigenvalues or *fundamental tones* $\mu_1(A, \Gamma[a])$ of $\mathcal{D}$ on subdomains $A \subset \Omega$ with respect to the space $L^2(A, \Gamma[a])$ associated to the energy measures $\Gamma[a] := |\nabla a|^2 \cdot dx$ of multipliers a of $H^1(A)$.

In a first result we prove that on the whole space $\mathbb{R}^n$ and for $n \geq 3$, the Möbius group $G(\mathbb{R}^n)$ acts isometrically on the multiplier algebra $\mathcal{M}(H^1(\mathbb{R}^n))$, leaves invariant the Dirichlet integral $\mathcal{D}$ and transform unitarily the space $L^2(\mathbb{R}^n, \Gamma[a])$ onto the space $L^2([R]^n, \Gamma[a \circ \gamma])$. Hence the spectrum of $\mathcal{D}$ on $L^2(\mathbb{R}^n, \Gamma[a])$ is constant along the orbit of $a \in \mathcal{M}(H^1(\mathbb{R}^n))$ in the multiplier algebra under the action of the Möbius group. This result is then localized in various ways.

The second main result shows that the conformality of an homeomorphism γ between Euclidean domains Ω and $\gamma(\Omega)$ can be detected spectrally from the invariance of the fundamental tone of the Dirichlet integral $\mathcal{D}$ on all subdomains $A \subseteq \Omega$ with respect to the energy measure of all finite energy multipliers $a \in \mathcal{FM}(H^1(\mathbb{R}^n))$

$$\mu_1(\gamma(a), \Gamma[a]) = \mu_1(A, \gamma[a \circ \gamma]).$$

Similar spectral characterizations of the class of quasi-conformal and quasi-regular maps are also proved.

The method used to prove the first mean result combines the known conformal invariance of the Hardy-Littlewood- Sobolev functional, the isometric action of the Möbius group $G(\mathbb{R}^n)$ on the multiplier algebra $\mathcal{M}(H^1(\mathbb{R}^n))$ and the unitary conformal flow of the energy measures $G(\mathbb{R}^n) \ni \gamma \mapsto \Gamma[a \circ \gamma]$ of multipliers.

The second main result is based on i) a property of *persistence of the spectral gap* of $\mathcal{D}$ under a change of reference measure from the Lebesgue one dx to the energy measure $\Gamma[a]$ of a multiplier (the ratio of these spectral gaps is bounded in terms of the multiplier seminorm $\eta(a)$ of $a \in \mathcal{M}(H^1(\mathbb{R}^n)))$ and on ii) an upper bound on the fundamental tone of $\mathcal{D}$ with respect to an energy measure $\Gamma[a]$, involving the conformal volume of the domain (results due to Li-yau in [5] for surfaces and to Colbois-El Soufi-Savo in [3] for general Riemannian manifolds).

Finally, the results of this work suggest that the natural setting to discuss geometric properties of the spectrum is the one of the Potential Theory of Dirichlet forms [1] and their multipliers [2]. In particular, a distinguished role is played by the *extended Dirichlet space* $H^1_e(\mathbb{R}^n)$ and its multiplier algebra and by the process of *random time change in Dirichlet spaces* as elaborated in [1].

References

[1] Z.Q. Chen, M. Fukushima, *Symmetric Markov processes, time change, and boundary theory*, London Mathematical Society Monographs Series, 35, Princeton University Press, Princeton, NJ, 2012.

[2] F. Cipriani, J.-L. Sauvageot, Variations in noncommutative potential theory: finite energy states, potentials and multipliers, *Trans. Amer. Math. Soc.* **367** (2015), no. 7, 4837–4871.

[3] B. Colbois, A. El Soufi, A. Savo, Eigenvalues of the Laplacian on a compact manifold with density, *Comm. Anal. Geom.* **23** (2015), no. 3, 639–670.

[4] M. Kac, Can one hear the shape of a drum?, *Amer. Math. Monthly* **73** no. 4, part II (1966), no. 2, 1–23.

[5] P.Li, S.-T. Yau, A New Conformal Invariant and Its Applications to the Willmore Conjecture and the First Eigenvalue of Compact Surfaces, *Invent. Math.* **69** (1982), 269–291.

[6] J. Milnor, Eigenvalues of the Laplace operator on certain manifolds, *Proc. Nat. Acad. Sci. U.S.A* **51** (1964), 542.

[7] H. Weyl, Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung), *Math. Ann.* **71** (1912), no. 4, 441–479.

The Dirichlet–Ferguson Diffusion on the Space of Probability Measures over a Closed Riemannian Manifold

Lorenzo Dello Schiavo

Wasserstein geometry. In the last two decades, the space $\mathscr{P}$ of all Borel probability measures over a Riemannian manifold (M, g), endowed with the L^2-Kantorovich–Rubinshtein distance W_2, has proven both a powerful tool and an interesting geometric object in its own right. Since the fundamental works of Y. Brenier, R. J. McCann, F. Otto, C. Villani, N. Gigli, and many others, several geometric notions have been introduced, including those of geodesics, tangent space $T_\mu\mathscr{P}$ at a point μ in $\mathscr{P}$ and gradient $\boldsymbol{\nabla} u(\mu)$ of a scalar-valued function u at μ. Indeed, the metric space $\mathscr{P}_2 := (\mathscr{P}, W_2)$ may — to some extent — be regarded as a kind of infinite-dimensional Riemannian manifold. Furthermore, provided that (M, g) be a closed manifold with non-negative sectional curvature, $\mathscr{P}_2$ has non-negative lower curvature bound in the sense of Alexandrov.

Volume measures on $\mathscr{P}_2$. The question of the existence of a Riemannian volume measure on $\mathscr{P}_2$, say $\mathrm{dvol}_{\mathscr{P}_2}$, has been insistently posed and remains to date not fully answered. A first natural requirement that one might ask of such a measure — if any — is an integration-by-parts formula for the gradient, which would imply the closability of the form

$$(\mathcal{E}) \qquad \mathcal{E}(u, v) := \int_{\mathscr{P}} \left\langle \boldsymbol{\nabla} u(\mu) \mid \boldsymbol{\nabla} v(\mu) \right\rangle_{T_\mu\mathscr{P}} \mathrm{dvol}_{\mathscr{P}_2}(\mu) \quad .$$

Further requirements are the validity of a Rademacher-type property, i.e. the $\mathrm{dvol}_{\mathscr{P}_2}$-a.e. differentiability of W_2-Lipschitz functions, and of its converse, the so-called Sobolev-to-Lipschitz property. Together, these properties would grant the identification of W_2 with the intrinsic distance induced by $\mathcal{E}$.

Diffusions processes on $\mathscr{P}$. When the Dirichlet form $(\mathcal{E})$ is regular, it is associated with a corresponding Markov process $\mu_\bullet$ deserving the name of "Brownian motion" on $\mathscr{P}_2$. Several processes constructed in this fashion have been studied on $\mathscr{P}_2$ when M is a one-dimensional manifold, possibly with boundary. Indeed, in this case — and only in this case — $\mathscr{P}_2$ is a compact convex subset in a separable Hilbert space, and the problem may be addressed by finite-dimensional approximation techniques involving orthonormal bases.

One-dimensional base spaces. In the case when $M = \mathbb{S}^1$, the unit circle, or $M = I$, the closed unit interval, M.-K. von Renesse and K.-T. Sturm proposed the *entropic measure* $\mathbb{P}_\beta$ [11] as a candidate for $\mathrm{dvol}_{\mathscr{P}_2}$ and constructed the associated *Wasserstein diffusion* $\mu_\bullet^{\mathrm{WD}}$. Whereas the construction of the entropic measure in the case when M is an arbitrary closed Riemannian manifold was subsequently achieved by K.-T. Sturm in [13], many of its properties, in particular the closability of the associated form $(\mathcal{E})$, remain unknown. Similar constructions to the Wasserstein diffusion — up to now confined to one-dimensional base spaces — include J. Shao's

Dirichlet–Wasserstein diffusion [12], when $M = \mathbb{S}^1$ or I; V. V. Konarovskyi's *modified massive Arratia flow* [7, 9], when $M = I$; and Konarovskyi and von Renesse's *coalescing-fragmentating Wasserstein dynamics* [8], when $M = \mathbb{R}$.

Multi-dimensional base spaces. In [4], we provide two constructions of a Markov diffusion $\mu_\bullet^{\mathrm{DF}}$ with state space $\mathscr{P}$ when M is an arbitrary closed manifold of dimension $d \geq 2$. On the one hand, combining results by Bendikov–Saloff-Coste [2] and Albeverio–Daletskii–Kondratiev [1] about elliptic diffusions on infinite products, we characterize $\mu_\bullet^{\mathrm{DF}}$ as the super-process constituted by up to countable independent massive Brownian particles with volatility equal to the inverse of their mass. Thus, we may regard $\mu_\bullet^{\mathrm{DF}}$ as a possible counterpart over M of Konarovskyi's Modified Massive Arratia Flow [7] over the unit interval. Here, no coalescence occurs by reasons of the dimension of M. On the other hand, we show that $\mu_\bullet^{\mathrm{DF}}$ is associated with the symmetric strongly local regular Dirichlet form $(\mathcal{E})$ when $\mathrm{dvol}_{\mathscr{P}_2}$ is the Dirichlet–Ferguson random measure $\mathcal{D}$ introduced in [6]. In this case, the form additionally satisfies the Rademacher property, so that $\mu_\bullet^{\mathrm{DF}}$ is a possible candidate for a canonical diffusion process on $\mathscr{P}_2$.

Some open questions. Several open questions remain in addressing the existence and properties of natural "Riemannian volume measures" on $\mathscr{P}_2$. Firstly, that one may consider different measures $\mathrm{dvol}_{\mathscr{P}_2}$ from those listed above. Secondly, that there is an interplay between such measures and some representations of infinite-dimensional Lie groups on $L^2(\mathrm{dvol}_{\mathscr{P}_2})$. Thirdly, that for every such measure one may study synthetic curvature bounds for the metric measure space $(\mathscr{P}, W_2, \mathrm{dvol}_{\mathscr{P}_2})$.

Other measures. For the choice $\mathrm{dvol}_{\mathscr{P}_2} = \mathcal{D}$, it is possible to show that the set of measures μ_0 on M absolutely continuous w.r.t. dvol_g is polar for the form $(\mathcal{E})$. In particular, $\mu_\bullet^{\mathrm{DF}}$ may not have the Feller property on the whole of $\mathscr{P}_2$, in which case it would not be possible to start the process at any such measure μ_0. This prompts to investigate the closability and other properties of the (pre-)Dirichlet form $(\mathcal{E})$ with different choices for $\mathrm{dvol}_{\mathscr{P}_2}$, possibly concentrated on the set of probability measures on M with density w.r.t. dvol_g. A meaningful requirement for these measures is their quasi-invariance w.r.t. the natural action on $\mathscr{P}$ of the (Lie) group of diffeomorphisms $\mathrm{Diff}(M)$, granting the validity of the Rademacher property, [5].

Representations of large groups. The (quasi-)invariance of $\mathrm{dvol}_{\mathscr{P}_2}$ under the action

$$(\circlearrowleft) \qquad \psi_\cdot \colon \mu \mapsto \psi_\sharp \mu := \mu \circ \psi^{-1} , \qquad \psi \in \mathrm{Diff}(M)$$

induces a (quasi-)regular representation of $\mathrm{Diff}(M)$ acting unitarily on the Hilbert space $L^2(\mathrm{dvol}_{\mathscr{P}_2})$. The latter space is sufficiently large for the representation to capture important properties of the group: for instance the fact that the Dirichlet–Ferguson measure $\mathcal{D}$ is not quasi-invariant w.r.t. $(\circlearrowleft)$ is related to the fact that the natural action of $\mathrm{Diff}(M)$ on M is n-transitive for every finite n, but not σ-transitive. The invariance properties of other choices for $\mathrm{dvol}_{\mathscr{P}_2}$ remain to date an open problem, related to a long-standing program in the representation of infinite-dimensional Lie groups initiated by I. M. Gel'fand, M. I. Graev and A. M. Vershik, see [10] and Ref.s therein.

Curvature properties. As an effect of the dimension of the base space M, the process $\mu_\bullet^{\mathrm{DF}}$ is not ergodic, having in fact a continuum of disjoint invariant sets. For the metric measure space $(\mathscr{P}, W_2, \mathcal{D})$ this partially rules out the study of synthetic Ricci-curvature lower bounds in the sense of Bakry–Émery or Lott–Sturm–Villani. Other negative results are known, including the fact that, for the entropic measure, the space $(\mathscr{P}(\mathbb{S}^1), W_2, \mathbb{P}_\beta)$ does not satisfy any curvature-dimension condition, [3]. The validity of the (Riemannian) curvature-dimension condition for $(\mathscr{P}, W_2, \mathrm{dvol}_{\mathscr{P}_2})$ for some appropriate choice of $\mathrm{dvol}_{\mathscr{P}_2}$ would provide a new class of examples of non-flat, infinite-dimensional spaces with synthetic Ricci-curvature lower bounds.

References

[1] Albeverio, S., Daletskii, A. Yu., and Kondrat'ev, Yu. G. Stochastic Analysis on Product Manifolds: Dirichlet Operators on Differential Forms. *J. Funct. Anal.*, 176:280–316, 2000.

[2] Bendikov, A. and Saloff-Coste, L. Elliptic diffusions on infinite products. *J. reine angew. Math.*, 493:171–220, 1997.

[3] Chodosh, O. A lack of Ricci bounds for the entropic measure on Wasserstein space over the interval. *J. Funct. Anal.*, 262(10):4570–4581, 2012.

[4] Dello Schiavo, L. The Dirichlet–Ferguson Diffusion on the Space of Probability Measures over a Closed Riemannian Manifold. *arXiv:1811.11598*, 2018.

[5] Dello Schiavo, L. A Rademacher-type theorem on L^2-Wasserstein spaces over closed Riemannian manifolds. *J. Funct. Anal.*, 2019.

[6] Ferguson, T. S. A Bayesian analysis of some nonparametric problems. *Ann. Statist.*, 1:209–230, 1973.

[7] Konarovskyi, V. V. A System of Coalescing Heavy Diffusion Particles on the Real Line. *Ann. Probab.*, 45(5):3293–3335, 2017.

[8] Konarovskyi, V. V. and Renesse, M.-K. von. Reversible Coalescing-Fragmentating Wasserstein Dynamics on the Real Line. *arXiv:1709.02839*, page 59, 2017.

[9] Konarovskyi, V. V. and Renesse, M.-K. von. Modified Massive Arratia flow and Wasserstein diffusion. *Comm. Pur. Appl. Math.*, 2018. (to appear).

[10] Kondratiev, Yu. G., Lytvynov, E. W., and Vershik, A. M. Laplace operators on the cone of Radon measures. *J. Funct. Anal.*, 269(9):2947–2976, 2015.

[11] Renesse, M.-K. von and Sturm, K.-T. Entropic measure and Wasserstein diffusion. *Ann. Probab.*, 37(3):1114–1191, 2009.

[12] Shao, J. A New Probability Measure-Valued Stochastic Process with Ferguson-Dirichlet Process as Reversible Measure. *Electron. J. Probab.*, 16(9):271–292, 2011.

[13] Sturm, K.-T. Entropic Measure on Multidimensional Spaces. In Dalang, R., Dozzi, M., and Russo, F., editors, *Seminar on Stochastic Analysis, Random Fields and Applications VI*, volume 63 of *Progr. Probab.*, pages 261–277. Birkhäuser/Springer Basel AG, Basel, 2011.

Uniform volume doubling and functional inequalities on compact Lie groups

Nathaniel Eldredge
(joint work with Maria Gordina, Laurent Saloff-Coste)

Summary: We say a compact connected Lie group K is *uniformly doubling* if there is a uniform upper bound for the volume doubling constants $D_{K,g}$ of all left-invariant Riemannian metrics g on K. In this setting, many interesting functional inequalities for the Laplacian Δ_g hold with constants depending only on $D_{K,g}$, and so in a uniformly doubling group, they hold with constants uniform over all g. Abelian Lie groups trivially have this property, but in our paper [4], we present a first example of a non-abelian uniformly doubling group: the special unitary group SU(2).

Recall that the volume doubling constant $D_{K,g}$ is defined as

$$D_{K,g} := \sup_{x\in K, r>0} \frac{\mathrm{Vol}_g(B_g(x,2r))}{\mathrm{Vol}_g(B_g(x,r))}$$

where Vol_g denotes the Riemannian volume induced by g (here a multiple of Haar measure) and B_g is the ball of the Riemannian distance. It can be shown, via the group invariance, that (K,g) satisfies a Poincaré inequality on balls, with constant depending only on $D_{K,g}$ [7, 11, 13]; namely, for all $f \in C^\infty(K)$ we have

$$\int_{B(x,r)} |f - f_{x,r}|^2 \, d\mathrm{Vol} \le 2D_{K,g} r^2 \int_{B(x,2r)} |\nabla f|^2 \, d\mathrm{Vol}$$

where $f_{x,r} = \fint_{B(x,r)} f \, d\mathrm{Vol}$ is the average of f over $B(x,r)$. It is well known that doubling and Poincaré together imply several other interesting functional inequalities, which in this case will therefore hold with constants depending only on $D_{K,g}$:

- Heat kernel estimates [3, 6, 10, 11, 12] of the form

$$p_t(x,y) \le C \frac{(1 + d(x,y)^2/4t)^\kappa}{V(\sqrt{t})} \exp\left(-\frac{d(x,y)^2}{4t}\right)$$
$$p_t(x,y) \ge \frac{c}{V(\sqrt{t})} \exp\left(-A\frac{d(x,y)^2}{t}\right)$$

  where $V(r) = \mathrm{Vol}_g(B_g(e,r))$ denotes the volume of a ball of radius r;
- Parabolic Harnack inequalities [5, 9];
- Spectral gap and Weyl eigenvalue estimates [8], of the form

$$c\frac{\mathrm{Vol}(K)}{V(s^{-1/2})} \le \mathfrak{W}(s) \le C\frac{\mathrm{Vol}(K)}{V(s^{-1/2})}.$$

  where $\mathfrak{W}(s)$ is the number of eigenvalues less than s;
- Riesz transform bounds [1, 3] of the form

$$c\|\nabla f\|_{L^p} \le \|\Delta^{1/2} f\|_{L^p} \le C\|\nabla f\|_{L^p}$$

As such, in a uniformly doubling group, the above inequalities would hold for all $g \in \mathfrak{L}(K)$ with constants depending only on D_K, independent of g.

The main result of our paper [4] is that the special unitary group SU(2) is uniformly doubling; to our knowledge, this is the first example of a non-abelian group known to have this property. (It is trivially satisfied in an abelian Lie group.) The proof is based on explicit estimation of the volume function $V(r)$ in terms of a convenient parametrization of metrics $g \in \mathfrak{L}(\mathrm{SU}(2))$, which also provides for each g an orthogonal basis satisfying simple Lie bracket relations. We show that at different scales, the g-balls of SU(2) can resemble those of Euclidean space $\mathbb{R}^3$, the sub-Riemannian geometry on the Heisenberg group $\mathbb{H}^3$, or the sphere S^2.

The proof also gives uniform doubling results for the left-invariant sub-Riemannian geometries on SU(2), implying that their corresponding sub-Laplacians satisfy the same functional inequalities mentioned above (suitably renormalized), with the same uniform bounds on the constants.

We remark that in the recent paper [2], the authors showed that the Sasakian left-invariant Riemannian metrics on SU(2), which form a proper subfamily of $\mathfrak{L}(\mathrm{SU}(2))$ uniformly satisfy a measure contraction property $MCP(\kappa, N)$. This raises the question as to whether this property could hold uniformly over all $g \in \mathfrak{L}(\mathrm{SU}(2))$.

We conjecture that *every* compact connected Lie group may be uniformly doubling, and seek to obtain more examples. As work in progress, we consider the unitary group U(2); because of its close connection to SU(2), we believe that similar methods may show that it is uniformly doubling, but the computations become considerably more complex. However, to resolve the question for large classes of groups, we expect that new methods will be required.

References

[1] Pascal Auscher, Thierry Coulhon, Xuan Thinh Duong, and Steve Hofmann. Riesz transform on manifolds and heat kernel regularity. *Ann. Sci. École Norm. Sup. (4)*, 37(6):911–957, 2004.

[2] Fabrice Baudoin, Erlend Grong, Kazumasa Kuwada, and Anton Thalmaier. Sub-Laplacian comparison theorems on totally geodesic Riemannian foliations. *Calc. Var. Partial Differential Equations*, 58(4):Art. 130, 38, 2019.

[3] Thierry Coulhon and Adam Sikora. Gaussian heat kernel upper bounds via the Phragmén-Lindelöf theorem. *Proc. Lond. Math. Soc. (3)*, 96(2):507–544, 2008.

[4] Nathaniel Eldredge, Maria Gordina, and Laurent Saloff-Coste. Left-invariant geometries on SU(2) are uniformly doubling. *Geom. Funct. Anal.*, 28(5):1321–1367, 2018.

[5] A. A. Grigorʹyan. The heat equation on noncompact Riemannian manifolds. *Mat. Sb.*, 182(1):55–87, 1991.

[6] Alexander Grigorʹyan. Heat kernel upper bounds on a complete non-compact manifold. *Rev. Mat. Iberoamericana*, 10(2):395–452, 1994.

[7] Bruce Kleiner. A new proof of Gromov's theorem on groups of polynomial growth. *J. Amer. Math. Soc.*, 23(3):815–829, 2010.

[8] P. Maheux and L. Saloff-Coste. Analyse sur les boules d'un opérateur sous-elliptique. *Math. Ann.*, 303(4):713–740, 1995.

[9] L. Saloff-Coste. A note on Poincaré, Sobolev, and Harnack inequalities. *Internat. Math. Res. Notices*, 2:27–38, 1992.

[10] Laurent Saloff-Coste. Uniformly elliptic operators on Riemannian manifolds. *J. Differential Geom.*, 36(2):417–450, 1992.
[11] Laurent Saloff-Coste. *Aspects of Sobolev-type inequalities*, volume 289 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2002.
[12] K. T. Sturm. Analysis on local Dirichlet spaces. III. The parabolic Harnack inequality. *J. Math. Pures Appl. (9)*, 75(3):273–297, 1996.
[13] Nicolas Th. Varopoulos. Fonctions harmoniques sur les groupes de Lie. *C. R. Acad. Sci. Paris Sér. I Math.*, 304(17):519–521, 1987.

A variational characterization of the Sine$_\beta$ process

Matthias Erbar
(joint work with Martin Huesmann, Thomas Leblé)

We prove that, at every positive temperature, the infinite-volume free energy of the one dimensional log-gas, or beta-ensemble, has a unique minimiser, which is the Sine-beta process arising from random matrix theory. We rely on a quantitative displacement convexity argument at the level of point processes, and on the screening procedure introduced by Sandier-Serfaty.

The one-dimensional log-gas in finite volume can be defined as a system of particles interacting through a repulsive pairwise potential proportional to the logarithm of the distance, and confined by some external field. For a fixed value of $\beta > 0$, called the *inverse temperature* parameter, and for $N \geq 1$, we consider the probability measure $\mathbb{P}_{N,\beta}$ on $\vec{X}_N = (x_1, \dots, x_N) \in \mathbb{R}^N$ defined by the density

$$(1) \qquad d\mathbb{P}_{N,\beta}(\vec{X}_N) := \frac{1}{\mathrm{Z}_{N,\beta}} \exp\left(-\beta\left(\sum_{i<j} -\log|x_i - x_j| + \sum_{i=1}^N N\frac{x_i^2}{2}\right)\right),$$

with respect to the Lebesgue measure on $\mathbb{R}^N$. The quantity $\mathrm{Z}_{N,\beta}$ is a normalization constant, the *partition function*. We call $\mathbb{P}_{N,\beta}$ the *canonical Gibbs measure* of the log-gas. Part of the motivation for studying log-gases comes from Random Matrix Theory (RMT), for which $\mathbb{P}_{N,\beta}$ describes the joint law of N eigenvalues in certain classical models: the Gaussian orthogonal, unitary, symplectic ensemble respectively for $\beta = 1, 2, 4$, and the "tridiagonal model" discovered in [DE02] for arbitrary β. We refer to the book [For10] for a comprehensive presentation of the connection between log-gases and random matrices. Log-gases are also interesting from a statistical physics point of view, as a toy model with singular, long-range interaction.

Questions about such systems usually deal with the large N limit (also called thermodynamic, or infinite-volume limit) of the system, as encoded by certain observables. For example, in order to understand the "global" behavior, one may look at the empirical measure $\frac{1}{N}\sum_{i=1}^N \delta_{x_i}$, and asks about the typical behavior of this *random* probability measure on $\mathbb{R}$ as N tends to infinity. By now, this is fairly well understood, we refer e.g. to the recent lecture notes [Ser17] and the references therein. In the present work, we are rather interested in the asymptotic behavior at *microscopic* scale.

Let $\mathcal{C}_{N,0}$ be the point configuration $\mathcal{C}_{N,0} := \sum_{i=1}^{N} \delta_{Nx_i}$, where $\vec{X}_N = (x_1, \dots, x_N)$ is distributed according to $\mathbb{P}_{N,\beta}$. The limit in law of $\mathcal{C}_{N,0}$ as $N \to \infty$ was constructed in [VV09] and named the Sine_β process. We refer to [KS09] for a different construction of a process that turns out to be the same, and to [VV17a, VV17b] for recent developments concerning Sine_β. This process is the universal behavior of log-gases (in the bulk), in the sense that replacing the $\frac{x_i^2}{2}$ term in (1) by a general potential $V(x_i)$ yields the same microscopic limit, up to a scaling on the average density of points (our convention is that Sine_β has intensity 1) and mild assumptions on V, see e.g. [BEY14].

In [LS17], the infinite-volume free energy of the log-gas (and of other related systems) was introduced as the weighted sum $\mathcal{F}_\beta := \beta\mathcal{W} + \mathcal{E}$, where the functionals $\mathcal{W}, \mathcal{E}$ and the free energy $\mathcal{F}_\beta$ are defined on the space of stationary random point processes. The functional $\mathcal{W}$ corresponds to the "renormalized energy" introduced in [SS12], and $\mathcal{E}$ is the usual *specific relative entropy.* The free energy $\mathcal{F}_\beta$ appears in [LS17] as the rate function for a large deviation principle concerning the behavior of log-gases at the microscopic level. If $\vec{X}_N = (x_1, \dots, x_N)$ is an N-tuple of particles distributed according to the Gibbs measure (1) of a log-gas, they are known to typically arrange themselves on an interval approximately given by $[-2, 2]$. For x in this interval, we let $\mathcal{C}_{N,\mathrm{x}}$ be the point configuration $(x_1, \dots, x_N)$ "seen from x", namely $\mathcal{C}_{N,\mathrm{x}} := \sum_{i=1}^{N} \delta_{N(x_i - \mathrm{x})}$. We may then consider the *empirical field* $\mathrm{Emp}_N(\vec{X}_N)$ of the system in the state $\vec{X}_N$, defined by averaging $\mathcal{C}_{N,\mathrm{x}}$ over x, which yields a probability measure on (finite) point configurations in $\mathbb{R}$, and it was proven in [LS17] that its law satisfies a large deviation principle, at speed N, with a rate function built using $\mathcal{F}_\beta$. We refer to the paper cited above for a precise statement, here it suffices to say that *understanding the minimisers* of $\mathcal{F}_\beta$ gives an *understanding of the typical microscopic behavior* of a finite N log-gas at temperature β, when N is large.

For any β in $(0, +\infty)$, the functional $\mathcal{F}_\beta$ is known to be lower semi-continuous, with compact sub-level sets. In particular, it admits a compact subset of minimisers. However, the question of uniqueness of minimisers for $\mathcal{F}_\beta$ remained open, and is our main result in this work.

Theorem 1. *For any β in $(0, +\infty)$, the free energy $\mathcal{F}_\beta$ has a unique minimiser.*

Since it was proven in [LS17, Corollary 1.2] that Sine_β minimises $\mathcal{F}_\beta$, we deduce that for any β in $(0, +\infty)$, the Sine_β process is the unique minimiser of $\mathcal{F}_\beta$. This provides a variational characterization of Sine_β.

Let us briefly highlight the main ideas of the proof, where we will leverage ideas from optimal transport. Since the free energy $\mathcal{F}_\beta$ is affine, it is not strictly convex for the usual linear interpolation of probability measures. We use instead the notion of *displacement convexity*, which was introduced in [McC97] to remedy situations where energy functionals are not convex in the usual sense.

We start with two stationary point processes $\mathsf{P}^0, \mathsf{P}^1$ such that $\mathsf{P}^0 \neq \mathsf{P}^1$, and assume that both are minimisers of $\mathcal{F}_\beta$. We cannot argue via displacement convexity directly on the level of $\mathsf{P}^0, \mathsf{P}^1$ since they are probability measures on *infinite* point configurations. Instead, we use transport theory between *finite* measures together with a careful approximation argument relying on screening of electric fields. More precisely,

$$\mathcal{F}_\beta(\mathsf{P}) = \lim_{R\to\infty} \frac{1}{|\Lambda_R|} \left(\beta \mathcal{W}_R(\mathsf{P}) + \mathcal{E}_R(\mathsf{P})\right),$$

where $\mathcal{W}_R, \mathcal{E}_R$ are quantities (the energy, and the relative entropy) depending on the restriction of P to the line segment $\Lambda_R := [-R, R]$. Then we can approximate $\mathsf{P}^0, \mathsf{P}^1$ by finite point processes $\mathsf{P}^0_R, \mathsf{P}^1_R$ which are the restriction of $\mathsf{P}^0, \mathsf{P}^1$ to Λ_R carefully modified at the extremities of Λ_R via a version of the "screening procedure" of Sandier-Serfaty in order to obain processes with exactly $2R$ point such that entropy and interaction energy are well approximated. Viewing $\mathsf{P}^0_R, \mathsf{P}^1_R$ as probability measures on $[-R, R]^{2R}$, let T_R be the optimal transport map which pushes P^0_R onto P^1_R and let $\mathsf{P}^{\mathsf{h}}_R$ be the half-interpolate along the displacement T_R, i.e. the push-forward of P^0_R by $\frac{1}{2}(\mathrm{Id} + \mathsf{T}_R)$. Since relative entropy is displacement convex, we have

$$\mathcal{E}_R[\mathsf{P}^{\mathsf{h}}_R] \leq \frac{1}{2}\left(\mathcal{E}_R(\mathsf{P}^0) + \mathcal{E}_R(\mathsf{P}^1)\right).$$

Moreover, the interaction potential $-\log|x-y|$ is strictly convex, hence the energy $\mathcal{W}_R$ is also displacement convex. More precisely, we have

$$\mathcal{W}_R[\mathsf{P}^{\mathsf{h}}_R] \leq \frac{1}{2}\left(\mathcal{W}_R(\mathsf{P}^0) + \mathcal{W}_R(\mathsf{P}^1)\right) - \mathrm{Gain}_R,$$

where $\mathrm{Gain}_R > 0$ is some quantitative positive gain due to the *strict* convexity of the interaction. With some work, using the fact that $\mathsf{P}^0, \mathsf{P}^1$ are stationary, we are able to show that the gain is at least proportional to R.

We turn $\mathsf{P}^{\mathsf{h}}_R$ into a process on the full line by pasting independent copies of itself on disjoint intervals of length $2R$. The relative entropy is additive, and we can show that the interaction of two independent copies is almost zero. Thanks to the quantitative convexity estimate, we obtain a global candidate P^{h} for which

$$\mathcal{F}_\beta(\mathsf{P}^{\mathsf{h}}) < \frac{1}{2}\left(\mathcal{F}_\beta(\mathsf{P}^0) + \mathcal{F}_\beta(\mathsf{P}^1)\right),$$

which is the desired contradiction, hence the minimiser of $\mathcal{F}_\beta$ is unique.

References

[DE02] I. Dumitriu and A. Edelman. Matrix models for beta ensembles. *Journal of Mathematical Physics*, 43(11):5830–5847, 2002.

[For10] P. Forrester. *Log-gases and random matrices (LMS-34)*. Princeton University Press, 2010.

[McC97] R. McCann. A Convexity Principle for Interacting Gases. *Advances in Mathematics*, 128(1):153 – 179, 1997.

[KS09] R. Killip and M. Stoiciu. Eigenvalue statistics for CMV matrices: from Poisson to clock via random matrix ensembles. *Duke Math. J.*, 146(3):361–399, 2009.

[LS17] T. Leblé and S. Serfaty. Large deviation principle for empirical fields of Log and Riesz gases. *Inventiones mathematicae*, 210(3):645–757, 2017.

[VV09] B. Valkó and B. Virág. Continuum limits of random matrices and the Brownian carousel. *Invent. math.*, 177(3):463–508, 2009.

[VV17a] B. Valkó and B. Virág. The Sine$_\beta$ operator. *Invent. math.*, 209(1):275–327, 2017.
[VV17b] B. Valkó and B. Virág. Operator limit of the circular beta ensemble. *arXiv preprint arXiv:1710.06988*, 2017.
[Ser17] S. Serfaty. Microscopic description of log and Coulomb gases. *arXiv preprint arXiv:1709.04089*, 2017.
[SS12] É. Sandier and S. Serfaty. From the Ginzburg-Landau model to vortex lattice problems. *Communications in Mathematical Physics*, 313(3):635–743, 2012.
[BEY14] P. Bourgade, L. Erdös, and H.-T. Yau. Universality of general β-ensembles. *Duke Math. J.*, 163(6):1127–1190, 2014.

Mott's law for the Miller-Abrahams random resistor network and for Mott's random walk

Alessandra Faggionato

Mott's law [10, 11] is a physical law concerning the low temperature decay of conductivity in doped semiconductors and, in general, in strongly disordered d–dimensional solids ($d \geq 2$) in the regime of Anderson localization. It states that, for β large,

$$\sigma(\beta) \asymp A(\beta) \exp\left\{ - c_0 \beta^{\frac{1+\alpha}{1+\alpha+d}} \right\}, \tag{1}$$

where $\beta = 1/k_B T$ is the inverse temperature, $\sigma(\beta)$ is the conductivity, $A(\beta)$ depends only weakly on β, $c_0 > 0$ and $\alpha \geq 0$ are β–independent physical parameters of the solid.

We recall that doped semiconductors are crystalline solids with inserted atoms of a different type, called *impurities*. Due to Anderson localization, the wavefunctions of conduction electrons are localized around impurities and can hop by quantum tunneling. In the regime of low impurity density one can model the electron transport by independent random walks, encoding the electron interactions into the jump rates. The final object is therefore a suitable random walk on a marked simple point process, which we call *Mott's random walk*.

We now give the rigorous definition of Mott's random walk. Let $\{x_i\}$ be the realization of a stationary and ergodic simple point process on $\mathbb{R}^d$ having finite mean density. We mark each point x_i with a random variable E_i, called *energy mark*. The random variables (E_i) are i.i.d. with common distribution ν. Physically relevant distributions ν are of the form

$$\nu_\alpha(dE) \propto |E|^\alpha \mathbb{I}(|E| \leq C_0) dE\,, \tag{2}$$

for suitable constants $C_0 > 0$ and $\alpha \geq 0$. Then $\omega := \{(x_i, E_i)\}$ is the realization of the marked simple point process. We call $\mathbb{P}$ its law.

Given ω as above, Mott's random walk $(X_t^\omega)_{t\geq 0}$ is the continuous time random walk with state space $\{x_i\}$ and probability rate for a jump from x_i to $x_j \neq x_i$ given by

$$c_{x_i,x_j}(\omega) = \exp\left\{ -|x_i - x_j| - \beta(|E_i| + |E_j| + |E_i - E_j|) \right\}. \tag{3}$$

In [1] we have proved (under suitable conditions and for $d \geq 2$) that for $\mathbb{P}$–a.a. $\omega = \{(x_i, E_i)\}$, as $\epsilon \downarrow 0$, the diffusively rescaled Mott's random walk

$$(4) \qquad \left(\epsilon X^{\omega}_{t/\epsilon^2}\right)_{t\geq 0}$$

weakly converges to a Brownian motion with diffusion matrix $D(\beta)$, which is non-random and strictly positive. Moreover, $D(\beta)$ admits the following variational characterization: for all $a \in \mathbb{R}^d$ it holds

$$(5) \qquad a \cdot D(\beta) a = \inf_{f \in L^\infty(\mathbb{P}_0)} \frac{1}{2} \int d\mathbb{P}_0(\omega) \sum_{x_i \in \hat{\omega}} c_{0,x_i}(\omega) \left(a \cdot x_i - \nabla_{x_i} f(\omega)\right)^2 .$$

Above $\mathbb{P}_0$ denotes the Palm distribution associated to $\mathbb{P}$, $\hat{\omega} := \{x_i\}$ if $\omega = \{(x_i, E_i)\}$, $\nabla_{x_i} f(\omega) := f(\tau_{x_i}\omega) - f(\omega)$ and finally $\tau_x \omega := \{(x_i - x, E_i)\}$ if $\omega = \{(x_i, E_i)\}$ and $x \in \mathbb{R}^d$. From now on we restrict to isotropic media, thus implying that $D(\beta) = d(\beta)\mathbb{I}$. Assuming that Mott's random walk satisfies Einstein's relation (as proved in [4] for $d = 1$), Mott's law (1) can be restated by replacing $\sigma(\beta)$ with $d(\beta)$ in (1). We point out that in [5, 8] we proved lower and upper bounds on the diffusion coefficient $d(\beta)$ in agreement with Mott's law: for suitable positive constants $c_1, c_2 > 0$ and for β large it holds

$$(6) \qquad \exp\left\{ - c_1 \beta^{\frac{1+\alpha}{1+\alpha+d}}\right\} \leq d(\beta) \leq \exp\left\{ - c_2 \beta^{\frac{1+\alpha}{1+\alpha+d}}\right\} .$$

In our talk we discuss the derivation of Mott's law for $d(\beta)$ recently obtained in [2]. More precisely, under suitable conditions, we have proved that

$$(7) \qquad \lim_{\beta \to \infty} \beta^{-\frac{1+\alpha}{d+1+\alpha}} \ln d(\beta) = c ,$$

where $c < 0$ admits a percolative characterization. The proof uses previous results in homogenization and percolation theory recently obtained in [3, 6, 7] and the variational characterization (5).

By using 2-scale homogenization, we have proved in [3] that $d(\beta)$ coincides $\mathbb{P}$–a.s. with $\lim_{L\to\infty} L^{2-d} \sigma_L(\omega, \beta)$, where $\sigma_L(\omega, \beta)$ is the effective conductivity along a given direction of the so called Miller-Abrahams random resistor network in a box of size L (cf. [9, 10]). As a consequence, the limit (7) implies also Mott's law for the infinite volume rescaled conductivity of the Miller-Abrahams random resistor network.

References

[1] P. Caputo, A. Faggionato, T. Prescott. *Invariance principle for Mott variable range hopping and other walks on point processes.* Ann. Inst. H. Poincaré Probab. Statist. **49** (2013), 654–697.

[2] A. Faggionato. *Mott's law for the Miller–Abrahams random resistor network and for Mott's random walk.* In preparation.

[3] A. Faggionato. *Miller–Abrahams random resistor network, Mott's random walk and 2-scale homogenization.* In preparation.

[4] *Einstein relation and linear response in one-dimensional Mott variable-range hopping.* Ann. Inst. H. Poincaré Probab. Statist. **55** (2019), 1477–1508.

[5] A. Faggionato, P. Mathieu. *Mott law as upper bound for a random walk in a random environment.* Comm. Math. Phys. **281** (2008), 263–286.

[6] A. Faggionato, H.A. Mimun. *Connection probabilities in Poisson random graphs with uniformly bounded edges.* ALEA **16** (2019), 463–486.
[7] A. Faggionato, H.A. Mimun. *Left-right crossings in the Miller-Abrahams random resistor network on a Poisson point process.* Forthcoming
[8] A. Faggionato, H. Schulz-Baldes, D. Spehner. *Mott law as lower bound for a random walk in a random environment.* Comm. Math. Phys. (2006) **263**, 21–64.
[9] A. Miller, E. Abrahams. *Impurity Conduction at Low Concentrations.* Phys. Rev. **120** (1960), 745–755.
[10] M. Pollak, M. Ortuño, A. Frydman. *The electron glass.* First edition, Cambridge University Press, United Kingdom, 2013.
[11] S. Shklovskii, A.L. Efros. *Electronic Properties of Doped Semiconductors.* Springer Verlag, Berlin, 1984.

Existence and uniqueness of the Liouville quantum gravity metric for $\gamma \in (0, 2)$

Ewain Gwynne
(joint work with Jason Miller)

Fix $\gamma \in (0, 2)$, let $U \subset \mathbb{C}$ be an open domain, and let h be the Gaussian free field (GFF) on U, or some minor variant thereof. The γ-*Liouville quantum gravity (LQG)* surface described by (U, h) is formally the random two-dimensional Riemannian manifold with metric tensor

$$e^{\gamma h}\,(dx^2 + dy^2), \tag{1}$$

where $dx^2 + dy^2$ is the Euclidean Riemannian metric tensor.

LQG surfaces were first introduced in the physics literature by Polyakov in the 1980's as canonical models of random two-dimensional Riemannian manifolds. Another motivation to study LQG surfaces is that they describe the scaling limit of random planar maps. The special case when $\gamma = \sqrt{8/3}$ corresponds to uniform random planar maps, including uniform triangulations, quadrangulations, etc. Other values of γ correspond to random planar maps sampled with probability proportional to the partition function of an appropriate statistical mechanics model on the map.

The definition (1) of the LQG metric tensor does not make literal sense since h is only a distribution, not a function, so it does not have well-defined pointwise values and hence cannot be exponentiated pointwise. Nevertheless, it is well known that one can make sense of the associated volume form $\mu_h = e^{\gamma h(z)}\,dz$ (where dz denotes Lebesgue measure) as a random measure on U via various regularization procedures [10, 5].

In order for γ-LQG to be a reasonable model of a "random two-dimensional Riemannian manifold", one also needs to construct the Riemannian distance function associated with an LQG surface, which should be a random metric D_h on U. Certain special LQG surfaces equipped with this distance function should describe the scaling limits of random planar maps equipped with the graph distance with respect to the Gromov-Hausdorff topology. Constructing a distance function on γ-LQG is a much more difficult problem than constructing the measure μ_h. Indeed, any natural regularization schemes for LQG distances involves minimizing over a large collection of paths, which results in a substantial degree of non-linearity.

Prior to our work, the γ-LQG distance function has only been constructed in the special case when $\gamma = \sqrt{8/3}$ in a series of works by Miller and Sheffield [13, 14]. In this case, for certain special choices of the pair (U, h), the random metric space (U, D_h) agrees in law with a *Brownian surface*, such as the Brownian map [11, 12]. These Brownian surfaces are continuum random metric spaces which arise as the scaling limits of uniform random planar maps with respect to the Gromov-Hausdorff topology.

We construct the γ-LQG distance function for all $\gamma \in (0, 2)$ via direct regularization of the Riemannian distance function associated with (1). We now describe how this distance function is constructed. It is shown in [2, 3] that for each $\gamma \in (0, 2)$, there is an exponent $d_\gamma > 2$ which describes distances in various approximations of γ-LQG (e.g., random planar maps). *A posteriori*, once the γ-LQG distance function is constructed, one can show that d_γ is its Hausdorff dimension [9]. The value of d_γ is not known explicitly except in the case when $\gamma = \sqrt{8/3}$, in which case we know that $d_{\sqrt{8/3}} = 4$. For $\gamma \in (0, 2)$, we define

$$\xi = \xi_\gamma := \frac{\gamma}{d_\gamma}. \tag{2}$$

For $\varepsilon > 0$, let $\{h_\varepsilon\}_{\varepsilon>0}$ be a family of continuous functions which approximate the GFF h as $\varepsilon \to 0$ (for technical convenience we take h_ε to be the convolution of h with the heat kernel $p_\varepsilon(z) := \frac{1}{2\pi\varepsilon} e^{-|z|^2/2\varepsilon}$). For $z, w \in \mathbb{C}$ and $\varepsilon > 0$, we define

$$D_h^\varepsilon(z, w) := \inf_{P: z \to w} \int_0^1 e^{\xi h_\varepsilon(P(t))} |P'(t)| \, dt \tag{3}$$

where the infimum is over all piecewise continuously differentiable paths from z to w.

Let $\mathfrak{a}_\varepsilon$ be the median of the D_h^ε-distance between the left and right boundaries of the unit square in the case when h is a whole-plane GFF normalized so that its average over the unit circle is zero. It was shown by Ding, Dubédat, Dunlap, and Falconet [1] that the laws of the functions $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ are tight w.r.t. the local uniform topology on $\mathbb{C} \times \mathbb{C}$, and every possible subsequential limit is a metric on $\mathbb{C}$ which induces the Euclidean topology. In [7], building on [1, 8, 4, 6] we proved that the subsequential limit is unique that that the convergence in fact occurs in probability.

Theorem 1 (Convergence of LFPP)**.** *The random distance functions $\mathfrak{a}_\varepsilon^{-1} D_h^\varepsilon$ converge in probability w.r.t. the local uniform topology on $\mathbb{C} \times \mathbb{C}$ to a random distance function D_h on $\mathbb{C}$ which is a.s. determined by h.*

We define the limiting distance function in Theorem 1 to be the γ-LQG distance function. In addition to proving the existence of the limit we also show that the limiting distance function is uniquely characterized by a list of natural properties that any reasonable notion of a γ-LQG distance function should satisfy, so is in some sense the only "correct" distance function on γ-LQG.

Our proofs are completely elementary in the sense that they use only basic properties of the Gaussian free field.

References

[1] J. Ding, J. Dubédat, A. Dunlap, and H. Falconet. Tightness of Liouville first passage percolation for $\gamma \in (0,2)$. Apr 2019.

[2] J. Ding and E. Gwynne. The fractal dimension of Liouville quantum gravity: universality, monotonicity, and bounds. *ArXiv e-prints*, July 2018.

[3] J. Ding, O. Zeitouni, and F. Zhang. Heat kernel for Liouville Brownian motion and Liouville graph distance. *Communications in Mathematical Physics*, to appear, 2018.

[4] J. Dubédat, H. Falconet, E. Gwynne, J. Pfeffer, and X. Sun. Basic properties of Liouville quantum gravity metrics for $\gamma \in (0,2)$. 2019.

[5] Bertrand Duplantier and Scott Sheffield. Liouville quantum gravity and KPZ. *Invent. Math.*, 185(2):333–393, 2011.

[6] E. Gwynne and J. Miller. Confluence of geodesics in Liouville quantum gravity for $\gamma \in (0,2)$. 2019.

[7] E. Gwynne and J. Miller. Existence and uniqueness of the Liouville quantum gravity metric for $\gamma \in (0,2)$. 2019.

[8] E. Gwynne and J. Miller. Local metrics of the Gaussian free field. 2019.

[9] E. Gwynne and J. Pfeffer. The KPZ formula for the Liouville quantum gravity metric. In preparation, 2019.

[10] Jean-Pierre Kahane. Sur le chaos multiplicatif. *Ann. Sci. Math. Québec*, 9(2):105–150, 1985.

[11] Jean-François Le Gall. Uniqueness and universality of the Brownian map. *Ann. Probab.*, 41(4):2880–2960, 2013.

[12] Grégory Miermont. The Brownian map is the scaling limit of uniform random plane quadrangulations. *Acta Math.*, 210(2):319–401, 2013.

[13] J. Miller and S. Sheffield. Liouville quantum gravity and the Brownian map I: The QLE(8/3,0) metric. *ArXiv e-prints*, July 2015.

[14] J. Miller and S. Sheffield. Liouville quantum gravity and the Brownian map II: geodesics and continuity of the embedding. *ArXiv e-prints*, May 2016.

Differential Harnack inequalities on path space

Robert Haslhofer

(joint work with Eva Kopfer and Aaron Naber)

Consider a Riemannian manifold (M^n, g) and for $f : M \to \mathbb{R}$ denote by $f_t : M \to \mathbb{R}$ the solution of the heat equation $(\partial_t - \Delta)f_t = 0$ with $f_0 = f$. The classical Li-Yau differential Harnack inequality [3] tells us that if f is nonnegative and $\mathrm{Rc} \geq 0$, then we have

$$\frac{\Delta f_t}{f_t} - \frac{|\nabla f_t|^2}{f_t^2} + \frac{n}{2t} \geq 0\,. \tag{1}$$

Hamilton [1], under the more restrictive assumption that $\sec \geq 0$ and $\nabla \mathrm{Rc} = 0$, proved the Hessian version of (1) given by

$$\frac{\nabla^2 f_t}{f_t} - \frac{\nabla f_t \otimes \nabla f_t}{f_t^2} + \frac{g}{2t} \geq 0\,. \tag{2}$$

In [2], we found differential Harnack inequalities on path space, which can be viewed as generalizations of the above classical inequalities on manifolds.

Recall that path space P_xM is the space of all continuous curves $\gamma : [0,\infty) \to M$ starting at x. It comes equipped with the Wiener measure $\mathbb{P}_x$ of Brownian motion, which is characterized by the formula

$$\mathbb{P}_x[\gamma_{t_1} \in U_1, \ldots, \gamma_{t_k} \in U_k] = \int_{U_1\times\ldots\times U_k} \rho_{t_1}(x, dy_1) \ldots \rho_{t_k - t_{k-1}}(y_{k-1}, dy_k), \tag{3}$$

where $\rho_t(x, dy)$ denotes the heat kernel measure.

We consider the following new notions of gradients, Hessians and Laplacians on path space. Let $\varphi : [0,\infty) \to \mathbb{R}$ be an H_0^1-function. The φ-gradient $\nabla_\varphi F : P_xM \to T_xM$ is defined by

$$\langle \nabla_\varphi F, v\rangle = \frac{d}{ds}\Big|_{s=0} F(\gamma_s)\,, \tag{4}$$

where γ_s is a family of curves with $\partial_s|_{s=0}\gamma_s = \varphi V$. Here, V denotes the vector field along γ obtained by parallel translating v. Similarly, the φ-Hessian $\mathrm{Hess}_\varphi F : P_xM \to T_x^*M \otimes T_x^*M$ is defined by

$$\mathrm{Hess}_\varphi F(v,v) = \frac{d^2}{ds^2}\Big|_{s=0} F(\gamma_s)\,, \tag{5}$$

where γ_s is a family of curves with $\partial_s|_{s=0}\gamma_s = \varphi V$ and $\nabla_{\varphi V}\big(\partial_s\gamma_s\big) = 0$. Finally, the φ-Laplacian $\Delta_\varphi F : P_xM \to \mathbb{R}$ obtained by tracing the φ-Hessian:

$$\Delta_\varphi F = \mathrm{tr}\big(\mathrm{Hess}_\varphi F\big). \tag{6}$$

Let us now state our main theorem in the context of Ricci flat spaces:

Theorem. *Let M be a Ricci-flat manifold, and let $F : P_xM \to \mathbb{R}^+$ be a nonnegative function on path space. Then, for all $\varphi \in H_0^1(\mathbb{R}^+)$ we have the inequality*

$$\frac{\mathbb{E}_x\left[\Delta_\varphi F\right]}{\mathbb{E}_x[F]} - \frac{\big|\mathbb{E}_x\left[\nabla_\varphi F\right]\big|^2}{\mathbb{E}_x[F]^2} + \frac{n}{2}||\varphi||^2 \geq 0. \tag{7}$$

For illustration, consider a function $F : P_0M \to \mathbb{R}^+$ which only depends on the value of the curve at a single time. Namely, let $F(\gamma) = f(\gamma_t)$, where $f : M \to \mathbb{R}^+$ and $t > 0$ are fixed. Let $\varphi(s) = \frac{s}{t}$ for $s \leq t$ and $\varphi(s) = 1$ for $s \geq t$. First, note that $||\varphi||^2 = \frac{1}{t}$. Next, it is an instructive exercise to compute

$$\begin{aligned} \nabla_\varphi F(\gamma) &= P_t(\gamma)\nabla f(\gamma_t)\,, \\ \Delta_\varphi F(\gamma) &= \Delta f(\gamma_t)\,, \end{aligned} \tag{8}$$

where $P_t(\gamma) : T_{\gamma(t)}M \to T_xM$ denotes parallel transport. Finally, using this and the Feynman-Kac formula we can derive the equalities

$$\begin{aligned} \mathbb{E}_x[F] &= \int_M f(y)\rho_t(x,dy) = f_t(x)\,, \\ \mathbb{E}_x[\Delta_\varphi F] &= \Delta f_t(x)\,, \\ \mathbb{E}_x[\nabla_\varphi F] &= \nabla f_t(x)\,. \end{aligned} \tag{9}$$

Plugging all of this into (7) we obtain precisely the Li-Yau Harnack inequality (1).

Plugging in a (smeared) delta function, our main theorem implies

$$-\Delta_\varphi \ln \mathbb{P}_x \leq \frac{n}{2} \tag{10}$$

for all normalized φ, which can be viewed as Laplace comparison theorem for the Wiener measure on the path space of Ricci-flat manifolds.

Finally, we also have a differential Matrix Harnack inequality on path space of general manifolds, meant to generalize Hamilton's Matrix Harnack (2):

Theorem. *Let $F : P_xM \to \mathbb{R}^+$ be a nonnegative Σ_T-measurable function on path space. Then, for every $\varphi \in H_0^1(\mathbb{R}^+)$ we have the inequality*

$$\begin{aligned}\frac{\mathbb{E}_x[\mathrm{Hess}_\varphi F]}{\mathbb{E}_x[F]} &- \frac{\mathbb{E}_x[\nabla_\varphi F] \otimes \mathbb{E}_x[\nabla_\varphi F]}{\mathbb{E}_x[F]^2} \\ &+ \left(\frac{1}{2} + C_T(\mathrm{Rc}) + C_T(\mathrm{Rm}, \nabla \mathrm{Rc})\frac{\mathbb{E}_x[F^2]^{1/2}}{\mathbb{E}_x[F]}\right) ||\varphi||^2 g_x \geq 0,\end{aligned} \tag{11}$$

where $C_T(\mathrm{Rc}) < \infty$ and $C_T(\mathrm{Rm}, \nabla\mathrm{Rc}) < \infty$ are constants, which converge to 0 as $|\mathrm{Rc}| \to 0$ and $|\mathrm{Rm}| + |\nabla\mathrm{Rc}| \to 0$, respectively.

References

[1] R. Hamilton, *A Matrix Harnack estimate for the heat equation*, Comm. Anal. Geom. 1(1):113–126, 1993.
[2] R. Haslhofer, E. Kopfer, A. Naber, *Differential Harnack inequalities on path space*, preprint, 2019.
[3] P. Li, S.-T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. 156(3-4):153–201, 1986.

Transport inequalities on the Poisson space

Ronan Herry
(joint work with N. Gozlan, G.Peccati)

We establish new transport-entropy inequalities for various random point measures including the important case of the Poisson random measures. Roughly speaking, understanding what happens to transport-entropy inequalities in the framework of random point processes serves as the basic motivation behind this work. The investigation of transport-entropy inequalities starts in the nineties with works by Marton [10, 11] and by Talagrand [16], in connection with the concentration of measure phenomenon for product measures. We refer to [9, Chapter 6], [17, Chapter 22], [3, Chapter 9] and [7, 6] for general introductions and surveys on these intimately related topics. For simplicity, we only present what happens to *Marton's inequality* on the Poisson space.

The *relative entropy* H of ν_1 with respect to ν_2 is defined by

$$H(\nu_1|\nu_2) = \int \log \frac{d\nu_1}{d\nu_2}\, \mathrm{d}\nu_1, \tag{1}$$

if $\nu_1 \ll \nu_2$, and $+\infty$ otherwise. The *Maton cost* which is a variant of the Monge-Kantorovich costs is defined as follows:

$$M^2(\nu_1|\nu_2) = \inf \mathbb{E}\left[\mathbb{P}(X_1|X_2)^2\right], \tag{2}$$

where the infimum runs over the set of all couples (X_1, X_2) of random variables such that $X_1 \sim \nu_1$ and $X_2 \sim \nu_2$. We refer to [8] for the presentation of the unifying framework of generalized transport costs which contains in particular Monge-Kantorovich as well as Marton transport costs. Marton's transport cost also admits the following explicit expression: if ν_1 and ν_2 are absolutely continuous with respect to some measure μ on Z, with $\nu_1 = f_1\mu$ and $\nu_2 = f_2\mu$, then by [10]

$$M^2(\nu_1|\nu_2) = \int \left[1 - \frac{f_1}{f_2}\right]_+^2 f_2 \, \mathrm{d}\mu. \tag{3}$$

Contrary to Talagrand's inequality which is satisfied only by some specific probability measures, according to [10] Marton's inequality holds for any probability measures. A classical version of Marton's universal transport inequality reads as follows: for any probability measure μ on Z, it holds

$$M^2(\nu_1|\nu_2) \le 4H(\nu_1|\mu) + 4H(\nu_2|\mu), \tag{4}$$

for all $\nu_1, \nu_2 \in P(Z)$. One can understand this inequality as a reinforcement of the classical Csiszar-Kullback-Pinsker inequality (see *e.g* [7] and the references therein) comparing the squared total variation distance to relative entropy. We refer to [5, 13, 14] for subsequent refinements of Marton's inequality. Similarly to the classical Talagrand's inequality, Marton's inequality has interesting consequences in terms of concentration of measure. As [10] shows, Eq. 4 gives back the universal concentration of measure inequalities for product measures involving the so-called "convex distance" discovered by Talagrand in [15]. To avoid entering into too technical details in this introduction, let us recall a more concrete application of Eq. 4 in terms of deviation inequalities for convex functions. Namely, if we equip $Z = \mathbb{R}^p$ with the standard Euclidean norm and $\mu \in P(\mathbb{R}^p)$ has a bounded support whose diameter is denoted by D, then for any $n \ge 1$, and for any vector $(X_1, \dots, X_n)$ of i.i.d random variables with common law μ, it holds

$$\mathbb{P}(f(X_1, \dots, X_n) \ge t) \le e^{-t^2/4D^2}, \qquad \forall t \ge 0, \tag{5}$$

for all *convex* or *concave* function $f\colon (\mathbb{R}^p)^n \to \mathbb{R}$ which is of mean 0 with respect to $\mu^{\otimes n}$ and 1-Lipschitz with respect to the Euclidean norm on $(\mathbb{R}^p)^n$.

We consider a Π_ν that is the law of a Poisson point process with intensity ν, which, for simplicity, we assume finite. In our work, we highlight a general principle leading to transport-entropy inequalities for point processes: Π_ν inherits the transport inequalities satisfied by its intensity measure ν. We now state a representative result illustrating this general rule in the setting of Eq. 4. We introduce the following cost: for any $\Pi_1, \Pi_2 \in P(M_b(Z))$,

$$\mathbb{M}^2(\Pi_1|\Pi_2) = \inf \mathbb{E}\left[\int \mathbb{E}\left[\left[1 - \frac{\eta_1(x)}{\eta_2(x)}\right]_+ \middle| \eta_2\right]^2 \eta_2(\mathrm{d}x)\right],$$

where $\eta_i(x)$ is a slightly abusive notation for $\eta_i(\{x\})$, and where the infimum runs over the set of couples (η_1, η_2) of random measures such that $\eta_1 \sim \Pi_1$ and $\eta_2 \sim \Pi_2$.

Theorem 1. *For any $\nu \in M_b(Z)$, the Poisson point process Π_ν satisfies the following inequality: for all $\Pi_1, \Pi_2 \in P(M_b(Z))$,*

$$\mathbb{M}^2(\Pi_1|\Pi_2) \leq 4H(\Pi_1|\Pi_\nu) + 4H(\Pi_2|\Pi_\nu).$$

In the setting of Poisson point processes, [12] uses the Talagrand convex distance to prove concentration of measure results for Poisson random measures. We can recover, in the spirit of Marton's work, the results of [12] using Thm. 1. Building on the ideas of [4], [1] considers a different approach towards concentration on measure for Poisson point processes. They obtain various general conditions on a functional $F\colon M_b(Z) \to \mathbb{R}$ in order for the random variable $F(\eta)$ to satisfy a deviation inequality. Since the space $M_b(Z)$ does not come with a natural distance, a rather involved technical condition replaces the condition of being Lipschitz that is classical in the theory of concentration of measure. However, [1, 2] shows that the so-called *geometric U-statistics* always satisfy this condition, and hence always satisfy some concentration of measure. Based on Thm. 1, we recover a deviation inequality for U-statistics in the spirit of [1, 2] with a simple argument. We also obtain from Thm. 1 a modified logarithmic Sobolev inequality on the Poisson space for convex non-decreasing functionals.

References

[1] S. Bachmann and G. Peccati, *Concentration bounds for geometric Poisson functionals: logarithmic Sobolev inequalities revisited*, Electron. J. Probab. **21** (2016), Paper No. 6, 44.

[2] S. Bachmann and M. Reitzner, *Concentration for Poisson U-statistics: subgraph counts in random geometric graphs*, Stochastic Process. Appl. **128** (2018), 3327–3352.

[3] D. Bakry and I. Gentil and M. Ledoux, *Analysis and geometry of Markov diffusion operators*, Grundlehren der Mathematischen Wissenschaften **348** (2001). Springer, Cham.

[4] S. Boucheron and G. Lugosi and P. Massart, *Concentration inequalities using the entropy method*, Ann. Probab. **31** (2003) 1583–1614.

[5] A. Dembo, *Information inequalities and concentration of measure*, Ann. Probab. **25** (1997), 927–939.

[6] N. Gozlan, *Transport inequalities and concentration of measure*, ESAIM Proc. Surveys **51** (2015), 1–23.

[7] N. Gozlan and C. Léonard, *Transport inequalities. A survey*, Markov Process. Related Fields **16** (2010), 635–736.

[8] N. Gozlan and C. Roberto and P.-M. Samson and P. Tetali, *Kantorovich duality for general transport costs and applications*, J. Funct. Anal. **273** (2017), 3327–3405.

[9] M. Ledoux, *The concentration of measure phenomenon*, Mathematical Surveys and Monographs **89** (2001). American Mathematical Society, Providence, RI.

[10] K. Marton, *A measure concentration inequality for contracting Markov chains*, Geom. Funct. Anal. **6** (1996), 556–571.

[11] K. Marton, *Bounding $\overline{d}$-distance by informational divergence: a method to prove measure concentration*, Ann. Probab. **24** (1996), 857–866.

[12] M. Reitzner, *Poisson point processes: large deviation inequalities for the convex distance*, Electron. Commun. Probab. **18** (2013), no. 96, 7.

[13] P.-M. Samson, *Concentration inequalities for convex functions on product spaces*, Progr. Probab. **56** (2003), 33–52.

[14] P.-M. Samson, *Infimum-convolution description of concentration properties of product probability measures, with applications*, Ann. Inst. H. Poincaré Probab. Statist. **43** (2007), 321–338.
[15] M. Talagrand, *Concentration of measure and isoperimetric inequalities in product spaces*, Inst. Hautes Études Sci. Publ. Math. **81** (1995), 73–205.
[16] M. Talagrand, *Transportation cost for Gaussian and other product measures*, Geom. Funct. Anal. **6** (1996), 587–600.
[17] C. Villani, *Optimal transport*, Grundlehren der Mathematischen Wissenschaften **338** (2009). Springer-Verlag, Berlin.

Quantum Optimal Transport

David F. Hornshaw

We consider the following problem:

How do we transport qubits of arbitrary size "optimally"?

For all $n \in \mathbb{N}$, we understand an n-qubit to be a state on the finite-dimensional C^*-algebra $A_j := \otimes_{n=1}^{j} M_2(\mathbb{C})$. The arbitrary size condition requires us to pass to the limit, i.e. the CAR-algebra $A := \otimes_{n=1}^{\infty} M_2(\mathbb{C})$ and its state space $\mathcal{S}(A)$. The latter consists of all positive functionals on A of norm one.

The principal idea considered in several papers by E. Carlen and J. Maas for the finite-dimensional case is to consider a noncommutative, or quantum, version of a dynamic transport metric (cf. [2],[3]) in the spirit of Benamou-Brenier (cf. [1]). In fact, we consider the more general setting of AF-C^*-algebras in [4] for which the above is but one illustrative example.

We thus give an extension of quantum transport metrics to different infinite-dimensional cases based on the finite-dimensional ones. Returning to our example above, we require: a module derivation ∇ defined on A mapping into another AF-C^*-algebra and an energy functional $\mathcal{E}$ on a set of admissible paths.

We moreover require both to be compatible with the approximately finite-dimensional structures in use. Here, the majority of technical work takes place and we obtain a crucial locality property for ∇. In essence and upon choosing inner products induced by faithful traces on domain and codomain, locality ensures $\nabla(A_j) \subset A_j$ and $\nabla^*(A_j) \subset A_j$ for all $j \in \mathbb{N}$.

Locality allows us to control the continutiy equation given by ∇ by controlling all finite-dimensional ones given by $\nabla_{|A_j}$. This shows compatibility of admissible paths w.r.t. the finite-dimensional structure.

Energy functionals $\mathcal{E}_j$ are given on each finite-dimensional problem, where it is essential to use quasi-entropies as functionals in the integrand to mirror division by a density. Doing so allows us to apply different monotonicity properties of quasi-entropies upon restriction to C^*-subalgebras. We obtain a Γ-limit type energy functional $\mathcal{E} = \Gamma\text{-}\lim_{j\in\mathbb{N}} \mathcal{E}_j$ in the limit.

The answer to our starting question thus becomes: by defining first a finite-dimensional dynamic transport distance of states on A_j for all $j \in \mathbb{N}$, then extending to infinity using a locality property of a suitable module derivation. Examples include iteration of infinite tensor products

$$\nabla_j := \nabla_{j-1} \otimes I + I \otimes \nabla_1$$

starting with $\nabla_1(x) := i([\sigma_1, x], [\sigma_2, x], [\sigma_3, x])$. Here, we consider the Pauli matrices σ_k, $1 \leq k \leq 3$, as directions and commutators as directional derivatives.

Commonly, properties of finite-dimensional quantum dynamic transport lift readily. We thus recover relations to noncommutative relative entropy, lower Ricci bounds and implied functional inequalities.

References

[1] J.-D. Benamou and Y. Brenier, *A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem*, Numer. Math. **84** (2000), 375–393.

[2] E. A. Carlen and J. Maas, *Gradient flow and entropy inequalities for quantum Markov semigroups with detailed balance*, J. Funct. Anal. **273** (2017), 1810–1869.

[3] E. A. Carlen and J. Maas, *Non-commutative calculus, optimal transport and functional inequalities in dissipative quantum systems*, J. Statist. Phys. (2019) *Online First*, 1–60.

[4] D. Hornshaw, *Quantum optimal transport for approximately finite-dimensional C^*-algebras*, *Preprint* (2019) arXiv:1910.03312.

Brownian disks and excursion theory for Brownian motion indexed by the Brownian tree

Jean-François Le Gall

Brownian disks are models of random geometry that arise as scaling limits of random planar maps with a boundary when the size of the boundary tends to infinity. More precisely, for every $p \geq 1$, consider a random Boltzmann quadrangulation $\mathbf{Q}_p$ with a boundary of size $2p$: the probability that $\mathbf{Q}_p$ is equal to a given (deterministic) quadrangulation Q with boundary size $2p$ is proportional to 12^{-n}, where n is the number of faces of Q. Then, if one equips the vertex set $V(\mathbf{Q}_p)$ with the graph distance rescaled by the factor $(2p/3)^{-1/2}$, the resulting random metric space converges in distribution as $p \to \infty$, in the Gromov-Hausdorff sense, to a limiting random metric space $(\mathbf{D}_1, \Delta)$ called the Brownian disk with boundary size 1 (by scaling, one can then consider the boundary disk $\mathbf{D}_z$ with boundary size z, and one can also define the Brownian disk with fixed boundary size and volume).

The main goal of the lecture is to present a new construction of the Brownian disk and to develop certain applications. Similarly as for the Brownian sphere (also called the Brownian map), this construction is based on the random process called Brownian motion indexed by the Brownian tree. The Brownian tree is a variant of David Aldous' CRT, and is most conveniently viewed as the compact $\mathbf{R}$-tree $(\mathcal{T}_e, d_e)$ coded by a Brownian excursion $(e_s)_{0 \leq s \leq \sigma}$ under the Itô measure. Conditionally on $\mathcal{T}_e$, one may consider the centered Gaussian process $(Y_a)_{a \in \mathcal{T}_e}$ such that $Y_\rho = 0$, where ρ denotes the root of $\mathcal{T}_e$, and $E[|Y_a - Y_b|^2] = d_e(a, b)$ for every $a, b \in \mathcal{T}_e$. The process $(Y_a)_{a \in \mathcal{T}_e}$ is then called Brownian motion indexed by the Brownian tree, or one may say that the quantities Y_a are Brownian labels assigned to the vertices of $\mathcal{T}_e$.

The excursion theory developed in [1] provides a description of the connected components of the set $\{a \in \mathcal{T}_e : Y_a > 0\}$ and the labels (values of the process Y) on these components. This description involves an excursion measure $\mathbf{N}^*$, which (informally) is supported on the space of compact $\mathbf{R}$-trees equipped with nonnegative continuous labels — a more rigorous presentation depends on the concept of a snake trajectory as developed in [1]. Under $\mathbf{N}^*$, one can make sense of a "boundary size" $\mathcal{Z}_0$ which measures (in some sense) the number of vertices with zero label. For every $z > 0$, one can also define the conditional probability measure $\mathbf{N}^{*,z} = \mathbf{N}^*(\cdot \mid \mathcal{Z}_0 = z)$.

A basic result of [2] provides a construction of the Brownian disk with boundary size z, which is similar to the construction of the Brownian sphere but involves the probability measure $\mathbf{N}^{*,z}$. Roughly speaking, if we consider a random $\mathbf{R}$-tree equipped with nonnegative labels distributed according to the measure $\mathbf{N}^{*,z}$, the Brownian disk is obtained by identifying pairs (a, b) of points in this tree that have the same positive label, and such that one can go from a to b "around the tree" encountering only points with greater label. A nice feature of this construction is the fact that labels exactly correspond to distances from the boundary in the Brownian disk: This allows for a simple approach to the definition of a uniform measure on the boundary.

The preceding construction also makes it possible to identify various subsets of the Brownian sphere as Brownian disks. In particular, connected components of the complement of a ball in the Brownian sphere equipped with their intrinsic metrics are independent Brownian disks conditionally on their boundary sizes and volumes [2]. A similar result holds for the connected components of the set of points in a Brownian disk whose distance to the boundary is greater than a fixed constant $h \geq 0$. Moreover, the collection of the boundary sizes of these components, viewed as a process indexed by h, evolves like a growth-fragmentation process whose law is determined explicitly [3]. This is the continuous version of an earlier result of Bertoin, Curien and Kortchemski dealing with large triangulations with a boundary.

References

[1] C. Abraham, J.-F. Le Gall, *Excursion theory for Brownian motion indexed by the Brownian tree*, J. Eur. Math. Soc. (JEMS) **2** (2018), 2951–3016.

[2] J.-F. Le Gall, *Brownian disks and the Brownian snake*, Ann. Inst. H. Poincaré Probab. Stat. **55** (2019), 237–313.

[3] J.-F. Le Gall, A. Riera, *Growth-fragmentation processes in Brownian motion indexed by the Brownian tree.* Ann. Probab., to appear.

Optimal transport: discrete to continuous

Eva Kopfer
(joint work with Peter Gladbach, Jan Maas, Lorenzo Portinale)

Dynamic discrete optimal transport has been introduced to establish a natural notion of generalized lower Ricci bounds and gradient flows in the space of probability measures. We provide convergence results for these transport metrics and will see that they are not always compatible with the well-established 2-Wasserstein metric.

Lott and Villani in [8] and Sturm in [10] independently introduced the notion of lower Ricci bounds on metric measure spaces (X, d, m). The defining property is given by the convexity of the relative entropy $\mathrm{Ent}(\,\cdot\,|m)$ along Wasserstein geodesics. These spaces are said to satisfy the curvature condition $\mathrm{CD}(K,\infty)$, where K is a lower bound of the synthetic Ricci curvature.

On these $\mathrm{CD}(K,\infty)$-spaces, or more restrictively on $\mathrm{RCD}(K,\infty)$-spaces, a powerful analysis has been developed in a series of papers, like e.g. [1, 2]. One achievement is given by the fact that the heat flow can be unambiguously defined as the L^2-gradient flow of the Cheeger energy or the Wasserstein gradient flow of the relative entropy. On $\mathrm{RCD}(K,\infty)$ spaces this heat flow is required to be linear, which constitutes a bridge between the Lott–Sturm–Villani and the Bakry–Émery approach of synthetic lower Ricci bounds, leading to a characterization by Bakry-Émery gradient estimates and Bochner inequalities.

Due to the lack of geodesics, the picture is completely different when X is a discrete space. In particular, the Wasserstein metric W_2 is not the right object for studying evolution equations of measures or synthetic lower Ricci curvature bounds. This observation motivated several authors to define a Riemannian-like distance $\mathcal{W}$ on $\mathcal{P}(X)$ over the discrete set X endowed by a Markov kernel Q with stationary, reversible measure π. The distance is a discrete variant of the Benamou–Brenier formula, where the Markov kernel encodes the geometrical structure of the space X: for two probability measures $\rho_0, \rho_1 \in \mathcal{P}(X)$ we define

$$\mathcal{W}(\rho_0,\rho_1)^2 = \inf\left\{\int_0^1 \frac{1}{2}\sum_{x,y\in X}(\pi(x)Q(x,y))^{-1}\frac{V_t(x,y)^2}{\theta(\frac{\rho_t(x)}{\pi(x)},\frac{\rho_t(y)}{\pi(y)})}\,dt\right\}$$

where the infimum is taken among all curves $(\rho_t, V_t)_{t\in[0,1]} \subset \mathcal{P}(X)\times\mathbb{R}^{X\times X}$ solving the discrete continuity equation

$$\partial_t\rho_t(x) + \sum_{y\in X} V_t(x,y) = 0 \qquad \forall x \in X$$

and connecting ρ_0 with ρ_1. Here, $\theta\colon \mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ is an arbitrary symmetric mean function.

If θ is given by the logarithmic mean, this new distance $\mathcal{W}$ is able to model the discrete heat flow as the gradient flow of the entropy [9], and to give meaning to synthetic lower Ricci bounds according to Lott, Sturm, and Villani [3], as W_2

does in the continuous setting. This suggests that $\mathcal{W}$ is an adequate discrete counterpart of W_2.

Gigli and Maas showed in [4] that the spaces of probability measures $\mathcal{P}(\mathbb{T}_N^d)$ on the periodic lattice with mesh size $\frac{1}{N}$ endowed with the renormalized distance $\mathcal{W}_N$ converge to the usual Wasserstein space $(\mathcal{P}(\mathbb{T}^d), W_2)$ on the flat torus in the sense of Gromov–Hausdorff.

In [5] we consider the more general setting of finite volume discretizations $\mathcal{T} = \{K_x\}_{x\in X}$ of a bounded, convex domain $\Omega \subset \mathbb{R}^n$. The conductance is then given by

$$\pi(x)Q(x,y) = \frac{|(K_x|K_y)|}{|x-y|},$$

where $|(K_x|K_y)|$ is the Hausdorff measure of the interface of the two cells. As the mesh size goes to 0 we prove that Gromov–Hausdorff convergence of the metric spaces $(\mathcal{P}(\mathcal{T}), \mathcal{W})$ to the usual Wasserstein space $(\mathcal{P}(\Omega), W_2)$ holds if and only if the so-called *isotropy condition*

$$\frac{1}{2}\sum_y \frac{|(K_x|K_y)|}{|x-y|}(x-y)\otimes(x-y) = |K_x| id \qquad \forall x \in X$$

holds in an asymptotic sense. This condition puts a strong geometric constraint on the family of discretizations. In particular general Voronoi tessellations will not satisfy the isotropy condition.

More generally in [7] we compute by means of Γ-convergence the homogenized limit functional for general convex cost functions $f_{x,y}\colon \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to [0,\infty)$ on periodic network graphs (X, E) in arbitrary dimensions. The homogenized cost $f_{\hom}\colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ is then given by the cell formula

$$\begin{aligned} f_{\hom}(m,V) = \inf\Big\{\frac{1}{2}\sum_{[x],[y]\in X/\mathbb{Z}^d} & f_{[x][y]}(\rho_{[x]},\rho_{[y]},V_{[x],[y]}) : \\ & \sum_{[x]\in X/\mathbb{Z}^d}\rho_{[x]} = m, \quad \sum_{[(x,y)]\in E/\mathbb{Z}^d} V_{[x],[y]}(y-x) = V, \\ & \text{for all } x \sum_{[y]\sim[x]} V_{[x],[y]} = 0\Big\}. \end{aligned}$$

For one-dimensional periodic lattices $\mathcal{T}$ with usual quadratic cost this homogenized cost boils down to

$$f_{\hom}(m,V) = \inf\Big\{\frac{1}{2}\sum_{[x],[y]\in X/\mathbb{Z}^d\ n.n} \frac{|x-y|}{\theta(\frac{\rho(x)}{\pi(x)},\frac{\rho(y)}{\pi(y)})} : \rho\in\mathcal{P}(X)\Big\}\frac{V^2}{m}.$$

In particular we have shown in [6] that in this setting Gromov-Hausdorff convergence of $(\mathcal{P}(\mathcal{T}), \mathcal{W})$ to $(\mathcal{P}(\mathbb{T}^1), W_{\hom})$ holds as the mesh size goes to 0, with

$$W_{\hom}(\rho_0, \rho_1)^2 := \inf \left\{ \int_0^1 f_{\hom}(m_t, V_t)\, dt \right\},$$

where the infimum is taken among all solutions to the continuity equation

$$\partial_t m_t + \nabla \cdot V_t = 0.$$

References

[1] L. Ambrosio, N. Gigli, and G. Savaré. Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below. *Invent. Math.*, 195(2):289–391, 2014.

[2] Matthias Erbar, Kazumasa Kuwada, and Karl-Theodor Sturm. On the equivalence of the entropic curvature-dimension condition and Bochner's inequality on metric measure spaces. *Invent. Math.*, 201(3):993–1071, 2015.

[3] M. Erbar and J. Maas. Ricci curvature of finite markov chains via convexity of the entropy. *Arch. Ration. Mech. Anal.*, 206(3):997–1038, 2012.

[4] N. Gigli and J. Maas. Gromov–Hausdorff Convergence of Discrete Transportation Metrics. *SIAM J. Math. Anal.*, 45(2):879–899, 2013.

[5] P. Gladbach, E. Kopfer, and J. Maas. Scaling limits of discrete optimal transport. *arXiv:1809.01092*, 2018.

[6] P. Gladbach, E. Kopfer, J. Maas, and L. Portinale. Homogenisation of one-dimensional discrete optimal transport. *arXiv:1905.05757*, 2019.

[7] P. Gladbach, E. Kopfer, J. Maas, and L. Portinale. Homogenization of transport costs on network graphs. *In progress.*

[8] J. Lott and C. Villani. Ricci curvature for metric-measure spaces via optimal transport. *Ann. Math.*, pages 903–991, 2009.

[9] J. Maas. Gradient flows of the entropy for finite Markov chains. *JFA*, 261(8):2250–2292, 2011.

[10] Karl-Theodor Sturm. On the geometry of metric measure spaces. I and II. *Acta Math.*, 196(1):133–177, 2006.

Ahlfors regular conformal dimension of compact metric spaces and parabolic index of infinite graphs

Jun Kigami

Let (X, d) be a metric space. Immediately by definition, the Hausdorff dimension $\dim_H(X, d)$ depends on the metric d. For example, for any $\alpha \in (0, 1]$, d^α is a metric and

$$\dim_H(A, d^\alpha) = \frac{1}{\alpha} \dim_H(A, d).$$

Hence the Hausdorff dimension can be arbitrary large. On the other hand, lowering the Hausdorff dimension is not so easy. One of the interesting question is how low the Hausdorff dimension can be with a Ahlfors regular quasisymmetric modification of a metric. Such a lower bound is called the Ahlfors regular conformal dimension. The exact definition is

$$\dim_{AR}(X, d) = \inf\{\alpha | \text{there exists a metric } \rho \text{ and a measure } \mu \text{ such that } \rho \text{ and } d \text{ are quasisymmetric and } \mu \text{ is } \alpha\text{-Ahlfors regular with respect to } \rho.\},$$

where μ is said to be α-Ahlfors regular with respect to ρ if and only if there exist $c_1, c_2 > 0$ such that

$$c_1 r^\alpha \leq \mu(B_\rho(x,r)) \leq c_2 r^\alpha,$$

where $B_\rho(x,r) = \{y | y \in X, \rho(x,y) < r\}$. It is known that $\dim_{AR}(\mathbb{R}^n, d_E) = n$ for any $n \geq 1$, where d_E is the Euclidean metric. Also by Tyson and Wu[1], we know $\dim_{AR}(SG, d_E) = 1$, where SG is the Sierpinski gasket. In this talk we are going to show an analytic characterization of the Ahlfors regular conformal dimension.

When you study the Ahlfors regular conformal dimension for example, the main problem is how to manipulate metrics and measures on a space. To deal with such a problem, we introduce the notions of a partition and a weight function of a compact metrizable space X. A partition of a space X consists of a tree T and a map $K : T \to$ the collection of compact subsets of X. A tree T is defined as follows: $T = \cup_{m \geq 0} T_m$, where $T_m \subseteq \mathbb{N}^m$ for any $m \geq 0$, $T_0 = \{\phi\}$ and there exists $k_p \in \mathbb{N}$ for any $p \in T$ such that

$$T_{m+1} = \bigcup_{p \in T_m} \{p1, p2, \ldots, pk_p\}$$

for any $m \geq 0$. Let $\mathcal{C}(X)$ be the collection of compact subsets of X. Then $K : T \to \mathcal{C}(X)$ is called a partition of X if and only if
(P1) $K(\phi) = X$,d (P2) for any $p \in T_m$, $K(p) = \cup_{i=1}^{k_p} K(pi)$ and
(P3) If $(p_0, p_1, p_2, \ldots)$ satisfy $p_{i+1} = p_i j$ for some $1 \leq j \leq k_{p_i}$ for any $m \geq 0$, then

$$\bigcap_{i \geq 1} K(p_i) = \text{a single point}$$

Typical example of a partition is a self-similar set. For example, let $p_1 = 0, p_2 = \frac{1}{2}, p_3 = 1, p_4 = 1 + \frac{1}{2}\sqrt{-1}, p_5 = 1 + \sqrt{-1}, p_6 = \frac{1}{2} + \sqrt{-1}, p_7 = \sqrt{-1}$ and $p_8 = \frac{1}{2}\sqrt{-1}$. Define $F_i : \mathbb{C} \to \mathbb{C}$ by

$$F_i(z) = \frac{1}{2}(z - p_i) + p_i.$$

Then the Sierpinski Carpet (SC for short) is the unique nonempty compact set $K \subseteq \mathbb{C}$ satisfying

$$K = \bigcup_{i=1}^{8} F_i(K).$$

In this case, let $T_m = \{1, 2, \ldots, 8\}^m$ and let $K(i_1 \ldots i_m) = F_{i_1 i_2 \ldots i_m}(K)$, where $F_{i_1 i_2 \ldots i_m} = F_{i_1} \circ F_{i_2} \circ \ldots \circ F_{i_m}$. Then K is a partition of the Sierpinski carpet. In this case we write $K_{i_1 \ldots i_m} = K(i_1 \ldots i_m)$.

A weight function g on a partition K is a function $g : T \to (0,1]$ satisfying the following conditions (W1), (W2) and (W3):
(W1) $g(\phi) = 1$, (W2) $g(pj) \leq g(p)$ for any $p \in T$ and any $j \in \{1, \ldots, k(p)\}$,
(W3) $\lim_{m \to \infty} \sup_{p \in T_m} g(p) = 0$.

There are two special classes of weight functions. One comes from the metrics and the other comes from measures. Namely, let ρ be a metric on X satisfying $\mathrm{diam(X}, \rho) = 1$. Define $g_\rho : T \to (0,1]$ by $g_\rho(p) = \mathrm{diam(K(p)}, \rho)$. Then g_ρ is a weight function. Or, let μ be a Borel regular probability measure on X satisfying

$\mu(O) > 0$ for any nonempty open subset O and $\mu(\{x\}) = 0$ for any $x \in X$. Define $g_\mu(p) = \mu(K(p))$. Then g_μ is a weight function. The collection of weight functions naturally contains metrics and measures. In [2], the theory of weight functions has been developed in order to manipulate metrics and measures.

In [3], Carassco-Piaggio has shown a characterization of the Afhlfors regular conformal dimension using discrete modulus of curves. Here we can translate his result by using the theory of weight functions. and have obtained the following characterization using p-energy of functions. For $p \geq 1$, $w \in T_m$ and $k \geq 0$, define

$$\mathcal{E}_{p,w,k} = \inf\{ \sum_{x,y \in T_{m+k}, K_x \cap K_y \neq \emptyset} |u(x) - u(x)|^p \big| u : T_{m+k} \to [0,1],$$
$$u(x) = 1 \text{ if } x = wi_{m+1} \dots i_{m+k},$$
$$u(x) = 0 \text{ if } x = vj_{m+1} \dots j_{m+k} \text{ for some } v \in T_m \text{ with } K_v \cap K_w = \emptyset\}$$

Define

$$\overline{\mathcal{E}}_p = \limsup_{k \to \infty} \sup_{w \in T} \mathcal{E}_{p,w,k}.$$

Then the analytic characterization of the Ahlfors regular conformal dimension is

$$\dim_{AR}(X, d) = \inf\{p | \overline{\mathcal{E}}_p = 0\}.$$

On infinite graphs, non-linear potential theory was developed by Yamasaki et al [5, 4] in 1970's (and completely forgotten for some time.) Let $G = (V, E)$ be a (non-directed) graph, locally finite and connected. For $u : V \to \mathbb{R}$, define

$$E^p(u) = \frac{1}{2} \sum_{(x,y) \in E} |u(x) - u(y)|^p$$

and

$$\mathcal{F}^p(G) = \{u | u : V \to \mathbb{R}, E^p(u) < +\infty\}$$

Choose $x_* \in V$ as the reference point, define

$$||u||_p = |u(x_*)| + E^p(u)^{\frac{1}{p}}$$

It is known that $(\mathcal{F}^p, || \cdot ||_p)$ is a reflective Banach space. Define

$$\ell_0(V) = \{u | u : V \to \mathbb{R}, u(x) = 0 \text{ except finite points}\}$$
$$\mathcal{F}_0^p(G) = \text{ the closure of } \ell_0(V) \text{ w.r.t. } || \cdot ||_p$$

By the results from [5] and [4],

$$1 \in \mathcal{F}_0^p(G) \Leftrightarrow \mathcal{F}_0^p(G) = \mathcal{F}^p(G)$$

In view of this result, G called p-transient (or p-hyperbolic) if and only if $1 \notin \mathcal{F}_0^p(V)$. Otherwise, G called p-recurrent (or p-parabolic). Then there exists $p_*(G) > 0$ such that G is p-recurrent for $p > p*(G)$ and G is p-transient for $p < p_*(G)$. This value $p_*(G)$ is called the parabolic index of G. Yamasaki et al have shown that $p_*(\mathbb{Z}^n) = n$ for any $n \geq 1$, where $\mathbb{Z}^n$ is provided the standard graph structure.

Now we consider the partition associated with self-similar set like the Sierpinski carpet as we have seen above. Let $G_n = (T_n, E_n)$, where $E_n = \{(x,y)|x \neq y \in T_n, K(x) \cap K(y) \neq \emptyset\}$. For any $(j_1, j_2, \ldots) \in \{1, \ldots, N\}^{\mathbb{N}}$, where N is the number of contractions, we identify G_m as a subset of G_{m+1} through the following injective map $\pi_{j_m} : G_m \to G_{m+1}$ define by $\pi_{j_m} = j_m i_1 \ldots i_m$ for any $i_1 \ldots i_m \in G_m$. The we may think of $\{G_m\}_{m\geq 1}$ as an increasing sequence of graphs and we can obtain an infinite graph $G = \cup_{m\geq 1} G_m$ as a limit. This G is called the blowup associated with $(j_1, j_2, \ldots)$. Since both Ahlfors regular conformal dimension and parabolic index are defined through p-energy of a function, one may naturally expect some relation. In fact,

Theorem(R. Shimizu) For any blowup G of K,

$$p_*(G) \leq \dim_{AR}(X, d)$$

In particular, for the SG, $p_*(G) = \dim_{AR}(X, d) = 1$.

References

[1] J. Tyson and J.-M. Wu, *Quasiconformal dimensions of self-similar fractals*, Rev. Mat. Iberoamericana **22**(2006), 205–258.

[2] J. Kigami, *Weighted partition of a compact metrizable space, its hyperbolicity and Ahlfors regular conformal dimension*, preprint, arXiv:1806.06558.

[3] M. Carrasco Piaggio, *On the conformal gauge of a compact metric space*, Ann. Sci. Ecole Norm. Sup. **46**(2013), 495–548.

[4] T. Nakamura and M. Yamasaki, *Generalised extremal length of an infinite network*, Hiroshima Math. J. **6**(1976), 95–111.

[5] M. Yamasaki, *Extremum problems on an infinite network*, Hiroshima Math. J. **5**(1975), 223–250.

Entropy inequalities for quantum Markov semigroups

Jan Maas

We study a class of ergodic quantum Markov semigroups on finite-dimensional unital C^*-algebras. These semigroups have a unique stationary state σ, and we are concerned with those that satisfy a quantum detailed balance condition with respect to σ. We show that the evolution on the set of states that is given by such a quantum Markov semigroup is gradient flow for the relative entropy with respect to σ in a particular Riemannian metric. This result is a quantum analogue of the classical Jordan–Kinderlehrer–Otto Theorem, which asserts that the diffusion equation is the gradient flow of the 2-Wasserstein metric from optimal transport. Our metric is a non-commutative analog of the 2-Wasserstein metric, defined in terms of a Benamou–Brenier formula:

$$\mathcal{W}^2(\rho_0, \rho_1) := \inf_{\rho, A} \left\{ \int_0^1 \sum_j \mathrm{Tr}\big[(\partial_j A)^* \rho \bullet_j \partial_j A\big] \, \mathrm{d}t \ : \ \partial_t \rho + \sum_j \partial_j^\dagger (\rho \bullet_j \partial_j A) = 0 \right\} .$$

Here the infimum runs over all curves $(\rho_t)_{t\in[0,1]}$ connecting a given pair of density matrices ρ_0 and ρ_1, and all matrix-valued curves $(A_t)_{t\in[0,1]}$ satisfying the stated continuity equation. The partial derivatives $\partial_j A := [V_j, A]$ denote commutators with the Kraus operators V_j, and the tilted non-commutative multiplication $\rho \bullet_j B := \int_0^1 \big(e^{-\omega_j/2}\rho\big)^{1-s} B\big(e^{\omega_j/2}\rho\big)^s \, \mathrm{d}s$ is defined in terms of suitable Bohr frequencies $\omega_j \in \mathbb{R}$.

In several interesting cases we are able to show that the relative entropy is strictly and uniformly convex with respect to the Riemannian metric introduced here. As a consequence, we obtain modified logarithmic Sobolev inequalities, which yield sharp rates of convergence to equilibrium for Bose and Fermi Ornstein-Uhlenbeck semigroups. We also obtain Talagrand inequalities that provide upper bounds for the transport distance in terms of quantum relative entropy. These results are quantum counterparts of seminal classical results by Bakry–Émery and Otto–Villani. The proofs of the geodesic convexity results combine intertwining properties of the semigroups with trace inequalities.

This is based on joint work with Eric Carlen.

References

[1] E. A. Carlen and J. Maas, *Gradient flow and entropy inequalities for quantum Markov semigroups with detailed balance*, J. Funct. Anal. **273** (2017), 1810–1869.

[2] E. A. Carlen and J. Maas, *Non-commutative calculus, optimal transport and functional inequalities in dissipative quantum systems*, J. Statist. Phys. (2019) *Online First*, 1–60.

[3] M. Mittnenzweig and A. Mielke, *An entropic gradient structure for Lindblad equations and couplings of quantum systems to macroscopic models*, J. Statist. Phys. **167** (2017) *Online First*, 205–233.

[4] C. Rouzé and N. Datta, *Relating relative entropy, optimal transport and Fisher information: A quantum HWI inequality*, arXiv:1709.07437.

Functional inequalities on sub-Riemannian manifolds via QCD

Emanuel Milman

We are interested in obtaining good quantitative estimates for functional inequalities on domains of a given diameter in various sub-Riemannian manifolds (equipped with their natural sub-Riemannian metric and volume measure).

In the Riemannian setting, a natural requirement on the domain is geodesic convexity, but this is not a viable option in the sub-Riemannian one, as geodesically convex domains are known to be scarce; for instance, it was shown by Monti and Rickly that a geodesically convex set in the Heisenberg group $\mathcal{H}^1$ containing three distinct points which do not lie on a common geodesic is necessarily $\mathcal{H}^1$ itself. Consequently, we treat arbitrary domains Ω but take their two-point geodesic hull $\mathrm{geo}(\Omega) = \cup\{\gamma \, ; \, \gamma \in \mathrm{Geo}(X), \gamma_0, \gamma_1 \in \Omega\}$ (which need not be geodesically convex) on the energy side of the inequality.

It is well-known from the work of Juillet that strictly sub-Riemannian manifolds do not satisfy any type of Curvature-Dimension condition $\mathsf{CD}(K, N)$, introduced by Lott-Sturm-Villani some 15 years ago, so we must follow a different path. Fortunately, it is known that many natural sub-Riemannian structures satisfy

the weaker Measure Contraction Property $\mathsf{MCP}(0, N)$ of Sturm and Ohta for an appropriate generalized dimension N.
We use a recent interpolation inequality of Barilari and Rizzi to upgrade the $\mathsf{MCP}(0, N)$ information to a new property we call Quasi Curvature-Dimension (QCD). The $\mathsf{QCD}(Q, K, N)$ condition is a "quasi-convex" relaxation of the $\mathsf{CD}(K, N)$ condition, where the defining interpolation inequality need only hold up to a factor of $Q \geq 1$. Our starting observation is that while ideal (strictly) sub-Riemannian manifolds do not satisfy any type of CD condition, they satisfy the $\mathsf{QCD}(Q, 0, N)$ condition with $Q = 2^{N-n}$ (where n is the topological dimension). As a consequence, we show that these spaces satisfy numerous functional inequalities with exactly the same quantitative dependence as their CD counterparts, up to a factor of Q. This is obtained by extending the localization paradigm to completely general interpolation inequalities, and a one-dimensional comparison of QCD densities with their "CD upper envelope".

We thus obtain the best known quantitative estimates for (say) the L^p-Poincaré and log-Sobolev inequalities on domains in the ideal sub-Riemannian setting, which in particular are independent of the topological dimension. For instance, the Li–Yau / Zhong–Yang spectral-gap estimate holds on domains Ω of diameter at most D in the Heisenberg groups $\mathcal{H}^d$ of arbitrary dimension (endowed with the Lebesgue measure $\mathfrak{m}$) up to a factor of 4:

$$\int_\Omega f \mathfrak{m} = 0 \;\;\Rightarrow\;\; \frac{1}{4}\frac{\pi^2}{D^2}\int_\Omega f^2 \mathfrak{m} \leq \int_{\mathrm{geo}(\Omega)} |\nabla_{\mathcal{H}^d} f|^2 \mathfrak{m}.$$

Similar estimates are obtained for the L^p-Poincaré and log-Sobolev inequalities; up to the above factor of 4, these estimates are best-possible. Analogous estimates hold, with $\frac{1}{4}$ above replaced by an appropriate numeric constant, on ideal generalized H-type Carnot groups, the Grushin plane, Sasakian and 3-Sasakian manifolds with appropriate curvature lower bounds, and general ideal Carnot groups.

A bridge between elliptic and parabolic Harnack inequalities

Mathav Murugan
(joint work with Martin T. Barlow, Naotaka Kajino)

The notion of conformal walk dimension serves as a bridge between the elliptic and parabolic Harnack inequalities. The importance of this notion is due to the fact that the finiteness of the conformal walk dimension characterizes the elliptic Harnack inequality. The conformal walk dimension is the infimum of all possible values of the walk dimension that can be attained by a time-change of the process and by a quasisymmetric change of the metric. Two natural questions arise (a) What are the possible values of the conformal walk dimension? (b) When is the infimum attained? We discuss the answer to (a) and mention partial progress towards (b).

We recall the relevant definitions now. The setting for this work is complete, locally compact, geodesic metric space (X, d) with a Radon measure m with full support and a strongly local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$. This defines a Hunt process on X that is symmetric with respect to the measure m, a sheaf of harmonic functions (solutions to the corresponding Laplace equation), and a sheaf of caloric functions (solutions to the corresponding heat equation).

We say that the *elliptic Harnack inequality* (EHI) holds if there exists $C, A > 1$ such that for all $h \geq 0$ harmonic in $B(x, Ar)$

$$\sup_{B(x,r)} h \leq C \inf_{B(x,r)} h.$$

We say the *parabolic Harnack inequality* PHI(β) holds if there exists $C, A > 1$ such that for all $u \geq 0$ caloric in $u : (0, T) \times B(x, AR)$ with $T = R^\beta$, we have

$$\sup_{Q_-} u \leq C \inf_{Q_+} u,$$

where $Q_- = (T/4, T/2) \times B(x, R), Q_+ = (3T/4, T) \times B(x, R)$. Here β signifies the space-time scaling exponent of the process, and is known as the *walk dimension.*

We say that a metric $\theta : X \times X \to [0, \infty)$ on X is *quasisymmetric* to d, there exists a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that

$$\frac{\theta(x,y)}{\theta(x,z)} \leq \eta\left(\frac{d(x,y)}{d(x,z)}\right) \quad \text{for all } x, y, z \in X,\ x \neq z.$$

Quasisymmetry is a generalization of conformal maps to the context of metric spaces. It was introduced by Ahlfors and Beurling on real line as boundary extensions of quasiconformal self maps on the upper half-space. The above definition is due to Tukia and Väisälä. The *conformal gauge* of a metric space (X, d) is defined as the set of metrics that are quasisymmetric to d. The *conformal walk dimension* d_{cw} of a strongly local Dirichlet space $(\mathcal{E}, \mathcal{F}, L^2(m))$ on (X, d) is defined as the infimum of all $\beta > 0$ such that PHI(β) is satisfied for the time-changed Dirichlet space $(\mathcal{E}, \mathcal{F}_e \cap L^2(\mu), L^2(\mu))$ on (X, θ), where θ belongs to the conformal gauge of (X, d) and μ is a smooth measure with full quasi-support. In other words, we seek to minimize the walk dimension by reparametrizing space (by choosing a different metric in the conformal gauge) and by reparametrizing time (by doing a time change with respect to a smooth measure with full quasi-support). The following theorems clarifies the relationship between elliptic and parabolic Harnack inequalities.

Theorem 1. [2] *In the above setting, we have the following characterization of EHI:*

$$\text{EHI} \iff d_{\text{cw}} < \infty.$$

The above theorem states that it is possible to upgrade spatial regularity of process (given by EHI) to space time regularity (given by PHI(β)). It is well known that the walk dimension is always at least two; that is $d_{cw} \geq 2$. An important consequence of Theorem 1 along with the stability of parabolic Harnack inequality is the stability of elliptic Harnack inequality [3, 7, 8, 1, 4]. The following theorem identifies the value of conformal walk dimension.

Theorem 2. [6] *We have the following equivalence.*

$$\text{EHI} \iff d_{\text{cw}} = 2.$$

In other words, Theorem 2 states that we can always upgrade from EHI to PHI($2+\epsilon$) for all $\epsilon > 0$ after a time-change and a quasisymmetric change of metric. Next, we address the following question: when is the infimum in the definition of d_{cw} attained? In other words, when is it possible to upgrade from EHI to PHI(2)? This leads to a natural uniformization question.

Gaussian uniformization problem (attainment problem for d_{cw}): Given a strongly local regular Dirichlet space on (X, d) that satisfies EHI, is the value $d_{\text{cw}} = 2$, attained? If so, construct a metric θ in the conformal gauge and a smooth measure μ with full quasi-support such that PHI(2) holds.

We describe some partial results towards the Gaussian uniformization problem. The first result states that if the conformal walk dimension is attained, the metric is determined by the measure up to a bi-Lipschitz equivalence.

Theorem 3. [6] *If $d_{\text{cw}} = 2$ is attained for some metric θ and a measure μ, then θ is bi-Lipschitz equivalent to the intrinsic metric $d_{\text{int}}(\mu)$.*

As a consequence of the above theorem, in order to find an 'optimal' metric and measure that satisfies PHI(2), it is enough to search for an optimal measure. The following theorem describes that any such measure is a minimal energy dominant measure. In particular, any optimal measure is determined uniquely up to a absolutely continuous change of measure.

Theorem 4. [6] *If $d_{\text{cw}} = 2$ is attained for some metric θ and a measure μ, then μ is a minimal energy dominant measure, that is μ satisfies the following properties: (a) (Energy dominance) $\Gamma(f, f) \ll \mu$ for every $f \in \mathcal{F}$, where $\Gamma(f, f)$ is the energy measure of f; (b)(Minimality) If $\tilde{\mu}$ satisfies (a), then $\mu \ll \tilde{\mu}$.*

As a consequence of the above result, any two 'optimal' measures must be mutually absolutely continuous. In fact, they satisfy a stronger A_∞ relation. We recall the A_∞ relation among measures. Let (X, d, m) be a complete metric measure space such that m is a doubling measure. Let m' be another doubling Borel measure on X. Then m' is said to be *A_∞-related* to m is for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$m(E) < \delta m(B) \quad \text{implies} \quad m'(E) < \epsilon m'(B)$$

whenever E is a measurable subset of a ball B. The following theorem describes A_∞ relation between any two optimal measures.

Theorem 5. [6] *If (X, d_1, μ_1) and (X, d_2, μ_2) are two metrics and measures that attain $d_{\text{cw}} = 2$, then μ_1 and μ_2 are A_∞ related in (X, d_1) (and also in (X, d_2)).*

The above theorem and its proof are motivated by a similar result on the attainment of Ahlfors regular conformal dimension in Loewner spaces by Heinonen and Koskela [5]. In the setting of self-similar sets (examples include fractals like

Sierpinksi gasket, Sierpinksi carpet, Vicsek set), the following theorem further narrows down the candidates for optimal measures to energy measures of harmonic functions.

Theorem 6. [6] *Let X be a self-similar set and X^∂ be its 'natural boundary'. If the conformal walk dimension is attained, then the conformal walk dimension is also attained by the energy measure of a harmonic function; that is $\mu = \Gamma(h,h)$ where h is a harmonic function on $X \setminus X^\partial$.*

As an easy consequence of the above theorem, Vicsek tree doesn't attain the conformal walk dimension since the energy measure of any harmonic function on $X \setminus X^\partial$ fails to have full support. On the other hand, Kigami showed that Sierpinski gasket attains the conformal walk dimension. We do not know if Sierpinksi carpet attains the conformal walk dimension.

References

[1] M.T. Barlow, R.F. Bass, T. Kumagai. Stability of parabolic Harnack inequalities on metric measure spaces, *J. Math. Soc. Japan* (2) **58** (2006), 485–519.

[2] M. T. Barlow, M. Murugan. Stability of elliptic Harnack inequality, *Ann. of Math.* (2) **187** (2018), 777–823

[3] A. Grigor'yan. The heat equation on noncompact Riemannian manifolds. (in Russian) *Matem. Sbornik.* **182** (1991), 55–87. (English transl.) *Math. USSR Sbornik* **72** (1992), 47–77.

[4] A. Grigor'yan, J. Hu, K.-S. Lau. Generalized capacity, Harnack inequality and heat kernels of Dirichlet forms on metric spaces. *J. Math. Soc. Japan* **67** 1485–1549 (2015).

[5] J. Heinonen, P. Koskela. Quasiconformal maps in metric spaces with controlled geometry, *Acta Math.* **181** (1998), no. 1, 1–61.

[6] N. Kajino, M. Murugan. A bridge between elliptic and parabolic Harnack inequalities (in preparation)

[7] L. Saloff-Coste. *A note on Poincaré, Sobolev, and Harnack inequalities*, Inter. Math. Res. Notices **2** (1992), 27–38.

[8] K.-T. Sturm. Analysis on local Dirichlet spaces III. The parabolic Harnack inequality. *J. Math. Pures. Appl. (9)* **75** (1996), 273–297.

Scaling limits for planar aggregation with subcritical fluctuations

James Norris

(joint work with Amanda Turner, Vittoria Silvestri)

We study a family of planar random growth processes in which clusters grow by the successive aggregation of particles. Clusters are encoded as a composition of conformal maps, following an approach first introduced by Carleson and Makarov, and Hastings and Levitov. The specific models that we study fall into the class of Laplacian growth models in which the growth rate of the cluster boundary is determined by the density of harmonic measure of the boundary as seen from infinity. In our case, the location of each successive particle is distributed according to the density of harmonic measure raised to some power. This set-up is closely related to the physically occurring dielectric-breakdown models, which include the Eden model for biological growth and diffusion-limited aggregation.

We establish scaling limits of the growth processes in the scaling regime where the size of each particle converges to zero as the number of particles becomes large. We show that, when the power of harmonic measure is chosen within a particular range, the macroscopic shape of the cluster converges to a disk, but that as the power approaches the edge of this range the fluctuations approach a critical point, which is a limit of stability. This phase transition in fluctuations can be interpreted as the beginnings of a macroscopic phase transition, from disks to non-disks.

Our clusters will grow from the unit disk by the aggregation of many small particles. Let

$$K_0 = \{z \in \mathbb{C} : |z| \leq 1\}, \quad D_0 = \{z \in \mathbb{C} : |z| > 1\}.$$

We fix a non-empty subset P of D_0 and set

$$K = K_0 \cup P, \quad D = D_0 \setminus P.$$

We assume that P is chosen so that K is compact and simply connected. Then we call P a basic particle.

We will call a conformal map F, defined on D_0 and having values in D_0, a basic map if it is univalent and satisfies

$$F(\infty) = \infty, \quad F'(\infty) \in (1, \infty).$$

By the Riemann mapping theorem, there is a one-to-one correspondence between basic particles and basic maps given by

$$P = \{z \in D_0 : z \notin F(D_0)\}.$$

The logarithmic capacity of P is given by

$$c = \log F'(\infty).$$

Then $c > 0$ and F has the form

$$F(z) = e^c \left(z + \sum_{k=0}^{\infty} a_k z^{-k} \right)$$

for some sequence $(a_k : k \geq 0)$ in $\mathbb{C}$.

Given a sequence of attachment angles $(\Theta_n : n \geq 1)$, set

$$F_n(z) = e^{i\Theta_n} F(e^{-i\Theta_n} z).$$

Define a process $(\Phi_n : n \geq 0)$ of conformal maps on D_0 as follows: set $\Phi_0(z) = z$ and for $n \geq 1$ define recursively

$$\Phi_n = \Phi_{n-1} \circ F_n = F_1 \circ \cdots \circ F_n.$$

Then Φ_n encodes a compact set $K_n \subseteq \mathbb{C}$, given by

$$K_n = K_0 \cup \{z \in D_0 : z \notin \Phi_n(D_0)\}$$

and Φ_n is the unique conformal map $D_0 \to D_n$ such that

$$\Phi_n(\infty) = \infty, \quad \Phi_n'(\infty) \in (0, \infty)$$

where $D_n = \mathbb{C} \setminus K_n$. It is straightforward to see that K_n may be written as the following disjoint union

$$K_n = K_0 \cup (e^{i\Theta_1} P) \cup \Phi_1(e^{i\Theta_2} P) \cup \cdots \cup \Phi_{n-1}(e^{i\Theta_n} P).$$

We think of the compact set K_n as a cluster, formed from the unit disk K_0 by the addition of n particles.

By choosing the sequences $(\Theta_n : n \geq 1)$ in different ways, we can obtain a wide variety of growth processes. Set

$$h_n(\theta) = \frac{|\Phi'_{n-1}(e^{\sigma+i\theta})|^{-\eta}}{Z_n}, \quad Z_n = \frac{1}{2\pi} \int_0^{2\pi} |\Phi'_{n-1}(e^{\sigma+i\theta})|^{-\eta} d\theta$$

and consider a sequence of random variables $(\Theta_n : n \geq 1)$ whose distribution given by

$$\mathbb{P}(\Theta_n \in B | \mathcal{F}_{n-1}) = \frac{1}{2\pi} \int_0^{2\pi} 1_B(\theta) h_n(\theta) d\theta$$

where $\mathcal{F}_n = \sigma(\Theta_1, \ldots, \Theta_n)$. We refer to the model so obtained as the aggregate Loewner evolution or ALE(η) model with basic map F and regularization parameter σ. We focus on the case where $\eta \in (-\infty, 1]$ and establish scaling limits in the small-particle regime, where $c \to 0$ and $\sigma \to 0$, while allowing $n \to \infty$ to obtain clusters of macroscopic logarithmic capacity.

Let F be a basic map of logarithmic capacity $c \in (0, 1]$. We say that F has regularity $\Lambda \in [0, \infty)$ if, for all $|z| > 1$,

$$\left| \log\left(\frac{F(z)}{z}\right) - c\frac{z+1}{z-1} \right| \leq \frac{\Lambda c^{3/2} |z|}{|z-1|(|z|-1)}.$$

Here and below we choose the branch of the logarithm so that $\log(F(z)/z)$ is continuous on $\{|z| > 1\}$ with limit c at ∞. Our results concern the limit $c \to 0$ with Λ fixed, but are otherwise universal in the choice of particle. We show that, for $\eta \in (-\infty, 1]$, in this limit, provided the regularisation parameter σ does not converge to 0 too fast, the cluster K_n converges to a disk of radius e^{cn}, and the fluctuations, suitably rescaled, converge to the solution of a certain stochastic partial differential equation. We show the following statement.

Let $\eta \in (-\infty, 1]$, $\Lambda \in [0, \infty)$ and $\varepsilon \in (0, 1/3)$ be given. Let $(\Phi_n : n \geq 0)$ be an ALE(η) process with basic map F and regularization parameter σ. Assume that F has logarithmic capacity c and regularity Λ, and that $e^\sigma \geq 1 + c^{1/3-\varepsilon}$. For all $\eta \in (-\infty, 1)$, $m \in \mathbb{N}$ and $T \in [0, \infty)$, there is a constant $C = C(\eta, \varepsilon, \Lambda, m, T) < \infty$ with the following property. There is an event Ω_1 of probability exceeding $1 - c^m$ on which, for all $n \leq T/c$ and all $|z| = r \geq 1 + c^{1/3-\varepsilon}$,

$$|\Phi_n(z) - e^{cn} z| \leq C\left(c^{1/2-\varepsilon} + \frac{c^{1-\varepsilon}}{(e^\sigma - 1)^2}\right).$$

Moreover, in the case where $\eta = 1$, provided $\varepsilon \in (0, 1/5)$ and $e^\sigma \geq 1 + c^{1/5-\varepsilon}$, there is also a constant $C = C(\varepsilon, \Lambda, m, T) < \infty$ with the following property. There *is an event* Ω_1 *of probability exceeding* $1 - c^m$ *on which, for all* $n \leq T/c$ *and all* $|z| = r \geq 1 + c^{1/5-\varepsilon}$,

$$|\Phi_n(z) - e^{cn}z| \leq C\left(c^{1/2-\varepsilon}\left(\frac{r}{r-1}\right)^{1/2} + \frac{c^{1-\varepsilon}}{(e^\sigma - 1)^3}\right).$$

We also establish the following characterization of the limiting fluctuations, which shows in particular that they are universal within the class of particles considered.

Let $\eta \in (-\infty, 1]$, $\Lambda \in [0, \infty)$ *and* $\varepsilon \in (0, 1/6)$ *be given. Let* $(\Phi_n : n \geq 0)$ *be an ALE(*η*) process with basic map* F *and regularization parameter* σ*. Assume that* F *has logarithmic capacity* c *and regularity* Λ*. Assume further that*

$$\sigma \geq \begin{cases} c^{1/4-\varepsilon}, & \text{if } \eta \in (-\infty, 1), \\ c^{1/6-\varepsilon}, & \text{if } \eta = 1. \end{cases}$$

Set $n(t) = \lfloor t/c \rfloor$*. Then, in the limit* $c \to 0$ *with* $\sigma \to 0$*, uniformly in* F,

$$(e^{-cn(t)}\Phi_{n(t)}(z) - z)/\sqrt{c} \to \mathcal{F}(t, z)$$

in distribution on $D([0, \infty), \mathcal{H})$*, where* $\mathcal{H}$ *is the set of holomorphic functions on* $\{|z| > 1\}$ *vanishing at* ∞*, equipped with the metric of uniform convergence on compacts, and where* $\mathcal{F}$ *is given by the following stochastic PDE driven by the analytic extension* ξ *in* D_0 *of space-time white noise on the unit circle,*

$$d\mathcal{F}(t, z) = (1 - \eta)z\mathcal{F}'(t, z)dt - \mathcal{F}(t, z)dt + \sqrt{2}d\xi(t, z).$$

Analysis and geometry on tamed spaces

Chiara Rigoni

(joint work with Matthias Erbar, Karl-Theodor Sturm, and Luca Tamanini)

Synthetic lower bounds for the Ricci curvature as introduced in the foundational papers by Lott and Villani on one side [6], and Sturm on the other [7], [8] initiated the development of the theory of metric measure spaces $(X, \mathsf{d}, \mathfrak{m})$ with lower bounded Ricci curvature. The crucial property of any such definition is the compatibility with the smooth Riemannian case and the stability with respect to measured Gromov-Hausdorff convergence. The theory is particularly rich if one assumes in addition that the spaces are infinitesimally Hilbertian. For such spaces, Ambrosio, Gigli, and Savaré in a series of seminal papers [1], [2], [3] developed a powerful first order calculus, based on (minimal weak upper) gradients of functions and on gradient flows for semiconvex functionals, and also a second order calculus can be established [4].

In this talk we introduce distributional valued lower Ricci bounds for metric measure spaces, extending the theory of synthetic lower Ricci bounds far beyond uniform bounds. In order to do it we present a formulation of the Bochner's inequlity, or Bakry-Émery inequality, $\mathsf{BE}_2(\kappa, \mathsf{N})$, where $N \geq 1$ and κ can be:

- a signed measure such that its total variation is Kato, or
- a 'distributional' lower bound, i.e., $\kappa \in W^{-1,\mathrm{Kato}}(X) := \{\kappa \in W^{-1,1}_{\mathrm{loc}}(X) : \psi = (I - \Delta)^{-1}\kappa \in L^\infty(X, \mathfrak{m}),\ |\nabla\psi|^2 \text{ Kato}\}$.

Indeed, it well known that Bochner's inequality is one of the most fundamental estimates in geometric analysis on Riemannian manifolds. It states that

$$\frac{1}{2}\Delta|\nabla u|^2 - \langle \nabla u, \nabla \Delta u\rangle \geq K \cdot |\nabla u|^2 + \frac{1}{N} \cdot (\Delta u)^2$$

for any smooth function u on a Riemannian manifold, provided K is a lower bound for the Ricci curvature on and N is an upper bound for the dimension. This inequality can be suitably generalized in the setting of metric measure spaces, requiring that the following inequality holds

$$(\mathsf{BE}_2(\mathsf{K},\mathsf{N})) \qquad \begin{aligned} &\frac{1}{2}\int \Delta\varphi|\nabla f|^2\,\mathsf{d}\mathfrak{m} - \int \varphi\langle\nabla f, \nabla\Delta f\rangle\,\mathsf{d}\mathfrak{m} \\ &\qquad\geq K\int \varphi|\nabla f|^2\,\mathsf{d}\mathfrak{m} + \frac{1}{N}\int \varphi(\Delta u)^2\,\mathsf{d}\mathfrak{m} \end{aligned}$$

for any $f \in D(\Delta)$ with $\Delta f \in W(X)$, and for any $\varphi \in D(\Delta)$, bounded and nonnegative, with $\Delta\varphi \in L^\infty(X, \mathfrak{m})$. An important result by Erbar, Kuwada, and Sturm [5] ensures that the validity of $\mathsf{BE}_2(\mathsf{K},\mathsf{N})$ is actually equivalent to the $\mathsf{RCD}(\mathsf{K},\mathsf{N})$ condition introduced in terms of optimal transport.

The approach we follow in order to introduce a distributional lower bound on the Ricci curvature is based on the theory of Dirichlet forms. Let $(X, \mathsf{d}, \mathfrak{m})$ be a complete and separable metric space equipped with a with a (non-negative) Borel measure which is finite on bounded sets. Let us also assume that $(X, \mathsf{d}, \mathfrak{m})$ is infinitesimally Hilbertian. In this setting locality properties and calculus rules for differential, gradient and Laplacian hold true, and the Cheeger energy Ch admits a Carré du champ given by the pointwise scalar product on the Hilbert tangent module. Hence, we perturb the Cheeger energy associated to the space by a term involving κ, namely we consider the perturbed energy given by

$$\mathcal{E}^\kappa(f) := \mathsf{Ch}(f) + 2\int f^2\,\mathsf{d}\kappa, \quad \text{for all } f \in D(\mathsf{Ch}).$$

Associated to this energy we have the Schrödinger operator L^κ which is such that for any $f, g \in D(\mathsf{L}^\kappa)$ it holds

$$-\int f\mathsf{L}^{2\kappa}g\,\mathsf{d}\mathfrak{m} = -\int f\mathsf{L}g\,\mathsf{d}\mathfrak{m} + 2\int f\cdot g\,\mathsf{d}\kappa.$$

Then we can introduce the following:

Definition. Let $(X, \mathsf{d}, \mathfrak{m})$ be an infinitesimally Hilbertian space. We say that $(X, \mathsf{d}, \mathfrak{m})$ is a tamed space if the L^2-Bochner inequality

$$(\mathsf{BE}_2(\kappa, N)) \qquad \frac{1}{2}\int \mathsf{L}^{2\kappa}\varphi\Gamma(f)\,\mathsf{d}\mathfrak{m} - \int \varphi\Gamma(f, \mathsf{L}f)\,\mathsf{d}\mathfrak{m} \geq \frac{1}{N}\int \varphi(\mathsf{L}f)^2\,\mathsf{d}\mathfrak{m},$$

holds for any $f \in D(\mathsf{L})$ such that $\mathsf{L}f \in D(\mathcal{E})$, and any $\varphi \in D(\mathsf{L}^{2\kappa}) \cap L^\infty(X, \mathfrak{m})$ with $\mathsf{L}^{2\kappa} f \in L^\infty(X, \mathfrak{m})$.

In particular, it turns out that $\mathsf{BE}_2(\kappa, N)$ is equivalent to the validity of the following gradient estimate $\mathsf{GE}_2(\kappa, N)$:

$$\Gamma(P_t f)^{1/2} + \frac{1}{N}\int_0^t P_s^{2\kappa}\Big(\mathbb{1}_{\{\Gamma(P_{t-s}f)>0\}}\frac{(\mathsf{L}P_{t-s}f)^2}{\Gamma(P_{t-s}f)^{1/2}}\Big)\,\mathsf{d}s \leq P_t^{2\kappa}\Gamma(f)^{1/2},$$

where $\{P_t^{2\kappa}\}_{t\geq 0}$ is the semigroup associated to $\mathsf{L}^{2\kappa}$.

References

[1] L. Ambrosio, N. Gigli and G. Savaré. *Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below.* Invent. Math. **195** (2014), no. 2, 289–391.

[2] L. Ambrosio, N. Gigli and G. Savaré. *Metric measure spaces with Riemannian Ricci curvature bounded from below.* Duke Math. J. **163** (2014), no. 7, 1405–1490.

[3] L. Ambrosio, N. Gigli and G. Savaré. *Bakry-Émery curvature- dimension condition and Riemannian Ricci curvature bounds.* Ann. Probab. **43** (2015), no. 1, 339–404.

[4] N. Gigli. *Nonsmooth differential geometry – an approach tailored for spaces with Ricci curvature bounded from below.* Mem. Amer. Math. Soc. **251** (2018), no. 1196, v+161 pp.

[5] M. Erbar, K. Kuwada and K.-T. Sturm. *On the equivalence of the entropic curvature-dimension condition and Bochner's inequality on metric measure spaces.* Invent. Math. **201** (2015), no. 3, 993–1071.

[6] J. Lott and C. Villani. *Ricci curvature for metric-measure spaces via optimal transport.* Ann. of Math. (2) **169** (2009), no. 3, 903–991.

[7] K.-T. Sturm. *On the geometry of metric measure spaces. I.* Acta Math. **196** (2006), no. 1, 65–131.

[8] K.-T. Sturm. *On the geometry of metric measure spaces. II.* Acta Math. **196** (2006), no. 1, 133–177.

Local limit theorem for the killed Green kernel of a random walks among random conductances

Annika Rothhardt, Anna-Lisa Sokol

(joint work with Martin Slowik)

Consider the d-dimensional Euclidean lattice, $(\mathbb{Z}^d, E^d)$, for $d \geq 2$, where we denote by E^d the set of all non-oriented nearest neighbour bonds. The graph $(\mathbb{Z}^d, E^d)$ is endowed with a family $\omega = \{\omega(e) : e \in E^d\} \in \Omega := (0, \infty)^{E^d}$ of non-negative weights. For any $\omega \in \Omega$, we refer to $\omega(e)$ as the *conductance* of the edge e. We also write $x \sim y$ if $\{x, y\} \in E^d$. A *space shift* by $z \in \mathbb{Z}^d$ is a map $\tau_z : \Omega \to \Omega$

$$(\tau_z\,\omega)(\{x, y\}) \;:=\; \omega(\{x + z, y + z\}), \qquad \forall\,\{x, y\} \in E^d.$$

Further, consider a probability measure, $\mathbb{P}$, on the measurable space $(\Omega, \mathcal{F})$ where $\mathcal{F}$ denotes the Borel-σ-algebra on Ω, and we write $\mathbb{E}$ to denote the corresponding expectation with respect to $\mathbb{P}$.

Assumption 1. *Assume that $\mathbb{P}$ satisfies the following conditions:*

(i) $\mathbb{P}$ *is ergodic and stationary with respect to space shifts.*

(ii) $\mathbb{E}\big[\omega(e)\big] < \infty$ *for all* $e \in E_d$.

For any given realization $\omega \in \Omega$, we consider the homogeneous Markov process, $X \equiv (X_t : t \geq 0)$ on $\mathbb{Z}^d$ in the random environment ω with generator $\mathcal{L}^\omega$ acting on bounded functions $f \colon \mathbb{Z}^d \to \mathbb{R}$ as

$$(\mathcal{L}^\omega f)(x) \;:=\; \sum_{y \sim x} \omega(x,y)\,\big(f(y) \,-\, f(x)\big). \tag{1}$$

We study the Green's function, $g_A(x,y)$, of the random walk, X, which is killed upon exiting a finite set $A \subset \mathbb{Z}^d$. For any $d \geq 2$, it is given by

$$g_A(x,y) \;=\; \mathrm{E}_x^\omega\bigg[\int_0^{\tau_A} \mathbb{1}_{X_t = y}\,\mathrm{d}t\bigg] \;=\; \int_0^\infty \mathrm{P}_x^\omega\big[X_y = y,\, t < \tau_A\big]\,\mathrm{d}t,$$

Our result relies on the following integrability condition.

Assumption 2 (Integrability condition)**.** *For some* $p, q \in [1, \infty]$ *with*

$$\frac{1}{p} + \frac{1}{q} \;<\; \frac{2}{d}, \tag{2}$$

assume that the following integrability condition holds

$$\mathbb{E}\big[\omega(e)^p\big] \;<\; \infty \qquad \textit{and} \qquad \mathbb{E}\big[\omega(e)^{-q}\big] \;<\; \infty, \tag{3}$$

where we used the convention that $0/0 = 0$.

Our main result is the *local limit theorem* of the killed Green's function.

Theorem 1. *Given Assumptions (0.1) and (0.2) hold. For every* $\varepsilon > 0$ *and every* $\delta > 0$ *the following holds.*

$$\lim_{n \to \infty} \sup_{\substack{x,y \in B_{1-\delta} \\ |x-y| > \varepsilon}} \Big| n^{d-2} g_{B(n)}([nx],[ny]) - g_{B_1}^{\mathrm{BM}}(x,y)\Big| \;=\; 0.$$

where B_1 *denotes the unit ball in* $\mathbb{R}^d$ *and* $B(n) := nB_1 \cap \mathbb{Z}^d$.

To prove this we exploit that a *quenched functional central limit theorem* holds for the random walk X and show Hölder-regularity for the solution of the Poisson equation of $\mathcal{L}^\omega$.

References

[1] S. Andres, J.-D. Deuschel and M. Slowik,, *Invariance principle for the random conductance model in a degenerate ergodic environment*, The Annals of Probability 43 (2015), 1866-1891.

[2] S. Andres, J.-D. Deuschel and M. Slowik, *Harnack inequalities on weighted graphs and some applications to the random conductance model*, Probability Theory and Related Fields 184 (2016), 931-977.

[3] A. Chiarini and M. Slowik, *Quenched local limit theorem for the random conductance model with time-dependent conductances on random graphs*, in preperation
[4] Karatzas, I., Shreve S., *Brownian Motion and Stochastic Calculus*, Springer-Verlag, New York.

Scaling limit of uniform spanning tree in three dimensions

Daisuke Shiraishi
(joint work with Omer Angel, David Croydon and Sarai Hernandez-Torres)

Notation: Let $\mathcal{U}_n$ be the uniform spanning tree in $2^{-n}\mathbb{Z}^3$ equipped with the graph distance $d_{\mathcal{U}_n}$. We suppose that the metric space $(\mathcal{U}_n, d_{\mathcal{U}_n})$ is always pointed at the origin. We write $\mu_{\mathcal{U}_n}$ for the counting measure in $2^{-n}\mathbb{Z}^3$ which places a unit mass on each vertex of $\mathcal{U}_n$. To retain information of the uniform spanning tree with respect to the Euclidean topology, we consider a map $\phi_{\mathcal{U}_n} : \mathcal{U}_n \to \mathbb{R}^3$, which we take to be simply the identity map. Now we have a random quintuplet $(\mathcal{U}_n, d_{\mathcal{U}_n}, 0, \mu_{\mathcal{U}_n}, \phi_{\mathcal{U}_n})$. After properly rescaling this random quintuplet, we will consider its weak limit as $n \to \infty$ with respect to the spatial Gromov-Hausdorff-Prokhorov topology (see [1] for this topology). Thus, the first task is to determine the correct rescaling factor.

Some results for loop-erased random walk: Wilson's algorithm (see [4] for the algorithm) guarantees that the uniform spanning tree $\mathcal{U}_n$ generates by using loop-erased random walks as follows (see [2] for the loop-erased random walk). We write $2^{-n}\mathbb{Z}^3 = \{x_i\}_{i=1}^{\infty}$ for a sequence of all points in $2^{-n}\mathbb{Z}^3$, where the order of points is arbitrary. Let S^x be the simple random walk on in $2^{-n}\mathbb{Z}^3$ started at x. Then Wilson's algorithm is as follows:

- Consider $\mathrm{LE}\big(S^{x_1}[0,\infty)\big)$, the loop-erasure of $S^{x_1}[0,\infty)$. We moreover let $\mathcal{U}_n^1 = \mathrm{LE}\big(S^{x_1}[0,\infty)\big)$.
- Given $\mathcal{U}_n^k$ $(k \geq 1)$, we consider $S^{x_{k+1}}$ (which is independent of $\mathcal{U}_n^k$) until it hits $\mathcal{U}_n^k$. We denote the first hitting time by t_{k+1}. Let $\mathcal{U}_n^{k+1} = \mathcal{U}_n^k \cup \mathrm{LE}\big(S^{x_{k+1}}[0, t_{k+1}]\big)$.
- Let $\mathcal{U}_n^\infty = \bigcup_{k\geq 1} \mathcal{U}_n^k$.

Then for any choice of $\{x_i\}$, it follows that $\mathcal{U}_n^\infty$ has the same distribution as that of $\mathcal{U}_n$.

With this in mind, to determine the correct rescaling factor of $d_{\mathcal{U}_n}$, we need to estimate the length (the number of steps) for loop-erased random walks. For this, the following result was proved in [3].

Theorem 1. *(Corollary 1.3 of* [3]*) Let S be the simple random walk on $2^{-n}\mathbb{Z}^3$ started at the origin. We write M_n be the first time that the loop-erasure of $S[0,\infty)$ exits from $\mathbb{D} := \{x \in \mathbb{R}^3 : |x| < 1\}$. Then there exist universal constants $\beta \in (1, 5/3]$, $c, C \in (0,\infty)$ such that for all $n \geq 1$,*

$$c2^{\beta n} \leq E(M_n) \leq C2^{\beta n}. \tag{1}$$

Remark 1. The constant β in the theorem above is called the growth exponent for loop-erased random walk in three dimensions. Numerical simulation suggests that $\beta = 1.624 \pm 0.001$ (see [5]).

By Theorem 1 plus some extra works, one can prove the tightness of

$$2^{-\beta n} d_{\mathcal{U}_n}(0, y_0)$$

where $y_0 = (1, 0, 0)$. Therefore, the correct rescaling factor for $d_{\mathcal{U}_n}$ should be $2^{-\beta n}$. Our main results (which will be stated in the next section) ensure that this is the case.

Main results: The first result shows the existence of the scaling limit of uniform spanning trees in three dimensions.

Theorem 2. *As $n \to \infty$, the random quintuplet $(\mathcal{U}_n, 2^{-\beta n} d_{\mathcal{U}_n}, 0, 2^{-3n} \mu_{\mathcal{U}_n}, \phi_{\mathcal{U}_n})$ converges weakly with respect to the spatial Gromov-Hausdorff-Prokhorov topology.*

Let $(\mathcal{T}, d_{\mathcal{T}}, \rho_{\mathcal{T}}, \mu_{\mathcal{T}}, \phi_{\mathcal{T}})$ be the weak convergence limit of

$$(\mathcal{U}_n, 2^{-\beta n} d_{\mathcal{U}_n}, 0, 2^{-3n} \mu_{\mathcal{U}_n}, \phi_{\mathcal{U}_n}).$$

We obtain various topological properties of the limit as follows.

Theorem 3. *With probability one, it follows that:*

- $(\mathcal{T}, d_{\mathcal{T}})$ *is a complete, locally finite real tree with precisely one topological end at infinity;*
- *the Hausdorff dimension of* $(\mathcal{T}, d_{\mathcal{T}})$ *is* $3/\beta$*;*
- $\max_{v \in \mathcal{T}} deg_{\mathcal{T}}(v) = 3$*, where* $deg_{\mathcal{T}}(v)$ *stands for the degree of* v *in the tree* $\mathcal{T}$*;*
- *given* $R > 0$*, there exist a random* $r_0(\mathcal{T}) > 0$ *and deterministic* $c_1, c_2, C \in (0, \infty)$ *such that*
$$c_1 r^{\frac{3}{\beta}} (\log r^{-1})^{-C} \le \mu_{\mathcal{T}}\big(B_{\mathcal{T}}(x, r)\big) \le c_2 r^{\frac{3}{\beta}} (\log r^{-1})^{C},$$
for every $x \in B_{\mathcal{T}}(\rho_{\mathcal{T}}, R)$ *and* $r \in (0, r_0(\mathcal{T}))$*, where* $B_{\mathcal{T}}(x, r) = \{y \in \mathcal{T} : d_{\mathcal{T}}(x, y) < r\}$*;*
- $\mu_{\mathcal{T}} = \mathcal{L} \circ \phi_{\mathcal{T}}$*, where* $\mathcal{L}$ *stands for Lebesgue measure on* $\mathbb{R}^3$*.*

The final result deals with the simple random walk on the uniform spanning tree. To state the theorem, we will introduce some notations.

Let $\mathcal{U}$ be the uniform spanning tree in $\mathbb{Z}^3$ equipped with the graph distance $d_{\mathcal{U}}$. We write $X^{\mathcal{U}} = \Big(\big(X^{\mathcal{U}}(k)\big)_{k \ge 0}, \big(P_x^{\mathcal{U}}\big)_{x \in \mathbb{Z}^3} \Big)$ for the simple random walk on $\mathcal{U}$. Let $\tau_{x,R}^{\mathcal{U}} := \inf \big\{ k \ge 0 \bigm| d_{\mathcal{U}}(x, X^{\mathcal{U}}(k)) > R \big\}$ and let $\tau_{x,R}^{E} := \inf \big\{ k \ge 0 \bigm| |x - X^{\mathcal{U}}(k)| > R \big\}$. We denote the heat kernel of $X^{\mathcal{U}}$ by $p_k^{\mathcal{U}}(x, y) = P_x^{\mathcal{U}}\big(X^{\mathcal{U}}(k) = y\big)/\mu_y^{\mathcal{U}}$, where $\mu_y^{\mathcal{U}}$ stands for the degree of y in $\mathcal{U}$. Finally, we write $X^{\mathcal{T}}$ for the Brownian motion on the space $(\mathcal{T}, d_{\mathcal{T}}, \mu_{\mathcal{T}})$ started at $\rho_{\mathcal{T}}$. Then we have the following theorem.

Theorem 4. *It follows that:*

- *For a.s. realization of* $\mathcal{U}$ *and all* $x \in \mathcal{U}$,
$$\lim_{R\to\infty} \frac{\log E_x^{\mathcal{U}}(\tau_{x,R}^{\mathcal{U}})}{\log R} = \frac{3+\beta}{\beta}, \quad \lim_{R\to\infty} \frac{\log E_x^{\mathcal{U}}(\tau_{x,R}^{E})}{\log R} = 3+\beta,$$
and
$$-\lim_{k\to\infty} \frac{2\log p_{2k}^{\mathcal{U}}(x,x)}{\log k} = \frac{6}{3+\beta};$$
- *as* $n \to \infty$, *the annealed laws of the rescaled processes*
$$\left(2^{-n} X^{\mathcal{U}}\left(2^{(3+\beta)n}t\right)\right)_{t\geq 0}$$
converge to the annealed law of $\left(\phi_{\mathcal{T}}(X_t^{\mathcal{T}})\right)_{t\geq 0}$.

References

[1] M. T. Barlow, D. A. Croydon, T. Kumagai. Subsequential scaling limits of simple random walk on the two-dimensional uniform spanning tree. *Annals of Probability.* Vol. **45**, Number 1, 4-55, 2017.

[2] G. Lawler. Loop-erased random walk. *Perplexing problems in probability: Festschrift in honor of Harry Kesten (M. Bramson and R. T. Durrett, eds.), Progr. Probab.*, vol. **44**, Birkhauser Boston, Boston, MA, 1999.

[3] X. Li, D. Shiraishi. One-point function estimates for loop-erased random walk in three dimensions. *Elect. J. Probab.* **24**, 46. 2019.

[4] D. B. Wilson. Generating random spanning trees more quickly than the cover time. *Proceedings of the twenty-eighth annual ACM symposium on Theory of computing*, pages 296-303. ACM, 1996.

[5] D. B. Wilson. Dimension of loop-erased random walk in three dimensions. *Phys. Rev. E*, **82**, no. 6, 2010.

Riesz transform on manifolds with ends

Adam Sikora

(joint work with Andrew Hassell and Daniel Nix)

We study boundedness of the Riesz transform on class of manifolds obtained by taking connected sum of several N-dimensional manifolds such as $\mathbb{R}^{n_i} \times \mathcal{M}_i$, $i = 1, \dots, l$ where $\mathcal{M}_i$ are compact Riemannian manifolds of dimension $N - n_i$. In many cases one can also include in the sum product of compact manifolds with Lie groups with polynomial growth or divergence form operator with periodic coefficients acting on $\mathbb{R}^{n_i}$. The most interesting case is the situation when when the Euclidean dimensions n_i are not all equal. Then the ends have different 'asymptotic global dimension', this implies that the Riemannian manifold $\mathcal{M}$ does not satisfy the doubling condition. In [4] Grigor'yan and L. Saloff-Coste studied heat kernel estimates on such connected sums of Euclidean ends. Our aim is to fully describe the range of exponents $1 \leq p \leq \infty$ for which the corresponding Riesz transform act as a bounded operator on $L^p(\mathcal{M})$.

L^p boundedness of the Riesz transform is a central topic in harmonic analysis and heat kernels theory and it has been studied for almost 100 years, starting with the classical work of Riesz [7]. For complete Riemannian Manifolds the Riesz transform can be defined as

$$R = \nabla \Delta^{-1/2}.$$

where ∇ the gradient corresponding to the Riemannian structure and Δ is the Laplace-Beltrami operator. In 1983 Strichartz formulated the question of continuity of the Riesz transform in the above mentioned Riemannian manifolds setting, see [8].

In the considered setting of manifolds with ends, under assumption that $n_i \geq 3$ for each i, a comprehensive answer to the Strichartz question is provided by the following theorem obtained in [6], compare also [1, Proposition 3.3]

Theorem 1. *Let $\mathbb{R}^{n_1} \times \mathcal{M}_1, \dots, \mathbb{R}^{n_l} \times \mathcal{M}_l$ be a set of $l \geq 2$ manifolds which are products of a Euclidean factor of dimension n_i with a compact Riemannian manifold $\mathcal{M}_i$, with the product metric, for $1 \leq i \leq l$. Suppose that $\mathcal{M} = (\mathbb{R}^{n_1} \times \mathcal{M}_1)\# \dots \#(\mathbb{R}^{n_l} \times \mathcal{M}_l)$ is a manifold with $l \geq 2$ Euclidean ends with $n_i \geq 3$ for each i. Then the Riesz transform $\nabla\Delta^{-1/2}$ defined on $\mathcal{M}$ is bounded on $L^p(\mathcal{M})$ if and only if $1 < p < \min\{n_1, \dots, n_l\}$. In addition, the Riesz transform $\nabla\Delta^{-1/2}$ is of weak type $(1, 1)$.*

Including the case $n_i = 2$ for some i, is not possible by simple modification of the prof of Theorem 1. In fact, distinction between transient and recurrent ends seems to essentially impact the expected heat kernel and resolvent behaviour and asymptotics. It is illuminating in this context to compare the papers [4] and [3]. The following result which includes possibility of an end of asymptotic dimension equals 2 was descried in [5]

Theorem 2. *Let $\mathbb{R}^{n_-} \times \mathcal{M}_-$ and $\mathbb{R}^{n_+} \times \mathcal{M}_+$ be two manifolds which are products of a Euclidean factor of dimension $n_\pm$ with a compact Riemannian manifold $\mathcal{M}_\pm$, with the product metric. Suppose that $\mathcal{M} = (\mathbb{R}^{n_-} \times \mathcal{M}_-)\#(\mathbb{R}^{n_+} \times \mathcal{M}_+)$ is a connected sum, with $n_- = 2$ and $n_+ \geq 3$. Then the Riesz transform defined on $\mathcal{M}$ is bounded on $L^p(\mathcal{M})$ if and only if $1 < p \leq 2$. That is, there exists C such that*

$$\big\| \, |\nabla\Delta^{-1/2} f| \, \big\|_p \leq C\|f\|_p, \ \forall f \in L^p(X, \mu)$$

if and only if $1 < p \leq 2$. In addition the Riesz transform operator is of weak type $(1, 1)$.

The approach developed in [6, 5] is in a sense continuation of the strategy described in [2], where the Riesz transform is studied for connected sum of several copies of $\mathbb{R}^n$. A significant difference between these settings is that the manifolds considered in [2] still satisfy the doubling condition.

The most essential part of the proofs of Theorems 1 and 2 is to calculate the kernel of the resolvent operator $(\Delta + k^2)^{-1}$ for low energy limits $k \to 0$ in a way which allows to estimate its gradient. Then one can use such estimates to investigate the Riesz transform using the well-known formula

$$\nabla\Delta^{-1/2} = \frac{2}{\pi}\nabla\int_0^\infty (\Delta + k^2)^{-1}\,dk.$$

In both papers [6] and [5] a crucial step to calculate the required resolvent kernel or its parametrix is to describe the behaviour of solutions (or approximate solutions) u to the resolvent equation

$$(\Delta + k^2)u = v, \quad v \in C_c^\infty(\mathcal{M}), \tag{1}$$

in the low energy limit $k \to 0$. However, there a significant difference between the asymptotic behaviour of the solution u of (1) in the setting of Theorem 1 considered in [6] and Theorem 2 studied in [5].

References

[1] G. Carron, Riesz transforms on connected sums. *Ann. Inst. Fourier (Grenoble)* 57(7):2329–2343, (2007).

[2] G. Carron, T. Coulhon and A. Hassell, Riesz transform and L^p-cohomology for manifolds with Euclidean ends. *Duke Math. J.* , 133:59–93, (2006).

[3] A. Grigor'yan, S. Ishiwata, L. Saloff-Coste, Heat kernel estimates on connected sums of parabolic manifolds. J. Math. Pures Appl. 113(9):155–194, (2018).

[4] A. Grigor'yan and L. Saloff-Coste, Heat kernel on manifolds with ends. *Ann. Inst. Fourier (Grenoble)* 59(5):1917–1997, (2009).

[5] A. Hassell, D. Nix, A. Sikora, Riesz transforms on a class of non-doubling manifolds II. *In preparation.*

[6] A. Hassell and A. Sikora, Riesz transforms on a class of non-doubling manifolds. *Comm. Partial Differential Equations* 44(11):1072–1099, (2019).

[7] M. Riesz, Sur les fonctions conjugués. *Math. Zeitschrift,* 27:218–244, (1928).

[8] R. Strichartz, Analysis of the Laplacian on the complete Riemannian manifold. *J. Funct. Anal.* 52:48–79, (1983).

Unique continuation and localization on the planar lattice

Charles K. Smart

Recall that the Anderson–Bernoulli model is a random linear operator on $\ell^2(\mathbb{Z}^d)$ given by

$$H = -\Delta + \beta V,$$

where Δ is the graph Laplacian, $\beta > 0$ is the noise strength, and $V : \mathbb{Z}^d \to \{0, 1\}$ is a Bernoulli potential. We discuss the following two results.

Theorem 1 (Ding–Smart [1]) If $d = 2$, then H almost surely has pure-point spectrum in $[0, \varepsilon]$.

Theorem 2 (Li–Zhang [2]) If $d = 3$, then H almost surely has pure-point spectrum in $[0, \varepsilon]$.

These results advance the state of the art by establishing localization for singular noise in dimensions larger than one. Following the program of Bourgain–Kenig, the key ingredients of these theorems are the following unique continuation results.

Theorem 3 (Ding–Smart [1]) The following holds for all $\alpha > 1 > \varepsilon > 0$ and sufficiently large $L > 0$. If $d = 2$, $|\bar{\lambda}| < \alpha$, and $Q = [-L, L]^2 \cap \mathbb{Z}^2$, then

$$\mathbb{P}[\mathcal{E}] \geq 1 - e^{-L^{1/4-\varepsilon}}$$

where $\mathcal{E}$ is the event that

$$H\psi = \lambda\psi \quad \text{in } Q \quad \text{and} \quad |\lambda - \bar{\lambda}| \leq e^{-L^{1/2+\varepsilon}}$$

implies

$$\#\{x \in Q : |\psi(x)| \geq e^{-L^{1+\varepsilon}} |\psi(0)|\} \geq L^{3/2-\varepsilon}.$$

Theorem 4 (Li–Zhang [2]) There is a $p > 0$ such that, for all $\alpha > 1 > \varepsilon > 0$, the following holds for sufficiently large $L > 0$. If $d = 3$, $|\Delta\psi| \leq \alpha|\psi|$ holds in $Q = [-L, L]^3 \cap \mathbb{Z}^3$, then

$$\#\{x \in Q : |\psi(x)| \geq e^{-L^{1+\varepsilon}} |\psi(0)|\} \geq L^{3/2+p}.$$

Both of these unique continuation theorems use ideas from recent work of Buhovsky–Logunov–Malinnikova–Sodin.

References

[1] Ding and Smart, *Localization near the edge for the Anderson Bernoulli model on the two dimensional lattice*, arXiv:1809.09041t

[2] Li and Zhang, *Anderson-Bernoulli Localization on the 3D lattice and discrete unique continuation principle*, arXiv:1906.04350

Configuration spaces over metric measure spaces

Kohei Suzuki

(joint work with Lorenzo Dello Schiavo)

The aim of this research is to explore foundations of infinite-dimensional analysis and geometry on configuration spaces over *non-smooth spaces*. The configuration space $\Upsilon(X)$ over the base space X is the set of all locally finite point measures on X, describing random dynamics of infinite particle systems on X. If X has a metric measure structure $(X, \mathsf{d}, \mathsf{m})$, there are two conceptually different objects on $\Upsilon(X)$ with the Poisson measure π_{m}: one is the Dirichlet form $(\mathcal{E}^{\Upsilon}, \mathscr{D}(\mathcal{E}^{\Upsilon}))$ generated by the square-field operator Γ^{Υ} lifted from X (cf. Ma–Röckner [5]); the other is the extended metric measure space $(\Upsilon(X), \mathsf{d}_2, \pi_{\mathsf{m}})$ induced by the L^2-transportation distance d_2:

$$(\mathcal{E}^{\Upsilon}, \mathscr{D}(\mathcal{E}^{\Upsilon})) \longleftrightarrow (\Upsilon(X), \mathsf{d}_2, \pi_{\mathsf{m}}).$$

As a main result, these two structures are identified under suitable assumptions on X whereby various non-smooth spaces are included. This result gives a natural differential geometric structure on $\Upsilon(X)$ over non-smooth X and various fundamental consequences are obtained. As a consequence, $\mathrm{EVI}(K, \infty)$ is established on $\Upsilon(X)$ over RCD spaces X, which gives a new family of infinite-dimensional examples satisfying Curvature-Dimension Conditions. Furthermore, the stability of Cheeger energies and Brownian motions on $\Upsilon(X)$ under the pointed measured Gromov (pmG) convergence of RCD spaces X is obtained.

Main Results.
Identification: One of the main results is the complete idenitification of analytic and geometric structures on configuration spaces $\Upsilon(X)$ with the Poisson measure π_{m} over metric measure spaces $(X, \mathsf{d}, \mathsf{m})$ under suitable assumptions, which can be summarised in the following diagram:

$$\begin{array}{ccccccc}
(\mathcal{E}^\Upsilon, \mathscr{D}(\mathcal{E}^\Upsilon)) \rightarrow & (\Upsilon, \mathsf{d}_{\mathcal{E}^\Upsilon}, \pi_{\mathsf{m}}) & \rightsquigarrow & (\mathsf{Ch}_{\mathsf{d}_{\mathcal{E}^\Upsilon}}, \mathscr{D}(\mathsf{Ch}_{\mathsf{d}_{\mathcal{E}^\Upsilon}})) & = (\mathcal{E}^\Upsilon, \mathscr{D}(\mathcal{E}^\Upsilon)) & & \\
& \| & & \| & & & \\
& (\Upsilon, \mathsf{d}_2, \pi_{\mathsf{m}}) & \rightsquigarrow & (\mathsf{Ch}_{\mathsf{d}_2}, \mathscr{D}(\mathsf{Ch}_{\mathsf{d}_2})) & \longrightarrow & (\Upsilon, \mathsf{d}_{\mathsf{Ch}_{\mathsf{d}_2}}, \pi_{\mathsf{m}}) = (\Upsilon, \mathsf{d}_2, \pi_{\mathsf{m}})
\end{array}$$

$\rightarrow$ Take intrinsic distance
$\rightsquigarrow$ Take Cheeger energy
$=$ Main results

The above diagram tells us that one can commute between $(\mathcal{E}^\Upsilon, \mathscr{D}(\mathcal{E}^\Upsilon))$ and $(\Upsilon(X), \mathsf{d}_2, \pi_{\mathsf{m}})$ through the two operations $\rightarrow$ and $\rightsquigarrow$.

The identification enables us to unify both of the analytic approach (Dirichlet form theory) and the geometric approach (metric measure theory) to investigate various fundamental problems on configuration spaces. One of the most fundamental consequence is that

$$(\mathsf{Ch}_{\mathsf{d}_2}, \mathscr{D}(\mathsf{Ch}_{\mathsf{d}_2})) \text{ turns out to be non-trivial and quadratic,}$$

which is revealed *only after the identification*. Furthermore, $(\Upsilon(X), \mathsf{d}_2, \pi_{\mathsf{m}})$ turns out to be an *extended* metric measure space in the sense of Ambrosio–Erbar–Savaré [2]. Further applications are explained in the following.
Curvature Bounds on $\Upsilon(X)$: As a main result, the Bakry–Émery (BE) Curvature-Dimension Condition is established on $\Upsilon(X)$ over BE(K, ∞) spaces X with some additional condition (VH) for volume growth and heat kernel estimates, which is satisfied, e.g. for RCD(K, N) spaces. Combined with the identification result, an Evolution-Variation Inequality (EVI) is obtained on $\Upsilon(X)$ over RCD spaces X:

$$\begin{array}{ccc}
(X, \Gamma, \mathsf{m}) \text{ is BE}(K,\infty) + \text{(VH)} & \Longrightarrow & (\Upsilon(X), \Gamma^\Upsilon, \pi_{\mathsf{m}}) \text{ is BE}(K, \infty) \\
& & \Big\downarrow \textit{Identification} \\
(X, \mathsf{d}, \mathsf{m}) \text{ is } \begin{cases} \text{RCD}(K, \infty) + \text{(VH)} \\ \qquad \text{or} \\ \text{RCD}(K, N) \end{cases} & \Longrightarrow & (\Upsilon(X), \mathsf{d}_2, \pi_{\mathsf{m}}) \text{ is EVI}(K, \infty).
\end{array}$$

This is a generalisation of Erbar–Huesmann [4] to the non-smooth setting.
Stability: As a main result, the stability of the Dirichlet forms is proved in the sense of the Kuwae–Shioya–Mosco (KSM) convergence by utilising the Fock space

structure of $L^2(\Upsilon, \pi_{\mathsf{m}})$. Combined with the identification result, the stability of Cheeger energies and Brownian motions on $\Upsilon(X)$ under the pmG convergence of $\mathrm{RCD}(K,N)$ spaces X is proved:

$$(X_n,\Gamma_n,\mathsf{m}_n) \xrightarrow{KSM} (X,\Gamma,\mathsf{m}) \implies (\Upsilon(X_n),\Gamma_n^{\Upsilon},\pi_{\mathsf{m}_n}) \xrightarrow{KSM} (\Upsilon(X),\Gamma^{\Upsilon},\pi_{\mathsf{m}})$$

$$\Big\downarrow \ \textit{Identification}$$

$$(X_n,\mathsf{d}_n,\mathsf{m}_n) \xrightarrow{pmG} (X,\mathsf{d},\mathsf{m}) \implies \begin{cases} (\mathsf{Ch}_{\mathsf{d}_2^n},\mathscr{D}(\mathsf{Ch}_{\mathsf{d}_2^n})) \xrightarrow{Mosco} (\mathsf{Ch}_{\mathsf{d}_2},\mathscr{D}(\mathsf{Ch}_{\mathsf{d}_2})) \\ (\{B_t^n\},\mathbb{P}_n^{\pi_{\mathsf{m}_n}}) \xrightarrow{weak} (\{B_t\},\mathbb{P}^{\pi_{\mathsf{m}}}). \end{cases}$$

Historical and Technical Remarks Comparison with the Smooth Case: In the smooth framework, the identification is obtained by the combination of Albeverio–Kondratiev–Röckner [3], Röckner–Schied [6] and Erbar–Huesmann [4]. Some main difficulties for extending these results to non-smooth spaces are

(1) No flow of diffeomorphism generated by vector fields;
(2) No core of C^1-functions.

(1) is essentially used for most proofs in the smooth setting and even for the construction of $(\mathcal{E}^{\Upsilon},\mathscr{D}(\mathcal{E}^{\Upsilon}))$. (2) is also used in various instances, in particular, to show $(\mathsf{Ch}_{\mathsf{d}_2},\mathscr{D}(\mathsf{Ch}_{\mathsf{d}_2})) = (\mathcal{E}^{\Upsilon},\mathscr{D}(\mathcal{E}^{\Upsilon}))$. Note that it is not sure if the semigroup has the Feller property on $\Upsilon(X)$ (even if X is smooth), due to which the continuous regularisation by semigroup actions cannot be used. So (2) is a serious problem in the non-smooth setting. For these reasons, more robust proofs are required in the non smooth setting, for which combinatorial computations, closed forms in infinite-product spaces, finite-dimensional approximation in the sense of Kuwae–Shioya, and the Varadhan-type short-time asymptotic in the sense of Hino–Ramirez come into play. The identification can be achieved by establishing the Rademacher theorem, the continuous-Sobolev-to-Lipschitz property and the topological upper regularity.

Comparison with standard metric measure geometry: The configuration space $(\Upsilon(X),\mathsf{d}_2,\pi_{\mathsf{m}})$ is out of the scope of standard metric measure geometry (e.g., Ambrosio–Gigli–Savaré [2]) since d_2 explodes on sets of positive measures, which typically happens in infinite-dimensional spaces (e.g. the Wiener space with the Cameron–Martin distance). As a consequence, the extended distance d_2 cannot be continuous on $\Upsilon(X)\times\Upsilon(X)$, only *lower semi-continuous.* This breaks various fundamental properties – which hold in usual metric measure spaces:

- Lipschitz functions are not necessarily continuous, nor Borel;
- d_2-metric balls are π_{m}-negligible, d_2-open balls are not open.

These facts require a number of careful arguments to establish our results compared to standard metric measure geometry.

References

[1] Ambrosio, L., Erbar, M. and Savaré, G. (2016). Optimal transport, Cheeger energies and contractivity of dynamic transport distances in extended spaces, *Nonlinear Anal.* **137**, 77–134.

[2] Ambrosio, L., Gigli, N. and Savaré, G. (2015): Bakry–Émery curvature–dimension condition and Riemannian Ricci curvature bounds, *Ann. Probab.* **43**, Number 1, 339–404.

[3] Albeverio, S., Kondratiev, Yu, G., and Röckner, M. (1998). Analysis and geometry on configuration spaces. *J. Funct. Anal.*, **154**(2), 444–500.

[4] M. Erbar and M. Huesmann. (2014). Curvature bounds for configuration spaces, *Calc. Var. Partial Differ. Equ.,* **201**(3), 397–430.

[5] Ma, Z.-M. and Röckner, M. (2000). Construction of Diffusions on Configuration Spaces. *Osaka J. Math.*, **37**, 273–314.

[6] Röckner and Schied, A. (1999). Rademacher's theorem on configuration spaces and applications, *J. Funct. Anal.,* **169**(2), 325–356.

Functional inequalities on path space of sub-Riemannian manifolds and horizontal Ricci curvature

Anton Thalmaier

(joint work with Li-Juan Cheng, Erlend Grong)

Sub-Riemannian geometry deals with the investigation of geometric structures intrinsically induced by the sub-Riemannian data (M, H, g_H) where M is a smooth manifold, H a subbundle of the tangent bundle (describing the "horizontal" directions), and g_H a metric tensor defined on the "horizontal" subbundle H. The subbundle H is assumed to be bracket-generating, meaning that its sections and their iterated brackets span the entire tangent bundle. We describe recent work related to the concept of "horizontal Ricci curvature". Our approach relies on a study of sub-Riemannian Brownian motions and stochastic analysis on path space over sub-Riemannian manifolds. Analogously to the work of Aaron Naber [6] (see also [5]) we show that certain functional inequalities and gradient estimates on the path space over M are equivalent to boundedness of the horizontal Ricci tensor [3]. To this end, we adopt the methods of [1, 2] to the sub-Riemannian setting.

We work with a connection ∇ on M which is compatible with (H, g_H) in the sense that parallel transport along smooth curves in M takes orthonormal frames in H to orthonormal frames in H. Since H is bracket-generating, compatible connections ∇ always have torsion $\mathbf{T}$:

$$\nabla_A B - \nabla_B A - [A, B] = \mathbf{T}(A, B), \quad A, B \in \Gamma(H).$$

To construct canonical connections one starts with a partial connection $\nabla\colon \Gamma(H)\times\Gamma(H) \to \Gamma(H)$, $(A, B) \mapsto \nabla_A B$ on H and extends it to a full connection in an appropriate way. A connection ∇ on M compatible with (H, g_H) is uniquely determined by its torsion. Choosing a complement V for H, that is $TM = H \oplus V$, there is a unique such connection with $\mathbf{T}(H, H) \subset V$.

Let $\mathbf{R}$ be the curvature of a compatible connection ∇ and $\mathrm{Ric}\colon TM \to TM$ the corresponding Ricci operator given by

$$\mathrm{Ric}(v) = \mathrm{trace}_H \mathbf{R}(v, \times)\times$$

where the trace is taken over H with respect to the inner product g_H. Our object of interest is the horizontal Ricci curvature $\mathrm{Ric}^H = \mathrm{Ric}|_H \in \mathrm{End}(H)$ defined as the restriction Ric of H. We consider the corresponding sub-Laplacian

$$\Delta^H = \mathrm{trace}_H \nabla^2_{\times,\times}$$

defined as horizontal trace of the Hessian ∇^2. Diffusion processes on M with generator $\frac{1}{2}\Delta^H$ are called sub-Riemannian Brownian motions, cf. [4].

For fixed $T > 0$, let $W^T = C([0,T]; M)$ be the path space over M equipped with the measure induced by the sub-Riemannian Brownian motion with starting point $x \in M$, and let

$$\mathcal{F}C^\infty_{0,T} = \left\{W^T \ni \gamma \mapsto f(\gamma_{t_1}, \ldots, \gamma_{t_n})\colon\ 0 < t_1 < \ldots < t_n \le T,\ f \in C_c^\infty(M^n)\right\}$$

be the class of smooth cylindrical functions on W^T. We consider the Cameron-Martin space

$$\mathbb{H} = \left\{ h\colon [0,T] \to H_x \text{ absolutely continuous} \;\middle|\; \int_0^T |\dot h(t)|^2_{g_H}\, dt < \infty \right\}$$

which becomes a Hilbert space with inner product

$$\langle h_1, h_2\rangle_{\mathbb{H}} = \int_0^T \langle \dot h_1(t), \dot h_2(t)\rangle_{g_H}\, dt.$$

For $F \in \mathcal{F}C^\infty_{0,T}$ we define a directional derivative $D_h F$ in the direction of $h \in \mathbb{H}$ and associated derivative operators D_t on $\mathcal{F}C^\infty_{0,T}$ such that

$$D_h F = \int_0^T \langle D_t F, \dot h_t\rangle_{g_H}\, dt.$$

The definition of D_h incorporates explicitly the torsion of the connection, for details see [3].

Theorem *(Characterization of* Ric^H *by functional inequalities on path space)* For a non-negative constant K the following conditions are equivalent:

(1) the horizontal Ricci curvature Ric^H is bounded by K, i.e.

$$-K \le \mathrm{Ric}^H \le K;$$

(2) *(Gradient estimate)* for any smooth cylindrical function $F \in \mathcal{F}C^\infty_{0,T}$ on path space the following estimate holds:

$$|D_0 \mathbb{E}_x[F]|_{g_H} \le \mathbb{E}_x\Big[|D_0 F|_{g_H} + \frac{K}{2}\int_0^T e^{\frac{K}{2}s} |D_s F|_{g_H}\, ds\Big];$$

(3) *(Log-Sobolev inequality)* for any $F \in \mathcal{F}C_{0,T}^\infty$ and $t > 0$ in $[0, T]$,

$$\mathbb{E}_x\Big[\mathbb{E}_x[F^2|\mathcal{F}_t]\log \mathbb{E}_x[F^2|\mathcal{F}_t]\Big] - \mathbb{E}_x[F^2]\log \mathbb{E}_x[F^2]$$
$$\le 2\int_0^t e^{\frac{K}{2}(T-r)}\left(\mathbb{E}_x|D_r F|_{g_H}^2 + \frac{K}{2}\int_r^T e^{\frac{K}{2}(s-r)}\mathbb{E}_x|D_s F|_{g_H}^2\,ds\right)dr;$$

(4) *(Poincaré inequality)* for any $F \in \mathcal{F}C_{0,T}^\infty$ and $t > 0$ in $[0, T]$,

$$\mathbb{E}_x\Big[\mathbb{E}_x[F|\mathcal{F}_t]^2\Big] - \mathbb{E}_x[F]^2$$
$$\le \int_0^t e^{\frac{K}{2}(T-r)}\left(\mathbb{E}_x|D_r F|_{g_H}^2 + \frac{K}{2}\int_r^T e^{\frac{K}{2}(s-r)}\mathbb{E}_x|D_s F|_{g_H}^2\,ds\right)dr.$$

Here $\mathbb{E}_x$ denotes the expectation with respect to the probability measure on path space induced by the sub-Riemannian Brownian motion on M starting at $x \in M$, and $(\mathcal{F}_t)$ denotes its natural filtration.

The theorem above can be extended to a characterization of $K_1 \le \mathrm{Ric}^H \le K_2$ with arbitrary constants $K_1 \le K_2$ by redefining D_h appropriately, see [3].

References

[1] L.-J. Cheng and A. Thalmaier, *Characterization of pinched Ricci curvature by functional inequalities*, J. Geom. Anal. **28** (2018), no. 3, 2312–2345.

[2] L.-J. Cheng and A. Thalmaier, *Spectral gap on Riemannian path space over static and evolving manifolds*, J. Funct. Anal. **274** (2018), no. 4, 959–984.

[3] L.-J. Cheng, E. Grong and A. Thalmaier, *Functional inequalities on path space of sub-Riemannian manifolds and applications*, arXiv:1912.03575 (2019).

[4] E. Grong and A. Thalmaier, *Stochastic completeness and gradient representations for sub-Riemannian manifolds*, Potential Analysis **51** (2019), no. 2, 219–254.

[5] R. Haslhofer and A. Naber, *Characterizations of the Ricci flow*, J. Eur. Math. Soc. (JEMS) **20** (2018), no. 5, 1269–1302.

[6] Aaron Naber, *Characterizations of bounded Ricci curvature on smooth and nonsmooth spaces*, arXiv:1306.6512v4 (2015).

Heat kernel estimates for symmetric Dirichlet forms

Jian Wang

(joint work with Zhen-Qing Chen, Takashi Kumagai)

Let (M, d, μ) be a metric measure space such that (M, d) is a locally compact separable metric space, and μ is a positive Radon measure on M with full support. We consider the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(M, \mu)$ as follows

$$\mathcal{E}(f,g) = \mathcal{E}^{(c)}(f,g) + \int_{M\times M\setminus \mathrm{diag}} (f(x) - f(y)(g(x) - g(y))\, J(dx, dy), \qquad f, g \in \mathcal{F},$$

where $(\mathcal{E}^{(c)}, \mathcal{F})$ is the strongly local part of $(\mathcal{E}, \mathcal{F})$ (namely $\mathcal{E}^{(c)}(f,g) = 0$ for all $f, g \in \mathcal{F}$ having $(f - c)g = 0$ μ-a.e. on M for some constant $c \in \mathbb{R}$) and $J(\cdot,\cdot)$ is a symmetric Radon measure $M \times M \setminus \mathrm{diag}$.

Denote the ball centered at x with radius r by $B(x,r)$ and $\mu(B(x,r))$ by $V(x,r)$.

Definition 1. (i) We say that (M,d,μ) satisfies the *volume doubling property* (VD), if there exists a constant $C_\mu \geq 1$ such that for all $x \in M$ and $r > 0$,

$$V(x,2r) \leq C_\mu V(x,r).$$

(ii) We say that (M,d,μ) satisfies the *reverse volume doubling property* (RVD), if there exist constants $l_\mu, c_\mu > 1$ such that for all $x \in M$ and $r > 0$,

$$V(x,l_\mu r) \geq c_\mu V(x,r).$$

Let $\mathbb{R}_+ := [0,\infty)$, and $\phi_c : \mathbb{R}_+ \to \mathbb{R}_+$ (resp. $\phi_j : \mathbb{R}_+ \to \mathbb{R}_+$) be a strictly increasing continuous function with $\phi_c(0) = 0$ (resp. $\phi_j(0) = 0$), $\phi_c(1) = 1$ (resp. $\phi_j(1) = 1$) and satisfying that there exist constants $c_{1,\phi_c}, c_{2,\phi_c} > 0$ and $\beta_{2,\phi_c} \geq \beta_{1,\phi_c} > 1$ (resp. $c_{1,\phi_j}, c_{2,\phi_j} > 0$ and $\beta_{2,\phi_j} \geq \beta_{1,\phi_j} > 0$) such that

$$\begin{gathered} c_{1,\phi_c}\left(\frac{R}{r}\right)^{\beta_{1,\phi_c}} \leq \frac{\phi_c(R)}{\phi_c(r)} \leq c_{2,\phi_c}\left(\frac{R}{r}\right)^{\beta_{2,\phi_c}} \quad \text{for all } 0 < r \leq R. \\ \left(\text{resp. } c_{1,\phi_j}\left(\frac{R}{r}\right)^{\beta_{1,\phi_j}} \leq \frac{\phi_j(R)}{\phi_j(r)} \leq c_{2,\phi_j}\left(\frac{R}{r}\right)^{\beta_{2,\phi_j}} \quad \text{for all } 0 < r \leq R.\right) \end{gathered} \tag{1}$$

We always assume that

$$\phi_c(r) \leq \phi_j(r) \text{ for } r \in (0,1] \quad \text{and} \quad \phi_c(r) \geq \phi_j(r) \text{ for } r \in [1,\infty). \tag{2}$$

Since $\beta_{1,\phi_c} > 1$, there exists a strictly increasing continuous function $\bar{\phi}_c(r) : \mathbb{R}_+ \to \mathbb{R}_+$ such that there are constants $c_2 \geq c_1 > 0$ so that

$$c_1 \frac{\phi_c(r)}{r} \leq \bar{\phi}_c(r) \leq c_2 \frac{\phi_c(r)}{r} \quad \text{for all } r > 0.$$

Given ϕ_c and ϕ_j satisfying (2), we set

$$\phi(r) := \phi_c(r) \wedge \phi_j(r) = \begin{cases} \phi_c(r), & r \in (0,1], \\ \phi_j(r), & r \in [1,\infty). \end{cases}$$

Definition 2. We say that the (*weak*) *Poincaré inequality* PI(ϕ) holds if there exist constants $C > 0$ and $\kappa \geq 1$ such that for any ball $B_r = B(x,r)$ with $x \in M$ and for any $f \in \mathcal{F}_b$,

$$\int_{B_r} (f - \overline{f}_{B_r})^2 \, d\mu \leq C\phi(r) \left(\int_{B_{\kappa r}} \Gamma_c(f,f) + \int_{B_{\kappa r} \times B_{\kappa r}} (f(y) - f(x))^2 \, J(dx,dy) \right),$$

where Γ_c is the energy measure of local bilinear form of $(\mathcal{E},\mathcal{F})$, and $\overline{f}_{B_r} = \frac{1}{\mu(B_r)} \int_{B_r} f \, d\mu$ is the average value of f on B_r.

Let $U \subset V$ be open sets of M with $U \subset \overline{U} \subset V$. We say a non-negative bounded measurable function φ is a cut-off function for $U \subset V$, if $\varphi \geq 1$ on U, $\varphi = 0$ on V^c and $0 \leq \varphi \leq 1$ on M.

Definition 3. We say that condition $\mathrm{CS}(\phi)$ holds if there exist constants $C_0 \in (0,1]$ and $C_1, C_2 > 0$ such that for every $0 < r \le R$, almost all $x_0 \in M$ and any $f \in \mathcal{F}$, there exists a cut-off function $\varphi \in \mathcal{F}_b$ for $B(x_0,R) \subset B(x_0,R+r)$ so that the following holds:

$$\begin{aligned}&\int_{B(x_0,R+(1+C_0)r)} f^2\, d\Gamma(\varphi,\varphi)\\ &\le C_1\Big(\int_{B(x_0,R+r)} \varphi^2\, d\Gamma_c(f,f)\\ &\qquad + \int_{B(x_0,R+r)\times B(x_0,R+(1+C_0)r)} \varphi^2(x)(f(x)-f(y))^2\, J(dx,dy)\Big)\\ &\quad + \frac{C_2}{\phi(r)}\int_{B(x_0,R+(1+C_0)r)} f^2\, d\mu.\end{aligned}$$

Define

$$p^{(c)}(t,x,y) = \frac{1}{V(x,\phi_c^{-1}(t))}\exp\left(-\frac{d(x,y)}{\bar\phi_c^{-1}(t/d(x,y))}\right), \quad t>0, x,y\in M_0,$$

This kernel arises in the two-sided estimates of heat kernel for strongly local Dirichlet forms; see [3]. Set

$$p^{(j)}(t,x,y) := \frac{1}{V(x,\phi_j^{-1}(t))} \wedge \frac{t}{V(x,d(x,y))\phi_j(d(x,y))}.$$

This is just two-sided estimates for symmetric pure jump processes with scaling function ϕ_j as in [1].

Definition 4. (i) We say that $\mathrm{HK}(\phi_c,\phi_j)$ holds if there exists a heat kernel $p(t,x,y)$ associated with $(\mathcal{E},\mathcal{F})$ and the following estimates hold for all $t>0$ and all $x,y \in M_0$,

$$\begin{aligned}&c_1\Big(\frac{1}{V(x,\phi_c^{-1}(t))} \wedge \frac{1}{V(x,\phi_j^{-1}(t))} \wedge \big(p^{(c)}(c_2 t,x,y) + p^{(j)}(t,x,y)\big)\Big)\\ &\le\ p(t,x,y) \qquad (3)\\ &\le c_3\Big(\frac{1}{V(x,\phi_c^{-1}(t))} \wedge \frac{1}{V(x,\phi_j^{-1}(t))} \wedge \big(p^{(c)}(c_4 t,x,y) + p^{(j)}(t,x,y)\big)\Big),\end{aligned}$$

where $c_k > 0$, $k = 1,\cdots,4$, are constants independent of $x,y\in M_0$ and $t>0$.

(ii) We say $\mathrm{HK}_-(\phi_c,\phi_j)$ holds if the upper bound in (3) holds but the lower bound is replaced by the following: there are constants $c_0, c_1 > 0$ so that for all $x,y\in M_0$,

$$\begin{aligned}p(t,x,y) \ge c_0\Big(&\frac{1}{V(x,\phi^{-1}(t))}\mathbf{1}_{\{d(x,y)\le c_1\phi^{-1}(t)\}}\\ &+ \frac{t}{V(x,d(x,y))\phi_j(d(x,y))}\mathbf{1}_{\{d(x,y)>c_1\phi^{-1}(t)\}}\Big).\end{aligned}$$

With the notations above, we now can state the following stable characterizations of two-sided heat kernel estimates for diffusions with jumps.

Theorem 1. *Assume that the metric measure space* (M, d, μ) *satisfies* VD *and* RVD, *and that the scale functions* ϕ_c *and* ϕ_j *satisfy* (1) *and* (2). *Let* $\phi := \phi_c \wedge \phi_j$. *The following are equivalent:*

(i) $\mathrm{HK}_-(\phi_c, \phi_j)$.
(ii) J_{ϕ_j}, $\mathrm{PI}(\phi)$ *and* $\mathrm{CS}(\phi)$.

If, additionally, (M, d, μ) *is connected and satisfies the chain condition, then all the conditions above are equivalent to:*

(iii) $\mathrm{HK}(\phi_c, \phi_j)$.

[2, Theorem 1.14] contains more equivalent characterizations of $\mathrm{HK}_-(\phi_c, \phi_j)$. We emphasize again that the connectedness and the chain condition of the underlying metric measure space (M, d, μ) are only used to derive optimal lower bounds off-diagonal estimates for heat kernel when the time is small (i.e., from $\mathrm{HK}_-(\phi_c, \phi_j)$ to $\mathrm{HK}(\phi_c, \phi_j)$), while for the equivalence between (i) and (ii) in the result above, the metric measure space (M, d, μ) is only assumed to satisfy the general VD and RVD; that is, neither do we assume M to be connected nor (M, d) to be geodesic.

References

[1] Z.-Q. Chen, T. Kumagai and J. Wang, *Stability of heat kernel estimates for symmetric non-local Dirichlet forms.* To appear in *Memoirs Amer. Math. Soc.*, available at arXiv:1604.04035.
[2] Z.-Q. Chen, T. Kumagai and J. Wang, *Heat kernel and parabolic Harnack inequalities for symmetric Dirichlet forms*, available at arXiv:1908.07650
[3] B.M. Hambly and T. Kumagai, *Transition density estimates for diffusion processes on post critically finite self-similar fractals*, Proc. London Math. Soc. **78** (1999), 431–458.

Regularity for nonlocal parabolic operators

Marvin Weidner
(joint work with Jamil Chaker, Moritz Kassmann)

We study weak solutions $u : I \times \mathbb{R}^d \to \mathbb{R}$ to

$$\partial_t u - Lu = f \quad \text{in} \quad I \times \Omega =: Q, \tag{1}$$

where $f \in L^\infty(Q)$ and $\Omega \subset \mathbb{R}^d$ is a bounded domain, $I \subset \mathbb{R}$ is a bounded open interval and L is a linear, nonlocal operator of the form

$$Lu(t,x) := 2 \text{ p.v.} \int_{\mathbb{R}^d} (u(t,y) - u(t,x))\mu_t(x, dy). \tag{2}$$

Here, $\mu_t(x, dy) = a(t,x,y)\mu(x,dy)$, where $a : \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d \to [1/2, 1]$ is a measurable function that is symmetric in the second and third variable and $(\mu(x,\cdot))_{x\in\mathbb{R}^d}$ is an admissible family of measures.

Our goal is to derive Hölder regularity estimates for weak solutions to (1) with $f = 0$ given an admissible family of measures $(\mu(x,\cdot))_{x\in\mathbb{R}^d}$.
More specifically, we say that $(\mathbf{HR}(\alpha))$ is satisfied by $(\mu(x,\cdot))_{x\in\mathbb{R}^d}$ if for every weak solution u to (1) in Q with $f = 0$, it holds that for every $Q' \subset\subset Q$

$$\sup_{(t,x),(s,y)\in Q'} \frac{|u(t,x)-u(s,y)|}{\left(|x-y|+|t-s|^{1/\alpha}\right)^{\gamma}} \le \frac{\|u\|_{L^\infty(I\times\mathbb{R}^d)}}{\eta^\gamma}, \tag{3}$$

where $\eta = \eta(Q,Q')$ and $\gamma = \gamma(d) \in (0,1)$ is the Hölder exponent.

A very prominent example of a family of measures that satisfies the Hölder regularity estimate is given by $\mu^\alpha(x,dy) = k(x,y)dy$, where $\alpha \in (0,2)$ and $k : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ is a symmetric, measurable function with

$$k(x,y) \asymp c(d,\alpha)|x-y|^{-d-\alpha}, \tag{4}$$

where $c(d,\alpha) > 0$ is a normalizing constant. Note that if $k(x,y) = c(d,\alpha)|x-y|^{-d-\alpha}$, we have $L = -(-\Delta)^{\alpha/2}$. It was shown in [1] that in this case $(\mathbf{HR}(\alpha))$ holds true. In [2] and [3] the authors use Moser iteration to derive $(\mathbf{HR}(\alpha))$ for families of measures $(\mu(x,\cdot))_{x\in\mathbb{R}^d}$ that do not need to be comparable to $(\mu^\alpha(x,\cdot))_{x\in\mathbb{R}^d}$ in the sense of (4) but (among other assumptions) require to have comparable energies on small scales, i.e. there exists $\Lambda \ge 1$ such that for every $x_0 \in \mathbb{R}^d$, $\rho \in (0,2)$ and every $v \in L^2(B_\rho(x_0))$:

$$\Lambda^{-1}\mathcal{E}^{\mu}_{B_\rho(x_0)}(v,v) \le \mathcal{E}^{\mu^\alpha}_{B_\rho(x_0)}(v,v) \le \Lambda\mathcal{E}^{\mu}_{B_\rho(x_0)}(v,v),$$

where we write

$$\mathcal{E}^\mu_D(v,v) := \int_D\int_D (v(x)-v(y))^2\mu(x,dy)dx,\ \ D\subset\mathbb{R}^d$$

for the restriction of the energy form associated to L. This allows us to consider families $(\mu(x,\cdot))_{x\in\mathbb{R}^d}$ that are singular with respect to Lebesgue measure, as for example

$$\mu^\alpha_{axes}(x,dy) := \sum_{k=1}^d \Big(|x_k-y_k|^{-1-\alpha}\, dy_k \prod_{i\neq k}\delta_{\{x_i\}}(dy_i)\Big), \tag{5}$$

where $\alpha \in (0,2)$. Note that $\mu^\alpha_{axes}(x,\cdot)$ assigns mass only to the coordinate axes with respect to $x \in \mathbb{R}^d$. The corresponding operator L according to (2) is given by $L = -\sum_{k=1}^d(-\partial_k\partial_k)^{\alpha/2}$.

In our main result, we take (5) as a starting point, but generalize to a highly anisotropic situation where the order of differentiability α depends on the coordinate direction. Given $\alpha_1,\dots,\alpha_d \in (0,2)$, we set $\alpha_{max} := \{\alpha_k | 1\le k\le d\}$ and define

$$\mu^{\alpha_1,\dots,\alpha_d}_{axes}(x,dy) := \sum_{k=1}^d \Big(|x_k-y_k|^{-1-\alpha_k}\, dy_k \prod_{i\neq k}\delta_{\{x_i\}}(dy_i)\Big). \tag{6}$$

In order to deal with the anisotropy of $\mu_{axes}^{\alpha_1,\dots,\alpha_d}(x,\cdot)$ let us introduce rectangles $M_\rho(x)$ with radius $\rho > 0$ and center $x \in \mathbb{R}^d$ as follows:

$$M_\rho(x) := \bigtimes_{k=1}^{d} \left(x_k - \rho^{\frac{\alpha_{\max}}{\alpha_k}}, x_k + \rho^{\frac{\alpha_{\max}}{\alpha_k}} \right).$$

Theorem 1. *Let $(\mu(x,\cdot))_{x\in\mathbb{R}^d}$ be symmetric, i.e. for every two measurable sets $A, B \in \mathcal{B}(\mathbb{R}^d)$:*

$$\int_A \int_B \mu(x,dy)dx = \int_B \int_A \mu(x,dy)dx.$$

Furthermore, assume that there exists $\Lambda \geq 1$ such that the following holds true:

(i) *tail estimate: For every $x_0 \in \mathbb{R}^d$ and $\rho \in (0,2)$:*

$$\mu(x_0, \mathbb{R}^d \setminus M_\rho(x_0)) \leq \Lambda \rho^{-\alpha_{\max}}.$$

(ii) *comparability assumption: For every $x_0 \in \mathbb{R}^d$, $\rho \in (0,2)$ and every $v \in L^2(M_\rho(x_0))$:*

$$\Lambda^{-1}\mathcal{E}^{\mu}_{M_\rho(x_0)}(v,v) \leq \mathcal{E}^{\mu_{axes}^{\alpha_1,\dots,\alpha_d}}_{M_\rho(x_0)}(v,v) \leq \Lambda \mathcal{E}^{\mu}_{M_\rho(x_0)}(v,v).$$

Then, $(\mu(x,\cdot))_{x\in\mathbb{R}^d}$ satisfies $(\boldsymbol{HR}(\alpha_{max}))$.

Note that by choosing $\alpha_1, = \dots, \alpha_d = \alpha$ for some $\alpha \in (0,2)$, Theorem 1 implies that $(\mu_{axes}^{\alpha}(x,\cdot))_{x\in\mathbb{R}^d}$ and $(\mu^{\alpha}(x,\cdot))_{x\in\mathbb{R}^d}$ satisfy $(\mathbf{HR}(\alpha))$, so we reobtain the Hölder regularity for the two examples from above. An interesting feature of the comparability assumption is that it allows to consider kernels of the form

$$\mu(x,dy) \asymp |x-y|^{-d-\gamma}\mathbf{1}_\Gamma(x-y)dy$$

where γ could be any (!) positive number.

References

[1] Chen, Zhen-Qing and Kumagai, Takashi, *Heat kernel estimates for stable-like processes on d-sets*, Stochastic Process. Appl. **108** (2003), 27–62.

[2] Felsinger, Matthieu and Kassmann, Moritz, *Local regularity for parabolic nonlocal operators*, Comm. Partial Differential Equations. **38** (2013), 1539–1573.

[3] Kassmann, Moritz and Schwab, Russell W., *Regularity results for nonlocal parabolic equations*, Riv. Math. Univ. Parma (N.S.). **5** (2014), 183–212.

[4] Chaker, Jamil and Kassmann, Moritz and Weidner, Marvin, *Regularity for nonlocal parabolic operators*, work in progress.

The space of algebraic measure trees and the Aldous chain in the diffusion limit

Anita Winter
(joint work with Wolfgang Löhr, Leonid Mytnik)

In [1] a symmetric Markov chain on cladograms is investigated and bounds on its mixing and relaxation times are given. The latter bound was sharpened in [6].

In this talk we encode cladograms as binary, algebraic measure trees which can be considered as (continuum) metric trees for which one ignores the metric distances and rather focuses on the tree structure. We show that any separable algebraic tree can be represented by a metric tree. We further consider algebraic measure trees which are algebraic trees additionally equipped with a sampling (probability) measure. This measure gives rise to the branch point distribution which turns out to be the length measure of an intrinsic choice of such a metric tree representation.

Further, we provide a notion of convergence of algebraic measure trees which resembles the idea of the Gromov-weak topology which itself is defined through weak convergence of sample distance matrices. Binary algebraic (measure) trees are of particular interest due to their close connection to triangulations of the circle. We will rely on this connection to show that the subspace of binary algebraic measure trees is compact and that in this subspace weak convergence of sample shapes and sample branch point distribution distance matrices are equivalent.

This allows us to show that the Aldous Markov chain on cladograms with a fixed number of leaves converges in distribution as the number of leaves goes to infinity. We give a rigorous construction of the limit, whose existence was conjectured by Aldous and which we therefore refer to as Aldous diffusion, as a solution of a well-posed martingale problem. We show that the Aldous diffusion is a Feller process with continuous paths, and the algebraic measure Brownian CRT is its unique invariant distribution.

Our approach is complement to a long series of papers by Forman, Pal, Rizzolo and Winkel on the construction of the Aldous diffusion (e.g., [2, 3]), which gives a detailed description of the Aldous diffusion in equilibrium but (so far) don't state convergence.

References

[1] D. Aldous, *Mixing Time for a Markov Chain on Cladograms*, Combinatorics, Probability and Computing **9** (2000), 191–204.

[2] N. Forman, S. Pal, D. Rizzolo and M. Winkel, *Diffusions on the space of interval partitions: Construction from marked Levy processes*, arXiv:1909.02584.

[3] N. Forman, S. Pal, D. Rizzolo and M. Winkel, *Diffusions on the space of interval partitions: Poisson-Dirichlet stationary distributions*, arXiv:1910.07626.

[4] W. Löhr and A. Winter, *Spaces of algebraic measure trees and triangulations of the circle*, arXiv:1811.11734.

[5] W. Löhr, L. Mytnik and A. Winter, *The Aldous chain on cladograms*, arxiv:1805.12057.
[6] J. Schweinsberg, *An $O(n^2)$ bound for the relaxation time of a Markov chain on cladograms*, Random Structures Algorithms, **20(1)** (2002), 59–70.

Reporters: David Hornshaw, Chiara Rigoni

Participants

Dr. Patricia Alonso Ruiz
Department of Mathematics
Texas A & M University
College Station, TX 77843-3368
UNITED STATES

Dr. Sebastian Andres
Centre for Mathematical Sciences
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM

Prof. Dr. Fabrice Baudoin
Department of Mathematics
University of Connecticut
196 Auditorium Road
Storrs, CT 06269-3009
UNITED STATES

Prof. Dr. Noam Berger
Fakultät für Mathematik
Technische Universität München
Boltzmannstrasse 3
85747 Garching bei München
GERMANY

Prof. Dr. Marek Biskup
Department of Mathematics
UCLA
405 Hilgard Avenue
Los Angeles, CA 90095-1555
UNITED STATES

Filip Bosnic
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld
GERMANY

Elia Brue
Scuola Normale Superiore di Pisa
Piazza dei Cavalieri 7
56126 Pisa
ITALY

Dr. Li Chen
Department of Mathematics
University of Connecticut
341 Mansfield Road
Storrs, CT 06269-1009
UNITED STATES

Prof. Dr. Zhen-Qing Chen
Department of Mathematics
University of Washington
Seattle, WA 98195
UNITED STATES

Prof. Dr. Fabio E.G. Cipriani
Dipartimento di Matematica
Politecnico di Milano
Piazza Leonardo da Vinci, 32
20133 Milano
ITALY

Prof. Dr. David Croydon
Department of Mathematics
Graduate School of Science
Kyoto University
Kitashirakawa, Oiwake-cho, Sakyo-ku
Kyoto 606-8502
JAPAN

Lorenzo Dello Schiavo
Institut für angewandte Mathematik
(IaM)
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Jean Dominique Deuschel
Institut für Mathematik
Sekr. MA 7-4
Technische Universität Berlin
Straße des 17. Juni 136
10623 Berlin
GERMANY

Prof. Dr. Nathaniel Eldredge
School of Mathematical Sciences
University of Northern Colorado
501, 20th Street
Greeley, CO 80639
UNITED STATES

Dr. Matthias Erbar
Institut für angewandte Mathematik (IaM)
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Alessandra Faggionato
Dipartimento di Matematica
Università di Roma "La Sapienza"
Istituto "Guido Castelnuovo"
Piazzale Aldo Moro, 2
00185 Roma
ITALY

Prof. Dr. Nina Gantert
Fakultät für Mathematik
Technische Universität München
Boltzmannstrasse 3
85748 Garching bei München
GERMANY

Prof. Dr. Masha Gordina
Department of Mathematics
University of Connecticut
196 Auditorium Road
Storrs, CT 06269-3009
UNITED STATES

Dr. Ewain Gwynne
Centre for Mathematical Sciences
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM

Dr. Karen Habermann
Laboratoire Jacques-Louis Lions
Sorbonne Université
Boite courrier 187
4, Place Jussieu
75252 Paris Cedex 05
FRANCE

Prof. Robert Haslhofer
Department of Mathematics
University of Toronto
Room BA 6208
40 St George Street
Toronto ONT M5S 2E4
CANADA

Dr. Ronan Herry
Institut für angewandte Mathematik (IaM)
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

David Hornshaw
Institut für angewandte Mathematik (IaM)
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Naotaka Kajino
Department of Mathematics
Kobe University
Rokkodai-cho 1-1, Nada-ku
Kobe 657-8501
JAPAN

Prof. Dr. Moritz Kaßmann
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld
GERMANY

Prof. Dr. Jun Kigami
Graduate School of Informatics
Kyoto University
Yoshida-honmachi, Sakyo-ku
Kyoto 606-8501
JAPAN

Dr. Eva Kopfer
Institut für angewandte Mathematik
(IaM)
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Takashi Kumagai
Research Institute for Mathematical
Sciences
Kyoto University
Kitashirakawa, Sakyo-ku
Kyoto 606-8502
JAPAN

Prof. Dr. Jean-Francois Le Gall
Département de Mathématiques
Université de Paris-Sud
Bât. 307
91405 Orsay Cedex
FRANCE

Prof. Jan Maas
IST Austria
Am Campus 1
3400 Klosterneuburg
AUSTRIA

Prof. Dr. Pierre Mathieu
Centre de Mathématiques et
d'Informatique
Université d'Aix-Marseille
39, Rue Joliot-Curie
13453 Marseille Cedex 13
FRANCE

Prof. Dr. Tai Alexis Melcher
Department of Mathematics
University of Virginia
Kerchof Hall
P.O.Box 400137
Charlottesville, VA 22904-4137
UNITED STATES

Prof. Dr. Grégory Miermont
UMPA
ENS de Lyon
46, Allée d'Italie
69364 Lyon Cedex 07
FRANCE

Prof. Dr. Emanuel Milman
Department of Mathematics
TECHNION
Israel Institute of Technology
Amado 630
Haifa 32000
ISRAEL

Dr. Mathav Murugan
Department of Mathematics
University of British Columbia
121-1984 Mathematics Road
Vancouver BC V6T 1Z2
CANADA

Prof. Dr. James R. Norris
Statistical Laboratory
Centre for Mathematical Sciences
Wilberforce Road
Cambridge CB3 OWB
UNITED KINGDOM

Dr. Chiara Rigoni
Institut für angewandte Mathematik
(IaM)
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Annika Rothhardt
Fachbereich Mathematik
Technische Universität Berlin
Straße des 17. Juni 136
10623 Berlin
GERMANY

Prof. Dr. Laurent Saloff-Coste
Department of Mathematics
Cornell University
Malott Hall
Ithaca, NY 14853-7901
UNITED STATES

Prof. Dr. Daisuke Shiraishi
Department of Mathematics
Graduate School of Science
Kyoto University
Kitashirakawa, Oiwake-cho, Sakyo-ku
Kyoto 606-8502
JAPAN

Dr. Adam B. Sikora
Department of Mathematics and
Statistics
Macquarie University
Balaclava Road
Macquarie Park, NSW 2109
AUSTRALIA

Dr. Martin Slowik
Fachbereich Mathematik
Technische Universität Berlin
Straße des 17. Juni 135
10623 Berlin
GERMANY

Prof. Dr. Charles K. Smart
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago, IL 60637-1514
UNITED STATES

Anna-Lisa Sokol
Fachbereich Mathematik
Technische Universität Berlin
Sekr. MA 8-5
Straße des 17. Juni 136
10623 Berlin
GERMANY

Prof. Dr. Karl-Theodor Sturm
Institut für angewandte Mathematik
(IaM)
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Kohei Suzuki
Scuola Normale Superiore
Piazza dei Cavalieri, 7
56100 Pisa
ITALY

Prof. Dr. Anton Thalmaier
Unite de Recherche en Mathématiques
Université du Luxembourg
Campus Belval, Maison du Nombre
6, Avenue de la Fonte
4364 Esch-sur-Alzette
LUXEMBOURG

Prof. Dr. Jian Wang
College of Mathematics and Informatics
Fujian Normal University
North 606, Technology Building
Qishan Campus
No. 1 Keji Road, Shangjie, Minhou
Fuzhou 350 117
CHINA

Marvin Weidner
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld
GERMANY

Prof. Dr. Anita Winter
Fakultät für Mathematik
Universität Duisburg-Essen
45117 Essen
GERMANY