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Amorphic Complexity of Group Actions with
Applications to Quasicrystals

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Amorphic complexity of group actions with applications to quasicrystals

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In this article, we define amorphic complexity for actions of locally compact σ-compact amenable groups on compact metric spaces. Amorphic complexity, originally introduced for $\mathbb{Z}$-actions, is a topological invariant which measures the complexity of dynamical systems in the regime of zero entropy. We show that it is tailor-made to study strictly ergodic group actions with discrete spectrum and continuous eigenfunctions. This class of actions includes, in particular, Delone dynamical systems related to regular model sets obtained via Meyer's cut and project method. We provide sharp upper bounds on amorphic complexity of such systems. In doing so, we observe an intimate relationship between amorphic complexity and fractal geometry.

1 Introduction

The study of low-complexity notions for group actions is both a timely and a classical topic. Its roots go back to Halmos, McKay, and von Neumann who classified actions with discrete spectrum, as well as Auslander, Ellis, Furstenberg, and Veech who set the foundations of the theory of equicontinuous actions and their extensions. Recent years have seen plenty of progress in illuminating the richness of possible dynamical behaviour of minimal actions of general groups in the low complexity regime, see for example [Kri07, CP08, CM16, ST17, Gla18, ŁS18, FK20]. As a matter of fact, the investigation of this regime not only contributes to the understanding of group actions as such but is of fundamental importance in the understanding of aperiodic order—with further applications to geometry, number theory and harmonic analysis [Mey72, BG13]—and the diffraction spectra of so-called Delone sets, that is, mathematical models of physical quasicrystals. The latter results from the observation that diffraction spectra of Delone sets can be studied by means of certain associated Delone dynamical systems [LM06, BLM07, Len09], see also [BG13] for further information

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and references. Analysing these Delone dynamical systems, it is most natural to ask when two such systems are conjugate [KS14]. The standard operating procedure to answer this question clearly is to utilize dynamical invariants and one might be tempted to study topological entropy of Delone dynamics. However, the physically most interesting case of pure point diffraction turns out to necessarily come with zero entropy [BLR07]. There is hence a need for finer topological invariants which can distinguish zero entropy systems.

In this article, we propose amorphic complexity—a notion recently introduced for $\mathbb{Z}$-actions [FGJ16]—as a promising candidate for this purpose. To that end, we extend amorphic complexity to actions of locally compact, σ-compact and amenable groups. We will see that amorphic complexity is tailor-made to study strictly ergodic systems with discrete spectrum and continuous eigenfunctions, that is, minimal mean equicontinuous systems [FGL21, Corollary 1.6]. Most importantly, however, we show that amorphic complexity is not only theoretically well-behaved but also well-computable in specific examples. This is particularly true due to a neat connection to fractal geometry. We elaborate on this in the last section of this article where we apply our findings to model sets—particular Delone sets constructed by means of Meyer's cut and project method [Mey72].

Before we introduce amorphic complexity and discuss our main results in more detail, let us briefly clarify some basic terminology. A triple (X, G, α) is called a *(topological) dynamical system* if X is a compact metric space (endowed with a metric d), G is a topological group and α is a continuous action of G on X by homeomorphisms (continuity of α is understood as continuity of the map $G \times X \ni (g, x) \mapsto \alpha(g)(x) \in X$). In the following, we use the shorthand gx instead of $\alpha(g)(x)$ for the action of $g \in G$ on $x \in X$ via α. Likewise, we may occasionally keep the action α implicit and simply refer to (X, G) as a dynamical system.

As mentioned before, we throughout assume that G is locally compact, σ-compact and amenable. Recall that there is hence a *(left) Følner sequence*, that is, a sequence $(F_n)_{n \in \mathbb{N}}$ of compact subsets of G having positive Haar measure such that

$$\lim_{n \to \infty} \frac{m(KF_n \triangle F_n)}{m(F_n)} = 0 \quad \text{for every compact } K \subseteq G, \tag{1}$$

where $\triangle$ denotes the symmetric difference and m is a *(left) Haar measure* of G (we may synonymously write $|F|$ for the Haar measure $m(F)$ of a measurable set $F \subseteq G$) [EG67, Theorem 3.2.1]. We will also make use of the existence of *right Følner sequences* which fulfil a condition analogous to (1) with the left Haar measure and the multiplication from the left replaced by the right Haar measure and multiplication from the right, respectively. However, we would like to stress that in the following, each Følner sequence is assumed to be a left Følner sequence if not stated otherwise. Given a (left or right) Følner sequence $\mathcal{F} = (F_n)$, the *(upper) asymptotic density* of a measurable subset $E \subseteq G$ with respect to $\mathcal{F}$ is defined as

$$\mathrm{ad}_{\mathcal{F}}(E) = \varlimsup_{n \to \infty} \frac{|E \cap F_n|}{|F_n|}. \tag{2}$$

Let us next turn to the definition of amorphic complexity of a dynamical system (X, G) with respect to a Følner sequence $\mathcal{F} = (F_n)_{n \in \mathbb{N}}$ in G. Given $x, y \in X$, $\delta > 0$, we set

$$\Delta(X, G, \delta, x, y) = \{t \in G \mid d(tx, ty) \geqslant \delta\}.$$

For $\nu \in (0, 1]$, we say that x and y are (δ, ν)-*separated* with respect to $\mathcal{F}$ if

$$\mathrm{ad}_{\mathcal{F}}(\Delta(X, G, \delta, x, y)) = \overline{\lim_{n \to \infty}} \frac{|\Delta(X, G, \delta, x, y) \cap F_n|}{|F_n|} \geqslant \nu.$$

Accordingly, a subset $S \subseteq X$ is said to be (δ, ν)-*separated* with respect to $\mathcal{F}$ if all distinct points $x, y \in S$ are (δ, ν)-separated. This already yields the first key notion in this work: the *(asymptotic) separation number of (X, G) with respect to $\delta > 0$ and $\nu \in (0, 1]$*, denoted by $\mathrm{Sep}_{\mathcal{F}}(X, G, \delta, \nu)$, is the supremum over the cardinalities of all (δ, ν)-separated sets in X.

In general, the asymptotic separation numbers do not have to be finite (even though X is compact) which immediately gives the following dichotomy: if $\mathrm{Sep}_{\mathcal{F}}(X, G, \delta, \nu)$ is finite for all $\delta, \nu > 0$, we say (X, G) has *finite separation numbers* with respect to $\mathcal{F}$ otherwise, we say it has *infinite separation numbers*. Our first main result—consisting of the following two theorems whose proofs are given in Section 3—identifies canonical classes of group actions with infinite and finite separation numbers, respectively. First, we give two criteria for infinite separation numbers.

Theorem 1.1. *If (X, G) is weakly mixing with respect to a non-trivial G-invariant probability measure, then (X, G) has infinite separation numbers with respect to every Følner sequence. Likewise, if G allows for a uniform lattice and (X, G) has positive topological entropy, then (X, G) has infinite separation numbers with respect to every Følner sequence.*

In the opposite direction, it turns out that in the minimal case finite separation numbers can be used to characterize mean equicontinuity.

Theorem 1.2. *Let G be a unimodular group, meaning it has a sequence which is a left and a right Følner sequence (this holds, in particular, if G is abelian). Further, suppose (X, G) is a minimal dynamical system. Then (X, G) has finite separation numbers with respect to every Følner sequence if and only if (X, G) is mean equicontinuous.*

It is worth mentioning that the class of mean equicontinuous systems comprises all Delone dynamical systems associated to regular model sets, see also Section 5. For further examples of mean equicontinuous actions of groups different from $\mathbb{Z}$, we refer the reader to the literature [Rob96, Rob99, Cor06, Vor12, GR17, Gla18, ŁS18, FK20, GL20, FGL21].

If (X, G) has finite separation numbers, we are in a position to obtain finer information by studying the scaling behaviour of the separation numbers as the separation frequency ν tends to zero. Here, we may in principle consider arbitrary growth rates. So far, however, previous results indicate that polynomial growth is the most relevant, see [FGJ16, GJ16, FG20] for $G = \mathbb{Z}$. With this in mind, we define the *lower* and *upper amorphic complexity* of (X, G) with respect to $\mathcal{F}$ as

$$\underline{\mathrm{ac}}_{\mathcal{F}}(X, G) = \sup_{\delta > 0} \varliminf_{\nu \to 0} \frac{\log \mathrm{Sep}_{\mathcal{F}}(X, G, \delta, \nu)}{-\log \nu} \quad \text{and} \quad \overline{\mathrm{ac}}_{\mathcal{F}}(X, G) = \sup_{\delta > 0} \varlimsup_{\nu \to 0} \frac{\log \mathrm{Sep}_{\mathcal{F}}(X, G, \delta, \nu)}{-\log \nu}.$$

In case that both values coincide, we call $\mathrm{ac}_{\mathcal{F}}(X, G) = \underline{\mathrm{ac}}_{\mathcal{F}}(X, G) = \overline{\mathrm{ac}}_{\mathcal{F}}(X, G)$ the *amorphic complexity* of (X, G) with respect to $\mathcal{F}$. It is convenient to set $\mathrm{ac}_{\mathcal{F}}(X, G) = \infty$ if (X, G) has infinite separation numbers with respect to $\mathcal{F}$. We discuss the most basic properties of amorphic complexity—including its invariance under conjugacy—in Section 2.

Our second main result deals with the problem as to which extent the asymptotic separation numbers and amorphic complexity depend on the particular Følner sequence $\mathcal{F}$. In general, we cannot rule out different amorphic complexities with respect to different Følner sequences. In fact, this problem already occurs when $G = \mathbb{Z}$, see Section 4. With the next theorem, however, we provide a sufficient criterion for the independence from $\mathcal{F}$. Here, we say a dynamical system (X, G) is *pointwise uniquely ergodic* if every orbit closure is uniquely ergodic. A strengthening of the following statement and its proof can be found in Section 4.

Theorem 1.3. *Let (X, G) be a dynamical system whose product (X^2, G) is pointwise uniquely ergodic. Then (X, G) has infinite separation numbers with respect to some Følner sequence if and only if it has infinite separation numbers with respect to all Følner sequences. Moreover, $\overline{\mathrm{ac}}_{\mathcal{F}}(X, G)$ and $\underline{\mathrm{ac}}_{\mathcal{F}}(X, G)$ are independent of the particular Følner sequence $\mathcal{F}$.*

It is worth mentioning that mean equicontinuous systems verify the assumptions of the above theorem [FGL21, Theorem 1.2].

With our third main result, we apply amorphic complexity to the dynamics of regular model sets. Before we come to the precise formulation, we need to introduce some terminology. In doing so, we restrict to a rather brief description of the most essential notions and refer the reader to Section 5 for the details. A *cut and project scheme* is a triple $(G, H, \mathcal{L})$, where G and H are locally compact abelian groups and $\mathcal{L}$ is an irrational lattice in $G \times H$. Together with a compact subset $W = \overline{\mathrm{int}(W)} \subseteq H$—referred to as a *window*—$(G, H, \mathcal{L})$ defines a particular instance of a Delone set, a so-called *model* set

$$\curlywedge(W) = \pi_G((G \times W) \cap \mathcal{L}),$$

where $\pi_G : G \times H \to G$ denotes the canonical projection. We call W as well as $\curlywedge(W)$ *regular* if ∂W is of zero Haar measure and say W is *irredundant* if $\{h \in H \mid h + W = W\} = \{0\}$. Now, as $\curlywedge(W)$ is a subset of G, G naturally acts on $\curlywedge(W)$ by translations. It turns out that the closure of all translated copies of $\curlywedge(W)$ is compact (in a suitable topology on subsets of G). Denoting this closure by $\Omega(\curlywedge(W))$, we arrive at the Delone dynamical system $(\Omega(\curlywedge(W)), G)$ associated to the model set $\curlywedge(W)$. We obtain

Theorem 1.4. *Let $(G, H, \mathcal{L})$ be a cut and project scheme with $W \subseteq H$ a regular irredundant window and suppose G and H are second countable. Then for every Følner sequence $\mathcal{F}$ in G, we get*

$$\overline{\mathrm{ac}}_{\mathcal{F}}(\Omega(\curlywedge(W)), G) \leqslant \frac{\overline{\dim}_{\mathrm{B}}(H)}{\overline{\dim}_{\mathrm{B}}(H) - \overline{\dim}_{\mathrm{B}}(\partial W)},$$

assuming that $\overline{\dim}_{\mathrm{B}}(H)$ is finite.

Here, $\overline{\dim}_{\mathrm{B}}(\cdot)$ denotes the upper box dimension, see Section 5 for the details. Let us remark that we further show that the above estimates are sharp in that they are realised by particular model sets. In conclusion, we obtain that every value in $[1, \infty)$ can be attained by amorphic complexity of minimal systems.

Motivated by the above results, we finish with the following question.

Given a locally compact, σ-compact and amenable group acting minimally on a compact metric space. Which values can amorphic complexity attain?

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In particular, for minimal $\mathbb{Z}$- or $\mathbb{R}$-actions, we conjecture that amorphic complexity cannot take values in $(0,1)$. Indeed, this complexity gap was recently established for subshifts associated to primitive constant length substitutions [FG20] and is a classical phenomenon which is well known to occur for polynomial entropy of minimal symbolic subshifts. For non-minimal $\mathbb{Z}$-actions, however, it was recently shown that all values in $(0,1)$ can be obtained by amorphic complexity, see [Kul20, Kul].

2 Basic properties of amorphic complexity

In this section, we collect the most basic properties of amorphic complexity. In particular, given a group G which allows for a lattice $\mathcal{L}$, we discuss how amorphic complexity of a G-action relates to amorphic complexity of the associated $\mathcal{L}$-action.

The proof of the following statement is verbatim as the proofs of [FGJ16, Proposition 3.4 & Proposition 3.9].

Proposition 2.1. *Let (X,G) and (Y,G) be dynamical systems. We have:*

(a) If (Y,G) is a factor of (X,G), then

$$\underline{\mathrm{ac}}_{\mathcal{F}}(Y,G) \leqslant \underline{\mathrm{ac}}_{\mathcal{F}}(X,G) \quad \text{and} \quad \overline{\mathrm{ac}}_{\mathcal{F}}(Y,G) \leqslant \overline{\mathrm{ac}}_{\mathcal{F}}(X,G).$$

In particular, (upper and lower) amorphic complexity is a topological invariant.

(b) We have that

$$\underline{\mathrm{ac}}_{\mathcal{F}}(X \times Y,G) \geqslant \underline{\mathrm{ac}}_{\mathcal{F}}(X,G) + \underline{\mathrm{ac}}_{\mathcal{F}}(Y,G), \quad \overline{\mathrm{ac}}_{\mathcal{F}}(X \times Y,G) \leqslant \overline{\mathrm{ac}}_{\mathcal{F}}(X,G) + \overline{\mathrm{ac}}_{\mathcal{F}}(X,G).$$

In particular, if $\mathrm{ac}_{\mathcal{F}}(X,G)$ and $\mathrm{ac}_{\mathcal{F}}(Y,G)$ exist, then $\mathrm{ac}_{\mathcal{F}}(X \times Y,G)$ exists as well.

Before we proceed with further properties of amorphic complexity, we take a closer look at certain particularly well-behaved Følner sequences. Recall that a *van Hove sequence* $(A_n)_{n\in\mathbb{N}}$ in G is a sequence of compacta $A_n \subseteq G$ of positive Haar measure such that

$$\lim_{n\to\infty} \frac{m(\partial_K A_n)}{m(A_n)} = 0,$$

for every compact set $K \subseteq G$ with $e \in K$, where $\partial_K A_n := KA_n \backslash \operatorname{int}\left(\bigcap_{g \in K} gA_n\right)$ (see [Tem92, Appendix 3] and [Str05] for further reference). It is not hard to see that every van Hove sequence is a Følner sequence. In fact, it holds

Proposition 2.2 ([Tem92, Appendix 3.K]). *Let G be a locally compact σ-compact amenable topological group. A sequence (A_n) of compact subsets of G is a van Hove sequence if and only if it is a Følner sequence and*

$$\lim_{n \to \infty} \frac{m(\partial_U A_n)}{m(A_n)} = 0, \tag{3}$$

for some open neighbourhood U of the neutral element e in G.

Remark 2.3. In particular, if G is discrete, then every Følner sequence in G is, in fact, a van Hove sequence.

It is well known that every locally compact σ-compact amenable group allows for a van Hove sequence. For the convenience of the reader, we prove the following (possibly well-known) refinement of this statement which we need in the sequel.

Proposition 2.4. *Let G be a locally compact σ-compact amenable topological group. Suppose (F_n) is a Følner sequence in G and B is a compact neighbourhood of e. Then $A_n := BF_n$ defines a van Hove sequence in G with $\operatorname{ad}_{(A_n)}(E) = \operatorname{ad}_{(F_n)}(E)$ for every measurable $E \subseteq G$.*

Proof. The last part follows from $E \cap A_n \subseteq (E \cap F_n) \cup (F_n \triangle A_n)$ and

$$0 \leqslant \lim_{n \to \infty} m(A_n \triangle F_n)/m(A_n) \leqslant \lim_{n \to \infty} m(BF_n \triangle F_n)/m(F_n) = 0, \tag{4}$$

which is a consequence of the fact that (F_n) is a Følner sequence and $F_n \subseteq BF_n = A_n$.

For the first part, we make use of Proposition 2.2. To that end, observe that for every (compact) $K \subseteq G$ we have $KA_n \triangle A_n \subseteq (KA_n \triangle F_n) \cup (F_n \triangle A_n) = (KBF_n \triangle F_n) \cup (F_n \triangle A_n)$. Due to (4) and the fact that (F_n) is a Følner sequence, this gives that (A_n) is a Følner sequence, too. To see (3), we need the following

Claim 2.5. *There is a relatively compact open neighbourhood U of e such that $F_n \subseteq \operatorname{int}\left(\bigcap_{g \in U} gA_n\right)$ for each $n \in \mathbb{N}$.*

Proof of the claim. First, observe that $\operatorname{int}\left(\bigcap_{g \in U} gBF_n\right) \supseteq \operatorname{int}\left(\bigcap_{g \in U} gB\right)F_n$. To prove the claim, it hence suffices to show that there is U with $e \in \operatorname{int}\left(\bigcap_{g \in U} gB\right)$.

For a contradiction, suppose $e \in \overline{\bigcup_{g \in U} gB^c}$ for every U in the open neighbourhood filter $\mathcal{U}$ of e. In other words, suppose there is a net $(g_U)_{U \in \mathcal{U}}$ with $g_U \in U$ (so that $g_U \to e$) and a net $(h_U)_{U \in \mathcal{U}}$ in B^c such that $g_U h_U \to e$. This, however, implies $h_U \to e$ which contradicts $e \in \operatorname{int}(B)$. Therefore, there is $U \in \mathcal{U}$ with $e \in \operatorname{int}\left(\bigcap_{g \in U} gB\right)$. Clearly, U can be chosen open and relatively compact. ○

Now, pick some U as in the above claim. As (F_n) is a Følner sequence, we have

$$m(\partial_U A_n)/m(A_n) \leqslant m(UA_n \backslash F_n)/m(F_n) \leqslant m(\overline{U}BF_n \backslash F_n)/m(F_n) \xrightarrow{n \to \infty} 0.$$

Finally, it follows from Proposition 2.2 that (A_n) is a van Hove sequence. □

For the next statement, recall that a *uniform lattice* $\mathcal{L}$ in G is a discrete subgroup of G such that there exists a measurable precompact subset $C \subseteq G$, referred to as *fundamental domain*, with $G = \bigsqcup_{\lambda \in \mathcal{L}} C\lambda$ and $m(C) > 0$. With the lattice $\mathcal{L}$ being a subgroup of G, we have a naturally defined dynamical system $(X, \mathcal{L})$ and it turns out that amorphic complexity is well behaved when going from (X, G) over to $(X, \mathcal{L})$.

Lemma 2.6. *Assume (X, G) is a dynamical system and G allows for a uniform lattice $\mathcal{L}$. Then for every Følner sequence $\mathcal{F}$ in G there is a Følner sequence $\mathcal{F}'$ in $\mathcal{L}$ such that*

$$\underline{\mathrm{ac}}_{\mathcal{F}}(X, G) = \underline{\mathrm{ac}}_{\mathcal{F}'}(X, \mathcal{L}) \qquad \text{and} \qquad \overline{\mathrm{ac}}_{\mathcal{F}}(X, G) = \overline{\mathrm{ac}}_{\mathcal{F}'}(X, \mathcal{L}).$$

Furthermore, (X, G) has infinite separation numbers with respect to $\mathcal{F}$ if and only if $(X, \mathcal{L})$ has infinite separation numbers with respect to $\mathcal{F}'$.

Proof. We denote the Haar measure on G by m and that on $\mathcal{L}$ by $|\cdot|$. Let $C \subseteq G$ be a fundamental domain as in the above definition of a uniform lattice. First, observe that for all $\delta > 0$ there are $\delta_\delta^-, \delta_\delta^+ > 0$ such that for all $x, y \in X$ and $c \in C$ we have $d(c^{-1}x, c^{-1}y) \geqslant \delta_\delta^-$ whenever $d(x, y) \geqslant \delta$ and $d(cx, cy) \geqslant \delta_\delta^+$ whenever $d(x, y) \geqslant \delta_\delta^-$. This straightforwardly follows from the precompactness of C.

Further, due to Proposition 2.4, we may assume without loss of generality that $\mathcal{F}$ is a van Hove sequence. Under this assumption, there are van Hove sequences $\mathcal{F}' = (F_n')$ and $\mathcal{F}'' = (F_n'')$ in $\mathcal{L}$ with $\lim_{n \to \infty} |F_n'|/|F_n''| = 1$ such that CF_n' and CF_n'' are von Hove sequences in G and $CF_n' \subseteq F_n \subseteq CF_n''$, see for example [Hau20, Lemma 3.2]. We will show that for all $x, y \in X$ and $\delta > 0$ we have

$$\mathrm{ad}_{\mathcal{F}}(\Delta(X, G, \delta, x, y)) \leqslant \mathrm{ad}_{\mathcal{F}'}(\Delta(X, \mathcal{L}, \delta_\delta^-, x, y)) \leqslant \mathrm{ad}_{\mathcal{F}}(\Delta(X, G, \delta_\delta^+, x, y)). \qquad (5)$$

Clearly, this implies that for all $\nu \in (0, 1)$ and all $\delta > 0$

$$\mathrm{Sep}_{\mathcal{F}}(X, G, \delta, \nu) \leqslant \mathrm{Sep}_{\mathcal{F}'}(X, \mathcal{L}, \delta_\delta^-, \nu) \leqslant \mathrm{Sep}_{\mathcal{F}}(X, G, \delta_\delta^+, \nu)$$

and thus proves the statement.

By definition of δ_δ^- and δ_δ^+ and since C is a fundamental domain, we have

$$\Delta(X, G, \delta, x, y) \subseteq C\Delta(X, \mathcal{L}, \delta_\delta^-, x, y) \subseteq \Delta(X, G, \delta_\delta^+, x, y).$$

Hence, utilizing the fact that for any subset $F \subseteq \mathcal{L}$ we have $m(CF) = |F| \cdot m(C)$, we obtain (5) from the following computation

$$\begin{aligned}
\mathrm{ad}_{\mathcal{F}}(\Delta(X, G, \delta, x, y)) &= \varlimsup_{n \to \infty} m(\Delta(X, G, \delta, x, y) \cap F_n)/m(F_n) \\
&\leqslant \varlimsup_{n \to \infty} m(C\Delta(X, \mathcal{L}, \delta_\delta^-, x, y) \cap CF_n'')/m(CF_n') \\
&= \varlimsup_{n \to \infty} m(C\Delta(X, \mathcal{L}, \delta_\delta^-, x, y) \cap CF_n'')/m(CF_n'') \cdot |F_n''|/|F_n'| \\
&= \mathrm{ad}_{\mathcal{F}''}(\Delta(X, \mathcal{L}, \delta_\delta^-, x, y)) = \mathrm{ad}_{\mathcal{F}'}(\Delta(X, \mathcal{L}, \delta_\delta^-, x, y)) \\
&= \varlimsup_{n \to \infty} m(C\Delta(X, \mathcal{L}, \delta_\delta^-, x, y) \cap CF_n')/m(CF_n'') \\
&\leqslant \varlimsup_{n \to \infty} m(\Delta(X, G, \delta_\delta^+, x, y) \cap F_n)/m(F_n) \\
&= \mathrm{ad}_{\mathcal{F}}(\Delta(X, G, \delta_\delta^+, x, y)). \qquad \square
\end{aligned}$$

Remark 2.7.

(a) If (F_n) is a van Hove sequence, then the sets F'_n and F''_n in the above proof are explicitly given by $F'_n = \{h \in \mathcal{L} \mid Ch \subseteq F_n\}$ and $F''_n = \{h \in \mathcal{L} \mid Ch \cap F_n \neq \varnothing\}$, see the proof of [Hau20, Lemma 3.2].

(b) Let us briefly comment on the necessity of the passage through Proposition 2.4 in the above proof. As mentioned in Remark 2.3, a Følner sequence in a discrete group is necessarily a van Hove sequence. Consequently, given a Følner sequence (F'_n) in the lattice $\mathcal{L}$ of G, (F'_n) is actually a van Hove sequence and therefore, one can show that (CF'_n) defines a van Hove sequence in G. Accordingly, whenever we seek to bound a Følner sequence (F_n) in G from below and above by sequences (CF'_n) and (CF''_n) similarly as in the previous proof, we actually bound (F_n) by van Hove sequences. It turns out that this implies that (F_n) itself must be van Hove. These observations are straightforward (though slightly tedious) to check.

3 On finiteness of separation numbers

This section deals with the scope of amorphic complexity. In particular, we identify mean equicontinuous systems as those systems where separation numbers are finite with respect to every Følner sequence and amorphic complexity may hence be finite itself. Moreover, we show that positive entropy as well as weak mixing imply infinite separation numbers.

3.1 Mean equicontinuity and finite separation numbers

We next discuss a natural class of dynamical systems with finite separation numbers: the class of mean equicontinuous systems, see [Aus59, Rob96, HJ97, Rob99, Cor06, Vor12, DG16, Gla18, ŁS18, FG20, FK20, GL20, FGL21] for numerous examples. In our discussion of mean equicontinuity, we follow the terminology of [FGL21]. Given a left or right Følner sequence $\mathcal{F}$, a system (X, G) is *(Besicovitch) $\mathcal{F}$-mean equicontinuous* if for all $\varepsilon > 0$ there is $\delta > 0$ such that for all $x, y \in X$ with $d(x, y) < \delta$ we have

$$D_{\mathcal{F}}(x, y) := \varlimsup_{n \to \infty} 1/m(F_n) \int\limits_{F_n} d(tx, ty) \, dm(t) < \varepsilon.$$

In this case, $D_{\mathcal{F}}$ clearly defines a continuous pseudometric on X. Thus, by identifying points $x, y \in X$ with $D_{\mathcal{F}}(x, y) = 0$, we obtain a compact metric space which we denote by $X/D_{\mathcal{F}}$.

Before we proceed, let us briefly recall the concept of the (upper) box dimension of a compact metric space (M, d). Given $\varepsilon > 0$, we call a subset S of M ε-*separated* if for all $s \neq s' \in S$ we have $d(s, s') \geq \varepsilon$ and denote by M_ε the maximal cardinality of an ε-separated subset of M. It is well known and easy to see that $M_\varepsilon < \infty$ due to compactness. With this notation, the *upper box dimension* of M is defined as

$$\overline{\dim}_B(M) = \varlimsup_{\varepsilon \to 0} \frac{\log M_\varepsilon}{-\log \varepsilon}.$$

Now, for $\mathcal{F}$-mean equicontinuous (X, G), we have

$$D_{\mathcal{F}}(x,y) \geqslant \varlimsup_{n \to \infty} 1/m(F_n) \int_{F_n} \mathbf{1}_{[\delta, \infty)}(d(tx, ty)) \cdot d(tx, ty) \, dm(t) \geqslant \delta \cdot \mathrm{ad}_{\mathcal{F}}(\Delta(X, G, \delta, x, y))$$

for all $\delta > 0$ and $x, y \in X$ and hence, $(X/D_{\mathcal{F}})_{\delta\nu} \geqslant \mathrm{Sep}_{\mathcal{F}}(X, G, \delta, \nu)$. It follows

Proposition 3.1. *If (X, G) is $\mathcal{F}$-mean equicontinuous for some left or right Følner sequence $\mathcal{F}$, then it has finite separation numbers with respect to $\mathcal{F}$ and*

$$\overline{\mathrm{ac}}_{\mathcal{F}}(X, G) \leqslant \overline{\dim}_B(X/D_{\mathcal{F}}).$$

It is important to note that if $\mathcal{F}$ is a left Følner sequence, then $D_{\mathcal{F}}$ is not necessarily invariant. In particular, the equivalence relation defined by $D_{\mathcal{F}}$ may not define a factor of (X, G) even if $D_{\mathcal{F}}$ is continuous. However, it is easy to see that $D_{\mathcal{F}}$ is invariant if $\mathcal{F}$ is a right Følner sequence. We utilize this observation below.

In any case, it is certainly desirable to have an invariant pseudometric which does not depend on a particular (right) Følner sequence. To that end, we may consider

$$D(x, y) := \sup\{D_{\mathcal{F}}(x, y) \mid \mathcal{F} \text{ is a left Følner sequence}\}$$

which is, in fact, invariant (see [FGL21, Proposition 3.12]). We say (X, G) is *(Weyl) mean equicontinuous* if D is continuous.

Proposition 3.2 ([FGL21, Proposition 5.8]). *Suppose (X, G) is $\mathcal{F}$-mean equicontinuous for some right Følner sequence $\mathcal{F}$. Then (X, G) is mean equicontinuous.*

Given a left or right Følner sequence $\mathcal{F}$, a system (X, G) is called *$\mathcal{F}$-mean sensitive* if there exists $\eta > 0$ such that for every open set $U \subseteq X$ we can find $x, y \in U$ with $D_{\mathcal{F}}(x, y) \geqslant \eta$. Moreover, we say (X, G) is *(Weyl) mean sensitive* if there exists $\eta > 0$ such that for every open set $U \subseteq X$ we can find $x, y \in U$ with $D(x, y) \geqslant \eta$. We have the following direct generalisation of the equivalence of (1) and (3) in [LTY15, Proposition 5.1] whose proof extends almost literally to the current setting.

Proposition 3.3. *The system (X, G) is $\mathcal{F}$-mean sensitive (with respect to a left or right Følner sequence $\mathcal{F}$) if and only if there is $\eta > 0$ such that for every $x \in X$ we have that $\{y \in X \mid D_{\mathcal{F}}(x, y) \geqslant \eta\}$ is residual in X.*

Clearly, if $\mathrm{ad}_{\mathcal{F}}(\Delta(X, G, \eta/2, x, y)) < \eta/2$, then $D_{\mathcal{F}}(x, y) \leqslant \eta/2 + (1 - \eta/2) \cdot \eta/2 < \eta$ (assuming, without loss of generality, that the maximal distance of points in X is 1).

Corollary 3.4. *If a dynamical system (X, G) is $\mathcal{F}$-mean sensitive (for a left or right Følner sequence $\mathcal{F}$), then it has infinite separation numbers with respect to $\mathcal{F}$.*

In the following, we take a closer look at the relation between mean equicontinuity and mean sensitivity in the minimal case. The proof of the next statement is similar to the one for $\mathbb{Z}$-actions [LTY15, Proposition 4.3 & Theorem 5.4–5.5], see also [ZHL19, Corollary 5.6] for a similar statement for countable amenable groups. For the convenience of the reader, we provide a direct proof in the current setting.

Lemma 3.5. *Let (X, G) be minimal. Then (X, G) is either mean equicontinuous or mean sensitive. Furthermore, if (X, G) is mean sensitive, then it is $\mathcal{F}$-mean sensitive for every right Følner sequence $\mathcal{F}$.*

Proof. Suppose (X, G) is not mean equicontinuous. That is, there is $x \in X$ and $\eta > 0$ such that for all $\delta > 0$ there is $y_\delta \in B_\delta(x)$ with $D(x, y_\delta) > \eta$. Now, given any open set $U \subseteq X$, there is $g \in G$ and $\delta_0 > 0$ such that $gB_{\delta_0}(x) \subseteq U$. Since D is invariant, we have $D(gx, gy_{\delta_0}) = D(x, y_{\delta_0}) > \eta$ which proves the first part.

For the second part, observe that Proposition 3.2 gives that for every right Følner sequence $\mathcal{F}$ there exist $x \in X$ and $\eta > 0$ such that for all $\delta > 0$ there is $y \in B_\delta(x)$ with $D_\mathcal{F}(x, y) > \eta$. Since $\mathcal{F}$ is assumed to be a right Følner sequence, $D_\mathcal{F}$ is invariant and we can argue similarly as for D to obtain $\mathcal{F}$-mean sensitivity. $\qquad\square$

Remark 3.6. Recall that G acts *effectively* on X if for all $g \in G$ there is $x \in X$ such that $gx \neq x$. According to [FGL21, Corollary 7.3], G is necessarily maximally almost periodic (see [FGL21] and references therein) if G allows for a minimal, mean equicontinuous and effective action on a compact metric space X. Hence, Lemma 3.5 gives that every minimal effective action by a group which is not maximally almost periodic (such as the *continuous Heisenberg group* $H_3(\mathbb{R})$) is mean sensitive.

Recall that a locally compact σ-compact amenable group G is *unimodular* if and only if G allows for a *two-sided Følner sequence*, that is, a sequence $\mathcal{F}$ which is both a left and a right Følner sequence. In conclusion to the above statements, we obtain

Corollary 3.7. *Suppose G is unimodular and (X, G) is minimal. Then (X, G) is mean equicontinuous if and only if the separation numbers of (X, G) are finite with respect to every left Følner sequence.*

Proof. By definition, mean equicontinuity implies $\mathcal{F}$-mean equicontinuity with respect to every left Følner sequence. Hence, the "only if"-part follows from Proposition 3.1.

For the other direction, let $\mathcal{F}$ be a two-sided Følner sequence. Since we assume the separation numbers with respect to $\mathcal{F}$ to be finite, we have that (X, G) is not $\mathcal{F}$-mean sensitive. Since $D_\mathcal{F}$ is invariant, we can argue similarly as in Lemma 3.5 to obtain that (X, G) is $\mathcal{F}$-mean equicontinuous. Utilizing Proposition 3.2, we obtain the desired statement. $\quad\square$

3.2 Entropy, mixing and infinite separation numbers

In this section, we discuss how chaotic behaviour—more specifically: weak mixing or positive entropy—implies infinite separation numbers. Here, we occasionally have to assume that a Følner sequence we consider is *tempered*, that is, there is $C > 0$ such that for all n we have $m(\bigcup_{k<n} F_k^{-1} F_n) < C \cdot m(F_n)$. It is well known that every Følner sequence allows for a tempered subsequence, see [Lin01, Proposition 1.4].

In line with [GW16], we call an invariant measure μ of (X, G) *weakly mixing* if for every system (Y, G) and all of its ergodic measures ν we have that $\mu \times \nu$ is ergodic for $(X \times Y, G)$. Hence, if μ is weakly mixing, $\mu^m = \times_{k=1}^m \mu$ is ergodic for (X^m, G) and all $m \in \mathbb{N}$.

Theorem 3.8. *Let (X, G) be a dynamical system with a weakly mixing measure μ and suppose the support of μ is not a singleton. Then (X, G) has infinite separation numbers with respect to every Følner sequence.*

Proof. For a tempered Følner sequence, the proof is similar to that of the respective statement for $\mathbb{Z}$-actions ([FGJ16, Theorem 2.2]) if we replace Birkhoff's Ergodic Theorem by Lindenstrauss' Pointwise Ergodic Theorem [Lin01, Theorem 1.2]. Here, we have to make use of the ergodicity of μ^m just as in [FGJ16].

Now, given an arbitrary Følner sequence, we can always go over to a tempered subsequence (see [Lin01, Proposition 1.4]). This gives infinite separation numbers for a subsequence and hence, due to the lim sup in (2), infinite separation numbers for the original sequence. $\qquad \square$

We next turn to systems with positive topological entropy. Our goal is to show

Theorem 3.9. *Suppose G allows for a uniform lattice and the dynamical system (X, G) has positive topological entropy. Then (X, G) has infinite separation numbers with respect to every Følner sequence in G.*

Remark 3.10. Observe that the proof of a similar statement for $\mathbb{Z}$-actions (see [FGJ16, Theorem 2.3]) utilised results that are only available for $G = \mathbb{Z}$. The present approach provides an alternative to the somewhat implicit argument in [FGJ16].

Remark 3.11. We do not make explicit use of the actual definition of entropy in the following and rather utilize results from the theory of topological independence. Therefore, we refrain from discussing the basics of entropy theory in the present work. Interested readers are referred to e.g. [OW87, KL16, Bow20, Hau20] for a background and further references.

In order to prove Theorem 3.9, we first restrict to actions of countable discrete (and, as throughout assumed, amenable) groups.

Definition 3.12 (cf. [KL16, Definition 8.7]). Let (X, G) be a dynamical system and suppose G is countable and discrete. Given a pair $\mathbf{A} = (A_0, A_1)$ of subsets of X, we say that a set $J \subseteq G$ is an *independence set* for $\mathbf{A}$ if for every non-empty finite subset $I \subseteq J$ and every $(s_g)_{g \in I} \in \{0, 1\}^I$ there exists $x \in X$ with $gx \in A_{s_g}$ for every $g \in I$.

Theorem 3.13 ([KL16, Theorem 12.19 & Proposition 12.7]). *Suppose G is discrete and countable and (X, G) is a dynamical system. If (X, G) has positive topological entropy, then there is a pair $\mathbf{A} = (A_0, A_1)$ of disjoint compact subsets of X and $d > 0$ such that for every tempered Følner sequence $\mathcal{F} = (F_n)$ in G there is an independence set J of $\mathbf{A}$ with $\mathrm{ad}_{\mathcal{F}}(J) = \lim_{n \to \infty} |F_n \cap J| / |F_n| \geqslant d > 0$.*

Let $\mathbf{A}$, $\mathcal{F}$ and $J \subseteq G$ be as in the above statement. Observe that due to the compactness of A_0 and A_1 we actually have that for every $s = (s_j)_{j \in J} \in \{0, 1\}^J$ there exists $x \in X$ which *follows* s, that is, $jx \in A_{s_j}$ for every $j \in J$.

Lemma 3.14. *Let G be a countable discrete group and suppose (X, G) has positive topological entropy. Then (X, G) has infinite separation numbers with respect to every Følner sequence in G. In fact, there are $\delta > 0$ and $\nu \in (0, 1]$ such that for every Følner sequence there is an uncountable (δ, ν)-separated set.*

Proof. Let $\mathbf{A} = (A_0, A_1)$ and $d > 0$ be as in Theorem 3.13. Given a Følner sequence $\mathcal{F}$, we may assume without loss of generality that $\mathcal{F}$ is tempered. By Theorem 3.13, we have

an associated independence set $J \subseteq G$ for $\mathbf{A}$ with $\mathrm{ad}_{\mathcal{F}}(J) \geqslant d$. Set $\delta = \mathrm{dist}(A_0, A_1)$ and $\nu = d/2 \leqslant \mathrm{ad}_{\mathcal{F}}(J)/2$. Our goal is to show that there is an infinite subset $S \subseteq \{0,1\}^J$ such that whenever $x, y \in X$ follow distinct elements in S, then $\mathrm{ad}_{\mathcal{F}}(\Delta(X, G, \delta, x, y)) \geqslant \nu$.

To that end, we first define a sequence $(G_n)_{n \in \mathbb{N}}$ of pairwise disjoint non-empty finite subsets of G such that for every infinite set $M \subseteq \mathbb{N}$ we have

$$\mathrm{ad}_{\mathcal{F}}(\bigcup_{n \in M} G_n) \geqslant 1 - \nu. \tag{6}$$

We may do so by starting with $G_1 = F_1$. Assuming we have already chosen $G_1, \ldots, G_n$ for some $n \in \mathbb{N}$, let $N = N(n) \in \mathbb{N}$ be large enough to guarantee that

$$|F_N \backslash (G_1 \cup \ldots \cup G_n)| \geqslant (1 - \nu)|F_N|$$

and set $G_{n+1} = F_N \backslash (G_1 \cup \ldots \cup G_n)$. Note that this gives that (G_n) satisfies (6) for every infinite $M \subseteq \mathbb{N}$ because

$$\mathrm{ad}_{\mathcal{F}}(\bigcup_{n \in M} G_n) \geqslant \varlimsup_{\substack{n \to \infty \\ n \in M}} \frac{|F_{N(n-1)} \cap G_n|}{|F_{N(n-1)}|} \geqslant 1 - \nu,$$

for any infinite $M \subseteq \mathbb{N}$.

Now, let E be an uncountable family of subsets of $\mathbb{N}$ such that $M \triangle M'$ is infinite for distinct $M, M' \in E$. Given $M \in E$, we define $s^M \in \{0,1\}^J$ by

$$s_j^M = \begin{cases} 1 & \text{if } j \in G_n \text{ and } n \in M, \\ 0 & \text{otherwise.} \end{cases}$$

Set $S = \{s^M \in \{0,1\}^J \mid M \in E\}$. Given $s \in S$, choose some $x(s) \in X$ which follows s (recall the discussion before the statement). It is straightforward to see that for distinct $M, M' \in E$, we have for $x = x(s^M)$ and $x' = x(s^{M'})$ that

$$\Delta(X, G, \delta, x, x') = \{g \in G \mid d(gx, gx') \geqslant \delta\} \supseteq \{g \in J \mid s_g^M \neq s_g^{M'}\}$$
$$= J \cap \big(\bigcup_{n \in M \triangle M'} G_n \big).$$

Using (6), we obtain

$$\mathrm{ad}_{\mathcal{F}}\big(J \cap \bigcup_{n \in M \triangle M'} G_n\big) \geqslant \mathrm{ad}_{\mathcal{F}}(J)/2 \geqslant \nu.$$

Hence, $\{x(s) \in X \mid s \in S\}$ is the uncountable (δ, ν)-separated set we sought. $\square$

Proof of Theorem 3.9. Let us denote by $\mathcal{L}$ a lattice (as provided by the assumptions) in G. Note that since G is σ-compact, we have that $\mathcal{L}$ is countable.

Due to [Hau20, Theorem 5.2], positive topological entropy of (X, G) implies positive topological entropy of $(X, \mathcal{L})$. Hence, Lemma 3.14 gives that $(X, \mathcal{L})$ has infinite separation numbers with respect to every Følner sequence. Due to Lemma 2.6, this implies infinite separation numbers of (X, G) with respect to every Følner sequence. $\square$

4 Independence of Følner sequences

In general, amorphic complexity might depend on the particular Følner sequence with respect to which we compute the separation numbers. For $G = \mathbb{Z}$, this can be seen by considering the example in [FGJ16, page 541]. There, $\mathrm{ac}_{\mathcal{F}}(X, \mathbb{Z}) = \infty$ for $\mathcal{F} = ([0, n))_{n \in \mathbb{N}}$ while $\mathrm{ac}_{\mathcal{F}'}(X, \mathbb{Z}) = 0$ for $\mathcal{F}' = ((-n, 0])_{n \in \mathbb{N}}$.

The goal of this section is to show

Theorem 4.1. *Let (X, G) be a dynamical system whose product (X^2, G) is pointwise uniquely ergodic. Then $\overline{\mathrm{ac}}_{\mathcal{F}}(X, G)$ and $\underline{\mathrm{ac}}_{\mathcal{F}}(X, G)$ are independent of the particular (left) Følner sequence $\mathcal{F}$.*

Remark 4.2. Notice that due to [FGL21, Theorem 1.2], the above gives that amorphic complexity of mean equicontinuous systems is independent of the particular Følner sequence.

In fact, we have the following stronger statement which immediately yields Theorem 4.1.

Theorem 4.3. *Let (X, G) be a dynamical system whose product (X^2, G) is pointwise uniquely ergodic. The following holds.*

(i) *Suppose there is a Følner sequence $\mathcal{F}$ such that $\mathrm{Sep}_{\mathcal{F}}(X, G, \delta, \nu) = \infty$ for some $\delta, \nu \in (0, 1)$. Then there exists $\delta_0 > 0$ such that $\mathrm{Sep}_{\mathcal{F}'}(X, G, \delta', \nu) = \infty$ for every Følner sequence $\mathcal{F}'$ and every $\delta' \in (0, \delta_0]$.*

(ii) *Let $\mathcal{F}^0$ and $\mathcal{F}^1$ be Følner sequences and suppose $\mathrm{Sep}_{\mathcal{F}^0}(X, G, \delta, \nu) < \infty$ for all $\nu, \delta \in (0, 1)$. Then there is a cocountable set $A \in (0, 1)$ such that for all $\delta \in A$ we have $\mathrm{Sep}_{\mathcal{F}^0}(X, G, \delta, \nu) = \mathrm{Sep}_{\mathcal{F}^1}(X, G, \delta, \nu)$ for all but countably many ν.*

Proof. Without loss of generality, we may assume that $\mathrm{diam}(X) = 1$. We start by providing some general observations. Given $\delta \in (0, 1)$, let (h_ℓ) and (H_ℓ) be sequences of non-decreasing continuous self-maps on $[0, 1]$. For large enough $\ell \in \mathbb{N}$, assume that $h_\ell(z) = 0$ for $z \in [0, \delta]$ and $h_\ell(z) = 1$ for $z \in [\delta + 1/\ell, 1]$ as well as $H_\ell = 0$ on $[0, \delta - 1/\ell]$ and $H_\ell = 1$ on $[\delta, 1]$. Clearly, $h_\ell(z) \leqslant \mathbf{1}_{[\delta, 1]}(z) \leqslant H_\ell(z)$ for all $z \in [0, 1]$ and large enough $\ell \in \mathbb{N}$. Hence, for all $x, y \in X$, every Følner sequence $\mathcal{F} = (F_n)$, and sufficiently large ℓ, we have

$$
\begin{aligned}
\int_{X^2} h_\ell(d(v, w)) d\mu_{(x,y)}(v, w) &= \lim_{n \to \infty} 1/|F_n| \cdot \int_{F_n} h_\ell(d(sx, sy)) dm(s) \\
&\leqslant \lim_{n \to \infty} 1/|F_n| \cdot \int_{F_n} \mathbf{1}_{[\delta, 1]}(d(sx, sy)) dm(s) = \mathrm{ad}_{\mathcal{F}}(\Delta(X, G, \delta, x, y)) \\
&\leqslant \lim_{n \to \infty} 1/|F_n| \cdot \int_{F_n} H_\ell(d(sx, sy)) dm(s) = \int_{X^2} H_\ell(d(v, w)) d\mu_{(x,y)}(v, w),
\end{aligned}
\tag{7}
$$

where we used the pointwise unique ergodicity of (X^2, G) and where $\mu_{(x,y)}$ denotes the unique invariant measure on the orbit closure of $(x, y) \in X^2$. Sending $\ell \to \infty$, we obtain equality in (7) unless

$$
\mu_{(x,y)}(\{(v, w) \in X^2 \mid d(v, w) = \delta\}) > 0.
\tag{8}
$$

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In other words, if (8) does not hold, then $\mathrm{ad}_{\mathcal{F}}(\Delta(X, G, \delta, x, y))$ is actually independent of the Følner sequence $\mathcal{F}$. Notice that given (x, y), there can be at most countably many δ which verify (8).

Let us prove statement (i). Suppose $\mathcal{F}$ is a Følner sequence and $\mathrm{Sep}_{\mathcal{F}}(X, G, \delta, \nu) = \infty$ for some $\delta, \nu \in (0, 1)$. Let $\mathcal{S}$ be a countable family of finite (X, G, δ, ν)-separated sets (with respect to $\mathcal{F}$) such that $\sup_{S \in \mathcal{S}} \#S = \infty$. Further, let $C \subseteq (0, 1)$ be the collection of all $\delta \in (0, 1)$ such that for some $S \in \mathcal{S}$ there are $(x, y) \in S^2$ such that (8) holds. As C is at most countable, there exists $\delta_0 \in (0, \delta]$ such that for any $S \in \mathcal{S}$ we have

$$\mathrm{ad}_{\mathcal{F}'}(\Delta(X, G, \delta_0, x, y)) = \mathrm{ad}_{\mathcal{F}}(\Delta(X, G, \delta_0, x, y)) \geqslant \mathrm{ad}_{\mathcal{F}}(\Delta(X, G, \delta, x, y)) \geqslant \nu$$

for all $x \neq y \in S$ and any Følner sequence $\mathcal{F}'$ where we used that $|\Delta(X, G, \cdot, x, y)|$ is non-increasing. It straightforwardly follows that each S is (X, G, δ', ν)-separated with respect to any Følner sequence $\mathcal{F}$ and every $\delta' \in (0, \delta_0]$. As S can be chosen arbitrarily large, this proves the first assertion.

Let us consider (ii). First, observe that due to (i), we have $\mathrm{Sep}_{\mathcal{F}^1}(X, G, \delta, \nu) < \infty$ for all $\delta, \nu \in (0, 1)$. Given $\delta \in (0, 1)$, we call $\nu \in (0, 1)$ δ-singular if $\mathrm{Sep}_{\mathcal{F}^i}(X, G, \delta, \nu) < \mathrm{Sep}_{\mathcal{F}^i}(X, G, \delta - \varepsilon, \nu)$ for all $\varepsilon > 0$ and some $i \in \{0, 1\}$. Otherwise, we say ν is δ-regular. The collection of all δ-singular elements of $(0, 1)$ is denoted by B_δ. We say δ is singular if B_δ is uncountable. Otherwise, we call $\delta \in (0, 1)$ regular. The collection of all singular δ in $(0, 1)$ is denoted by B. We set $A = (0, 1) \backslash B$.

Next, we show that for all $\delta \in (0, 1)$ and each $\nu \in B_\delta^c$ we have $\mathrm{Sep}_{\mathcal{F}^0}(X, G, \delta, \nu) = \mathrm{Sep}_{\mathcal{F}^1}(X, G, \delta, \nu)$. To prove (ii), it then remains to show that B is countable.

Given $\delta \in (0, 1)$, let $\nu \in (0, 1)$ be δ-regular. By definition, there is $\varepsilon > 0$ such that $\mathrm{Sep}_{\mathcal{F}^i}(X, G, \delta, \nu) = \mathrm{Sep}_{\mathcal{F}^i}(X, G, \delta', \nu)$ for all $\delta' \in (\delta - \varepsilon, \delta)$ and $i = 0, 1$. Let $S \subseteq X$ be δ-ν-separated w.r.t. $\mathcal{F}^0$ and suppose S is of maximal cardinality. As S is finite, the collection of all $\delta \in (0, 1)$ which verify (8) for some pair $(x, y) \in S^2$ is countable. There is hence $\delta' \in (\delta - \varepsilon, \delta)$ which does not verify (8) for any $(x, y) \in S^2$. Clearly, S is δ'-ν-separated for $\mathcal{F}^0$. By the above, S is also δ'-ν-separated for $\mathcal{F}^1$. Hence,

$$\mathrm{Sep}_{\mathcal{F}^1}(X, G, \delta, \nu) = \mathrm{Sep}_{\mathcal{F}^1}(X, G, \delta', \nu) \geqslant \mathrm{Sep}_{\mathcal{F}^0}(X, G, \delta', \nu) = \mathrm{Sep}_{\mathcal{F}^0}(X, G, \delta, \nu).$$

By changing the roles of $\mathcal{F}^0$ and $\mathcal{F}^1$, we obtain the converse inequality and accordingly $\mathrm{Sep}_{\mathcal{F}^0}(X, G, \delta, \nu) = \mathrm{Sep}_{\mathcal{F}^1}(X, G, \delta, \nu)$ for all δ-regular ν.

It remains to show that B is countable. To that end, we need the following

Claim 4.4. *If $\delta \in (0, 1)$ is singular, then B_δ has non-empty interior.*

Proof of the claim. Let $\nu \in (0, 1)$ be δ-singular and $\nu' \in (0, \nu)$ be δ-regular. Observe that due to the monotonicity in both arguments of $\mathrm{Sep}_{\mathcal{F}^i}(X, G, \cdot, \cdot)$, there has to be a *jump point* ν_0 between ν and ν' (possibly coinciding with ν or ν'), i.e., a point ν_0 such that for $i = 0$ or $i = 1$ we have $\mathrm{Sep}_{\mathcal{F}^i}(X, G, \delta, \nu_0 - \varepsilon) > \mathrm{Sep}_{\mathcal{F}^i}(X, G, \delta, \nu_0)$ for all $\varepsilon > 0$. As $\mathrm{Sep}_{\mathcal{F}^i}(X, G, \delta, \cdot)$ is non-increasing and integer-valued, each compact subinterval of $(0, 1)$ can contain at most finitely many such jump points. Therefore, the set of δ-singular points is a union of isolated points and intervals. Since a subset of $(0, 1)$ with only isolated points is at most countable, this proves the claim. $\diamond$

Now, for a contradiction, assume that B is uncountable. By the above claim, B_δ contains an interval I_δ whenever $\delta \in B$. Thus, there must be an uncountable set $B' \subseteq B$ with $\bigcap_{\delta \in B'} I_\delta \neq \varnothing$. Accordingly, there is $\nu \in (0, 1)$ such that ν is δ-singular for all $\delta \in B'$. As $\mathrm{Sep}_{\mathcal{F}^i}(X, G, \cdot, \nu)$ is non-increasing, there can be at most countably many δ with $\mathrm{Sep}_{\mathcal{F}^i}(X, G, \delta - \varepsilon, \nu) > \mathrm{Sep}_{\mathcal{F}^i}(X, G, \delta, \nu)$ for all $\varepsilon > 0$. This contradicts the uncountability of B'. Hence, B is at most countable. This finishes the proof. $\qquad\square$

5 Application to regular model sets

In this section, we study amorphic complexity of (the dynamical hull of) model sets. Given a model set, our third main result provides an upper bound for its amorphic complexity which may be understood as a measure of its amorphicity. We start by collecting a number of preliminary facts concerning Delone sets, cut and project schemes and their associated dynamics.

5.1 Delone dynamical systems and model sets

From now on, in what follows, G is a locally compact second countable abelian group with Haar measure m_G. Further, in all of the following, we switch to additive notation for the group operation in G, accounting for its commutativity. By the Birkhoff-Kakutani Theorem, G is metrizable and the metric d_G can be chosen to be invariant under G. In fact, open balls with respect to d_G are relatively compact [Str74] so that G is automatically σ-compact.

A set $\Gamma \subseteq G$ is called r-*uniformly discrete* if there exists $r > 0$ such that $d_G(g, g') > r$ for all $g \neq g' \in \Gamma$. Moreover, Γ is called R-*relatively dense* (or R-*syndetic*) if there exists $R > 0$ such that $\Gamma \cap B_G(g, R) \neq \varnothing$ for all $g \in G$, where $B_G(g, R)$ denotes the open d_G-ball of radius R centred at g. We call Γ a *Delone set* if it is uniformly discrete and relatively dense. The collection of all Delone sets in G will be denoted by $\mathcal{D}(G)$.

Given $\rho > 0$ and $g \in \Gamma$, the tuple $(B_G(0, \rho) \cap (\Gamma - g), \rho)$ is called a *(ρ-)patch* of Γ. The set of all patches of Γ is denoted by $\mathcal{P}(\Gamma)$. A Delone set Γ is said to have *finite local complexity* (FLC) if for all $\rho > 0$ the number of its ρ-patches is finite. For $\Gamma, \Gamma' \in \mathcal{D}(G)$, set

$$\mathrm{dist}(\Gamma, \Gamma') = \inf\left\{\varepsilon > 0 \mid \exists g \in B_G(0, \varepsilon) : (\Gamma - g) \cap B_G(0, 1/\varepsilon) = \Gamma' \cap B_G(0, 1/\varepsilon)\right\}.$$

Then $d(\Gamma, \Gamma') = \min\{1/\sqrt{2}, \mathrm{dist}(\Gamma, \Gamma')\}$ defines a metric on $\mathcal{D}(G)$ (see [LMS02, Section 2]). Moreover, for any Delone set $\Gamma \subseteq G$ with FLC the *dynamical hull* of Γ, defined as

$$\Omega(\Gamma) = \overline{\{\Gamma - g \mid g \in G\}},$$

where the closure is taken with respect to d, is compact [Sch99, Proposition 2.3]. The dynamical system $(\Omega(\Gamma), G)$, given by the translation action of G on the hull $\Omega(\Gamma)$, is called a *Delone dynamical system*.

The method of choice to construct Delone sets is to utilize a *cut and project scheme* (CPS). A CPS consists of a triple $(G, H, \mathcal{L})$ of two locally compact abelian groups G (*external group*) and H (*internal group*) and a uniform lattice $\mathcal{L} \subseteq G \times H$ which is *irrational*, that is, the natural projections $\pi_G : G \times H \to G$ and $\pi_H : G \times H \to H$ satisfy

(i) the restriction $\pi_G|_{\mathcal{L}}$ is injective;

(ii) the image $\pi_H(\mathcal{L})$ is dense.

If not stated otherwise, we throughout assume that G and H are second countable. As a consequence of (i), if we let $L = \pi_G(\mathcal{L})$ and $L^* = \pi_H(\mathcal{L})$, the *star map*

$$* : L \to L^* : l \mapsto l^* = \pi_H \circ \pi_G|_{\mathcal{L}}^{-1}(l)$$

is well defined and surjective. Given a precompact set $W \subseteq H$ (referred to as *window*), we define the point set

$$\curlywedge(W) = \pi_G\left(\mathcal{L} \cap (G \times W)\right) = \{l \in L \mid l^* \in W\}.$$

If W is compact and *proper* (that is, $\overline{\text{int}(W)} = W$), then $\curlywedge(W)$ is a Delone set and has FLC [Rob07]. In this case, we call $\curlywedge(W)$ a *model set*. If further $m_H(\partial W) = 0$, then we call the window, as well as the resulting model set, *regular*. Otherwise, we refer to W and $\curlywedge(W)$ as *irregular*. Delone dynamical systems associated to regular model sets are mean equicontinuous, see [FGL21, Remark 6.2 & Corollary 6.3].

We say that a subset $A \subseteq H$ is *irredundant* if $\{h \in H \mid h + A = A\} = \{0\}$. Clearly, if ∂W is irredundant, then so is W. A CPS is called *Euclidean* if $G = \mathbb{R}^N$ and $H = \mathbb{R}^M$ for some $M, N \in \mathbb{N}$, and *planar* if $N = M = 1$. Note that in the Euclidean case, any compact window is irredundant. Further, observe that if W is not irredundant, it is possible to construct a CPS $(G, H', \mathcal{L}')$ with irredundant window $W' \subseteq H'$ such that for each $\Lambda \in \Omega(\curlywedge(W))$ with $\curlywedge(\text{int}(W)) \subseteq \Lambda \subseteq \curlywedge(W)$ we have $\curlywedge(\text{int}(W')) \subseteq \Lambda \subseteq \curlywedge(W')$ (compare [LM06, Section 5] and [BLM07, Lemma 7]).

As $\mathcal{L}$ is a uniform lattice in $G \times H$, the quotient $\mathbb{T} := (G \times H)/\mathcal{L}$ is a compact abelian group. A natural action of G on $\mathbb{T}$ is given by $(u, [s, t]_{\mathcal{L}}) \mapsto [s + u, t]_{\mathcal{L}}$. Here, $[s, t]_{\mathcal{L}}$ denotes the equivalence class of $(s, t) \in G \times H$ in $\mathbb{T}$. Observe that due to the assumptions on $(G, H, \mathcal{L})$, this action is equicontinuous, minimal and has hence a unique invariant measure $\mu_{\mathbb{T}}$. Furthermore, if $W \subseteq H$ is irredundant, $(\mathbb{T}, G)$ is the maximal equicontinuous factor of $(\Omega(\curlywedge(W)), G)$ [BLM07]. The respective factor map β is also referred to as *torus parametrization*.

Given an irredundant window W, the fibres of the torus parametrization are characterized as follows: for $\Gamma \in \Omega(\curlywedge(W))$, we have

$$\Gamma \in \beta^{-1}([s, t]_{\mathcal{L}}) \quad \Leftrightarrow \quad \curlywedge(\text{int}(W) + t) - s \subseteq \Gamma \subseteq \curlywedge(W + t) - s \tag{9}$$

as well as

$$\Gamma \in \beta^{-1}([0, t]_{\mathcal{L}}) \quad \Leftrightarrow \quad \exists\, (t_j) \in L^{*\mathbb{N}} \text{ with } \lim_{j \to \infty} t_j = t \text{ and } \lim_{j \to \infty} \curlywedge(W + t_j) = \Gamma.$$

In the following, we denote by $\text{Vol}(\mathcal{L})$ the volume of a fundamental domain of $\mathcal{L}$. Note that $\text{Vol}(\mathcal{L})$ is well defined.

Proposition 5.1 ([HR15, Proposition 3.4]). *Let $(G, H, \mathcal{L})$ be a CPS and $W \subseteq H$ be precompact. Then for every van Hove sequence $\mathcal{F} = (F_n)$ in G we have*

$$\frac{m_H(\text{int}(W))}{\text{Vol}(\mathcal{L})} \leqslant \varliminf_{n \to \infty} \frac{\sharp(\curlywedge(W) \cap F_n)}{m_G(F_n)} \leqslant \varlimsup_{t \to \infty} \frac{\sharp(\curlywedge(W) \cap F_n)}{m_G(F_n)} \leqslant \frac{m_H(W)}{\text{Vol}(\mathcal{L})}.$$

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Let us collect three more statements which follow easily from the definition of the metric d on $\mathcal{D}(G)$. Similarly to the notion of (δ, ν)-separation of elements of a dynamical system (see Section 1), given a van Hove sequence $\mathcal{F}$ in G, we set

$$\nu_{\mathcal{F}}(\delta, \Gamma, \Gamma') = \mathrm{ad}_{\mathcal{F}}(\{g \in G \mid d(g\Gamma, g\Gamma') \geqslant \delta\}),$$

where $\delta > 0$ and $\Gamma, \Gamma' \in \mathcal{D}(G)$.

Proposition 5.2. *For every van Hove sequence $\mathcal{F} = (F_n)$ in G we have*

$$\nu_{\mathcal{F}}(\delta, \Gamma, \Gamma') \leqslant m_G(B_G(0, 1/\delta)) \varlimsup_{n \to \infty} \frac{\sharp((\Gamma \Delta \Gamma') \cap F_n)}{m_G(F_n)},$$

with $\delta > 0$ and $\Gamma, \Gamma' \in \mathcal{D}(G)$.

Accordingly, together with Proposition 5.1, we get

Corollary 5.3. *If $m_H(\partial W) = 0$ and $\wedge(\mathrm{int}(W)) \subseteq \Gamma \subseteq \wedge(W)$, then $\nu_{\mathcal{F}}(\delta, \Gamma, \Gamma') = \nu_{\mathcal{F}}(\delta, \wedge(W), \Gamma')$ for all van Hove sequences $\mathcal{F}$, $\delta > 0$ and $\Gamma' \in \mathcal{D}(G)$.*

Finally, observe that

Proposition 5.4. *Suppose $\delta > 0$, $\Gamma, \Gamma' \in \mathcal{D}(G)$ and $g \in B_G(0, \delta/2)$. If $d(\Gamma, \Gamma') \geqslant \delta$, then $d(\Gamma, \Gamma' + g) \geqslant \delta/2$.*

5.2 Upper bound on the amorphic complexity of regular model sets

We next come to our third main result. First, recall that for a locally compact σ-compact group H, the upper box dimension is given by

$$\overline{\dim}_{\mathrm{B}}(H) = \varlimsup_{\varepsilon \to 0} \frac{\log m_H(B_H(h, \varepsilon))}{\log \varepsilon},$$

where $h \in H$ is arbitrary. Observe that $\overline{\dim}_{\mathrm{B}}(H)$ is well defined because of the invariance of the metric d_H and the Haar measure m_H. Note further that the above definition, as well as the definition of the (upper) box dimension of compact sets in Section 3.1, are special cases of a more general concept of box dimension. We refrain from reproducing the slightly technical (and standard) general definition here and refer the interested reader to [Edg98, Section 1.4] instead.

We will also make use of *Minkowski's characterisation* of the box dimension of a given compact set $M \subseteq H$ by

$$\overline{\dim}_{\mathrm{B}}(M) = \overline{\dim}_{\mathrm{B}}(H) - \varliminf_{\varepsilon \to 0} \frac{\log m_H(B_H(M, \varepsilon))}{\log \varepsilon}.$$

The proof of this fact in our setting is similar to the one in the Euclidean space, see for instance [Fal03, Proposition 3.2].

Finally, in order to derive upper bounds on amorphic complexity, it is convenient to make use of an alternative characterisation which utilises spanning sets instead of separating sets—similar as in the derivation of upper bounds for topological entropy (or box dimension).

Given $\delta > 0$ and $\nu \in (0, 1]$, we say a subset $S \subseteq X$ is (δ, ν)-*spanning* with respect to a Følner sequence $\mathcal{F}$ if for all $x \in X$ there exists $s \in S$ such that $\mathrm{ad}_{\mathcal{F}}(\Delta(X, G, \delta, x, s)) < \nu$. We denote by $\mathrm{Span}_{\mathcal{F}}(X, G, \delta, \nu)$ the smallest cardinality among the (δ, ν)-spanning sets with respect to $\mathcal{F}$. It is not difficult to see that $\mathrm{Span}_{\mathcal{F}}(X, G, \delta, \nu)$ instead of $\mathrm{Sep}_{\mathcal{F}}(X, G, \delta, \nu)$ can equivalently be used in defining amorphic complexity, see also [FGJ16, Lemma 3.1 & Corollary 3.2].

Theorem 5.5. *Suppose $(G, H, \mathcal{L})$ is a cut and project scheme, where G and H are locally compact second countable abelian groups. Furthermore, let $W \subseteq H$ be compact, proper, regular and irredundant and assume that $\overline{\dim}_{\mathrm{B}}(H)$ is finite. Then*

$$\overline{\mathrm{ac}}_{\mathcal{F}}(\Omega(\curlywedge(W)), G) \; \leqslant \; \frac{\overline{\dim}_{\mathrm{B}}(H)}{\overline{\dim}_{\mathrm{B}}(H) - \overline{\dim}_{\mathrm{B}}(\partial W)}, \tag{10}$$

for any Følner sequence $\mathcal{F}$.

Proof. As W is regular and hence $(\Omega(\curlywedge(W)), G)$ mean equicontinuous, we may assume without loss of generality that $\mathcal{F}$ is van Hove, see Remark 4.2 and Theorem 4.3. We first choose compact sets $A \subseteq G$ and $B \subseteq H$ such that $W \subseteq B$ and $\pi(A \times B) = \mathbb{T}$, where $\pi : G \times H \to \mathbb{T} = (G \times H)/\mathcal{L}$ is the canonical projection.

Given $(g, h) \in A \times B$, let $\hat{\Gamma}_{g,h} = \curlywedge(W + h) - g$. Observe that $\hat{\Gamma}_{g,h}$ may not be an element of $\Omega(\curlywedge(W))$. While for our asymptotic estimates this will be of no problem (due to Corollary 5.3), its explicit definition makes $\hat{\Gamma}_{g,h}$ more convenient to deal with in computations.

Claim 5.6. *Let $\delta > 0$. If $d_G(g, g') \leqslant \delta/2$ and $d(\hat{\Gamma}_{g,h}, \hat{\Gamma}_{g',h'}) \geqslant \delta$, then*

$$[-g, -h]_{\mathcal{L}} \in \pi\left(B_G(0, 2/\delta) \times (W \Delta(W + h' - h))\right) =: D(\delta, h' - h).$$

Proof of the claim. By Proposition 5.4, we know that $d(\hat{\Gamma}_{g,h}, \hat{\Gamma}_{g,h'}) \geqslant \delta/2$. Hence, there exists $(\ell, \ell^*) \in \mathcal{L}$ with $\ell \in B_G(g, 2/\delta)$ and $\ell \in \hat{\Gamma}_{g,h} \Delta \hat{\Gamma}_{g,h'}$. The latter implies that $\ell^* \in (W + h) \Delta(W + h')$.

Equivalently, this means that $\ell - g \in B_G(0, 2/\delta)$ and $\ell^* - h \in W \Delta(W + h' - h)$, so that

$$[-g, -h]_{\mathcal{L}} = [\ell - g, \ell^* - h]_{\mathcal{L}} \in \pi\left(B_G(0, 2/\delta) \times W \Delta(W + h' - h)\right).$$

This proves the claim. $\circ$

We can now apply the claim to estimate the separation frequency of a pair $\hat{\Gamma}_{g,h}$ and $\hat{\Gamma}_{g',h'}$.

$$
\begin{aligned}
\nu_{\mathcal{F}}(\delta, \hat{\Gamma}_{g,h}, \hat{\Gamma}_{g',h'}) \;&=\; \varlimsup_{n \to \infty} \frac{1}{m_G(F_n)} \int_{F_n} \mathbf{1}_{[\delta, \infty)}(d(\hat{\Gamma}_{g,h} - t, \hat{\Gamma}_{g',h'} - t))\, dt \\
&=\; \varlimsup_{n \to \infty} \frac{1}{m_G(F_n)} \int_{F_n} \mathbf{1}_{[\delta, \infty)}(d(\hat{\Gamma}_{g+t,h}, \hat{\Gamma}_{g'+t,h'}))\, dt \\
&\leqslant\; \varlimsup_{n \to \infty} \frac{1}{m_G(F_n)} \int_{F_n} \mathbf{1}_{D(\delta, h'-h)}([-g - t, h]_{\mathcal{L}})\, dt \\
&\overset{(*)}{=}\; \mu_{\mathbb{T}}(D(\delta, h' - h)) \\
&\leqslant\; m_G(B_G(0, 2/\delta)) \cdot m_H(W \Delta(W + h' - h)) \\
&\leqslant\; m_G(B_G(0, 2/\delta)) \cdot m_H(\overline{B_H(\partial W, d(0, h' - h))}),
\end{aligned}
$$

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where the equality $(*)$ follows from the unique ergodicity of $(\mathbb{T}, G)$ and the fact that $\mu_{\mathbb{T}}(\partial D(\delta, h' - h)) = 0$.

Now, suppose that $\delta > 0$ and $\nu > 0$ are given. Let

$$\varepsilon = \inf \left\{ \eta > 0 \mid m_H \left(B_H \left(\partial W, \eta \right) \right) \geqslant \nu / m_G \left(B_G(0, 2/\delta) \right) \right\}.$$

Then we have $m_H(B_H(\partial W, \varepsilon)) \leqslant \nu / m_G(B_G(0, 2/\delta))$ but at the same time $m_H \left(\overline{B_H(\partial W, \varepsilon)} \right) \geqslant \nu / m_G(B_G(0, 2/\delta))$ due to the regularity of Haar measure. Consequently, if $d_G(g, g') < \delta/2$ and $d_H(h, h') < \varepsilon$, then the first inequality combined with the above estimate yields that $\hat{\Gamma}_{g,h}$ and $\hat{\Gamma}_{g',h'}$ cannot be (δ, ν)-separated.

For $g \in G$ and $h \in H$, let $\Gamma_{g,h}$ denote some element of $\Omega(\wedge(W))$ with $\wedge(\text{int}(W) + h) - g \subseteq \Gamma_{g,h} \subseteq \hat{\Gamma}_{g,h}$, see (9). We cover A by $N = N_{\delta/2}(A)$ balls of radius $\delta/2$ and B by $M = N_{\varepsilon}(B)$ balls of radius ε and denote by $(g_n)_{n=1}^{N}$ and $(h_m)_{m=1}^{M}$ the midpoints of these balls. Then the set $\{\Gamma_{g_n, h_m} \mid n = 1, \ldots, N, \ m = 1, \ldots, M\}$ is (δ, ν)-spanning due to the above and Corollary 5.3. We obtain the estimate

$$
\begin{aligned}
\overline{\text{ac}}_{\mathcal{F}}(\Omega(\wedge(W)), G) &= \sup_{\delta > 0} \overline{\lim_{\nu \to 0}} \frac{\log \text{Span}_{\mathcal{F}}(\Omega(\wedge(W)), G, \delta, \nu)}{-\log \nu} \\
&\leqslant \sup_{\delta > 0} \overline{\lim_{\varepsilon \to 0}} \frac{\log(N_{\delta/2}(A) \cdot N_{\varepsilon}(B))}{-\log m_H \left(B_H(\partial W, \varepsilon) \right)} \\
&= \overline{\lim_{\varepsilon \to 0}} \frac{\log N_{\varepsilon}(B) / - \log \varepsilon}{\log m_H \left(B_H(\partial W, \varepsilon) \right) / \log \varepsilon} \\
&\leqslant \frac{\overline{\dim}_{\text{B}}(H)}{\overline{\dim}_{\text{B}}(H) - \overline{\dim}_{\text{B}}(\partial W)},
\end{aligned}
$$

where we used Minkowski's characterisation in the last step. This completes the proof. $\square$

Remark 5.7. It is not too difficult to see that the above result is optimal in the sense that equality is attained for some examples while at the same time, it cannot hold in general.

(a) In order to see that amorphic complexity can be smaller than the bound provided by (10), let $H = \mathbb{R}$ and suppose $C \subseteq \mathbb{R}$ is an arbitrary Cantor set of dimension $d \in [0, 1)$. Let W be a window given by the union of C with a countable number of gaps (that is, bounded connected components of $\mathbb{R} \backslash C$) such that $\partial W = C$. Clearly, this can be done such that for each n, we have that W contains less than n intervals of size 2^{-n} or bigger. If $\varepsilon \in (2^{-n}, 2^{-n+1}]$, then each of these intervals contributes at most 2ε to $m_H(W \triangle (W + \varepsilon))$, whereas the union of the other intervals contributes at most ε in total (and ∂W does not contribute since it is of zero measure). Hence, we obtain $m_H(W \triangle (W + \varepsilon)) \leqslant 2\varepsilon n \leqslant 2\varepsilon(-\log \varepsilon / \log 2 + 1)$. Accordingly, the computation in the proof of Theorem 5.5 yields $\overline{\text{ac}}_{\mathcal{F}}(\Omega(\wedge(W)), G) \leqslant 1 < \frac{1}{1-d}$.

(b) The most straightforward examples in which equality is attained in (10) are given by CPS with $H = \mathbb{R}$. We refrain from discussing the technicalities (which are in spirit similar to those in the proof of the above theorem) and simply sketch the main ingredients of the construction. For $\gamma > 2$, consider a middle segment Cantor set C_{γ}

which is constructed by always removing the middle $(1-2/\gamma)$-th part of intervals in the canonical construction of Cantor sets. Observe that C_γ is of dimension $\overline{\dim}_B(C_\gamma) = \log 2/\log \gamma$ with gaps of size $(1 - 2/\gamma) \cdot \gamma^{-n}$. If W is the window that is obtained by including all gaps of size $(1 - 2/\gamma) \cdot \gamma^{-n}$ with n odd, it can be readily checked that

$$\lim_{\varepsilon \to 0} \frac{\log m_H(W \Delta(W + \varepsilon))}{\log \varepsilon} = (1 - \log 2/\log \gamma).$$

We may assume without loss of generality to be given an element (u, v) of some set of generators of $\mathcal{L}$ with $C_\gamma \subseteq [0, v]$. Let $h_1, \ldots, h_{\lfloor 1/\varepsilon \rfloor} \in H$ be equidistributed in $[0, v] \subseteq H$. Similarly to the estimates in the proof of Theorem 5.5, it can be checked that for small enough δ, we have that $\{\Gamma_{0,h_1}, \ldots \Gamma_{0,h_{\lfloor 1/\varepsilon \rfloor}}\}$ is (δ, ν)-separated with $\nu = m_G(B_G(0, 1/\delta))m_H(W \Delta(W + \varepsilon))$ as ε (and hence ν) tends to zero. Then one obtains $\overline{\mathrm{ac}}_\mathcal{F}(\Omega(\curlywedge(W)), G) = \sup_{\delta>0} \overline{\lim}_{\nu\to 0} \frac{\log \mathrm{Sep}_\mathcal{F}(\Omega(\curlywedge(W)),G,\delta,\nu)}{-\log \nu} \geqslant \frac{1}{1-\overline{\dim}_B(C_\gamma)}$.

(c) Note that the construction sketched in (b) yields uncountably many regular model sets that lie in different conjugacy classes. In fact, it shows that any value in $[1, \infty)$ can be realised as the amorphic complexity of a regular model set.

(d) The above considerations indicate that while the structure of the boundary of the window inflicts some upper bound on the complexity of the dynamics of the resulting model set, it greatly depends on the interior of the window whether this bound is actually attained or not. This coincides with similar observations concerning the topological entropy of irregular model sets [JLO19].

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