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Quantum symmetry

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The symmetry of objects plays a crucial role in many branches of mathematics and physics. It allowed, for example, the early prediction of the existence of new small particles. "Quantum symmetry" concerns a generalized notion of symmetry. It is an abstract way of characterizing the symmetry of a much richer class of mathematical and physical objects. In this snapshot we explain how quantum symmetry emerges as matrix symmetries using a famous example: Mermin's magic square. It shows that quantum symmetries can solve problems that lie beyond the reach of classical symmetries, showing that quantum symmetries play a central role in modern mathematics.

1 Introduction

In the first half of the 20th century the principles of quantum mechanics were discovered. These principles describe how very small particles behave. These are particles at the scale of atoms or even smaller: electrons, neutrons, protons, and many others. The theory required a startling new intuition about physical concepts which had been thought for a long time to have been well understood: movement, place, energy. Such seemingly easy concepts – visible to the human eye on a daily basis – had to be re-invented. At the time, new models to understand quantum mechanics had been proposed by Erwin Schrödinger (1887–1961), Werner Heisenberg (1901–1976), and many others. Although nowadays these models are widely accepted to be correct, initially they led to huge debates amongst leading mathematical physicists including Niels Bohr (1885–1962) and Albert Einstein (1879–1961).

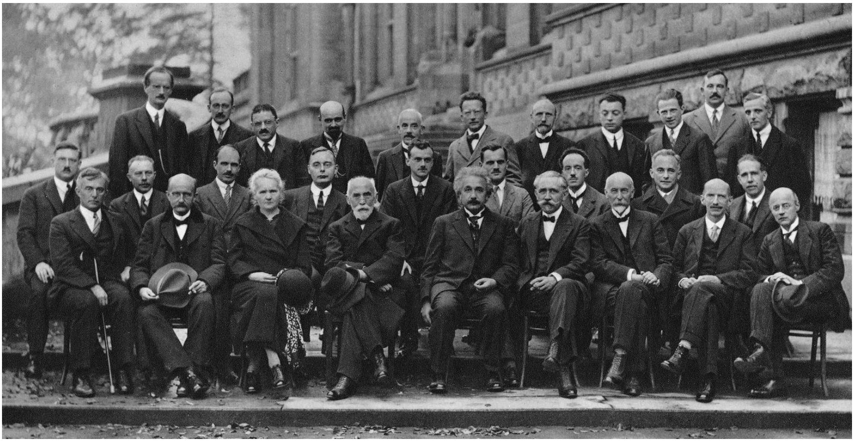

Figure 1: Solvay conference (Brussels, 1927) in mathematical physics where the foundations of quantum mechanics were laid. Amongst the attendants are Bohr, Einstein, and Heisenberg.

In this snapshot, the feature of quantum mechanics that we will concentrate on is that it is based on "matrix analysis", an approach usually attributed to Heisenberg. We will give some more details shortly, but briefly, an $n \times m$ *matrix* is a rectangular array of numbers, arranged into n rows and m columns. The use of matrices leads to surprising new phenomena. We begin showing this by considering a famous puzzle that illustrates how useful matrices can be: Mermin's magic square. We then discuss how sets can be seen as a special case of this matrix analysis. The latter step is a parallel between the step from deterministic mechanics (what you see in daily life) to quantum mechanics (what goes on in the microscopic world). Meanwhile we aim to give some intuitive answers to the question of what happens to symmetries when one "goes quantum".

2 From classical to quantum: a puzzle you can't solve, or can you?

Let us begin by defining Mermin's magic square. Take a 3×3-grid, as shown in Figure 2. Then we ask: Is it possible to fill the grid with numbers 1 and -1 such that:

- The product of the numbers in each row yields 1.
- The product of the numbers in each column yields -1.

$\Rightarrow$

1	-1	-1
-1	-1	1
1	-1	?

Figure 2: On the left, an empty 3×3-grid. On the right, an attempt to fill the grid according to the rules.

On the right-hand side of Figure 2, you see an attempt to fill the grid. If you multiply the numbers in the first row or the second row, the result is 1. If you multiply the numbers in the first column and the second column, the result is -1. So far so good. However, now we want to decide whether there should be a 1 or a -1 at the place of the question mark. To make the product of the numbers on the third row result in a 1, we need to replace the question mark with a -1. However, if we want that the product of the numbers in the third column is a -1 then we must replace the question mark with a 1. So we conclude that we cannot complete the table according to the rules. We need to go back and change the other numbers as well. However, thinking further, we see that it is impossible to fill the table according to these rules. Why?

Let us try to compute the product of all numbers in the table. We know that, according to the rules, the product of each row must be 1. So if we take the total product we get $1 \cdot 1 \cdot 1 = 1$. On the other hand we know that the product of each column yields a -1. But then the total product must be $-1 \cdot -1 \cdot -1 = -1$. This is, of course, impossible.

Now let us look at the "quantum" situation. At this point we will require a little bit of linear algebra. We will consider 2×2 matrices, so square arrays of numbers arranged in 2 rows and 2 columns. We multiply two such matrices according to the following rule:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}. \tag{2.1}$$

We multiply a number λ with a 2×2-matrix as follows

$$\lambda \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}.$$

Let us introduce the special matrices

$$\mathbf{1} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{\text{-}1} := \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We also introduce the *Pauli matrices*, named after the physicist Wolfgang Pauli (1900–1958):

$$\sigma_x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Here the symbol i denotes the square root of -1, that is, the complex number with the property that $i^2 = -1$. If we multiply these matrices we find that

$$\sigma_x \sigma_y = i\sigma_z, \qquad \sigma_y \sigma_z = i\sigma_x, \qquad \sigma_z \sigma_x = -i\sigma_y.$$

We also find that

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbf{1}.$$

Extending the definition of a matrix given in the introduction, we allow the entries to be matrices themselves. In equation (2.1), if the entries $a, b, c, \ldots, h$ are matrices, each time that we see a product of some of these entries on the right hand side of the expression (2.1) we apply the matrix multiplication. We shall further use the notation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \sigma = \begin{pmatrix} a\sigma & b\sigma \\ c\sigma & d\sigma \end{pmatrix}, \tag{2.2}$$

where a, b, c, d are numbers and σ is a 2×2-matrix. The equation in (2.2) is called a *tensor product*. Let us compute the example $\sigma_x \otimes \sigma_y$ to make this a little clearer:

$$\begin{aligned} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} &= \begin{pmatrix} 0 \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & 1 \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ 1 \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & 0 \cdot \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{0} & \sigma_y \\ \sigma_y & \mathbf{0} \end{pmatrix}, \end{aligned}$$

where $\mathbf{0}$ denotes the matrix with all 0 entries.

Now we fill the magic square as shown in Figure 3. The entries are not numbers anymore, rather they are 2×2-matrices whose entries are again 2×2-matrices (that is, they are tensor products).

$\mathbf{1} \otimes \sigma_z$	$\sigma_z \otimes \mathbf{1}$	$\sigma_z \otimes \sigma_z$
$\sigma_x \otimes \mathbf{1}$	$\mathbf{1} \otimes \sigma_x$	$\sigma_x \otimes \sigma_x$
$-\sigma_x \otimes \sigma_z$	$-\sigma_z \otimes \sigma_x$	$\sigma_y \otimes \sigma_y$

Figure 3: The 3×3 grid filled with tensor products of matrices.

Now, after a certain amount of calculation, we observe that each row multiplies to the same matrix $\mathbf{1} \otimes \mathbf{1}$. Each column multiplies to the matrix $-\mathbf{1} \otimes \mathbf{1}$. To abbreviate the computations above we use the rule $(A \otimes B) \cdot (C \otimes D) = (AC \otimes BD)$, which can be checked from the descriptions above. So we get for the first row:

$$(\mathbf{1} \otimes \sigma_z) \cdot (\sigma_z \otimes \mathbf{1}) \cdot (\sigma_z \otimes \sigma_z) = (\mathbf{1} \cdot \sigma_z \cdot \sigma_z) \otimes (\sigma_z \cdot \mathbf{1} \cdot \sigma_z) = \sigma_z^2 \otimes \sigma_z^2 = \mathbf{1} \otimes \mathbf{1}.$$

The other rows and columns can be checked by a similar computation.

In conclusion, the 3×3 square given by Figure 3 shows that one is able to produce a solution to the problem if one permits matrices instead of just numbers. The reader should note that this should not be considered some artificial solution, because matrices and tensor products arise naturally in many areas of mathematics, including quantum mechanics, as we will see in the next section.

3 Operator theory: the language of quantum mechanics

In the 1920s, the foundations of quantum mechanics were laid by a group of mathematical physicists. Quantum mechanics deals with the way that small particles (such as electrons) move. These particles move in a "strange" way. They do not have determined positions or momenta, but they are rather determined by probability distributions. The chance that a particle can be found in a certain region is given by a probability distribution. Sometimes this phenomena is also described as particles having multiple locations at the same time. Only after measuring the position of the particle the probability distribution changes into a new probability distribution (with a high density at the region where the particle was measured).

As mentioned in the introduction, amongst the mathematical physicists working at the beginnings of quantum theory was Werner Heisenberg, who proposed a mathematical model to describe these probability distributions using matrix algebras. The 2×2-matrices of the previous section provide a simple example. If properly interpreted, these 2×2-matrices can be used to describe a particle that can only be in two places (in fact, even more information can be stored in a 2×2 matrix, but we keep it simple here). One could think of 2 pots, and secretly putting a marble in pot 1 with probability p or a marble in pot 2 with probability q where $p + q = 1$, as illustrated in Figure 4.

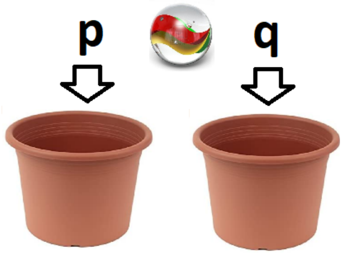

Figure 4: Marble in a pot with probability distribution

The corresponding matrix would be the diagonal matrix given by

$$\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}.$$

We will not try to explain the precise details of quantum mechanics and Heisenberg's matrix analysis. The main idea is that if the matrix has non-zero values only on the diagonal, like in the example with the pots, then those diagonal entries will be the probabilities that a particle is in a certain physical state. This state is indexed by the place where the probability appears on the diagonal. Now in more generality, most matrices can be "diagonalized" by choosing an appropriate set of what are called basis vectors. Each number on the diagonal of this diagonalized matrix is the probability that the state of the particle is described by the corresponding basis vector. The fact that different matrices have different basis vectors that diagonalize them leads to unexpected behavior which cannot described by classical probability and classical mechanics. Mermin's magic square is an example of this. To summarise, the take-home message is that properties of small particles can be described with the analysis of matrices.

Soon after Heisenberg (and many others) had discovered this matrix analysis it was John von Neumann (1903–1957) who realized that to describe infinite systems of particles one also needs infinite matrices (which means a matrix with infinitely many rows and columns). This then evolved into the theory of *operator algebras*. Operator algebras nowadays constitute a complete mathematical area. It still has strong roots in quantum mechanics but many surprising links to completely different areas have been found. These include links to knot theory, logic, complexity theory, number theory, and geometry (basically all parts of mathematics and parts of mathematical physics and computer science).

Figure 5: Werner Heisenberg (left) and John von Neumann (right).

4 Symmetry and Groups

The concept of symmetry plays an important role throughout all of mathematics. In the most elementary sense, an object is said to have a symmetry if after reflecting or rotating the object, it stays the same. Symmetry is applied widely in many real life structures; some examples from architecture can be seen in Figure 6.

Figure 6: Examples of symmetry in architecture.

The notion of symmetry can be translated into the notion of a *group*. A group is a set G with:

1. a distinguished element e in G called the *identity*;
2. a function $G \times G \to G : (a, b) \mapsto a \cdot b$ called the *multiplication*;
3. a function $G \to G : a \mapsto a^{-1}$ called the *inverse*;

that satisfy the rules that for all a, b, c in G we have

$$(a \cdot b) \cdot c = a \cdot (b \cdot c), \qquad e \cdot a = a \cdot e = a, \qquad a \cdot a^{-1} = a^{-1} \cdot a = e.$$

Let us also mention for later that if for all a, b in G we have that

$$a \cdot b = b \cdot a$$

the group is said to be *commutative*. A simple example of a group is formed by the positive real numbers $\mathbb{R}_{>0}$ with the usual multiplication, inverse and identity 1. Another example could be the set of symmetries of an equilateral triangle. These are three reflections, about the lines that join each corner to the midpoint of the opposite side, and three rotations, by $120°, 240°$, and $360° = 0°$. This last symmetry is the identity of the group. We invite the reader to check that all the group properties listed above are satisfied for this example, where the multiplication is composition of symmetries (that is, doing them one after another) and the inverse of a symmetry is whatever symmetry takes us back to the original figure (so, for instance, the inverse of rotation by $120°$ is rotation by $240°$, and every reflection is its own inverse). More generally, suppose that we have any figure and we let G denote its set of symmetries. Then G is a group with composition of symmetries as the multiplication. For example, the left-hand picture in Figure 6 has many symmetries, some of which are shown in Figure 7 One may reflect across a red line and then reflect across another red line without changing the image.

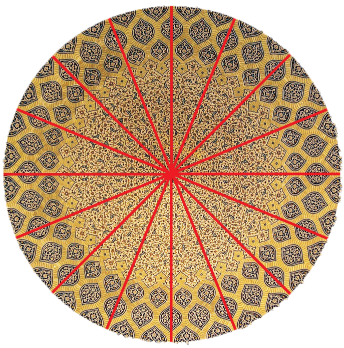

Figure 7: Reflecting along a red line gives a symmetry. Note that there are many more symmetries than the ones shown!

Symmetry and groups occur a lot in mathematics. As another example, we may consider the 2×2-matrices of the form

$$A_\theta := \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}, \qquad \theta \text{ in } \mathbb{R},$$

which corresponds to a rotation of the plane by $\theta°$. If we endow it with the multiplication above, $\mathbf{1}$ for the identity, and inverse of A_θ being $A_\theta^{-1} = A_{-\theta}$, then we find another group.

From symmetry to quantum symmetry

We shall heuristically describe what a "quantum group" is here. Recall that the upshot of Mermin's square was that one can solve certain problems if one replaces numbers by matrices. We will do the same here essentially, but we have to explain how the set G of a group is turned into a set of matrices. We note immediately that a quantum group is a generalization of a group as defined above. This means, in particular, that groups are quantum groups. So we are dealing with a generalized notion of symmetry. Let us explain what type of generalization we mean by taking the Gelfand-Naimark theorem as a starting point.

Gelfand-Naimark theorem

From this point we require a little bit of undergraduate level mathematics, but we will try to give the gist of the idea by first considering only finite sets. So, suppose that X is a finite set. There is a very important object associated to X which we call $C(X)$. It consists of all functions $X \to \mathbb{C}$, and has the structure of a *C^*-algebra*, which very imprecisely means it is a vector space[1] with a multiplication, an *involution* (that is, a function that is equal to its own inverse), and a *norm* (essentially, a way of measuring lengths or distances), which satisfy certain identities. So we have an assignment

$$\text{Finite sets} \;\to\; C^*\text{-algebras} \;:\; X \;\mapsto\; C(X).$$

Now the point is that $C(X)$ encodes all the relevant information of the set X. One can completely recover X from $C(X)$, as follows. We let $\Omega(C(X))$ denote the set of all non-zero linear[2] maps $\varphi : C(X) \to \mathbb{C}$ that satisfy $\varphi(fg) = \varphi(f)\varphi(g)$. Such maps exist! Indeed, take x in X and let $\varphi_x(f) = f(x)$ be the evaluation map. Then φ_x is in $\Omega(C(X))$. It is then a little exercise to show that the maps φ_x for all x in X actually make up the whole set $\Omega(C(X))$. In other words, there is a correspondence:

$$X \;\leftrightarrow\; \Omega(C(X)) \;:\; x \;\leftrightarrow\; \varphi_x.$$

[1] For the definition of a vector space, we refer to https://en.wikipedia.org/wiki/Vector_space.

[2] A map φ on a vector space is linear if it satisfies $\phi(\alpha\underline{v} + \beta\underline{w}) = \alpha\varphi(\underline{v}) + \beta\varphi(\underline{w})$, for all vectors $\underline{v}, \underline{w}$ and scalar multiples α, β.

So we have completely recovered the set X from $C(X)$. More generally, if X was a compact topological space[3] then essentially the same protocol allows one to recover the topology of X from $C(X)$, though the details are much more involved. This correspondence is known as the *Gelfand-Naimark theorem*, and it was proved in the 1940's. Basically it says that the structure of a compact topological space (think of the finite set if you wish) can completely be described by a particular C*-algebra. Hence the Gelfand-Naimark theorem translates topology into algebra and vice-versa.

One should really think of two languages here. To use topology or to use C*-algebras is something like using German or French. Sometimes it is much easier to explain something in one language than in the other. The same holds on the mathematical level. You may gain more information on the algebraic side by studying topology and vice-versa. This is one of the many strengths of the Gelfand-Naimark theorem.

4.1 From symmetry to quantum symmetry

Suppose that G is a group. For simplicity we assume that G is finite. Remember that G comes with a distinguished element e in G which we called the *identity*. Further G has two distinguished maps, namely the *multiplication* and the *inverse*. By the Gelfand-Naimark theorem we know that all the data about G as a set is contained in the set of functions $C(G) : G \to \mathbb{C}$. We consider the maps:

$$S : C(G) \to C(G) : f(\,\cdot\,) \mapsto f(\,\cdot^{-1}),$$

and

$$\Delta : C(G) \to C(G \times G) : f \mapsto ((a, b) \mapsto f(a \cdot b))\,.$$

For the identity we have the special map

$$\epsilon : C(G) \to \mathbb{C} : f \mapsto f(e)$$

given by the evaluation at the identity. These maps satisfy certain additional relations that encode the defining properties of a group. It turns out that the quadruple $(C(G), \Delta, S, \epsilon)$ encodes all the information of the group G, in exactly the same spirit as how the Gelfand-Naimark theorem encodes the properties of a set and lets us move from sets to algebras. We call $(C(G), \Delta, S, \epsilon)$ a *quantum group*.

Formally a quantum group can be defined as a C*-algebra with certain additional maps such as Δ, S and ϵ as above. If the C*-algebra is commutative,

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[3] For a nicely-explained introduction to compactness, we recommend the notes by Terry Tao, which can be found here: http://www.math.ucla.edu/%7Etao/preprints/compactness.pdf

then the quantum group is constructed from a group exactly as we described above. If the C^*-algebra is not commutative then it is a (possibly infinite dimensional) matrix algebra with additional maps that resemble the group multiplication, inverse and unit. In this sense we view quantum groups as a "matrix generalization" of a classical group.

The name quantum group is usually attributed to Fields medallist Vladimir Drinfeld. The formal definition of a C^*-algebraic (compact) quantum group is due to Woronowicz [1].

Conclusion

We have seen through Mermin's magic square that using matrix algebra and operator theory we can solve problems that we cannot solve with regular numbers. In the same vein, we have also seen that quantum groups are generalizations of groups by replacing a function algebra by a matrix algebra. The natural question is now: can we expect to resolve new problems using quantum groups that could not have been solved with ordinary groups? The answer is yes. New knot invariants have been discovered using the theory of quantum groups. Quantum groups provide the right type of symmetry in non-commutative geometry, and they also play a crucial role in random matrix theory and provide symmetries for the exchange of non-commutative probability variables. In general one could say that many parts of mathematical analysis has a matrix version and quantum groups supply their symmetries.

Image credits

Other images created by the author.

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