Mathematisches Forschungsinstitut Oberwolfach

Report No. 32/2021

DOI: 10.4171/OWR/2021/32

Differentialgeometrie im Grossen (hybrid meeting)

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4 July – 10 July 2021

Abstract. The field of classical differential geometry has expanded enormously over the last several decades, helped by the development of tools from neighboring fields such as partial differential equations, complex analysis and geometric topology. In the spirit of the previous meetings in the series, this meeting will bring together researchers from apparently separate subfields of differential geometry, but whose work is linked by common themes. In particular, this meeting will emphasize intrinsic geometric questions motivated by the classification and rigidity of global geometric structures and the interaction of curvature with the underlying geometry and topology.

Mathematics Subject Classification (2020): 53-XX.

Introduction by the Organizers

The workshop *Differentialgeometrie im Großen* was held July 4 - July 10, 2021. The participants were specialists in differential geometry and its neighboring fields, covering a broad spectrum of subareas which are in the focus of current developments.

The workshop was held in a hybrid mode due to the ongoing pandemic. There were 22 in-person participants and a further 26 virtual participants.

The lectures during the five days of the meeting were roughly organized according to different themes. Most of the talks were in the afternoon or evening, to accommodate virtual participants in North America.

The first day of the meeting began with a broad selection of talks on positively curved manifolds, the Ricci flow, rigidity in Carnot groups and hyperkähler manifolds. On the second day we started with talks in metric geometry and the structure of metric spaces, and ended with talks on negatively curved manifolds and the SYZ conjecture.

On Wednesday, the in-person participants enjoyed a hike in the morning. In the afternoon we had three talks on complex geometry. On Thursday we had short talks on Einstein metrics and intermediate Ricci curvature followed by talks on geometric group theory, hyperbolic structures and symmetric spaces. In the evening we saw a talk on geometric flows.

We ended the workshop on Friday with three talks on the Ricci flow and comparison geometry.

The meeting gave a good overview of the current developments in differential geometry, and highlighted some of the important developments in the field. The participants included researchers from all over the world, ranging from graduate students to scientific leaders in their areas. Despite a smaller number of in-person participants compared to a normal year, the atmosphere during the meeting was lively and open. In particular, attendees appreciated the opportunity to experience again in-person mathematical dialogue and activity, and greatly benefited from the ideal environment at Oberwolfach.

Workshop (hybrid meeting): Differentialgeometrie im Grossen

Table of Contents

Burkhard Wilking (joint with Lee Kennard, Michael Wiemeler)
Torus actions on positively curved manifolds 1689

Alix Deruelle (joint with Felix Schulze)
A relative entropy for expanders of the Ricci flow 1689

Bruce Kleiner (joint with Stefan Müller, László Székelyhidi, Xiangdong Xie)
Rigidity and flexibility of bilipschitz, quasiconformal, and Sobolev mappings in Carnot groups 1692

Song Sun
Collapsing geometry of hyperkähler 4-manifolds 1694

Paul Creutz (joint with Matthew Romney)
Triangulating metric surfaces 1695

Giuliano Basso
Improving conical bicombings 1696

Koji Fujiwara (joint with Panos Papasoglu)
Asymptotic dimension of planes and planar graphs 1697

Stephan Stadler (joint with Alexander Lytchak)
Curvature bounds of subsets in dimension two 1698

Michael Kapovich
On torsion in discrete isometry groups of negatively curved manifolds .. 1700

Yang Li
Metric SYZ conjecture and beyond 1703

Eleonora Di Nezza (joint with Tamas Darvas, Chihn Lu)
Pluripotential Theory: how to get singular Kähler-Eisntein metrics 1705

Hans-Joachim Hein (joint with Valentino Tosatti)
Smooth asymptotics for collapsing Calabi-Yau manifolds 1708

Ronan J. Conlon (joint with Alix Deruelle)
Steady gradient Kahler-Ricci solitons on crepant resolutions of Calabi-Yau cones 1711

Frieder Jäckel (joint with Ursula Hamenstädt)
A stability result for Einstein metrics of Tian 1714

Masoumeh Zarei (joint with Manuel Amann, Peter Quast)
Intermediate Ricci, homotopy, and submanifolds of symmetric spaces .. 1714

Maria Beatrice Pozzetti (joint with Jonas Beyrer)
On positive surface group representations in $\mathrm{SO}(p,q)$ 1717

Andrés Sambarino (joint with M. Bridgeman, B. Pozzetti and A. Wienhard)
Projective manifolds, hyperbolic manifolds and the Hessian of Hausdorff dimension 1720

Vladimir Marković
Non-uniqueness of minimal surfaces in Hermitian Lie groups 1720

Natasa Sesum
Ancient solutions in geometric flows 1721

Gang Tian (joint with Yan Li, Xiaohua Zhu)
Limiting behavior of Ricci flow on Fano manifolds 1724

Miles Simon (joint with Tobias Lamm)
Ricci flow of $W^{2,2}$ *metrics in four dimensions* 1726

John Lott
Comparison geometry of holomorphic bisectional curvature 1728

Abstracts

Torus actions on positively curved manifolds

Burkhard Wilking
(joint work with Lee Kennard, Michael Wiemeler)

We study positively curved manifolds with isometric torus actions.

Our first main result states that a fixed point component of a five-torus has the rational cohomology of a rank one symmetric space.

Using this result for 5-dimensional subgroups of large torus actions we can also recover the topology of the ambient manifold if we make slightly stronger assumptions. For example we show that if $T^7 \subset Iso(M^n)$, $\kappa > 0$, $b_{odd}(M) = 0$, then M has the rational cohomology of a rank one symmetric space.

A relative entropy for expanders of the Ricci flow

Alix Deruelle
(joint work with Felix Schulze)

A *Ricci soliton* is a triple (M^n, g, X) where (M^n, g) is a Riemannian manifold and a vector field X satisfying the equation

$$\mathrm{Ric}(g) - \frac{1}{2}\mathcal{L}_X(g) + \frac{\lambda}{2}g = 0,$$

for some $\lambda \in \{-1, 0, 1\}$. We call X the *soliton vector field.* A soliton is said to be *steady* if $\lambda = 0$, *expanding* if $\lambda = 1$, and *shrinking* if $\lambda = -1$. Moreover, if $X = \nabla^g f$ for some real-valued smooth function f on M called the potential function then $(M^n, g, \nabla^g f)$ is said to be a *gradient* soliton. In this paper, we focus on expanding gradient Ricci solitons whose equation reduces to

$$2\,\mathrm{Ric}(g) + g = \mathcal{L}_{\nabla^g f}(g). \tag{1}$$

Notice that equation (1) normalizes the metric and defines the potential function f up to an additive constant.

A Ricci soliton is said to be *complete* if (M^n, g) and the vector field $\nabla^g f$ are complete in the usual sense. To each expanding gradient Ricci soliton $(M^n, g, \nabla^g f)$, one may associate a self-similar solution of the Ricci flow as follows:

$$g(t) = t\varphi_t^* g,$$

where $(\varphi_t)_{t>0}$ is the one-parameter family of diffeomorphisms generated by the vector field $-\nabla^g f/t$ such that $\varphi_{t=1} = Id_M$. This solution is *Type III*, i.e. there exists a nonnegative constant C such that for any $t \in (0, +\infty)$,

$$t \sup_M |\,\mathrm{Rm}(g(t))| \leq C,$$

if the curvature is bounded on the manifold M^n. Therefore, it is likely that expanding gradient Ricci solitons are good candidates for singularity models for Type III solutions to the Ricci flow. To illustrate these heuristics more accurately, let us mention that the second author and Simon [1] have shown that expanding gradient Ricci solitons naturally arise as a blow-down of non-compact non-collapsed Type III solutions with non-negative curvature operator.

Given an expanding gradient Ricci soliton $(M^n, g, \nabla^g f)$ with quadratic Ricci curvature decay together with covariant derivatives, one can associate a unique tangent cone $(C(S), dr^2 + r^2 g_S, o)$ with a smooth Riemannian link (S, g_S): [2, 3, 4]. In particular, such an expanding gradient Ricci soliton is *asymptotically conical.* Moreover, the metric cone $(C(S), dr^2 + r^2 g_S, o)$ can be interpreted as the initial condition of the Ricci flow $(g(t))_{t>0}$ associated to the soliton in the sense that $(M^n, d_{g(t)}, p)_{t>0}$ converges to $(C(S), dr^2 + r^2 g_S, o)$ in the pointed Gromov-Hausdorff sense as $t \to 0$, if p is a critical point of the potential function f.

In this note, we investigate the uniqueness question among the class of asymptotically conical expanding gradient Ricci solitons coming out of a given metric cone over a smooth link.

Notice that uniqueness holds true when considering the class of complete expanding gradient Kähler-Ricci solitons coming out of Kähler cones: see [5] and [6, Corollary B] for a precise statement.

Observe that the uniqueness question is of interest even in the case of an asymptotic cone $(C(S), dr^2 + r^2 g_S, r\partial_r/2)$ which is Ricci flat and endowed with the radial vector field $r\partial_r/2$ since it is an exact expanding gradient Ricci soliton outside the tip. In particular, if a complete expanding gradient Ricci soliton comes out of $(C(S), dr^2 + r^2 g_S, r\partial_r/2)$ then uniqueness of the Ricci flow fails in case metric singularities are allowed. Now, even if we restrict our attention to complete expanding gradient Ricci solitons coming out of a given Ricci flat cone, Angenent-Knopf [7] have proved that uniqueness still fails for some Ricci flat cones in dimension greater than 4.

The first main result is a unique continuation result at infinity for two expanding gradient Ricci solitons coming out of the same cone and it can be informally stated as follows.

Theorem 1. *Let $(M_i^n, g_i, \nabla^{g_i} f_i)$, $i = 1, 2$, be two expanding gradient Ricci solitons coming out of the same cone $(C(S), g_C := dr^2 + r^2 g_S, \frac{r}{2}\partial_r)$ over a smooth link (S, g_S). Assume the soliton metrics g_1 and g_2 are gauged in such a way that their soliton vector fields coincide outside a compact set. Then the trace at infinity*

$$\lim_{r \to +\infty} r^n e^{\frac{r^2}{4}} (g_1 - g_2) =: \operatorname{tr}_\infty \left(r^n e^{\frac{r^2}{4}} (g_1 - g_2) \right)$$

exists in the $L^2_{loc}(C(S) \setminus \{o\})$-topology, it preserves the radial vector field ∂_r and its tangential part is divergence free with respect to the metric on the link in the weak sense. Moreover, g_1 and g_2 coincide pointwise outside a compact set if and only if their associated trace at infinity vanishes, i.e.

$$\operatorname{tr}_\infty \left(r^n e^{\frac{r^2}{4}} (g_1 - g_2) \right) \equiv 0\,.$$

The main tool to show the above result is as follows: we establish the existence of a suitable frequency function at infinity, where the method follows the work of Bernstein for mean curvature flow in codimension one [8], which itself is based on the fundamental work of Garofalo-Lin [9]. The main difficulty and crucial point in this approach comes from the fact that different to the case of mean curvature flow, where the graphical representation at infinity of one expander over the other yields a natural well-controlled gauge, in the current setting it is necessary to establish a suitable gauge at infinity between the two expanders. To establish the needed decay estimates for the frequency function it is necessary to simultaneously control the gauge together with the frequency function. The gauge we employ is a Bianchi gauge, motivated by the work of Kotschwar [10] for the comparison of two general solutions of Ricci flow. Due to self-similarity of our solutions the evolution equation for the Bianchi gauge turns into an ODE, which then results in an ODE-PDE system for the frequency function set-up.

Kotschwar-Wang [11] have employed Carleman estimates to prove the uniqueness of Ricci shrinkers smoothly asymptotic to a smooth cone. We expect that similar to work of Bernstein [8] for mean curvature flow, the methods in this paper can be adapted to give an alternative proof of the result of Kotschwar-Wang. But different to the case treated by Bernstein, the setup for Ricci shrinkers does not directly transform in the system treated in the current paper. A unique continuation result for expanders asymptotic to Ricci flat cones was obtained by the first author using Carleman estimates [12]. The results of Bernstein for mean curvature flow have been extended to the higher codimension case by Khan [13]. The unique continuation result of Bernstein [8] has been employed centrally by Bernstein-Wang [14] in their proof that the space of expanders smoothly asymptotic to smooth cones has the structure of a smooth Banach manifold. Frequency bounds for solutions to a general class of drift laplacians equations have been obtained by Colding-Minicozzi in [15].

In case the asymptotic cone is Ricci flat, the convergence rate was shown to hold pointwise in the smooth sense in [12]. For an arbitrary asymptotic cone, Theorem 1 shows that the same convergence rate holds for the L^2 norm on level sets of the distance function from the tip of the cone.

As an application of the decay estimates achieved via the frequency function, we show the existence of a relative entropy for two expanders asymptotic to the same cone.

Theorem 2 (A relative entropy for two expanders coming out of the same cone)**.** *Let $(M_1^n, g_1, \nabla^{g_1} f_1)$ and $(M_2^n, g_2, \nabla^{g_2} f_2)$ be two expanding gradient Ricci solitons coming out of the same cone $(C(S), g_C := dr^2 + r^2 g_S, \frac{r}{2}\partial_r)$ over a smooth link (S, g_S). Then the following limit exists for all $t > 0$ and is constant in time:*
(2)
$$\mathcal{W}(g_2(t), g_1(t)) := \lim_{R\to+\infty}\left(\int_{f_2(t)\leq R}\frac{e^{f_2(t)}}{(4\pi t)^{\frac{n}{2}}}\,d\mu_{g_2(t)} - \int_{f_1(t)\leq R}\frac{e^{f_1(t)}}{(4\pi t)^{\frac{n}{2}}}\,d\mu_{g_1(t)}\right).$$

Feldman-Ilmanen-Ni [16] have introduced a forward reduced volume and an expanding entropy (denoted by $\mathcal{W}_+$) that detect expanding gradient Ricci solitons on a *closed* manifold. The purpose of Theorem 2 is to provide a replacement of the aforementioned functionals in the non-compact setting.

In order to prove that the limit (2) in Theorem 2 is well-defined, we invoke the integral convergence rate for the difference of the soliton metrics $g_2 - g_1$ obtained in Theorem 1. Observe that comparing the solutions to their common initial cone metric only yields a quadratic decay and is therefore not sufficient to ensure the existence of the limit (2). We underline the fact that (2) is established by taking differences rather than by considering a renormalization.

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Rigidity and flexibility of bilipschitz, quasiconformal, and Sobolev mappings in Carnot groups

Bruce Kleiner

(joint work with Stefan Müller, László Székelyhidi, Xiangdong Xie)

The lecture covered results from a series of recent papers on geometric mapping theory in Euclidean space and Carnot groups; these were motivated by geometric group theory, analysis on metric spaces, differential geometry, and PDEs.

The first results concern rigidity of product structure in $\mathbb{R}^n$. A mapping $f : X_1 \times X_2 \to Y_1 \times Y_2$ between product sets **splits** (or preserves product structure) if it is of the form $f(x_1, x_2) = (f_1(x_1), f_2(x_2)$ for some mappings $f_i : X_i \to Y_i$, after possibly reindexing the factors Y_1, Y_2.

Theorem 1 (K-Müller-Szekelyhidi-Xie)**.** *If $n \geq 2$ and $f : \Omega \to \mathbb{R}^n \times \mathbb{R}^n$ is a $W^{1,2}_{loc}$-mapping such that $Df(x)$ is split and bijective for a.e. $x \in \Omega$, then f is split.*

The exponent 2 is sharp:

Theorem 2 (KMSX)**.** *For every $1 \leq p < 2$ there is a $W^{1,p}_{loc}$-mapping*

$$f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$$

such that $Df(x)$ is split and bijective for a.e. $x \in \Omega$, but f is not split.

The assertion in Theorem 1 is false when $n = 2$ because of the map $\mathbb{R}^2 \to \mathbb{R}^2$ which folds along the diagonal line $\{x_1 = x_2\}$. In fact, there are even bilipschitz counterexamples:

Theorem 3 (KMSX)**.** *There is a bilipschitz homeomorphism $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ such that*

- *$Df(x)$ is split and bijective for a.e. x.*
- *f is not split.*
- *f is area preserving: $detDf(x) = 1$ for a.e. x.*
- *There is a null set N such that $Df(x)$ takes only five values for $x \notin N$.*

The remaining results are joint with Stefan Müller and Xiangdong Xie, and are concerned with bilipschitz, quasiconformal, and Sobolev mappings in the Carnot group setting. Such mappings have been studied since the 1970s, and arise, for example, in geometric group theory as boundary homeomorphisms associated with bilipschitz mappings $X \to X'$ between negatively curved manifolds, or as blow-downs of quasi-isometries between finitely generated nilpotent groups. Since the work of Pansu on rigidity in 1989, they have been of interest to a broader community of differential geometers, people working on analysis in metric spaces, and on PDEs.

In what follows, all Carnot groups will be equipped with their Carnot-Caratheodory (i.e. sub-Riemannian) distance.

Theorem 4. *Let $\mathbb{H}$ be the Heisenberg group and $U_1, U_2 \subset \mathbb{H}$ be connected open subsets. Suppose*

$$f : U_1 \times U_2 \to U' \subset \mathbb{H} \times \mathbb{H}$$

is a quasisymmetric homeomorphism. Then f is split.

The same holds more generally for a $W^{1,3}_{loc}$-mapping provided the horizontal differential $d_H f$ is nonsingular almost everywhere. We do not know the optimal Sobolev exponent for rigidity. This is a special case of a general result for maps between products of Carnot groups.

Theorem 5. *Let $\mathbb{H}_n^{\mathbb{C}}$ be the n^{th} complex Heisenberg group.*

- *(Hypoellipticity) Any quasisymmetric homeomorphism $\mathbb{H}_n^{\mathbb{C}} \supset U \to U' \subset \mathbb{H}_n^{\mathbb{C}}$ is locally holomorphic or antiholomorphic.*
- *Any globally defined quasisymmetric homeomorphism $\mathbb{H}_n^{\mathbb{C}} \to \mathbb{H}_n^{\mathbb{C}}$ is affine.*

Reimann-Ricci proved the first assertion above, assuming f is C^2. The regularity result only requires that $f \in W^{1,2n+1}_{loc}$, and nondegeneracy of the horizontal differential $d_H f$ almost everywhere.

Theorem 6. *If G is a nonrigid Carnot group (in the sense of Ottazzi-Warhurst) other than $\mathbb{R}^n$ or a Heisenberg group, and $f : G \supset U \to U' \subset G$ is quasisymmetric, then $f \in W^{1,\infty}_{loc}$, and is locally bilipschitz. If $U = G$, then f is bilipschitz.*

The starting point for our results is theorem about Pansu pullback – the pullback of differential forms using the Pansu differential. Although Pansu pullback need not commute with the exterior derivative even for smooth contact diffeomorphisms, a partial analog does hold.

The lecture also mentioned:

- Applications to quasiregular mappings in Carnot groups.
- Sobolev mappings induce a chain mapping on Rumin complexes, in the setting of contact manifolds.
- A rigidity result characterizing Sobolev mappings $N \supset U \to N$ where $N \subset GL(n, \mathbb{R})$ is the Carnot group of upper trianglar matrices with 1s on the diagonal.

Collapsing geometry of hyperkähler 4-manifolds

Song Sun

In this talk I discussed the collapsing geometry of 4 dimensional hyperkähler metrics. I gave an overview of the background and explained various examples and related previous results. Then I talked about my recent joint work with Ruobing Zhang which has the following geometric consequences

- Any collapsing Gromov-Hausdorff limit of unit diameter hyperkähler metrics on the K3 manifold is either an interval, a singular special Kähler metric on S^2, or a flat quotient $T^3/\mathbb{Z}_2$.
- Any non-flat complete hyperkähler 4-manifold with finite energy must be asymptotic to a known model end, which belongs to 6 families given by $ALE, ALF, ALG, ALH, ALG^*$, and ALH^*.

Together with recent results studying gravitational instantons with given asymptotic structure, the above also proves Yau's compactification conjecture for gravitational instantons.

I expect that the ideas and techniques developed in this work will have further applications.

Triangulating metric surfaces

Paul Creutz
(joint work with Matthew Romney)

Our main result is the following very general triangulation theorem.

Theorem 1 ([2]). *Let X be a geodesic metric space homeomorphic to a closed surface and $\varepsilon > 0$. Then X may be decomposed into finitely many non-overlapping convex triangles, each of diameter at most ε*

Here, *triangle* means a subset of X homeomorphic to the closed disk whose boundary is the union of three geodesics. We remark that Theorem 1 does not give a triangulation of X in the usual sense. The difference is that we do not require adjacent triangles to intersect along entire edges.

Theorem 1 is a classical result when X is a surface of synthetically bounded total curvature. This is already a very general condition that is satisfied for example when X is a Riemannian manifold or a metric simplicial complex, or more generally satisfies an upper or lower curvature bound à la Alexandrov. However, in recent years there has also been an increasing interest in different classes of surfaces that do not fall into this setting. These include Finsler surfaces, surfaces satisfying a quadratic isoperimetric inequality, Ahlfors 2-regular quasispheres, fractal spheres and many more. In contrast to the classical bounded-curvature triangulation theorem, our result now also applies to all of these.

The main application of the classical triangulation theorem for surfaces of bounded curvature is to show that every such surface is a limit of smooth surfaces of uniformly bounded integral curvature. Theorem 1 allows for similar applications in much more general settings. In particular in the subsequent article [3] it is applied to prove an analogous approximation theorem for geodesic surfaces of locally finite Hausdorff measure. This generalized approximation theorem then has further remarkable applications concerning the intensively studied *uniformization problem*.

One key step in the classical proof of the triangulation theorem for surfaces of bounded curvature is to show that any point in such surface has polygonal neighbourhoods of arbitrary small perimeter. As has been noted before, this is the only step in the proof that relies on the bounded curvature assumption. Simple examples however show that this property does not hold for general geodesic surfaces, and indeed it is quite difficult to prove even under the the bounded-curvature assumption. Instead, our proof relies on a relatively short argument showing that every point in X has a neighborhood that may be covered by finitely many polygons, each of arbitrary small perimeter. Thus our approach also quite simplifies the classical proof of the bounded curvature case.

As mentioned above, in principle, this is the only major difference between our proof of Theorem 1 and that of the classical triangulation theorem in the bounded curvature setting. However it turns out that, despite its fundamental significance, a proper proof of the bounded-curvature triangulation theorem can only be found in the textbook [1]. Actually even the proof in [1] contains several technical errors. These errors are related to the fact that geodesics can be highly non-unique and hence intersect in complicated ways. Fixing these errors requires some serious technical work and thus we believe that our, now clean, proof of Theorem 1 might also be an important contribution to the literature on surfaces of bounded curvature.

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Improving conical bicombings

Giuliano Basso

A bicombing σ on a metric space X distinguishes for every pair of points a geodesic connecting them. More precisely, $\sigma\colon X \times X \times [0,1] \to X$ is called bicombing if for all x, $y \in X$, the curve $\sigma_{xy} := \sigma(x, y, \cdot)\colon [0,1] \to X$ is a constant speed geodesic from x to y. We say that σ is conical if

$$d(\sigma_{xy}(t), \sigma_{x'y'}(t)) \leq (1-t)d(x,x') + t\, d(y,y') \tag{1}$$

for all x, y, x', $y' \in X$ and all $t \in [0,1]$. Conical bicombings have been introduced by Descombes and Lang in [3]. They are a useful tool in metric fixed point theory and geometric group theory (see, for example, [5, 6, 7]). As it turns out, in some situations it is desirable to work with bicombings that satisfy conditions which are more restrictive than (1).

We say that a bicombing σ is *consistent* if σ_{pq} is a subsegment of σ_{xy} whenever p, $q \in \sigma_{xy}([0,1])$. There are many examples of conical bicombings which are not consistent. In fact, consistency seems to be a rather restrictive notion. For example, if E is a dual Banach space and $C \subset E$ is a closed convex subset with nonempty interior, then the bicombing on C consisting of all linear segments is the only consistent conical bicombing on C (see [2, Theorem 1.5]). The following question arises naturally:

Question 1. *Does every metric space with a conical bicombing also admit a consistent conical bicombing?*

In [3], Descombes and Lang showed that Question 1 has a positive answer for all proper metric spaces of finite combinatorial dimension in the sense of Dress (see [4]). In fact, they showed that such spaces have unique straight geodesics and the bicombing consisting of straight geodesics is a consistent concial bicombing. In [1], I proved the following result:

Theorem 2. *Let X be a proper metric space admitting a conical bicombing. Then X admits a consistent bicombing γ which consists of straight geodesics such that $t \mapsto d(\gamma_{xy}(t), \gamma_{x'y'}(t))$ is convex on $[0,1]$ whenever $d(x,y) = d(x',y')$.*

A geodesic $\gamma\colon [0,1] \to X$ is called straight if $d_z \circ \gamma$ is convex on $[0,1]$ for all $z \in X$, where $d_z := d(z, \cdot)$. Notice that if σ is a consistent conical bicombing, then any σ-geodesic is necessarily straight. Moreover, if a consistent bicombing σ is conical, then the function $t \mapsto d(\sigma_{xy}(t), \sigma_{x'y'}(t))$ is convex on $[0,1]$ for all x, y, x', $y' \in X$. At present, I do not know if the bicombing γ appearing in Theorem 2 is in fact conical. This would give a positive answer to Question 1 for all proper metric spaces.

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Asymptotic dimension of planes and planar graphs

Koji Fujiwara

(joint work with Panos Papasoglu)

The notion of *asymptotic dimension* introduced by Gromov [2] has become central in Geometric Group Theory mainly because of its relationship with the Novikov conjecture. The asymptotic dimension $\operatorname{asdim} X$ of a metric space X is defined as follows: $\operatorname{asdim} X \leq n$ if and only if for every $m > 0$ there exists $D(m) > 0$ and a covering $\mathcal{U}$ of X by sets of diameter $\leq D(m)$ ($D(m)$-bounded sets) such that any m-ball in X intersects at most $n+1$ elements of $\mathcal{U}$. We say $\operatorname{asdim} X \leq n$, *uniformly* if one can take $D(m)$ independently from X if it belongs to a certain family.

We prove the following, [1]: Let P be a geodesic metric space that is homeomorphic to $\mathbb{R}^2$. Then the asymptotic dimension of P is at most three, uniformly. More generally if P is a geodesic metric space such that there is an injective continuous map from P to $\mathbb{R}^2$, then the conclusion holds.

To be more precise, the following holds: Given $m > 0$ there is some $D(m) > 0$ such that there is a cover of P with sets of diameter $< D(m)$ and that any ball of radius m intersects at most 4 of these sets.

Moreover, we can take $D(m) = 3 \cdot 10^6 m$.

In the talk, I give an outline of the proof of our results. We fix a basepoint e in P and we consider 'annuli' around e of a fixed width (these are metric annuli so, if P is a plane with a Riemannian metric, topologically are generally discs with finitely many holes). Here, annuli are subsets defined as follows: Consider $f(x) = d(e, x)$. Fix $m > 0$. We will pick $N \gg m$ and consider for $k \in \mathbb{N}$ the "annulus"

$$A_k(N) = \{x | kN \leq f(x) < (k+1)N\}$$

We show that in the large scale these annuli resemble cacti. Generalizing a well known result for trees and $\mathbb{R}$-trees we show that cacti have asymptotic dimension at most 1. We then show that 'coarse cacti' also have asymptotic dimension 1. Finally, we decompose our space in 'layers' which are coarse cacti which implies that the asymptotic dimension of the space is at most 3.

An obvious open question is the following: Is the asymptotic dimension of a plane at most two for any geodesic metric?

Jørgensen-Lang [3] have answered the question affirmatively by now.

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Curvature bounds of subsets in dimension two

Stephan Stadler

(joint work with Alexander Lytchak)

This work concerns the intrinsic geometry of subsets in two-dimensional metric spaces with upper curvature bounds. The main geometric result is

Theorem 1. Let X be a two-dimensional contractible CAT(κ) space. Let $A \subset X$ be a closed, Lipschitz connected subset with $H_1(A) = 0$. Then A is a CAT(κ) space with respect to the induced intrinsic metric.

For $\kappa = 0$, this confirms a folklore conjecture, which appeared in print in [2, Conjecture 1]. Related statements and conjectures about subsets of non-positively curved spaces can be found in [1, Chapter 4].

Some special cases of Theorem 1 are known. In [3] and later in [8], it was shown that Jordan domains in the euclidean plane are CAT(0). A more general version appeared in [5]. In [7], Theorem 1 is proved for CAT(0) euclidean simplical complexes. Another special case plays a central role in [6].

The contractibility assumption is redundant for $\kappa \leq 0$; for $\kappa > 0$ it is satisfied if the diameter of X is less than $\frac{\pi}{\sqrt{\kappa}}$. For $\kappa > 0$ the statement is wrong without the contractibility assumption: A closed metric ball of radius $\pi > r > \frac{\pi}{2}$ in the round sphere $\mathbb{S}^2$ is contractible but not CAT(1) in its intrinsic metric.

Localizing the above result we deduce the following:

Corollary 1. Let Y be a metric space of curvature bounded above by κ and dimension two. Let $A \subset Y$ be closed, Lipschitz connected and locally simply connected. Then A has curvature bounded above by κ with respect to the intrinsic metric.

If $\kappa \leq 0$, the theorem of Cartan–Hadamard implies that the universal covering of A is contractible. Hence A is *aspherical* in the sense that all higher homotopy groups of A vanish:

Corollary 2. Let Y be a two-dimensional space of non-positive curvature and let $A \subset Y$ be a closed, Lipschitz connected and locally simply connected subset. Then A is aspherical with respect to the topology induced by the intrinsic metric.

Somewhat surprisingly, no topological assumption is needed for our next conclusion. Indeed, we obtain the following topological statement about *all* subsets of non-positively curved metric spaces of dimension two.

Theorem 2. Let Y be non-positively curved and two-dimensional. Let $A \subset Y$ be an arbitrary subset. Then all higher Lipschitz homotopy groups of A vanish: Every Lipschitz n-sphere in A with $n \geq 2$ bounds a Lipschitz ball in A.

This result is of geometric origin and is deduced from our main theorem. It has the following purely topological application:

Corollary 3. Let Y be a two-dimensional space of non-positive curvature. Then any neighborhood retract $A \subset Y$ is aspherical.

It is known, but surprisingly difficult to prove that *all* subsets of the euclidean plane are aspherical, [4]. This and the results above make the following generalization of the famous Whitehead Conjecture [9] plausible:

Conjecture 1. Let X be a two-dimensional aspherical space. Then any subset A of X is aspherical.

Possibly, a combination of the geometric ideas of the present paper and the purely topological methods of [4] may lead to the resolution of this conjecture for non-positively curved spaces.
We expect our results to simplify the description of geodesically complete two-dimensional $\mathrm{CAT}(\kappa)$ spaces obtained and announced in [6]. Moreover, we expect the results to facilitate a good understanding of two-dimensional $\mathrm{CAT}(\kappa)$ spaces beyond geodesic completeness. For instance, they might lead to a resolution of the following conjecture of potential relevance to geometric group theory:

Conjecture 2. Any compact two-dimensional non-positively curved space is homotopy equivalent to a finite, two-dimensional, non-positively curved euclidean complex.

We want to point out that all the results above trivially hold in dimension one, since any one-dimensional $\mathrm{CAT}(\kappa)$ space is covered by a tree. On the other hand, all results completely fail in dimension at least three: already the complement of an open ball in $\mathbb{R}^3$ is not non-positively curved and not aspherical.

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On torsion in discrete isometry groups of negatively curved manifolds

Michael Kapovich

In 1960 Attle Selberg wrote an influential paper on discrete subgroups of Lie groups, where, among other things he proved a result now known as "Selberg's Lemma"

Theorem 1. *Suppose that* Γ *is a finitely generated subgroup of* $G = SL_n(\mathbb{C})$*. Then* Γ *contains a torsion-free subgroup of finite index* Γ'*.*

Other proofs of this theorem can be found for instance in [4, 16]. The geometric meaning of Selberg's result is that if $X = SL_n(\mathbb{C})/SU(n)$ is the symmetric space of the group G (i.e. the space of positive-definite hermitian matrices with unit determinant) and $\Gamma < G$ is discrete, then, in general, the quotient space X/Γ is not a manifold but an orbifold $\mathcal{O}$ because of torsion (finite order elements) in Γ. What Selberg's Lemma guarantees is the existence of a finite-sheeted orbi-covering $\mathcal{O}' = X/\Gamma' \to \mathcal{O}$, where $\mathcal{O}'$ is a manifold.

In Selberg's paper his "lemma" was just a technical result, Selberg's main interest was a generalization of Calabi's infinitesimal rigidity theorem to other classes of locally-symmetric spaces. What he did not know is that by proving this lemma he also solved **Fenchel's Conjecture (1940s):**

Conjecture 2. *If* $\Gamma < PSL(2, \mathbb{R})$ *is a finitely-generated Fuchsian subgroup, then* Γ *contains a torsion-free finite index subgroup.*

Fenchel's conjecture was supposed to have been solved in 1952 by Ralph Fox,[6], following earlier work by Jakob Nielsen, [14], who also gave the conjecture its name. Except, the solution by Fox turned out to be partially incorrect (he made mistakes in some special cases) and his work was completed only by T. C. Chau in 1983,

[2], who, apparently, was unaware of Selberg's paper and the fact that Selberg proved a much more general result 23 years earlier. This amusing story is only a digression of our main theme.

Gregory Margulis knew Selberg's work quite well and in late 1960s Margulis proved his most famous superrigidity theorem (and its application, Arithmeticity Theorem), which is a far-reaching generalization of the earlier rigidity results. Margulis' work became well-known and he was, of course, invited to give a talk at the ICM in Vancouver, in 1974. And, of course, he was not allowed to go there. Instead, the text of Margulis' address was read by David Kazhdan (who was just allowed to immigrate from USSR to Israel in 1974). In his ICM address, [12], Margulis, among other things, explored similarities between discrete subgroups of Lie groups and discrete isometry groups of Riemannian manifolds of nonpositive curvature (Hadamard manifolds to be more precise) and posed a number of questions and conjectures. In particular, Margulis asked

Question 3. *Does an analogue of Selberg's Lemma hold for finitely generated subgroups of isometry groups of Hadamard manifolds?*

Recall that a *Hadamard manifold* is a complete simply-connected Riemannian manifold whose sectional curvature is ≤ 0. The relation of this question to Selberg's Lemma that each symmetric space of noncompact type, such as $X = SL_n(\mathbb{C})/SU(n)$, is Hadamard.

Note that if Γ is a group containing finite-index torsion-free subgroup, then the orders of torsion elements in Γ are bounded. Thus, if $\Gamma < SL(n,\mathbb{C})$ is finitely generated then orders of finite order elements in Γ are bounded. In particular, if Selberg's Lemma were to hold for Hadamard manifolds, then each finitely generated subgroup of the isometry group of a Hadamard manifold would have bounded torsion. It turns out that Margulis' question has negative answer:

Theorem 4. *(M. Kapovich,* [8]*.) For each $n \geq 4$ and $\epsilon > 0$ there exists a Hadamard manifold $X^n = X$ of sectional curvature $-1-\epsilon \leq K_X \leq -1$ and a discrete finitely generated isometry group $\Gamma_n = \Gamma$ of X such that torsion in Γ is unbounded.*

In particular, an analogue of Selberg's Lemma fails for Hadamard manifolds of dimension ≥ 4. At the same time, Selberg's Lemma does hold for discrete isometry groups of Hadamard manifolds in dimension ≤ 3. The key is

Theorem 5. *(M. Feighn, G. Mess,* [5]*) If X is a contractible 3-dimensional manifold and Γ is a group acting effectively and properly discontinuously on X, then the orbifold $\mathcal{O} = X/\Gamma$ contains a* compact core*, i.e. a compact suborbifold $\mathcal{O}_c$ whose fundamental group is isomorphic to that of $\mathcal{O}$.*

At the same time, if $\mathcal{O}_c$ is a *good* compact 3-dimensional orbifold (such as one in the above theorem), then $\pi_1(\mathcal{O}_c)$ contains a torsion-free subgroup of finite index, i.e. $\pi_1(\mathcal{O}_c)$ satisfies the conclusion of Selberg's Lemma. This deep result is a combination of work of many people:

1. John Hempel, [7], Darryl McCullough and AndyMiller, [13], who proved, following ideas of William Thurston, that this holds for 3-dimensional orbifolds satisfying Thurston's Geometrization Conjecture.

2. Orbifold Geometrization Theorem (Michel Boileau, Berhnard Leeb and Joan Porti, [1], again following ideas of William Thurston, and with important contributions of Darryl Cooper, Craig Hodgson and Steve Kerckhoff [3]; Bruce Kleiner and John Lott, [10], extending the work of Gregory Perelman), stating that all compact 3-dimensional orbifolds containing no bad 2-dimensional suborbifolds, satisfy Thurston's Geometrization Conjecture.

The proof of Theorem 4 is based on my earlier work with Leonid Potyagailo, [9, 15], where we constructed discrete finitely generated subgroups of $PO(4,1)$, the isometry group of the hyperbolic 4-space, which have infinitely many cusps. A suitable "cusp-closing" of the corresponding hyperbolic 4-manifolds results in 4-dimensional negatively curved orbifolds whose fundamental groups have unbounded torsion.

Conjecture 6. *1. Let Γ be a finitely generated discrete isometry group of a negatively curved Hadamard manifold X with negatively pinched sectional curvature, $-b^2 \le K_X \le -1$. Assume that the* critical exponent δ_Γ *of Γ is ≤ 2. Then X/Γ has only finitely many cusps and orders of torsion elements in Γ are bounded.*

2. There exists a sequence of 4-dimensional Hadamard manifolds X_k with negatively pinched sectional curvature, $-b_k^2 \le K_X \le -1$ and their discrete finitely generated isometry groups Γ_k such that:

(a) The sequence δ_{Γ_k} converges to 2 (from above).

(b) $\lim_{k\to\infty} b_k = 1$.

A very recent positive result in the direction of part 1 of this conjecture was proven by Beibei Liu and Shi Wang:

Theorem 7. *(B. Liu, S. Wang, [11]) The cusp finiteness part in the conjecture holds if $\delta_\Gamma < 1$.*

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Metric SYZ conjecture and beyond

Yang Li

A Calabi-Yau manifold is an n-dimensional compact Kähler manifold with a nowhere vanishing holomorphic volume form Ω. By the celebrated theorem of Yau, there is a unique Kähler Ricci-flat metric ω within the given Kähler class, characterized by the complex Monge-Ampère equation

$$\omega^n = \text{const } \Omega \wedge \overline{\Omega}.$$

The major open problem is to describe the behaviour of this canonical metric; in the case of interest, we fix the Kähler class, but allow the complex structure to vary (concretely, this means taking a family of projective varieties and vary the coefficients of the defining equations, and in the limit the varieties can become singular.) A particularly severe degeneration is called the large complex structure limit, the prototypical case being the Fermat family

$$X_s = \{Z_0 Z_1 \dots Z_{n+1} + e^{-s} \sum_0^{n+1} Z_i^{n+2} = 0\}, \quad s \gg 1.$$

As $s \to \infty$ the algebraic limit is the union of $(n+2)$ planes in $\mathbb{CP}^{n+1}$, but the metric behaviour is much more elusive (except when $n = 1$, in which case these are cubic elliptic curves carrying flat metrics).

The Strominger-Yau-Zaslow (SYZ) conjecture (subject to some interpretation) asks for a special Lagrangian T^n-fibration at least in the generic region of the Calabi-Yau manifold, where 'generic' could be taken to mean an open region with

0.99 of the total measure. The word 'special Lagrangian' means a submanifold satisfying

$$\omega|_L = 0, \quad \mathrm{Im}\ \Omega|_L = 0.$$

These are absolute minimizers of volume within their homology classes.

In the presentation we discuss two types of results. The first theorem, specific to the Fermat family, confirms the SYZ conjecture up to taking a subsequence of $s \to \infty$. The restriction to the Fermat family mainly comes from the combinatorial step in the arguments which uses the permutation group symmetry. The second theorem is in principle much more general, and says that assuming a certain conjecture in non-archimedean geometry, the SYZ conjecture will follow in large generality. Very roughly, there is a non-archimedean object called the Berkovich space which can be associated to a degeneration family, and there is a version of the Calabi conjecture on this Berkovich space, whose formulation involves only algebraic concepts such as intersection numbers and nef line bundles. This non-archimedean Calabi conjecture has been solved by Boucksom-Favre-Jonsson, but not much is known about the properties of the solution. The non-archimedean conjecture is that this solution is not too wild.

Thus our result morally fits into the general philosophy of 'existence questions of complex geometric PDE is controlled by a purely algebraic criterion', exemplified by the Yau-Tian-Donaldson (YTD) conjecture, which puts an equivalence between Kähler-Einstein metrics and K-stability. It must be conceded that while we have reduced the SYZ conjecture to a purely algebraic question, there is still little practical means to verify it in examples. This we hope will be overcome by further developments in algebraic geometry. A little more historical comparison with the YTD conjecture may be encouraging: even after the initial breakthrough of Chen-Donaldson-Sun, it was still difficult in practice to verify the K-stability condition in examples, and it takes algebraic geometers almost a decade to gradually improve the situation.

A few words concerning the proof: the key intermediate step is that in the generic region, the Calabi-Yau metric is up to C^∞ small errors approximated by a semiflat metric. By a nontrivial result of Savin, this can be reduced to a C^0 convergence result at the level of Kähler potentials, which in turn is achieved using tools from pluripotential theory.

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Pluripotential Theory: how to get singular Kähler-Eisntein metrics

Eleonora Di Nezza

(joint work with Tamas Darvas, Chihn Lu)

We recall some notions and facts also in order to fix notations.

Let (X, ω) be a compact Kähler manifold of dimension n and fix θ a smooth closed real $(1,1)$-form. A function $u : X \to \mathbb{R} \cup \{-\infty\}$ is called quasi-plurisubharmonic if locally $u = \rho + \varphi$, where ρ is smooth and φ is a plurisubharmonic function. We say that u is θ-plurisubharmonic (θ-psh for short) if it is quasi-plurisubharmonic and $\theta_u := \theta + dd^c u \geq 0$ in the weak sense of currents on X. We let $\mathrm{PSH}(X, \theta)$ denote the space of all θ-psh functions on X. The cohomology class $\{\theta\} \in H^{1,1}(X, \mathbb{R})$ is *big* if there exists $\psi \in \mathrm{PSH}(X, \theta)$ such that $\theta + dd^c \psi \geq \varepsilon \omega$ for some $\varepsilon > 0$.

A potential $u \in \mathrm{PSH}(X, \theta)$ has *analytic singularities* if it can be written locally as $u(z) = c \log \sum_{j=1}^k |f_j(z)|^2 + h(z)$, where $c > 0$, the $f_j's$ are holomorphic functions and h is smooth. By the fundamental approximation theorem of Demailly [5], if $\{\theta\}$ is big there are plenty of θ-psh functions with analytic singularities.

Given $u, v \in \mathrm{PSH}(X, \theta)$, we say that

- u is more singular than v, i.e., $u \preceq v$, if there exists $C \in \mathbb{R}$ such that $u \leq v + C$;
- u has the same singularity as v, i.e., $u \simeq v$, if $u \preceq v$ and $v \preceq u$.

The classes $[u]$ of this latter equivalence relation are called *singularity types*.

When θ is non-Kähler, elements of $\mathrm{PSH}(X, \theta)$ can be quite singular, and we distinguish the potential with the smallest singularity type in the following manner:

$$V_\theta := \sup\{u \in \mathrm{PSH}(X, \theta) \text{ such that } u \leq 0\}.$$

A function $u \in \mathrm{PSH}(X, \theta)$ is said to have minimal singularities if it has the same singularity type as V_θ, i.e., $[u] = [V_\theta]$.

Given $\theta^1, ..., \theta^n$ closed smooth $(1,1)$-forms representing big cohomology classes and $u_j \in \mathrm{PSH}(X, \theta^j)$, $j = 1, ...n$, following the construction of Bedford-Taylor [2, 1] in the local setting, it has been shown in [3] that the sequence of positive measures

$$\mathbb{1}_{\bigcap_j \{u_j > V_{\theta^j} - k\}} \theta^1_{\max(u_1, V_{\theta^1} - k)} \wedge \ldots \wedge \theta^n_{\max(u_n, V_{\theta^n} - k)} \tag{1}$$

has total mass (uniformly) bounded from above and is non-decreasing in $k \in \mathbb{R}$, hence converges weakly as $k \to +\infty$ to the so called *non-pluripolar product*

$$\theta^1_{u_1} \wedge \ldots \wedge \theta^n_{u_n}.$$

The resulting positive measure does not charge pluripolar sets. In the particular case when $u_1 = u_2 = \ldots = u_n = u$ and $\theta^1 = ... = \theta^n = \theta$ we will call θ^n_u the non-pluripolar measure of u_i, which generalizes the usual notion of volume form in case θ_u is a smooth Kähler form. As a consequence of Bedford-Taylor theory it can be seen that the measures in (1) all have total mass less than $\int_X \theta^n_{V_\theta} := \mathrm{vol}(\theta)$, in particular, after letting $k \to \infty$ we notice that $0 \leq \int_X \theta^n_u \leq \int_X \theta^n_{V_\theta}$. In what follows we are going to consider only the case of non-vanishing mass, i.e. $\int_X \theta^n_u > 0$.

It was recently proved in [6, Theorem 1.2] (and generalized in [7, Theorem 1.1]) that for any $u, v \in \mathrm{PSH}(X,\theta)$ the following monotonocity property holds for the total masses:

$$u \preceq v \Longrightarrow \int_X \theta_u^n \leq \int_X \theta_v^n,$$

$$u \simeq v \Longrightarrow \int_X \theta_u^n = \int_X \theta_v^n.$$

It is worth noticing that the reverse implication in the latter statement is not true, meaning that there are examples of θ-psh functions not having the same singularity type but having the same mass. One can then wonder if, given $u \in \mathrm{PSH}(X,\theta)$, there exists a *least singular* potential that is less singular than u but has the same full mass as u. As we will see this is indeed the case.

In joint works with Tamas Darvas and Chinh Lu [8], we introduce the *ceiling* operator $\mathcal{C} : \mathrm{PSH}(X,\theta) \mapsto \mathrm{PSH}(X,\theta)$ defined by

$$\mathcal{C}(u) = \sup\left\{v \in \mathrm{PSH}(X,\theta) \,:\, [u] \leq [v], v \leq 0 \text{ and } \int_X \theta_u^n = \int_X \theta_v^n\right\}.$$

This function is proved to θ-psh, less singular than u and with the same mass.

We then say that a potential $\phi \in \mathrm{PSH}(X,\theta)$ is a *model* potential if $\phi = \mathcal{C}(\phi)$, i.e., if ϕ is a fixed point of $\mathcal{C}$. Similarly, the corresponding singularity types $[\phi]$ are called model type singularities.

Examples of model potentials are functions with analytic singularities. As a more specific example we have that the potential V_θ is a model potential.

Fixing a model potential $\phi \in \mathrm{PSH}(X,\theta)$, it is natural to consider the set of ϕ-relative *full mass potentials*:

$$\mathcal{E}(X,\theta,\phi) := \left\{u \in \mathrm{PSH}(X,\theta), [u] \leq [\phi] \text{ such that } \int_X \theta_u^n = \int_X \theta_\phi^n\right\}.$$

Observe that when $\phi = V_\theta$, the relative class $\mathcal{E}(X,\theta,V_\theta)$ is nothing else than the full Monge-Ampère energy class $\mathcal{E}(X,\theta)$, previously introduced by [9].

In a series of works [7, 10] with Chinh Lu and Tamas Darvas, we studied solutions to complex Monge-Ampère equations with prescribed singularity. One starts with a potential $u \in \mathrm{PSH}(X,\theta)$ and a density $0 \leq f \in L^p(X)$, $p > 1$, and looks for a solution $u \in \mathrm{PSH}(X,\theta)$ such that $\theta_u^n = f\omega^n$ and $[u] = [\phi]$. The compatibility condition $\int_X \theta_\phi^n = \int_X f\omega^n > 0$ is necessary for the probability of this equation. Beyond this normalization condition, as it turns out, the necessary and sufficient condition for the well posedness is that ϕ is a model potential. The result we achieved states as

Theorem 1 ([10])**.** *Let $\lambda \geq 0$. Assume ϕ is a model potential and that μ is a non-pluripolar positive measure on X such that $\mu(X) = \int_X \theta_\phi^n > 0$. Then there exists a unique (up to constant when $\lambda = 0$) $u \in \mathcal{E}(X,\theta,\phi)$ such that $\theta_u^n = e^{\lambda u}\mu$.*

In addition to this, in the particular case when $\mu = f\omega^n$ with $f \in L^p(X, \omega^n)$, $p > 1$ we have that

$$\phi - C\Big(\lambda, p, \omega, \int_X \theta_\phi^n, \|f\|_{L^p}\Big) \leq u \leq \phi \leq 0.$$

When θ is Kähler and $\phi = 0$, the first part of the Theorem is due to [9] while the second part reduces to Kołodziej's L^∞-estimate [11] in the context of the Calabi-Yau theorem [12]. In the general case $\{\theta\}$ a big class and $\phi = V_\theta$, then the above result was proved in [4] and [13].

This result is a significant generalization of Kołodziej's L^∞ estimate [11] to our relative context.

Solutions of complex Monge-Ampère equations are linked to existence of special Kähler metrics. In particular, we can think of the solution to $\theta_u^n = f\omega^n$ as a potential with prescribed singularity type and prescribed Ricci curvature in the philosophy of the Calabi-Yau theorem. As an immediate application of the resolution of the Monge-Ampère equation $\theta_u^n = e^{\lambda u} f\omega^n$ with prescribed singularities $[u] = [\phi]$, we obtain existence of singular *Kähler-Einstein* (KE) metrics with prescribed singularity type on Kähler manifolds of general type.

Corollary 2 ([7]). *Let X be a smooth projective manifold with canonical ample ($K_X > 0$) and let h be a smooth Hermitian metric on K_X with $\theta := \Theta(h) > 0$. Suppose also that $\phi \in \mathrm{PSH}(X, \theta)$ is a model potential, has small unbounded locus and $\int_X \theta_\phi^n > 0$. Then there exists a unique singular KE metric $he^{-\phi_{KE}}$ on K_X ($\theta^n_{\phi_{KE}} = e^{\phi_{KE} + f_\theta}\theta^n$, where f_θ is the Ricci potential of θ satisfying* Ric $\theta = \theta + dd^c f_\theta$*), with $\phi_{KE} \in \mathrm{PSH}(X, \theta)$ having the same singularity type as ϕ.*

An analogous result also holds on Calabi-Yau manifolds. For the sake of completeness, we should mention that the existence of singular Kähler-Einstein metrics with prescribed singularities on a Fano manifold is studied in [14].

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Smooth asymptotics for collapsing Calabi-Yau manifolds

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(joint work with Valentino Tosatti)

Consider a surjective holomorphic map $f : X^{m+n} \to B^m$ of compact Kähler manifolds such that the total space is Calabi-Yau, i.e., $c_1(X) = 0$ in $H^2(X, \mathbb{R})$. By adjunction, the generic fiber of f is a smooth compact Calabi-Yau manifold Y^n. The underlying diffeomorphism type of Y^n is fixed but its complex structure will usually depend on the point of B whose preimage we are considering.

For arbitrary smooth Kähler forms ω_B, ω_X on B, X, consider the 1-parameter family $\mathfrak{k}_t = [f^*\omega_B] + e^{-t}[\omega_X]$ of Kähler classes on X, where $t \in [0, \infty)$. Note that, here, e^{-t} is just generic notation for a small parameter. Let ω_t denote the unique Ricci-flat Kähler metric on X representing the class $\mathfrak{k}_t$ according to Yau's theorem [13]. It is an interesting problem to understand the asymptotics of ω_t as $t \to \infty$.

This problem has been solved completely in two special cases where ω_t can be approximated by a *gluing construction*, meaning that there exist a covering of X by open regions and more or less explicit collapsing Ricci-flat model metrics in each region that agree sufficiently well on all overlaps. Case 1 is $X = K3$, $B = \mathbb{C}P^1$ and f an elliptic fibration (i.e., the generic fiber of f is a smooth elliptic curve) with 24 singular fibers, each of which is a torus pinched along a meridian [3]. Case 2 is X a suitable Calabi-Yau 3-fold, $B = \mathbb{C}P^1$ and f a Lefschetz pencil of $K3$ surfaces, i.e., the generic fiber of f is a smooth $K3$ surface and the singular fibers are $K3$ surfaces with at worst ordinary nodes, i.e., isolated quadratic (Morse) singularities [9]. In both cases, the picture that emerges is that *locally uniformly away from the singular fibers*, ω_t is increasingly well approximated by a model metric of the form $f^*\omega_{\mathrm{can}} + e^{-t}\omega_{\mathrm{SRF}}$. Here ω_{can} is a certain canonical Kähler metric on B with mild singularities along the singular values of f and with nonnegative Ricci curvature. Indeed, $\mathrm{Ric}(\omega_{\mathrm{can}})$ is equal to the Weil-Petersson Kähler form pulled back from the moduli space of polarized Calabi-Yau structures on Y. On the other hand, ω_{SRF} is a so-called semi-Ricci-flat form, i.e., a closed $(1,1)$-form on X that restricts to the unique Ricci-flat Kähler metric representing the class $[\omega_X|_Y]$ on each regular fiber Y. Life is far more complicated *near the singular fibers*. In particular, near the nodes of the singular fibers, Taub-NUT metrics [7] bubble off in Case 1 and one of Yang Li's 3-dimensional analogs of the Taub-NUT metric [8] in Case 2.

Over the last 10 years or so, a lot of effort has been invested into determining the asymptotic behavior of ω_t *locally uniformly away from the singular fibers* without making any particular assumptions about the structure of the singular fibers, in particular, *without using any gluing models near the singular fibers* (which would be unavailable in all but a few special cases such as the ones mentioned above).

That this may be possible was first demonstrated in [10], where by using Yau's estimates it was proved that ω_t converges to $f^*\omega_{\mathrm{can}}$ weakly as currents on X and that locally uniformly away from the singular fibers, ω_t is uniformly equivalent to $f^*\omega_{\mathrm{can}} + e^{-t}\omega_{\mathrm{SRF}}$ in the sense of eigenvalues of metric tensors. From that point on it has always seemed plausible that the problem can be treated as some kind of higher-order interior regularity problem (local on the base, global on the fibers) for the degenerate complex Monge-Ampère equation that governs the behavior of ω_t. Still using Yau's estimates, the results of [10] were improved to C^0_{loc} convergence $\omega_t \to f^*\omega_{\mathrm{can}}$ away from the singular fibers in [11] and to C^∞_{loc} convergence if the generic fiber is a quotient of a complex torus in [2]. In [12] it was also proved that the latter case is the only one where the sectional curvature of ω_t remains locally uniformly bounded. In our previous work [5], using the Liouville theorem of [4], we introduced a new nested blowup-and-contradiction method. This method allowed us to prove that $\omega_t \to f^*\omega_{\mathrm{can}}$ in $C^{0,\alpha}_{\mathrm{loc}}$ in general and in C^∞_{loc} if the regular fibers are biholomorphic to each other (but not biholomorphic to quotients of tori). The latter two statements seem to be beyond the reach of Yau's estimates.

The difficulty with all of these results is that ordinary $C^{k,\alpha}$ norms (for instance in local coordinates) are not well-adapted to the problem because the ellipticity of the PDE blows up like e^t in the fiber directions. It is more reasonable to introduce weighted $C^{k,\alpha}$ norms, where each fiber derivative is penalized by a weight of $e^{t/2}$. However, this idea raises several problems. First, these weighted $C^{k,\alpha}$ norms are very sensitive to coordinate changes as soon as $k + \alpha > 0$, i.e., they do not remain uniformly equivalent to their counterparts in different coordinate systems. Second, even with a lot of fine-tuning of the norms it has turned out to be impossible to go beyond weighted $C^{1,\alpha}$ estimates in general (unpublished), although in the isotrivial (i.e., all regular fibers are pairwise isomorphic) or torus fibered cases the results of [2, 5] do establish uniform weighted C^∞ estimates. Third, even in the latter two cases, it is not clear whether these estimates answer the original question.

The third problem was addressed in [1] in the torus fibered case, where using a fiberwise interpolation argument it was proved that the weighted C^∞ estimates of [2] imply that $\omega_t - (f^*\omega_{\mathrm{can}} + e^{-t}\omega_{\mathrm{SRF}})$ decays faster than any finite power of the small parameter e^{-t} in all $C^{k,\alpha}$ norms (weighted or unweighted), up to terms pulled back from the base that converge to zero at an unspecified rate. Using this idea, it is then also easy to show that the weighted C^∞ estimates of [5] imply the same statement in the isotrivial case. This finally led us to realize that the correct statement in general should be that $\omega_t - (f^*\omega_{\mathrm{can}} + e^{-t}\omega_{\mathrm{SRF}})$ admits an expansion according to finite powers of e^{-t} whose coefficient functions are determined by the local complex geometry of the fibration. Thus, in [6], we prove the following by a very technical refinement of our blowup method from [5].

Theorem (simplified). *Write* $\omega_t - (f^*\omega_{\mathrm{can}} + e^{-t}\omega_{\mathrm{SRF}}) = i\partial\bar{\partial}\psi_t$ *and decompose* ψ_t *into its fiberwise average,* $\underline{\psi_t}$*, with respect to the fiberwise Calabi-Yau volume form and the rest. Then for all* k *and* α*, over all sufficiently small balls* $\mathcal{B}$ *contained in the regular values of the fibration,* $\underline{\psi_t}$ *goes to zero in* $C^{k+2,\alpha}$*. Moreover, we can decompose* $\psi_t - \underline{\psi_t} = \psi_t' + \psi_t''$*, where the "obstruction part"* ψ_t' *lies in a fixed finite-rank* $C^\infty(\mathcal{B})$*-submodule of* $C^\infty(f^{-1}(\mathcal{B}))$ *determined by the local complex geometry of the fibration over* $\mathcal{B}$ *and by* k *and* α*, and where the "remainder"* ψ_t'' *is uniformly bounded in weighted* $C^{k+2,\alpha}$*. The obstruction part* ψ_t' *is only bounded in unweighted* $C^{k+2,\alpha}$*. However, both of these bounds depend only on the constants in the uniform* C^0 *equivalence* $\omega_t \sim f^*\omega_{\mathrm{can}} + e^{-t}\omega_{\mathrm{SRF}}$ *over any slightly larger ball* $\mathcal{B}' \supset \mathcal{B}$*.*

As a corollary, ω_t converges to $f^*\omega_{\mathrm{can}}$ in unweighted C^∞ locally uniformly away from the singular fibers. In addition, we also always have a uniform weighted $C^{1,\alpha}$ bound on $\omega_t - (f^*\omega_{\mathrm{can}} + e^{-t}\omega_{\mathrm{SRF}})$ for every $\alpha < 1$, but precisely at the scale of weighted C^2 there is a contribution

$$e^{-2t} i\partial\bar{\partial}\Delta_Y^{-2} g_{\mathrm{can}}^{\bar{\mu}\nu}(\langle A_\mu, A_{\bar{\nu}}\rangle - \underline{\langle A_\mu, A_{\bar{\nu}}\rangle})$$

to the obstruction part, which is uniformly bounded in unweighted $C^{k,\alpha}$ for all k, α but not in weighted $C^{2,\alpha}$ for any $\alpha > 0$. Here Δ_Y denotes the fiberwise Laplacian with respect to the unique fiberwise Calabi-Yau metric in the class $[\omega_X|_Y]$ and A_μ denotes the Kodaira-Spencer form in direction ∂_{z^μ} on Y (harmonic with respect to the fiberwise Calabi-Yau metric). The latter vanishes in the isotrivially fibered case. In the torus fibered case it is parallel, so that $\langle A_\mu, A_{\bar{\nu}}\rangle = \underline{\langle A_\mu, A_{\bar{\nu}}\rangle}$.

Simply for the sake of proving an expansion there probably exist easier and more direct approaches, but we discovered the correct statement by trying to understand and repairing the failure of our uniform weighted $C^{k,\alpha}$ estimates for $k \geq 2$, and this approach has a useful consequence: the expansion is purely local on the base, and the pieces of the expansion are uniformly controlled once there is uniform control on the constants in the C^0 equivalence $\omega_t \sim f^*\omega_{\mathrm{can}} + e^{-t}\omega_{\mathrm{SRF}}$. In our setting, this overcomes the notorious problem of controlling an asymptotic expansion uniformly in terms of additional parameters that the solution may depend on.

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Steady gradient Kahler-Ricci solitons on crepant resolutions of Calabi-Yau cones

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(joint work with Alix Deruelle)

A *steady gradient Kahler-Ricci soliton* is a triple (M, ω, X), where (M, ω) is a complete Kahler manifold with Kahler form ω and X is a complete real holomorphic vector field on M equal to $\nabla^g f$ for some smooth real-valued function $f : M \to \mathbb{R}$, such that the Ricci form ρ_ω of ω satisfies

$$\rho_\omega = \frac{1}{2}\mathcal{L}_X \omega. \tag{1}$$

The vector field X is called the *soliton vector field.* There are two points of view that one can take when considering steady gradient Kahler-Ricci solitons: the static and the dynamic. From the static point of view, equation (1) provides a natural generalisation of the Calabi-Yau condition, namely $\rho_\omega = 0$. From the dynamic point of view, steady gradient Kahler-Ricci solitons give rise to eternal solutions of the Kahler-Ricci flow evolving only by pullback by biholomorphisms. Indeed, given (M, ω, X) as above, let φ_t be the one-parameter family of biholomorphisms of M generated by the vector field $-\frac{X}{2}$ with $\varphi_0 = \mathrm{Id}$. Then $\omega_t := \varphi_t^*\omega$, $t \in (-\infty, \infty)$, defines such a solution. These particular solutions are important in that they may arise as singularity models of the flow.

Steady gradient Kahler-Ricci solitons that are not Calabi-Yau are necessarily non-compact [8]. Examples include Hamilton's cigar soliton [7] on $\mathbb{C}$ which was generalised by Cao [3] to $\mathbb{C}^n$ and $K_{\mathbb{P}^n}$. Further generalisations were then obtained by Dancer-Wang [6], Yang [13], and more recently by Schafer [10]. All examples mentioned thus far are highly symmetric and were constructed by solving an ODE. In [2], Biquard-Macbeth implement a gluing method to construct examples of complete steady gradient Kahler-Ricci solitons in small Kahler classes of an equivariant crepant resolution of $\mathbb{C}^n/\Gamma$, where Γ is a finite subgroup of $SU(n)$

acting freely on $\mathbb{C}^n \setminus \{0\}$. Our main result is the construction of a complete steady gradient Kahler-Ricci soliton in *every* Kahler class of a crepant resolution of a Calabi-Yau cone, unique up to the flow of the soliton vector field, converging at a polynomial rate to Cao's steady gradient Kahler-Ricci soliton on the cone.

Cao's construction of a steady gradient Kahler-Ricci soliton on $\mathbb{C}^n$ [3] allows for an ansatz to construct a one-parameter family of incomplete steady gradient Kahler-Ricci solitons $\tilde{\omega}_a$, $a \geq 0$, on any Calabi-Yau cone $(C_0,\, g_0)$. With this in mind, our main result can be stated as follows.

Theorem 1 (C.-Deruelle [4, Theorem A]). *Let $(C_0,\, g_0)$ be a Calabi-Yau cone of complex dimension $n \geq 2$ with complex structure J_0, Calabi-Yau cone metric g_0, radial function r, and trivial canonical bundle. Let $\pi : M \to C_0$ be a crepant resolution of C_0 with complex structure J such that the real holomorphic torus action on C_0 generated by $J_0 r\partial_r$ extends to M so that the holomorphic vector field $2r\partial_r$ on C_0 lifts to a real holomorphic vector field $X = \pi^*(2r\partial_r)$ on M. Set $t := \log(r^2)$ and define the Kahler form $\hat{\omega} := \frac{i}{2}\partial\bar{\partial}\left(\frac{nt^2}{2}\right)$ on C_0.*

Then in every Kahler class $\mathfrak{k}$ of M, up to the flow of X, there exists a unique complete steady gradient Kahler-Ricci soliton $\omega \in \mathfrak{k}$ with soliton vector field X and with $\mathcal{L}_{JX}\omega = 0$ such that for all $\varepsilon \in (0,\, 1)$, there exist constants $C(i,\, j,\, \varepsilon) > 0$ such that

$$(2) \qquad |\widehat{\nabla}^i \mathcal{L}_X^{(j)}(\pi_*\omega - \hat{\omega})|_{\hat{g}} \leq C(i,\, j,\, \varepsilon)t^{-\varepsilon - \frac{i}{2} - j} \quad \text{for all } i,\, j \in \mathbb{N}_0,$$

where $\hat{g}$ denotes the Kahler metric associated to $\hat{\omega}$ and $\widehat{\nabla}$ is the corresponding Levi-Civita connection. More precisely, for all $\varepsilon \in \left(0,\, \frac{1}{2}\right)$ and for all $a \geq 0$, there exist constants $C(i,\, j,\, \varepsilon,\, a) > 0$ such that

$$(3) \qquad |\widehat{\nabla}^i \mathcal{L}_X^{(j)}(\pi_*\omega - \tilde{\omega}_a - \hat{\zeta})|_{\hat{g}} \leq C(i,\, j,\, \varepsilon,\, a)t^{-2+\varepsilon - \frac{i}{2} - j} \quad \text{for all } i,\, j \in \mathbb{N}_0,$$

where $\hat{\zeta}$ is a real $(1,\, 1)$-form uniquely determined by $\mathfrak{k}$ that is invariant under the flow of X and JX, and $\tilde{\omega}_a$, $a \geq 0$, denotes Cao's family of incomplete steady gradient Kahler-Ricci solitons on C_0. If $\mathfrak{k}$ is compactly supported or if $n = 2$, then for all $a \geq 0$, there exists a smooth real-valued function $\varphi : M \to \mathbb{R}$ and constants $C(i,\, j,\, a) > 0$ such that for all $i,\, j \in \mathbb{N}_0$,

$$(4) \qquad \omega - \tilde{\omega}_a = i\partial\bar{\partial}\varphi, \qquad \text{where} \qquad |\widehat{\nabla}^i \mathcal{L}_X^{(j)}(t^{-n+1}e^{nt}\varphi)|_{\hat{g}} \leq C(i,\, j,\, a)t^{-\frac{i}{2} - j}.$$

A resolution for which the torus action on the cone extends to the resolution is called *equivariant*. Such a resolution of a complex cone always exists (cf. [9, Proposition 3.9.1]). However, for Calabi-Yau cones, it may not necessarily be crepant. On the other hand, if a crepant resolution of a Calabi-Yau cone is unique, then it is necessarily equivariant; apply the proof of [5, Lemma 2.13] to see this. Moreover, the steady solitons of Theorem 1 display so-called "cigar-paraboloid" asymptotics. Most notably, the volume of a ball of radius R in M grows at rate $O(R^{\frac{1}{2}\dim_{\mathbb{R}} M})$ and the curvature decays linearly. Furthermore, Cao's steady gradient Kahler-Ricci soliton $\tilde{\omega}_a$ on C_0 also converges to $\hat{\omega}$ at infinity, yielding the following more refined asymptotics:

$$|\widehat{\nabla}^i \mathcal{L}_X^{(j)}(\pi_*\omega - \hat{\omega})|_{\hat{g}} \leq \begin{cases} C(i,j)t^{-1-\frac{i}{2}-j}\log(t) & \text{if } j = 0, \\ C(i,j)t^{-1-\frac{i}{2}-j} & \text{if } j \geq 1. \end{cases}$$

Finally, the steady solitons of Theorem 1 are κ-collapsed, hence they cannot appear as blowup models of finite-time singularities of the Kahler-Ricci flow.

To prove Theorem 1, we first construct a background Kahler metric on M that is asymptotic at a polynomial rate to the steady gradient Kahler-Ricci soliton on the cone given by the ansatz of Cao. This metric serves as an "approximate" steady gradient Kahler-Ricci soliton on M. We then perturb this metric to an actual steady gradient Kahler-Ricci soliton by solving a complex Monge-Ampere equation with polynomially decaying data. This involves two steps. First, we solve the complex Monge-Ampere equation for compactly supported data by implementing the continuity method as in the seminal work of Aubin [1] and Yau [14] on the existence of Kahler-Einstein metrics on compact Kahler manifolds, although we work with functions that, together with their derivatives, decay exponentially at infinity in order to compensate for the non-compactness of M. The main difficulty in this step is obtaining an a priori C^0-estimate in the closedness part of the continuity method. To do this, we introduce, in line with Tian-Zhu [12] and their work on the uniqueness of shrinking gradient Kahler-Ricci solitons on compact Kahler manifolds, the I- and J-functionals on the space of Kahler potentials. Thanks to the exponential decay, these functionals are well-defined on the function spaces with which we work in this step. By considering their difference, we obtain an a priori weighted energy estimate. A localisation result for points where the global maximum and global minimum values of solutions along the continuity path occur then allows us to derive the desired a priori C^0-estimate. The second step involves an application of the implicit function theorem to solve the initial complex Monge-Ampere equation with polynomially decaying data by reducing to the compactly supported case. These ideas were subsequently adapted to the asymptotically cylindrical setting by Schafer [11].

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A stability result for Einstein metrics of Tian

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(joint work with Ursula Hamenstädt)

It is an easy and well known consequence of the convergence Theory for Riemannian manifolds that for all $n \geq 2, \lambda \in \mathbb{R}, D, i_0 > 0$ there exists $\varepsilon = \varepsilon(n, \lambda, D, i_0)$ such that any closed manifold M that admits a Riemannian metric g with $diam(M, g) \leq D$, $inj(M, g) \geq i_0$ and $|Ric(g) - \lambda g| \leq \epsilon$ also admits an Einstein metric with constant λ. We present an unpublished theorem of Tian that generalises the above result by dropping the upper diameter bound, but in addition assumes a certain L^2-bound on $Ric(g) + (n-1)g$. We then dicuss how to further generalise this result when the injectivity radius bound is also dropped.

Intermediate Ricci, homotopy, and submanifolds of symmetric spaces

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(joint work with Manuel Amann, Peter Quast)

In the spirit of combining Riemannian geometry, topology and algebra in the area of symmetric spaces we draw on the "generalized connectedness lemma" by Guijarro–Wilhelm (see [3], Theorem B), inspired by the following proposition, in order to suggest a new approach to the study of certain classes of submanifolds of symmetric spaces genuinely containing totally geodesic ones.

Proposition 1. [1, Proposition 0.1] Let P be a symmetric space of compact type. Then there exists a positive integer k_P such that for all integers k with $\dim P > k \geq k_P$ it holds that $\mathrm{Ric}_k > 0$.

Theorem 1 presents the details of this approach, but before stating the theorem, let us explain two points.

First, we need to recall the "Cartan type". We say that two classical irreducible symmetric spaces of compact type have the *same Cartan type* if they have a common *Cartan symbol* (CS) as given in Table 1.

TABLE 1. Cartan symbols of classical irreducible symmetric spaces of compact type described by infinitesimal symmetric pairs $(\mathfrak{g}, \mathfrak{k})$, see [4, Chap. X, §6]

CS	$\mathfrak{g}$	$\mathfrak{k}$	Conditions
A	$\mathfrak{su}_n \times \mathfrak{su}_n$	$\Delta\mathfrak{su}_n$	$n \geq 2$
A I	$\mathfrak{su}_n$	$\mathfrak{o}_n$	$n \geq 2$
A II	$\mathfrak{su}_{2n}$	$\mathfrak{sp}_n$	$n \geq 2$
A III	$\mathfrak{su}_{p+q}$	$\mathfrak{s}(\mathfrak{u}_p \oplus \mathfrak{u}_q)$	$1 \leq p \leq q$
BD	$\mathfrak{o}_n \times \mathfrak{o}_n$	$\Delta\mathfrak{o}_n$	$n \geq 3,\ n \neq 4$
BD I	$\mathfrak{o}_{p+q}$	$\mathfrak{o}_p \oplus \mathfrak{o}_q$	$1 \leq p \leq q,\ (p,q) \notin \{(1,1),\ (2,2)\}$
C	$\mathfrak{sp}_n \times \mathfrak{sp}_n$	$\Delta\mathfrak{sp}_n$	$n \geq 1$
C I	$\mathfrak{sp}_n$	$\mathfrak{u}_n$	$n \geq 1$
C II	$\mathfrak{sp}_{p+q}$	$\mathfrak{sp}_p \oplus \mathfrak{sp}_q$	$1 \leq p \leq q$
D III	$\mathfrak{o}_{4n}$	$\mathfrak{u}_{2n}$	$n \geq 2$

Second, Theorem 1 works with different notions of equivalence. In order to avoid a repetition and trying to merge these notions into one statement, we first fix a notion of equivalence before stating the theorem. Accordingly, we state that "Q is isomorphic to a symmetric space of compact type" if it is either homotopy equivalent, homeomorphic, diffeomorphic, or isometric (by abuse of notation, of course, the latter notion does incorporate the possibility of applying different scaling factors on the different de Rham factors of a symmetric space) to a symmetric space of compact type. Having chosen such a notion of equivalence, this determines the term "isomorphic" throughout the theorem.

Note further that in the assertion of the theorem foc_Q is the focal radius of Q, S_v is the shape operator of Q corresponding to a unit vector v normal to Q and the quantities k_P and C_P are the respective ones from Table 3 in [1] for higher rank spaces and respectively 1 and $(n-9)/2$ for the rank one spaces.

Theorem 1. Let Q be a compact connected embedded proper submanifold of an irreducible simply-connected compact classical symmetric space P with $\mathrm{Ric}_{k_P} \geq \delta$, for some $\delta > 0$. Assume further that Q is isomorphic to a symmetric space of compact type and satisfies the following condition: There exists some $r \in [0, \pi/2)$ with

$$\mathrm{foc}_Q > r$$

such that for every $x \in Q$, every unit normal vector $v \in T_xQ^\perp$, and any k_P-dimensional subspace W of T_xQ we have that

$$|\mathrm{trace}(\mathrm{S}_v\ |_W)| \leq \sqrt{\frac{\delta}{k_P}} k_P \cot\left(\pi/2 - \sqrt{\frac{\delta}{k_P}} r\right).$$

Then if codim $Q \leq C_P$, one of the following cases occurs:

1. If P is isometric to a sphere, then Q is isomorphic to a product of spheres whose dimensions are at least 10.
2. If $P \cong \frac{\mathbf{SO}(2+q)}{\mathbf{SO}(2)\times\mathbf{SO}(q)}$, $q \geq 10$, then Q is isomorphic to a symmetric space $\frac{\mathbf{SO}(2+q')}{\mathbf{SO}(2)\times\mathbf{SO}(q')}$, $q' < q$, possibly up to products with spheres of dimensions at least 10, or Q is isomorphic to a complex projective space $\mathbb{CP}^n$, $n \geq 5$, possibly up to products with spheres of dimensions at least 10.

   Similarly, if P^n is isometric to a complex projective space with $n \geq 11$, then Q is isomorphic to a complex projective space $\mathbb{CP}^r$, $r \geq 5$, possibly up to products with spheres of dimensions at least 10, or Q is isomorphic to a Grassmannian manifold $\frac{\mathbf{SO}(2+q)}{\mathbf{SO}(2)\times\mathbf{SO}(q)}$ with $q \geq 10$, possibly up to products with spheres of dimensions at least 10.
3. If P is isometric to a Grassmannian manifold (other than those appearing in Items 1 and 2), then Q is isomorphic to a symmetric space with the same Cartan type as P, possibly up to products with spheres of dimensions at least 10.
4. If P is none of the above symmetric spaces, then Q is isomorphic to a necessarily reducible symmetric space of the form $Q_1 \times S^{l_1} \times \ldots \times S^{l_r}$, where Q_1 has the same Cartan type as P and $l_i \geq 10$, for $1 \leq i \leq r$.

Remark 2. Note that if $\dim P$ is not "large" enough, then $C_P < 1$. Thus there is no (proper) submanifold Q of P with codim $Q \leq C_P < 1$. For example, for a compact rank one symmetric space P, if $\dim P \leq 10$, then $C_P < 1$. For this reason, we say that $\dim P$ is a *valid* dimension if $C_P \geq 1$. For instance the valid dimension of a compact rank one symmetric space is 11.

In Theorem 1, we implicitly assume that dim P is a valid dimension.

Our approach for proving Theorem 1 comprises three major steps.

(1) First we present a uniform method of how to compute the smallest k for which the symmetric space P has k-positive Ricci curvature. This will follow from a detailed analysis of associated root systems and isotropy orbits.
(2) Next, we use this information as one key ingredient in the generalized connectedness lemma by Guijarro–Wilhelm [3, Theorem B]. As an outcome in the respective cases we gain control on the degree of connectedness of the embedding, and we arrange codimensions in such a way that the map is 10-connected.
(3) In particular, this implies that the first 9 homotopy groups of the ambient manifold and the submanifold have to coincide. We hence collect and compute nearly all such homotopy groups for *all* irreducible simply-connected symmetric spaces, and compare them with the ones of the ambient space. This lets us conclude that Cartan types of "almost all" cases necessarily agree in the codimension ranges we consider.

As already announced this new generalized approach applying to more general submanifolds even has previously unknown consequences for totally geodesic submanifolds.

We denote by $\mathbf{Gr}(p, n)$ the simply-connected Grassmannian of oriented real p-planes in $\mathbb{R}^n$ (here by a slight abuse of notation), or complex p-planes in $\mathbb{C}^n$, or quaternionic ones in $\mathbb{H}^n$, respectively. Furthermore, C_P denotes the corresponding number from Table 3 in [1].

Theorem 2. Let $P = \mathbf{Gr}(p, n)$ be as above with $p \geq 3$, and let Q be a complete totally geodesic embedded submanifold of P which satisfies codim $Q \leq C_P$. Then Q has the same Cartan type as P.

Remark 3. It is interesting to note that restricting to the category of totally geodesic submanifolds in most cases our approach allows us to reprove and to actually *even improve* the estimate

$$\operatorname{ind} P \geq \operatorname{rk} P$$

stating that the codimension of a totally geodesic submanifold of P is at least as large as its rank, where $\operatorname{ind} P$ is the the lowest codimension of totally geodesic submanifids of P [2, Theorem 1.1].

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On positive surface group representations in $\mathbf{SO}(p, q)$

Maria Beatrice Pozzetti

(joint work with Jonas Beyrer)

Let Γ denote the fundamental group of a topological surface S of genus $g \geq 2$, and G denote a semisimple Lie group of noncompact type, such as $\mathsf{G} = \mathrm{PO}(p, q)$. We consider points in the character variety $\Xi(\Gamma, \mathsf{G}) := \mathrm{Hom}(\Gamma, \mathsf{G})/\!\!/\mathsf{G}$ as conjugacy classes of actions of Γ on the associated symmetric space $\mathcal{X} := \mathsf{G}/\mathsf{K}$, a manifold of non-positive curvature.

In this context is natural to restrict the attention to the points corresponding to conjugacy classes of injective representations $\rho : \Gamma \to \mathsf{G}$ with discrete image, as those correspond to properly discontinuous actions, and thus give rise to interesting classes of infinite volume locally symmetric spaces.

Question 1. Are there connected components of $\Xi(\Gamma, \mathsf{G})$ only consisting of injective representations with discrete image?

It is well known that this is the case for the group $\mathsf{G} = \mathrm{PSL}(2, \mathbb{R})$: indeed the Teichmüller space, regarded as the parameter space of marked hyperbolic structures on S, identifies with a connected components of $\Xi(\Gamma, \mathsf{G})$ through the holonomy representation. On the other hand, this can never be the case for $\mathsf{G} = \mathrm{PSL}(2, \mathbb{C})$ as the corresponding character variety is connected and contains the trivial representation, which is clearly not injective. Despite one might naively think that this can only be the case if $\mathsf{G} = \mathrm{PSL}(2, \mathbb{R})$, and thus the action can be cocompact, some other connected components with this property have been discovered: Hitchin and maximal representation. It is by now customary to call connected components of character varieties only consisting of conjugacy classes of injective homomorphisms with discrete image *Higher rank Teichmüller spaces*.

A recent breakthrough in the field was given by Guichard-Wienhard, who introduced the notion of Θ-*positive structure* [5]. Loosely speaking a semisimple Lie group G admits a Θ-positive structure if there exists a parabolic subgroup P_Θ for which it is possible to define a good notion of positivity for pairwise transverse triples in the flag manifold $\mathsf{G}/\mathsf{P}_\Theta$, which generalize the notion of positively oriented triples of elements in $\mathbb{S}^1 := \mathrm{PSL}(2, \mathbb{R})/\mathsf{P}$. With this notion at hand they formulated the following:

Conjecture 2 ([5]). *There exist Higher rank Teichmüller spaces if and only if the group* G *admits a* Θ*-positive structure.*

An example of a group G admitting a Θ-positive structure is $\mathrm{PO}(p, q)$, where the relevant flag manifold $\mathsf{G}/\mathsf{P}_\Theta = \mathcal{F}_{1,\ldots,p-1}$ consists of flags $V_1 < \ldots < V_{p-1}$ of isotropic subspaces V_i of dimension $1 \leq i \leq p-1$.

Theorem 3 ([2, 4]). *Conjecture 2 holds true for* $\mathsf{G} = \mathsf{PSO}(p, q)$.

Observe that Guichard-Labourie-Wienhard don't restrict themselves to the group $\mathsf{PSO}(p, q)$ and prove, more generally in [4], the existence of higher rank Teichmüller components for al groups G admitting a Θ-positive structure.

The goal of the talk is to give an idea of the proof of Theorem 3 presented in [2]. We say that a representation $\rho : \Gamma \to \mathrm{PSO}(p, q)$ is Θ-positive Anosov if it admits a continuous dynamics preserving transverse map $\xi : \mathbb{S}^1 \to \mathcal{F}_{1,\ldots,p-1}$ that maps positively oriented triples to positive triples. The proof of Theorem 3 presented in [2] shows, more specifically, that Θ-positive Anosov representations form connected components of the character variety. This, in turn, is obtained as a consequence of two geometric properties of Θ positive representations of independent interest, which we now describe.

The first result is a weight vs root collar lemma. To this aim we say that two elemets $g, h \in \Gamma$ are *linked* if their geodesic representative intersect in one (and thus any) hyperbolic structure on S. If we denote by $\lambda_1(\rho(g)), \ldots, \lambda_{p+q}(\rho(g))$ the generalized eigenvalues of the matrix $\rho(g)$ counted with multiplicity, and ordered so that their absolute values decrease, the *k-th root* $\alpha_k(\rho(g))$ is given by

$\log(\lambda_k/\lambda_{k+1}(\rho(g)))$ while its *k-th weight* $\omega_k(\rho(g))$ is given by $2\log(\lambda_1 \dots \lambda_k(\rho(g)))$. We then have

Theorem 4. *Let $\rho : \Gamma_g \to \mathrm{PO}(p,q)$ be Θ-positive Anosov. For any linked $g, h \in \Gamma$ it holds*

$$\left(e^{\alpha_k(\rho(g))} - 1\right)\left(e^{\omega_k(\rho(h))} - 1\right) \geq 1.$$

In particular this ensures that any limit ρ_∞ of a sequence of Θ-positive Anosov representations is k-proximal for any $k \leq p-1$, namely has eigenvalues gaps in this range.

In Theorem 4, the k-th weight $\omega_k(\rho(h))$ can be reinterpreted as the translation length of the element $\rho(h)$ on the symmetric space $\mathcal{X}$ endowed with an appropriate Finsler distance. The second geometric result shows some positivity of such length functions, namely the existence of a geodesic current μ_ρ^k such that $\omega_k(\rho(h))$ can be computed as intersection with μ_ρ^k. Thanks to results of [6] this is a consequence of the following positivity of the pullback of the natural crossratio cr_k defined on 4-tuples in $\mathrm{Is}_k(\mathbb{R}^{p,q})$. For this we denote by $\xi^k : \mathbb{S}^1 \to \mathrm{Is}_k(\mathbb{R}^{p,q})$ the maps induced by ξ, so that $\xi(x) = (\xi^1(x), \dots, \xi^{p-1}(x))$.

Theorem 5. *Let $\rho : \Gamma \to \mathrm{PO}(p,q)$ be Θ positive Anosov, and $(a,b,c,d) \in (\mathbb{S}^1)^4$ be positively oriented. Then*

$$\mathrm{cr}_k(\xi^k(a), \xi^k(b), \xi^k(c), \xi^k(d)) > 1.$$

Having Theorem 4 and 5 at hand we can apply [1, Theorem B] and construct transverse boundary maps for a limit representation, which we can then prove being strongly dynamics preserving.

In turn a key ingredient in the proofs of Theorem 4 and 5 is the following property of the boundary maps, proven in [7].

Theorem 6. *Let $\rho : \Gamma \to \mathrm{PO}(p,q)$ be Θ-positive Anosov, then*

(1) $\xi^k(\mathbb{S}^1)$ is a C^1 submanifold of $\mathrm{Is}_k(\mathbb{R}^{p,q})$ tangent to $\xi^{k+1}(x)/\xi^{k-1}(x)$ at $\xi^k(x)$ for all $1 \leq k \leq p-2$.

(2) $\xi^{p-1}(\mathbb{S}^1)$ is a Lipschitz submanifold of $\mathrm{Is}_{p-1}(\mathbb{R}^{p,q})$.

In order to prove an infinitesimal version of Theorem 5 (2) we need, among other things, to establish a good control of the almost everywhere defined derivative in case (2), analogue to case (1).

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Projective manifolds, hyperbolic manifolds and the Hessian of Hausdorff dimension

Andrés Sambarino
(joint work with M. Bridgeman, B. Pozzetti and A. Wienhard)

Let Γ be the fundamental group of a closed (real) hyperbolic n-manifold M. We study the second variation of the Hausdorff dimension of the limit set of convex co-compact morphisms acting on the complex-hyperbolic space $\rho : \Gamma \to Isom(\mathbb{H}^n_{\mathbb{C}})$, obtained by deforming a discrete and faithful representation of Γ that preserves a totally geodesic (and totally real) copy of the real-hyperbolic space $\mathbb{H}^n_{\mathbb{R}} \subset \mathbb{H}^n_{\mathbb{C}}$. This computation is based on the study of the space of convex projective structures on M and a natural metric on it induced by the Pressure form.

Non-uniqueness of minimal surfaces in Hermitian Lie groups

Vladimir Marković

Denote by Σ_g a surface of genus $g \geq 2$. Let G be a Lie group and consider an Anosov representation

$$\rho : \pi_1(\Sigma_g) \to G.$$

The main examples are:

(1) G is split and ρ is Hitchin,
(2) G is Hermitian and ρ is (Toledo) maximal.

Denote by $\mathbf{T}_g$ the Teichmüller space of marked complex structures on Σ_g. For each $S \in \mathbf{T}_g$ there exists a unique harmonic map $h : S \to G/K$ induced by ρ. Consider the associated energy functional

$$E_\rho : \mathbf{T}_g \to (0, \infty),$$

given as the energy of h. The work of Schoen-Yau implies that E_ρ is proper on $\mathbf{T}_g$. Thus, E_ρ achieves its global minimum and therefore it has at least one stationary point (minimal surface).

Conjecture 1. *The energy functional E_ρ has a unique stationary point (and thus a unique minimal surface).*

Schoen (applying the work of Micallef-Wolfson) proved the conjecture when $G = \mathrm{PSL}\mathbb{R} \times \mathrm{PSL}\mathbb{R}$. The conjecture was then proved for all rank 2 groups by Labourie (the real split case), and Collier-Tholozan-Toulisse (the Hermitian case). We disprove the Hermitian case [1] of the Uniqueness conjecture when the rank of G is three.

Theorem 2. *Let G be a Hermitian group of rank 3. For every large enough $g \geq 2$, there exists a maximal representation $\rho : \pi_1(\Sigma_g) \to G$ such that $E_\rho : \mathbf{T}_g \to (0, \infty)$ has at least two stationary points.*

It suffices to prove the theorem in the special case

$$G = \prod_{i=1}^{3} \mathrm{PSL}\mathbb{R}.$$

Theorem 3. *For every large enough $g \geq 2$, there exists a Fuchsian representation $\rho : \pi_1(\Sigma_g) \to \prod_{i=1}^{3} \mathrm{PSL}\mathbb{R}$ such that $E_\rho : \mathbf{T}_g \to (0, \infty)$ has at least two stationary points.*

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Ancient solutions in geometric flows

Natasa Sesum

Ancient non-collapsed solutions to mean curvature flow. Unlike in the case of ancient solutions to the curve shortening flow and Ricci flow on surfaces, ancient solutions in the three-dimensional Ricci flow or ancient solutions to the mean curvature flow are not given by explicit formula, which makes their classification even more challenging.

We say that $M_t = F(M^n, t)$, where $F(\cdot, t) : M^n \to \mathbb{R}^{n+1}$ is a solution to the *mean curvature flow equation* if

$$\frac{\partial}{\partial t} F = -H\, \nu,$$

where H is the mean curvature of M_t and ν is the outward unit normal vector.

We have already seen that G. Huisken in [12] showed that if $M_0 \subset \mathbb{R}^{n+1}$ is a closed convex embedded hypersurface, the mean curvature flow starting at M_0 converges to a round point. In more general situations, in higher dimensions, without the convexity assumption *local singularities* may likely occur. For example, if M_0 looks like a *dumbbel*, the *neck pinches off*, that is a blowup limit around a singularity is a round shrinking cylinder. Comparison argument with shrinking spheres implies that every *compact mean curvature flow develops singularities in finite time.* It is known that at the first singular time T, the $\limsup_{t\to T} \sup_{M_t} |A|(\cdot, t) = +\infty$.

In the context of mean curvature flow there are several notions of *weak solutions*, which enables one to continue the flow through singularities, without performing surgery. Such notions of weak solutions are still missing in general in the context of Ricci flow. As we have seen earlier, one of the ways to continue the flow past the singular time in both, the Ricci flow and the mean curvature flow is the flow with surgery. This includes cutting the hypersurface along necks, gluing in caps and continuing the flow of the pieces, while the components of known geometry and topology are discarded (see [13] and [9] for results of mean curvature flow with surgery in different settings). Furthermore, similar to the Ricci flow, the study of *ancient solutions*, could potentially help with surgeries in more general settings.

In [1, 2], with Angenent and Daskalopoulos we considered ancient compact ancient solutions to the *mean curvature flow* in dimensions $n \geq 2$, i.e. solutions that are defined for $t \in (-\infty, T)$, for some $T < +\infty$.

There is a notion of *α-noncollapsed solutions* to the mean convex MCF, which is the analogue to Perelman's *κ-noncollapsing condition for the Ricci flow.* This was first introduced by W. Sheng and X.J. Wang. The results that we will mention below hold in any dimension, but for simplicity we will *focus only on the case of surfaces in $\mathbb{R}^3$.*

In recent important work, S. Brendle and K. Choi (see [6] and references within) gave the complete classification of *noncompact ancient solutions* to mean curvature flow on surfaces that are strictly convex. More precisely, they showed that *any noncompact and complete ancient solution to mean curvature flow that is strictly convex and noncollapsed is the Bowl soliton, up to scaling and ambient isometries.* The *Bowl soliton* is the unique rotationally-symmetric, strictly convex solution to mean curvature flow that *translates with unit speed.*

In [1] and [2] we focused on ancient noncollapsed closed solutions to the mean curvature flow. X.J. Wang showed in this case the backward limit as $t \to -\infty$ of the rescaled flow is either a sphere or a cylinder $\mathbb{R} \times S^2$ of radius $\sqrt{2}$.

It is known that if the backward limit is a sphere, then the ancient solution has to be a family of shrinking spheres. Hence, to classify ancient compact noncollapsed solutions one may restrict to the ones which are non-self-similar. We will refer to them as *ancient ovals.* Based on *formal matched asymptotics*, S. Angenent conjectured the existence of an ancient oval solution defined on $t \in (-\infty, T)$ for some $T < +\infty$, which as $t \to -\infty$ becomes more and more oval in the sense that it looks like a round cylinder $\mathbb{R} \times S^2$ in the middle region, and like a rotationally symmetric translating soliton (the Bowl soliton) near the tips. A variant of this conjecture had been shown by B. White in 2003, but the approximate form of these solutions was not shown. More recently, R. Haslhofer and O. Hershkovits carried out B. White's construction in more detail. We will refer to them as *White ancient ovals.*

The main classification result of the authors with S. Angenent in [2] establishes the *uniqueness of ancient ovals.* More precisely, *any ancient oval in $\mathbb{R}^3$ is equal to the White oval solution up to ambient isometries, scaling, and translations in* *time.* A similar result was shown to hold in any dimension $n \geq 3$ (see in [2] for the detailed statement). The proof has two steps: one first shows the rotational symmetry of such solutions and then one establishes the uniqueness of rotationally symmetric ancient ovals. Analyzing the *asymptotic behavior as* $t \to -\infty$ of the rescaled ancient ovals plays an important role in our proof. This was done in [1], where the precise unique asymptotics were described in each of the three regions: the parabolic region, the intermediate region, and the tip region.

The classification of κ-solutions to 3-dimensional Ricci flow. We will next discuss recent works by the authors and S. Brendle on the classification of *noncollapsed solutions to the three-dimensional Ricci flow*, which finally resolves the conjecture by G. Perelman.

We say that $(M, g(t))$ is a Ricci flow solution starting at the initial metric g_0 if it satisfies the equation

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij}, \qquad g_{ij}(\cdot, 0) = g_{0ij}(\cdot),$$

where R_{ij} is the Ricci curvature.

In a recent important work, S. Brendle [4] resolved the *classification of ancient complete noncompact κ-noncollapsed* solutions, showing that they are either the round cylinders or steady Ricci solitons. After providing the classification of those solutions under the assumption of rotational symmetry, he shows that *any three dimensional κ-noncollapsed noncompact ancient Ricci flow solution has to be rotationally symmetric.*

Regarding the classification of *closed κ-noncollapsed ancient solutions*, we have recently modified Brendle's arguments from [4] to show that such solutions must be *rotationally symmetric.* What remained for the resolution of Perelman's conjecture was the classification of rotationally symmetric closed solutions. It turned out that such result could be approached using the techniques that the authors have developed in the case of the mean curvature flow, and has been resolved in a series of papers, [3] and [7]. All these lead to the *complete classification of closed ancient κ-noncollapsed solutions to the three-dimensional Ricci flow,* as envisioned by G. Perelman.

Our uniqueness result heavily relies on analyzing the limits, as $t \to -\infty$, of any given solution. We show that such a limit has to be either a round sphere or has a round cylinder as one of its backward limits. In the latter case, in [3], we show that all three dimensional κ-solutions, that have the round cylinder as one of its backward limits, have unique asymptotics. We describe the precise asymptotics in each of the three regions: the parabolic region, the intermediate region, and the tip region. The asymptotics are similar to those of two-dimensional mean curvature flow ovals. We use these precise asymptotics in [7] to prove the classification of κ-noncollapsed closed ancient solutions to the Ricci flow. More recently, in [8] we generalize those techniques to prove analogous classifciation result in higher dimensions under the assumption on positive isotropic curvature.

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Limiting behavior of Ricci flow on Fano manifolds

Gang Tian

(joint work with Yan Li, Xiaohua Zhu)

Consider the Kähler-Ricci flow on a Fano manifold M of complex dimension n:

$$\begin{aligned} \frac{\partial \omega(t)}{\partial t} &= -\operatorname{Ric}(\omega(t)) + \omega(t), \\ \omega(0) &= \omega_0, \end{aligned} \tag{1}$$

where ω_0 is a given Kähler metric with Kähler class $2\pi c_1(M)$. In 1986, H.D. Cao proved that (1) has a unique solution $\omega(t)$ for all $t \geq 0$. A long-standing problem is the limiting behavior of $\omega(t)$ as t goes to ∞. The following was written down in my 1997 paper in Invent. Math.:

Conjecture 1. *Any sequence of* $(M, \omega(t))$ *contains a subsequence converging to a length space* $(M_\infty, \omega_\infty)$ *in the Gromov-Hausdorff topology and* $(M_\infty, \omega_\infty)$ *is a smooth Kähler-Ricci soliton outside a closed subset* S *of codimension at least* 4.

This was often referred as the Hamilton-Tian conjecture.

First, I gave a brief tour on history of studying this conjecture. Since Perelman in 2003, many progresses were made by Tian-Zhu (2006), Tian-Zhang-Zhu (2013) and Dervan-Szekelyhidi (2020) under various assumptions on M. In general cases, it follows from Q. Zhang's upper estimate (2013) on volume ratio along Ricci flow and Perelman's non-collapsing result that $(M, \omega(t))$ converge to a length space $(M_\infty, \omega_\infty)$ in the Gromov-Hausdorff topology. Then the conjecture is reduced to studying the regularity of $(M_\infty, \omega_\infty)$ and convergence of $\omega(t)$. This problem has been solved due to works of Tian-Z.L. Zhang in 2016, R. Bamler in 2018 and Chen-Wang in 2020. In fact, in his Annals paper in 2018, Bamler proved a generalized version of the Hamilton-Tian conjecture.

Then we were led to studying the question: *What is the best regularity of* $(M_\infty, \omega_\infty)$? It was proved by Tian-Z.L.Zhang (Acta Math., 2016) by using the Hamilton-Tian conjecture that M_∞ is a normal variety and ω_∞ is a smooth Kähler-Ricci soliton outside its singular set which is an analytic subvariety. This was done by developing a version of the partial C^0-estimate for Kähler-Ricci flow. *Does this regularity answer the above question*? There was a folklore speculation that M_∞ *is actually smooth*, in other words, any solution $\omega(t)$ of (1) is of *type I*, that is, the curvature of $\omega(t)$ is uniformly bounded. Our main result is to disprove this speculation, i.e., there are solutions of type II, that is, the curvature of $\omega(t)$ may be unbounded. Therefore, the regularity achieved in my work with Z.L.Zhang is the best in general. We found solutions of type II by using G-manifolds.

Next in my talk, I recalled definition and some properties of Fano G-manifolds, where G is a reductive Lie group. Then we stated a criterion proved by Delcroix (2015) for a Fano G-manifold to admit Kähler-Einstein metrics. Delcroix proved his criterion by the continuity method. Later, alternative proof was given by Li-Zhou-Zhu (2017).

Next, I stated our theorem joint with Yan Li and Xiaohua Zhu:

Theorem 2. *Let* G *be a semi-simple complex Lie group and* M *be a Fano* G*-manifold which admits no Kähler-Einstein metrics, then any solution of the Kähler-Ricci flow on* M *with an initial metric* $\omega_0 \in 2\pi c_1(M)$ *must be of type II.*

I also remarked that the semi-simplicity condition can be dropped due to a very recent observation by Tian-Zhu. According to Delcroix, there are three Fano $SO_4(\mathbb{C})$-manifolds and three Fano $Sp_4(\mathbb{C})$-manifolds, and three of these six manifolds do not admit Kähler-Eisntein metrics. As an application of Theorem 2, we proved that on those Fano $\mathrm{SO}_4(\mathbb{C})$-manifolds and Fano $\mathrm{Sp}_4(\mathbb{C})$-manifold which do not admit Kähler-Einstein metrics, the Kähler-Ricci flow develops singularity of type II.

Then I discussed the ideas in the proof of Theorem 2. The most crucial step is to prove

Proposition 3. If $(M_\infty, \omega_\infty, J_\infty)$ be a smooth limit of Kähler metrics ω_i with Kähler class $2\pi c_1(M)$ on a Fano G-manifold M in the Cheeger-Gromov topology, then $(M_\infty, \omega_\infty)$ is also a Fano G-manifold.

I outlined a proof of this proposition and emphasised the role of the smoothness condition in our proof.

Finally, I briefly discussed my work with Yan Li and Xiaohua Zhu on extending Delcroix's criterion to Fano G-varieties which may have normal singularity. I also mentioned very recent work of Yan Li and Zhenye Li (2021) in which they identified limits of Kähler-Ricci flow on those Fano $SO_4(\mathbb{C})$-manifolds which do not admit Kähler-Einstein metrics. It turns out that those limits are not Fano G-varieties.

Ricci flow of $W^{2,2}$ metrics in four dimensions

Miles Simon
(joint work with Tobias Lamm)

In this talk we explain how to construct solutions to Ricci flow and Ricci DeTurck flow which are instantaneously smooth but whose initial values are (possibly) non-smooth Riemannian metrics whose components, in smooth coordinates, belong to certain Sobolev spaces.

For a given smooth Riemannian manifold (M, h), and an interval $I \subseteq \mathbb{R}$, a smooth family $g(t)_{t\in I}$ of Riemannian metrics on M is a solution to Ricci DeTurck h Flow if

$$\begin{aligned}\frac{\partial}{\partial t} g_{ij} =& g^{ab}({}^h\nabla_a {}^h\nabla_b g_{ij}) - g^{kl} g_{ip} h^{pq} R_{jkql}(h) - g^{kl} g_{jp} h^{pq} R_{ikql}(h) \\ &+ \tfrac{1}{2} g^{ab} g^{pq} \left({}^h\nabla_i g_{pa} {}^h\nabla_j g_{qb} + 2{}^h\nabla_a g_{jp} {}^h\nabla_q g_{ib} - 2{}^h\nabla_a g_{jp} {}^h\nabla_b g_{iq} \right.\\ &\left. -2{}^h\nabla_j g_{pa} {}^h\nabla_b g_{iq} - 2{}^h\nabla_i g_{pa} {}^h\nabla_b g_{jq}\right), \end{aligned} \tag{1}$$

in the smooth sense on $M \times I$, where here, and in the rest of the paper, ${}^h\nabla$ refers to the covariant derivative with respect to h. A smooth family $\ell(t)_{t\in I}$ of Riemannian metrics on M is a solution to Ricci flow if

$$\frac{\partial \ell}{\partial t} = -2\mathrm{Rc}(\ell) \tag{2}$$

in the smooth sense on $M \times I$. Ricci DeTurck flow and Ricci flow in the smooth setting are closely related : given a Ricci DeTurck flow $g(t)_{t\in I}$ on a compact manifold and an $S \in I$ there is a smooth family of diffeomorphisms $\Phi(t) : M \to M$, $t \in I$ with $\Phi(S) = Id$ such that $\ell(t) = (\Phi(t))^* g(t)$ is a smooth solution to Ricci flow. The diffeomorphisms $\Phi(t)$ solve the following ordinary differential equation:

$$\begin{aligned}\frac{\partial}{\partial t}\Phi^\alpha(x,t) =& V^\alpha(\Phi(x,t),t), \quad \text{for all } \ (x,t) \in M^n \times I, \\ \Phi(x,S) =& x. \end{aligned} \tag{3}$$

where $V^\alpha(y,t) := -g^{\beta\gamma}\left({}^g\Gamma^\alpha_{\beta\gamma} - {}^h\Gamma^\alpha_{\beta\gamma}\right)(y,t)$

There are a number of papers on solutions to Ricci DeTurck flow and Ricci flow starting from non-smooth Riemannian metric/distance spaces: Given a non-smooth starting space (M, g_0) or (M, d_0), it is possible in some settings, to find smooth solutions $g(t)_{t\in(0,T)}$ to (1), respectively $\ell(t)_{t\in(0,T)}$ to (2) defined for some $T > 0$, where the initial values are achieved in some weak sense. Here is a non-exhaustive list of papers, where examples of this type are constructed : [5, 3, 6, 7, 1, 2, 4]. The initial non-smooth data considered in these papers has certain structure, which when assumed in the smooth setting, leads to a priori estimates for solutions, which are then used to construct solutions in the class being considered. In some papers this initial structure comes from geometric conditions, in others from regularity conditions on the initial function space of the metric components in smooth coordinates. In the second instance, this is usually in the setting, that one has some C^0 control of the metric. That is, the metric is close in the L^∞ sense to the standard euclidean metric in smooth coordinates: $(1-\varepsilon)\delta \le g(0) \le (1+\varepsilon)\delta$ for a sufficiently small ε. In this talk, the structure of the initial metric $g(0)$ comes from the assumption, in the four dimensional compact setting, that the components in coordinates are in $W^{2,2}$, and uniformly bounded from above and below : $\frac{1}{a}\delta \le g(0) \le a\delta$ for some constant c. Closeness of the metric to δ is not assumed. With this initial structure, we show that a solution to Ricci DeTurck flow exists. In the non-compact setting, we further require that the $W^{2,2}$ norm on balls of radius one is uniformly small and a uniform bound from above and below in the L^∞ sense, both with respect to a geometrically controlled background metric. We also investigate the question of how the initial values are achieved, in the metric and distance sense, as time goes back to zero.

Using this solution $g(t)_{t\in[0,T)}$ to Ricci DeTurck flow, we consider the Ricci flow realted solution $(\Phi(t))^*(g(t))_{t\in(0,T)}$ as defined above, where $\Phi(S) = Id$ for some $S > 0$. The convergence as time goes back to zero in the distance and metric sense is investigated for this Ricci Flow solution. We require some new estimates on convergence in the L^p sense for solutions to Ricci flow, in order to show that there is indeed a limiting weak Riemannian metric, as time approaches to zero. We also show that the initial metric value of the Ricci flow that is achieved is isometric, in a weak sense, to the initial value $g(0)$ of the Ricci DeTurck flow solution.

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Comparison geometry of holomorphic bisectional curvature

John Lott

Holomorphic bisectional curvature is a Kähler analog of Riemannian sectional curvature. There is a well developed theory of Riemannian manifolds with lower sectional curvature bounds, including such topics as triangle comparison, Gromov-Hausdorff limits and Alexandrov spaces. We give Kähler analogs.

To state the first main result, we define a modified distance-squared function. Given $d \geq 0$ and $K \in \mathbb{R}$, define $d_K \geq 0$ by

$$d_K^2 = \begin{cases} -\frac{4}{K} \log \cos \left(d \sqrt{\frac{K}{2}} \right) & \text{if } K > 0, \\ d^2 & \text{if } K = 0, \\ \frac{4}{-K} \log \cosh \left(d \sqrt{\frac{-K}{2}} \right) & \text{if } K < 0. \end{cases} \tag{1}$$

(If $K > 0$ then we restrict to $d \leq \frac{\pi}{\sqrt{2K}}$.) Let M be a complete Kähler manifold. Given $p \in M$ and $K \in \mathbb{R}$, let $d_p \in C(M)$ be the distance from p and define $d_{K,p}$ using (1), replacing the d in the right-hand side by d_p.

We write $BK \geq K$ if the holomorphic bisectional curvatures of M are bounded below by $K \in \mathbb{R}$. We prove the following analog of triangle comparison.

Theorem 2. *Let M be a complete Kähler manifold. Given $K \in \mathbb{R}$, the manifold M has $BK \geq K$ if and only if it satisfies the following property. Let $i : \overline{D^2} \to M$ be an embedding of a disk into M, that is holomorphic on D^2. Let Σ be the image of i. Let dA denote the area form on Σ. Let z be the local coordinate on D^2 and let $\theta \in [0, 2\pi)$ be the local coordinate on $\partial \overline{D^2}$. Then*

$$d_{K,p}^2(0) \geq \frac{2}{\pi} \iint_\Sigma \log |z| \, dA + \frac{1}{2\pi} \int_{\partial \Sigma} d_{K,p}^2(\theta) d\theta, \tag{3}$$

where the "0" on the left-hand side denotes $i(0)$, the center of Σ.

Next, we consider noncollapsing sequences of complete pointed Kähler manifolds with $BK \geq K$. Lee and Tam showed that after passing to a subsequence, there is a pointed Gromov-Hausdorff limit that is a complex manifold [4]. Regarding its geometry, we show that (3) holds on the limit.

Theorem 4. *Let $\{(M_i, p_i, g_i)\}_{i=1}^\infty$ be a sequence of pointed n-dimensional complete Kähler manifolds with $BK \geq K$. Suppose that there is some $v_0 > 0$ so that for all i, we have $\mathrm{vol}(B(p_i, 1)) \geq v_0$. Then after passing to a subsequence, there is a pointed Gromov-Hausdorff limit $(X_\infty, p_\infty, d_\infty)$ with the following properties.*

(1) X_∞ is a complex manifold.
(2) Embedded holomorphic disks Σ in X_∞ satisfy (3), where dA is now the two dimensional Hausdorff measure coming from d_∞.

Some simple examples of such limit spaces come from two dimensional length spaces with Alexandrov curvature bounded below. The proof of Theorem 4 uses local Ricci flow techniques, as developed by Bamler-Cabezas-Rivas-Wilking [1], Hochard [2], Lee-Tam [3] and Simon-Topping [6].

We also show

- A complete Kähler manifold has $BK \geq K$ if and only if $\sqrt{-1}\partial\overline{\partial}d^2_{K,p}/2 \leq \omega$ as currents.
- If a Hermitian manifold satisfies (3) then it must be Kähler.
- A domain M in a model space (of constant holomorphic sectional curvature) satisfies (3) if and only if the length metric on M is the same as the restricted metric from the model space.

We give a notion of "$BK \geq K$" for possibly singular complex spaces. We use the notion of Kähler spaces from [5], which is formulated in terms of local potential functions $\{\phi_\alpha\}$. We define metric Kähler spaces and an associated complex Gromov-Hausdorff convergence, which may be of independent interest. We say that a metric Kähler space has "$BK \geq K$" if $\phi_\alpha - d^2_{K,p}/2$ is plurisubharmonic for all α and p. For normal complex spaces, this is equivalent to (3) being satisfied. The following properties hold:

- Given a sequence of metric Kähler spaces with "$BK \geq K$", if it converges in the pointed complex Gromov-Hausdorff sense then the limit space has "$BK \geq K$".
- Under the assumptions of Theorem 4, a subsequence converges in the pointed complex Gromov-Hausdorff sense.
- If a Kähler orbifold has $BK \geq K$ in the sense of curvature tensors then its underlying length space has "$BK \geq K$".

Finally, we discuss tangent cones of the limit spaces from Theorem 4. We show

- A tangent cone is a Kähler cone that is biholomorphic to $\mathbb{C}^n$.
- When the distance function from the vertex is radially homogeneous on $\mathbb{C}^n$, the tangent cone is an affine cone over a copy of $\mathbb{C}P^{n-1}$ with "$BK \geq 2$".

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