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Classical Algebraic Geometry (hybrid meeting)

Organized by
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Abstract. Progress in algebraic geometry often comes through the introduction of new tools and ideas to tackle the classical problems in the development of the field. Examples include new invariants that capture some aspect of geometry in a novel way, such as the derived category, and the extension of the class of geometric objects considered to allow constructions not previously possible, such as the transition from varieties to schemes or from schemes to stacks. Many famous old problems and outstanding conjectures have been resolved in this way over the last 50 years. While the new theories are sometimes studied for their own sake, they are in the end best understood in the context of the classical questions they illuminate. The goal of the workshop was to study new developments in algebraic geometry, with a view toward their application to the classical problems.

Mathematics Subject Classification (2010): 14-xx.

Introduction by the Organizers

The workshop *Classical Algebraic Geometry* held June 20–26, 2021 at the Mathematisches Forschungsinstitut Oberwolfach was organized by Olivier Debarre (Paris), David Eisenbud (Berkeley), Gavril Farkas (HU Berlin) and Ravi Vakil (Stanford). The workshop took place in hybrid format with 28 participants present in Oberwolfach, the rest participating online. There were 20 one-hour talks (12 in person and 8 in Zoom format), including one Zoom talk every evening to accommodate for the different time zones of the participants. On Tuesday an evening session of short presentations took place, allowing young participants to introduce their current work (and themselves). The extended abstracts give a detailed account of the broad variety of topics of the meeting. We have selected a few of the highlights of the meeting:

• János Kollár: *Projectivity criteria*
In one of the central talks of the workshop, János Kollár spoke on projectivity criteria for Moishezon manifolds. The culmination of the lecture was the recent theorem proved by his student Villalobos Paz, that a compact Moishezon manifold X is projective if and only if there are *no* rational curve R on X such that $-R$ lies in the closure of the Mori cone of curves $\overline{NE}(X)$. The start of the story was also the start of Mori's work at the dawn of the Minimal Model Program based on contracting rational curves one at a time. A corollary is that if X is *not* a projective variety, then it contains a rational curve. Villalobos Paz extends this result to algebraic spaces, a seemingly small extension, but a big problem because the Minimal Model Program steps of Birkar-Cascini-Hacon-McKernan are *not* étale-local in nature (that is, they do not commute with étale base change). Kollár described a second main theorem of Villalobos Paz in a similar vein, on the fact that nice enough — but not flat — proper morphisms of complex analytic spaces with projective fibers, and projective away from a point, are actually locally projective. As he often does, throughout his talk, Kollár gave a number of natural questions that will certainly guide the future developments of this subject.

• Hannah Larson: *Brill–Noether theory over the Hurwitz space*
Formulated in the late 19th century and rigorously proven by Griffiths and Harris in the 1980s, the Brill-Noether Theorem establishes which linear systems can appear on a general curve C of genus g. A similar question can be asked for a general curve C of genus g and prescribed gonality k, that is, endowed with a degree k map $f\colon C \to \mathbb{P}^1$. In a very impressive talk, Larson explained how in two important papers, she found and proved analogues of all the main theorems of Brill-Noether theory for the case of general curves of prescribed gonality. She determined the dimension of the Brill-Noether variety $W^r_d(C)$ (which in general will be different from the answer one gets for a general curve of genus g) and determined the dimensions of all of its components corresponding to the various scrollar invariants (splitting type) of the degree-k map in question. In particular, all these components are normal and Cohen-Macaulay. She then computed their cohomology classes in the intersection ring of the Jacobian of C and pointed out that, quite surprisingly, the system of cohomology classes of splitting strata lead to a very interesting problem in the theory of Coxeter groups.

• Carl Lian: *Counting linear series on curves: old and new results*
Carl Lian addressed in his talk the following basic question in enumerative geometry. Given a general curve C of genus g and fixed points $x_1, \ldots, x_n$ on C, how many linear systems $\ell \in G^r_d(C)$ corresponding to maps $f\colon C \to \mathbb{P}^r$ exist which satisfy prescribed incidence conditions at the points $x_1, \ldots, x_n$? He showed that when d is large and n is chosen so that one expects finitely many such maps, their number equals $(r+1)^g$. In the case $r = 1$ (in which case $n = g + 3$), his answer

recovers a recent result of Tevelev, who showed that the degree of the evaluation map $\mathrm{Pic}^{g+1}(C) \dashrightarrow M_{0,g+3}$ equals 2^g. This result, motivated by questions in the theory of scattering amplitudes, was the original motivation of Lian's study, who in his wonderful talk, explained how in the range when the degree d is smaller, the number of linear systems on C with prescribed incidences has an interpretation in terms of intersection on Schubert cycles on the Grassmannian $G(r+1, d+1)$ of r-planes in the projective space $\mathbb{P}^d$. These numbers can be explicitly computed when $r = 1$ and recover very recent results of Cela, Pandharipande and Schmitt.

• Stefan Schreieder: *Infinite torsion in the Griffiths group*
Stefan Schreieder gave a beautiful talk on the construction of smooth projective complex varieties for which the 2-torsion of the Griffiths group is not finitely generated. The Griffiths group of a smooth complex projective variety is the group of homologically trivial cycles modulo algebraic equivalence. This is a countable abelian group that measures the failure of injectivity of the cycle class map. Griffiths introduced these groups in 1969 and gave the first examples of nontrivial Griffiths groups using his transcendental Abel–Jacobi map. Clemens generalized this result by constructing in 1983 Griffiths groups that are not finitely generated modulo torsion. Schreieder gave a very pleasant introduction to and overview of the subject and defined a new theory which he called refined unramified cohomology. He then proceeded to explain how he used these new cohomology groups and a degeneration argument to show that the third Griffiths group of the product of a very general Enriques surface and the Jacobian of a very general plane quartic curve has infinitely many 2-torsion classes that are linearly independent modulo 2.

The young participants' presentations, listed below, covered a similarly wide range of topics. Based on past experience, it is likely that these researchers will establish themselves as leaders in their areas.

- Andrei Bud (HU Berlin)
  Maximal gonality on strata of differentials
- Andrea di Lorenzo (HU Berlin)
  Take five: five invariants of five stacks in five minutes
- Laure Flapan (Michigan State)
  Some cycles on hyperkähler manifolds
- Zhuang He (HU Berlin)
  Birational geometry of blow-ups of $\mathbb{P}^n$ *along points and lines*
- Ritvik Ramkumar (Berkeley)
  A moduli space for pairs of linear spaces in $\mathbb{P}^n$
- Emre Sertöz (MPIM Bonn)
  Separating periods of quartic surfaces
- Jieao Song (Paris, IMJ-PRG)
  Geometry of Debarre-Voisin varieties

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Workshop (hybrid meeting): Classical Algebraic Geometry

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Abstracts

On the coniveau of rationally connected threefolds

Claire Voisin

Let X be a smooth complex projective variety. A cohomology class $\alpha \in H^i(X, \mathbb{Z})$ is said to be of coniveau $\geq c$ if there exists a closed algebraic subset $Y \subset X$ with $\operatorname{codim} Y \geq c$ and $\alpha_{|X \setminus Y} = 0$. Equivalently α comes from a cohomology class with support on Y and the adequate version of Poincaré duality says that, denoting $j : Y \to X$ the inclusion of Y in X, one has

$$\alpha = PD_X(j_*\beta)$$

for some homology class $\beta \in H_{2n-i}(Y, \mathbb{Z})$, where $n = \dim X$, and PD_X denotes the Poincaré duality isomorphism

$$H_{2n-i}(X, \mathbb{Z}) \cong H^i(X, \mathbb{Z}).$$

When passing to rational coefficients, Deligne's theory of mixed Hodge structures [4] implies that, introducing a desingularization $\widetilde{Y} \stackrel{\tau}{\to} Y$ of Y and the morphism $\tilde{j} = j \circ \tau : \widetilde{Y} \to X$, one has

$$\operatorname{Im}(j_* : H_{2n-i}(Y, \mathbb{Q}) \to H_{2n-i}(X, \mathbb{Q})) = \operatorname{Im}(\tilde{j}_* : H_{2n-i}(\widetilde{Y}, \mathbb{Q}) \to H_{2n-i}(X, \mathbb{Q})).$$

This equality does not hold in general with $\mathbb{Z}$-coefficients. In the paper [2], Benoist and Ottem say that a class $\alpha \in H^i(X, \mathbb{Z})$ is of strong coniveau $\geq c$ if there exist a *smooth* projective variety $\widetilde{Y}$ of dimension $n - c$, a morphism $f : \widetilde{Y} \to X$ and a homology class $\tilde{\beta} \in H_{2n-i}(\widetilde{Y}, \mathbb{Z})$ such that $\alpha = PD_X(\tilde{j}_*\tilde{\beta})$. Passing to rational coefficients, we get from Deligne theorem that a rational cohomology class is of coniveau $\geq c$ if and only if it is of strong coniveau $\geq c$. With $\mathbb{Z}$-coefficients, we only have the obvious inclusion

$$\widetilde{N}^c H^i(X, \mathbb{Z}) \subset N^c H^i(X, \mathbb{Z})$$

where on the left, we have cohomology of strong coniveau $\geq c$ and on the right cohomology of coniveau $\geq c$, and Benoist-Ottem show that for any $i \geq 3$ there exist examples of degree i integral cohomology classes on smooth projective varieties X which are of coniveau ≥ 1 but not of strong coniveau ≥ 1.

My interest in these questions comes from the fact that, for any i, the group

$$H^i(X, \mathbb{Z})/\widetilde{N}^1 H^i(X, \mathbb{Z})$$

is a stable birational invariant of smooth projective varieties, so if a smooth projective variety X is stably rational, then $H^i(X, \mathbb{Z}) = \widetilde{N}^1 H^i(X, \mathbb{Z})$ for $i > 0$. The (stable version of) Lüroth problem asks whether rationally connected varieties (or unirational varieties) are stably rational; we have plenty of negative answers to that question, starting from dimension 3. However the invariants used to exhibit counterexamples are not so numerous and easy to compute. The natural question from this viewpoint is:

Let X be a rationally connected smooth projective variety. Is $H^i(X,\mathbb{Z}) = \widetilde{N}^1H^i(X,\mathbb{Z})$ for $i > 0$?

Note that, if we replace the strong coniveau by the coniveau, the answer to this question is yes, by a result of Colliot-Thélène and myself [3].

The result I presented in this talk is proved in [5].

Theorem 1. *Let X be a smooth rationally connected threefold. Then*

$$\widetilde{N}^1H^3(X,\mathbb{Z})/\text{Torsion} = H^3(X,\mathbb{Z})/\text{Torsion}.$$

I do not know if the torsion of $H^3(X,\mathbb{Z})$ is of strong coniveau 1 or not. This torsion is called the Artin-Mumford invariant of X (or Brauer group in this case). It was the first invariant ever used to exhibit unirational threefolds which are not stably rational, although we now know that there are other obstructions to stable rationality, even for unirational threefolds, see [6].

The main ingredients of the proof are (1) the fact that $H^3(X,\mathbb{Z}) = N^1H^3(X,\mathbb{Z})$ as already mentioned, (2) the injectivity of the Abel-Jacobi map on torsion codimension 2 cycles, established by Bloch and (3) Kollár-Miyaoka-Mori method of gluing very free rational curve to a given curve mapping to X in order to make it unobstructed.

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Irrational Gushel Mukai Threefolds

Giovanni Mongardi

(joint work with Olivier Debarre)

The rationality of algebraic varieties has long been an interesting and challenging field of study. After the cases of curves and surfaces were established, rationality (or rather, irrationality) of threefolds was studied using several techniques. The first was introduced by Clemens and Griffiths: they noticed that a rational threefold must have the intermediate Jacobian of an algebraic curve, therefore rationality can be disproven either by looking at singularities of the Theta divisor, or by looking at polarized automorphisms of the intermediate Jacobian. Here, we will use the latter.

Beauville established in [1, Theorem. 5.6(ii)] that a *general* Fano threefold with Picard number 1, index 1, and degree 10 (also known as a Gushel Mukai, or GM, threefold) is irrational, but not a single smooth example was known, although it is expected that all of these Fano threefolds are irrational. In this work, we construct an explicit two dimensional projective family of irrational GM threefolds. Our starting point was a very special EPW (for Eisenbud–Popescu–Walter) sextic hypersurface $Y_{\mathbb{A}} \subset \mathbb{P}^5$, constructed in [8], with a faithful action by the simple group $\mathbb{G} := \mathrm{PSL}(2, \mathbb{F}_{11})$ of order 660.

By [5], the intermediate Jacobians of the GM varieties of dimension 3 or 5 obtained from the sextic $Y_{\mathbb{A}}$ are all isomorphic to a fixed principally polarized abelian variety $(\mathbb{J}, \theta)$ of dimension 10. This applies in particular to $X^5_{\mathbb{A}}$, and the $\mathbb{G}$-action on $X^5_{\mathbb{A}}$ induces a faithful $\mathbb{G}$-action on $(\mathbb{J}, \theta)$. We use this fact to prove that the GM threefolds that we construct from $Y_{\mathbb{A}}$ are not rational: by the Clemens–Griffiths criterion ([4, Corollary 3.26]), it suffices to prove that their (common) intermediate Jacobian $(\mathbb{J}, \theta)$ is not a product of Jacobians of curves. For this, we follow Beauville and use the fact that $(\mathbb{J}, \theta)$ has "too many automorphisms" (because of the $\mathbb{G}$-action). Note that the GM threefolds themselves may have no nontrivial automorphisms. This is how we produce a complete 2-dimensional family of irrational GM threefolds, all mutually birationally isomorphic. The situation is reminiscent of that of the Klein cubic threefold $W \subset \mathbb{P}^4$: Klein proved in [6] that W has a faithful linear $\mathbb{G}$-action; one hundred years later, Adler proved that the automorphism group of W is exactly $\mathbb{G}$ and Roulleau showed that W is the only smooth cubic threefold with an automorphism of order 11. The intermediate Jacobian of W is a principally polarized abelian variety of dimension 5 isomorphic to the product of 5 copies of an elliptic curve with complex multiplication and Adler proved that it is the only abelian variety of dimension 5 with a faithful action of $\mathbb{G}$. This is the reason why we call our sextic $Y_{\mathbb{A}}$ the Klein EPW sextic.

Let $\xi \colon \mathbb{G} \to \mathrm{GL}(V_\xi)$ be one of the irreducible representation of $\mathbb{G}$ of dimension 5, where the action of one of the order 11 elements has eigenvalues which are all squares. Notice that the other dimension 5 representation is the dual one. From the existence of a unique (up to multiplication by a nonzero scalar) $\mathbb{G}$-equivariant symmetric isomorphism

$$w \colon \Lambda^2 V_\xi \to \Lambda^2 V_\xi^\vee$$

we infer that there is a unique $\mathbb{G}$-invariant quadric $\mathbf{Q}$ whose equation is

$$x_{12}x_{13} + x_{23}x_{24} + x_{34}x_{35} - x_{45}x_{14} + x_{15}x_{25} = 0.$$

This quadric does not contain the Grassmannian $\mathrm{Gr}(2, V_\xi)$, therefore it defines a Gushel-Mukai fivefold

$$X^5_{\mathbb{A}} := \mathbf{Q} \cap \mathrm{Gr}(2, V_\xi) \tag{1}$$

A computer check with Macaulay now ensures that the GM fivefold $X^5_{\mathbb{A}}$ defined by (1) is smooth.

The group $\mathbb{G}$ is simple and nonabelian, therefore the representation $\Lambda^5 V_\xi$ is trivial. The isomorphism w therefore induces an isomorphism of representations

$$v\colon \Lambda^2 V_\xi \to \Lambda^2 V_\xi^\vee \otimes \Lambda^5 V_\xi \cong \Lambda^3 V_\xi. \tag{2}$$

Since w is symmetric, v satisfies $v(x) \wedge y = x \wedge v(y)$ for all $x, y \in \Lambda^2 V_\xi$. Let $\chi_0 \colon \mathbb{G} \to V_{\chi_0}$ be the irreducible trivial representation and consider the $\mathbb{G}$-representation

$$V_6 := V_{\chi_0} \oplus V_\xi.$$

We can define a lagrangian subspace $\mathbb{A}$ in $\Lambda^3 V_6$ as follows:

$$\mathbb{A} := \{e_0 \wedge x + v(x) \mid x \in \Lambda^2 V_\xi\}.$$

The Gushel Mukai fivefold $X^5_{\mathbb{A}}$ is the one associated with the Lagrangian $\mathbb{A}$ and the hyperplane $V_\xi \subset V_6$. The ten dimensional intermediate jacobian $(\mathbb{J}, \theta)$ of $X^5_{\mathbb{A}}$ has therefore a faithful $\mathbb{G}$ action. If we take special hyperplanes, corresponding to singular points of the dual EPW sextic

$$Y_{\mathbb{A}^\perp} := \{[H] \in \mathbb{P}(V_6^\vee), \text{ such that } H \wedge \Lambda^2(V_6^\vee) \cap \mathbb{A}^\perp \neq 0\}$$

we obtain Gushel Mukai threefolds with intermediate Jacobian $(\mathbb{J}, \theta)$ by [5]. One of them, computed by Kuznetsov, is given by the intersection of $\mathrm{Gr}(2, V_\xi)$ with the following linear space

$$x_{03} + x_{12} = x_{04} - x_{23} = 0,$$

and the quadric with equation

$$x_{01}x_{02} - x_{13}x_{14} - x_{24}x_{34} = 0.$$

To prove the irrationality of the above threefold and all Gushel Mukai threefolds associated to $\mathbb{A}$, we will work on $(\mathbb{J}, \theta)$:

Proposition 1. *The principally polarized abelian variety $(\mathbb{J}, \theta)$ is indecomposable.*

Proof. If $(\mathbb{J}, \theta)$ is isomorphic to a product of $m \geq 2$ nonzero indecomposable principally polarized abelian varieties, such a decomposition is unique up to the order of the factors hence induces a map from $\mathbb{G}$ to the permutation group of the factors. Since the analytic representation is irreducible, the image of u is nontrivial and, the group $\mathbb{G}$ being simple, u is injective; but this is impossible because $\mathbb{G}$ contains elements of order 11 but the permutation group of the factors does not, because $m \leq 10$. □

We can now sketch the proof of our main result.

Theorem 2. *Any smooth GM threefold associated with the Lagrangian $\mathbb{A}$ is irrational.*

Proof. Let X be such a threefold. There is a $\mathbb{G}$-equivariant isomorphism

$$(\mathrm{Jac}(X), \theta_X) \to (\mathbb{J}, \theta).$$

We follow [2, 3]: to prove that X is not rational, we apply the Clemens–Griffiths criterion ([4, Corollary 3.26]); in view of the previous Proposition, it suffices to prove that $(\mathbb{J}, \theta)$ is not the Jacobian of a smooth projective curve.

Suppose $(\mathbb{J},\theta) \cong (\mathrm{Jac}(C),\theta_C)$ for some smooth projective curve C of genus 10. The group $\mathbb{G}$ then embeds into the group of automorphisms of $(\mathrm{Jac}(C),\theta_C)$; by the Torelli theorem, this group is isomorphic to $\mathrm{Aut}(C)$ if C is hyperelliptic and to $\mathrm{Aut}(C)\times\mathbb{Z}/2\mathbb{Z}$ otherwise. Since any morphism from $\mathbb{G}$ to $\mathbb{Z}/2\mathbb{Z}$ is trivial, we see that $\mathbb{G}$ is a subgroup of $\mathrm{Aut}(C)$. This contradicts the fact that the automorphism group of a curve of genus 10 has order at most 432 ([7]). □

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Tautological classes of matroids

Christopher Eur
(joint work with Andrew Berget, Hunter Spink, Dennis Tseng)

Methods inspired from the algebraic geometry of realizable matroids has recently led to fruitful developments in matroid theory. We introduce a new framework that recovers, unifies, and extends these developments. For notations and conventions regarding matroids, we point to [11] as a standard reference.

Let M be a matroid of rank r on a nonempty ground set $E=\{0,1,\dots,n\}$. Let $T=(\mathbb{C}^*)^E$ be the algebraic torus with the standard action on $\mathbb{C}^E$. A realization (over $\mathbb{C}$) of M is an r-dimensional linear subspace $L\subseteq\mathbb{C}^E$ such that the set of bases of M equals the subcollection $\{B\in\binom{E}{r}\mid L\cap\bigcap_{i\in B}H_i=\{0\}\}$ of size r subsets of E. Here H_i denotes the i-th coordinate hyperplane of $\mathbb{C}^E$. Let X_E be the permutohedral variety of dimension n, which is obtained from $\mathbb{P}^n$ by sequentially blowing up from lower to higher dimensions (the strict transforms of) all coordinate subspaces of $\mathbb{P}^n$. It is a toric variety with the open torus $T/\mathbb{C}^*$, the quotient of T by the diagonal copy of $\mathbb{C}^*$.

Given a realization $L\subseteq\mathbb{C}^E$ of a matroid M, we define two T-equivariant vector bundles $\mathcal{S}_L$ and $\mathcal{Q}_L$ on the permutohedral variety X_E as follows.

Definition 1. The **tautological subbundle** $\mathcal{S}_L$ (resp. the **tautological quotient bundle** $\mathcal{Q}_L$) is the unique torus-equivariant vector bundle whose fiber over a point $\bar{t}$ in the open torus $T/\mathbb{C}^*$ of X_E is $t^{-1}L$ (resp. $\mathbb{C}^E/t^{-1}L$).

The T-equivariant K-classes of $\mathcal{S}_L$ and $\mathcal{Q}_L$ depend only on the matroid M that L realizes, and one can thus define T-equivariant K-classes $[\mathcal{S}_M]$ and $[\mathcal{Q}_M]$ on X_E for an arbitrary, not necessarily realizable matroid M. The Chern classes of these tautological classes recover previously studied geometric models of matroids:

- The first Chern class $c_1(\mathcal{Q}_M)$ equals the nef divisor class on X_E corresponding to the base polytope [8] of the matroid M.
- The top Chern class $c_{|E|-r}(\mathcal{Q}_M)$ equals the Bergman class Δ_M of the matroid M as studied in [12, 9], which coincides with the homology class in X_E of the wonderful compactification [5] of a realization L of M when M has a realization.
- The products of Chern classes $c_i(\mathcal{S}_M)c_{|E|-r}(\mathcal{Q}_M)$ for $0 \le i \le r$ equal the Chern-Schwartz-MacPherson (CSM) classes of a matroid M introduced in [10], and coincides with the CSM classes of the associated hyperplane arrangement complement when the matroid has a realization.

The permutohedral variety X_E resolves the rational Cremona map $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$. Let α and β be divisor classes on X_E obtained as the pullbacks of the hyperplane class from each $\mathbb{P}^n$. We express the Tutte polynomial of a matroid, which is the universal deletion-contraction invariant of matroids, in terms of intersection multiplicities of α, β, and Chern classes of $\mathcal{S}_M$ and $\mathcal{Q}_M$.

Theorem 1. Let $\int_{X_E} : A^\bullet(X_E) \to \mathbb{Z}$ be the degree map on X_E, and $T_M(u,v)$ the Tutte polynomial of a rank r matroid M on ground set E. Then, one has

$$\sum_{i+j+k+\ell=n} \left(\int_{X_E} \alpha^i \beta^j c_k(\mathcal{S}_M^\vee) c_\ell(\mathcal{Q}_M) \right) x^i y^j z^k w^\ell =$$
$$(x+y)^{-1}(y+z)^r(x+w)^{|E|-r} T_M\Big(\frac{x+y}{y+z}, \frac{x+y}{x+w}\Big).$$

We also establish a log-concavity property for the Tutte polynomial. For a homogeneous polynomial $f \in \mathbb{R}[x_1, \dots, x_N]$ of degree d with nonnegative coefficients, we say that its coefficients form a *log-concave unbroken array* if, for any $1 \le i < j \le N$ and a monomial $x^{\boldsymbol{m}}$ of degree $d' \le d$, the coefficients of $\{x_i^k x_j^{d-d'-k} x^{\boldsymbol{m}}\}_{0 \le k \le d-d'}$ in f form a log-concave sequence with no internal zeros.

Theorem 2. The coefficients of the polynomial

$$t_M(x,y,z,w) = (x+y)^{-1}(y+z)^r(x+w)^{|E|-r} T_M\Big(\frac{x+y}{y+z}, \frac{x+y}{x+w}\Big)$$

form a log-concave unbroken array.

The two theorems together unify, recover, and extend several previous geometric interpretations for the Tutte polynomial and the log-concavity properties for the characteristic polynomial of a matroid, as given in [1, 3, 7, 9, 10, 2]. We prove

Theorem 1 by using the method of localization in torus-equivariant geometry, and Theorem 2 by using methods from tropical Hodge theory. Previous geometric frameworks for studying matroids were disjoint in the sense that one could not easily use both of these two fundamental methods within one framework.

In order to use Theorem 1 to recover previous K-theoretic interpretations of the Tutte polynomial of a matroid, we develop an exceptional Hirzebruch-Riemann-Roch type formula for permutohedral varieties.

Theorem 3. There exists a ring isomorphism $\zeta_{X_E} : K_0(X_E) \xrightarrow{\sim} A^\bullet(X_E)$ which satisfies

$$\chi\big([\mathcal{E}]\big) = \int_{X_E} (1 + \alpha + \cdots + \alpha^n) \cdot \zeta_E([\mathcal{E}])$$

for any $[\mathcal{E}] \in K_0(X_E)$. Denote by $\bigwedge^i$ for the i-th exterior power and $c(\mathcal{E}, u) := \sum_{i \geq 0} c_i(\mathcal{E}) u^i$ the Chern polynomial of $[\mathcal{E}]$. If $[\mathcal{E}]$ "has simple Chern roots" (which $\mathcal{S}_M^\vee$ and $\mathcal{Q}_M^\vee$ do) and rank $\operatorname{rk}(\mathcal{E})$, then we have

$$\sum_{i \geq 0} \zeta_{X_E}\big([\textstyle\bigwedge^i \mathcal{E}]\big) u^i = (u+1)^{\operatorname{rk}(\mathcal{E})} c(\mathcal{E}, \tfrac{u}{u+1}), \quad \text{and}$$

$$\sum_{i \geq 0} \zeta_{X_E}\big([\textstyle\bigwedge^i \mathcal{E}^\vee]\big) u^i = (u+1)^{\operatorname{rk}(\mathcal{E})} c(\mathcal{E}, 1)^{-1} c(\mathcal{E}, \tfrac{1}{u+1}).$$

The map ζ_E is not the Chern character map, and the Chow class $(1+\alpha+\cdots+\alpha^n)$ is not the Todd class of X_E. The proof of Theorem 3 is a purely algebraic. We use the localization methods for T-equivariant K-theory and T-equivariant Chow rings of toric varieties along with the Atiyah-Bott localization formula.

Question 1. Is there a geometric interpretation or a proof of this Hirzebruch-Riemann-Roch type formula (Theorem 3)?

Remark 1. One can show that the map ζ_E is the unique isomorphism that sends $[\mathcal{O}_{W_L}]$, the K-class of the structure sheaf of the wonderful compactification W_L associated to a realization L of a matroid, to the Chow class $[W_L]$ in X_E of W_L as a subvariety of X_E. However, this description of ζ_E makes it unclear why such isomorphism should even exist.

Theorem 3 also opens new questions about wonderful compactifications. For instance, it implies that for a realization $L \subseteq \mathbb{K}^E$ of a matroid M over an algebraically closed field $\mathbb{K}$ of arbitrary characteristic, one has that the Euler characteristic of the line bundle $\det \mathcal{Q}_M$ pulled back to W_L satisfies

$$\chi(\det \mathcal{Q}_M; W_L) = |\mu(M)|,$$

where $\mu(M)$ is the constant coefficient of the characteristic polynomial of M. Separately, one can show by computation that also $h^0(\det \mathcal{Q}_M; W_L) = |\mu(M)|$.

Question 2. Is $H^i(\det \mathcal{Q}_M; W_L) = 0$ for all $i > 0$?

When the realization L is over a field of characteristic zero, one can show via Kawamata-Viehweg vanishing theorem that $H^i(\det \mathcal{Q}_M; W_L) = 0$ for all $i > 0$.

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Compactifying Lagrangian fibrations

Giulia Saccà

Lagrangian fibration are the natural higher dimensional analogues of elliptic K3 surfaces and provide natural means to study, construct and classify compact hyper-Kähler manifolds. This short note describes some results from [13], a work in progress on compactification techniques for quasi-projective Lagrangian fibrations.

Definition 1. *A Lagrangian fibration is a proper surjective morphism $f : M \to B$ with connected fibers, where M is a Kähler manifold with a holomorphic symplectic form, B is a normal variety, and the general fiber of f is a Lagrangian submanifold.*

By the Arnol'd-Liouville theorem, the general fiber is a complex torus. In this note we will be mostly concerned with the case M is quasi-projective and f is a projective morphism, though we may also allow M to be a symplectic variety in the sense of [2]. By Mastushita [10], a Lagrangian fibration is equidimensional.

Example 2. *Let $S \to \mathbb{P}^1$ be an elliptic K3 surface. The composition $S^{[n]} \to \operatorname{Sym}^n S \to \operatorname{Sym}^n \mathbb{P}^1 = \mathbb{P}^n$ is a Lagrangian fibration.*

In [4], Donagi and Markman gave a necessary and sufficient condition, expressed in terms of the tangent map of the classifying morphism from U to the moduli space of ppav, for the total space of the family $f : J_U \to U$ of principally polarized abelian varieties to admit a holomorphic symplectic form making the fibration Lagrangian. For more details, see [4, §7.2]. This condition is the applied to construct several Lagrangian fibrations of geometric origin. In view of constructing examples of compact hyper-Kähler manifolds, we considering the following question:

Question 3. *Let $U \subset \mathbb{P}^n$ be an open subset and let $\pi_U : M_U \to U$ be a Lagrangian fibration. When does there exist a smooth hyper-Kähler compactification M of M_U, with a regular morphism $\pi : M \to \mathbb{P}^n$ extending π_U?*

Given an smooth projective fibration $M_U \to U$, with $U \subset \mathbb{P}^n$, Brosnan [3] noticed an obstruction to the existence of a smooth compactification of the total space, with a regular equidimensional morphism to $\mathbb{P}^n$. This obstruction is expressed in terms of the intermediate extension of the local systems coming from the smooth part of the fibration. Since it depends only on these local systems, it is the same for any other fibration which is locally isomorphic to $M_U \to U$. The following examples shows that 1) there are abelian group schemes which do not admit a hyper-Kähler compactification but have torsors that do admit a hyper-Kähler compactification; 2) there is an abelian group scheme which admits a hyper-Kähler compactification and has a non trivial torsor with a hyper-Kähler compactification in a different deformation class.

Example 4. *Let (S, H) be a general polarized K3 surface of genus g and let m and χ be integers, with $m \geq 1$. Consider the Mukai vector $v = (0, mH, \chi)$ and let M_v be the moduli space of H-semistable pure dimension one sheaves on S with Mukai vector v. This moduli space is smooth if and only if $\gcd(m, \chi) = 1$. If this is the case, M_v is a hyper-Kähler manifold of $K3^{[n]}$-type, for $n = v^2/2 + 1$. If this is not the case, then the moduli space has a symplectic resolution if and only if $m = g = 2$. This symplectic resolution is of $OG10$-type. Fix $m \geq 2$. Let $U \subset |mH|$ be the open subset parametrizing smooth curves, let $\mathcal{C}_U/U$ be the family of smooth curves, and for $\chi \in \mathbb{Z}$, let $\mathrm{Pic}^{\chi}(\mathcal{C}_U/U) \to U$ be the relative Picard variety of degree $d = \chi + H^2/2$. If $\gcd(\chi, m) = 1$, then M_v is a hyper-Kähler compactification of $\mathrm{Pic}^{\chi}(\mathcal{C}_U/U) \to U$. If $m = g = 2$ and χ is even, then a symplectic resolution of M_v, which is of $OG10$-type, is a hyper-Kähler compactification of $\mathrm{Pic}^{\chi}(\mathcal{C}_U/U) \to U$. Finally, if $\gcd(\chi, m) \neq 1$ and m and g are not both equal to 2, then $\mathrm{Pic}^{\chi}(\mathcal{C}_U/U) \to U$ has no hyper-Kähler compactification.*

We now come to the main results.

Theorem 5. [13] *Let $U \subset B$ be an open subset of a normal projective variety B and let $\pi_U : M_U \to U$ be a projective Lagrangian fibration with holomorphic symplectic form σ_{M_U}. Suppose the complement of U in B has codimension ≥ 2 and that there is a compactification of M_U with a holomorphic 2-form extending σ_{M_U}. Then there exists a $\mathbb{Q}$-factorial terminal symplectic compactification M of M_U and a projective Lagrangian fibration $\pi : M \to B$ extending π_U. Moreover, the following are equivalent:*

(1) M is smooth;
(2) M is birational to a smooth holomorphic symplectic variety;
(3) M is smoothable by a flat deformation.

Sketch of Proof. Let $\widetilde{M}$ be any smooth projective compactification of M_U with a regular morphism $\widetilde{\pi} : \widetilde{M} \to \mathbb{P}^n$. By assumption there exists a holomorphic 2-form on $\widetilde{M}$ which is generically non-degenerate, so the canonical bundle of $\widetilde{M}$ is effective, and trivial if and only if $\widetilde{M}$ is hyper-Kähler. If this is not the case, then the canonical divisor is supported on the complement of M_U and, since the complement of U in B has codimension ≥ 2, it is not nef over $\mathbb{P}^n$. Following Lai [7] (see also [5]) it is possible to run the minimal model program and reach, after a finite sequence of birational maps relative to $\mathbb{P}^n$, a $\mathbb{Q}$-factorial terminal symplectic compactification $\bar{M}$ of M_U with a regular morphism to $\mathbb{P}^n$. The equivalence (1)–(2) follows as in [12, Prop. 1.7] and (1)–(3) is the main result of [11]. □

A consequence of this result is that to answer Question 3 it is equivalent to show there exist a hyper-Kähler compactification with a regular morphism to $\mathbb{P}^n$ or a projective birational model that is hyper-Kähler. A comment about the assumption on the extension of the holomorphic form: using the techniques of [8, §2], it is possible to show in certain geometric settings that the holomorphic 2 form on a family of abelian varieties extends to a holomorphic form on any smooth projective compactification of the total space. The following answers a question asked to me by Voisin.

Theorem 6. [13] *Let $\pi : M \to \mathbb{P}^n$ be a Lagrangian fibration with integral fibers. Then there exists a projective hyper-Kähler manifold A, with a Lagrangian fibration $A \to \mathbb{P}^n$ that has a section and is étale locally isomorphic to $M \to \mathbb{P}^n$.*

Sketch of Proof. By [1, 9], there exists a fibration $A^\circ \to \mathbb{P}^n$ with a holomorphic symplectic form and which, étale locally over the base, is isomorphic to the smooth locus of π. We refer to this fibration as the relative Albanese fibration of $M \to \mathbb{P}^n$. Applying to A° a similar argument to that of Theorem 5, yields a projective, $\mathbb{Q}$-factorial, terminal, symplectic birational model $\overline{A}$ of A°, with a regular morphism to $\mathbb{P}^n$. By construction, $\overline{A}$ and M are birational étale locally over $\mathbb{P}^n$. To extend these birational maps to isomorphism, we use the fact that since the restriction map $H^2(M) \to H^2(M_t)$ has rank one and since the fibers of $M \to \mathbb{P}^n$ are irreducible, any line bundle on M that is ample on the general fiber is ample on every fiber. A similar argument was used in [14]. □

Example 4 shows that the integrality of the fibers is, in general, a necessary condition for the conclusion of this theorem to hold. One may also consider, instead of the relative Albanese fibration, the relative degree-0 Picard variety $\mathrm{Pic}^0(M/\mathbb{P}^n)$. In this case using the same techniques one can prove that there exists a $\mathbb{Q}$-factorial terminal symplectic compactification. While there are examples where it is know that the relative Picard variety has a symplectic compactification, at this time the author does not know a general statement.

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Odd cohomology vanishing and polynomial point counts on moduli spaces of stable curves

Sam Payne

(joint work with Jonas Bergström and Carel Faber)

In the late 1990s, Arbarello and Cornalba developed a method based in algebraic geometry, Hodge theory, and the combinatorial structure of the boundary to compute rational cohomology groups of moduli spaces of stable curves. As applications of this method, they computed $H^2(\overline{M}_{g,n})$, giving explicit generators and relations, and showed that $H^k(\overline{M}_{g,n})$ vanishes for $k \in \{1, 3, 5\}$, for all g and n [1]. It was already known that $H^{11}(\overline{M}_{1,11}) \cong \mathbb{Q}^2$.

Our first main result affirmatively answers the natural question they posed, regarding the vanishing of odd cohomology in the two intermediate degrees; we show that $H^7(\overline{M}_{g,n})$ and $H^9(\overline{M}_{g,n})$ vanish for all g and n.

The proof of this result follows the inductive procedure developed by Arbarello and Cornalba, leveraging a number of intermediate results that have been proved in the past twenty years to establish the needed base cases. For H^7, all of the needed cases can already be extracted from the existing literature. The required cases for $g = 2$ and 3 were established by Bergström, Petersen, and Tommasi [3, 4, 8], while the cases $g = 4$ and $n = 0, 1$ are special cases of vanishing results in the virtual cohomological dimension [6, 7]. These same references, plus [9], handle all but three of the cases needed for H^9.

In order to prove our first main result, we needed to show that $H^9(\overline{M}_{4,n})$ vanishes for $n \in \{1, 2, 3\}$. In fact, we have proved much more. Our second main result says that $\#\overline{M}_{4,n}(\mathbb{F}_q)$ is a polynomial in q, for $n \in \{1, 2, 3\}$. It then follows, via the Behrend Trace Formula for algebraic stacks [2], that the cohomology of $\overline{M}_{4,n}$ is pure Hodge-Tate, and in particular that it is supported in even degrees. Moreover, we have computed the polynomials and the isomorphism class of the S_n-representation (modulo a few details that remain to be checked).

Both of our main results confirm predictions of the Langlands Program, via the conjectural correspondence between irreducible motives appearing in Deligne-Mumford stacks that are smooth and proper over the integers and polarized algebraic cuspidal automorphic representations of conductor 1. The latter have recently been classified in weights less than 23 by Chenevier and Lannes [5]. The predictions work as follows.

Suppose X is smooth and proper over $\mathbb{Z}$. The classification of Chenevier and Lannes contains no representations of odd weight less than 11. Therefore, assuming the conjectural correspondence, one concludes that $H^k(X)$ vanishes for all odd $k < 11$. Furthermore, the only representation of weight 11 corresponds to the motive S_{12} and any space that contains this motive must also possess a holomorphic 11-form. Thus, if X has no holomorphic 11-form (e.g., if X is unirational) and if $\dim X \leq 12$ then the correspondence predicts that the cohomology of X should be pure Hodge-Tate. In particular, since $\dim \overline{M}_{4,n} = 9 + n$ and $\overline{M}_{4,n}$ is unirational in the relevant cases, the correspondence predicts that the cohomology of $\overline{M}_{4,n}$ is pure Hodge-Tate for $n \leq 3$. This is what we have now proved, unconditionally.

In the talk, I also explained some advances in the standard point counting sieve method, by systematically using inverse Hasse-Weil Zeta functions to collect terms and simplify cancellations. This simplification is based on a proposition of Vakil and Wood from [10], and was first used in the context of point counts on moduli spaces by Wennink [11].

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Towards the Kodaira dimension of moduli spaces of Abelian differentials

Martin Möller
(joint work with Matteo Costantini, Dawei Chen)

The moduli spaces of Abelian differentials $\Omega\mathcal{M}_{g,n}(\mu)$ parametrize n-pointed stable curves of genus g together with an abelian differential with zeros at the marked points of orders $\mu = (m_1, \ldots, m_n)$. These strata have attracted a lot of interest from the dynamics viewpoint since they admit and action of $\mathrm{SL}_2(\mathbb{R})$ that encodes the dynamics on polygonal billiard tables. Here we report on progress towards the algebraic geometric properties of the projectivized strata $\mathbb{P}\Omega\mathcal{M}_{g,n}(\mu) = \Omega\mathcal{M}_{g,n}(\mu)/\mathbb{C}^*$.

For low genus strata, i.e. $g \leq 9$ for all n and moreover for some strata for $g \leq 11$ but large n, the strata are unruled by results of Barros [1] and Bud [2]. For large genus the moduli space of curves $\mathcal{M}_{g,n}$ is of maximal Kodaira dimension and the moduli spaces of abelian differentials are finite of covers of such $\mathcal{M}_{g,n}$ if moreover n is large by results of [3]. This implies that $\Omega\mathcal{M}_{g,n}(\mu)$ has maximal Kodaira dimension if both g and n is large.

The main open question is thus the behaviour of the Kodaira dimension for g large in the cases with few points, including the special case of subcanonical points where $n = 1$ and thus $\mu = (2g-2)$. The connected components of strata have been classified by Kontsevich and Zorich [6]. For each μ there are up to three components, distinguished by odd or even spin parity and possibly components consisting entirely of hyperelliptic curves. The even and the hyperelliptic components are unirational for trivial reasons. For the first interesting case we are reasonably confident to conjecture:

Conjecture 1. *The odd components of the minimal strata* $\mathbb{P}\Omega\mathcal{M}_{g,n}(2g-2)^{\mathrm{odd}}$ *have maximal Kodaira dimension for* $g \geq 12$.

The strategy is the classical one of Harris and Mumford [7] writing the canonical bundle of a compactification as an ample plus an effective divisor. Two prerequisites have been provided by earlier work. The compactification from [4] is a smooth proper Deligne-Mumford stack and [5] gives a formula the canonical class on this stack.

The first step in our program is to show that the coarse moduli space associated with the compactified stack is actually a projective variety by exhibiting an ample line bundle. We denote that coarse moduli space by $\mathbb{P}\mathrm{MS}(\mu)$, an acronym for the multi-scale differentials whose moduli space compactifies the moduli space of Abelian differentials.

The second step is to understand the ramification divisor of the map from the compactified stack to the compact coarse moduli space. The third step is to control the singularities. Both these steps have been carried out. We give a flavour of the result in the interior. The complete result is a long case distinction.

Theorem 2. *The singularities of the coarse moduli space $\mathbb{P}\Omega M_{g,n}(\mu)$ with marked points are canonical except for the holomorphic stratum $\Omega\mathcal{M}_{1,2}(0,0)$ and the meromorphic strata $\mu = (m, 2-m)$ in genus $g = 2$ for $m \geq 4$ and $m \equiv 1 \mod 3$.*

However, the compactified coarse moduli space $\mathbb{P}\mathrm{MS}(\mu)$ has non-canonical singularities for all but finitely many μ.

The fourth step is to find an effective divisor with small slope. For this purpose we use a generalized Weierstrass divisor. At least for $\mu = (2g-2)^{\mathrm{odd}}$ and $\mu = (2g-3, 1)$ the slope beats for $g \geq 12$ the slope of the canonical class, which is the main reason for the conjecture. However, with respect to several other boundary divisors the Weierstrass divisor has pretty bad slope. So the fifth step will be to mix with known Brill-Noether divisors, pulled back from $\overline{\mathcal{M}}_{g,n}$ and estimate that the slope is good enough to control all boundary divisors, even with a compensation term stemming from non-canonical singularities.

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Counting linear series on curves: old and new results

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(joint work with Gavril Farkas)

One of the most celebrated achievements of 19th century algebraic geometry remains Castelnuovo's enumeration of covers of minimal degree of $\mathbb{P}^1$, clarified a century later by Griffiths-Harris.

Theorem 1 (Castelnuovo [3], Griffiths-Harris [9]). *Let C be a general curve of genus $2h \geq 0$. Then, the number of covers $f : C \to \mathbb{P}^1$ of degree $h+1$ is the Catalan number $C_h = \frac{1}{h+1}\binom{2h}{h}$.*

A modern formulation of the proof may be given in the language of limit linear series [5]. Namely, a map f is given by the data of $\mathcal{L} \in \mathrm{Pic}^{h+1}(C)$ such that $V = H^0(\mathcal{L})$ has dimension 2. If C degenerates to a singular curve C_0 given by $2h$ elliptic tails attached at nodes to $\mathbb{P}^1$, then V specializes to a limit linear series V_0 on C_0. The limit linear series V_0 is essentially determined by a 2-dimensional subspace $W_0 \subset H^0(\mathbb{P}^1, \mathcal{O}(h+1))$, ramified at the nodes on $\mathbb{P}^1$, and the number of such W_0 is computed by Schubert calculus.

Our first new result concerns enumerating covers with higher ramification, which has been studied in many special cases, e.g. in cycle class computations on moduli spaces of curves, see [10, 15, 7]. Let C be a general curve of genus g, and consider $f : C \to \mathbb{P}^1$ of degree d constrained to have ramification indices $d_1, \ldots, d_m \geq 2$ at *unspecified* points on C. (The situation where these points are *specified* is easier, see [16].) Upon a limit linear series degeneration as above, the interesting contributions come now from genus 1.

Theorem 2 (L. [12]). *Let (E, p_1) be a general elliptic curve. Let d, d_1, d_2, d_3, d_4 be integers such that $d_1 + d_2 + d_3 + d_4 = 2d + 4$. Then, the number of 4-tuples (p_2, p_3, p_4, f), where $p_i \in E$ are pairwise distinct points, and $f : E \to \mathbb{P}^1$ is a cover of degree d with ramification index d_i at each p_i, is*

$$N_{d_1,d_2,d_3,d_4} = \int_{Gr(2,d+1)} \left(\prod_{i=1}^{4} \sum_{a_i+b_i=d_i-2} \sigma_{a_i}\sigma_{b_i} \right) (8\sigma_{11} - 2\sigma_1^2).$$

A mysterious consequence is the invariance of N_{d_1,d_2,d_3,d_4} under the involution $d_i \mapsto d+2-d_i$, that is, $N_{d_1,d_2,d_3,d_4} = N_{d+2-d_1,\ldots,d+2-d_4}$, which we have only been able to verify by direct computation.

The main difficulty in the proof of Theorem 2 is removing the excess loci where the p_i become equal, which is achieved by a geometric construction inspired by Harris [10]. This method, however, produces a certain weighted count of linear series with base-points, so a lengthy inclusion-exclusion procedure is needed to obtain the main result. In particular, the formula for N_{d_1,d_2,d_3,d_4} is not obtained in a transparent way as an intersection of cycles on $\mathrm{Gr}(2, d+1)$, but suggests that such a computation should be possible.

In joint work with Farkas, we also consider a variant of Theorem 1 in which we impose incidence conditions on the map f. That is, we fix n general points each on C and $\mathbb{P}^1$, and require that f send the marked points on C to that on $\mathbb{P}^1$. While this problem was studied in the infancy of Gromov-Witten theory [2], it has recently received renewed interest after a result of Tevelev [18].

Theorem 3 (Cela-Pandharipande-Schmitt [4], Farkas-L. [8])**.** *Let $(C, p_1, \dots, p_n)$ be a general pointed curve of genus g, and let $q_1, \dots, q_n \in \mathbb{P}^1$ be general points. Let $L_{g,d}$ be the number of maps $f : C \to \mathbb{P}^1$ of degree d such that $f(p_i) = q_i$ for $i = 1, 2, \dots, n = 2d - g + 1$.*

(a) Suppose $d \geq g + 1$. Then

$$L_{g,d} = 2^g.$$

(b) Suppose d, g are arbitrary, and let $\ell = d - g - 1$. Then

$$\begin{aligned} L_{g,d} &= 2^g - 2\sum_{i=0}^{-\ell-2}\binom{g}{i} + (-\ell-2)\binom{g}{-\ell-1} + \ell\binom{g}{-\ell} \\ &= \int_{Gr(2,d+1)} \sigma_1^g \cdot \left[\sum_{a+b=n-3} \sigma_a \sigma_b\right]. \end{aligned}$$

Setting $n = 3$ in (b) recovers Theorem 1. The proof of Theorem 2 given by Cela-Pandharipande-Schmitt [4] employs recent advances in the intersection theory of Hurwitz spaces [17, 13], whereas that of [8] reduces, by degeneration as above, to a Schubert calculus computation in genus 0.

The analogous problem for maps of sufficiently high degree (after part (a)) to higher-dimensional projective spaces $\mathbb{P}^r$ is also addressed in [8]. For arbitrary d, one needs to carry out an intersection theory calculation on the moduli space of *complete collineations*, which is in progress. A parallel investigation in Gromov-Witten theory for more general target spaces (e.g. flag varieties) has been initiated by other authors; in the classical enumerative setting, one needs to develop Brill-Noether-type statements for maps to such targets, which is another future direction of interest.

One can recast all of our results for covers of $\mathbb{P}^1$ in terms of *Hurwitz correspondences*. Namely, let $\overline{\mathcal{H}}$ be the Hurwitz space of branched covers of $\mathbb{P}^1$ with some specified branching data at marked points, compatified by admissible covers [11, 1]. Then, $\overline{\mathcal{H}}$ admits two forgetful maps $\phi : \overline{\mathcal{H}} \to \overline{\mathcal{M}}_{g,n}$ and $\delta : \overline{\mathcal{H}} \to \overline{M}_{0,b}$ remembering the source and target, respectively. Our results give formulas for degrees of the maps ϕ and $(\phi, \delta) : \overline{\mathcal{H}} \to \overline{\mathcal{M}}_{g,n} \times \overline{M}_{0,b}$ in certain situations in which they are expected to be generically finite.

A result of Faber-Pandharipande [6] shows more generally that the cycle class $(\phi, \delta)_*([\mathcal{H}])$ always lies in the tautological ring, and moreover gives an algorithm to compute the class via virtual localization. However, the algorithm is intractable to implement in practice, and so the question remains of whether such classes are accessible by other means.

Our computations with limit linear series provide a hint toward an approach, at least over the locus $\mathcal{M}_{g,n}^{ct}$ of compact type curves. Namely, let $\pi : \mathcal{G}_d^r(\mathcal{C}_{g,n}) \to \mathcal{M}_{g,n}^{ct}$ be the stack of universal limit linear series on curves of compact type [14].

Then, cycle classes $\phi_*([\overline{\mathcal{H}}])$ may be related to pushforwards of certain tautological classes under π. The map π behaves similarly to a Grassmannian bundle, offering a prototype for the systematic computation of such pushforwards, but various technical issues would need to be surmounted. Our hope is that progress on this problem will lead not only to new enumerative results but also applications to the birational geometry of moduli spaces, via further-reaching computations of divisor classes related to linear series.

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Brill–Noether theory over the Hurwitz space

Hannah Larson

(joint work with E. Larson, I. Vogt)

Brill–Noether theory studies the maps of algebraic curves to projective space. Given a curve C, the data of a map $C \to \mathbb{P}^r$ of degree d is equivalent to a degree d line bundle L on C together with an $(r+1)$-dimensional space of global sections having no common zeros. This motivates the definition of the *Brill–Noether variety*

$$W^r_d(C) := \{L \in \mathrm{Pic}^d(C) : h^0(C, L) \geq r + 1\}.$$

For a particular curve C, describing the geometry of $W^r_d(C)$ may be very difficult. However, when C is general in moduli, the geometry of $W^r_d(C)$ is well-understood through the following theorem, due to several mathematicians in 1970s and 1980s.

Theorem 1. *For C a general curve of genus g, $W^r_d(C)$ is ...*

(1) *Of the expected dimension $\rho = g - (r+1)(g + r - d)$ and nonempty if and only if $\rho \geq 0$. (Griffith–Harris 1980* [11]*)*
(2) *Normal, Cohen–Macaulay, and smooth away from $W^{r+1}_d(C)$. (Geiseker 1982* [10]*)*
(3) *Of class*

$$[W^r_d(C)] = \prod_{\alpha=0}^{r} \frac{\alpha!}{(g-d+r+\alpha)!} \cdot \theta^{(r+1)(g-d+r)}.$$

(Independently by Kempf 1971 [13]*, and Kleiman–Laksov 1972* [14]*)*
(4) *Irreducible if $\rho > 0$. (Fulton–Lazarsfeld 1981* [9]*)*
(5) *When $\rho \geq 0$, the* universal $\mathcal{W}^r_d$ *has a* unique *irreducible component dominating the moduli space of curves. (Eisenbud–Harris 1987* [8]*)*

In nature, curves often come to us already equipped with a map to some projective space. The presence of such a map may force C to be special in moduli and fail the theorem above. It is therefore natural to ask: how does the presence of some unexpected map affect the geometry of maps of C to other projective spaces? The simplest case of this is to study $W^r_d(C)$ for curves C equipped with a degree k map $f : C \to \mathbb{P}^1$.

Although classical results fully describe $W^r_d(C)$ in the cases $k = 2, 3$ (Clifford's theorem for hyperelliptic curves and Maroni's work on trigonal curves), the picture is more complicated for $k \geq 4$. This problem received much attention in the 1990s and 2000s [4, 17, 1, 5, 6, 18, 7]. It was found that, in this setting, $W^r_d(C)$ often has multiple components of varying dimensions. In 2017, through work of Pfluger [19] and Jensen–Ranganathan [12], the dimension of the largest component of $W^r_d(C)$ was determined for C a general curve of gonality k. However, the dimensions of the other components remained a mystery.

The key to understanding these multiple components is to study the push forwards of line bundles from C to $\mathbb{P}^1$. This gives us a rank k vector bundle on $\mathbb{P}$, whose splitting type we keep track of. Explicitly, given a degree k cover $f : C \to \mathbb{P}^1$, we define *Brill–Noether splitting loci*

$$W^{\vec{e}}(C) := \{L \in \mathrm{Pic}^d(C) : f_*L \cong \mathcal{O}(e_1) \oplus \cdots \oplus \mathcal{O}(e_k) \text{ or a specialization thereof}\}.$$

(These loci were independently introduced and studied by Cook-Powell–Jensen in [2, 3]). Our main result describes the geometry of $W^{\vec{e}}(C)$ (which in turn determines the geometry of $W^r_d(C)$). To state the theorem, let

$$u(\vec{e}) := h^1(\mathbb{P}, \mathrm{End}(\mathcal{O}(e_1) \oplus \cdots \oplus \mathcal{O}(e_k))) = \sum_{i,j} \max\{0, e_i - e_j - 1\},$$

which is the dimension of the versal deformation space of $\mathcal{O}(e_1) \oplus \cdots \oplus \mathcal{O}(e_k)$.

Theorem 2 (H. Larson [16] and E. Larson, H. Larson, and I. Vogt [15])**.** *For C a general degree k, genus g cover of $\mathbb{P}^1$, $W^{\vec{e}}(C)$ is ...*

(1) Of the expected dimension $\rho' := g - h^1(\mathrm{End}(\mathcal{O}(\vec{e})))$

(2) Normal, Cohen–Macaulay, and smooth away from the union of the $W^{\vec{e}'}(C) \subset W^{\vec{e}}(C)$ having codimension 2 or more.

(3) Of class

$$[W^{\vec{e}}(C)] = \frac{N(\vec{e})}{u(\vec{e})!} \cdot \theta^{u(\vec{e})},$$

where $N(\vec{e})$ is the number of reduced words of a certain element $w_{\vec{e}}$ of the affine symmetric group.

(4) Irreducible if $\rho' > 0$.

(5) When $\rho' \geq 0$, the universal $\mathcal{W}^{\vec{e}}$ *has a* unique *irreducible component dominating the Hurwitz space of degree k, genus g covers.*

The basic approach is to degenerate C to a singular curve X, which is a chain of g elliptic curves, attached so that the difference of two nodes on the same component is exactly k-torsion. We then study the limit line bundles on X that have enough sections at each twist by $f^*\mathcal{O}(1)$ to possibly be a limit of line bundles with splitting type $\vec{e}$. The key technical difficulty is to prove a *regeneration theorem* in this context: we must show that all of these candidate limits are indeed limits and thereby enable a study of $W^{\vec{e}}(C)$ via degeneration.

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The Rank of Syzygies of Curves

Michael Kemeny

The algebraic approach to studying the projective geometry of an embedded variety $X \subseteq \mathbb{P}^n$ is to study the variety via it's *equations* and via the *relations* (and higher relations) amongst them. In order to do this, following Mumford [6], in practice one typically makes the assumption that X is *defined by quadrics*, i.e. the homogeneous ideal sheaf $I_{X/\mathbb{P}^n}$ is generated in degree two.

The assumption that a variety is defined by quadrics is often satisfied in cases of classical interest. For instance, we know it holds for canonical curves of Clifford index at least two by Petri's famous Theorem. In such a circumstance, it is very natural to ask what one can say about the generating set of quadrics. In particular, a fundamental invariant of a quadric is it's *rank*, so it's natural to ask what is the minimal rank r such that there exists a generating set of quadrics of rank at most r defining the variety $X \subseteq \mathbb{P}^n$.

In the case of canonical curves $C \subseteq \mathbb{P}^{g-1}$, it was conjectured in 1967 by Andreotti–Mayer [1] that any canonical curve of Clifford index at least two should be defined by quadrics of rank at most four. This was proven by Mark Green in 1984, [2].

In a seminar work [3], Mark Green introduced the study of *syzygies*, long a major topic in commutative algebra, into algebraic geometry. Set $S := \mathbb{C}[x_0, \dots, x_n]$, which we consider as a graded ring. Rather than just studying the ideal $I_X \subseteq S$

of an embedded variety $X \subseteq \mathbb{P}^n$, Green suggested to consider the entire *minimal free resolution*

$$\dots F_2 \to F_1 \to I_X \to 0,$$

of the graded S-module I, and to ask the question of which intrinsic geometric invariants of X could be read off from the graded pieces of the modules F_i. In the case of canonical curves, Green further made a very precise conjecture to the effect that one can read off the Clifford index from the resolution.

Following Green's philosophy of generalizing from an ideal I_X to the resolution $F_\bullet \to I_X$, Schreyer provided a notion of rank for linear generators of the modules F_i. Decompose the free modules F_i into their graded pieces by writing

$$F_i = \bigoplus_{j \geq 1} \mathrm{K}_{i,j}(X, L) \otimes_{\mathbb{C}} S(-i-j),$$

where the vector space $\mathrm{K}_{i,j}(X, L)$, called the $(i,j)^{th}$ Koszul cohomology group. For any $\alpha \neq 0 \in \mathrm{K}_{p,1}(X, L)$, we have a well-defined notion of *rank*. The following conjecture of Schreyer from the early 90s simultaneously unifies and generalises *both* Voisin's Theorem on generic Green's Conjecture [7], [8] *and* Green's Theorem on the ideal sheaf of a canonical curve:

Conjecture 1 (The Geometric Syzygy Conjecture)**.** *For a general curve of genus g, all linear syzygy spaces $\mathrm{K}_{p,1}(C, \omega_C)$ are spanned by syzygies of minimal rank $p+1$.*

In my talk at Oberwolfach, I explained a proof of The Geometric Syzygy Conjecture in full, based on the papers [4], [5].

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Effective cones of moduli spaces of stable rational curves and blown up toric surfaces

Ana-Maria Castravet

(joint work with Antonio Laface, Jenia Tevelev, Luca Ugaglia)

A question attributed to Fulton is whether the moduli space $\overline{M}_{0,n}$, of stable rational curves with n markings, has the property that every subvariety of codimension k is numerically equivalent to an effective combination of strata of codimension k (for all $0 < k < n-3$). The motivation comes from the stratification arising from the topological type of the curves parametrized, which makes $\overline{M}_{0,n}$ resemble a toric variety. On toric varieties such a property holds because the action of the open torus can be used to move subvarieties into the toric boundary.

For the case of curves ($k = n-4$), the answer to Fulton's question is known to be affirmative when $n \leq 7$ [7] and is not known for $n \geq 8$. The statement that any curve is a sum of 1-dimensional strata is known as the F-conjecture (after Fulton and Faber). Results of Gibney, Keel and Morrison [5] show that the F-conjecture, if true for for all n, implies the similar statement for all the moduli spaces $\overline{M}_{g,n}$ of genus g stable curves with n markings, for all g and n, thus giving an explicit, combinatorial description of ample divisors on the moduli spaces $\overline{M}_{g,n}$.

For the case of divisors ($k = 1$) Fulton's question can be reformulated as follows:

Is the cone $\overline{\mathrm{Eff}}(\overline{M}_{0,n})$ of effective divisors generated by boundary divisors?

If $M_{0,n}$ denotes the locus of irreducible stable rational curves with n markings, i.e., the configuration space parametrizing n distinct, labeled, points on $\mathbb{P}^1$, taken up to the action of PGL_2, then $M_{0,n} \subset \overline{M}_{0,n}$ is a dense open set, and the boundary divisors are the irreducible components of its complement. There is one boundary divisor for each partition $I \sqcup I^c = \{1, \ldots, n\}$, $2 \leq |I| \leq n-2$. The boundary divisors generate the Picard group and the Chow ring of $\overline{M}_{0,n}$.

As noted, the answer is affirmative on $\overline{M}_{0,5}$, a del Pezzo surface of degree 5, where the boundary divisors coincide with the (-1) curves, but it is negative already for $n = 6$, with a counterexample found by Keel and Vermeire [9]. As a consequence, Fulton's question for divisors (as well as cycles of dimension ≥ 2) has a negative answer for all $n \geq 6$, as forgetful maps $\overline{M}_{0,n} \to \overline{M}_{0,n-1}$ map strata to strata. We prove that in fact the effective cone is not finitely generated:

Theorem. [1] *The cone $\overline{\mathrm{Eff}}(\overline{M}_{0,n})$ is not polyhedral for $n \geq 10$, both in characteristic 0 and in characteristic p, for an infinite set of primes p of positive density, including all primes up to 2000.*

We detail the history of this question. By a theorem of Kapranov, for each marking $i \in \{1, \ldots, n\}$, the tautological line bundles ψ_i[1] induce birational morphisms $\overline{M}_{0,n} \to \mathbb{P}^{n-3}$ that are iterated blow-ups of $(n-1)$ points in linearly general position, proper transforms of all $\binom{n-1}{2}$ lines spanned by any two points, all $\binom{n-1}{3}$ planes spanned by any three points, etc. Moreover, any boundary divisor

[1]The fiber of ψ_i at $(C, p_1, \ldots, p_n) \in \overline{M}_{0,n}$ is the cotangent bundle $(T_{p_i}C)^\vee$.

is contracted by some Kapranov map. In particular, they span extremal rays of the effective cone $\overline{\mathrm{Eff}}(\overline{M}_{0,n})$. Each boundary divisor is rigid (no multiple moves), so they are uniquely determined by the extremal rays they span. For example, for $\overline{M}_{0,5}$, the Kapranov maps give all the different ways of identifying a del Pezzo surface of degree 5 as a blow-up of $\mathbb{P}^2$ at four points and each boundary divisor is an exceptional (-1) curve for such a blow-up. The Keel-Vermeire divisor on $\overline{M}_{0,6}$ can be contracted by a birational contraction, and therefore it generates an extremal ray of the effective cone $\overline{\mathrm{Eff}}(\overline{M}_{0,6})$. There are 15 Keel-Vermeire divisors on $\overline{M}_{0,6}$, but just one up to the action of S_6 that permutes the markings. Hassett and Tschinkel proved that $\overline{\mathrm{Eff}}(\overline{M}_{0,6})$ is generated by the boundary and Keel-Vermeire divisors [6]. For all $n \geq 6$, we introduced a generalization of the Keel-Vermeire divisors in [2], namely *hypertree divisors*, also contractible by birational contractions, therefore, rigid divisors, spanning extremal rays of the effective cone. As n grows, there are *many* hypertrees, even up to the S_n symmetry. Later on, Opie [8] and Doran, Giansiracusa and Jensen [4] found other extremal divisors for $n \geq 7$. Our main result explains this complexity. We use an enhancement of a technique introduced in [3], in order to reduce the question to a question about toric surfaces blown up at a general point (we may assume, the identity point of the open torus). The main result is:

Theorem. [1] *There exist projective toric surfaces $\mathbb{P}_\Delta$, given by* good polygons *Δ, for which the blow-up at the identity point of the open torus has an effective cone which is not polyhedral in characteristic* 0*. For some of these toric surfaces, the effective cone is not polyhedral in characteristic p for an infinite set of primes p of positive density.*

Good polygons are lattice polygons which give rise to what we call *arithmetic elliptic pairs of infinite order*, an interesting class of arithmetic threefolds, to which we apply tools from arithmetic geometry and Galois representations to conclude our analysis in characteristic p.

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Infinite torsion in Griffiths groups

Stefan Schreieder

For an algebraic scheme X of dimension d over $\mathbb{C}$, we consider the Chow group

$$\mathrm{CH}^i(X) := \mathrm{CH}_{d-i}(X)$$

of $(d-i)$-dimensional cycles on X modulo rational equivalence, cf. [2]. We further consider the quotient

$$A^i(X) := \mathrm{CH}^i(X)/\sim_{\mathrm{alg}}$$

modulo algebraic equivalence.

For an abelian group A, we define

$$H^i(X, A) := H^{BM}_{2d-i}(X_{an}, A),$$

where the right hand side denotes Borel–Moore homology of degree $2d-i$ of the underlying analytic space X_{an} with values in A. If X is proper, then Borel–Moore homology agrees with usual homology and if X is smooth and equi-dimensional, then

$$H^i(X, A) \cong H^i_{sing}(X_{an}, A)$$

agrees with the singular cohomology of X_{an} with values in A. The latter justifies our notation.

For any algebraic scheme X over $\mathbb{C}$, there is a cycle class map

$$\mathrm{cl}^i_X : A^i(X) \longrightarrow H^{2i}(X, \mathbb{Z}).$$

The kernel of this map is called the Griffiths group of X:

$$\mathrm{Griff}^i(X) := \ker(\mathrm{cl}^i_X : A^i(X) \to H^{2i}(X, \mathbb{Z})).$$

That is, the Griffiths group is the quotient of the group of homologically trivial cycles by the subgroup of algebraically trivial cycles. Since the Hilbert scheme that parametrizes subschemes of X has only countably many components, the group $A^i(X)$ is a countable abelian group and so the same holds for $\mathrm{Griff}^i(X)$. While Griffiths [3] showed famously that this group may be nontrivial, determining that group in explicit examples is notoriously difficult. For instance, not a single nontrivial abelian group is known to be the Griffiths group of an algebraic scheme (or of a smooth projective variety).

Some important results concerning the Griffiths group are as follows:

(1) For smooth complex projective varieties, the exponential sequence implies $\mathrm{Griff}^1(X) = 0$. Using different arguments, $\mathrm{Griff}^1(X) = 0$ is shown to hold for any algebraic scheme X over $\mathbb{C}$ in [6].

(2) If $X \subset \mathbb{P}^4_{\mathbb{C}}$ is a very general quintic hypersurface, then

$$\mathrm{Griff}^2(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^{\oplus \infty}$$

holds by results of Griffiths [3] and Clemens [1]. In particular, Griffiths groups are in general not finitely generated modulo torsion.

(3) Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be a very general quartic curve with Jacobian JC (which is a very general principally polarized abelian threefold). Then by a theorem of Totaro [10], which builds on work of Nori, Bloch-Esnault, Schoen and Rosenschon–Srinvias,

$$\mathrm{Griff}^2(JC) \otimes_{\mathbb{Z}} \mathbb{Z}/\ell \cong (\mathbb{Z}/\ell)^{\oplus \infty}$$

for any prime ℓ. In particular, Griffiths groups are in general not finitely generated modulo any prime ℓ.

Despite the above breakthroughs, it remained open whether the torsion subgroup $\mathrm{Griff}^i(X)_{tors}$ of Griffiths groups is in general finitely generated. A positive result in this direction is due to Merkurjev–Suslin:

Theorem 1 ([4]). *Let X be a smooth complex projective variety. Then the transcendental Abel–Jacobi map yields an injection*

$$\mathrm{Griff}^2(X)_{tors} \hookrightarrow \frac{H^3(X,\mathbb{Z})}{N^1 H^3(X,\mathbb{Z})} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}.$$

In particular, for any $n \geq 1$, the n-torsion subgroup of $\mathrm{Griff}^2(X)$ is finite.

The first result presented in this talk generalizes the above theorem. To state it, we define for any separated scheme of finite type over $\mathbb{C}$ the coniveau filtration N^* on $\mathrm{Griff}^i(X)$ by

$$N^j \,\mathrm{Griff}^i(X) := \mathrm{im}\left(\mathrm{colim}_{\to} \mathrm{Griff}^{i-j}(Z) \to \mathrm{Griff}^i(X)\right),$$

where $Z \subset X$ runs through all closed reduced subschemes with $\dim(Z) = \dim(X) - j$. Since $\mathrm{Griff}^1(Z) = 0$ by [6], we get a decreasing filtration

$$N^{i-1} = 0 \subset N^{i-2} \subset \cdots \subset N^1 \subset N^0 = \mathrm{Griff}^i(X).$$

The above filtration induces a filtration on the torsion subgroup $\mathrm{Griff}^i(X)_{tors}$ of $\mathrm{Griff}^i(X)$ by $N^j \,\mathrm{Griff}^i(X)_{tors} = N^j \cap \mathrm{Griff}^i(X)_{tors}$.

Theorem 2 (Sch. 2021). *Let X be a separated scheme of finite type over $\mathbb{C}$. Then for any positive integer n, there is a transcendental Abel–Jaocbi map*

$$\bar{\lambda}^i_{tr} : \mathrm{Griff}^i(X)_{tors} \longrightarrow \frac{H^{2i-1}(X,\mathbb{Z})}{N^1 H^{2i-1}(X,\mathbb{Z})} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$$

which is induced by Griffiths' transcendental Abel–Jaocbi map if X is smooth projective. The kernel of the above map is given by

$$\ker(\bar{\lambda}^i_{tr}) = N^1 \,\mathrm{Griff}^i(X)_{tors}.$$

The above theorem implies that any torsion class $z \in \mathrm{Griff}^i(X)_{tors}$ which has trivial Abel–Jacobi invariant is given by the pushforward of a homologically trivial cycle $z' \in \mathrm{Griff}^{i-1}(D)$ on some divisor D on X and one can show that z' may in fact be chosen to be torsion of the same order as z. This property is not at all obvious in any of the known explicit examples of torsion elements in the Griffiths group with trivial Abel–Jacobi invariants, see [9, 8, 7] (non-torsion classes in the Griffiths group with trivial Abel–Jacobi invariant have earlier been constructed by Nori [5]). Since Borel–Moore homology of algebraic schemes is finitely generated, we deduce from the above theorem that the cokernel of $N^1 \mathrm{Griff}^i(X)[n] \subset \mathrm{Griff}^i(X)[n]$ is finite for any $n \geq 1$. Applying the same argument to any codimension one subscheme $D \subset X$, we find that $N^1 \mathrm{Griff}^{i-1}(D)[n] \subset \mathrm{Griff}^{i-1}(D)[n]$ has finite cokernel as well, and so one might be tempted to try to prove finiteness of the n-torsion subgroup $\mathrm{Griff}^i(X)[n]$ by induction on the coniveau filtration. The main obstacle here is that there may be infinitely many choices of divisors $D \subset X$ as above.

The second result presented in this talk settles the long-standing question whether the torsion subgroup of Griffiths groups is finitely generated. Our result shows in particular that an inductive approach as outlined above does in fact not work; that is, $N^1 \mathrm{Griff}^i(X)_{tors}$ is in general not supported on a finite union of divisors $D \subset X$.

Theorem 3 (Sch. 2021)**.** *Let X be a very general Enriques surface over $\mathbb{C}$ and let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be a very general quartic curve with Jacobian JC. Then*

$$\mathrm{Griff}^3(X \times JC)[2] \cong (\mathbb{Z}/2)^{\oplus\infty}.$$

The above result is deduced from [10] together with the following injectivity theorem, which is the main result of [7].

Theorem 4 (Sch. 2021)**.** *Let Y be a smooth complex projective variety and let X be a very general Enriques surface over $\mathbb{C}$. Then the exterior product map*

$$A^i(Y)/2 \longrightarrow A^{i+1}(X \times Y)[2], \quad [z] \mapsto [z \times K_X]$$

is injective for any $i \geq 1$.

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Equivariant birational types

Yuri Tschinkel
(joint work with M. Kontsevich, V. Pestun, A. Kresch, B. Hassett)

Let X be a smooth projective variety of dimension n over an algebraically closed field k of characteristic zero. We assume that X is rational, i.e., birational to $\mathbb{P}^n$, and that it carries a regular and generically free action of a finite group G. A classical problem is to decide whether or not this action is *linearizable*, i.e., whether or not X is G-equivariantly birational to $\mathbb{P}^n$, with a (projectively) linear action of G. There is an extensive literature on this problem, already for $n = 2$, going back to Bertini, Castelnuovo, Kantor, and many others, and culminating in the work of Dolgachev–Iskovskikh [1].

Among the known equivariant birational invariants is:

(1) Existence of fixed points upon restriction to abelian subgroups of G.

A more sophisticated invariant was introduced in [9], for *abelian* G:

(2) Let $\mathfrak{p} \in X$ be a point fixed by G. Let $\{a_1, \ldots, a_n\}$ be its weights, i.e., characters of G in the tangent space to X at $\mathfrak{p}$, and

$$\det(a_1, \ldots, a_n) = a_1 \wedge \cdots \wedge a_n \in \wedge^n(G^\vee)$$

the *determinant*. Let $Y \to X$ be a G-equivariant blowup. Then Y contains a G-fixed point $\mathfrak{q}$ (in the preimage of $\mathfrak{p}$) with weights $\{b_1, \ldots, b_n\}$, such that

$$\det(b_1, \ldots, b_n) = \pm \det(a_1, \ldots, a_n),$$

i.e., this is an *equivariant birational invariant.*

Inspired by applications of ideas from motivic integration to the study of rationality properties of algebraic varieties [8, 4], and by keen interest in equivariant birational geometry, the following generalization of (2) was introduced in [3]:

Let G be *abelian*, and $X^G = \sqcup_\alpha F_\alpha$ the decomposition of the G-fixed locus into irreducible components. Recording the G-eigenvalues

$$[b_{1,\alpha}, \ldots, b_{n,\alpha}], \quad b_{j,\alpha} \in G^\vee = \mathrm{Hom}(G, k^\times),$$

in the tangent space $\mathcal{T}_{x_\alpha} X$, at some $x_\alpha \in F_\alpha$, we put, formally,

$$\beta(X) := \sum_\alpha [b_{1,\alpha}, \ldots, b_{n,\alpha}].$$

Let $\mathcal{S}_n(G)$ be the free abelian group generated by *unordered* tupels $[b_1, \ldots, b_n]$, with $b_i \in G^\vee$, such that $\sum_i \mathbb{Z} b_i = G^\vee$. Consider the quotient

$$\mathcal{S}_n(G) \to \mathcal{B}_n(G),$$

by the *blow-up* relations

$$\beta(Y) - \beta(X) = 0,$$

for every G-equivariant blowup $Y \to X$. It turns out, that *all* such relations can be encoded in a compact form:

(B) for all $b_1, b_2, b_3, \dots, b_n \in G^\vee$ we have

$$\begin{aligned} [b_1, b_2, b_3, \dots b_n] &= [b_1 - b_2, b_2, b_3, \dots, b_n] + [b_1, b_2 - b_1, b_3, \dots, b_n], \quad && b_1 \neq b_2, \\ &= [b_1, 0, b_3, \dots, b_n], && b_1 = b_2. \end{aligned}$$

Equivariant Weak Factorization yields:

(3) The class $\beta(X) \in \mathcal{B}_n(G)$ is a G-equivariant birational invariant [3].

The groups $\mathcal{B}_n(G)$ exhibit a rather intricate internal structure, they are equal to cohomology of certain congruence subgroups, carry Hecke operators etc., see [3]. First geometric applications of this new invariant can be found in [2].

The next development, in [5], addressed three issues:

- extension to *nonabelian* groups,
- considerations of *all* possible, and not just maximal, stabilizers, and
- inclusion of the function-field information of strata, with induced actions.

The geometric input data for the definitions in [5] are:

- $\mathrm{Bir}_d(k)$ – birationality classes of varieties of dimension d over k,
- $\mathrm{Alg}_N(K_0)$ – isomorphism classes of Galois algebras K over $K_0 \in \mathrm{Bir}_d(k)$ for a finite group N, subject to a certain **Assumption 1.**

Let $\mathrm{Burn}_n(G) = \mathrm{Burn}_{n,k}(G)$ be the $\mathbb{Z}$-module, generated by symbols

$$(H, N \curvearrowright K, \beta),$$

where

- $H \subseteq G$ is an abelian subgroup, with character group $H^\vee$, and $N := N_G(H)/H$,
- $K \in \mathrm{Alg}_N(K_0)$, with $K_0 \in \mathrm{Bir}_d(k)$, and $d \leq n$,
- $\beta = (b_1, \dots, b_{n-d})$, a sequence, up to order, of *nonzero* elements of $H^\vee$, that generate $H^\vee$.

The symbols are subject to **conjugation** and **blowup** relations:

(C): $(H, N \curvearrowright K, \beta) = (H', N' \curvearrowright K, \beta')$, when $H' = gHg^{-1}, N' = N_G(H')/H'$, and β and β' are related by conjugation by $g \in G$.

(B1): $(H, N \curvearrowright K, \beta) = 0$, when $b_1 + b_2 = 0$.

(B2): $(H, N \curvearrowright K, \beta) = \Theta_1 + \Theta_2$, where

$$\Theta_1 = \begin{cases} 0, & \text{if } b_1 = b_2, \\ (H, N \curvearrowright K, \beta_1) + (H, N \curvearrowright K, \beta_2), & \text{otherwise,} \end{cases}$$

with

$$\beta_1 := (b_1 - b_2, b_2, b_3, \dots, b_{n-d}), \quad \beta_2 := (b_1, b_2 - b_1, b_3, \dots, b_{n-d}),$$

and

$$\Theta_2 = \begin{cases} 0, & \text{if } b_i \in \langle b_1 - b_2 \rangle \text{ for some } i, \\ (\overline{H}, \overline{N} \curvearrowright \overline{K}, \bar{\beta}), & \text{otherwise,} \end{cases}$$

with

$$\overline{H}^\vee := H^\vee / \langle b_1 - b_2 \rangle, \quad \bar{\beta} := (\bar{b}_2, \bar{b}_3, \dots, \bar{b}_{n-d}), \quad \bar{b}_i \in \overline{H}^\vee,$$

and a specified action on a new algebra $\overline{K}$. The Burnside groups $\mathrm{Burn}_n(G)$ also have an intricate internal structure: they admit interesting filtrations, forgetful homomorphisms, restriction, induction, comparison homomorphisms, see [6].

The class of a G-variety X is computed on a *standard model* (X, D):

- X is smooth projective, D a normal crossings divisor,
- G acts freely on $U := X \setminus D$,
- $\forall g \in G$ and irreducible components D, either $g(D) = D$ or $g(D) \cap D = \emptyset$.

Passing to a standard model X, define the class:

$$[X \circlearrowleft G] := \sum_H \sum_F (H, N \curvearrowright k(F), \beta_F(X)) \in \mathrm{Burn}_n(G),$$

where the sum is over (conjugacy classes of) *abelian* subgroups $H \subseteq G$, and all $F \subset X$ with generic stabilizer H. The symbols record the generic eigenvalues of H in the normal bundle along F, as well as the $N = N_G(H)/H$-action on the function field of F, respectively the orbit of F. Note that, on a standard model, all stabilizers are abelian, and all symbols satisfy **Assumption 1**.

(4) The class $[X \circlearrowleft G] \in \mathrm{Burn}_n(G)$ is a well-defined G-equivariant birational invariant [5, Theorem 5.1].

Using this invariant, we found new examples of finite groups G admitting intransitive, nonbirational actions on $\mathbb{P}^2$, addressing a problem raised in [1, Section 9], and $\mathbb{P}^3$ [7].

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Antisymplectic involutions on projective hyper-Kähler manifolds

Emanuele Macrì
(joint work with Laure Flapan, Kieran G. O'Grady, Giulia Saccà)

In this talk, I reported on the study of fixed loci of antisymplectic involutions on projective hyper-Kähler manifolds, induced by an ample class of square 2 in the Beauville-Bogomolov-Fujiki lattice. I presented results on how to determine the number of connected components of the fixed loci and how to study their geometry in lower dimensions.

Let X be a projective hyper-Kähler (HK) manifold, namely X is simply connected and $H^0(X, \Omega_X^2) = \mathbb{C} \cdot \eta$, where η is a non-degenerate symplectic form.

Definition 1. Let $\tau\colon X \xrightarrow{\cong} X$ be an involution, $\tau^2 = \mathrm{id}$. We say that τ is *antisymplectic* if $\tau^*\eta = -\eta$.

An immediate observation is that if τ is an antisymplectic involution, then its fixed locus $\mathrm{Fix}(\tau) \subset X$ is a closed lagrangian submanifold.

The goal is to understand the geometry of $\mathrm{Fix}(\tau)$; see [1, 14]. The motivation comes from several viewpoint in the theory of HK manifolds, including understanding the correspondence with Fano manifolds (currently only observed in special examples [2, 4, 5, 6]) and in the existence of covering families of lagrangian submanifolds and applications to the study of Chow groups [16]. The rich geometry of these fixed loci can be already observed in the lower dimensional case; for example, EPW sextics [15] and cubic fourfolds [12].

Notice also that for *symplectic* involutions, namely if $\tau^*\eta = \eta$, the fixed loci are well understood for two of the main families of examples of HK manifolds [11]: their connected components are symplectic submanifolds in that case.

Let (X, λ) be a polarized hyper-Kähler manifold of dimension $2n$. We assume that X is of $\mathrm{K3}^{[n]}$*-type*, namely it is deformation equivalent to the Hilbert scheme of n points on a K3 surface.

Let q_X denote the Beauville-Bogomolov-Fujiki quadratic form on $H^2(X;\mathbb{Z})$. We assume that the polarization λ satisfies $q_X(\lambda) = 2$. If we denote by $\mathrm{div}(\lambda)$ the positive generator of the ideal $\{q(\lambda, w) : w \in H^2(X;\mathbb{Z})\} \subset \mathbb{Z}$, the *divisibility* of λ, then we must have $\mathrm{div}(\lambda) \in \{1, 2\}$; moreover, if $\mathrm{div}(\lambda) = 2$, then $4 \mid n$.

By the Global Torelli Theorem [17, 13, 10], to such polarization λ we can associate an antisymplectic involution

$$\tau_\lambda\colon X \xrightarrow{\cong} X$$

which acts on $H^2(X;\mathbb{Z})$ as reflection at λ:

$$\tau_{\lambda,*}(x) = -x + q_X(\lambda, x), \qquad x \in H^2(X;\mathbb{Z})).$$

Equivalently, we are looking at involutions τ for which the invariant part of the action on $H^2(X;\mathbb{Z})$ is of rank 1, generated by an ample class of square 2.

The main result in [8] determines the number of connected components of $\mathrm{Fix}(\tau_\lambda)$:

Theorem 2. *The fixed locus* $\mathrm{Fix}(\tau_\lambda)$ *has exactly* $\mathrm{div}(\lambda)$ *connected components.*

We can then start looking at the geometry of such fixed loci in lower dimension. We start with the divisibility 1 case; by Theorem 2 the fixed locus $F := \mathrm{Fix}(\tau_\lambda)$ is connected in this case. The case $n = 2$ is now well-known: the general (X, λ) in the moduli space is a double EPW sextic, with the double cover involution coinciding with the involution τ_λ. Then F is a surface of general type, whose invariants are all known; see [7]. In the cases $n = 3$ and $n = 4$ we do expect a similar behavior: the fixed locus F should be of general-type with an explicit formula for its canonical bundle in terms of $\lambda|_F$.

In the divisibility 2 case, again by Theorem 2 the fixed locus $\mathrm{Fix}(\tau_\lambda)$ has exactly two connected components. The first case $n = 4$ is already not completely clear: all (X, λ) in the moduli spaces are isomorphic to the Lehn-Lehn-Sorger-van Straten HK 8-fold associated to a cubic fourfold (not containing a plane), with the involution coinciding with the involution coming from realizing X as moduli space of equivalence classes of twisted cubic curves in the cubic Y; see [12] and [3, Appendix B]. One component to the fixed locus is then isomorphic to the cubic fourfold Y itself. The second component is the closure of the locus parameterizing twisted cubics contained in a cubic surface with four A_1-singularities, but the global geometry of this component is still unknown (although we suspect it being of general type).

The main result in [9] deals with the next case $n = 8$.

Theorem 3. *Let* $n = 8$ *and let* (X, λ) *be a polarized HK manifold of* $\mathrm{K3}^{[8]}$*-type such that* $q_X(\lambda) = 2$ *and* $\mathrm{div}(\lambda) = 2$*. Then one connected component* Y *of* $\mathrm{Fix}(\tau_\lambda)$ *is a prime Fano manifold of dimension* 8 *and index 3.*

The odd cohomology of Y *vanishes and its Hodge diamond is*

$$\begin{array}{lccccc}
H^8(Y;\mathbb{C}): & 1 & 22 & 253 & 22 & 1 \\
H^6(Y;\mathbb{C}): & & 1 & 22 & 1 & \\
H^4(Y;\mathbb{C}): & & 1 & 22 & 1 & \\
H^2(Y;\mathbb{C}): & & & 1 & & \\
H^0(Y;\mathbb{C}): & & & 1 & &
\end{array}$$

Some of the arguments in our proofs work for any n. In divisibility 2, we can always isolate a special component Y, by using the choice of a linearization of the action of the involution on the line bundle $\mathcal{O}_X(\lambda)$. Theorem 3 would then hold in any dimension, if we would be able to establish normality of a certain degeneration of the fixed component Y.

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On the irrationality of moduli spaces of K3 and abelian surfaces

Daniele Agostini
(joint work with Ignacio Barros, Kuan-Wen Lai)

In recent years there has been a renewed interest in quantitative measures of irrationality for algebraic varieties [2]. Beyond the usual Kodaira dimension, two such measures for a complex variety X are the *degree of irrationality* $\operatorname{irr}(X)$, defined as the minimum degree of a dominant map

$$X \dashrightarrow \mathbb{P}^{\dim X},$$

and the *covering gonality* $\operatorname{cov.gon}(X)$, the minimum gonality of a curve passing through a general point of X.

Such rationality questions are especially interesting in the case of moduli spaces. For example, the moduli space $\mathcal{M}_g$ of genus g curves is of general type for g large [6], however the only known bounds on the measures of irrationality are in terms of Hurwitz numbers [2], which grow very fast with the genus g.

In our work, we consider instead the moduli space $\mathcal{F}_g$ of polarized K3 surfaces of genus g. As for $\mathcal{M}_g$, this space is unirational for g small, but of general type as soon as g is large enough [4]. However, the degree of irrationality is bounded polynomially in terms of the genus:

Theorem 1 (A.-Barros-Lai, [1])**.** *For every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that*

$$\operatorname{irr}(\mathcal{F}_g) \leq C_\varepsilon \cdot g^{14+\varepsilon} \qquad \text{for all } g.$$

Furthermore, for infinitely many series, the bound can be improved:

Theorem 2 (A.-Barros-Lai, [1])**.** *Let $d := 2g-2 > 6$, n a positive integer and assume one of the following:*

(A) $d \equiv 0$ or $2 \pmod 6$ and is not divisible by any odd prime $p \equiv 2 \pmod 3$.
(B) $d \equiv 2$ or $4 \pmod 8$ and is not divisible by any prime $p \equiv 3 \pmod 4$.
(C) $\frac{d}{2} - n$ is a square.

Then there exists a constant $C > 0$, depending on n in case (C), *such that*

$$\operatorname{irr}(\mathcal{F}_g) \leq C \cdot g^{10}.$$

Another similar result, to appear soon, is about the moduli space $\mathcal{A}_d := \mathcal{A}_{(1,d)}$ of abelian surfaces with a $(1,d)$-polarization.

Theorem 3 (A.-Barros-Lai, *to appear*)**.** *For every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that*

$$\operatorname{irr}(\mathcal{A}_d) \leq C_\varepsilon \cdot d^{6+\varepsilon} \qquad \text{for all } d.$$

Moreover, let $n > 0$ be a positive integer and suppose that $\frac{d}{6} - \frac{\varepsilon}{2}$ is a square. Then there exists $C_n > 0$ such that

$$\operatorname{irr}(\mathcal{A}_d) \leq C_n \cdot d^2.$$

To illustrate the idea of the proof, we consider case (A) of Theorem 2. In this setting, Hassett [5] proved that there are dominant rational maps $\mathcal{F}_g \dashrightarrow \mathcal{C}_d$ of degree at most two, onto some special divisors in the moduli space $\mathcal{C}$ of cubic fourfolds. Hence $\operatorname{irr}(\mathcal{F}_g) \leq 2 \cdot \operatorname{irr}(\mathcal{C}_d)$. Furthermore, the $\mathcal{C}_d$ are (components of) Heegner divisors in the period domain $\mathcal{C}$, which admits an embedding $\mathcal{C} \hookrightarrow \mathbb{P}^N$ into a certain projective space. Thus, the irrationality degree of the $\mathcal{C}_d$ are bounded by the degrees $\deg(\mathcal{C}_d)$ of the closures $\overline{\mathcal{C}}_d$ inside $\mathbb{P}^N$: indeed, a general linear projection $\overline{\mathcal{C}}_d \to \mathbb{P}^{19}$ has degree exactly $\deg(\mathcal{C}_d)$. At this point, the fundamental input is provided by the work of Borcherds [3], which implies that the generating series

$$\sum_d \deg(\mathcal{C}_d) \cdot q^{\frac{d}{6}}$$

is a modular form of weight 11, hence its coefficients grow at most like a polynomial of degree 10.

The other cases of Theorem 2 are dealt with in a similar way. The same ideas extend to Theorem 1: there are [8] simultaneous maps $\mathcal{F}_g \dashrightarrow \mathcal{P}_\#$ of all moduli spaces into the same period space $\mathcal{P}_\#$, the catch is that the images are not divisors anymore but higher codimension cycles. However, we can use a generalization of Borcherds'result, known as Kudla's modularity conjecture [7], to conclude. Finally, Theorem 3 is obtained by considering moduli of lattice-polarized K3 surfaces of Kummer type.

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Torsion line bundles on finite covers

Aaron Landesman

We outline a new series of moduli spaces whose points are closely related to n-torsion line bundles on degree d covers of a base B. The story is especially clean when $d = 2$, in which case these spaces parameterize n-coverings of generically singular genus 1 curves over B, as described in [1]. For example, when $B = \mathbb{P}^1$, we construct a moduli space whose $\mathbb{P}^1$ points parameterize n-torsion line bundles on hyperelliptic curves.

This investigation is motivated by the Cohen-Lenstra heuristics in number theory, which work in the case that B is the spectrum of the integers, and describe the average number of n-torsion elements in class groups of quadratic number fields. It is a big open question in arithmetic statistics to count the asymptotic number of these n-torsion elements in quadratic fields, and it would be quite interesting if one were able to use these moduli spaces to approach that problem. A simple-to-state consequence of our approach is the following: under the correspondence between quadratic forms and line bundles on spectra of rings of integers of quadratic fields, a quadratic form q corresponds to an n-torsion line bundle if and only if there exists a degree n polynomial whose resultant with q is ± 1.

We now describe our moduli space in the case $d = 2$. Let S denote the secant variety to the rational normal curve in $\mathbb{P}^n$, and let $U \subset S$ denote the open subscheme where one removes the rational normal curve. Then, the relevant moduli space is the quotient of U by PGL_2, acting as automorphisms of the rational normal curve.

Let us explain why the secant variety to the rational normal curve is related to n-torsion line bundles on degree 2 covers. Starting with a degree 2 cover $g : X \to B$ and an n-torsion line bundle L on X, we produce an embedding $X \to \mathbb{P}(g_*L)$. The condition that this is n-torsion amounts to the existence of an isomorphism $L^{\otimes n} \simeq \mathcal{O}_X$. To understand the nth tensor power of L, we compose with the n-Veronese embedding to get a map $X \to \mathbb{P}(g_*L) \to \mathbb{P}(\mathrm{Sym}^n(g_*L))$. We have a surjection $\mathrm{Sym}^n(g_*L) \to g_*L^{\otimes n} \simeq g_*\mathcal{O}_X \to \mathrm{coker}(\mathcal{O}_B \to g_*\mathcal{O}_X)$ which picks out a line M in $\mathbb{P}(\mathrm{Sym}^n g_*L)$ and a point p on M which does not meet X. In fibers, we can think of X as two points on the rational normal curve. These two points span the line M, and the point p on M is then a point on the secant variety to the rational normal curve.

After explaining the above construction, we investigated in the case $d = 2$ and $n = 3$. For this, we reviewed the classical correspondence between 3-torsion line bundles on degree 2 covers and degree 3 covers. We used our construction to understand this classical correspondence as a corollary of the fact that the secant variety to the twisted cubic is all of $\mathbb{P}^3$.

We then proceeded to explain an analogous construction in degree $d = 3$. Here, instead of the moduli space being the 2-secant variety to the rational normal curve in $\mathbb{P}^n$, the relevant space was the space of lines contained in the 3-secant variety to the n-Veronese surface. Following this, we investigated the case $d = 3$ and $n = 2$, which recovers the classical Recillas correspondence. We used our construction to understand this as a corollary of the fact that any line in $\mathbb{P}^5$ not meeting the 2-Veronese surface lies on a unique 3-Secant 2-plane to the 2-Veronese surface.

To conclude, we returned to the case that $d = 2$ and gave two other equivalent descriptions of our moduli space. First, we exhibited it as a moduli space for families of generically singular genus 1 curves with a degree n line bundle and geometrically integral fibers. Second, we exhibited it as a moduli space of families of smooth divisors in the linear system $\mathcal{O}_Y(1)$, for $Y = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ the Hirzebruch surface.

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Cohomology of line bundles on the incidence correspondence

Claudiu Raicu
(joint work with Zhao Gao)

Let V be a vector space of dimension $n \geq 3$ over a field $\mathbf{k}$ and consider the variety $\text{Flag}(V)$ parametrizing complete flags in V. A fundamental question at the confluence of algebraic geometry and representation theory is the following.

Problem 1. *Determine the cohomology groups of line bundles on* $\text{Flag}(V)$*.*

In characteristic zero, Problem 1 is completely understood via Borel–Weil–Bott theory. In particular, for each line bundle there exists at most one non-vanishing cohomology group, which is an irreducible representation of $\text{GL}(V)$. This is no longer true in characteristic $p > 0$, where describing the structure of the cohomology groups as $\text{GL}(V)$-representations is a wide open problem. It is natural to consider a relaxation of Problem 1 where we only ask to detect which of the cohomology groups (do not) vanish.

Problem 2. *Characterize the (non)vanishing behavior of the cohomology groups of line bundles on* $\text{Flag}(V)$*.*

For H^0, the answer to Problem 2 is given by Kempf's vanishing theorem [6, 5]: a line bundle $\mathcal{L}$ associated to a weight $\lambda \in \mathbb{Z}^n$ has $H^0(\text{Flag}(V), \mathcal{L}) \neq 0$ if and only if λ is dominant (that is, if $\lambda_1 \geq \lambda_2 \geq \cdots$); moreover, for such line bundles all the higher cohomology groups vanish. A complete characterization for the (non)vanishing of $H^1(\text{Flag}(V), \mathcal{L})$ is given by Andersen [1]. By Serre duality, the aforementioned results characterize the (non)vanishing of $H^{d-1}(\text{Flag}(V), \mathcal{L})$ and $H^d(\text{Flag}(V), \mathcal{L})$, where $d = \dim(\text{Flag}(V))$, but the behavior of intermediate cohomology groups remains more mysterious. One can also consider Problems 1, 2 in the context of generalized (partial) flag varieties G/B (resp. G/P) where G is a connected reductive algebraic group and B (resp. P) is a Borel (resp. parabolic) subgroup. Despite some progress, these questions remain to a large extent unanswered.

Following up on work of Liu [7] and Liu–Polo [8], we restrict in [3] to line bundles corresponding to weights in $\mathbb{Z}^n$ of the form

$$\lambda_2 = \cdots = \lambda_{n-1} = 0. \tag{1}$$

These arise as pullbacks of line bundles on the incidence correspondence

$$X = \{(p, H) \in \mathbb{P}V \times \mathbb{P}V^\vee : p \in H\},$$

along the natural forgetful map $f : \mathrm{Flag}(V) \longrightarrow X$. We note here that X is an example of a partial flag variety, and that via the projection formula, the cohomology of a line bundle on X agrees with that of its pull-back to $\mathrm{Flag}(V)$. It is convenient to parametrize the weights in (1) as

$$\lambda = (e+1, 0, \cdots, 0, d+n-1), \tag{2}$$

for $d, e \in \mathbb{Z}$. It further suffices to consider only parameters satisfying $d \geq 0$, $e \geq -1$, since the remaining cases can be analyzed using Kempf vanishing and Serre duality. Our answer to Problem 2 for the line bundles coming from X is as follows (note that the case $n = 3$ is contained in [4]).

Theorem 3. *Suppose that* $\mathrm{char}(\mathbf{k}) = p > 0$*, let* $\mathcal{L} = \mathcal{L}(d, e)$ *denote the line bundle on* $\mathrm{Flag}(V)$ *associated to the weight* λ *in* (2)*, and assume that* $d \geq 0$, $e \geq -1$.

(a) $H^j(\mathrm{Flag}(V), \mathcal{L}) = 0$ *for* $j \leq n-3$ *or* $j \geq n$.

(b) $H^{n-1}(\mathrm{Flag}(V), \mathcal{L}) \neq 0$ *if and only if* $e \leq (t+n-2)q - n$*, where* $1 \leq t < p$ *and* $q = p^r$ *(or* $t = 0$ *and* $q = 1$ *if* $d = 0$*) are determined by the inequality*

$$tq \leq d < (t+1)q.$$

(c) $H^{n-2}(\mathrm{Flag}(V), \mathcal{L}(d,e)) = H^{n-1}(\mathrm{Flag}(V^\vee), \mathcal{L}(e+1, d-1))$*, whose (non) vanishing is determined by the recipe in part (b).*

A key ingredient in the proof of the nonvanishing part of Theorem 3 is the theory of Frobenius splittings due to Mehta and Ramanathan [9]. A second ingredient, which is also relevant for the vanishing part, comes from the identification

$$H^j(\mathrm{Flag}(V), \mathcal{L}) = H^{j-n+2}(\mathbb{P}V, \mathrm{D}^d\,\mathcal{R}(e)), \tag{3}$$

where $\mathcal{R} = \Omega^1_{\mathbb{P}V}(1)$ is the tautological rank $(n-1)$ subbundle on $\mathbb{P}V$, and

$$\mathrm{D}^d\,\mathcal{R} = (\mathrm{Sym}^d(\mathcal{R}^\vee)^\vee)$$

is a divided power of $\mathcal{R}$. We prove the vanishing for the cohomology groups on the right hand side of (3) using natural filtrations defined on divided/symmetric powers via Frobenius as in [2], along with considerations of Castelnuovo–Mumford regularity, and prove the following.

Theorem 4. *Suppose that* $\mathrm{char}(\mathbf{k}) = p > 0$*. For* $d \geq 1$*, let* $q = p^r$ *and* $1 \leq t < p$ *such that* $tq \leq d < (t+1)q$*. We have that the Castelnuovo–Mumford regularity of* $\mathrm{D}^d\,\mathcal{R}$ *is given by*

$$\mathrm{reg}(\mathrm{D}^d\,\mathcal{R}) = (t+n-2)q - n + 2.$$

A careful analysis of the arguments allows for some explicit calculations of cohomology for extreme values of the parameters d, e. For instance, if $d = tq$ and $e = (t+n-2)q - n$ in part (b) of Theorem 3, then

$$H^{n-1}(\mathrm{Flag}(V), \mathcal{L}) = F^q\left(\mathbb{S}_{t-1,t-1}V\right),$$

where $\mathbb{S}_{t-1,t-1}$ denotes the Schur functor associated to $(t-1, t-1, 0, \cdots)$, and F^q denotes the polynomial subfunctor of Sym^q generated by the q-th powers of linear forms. Other explicit calculations of cohomology are contained in [7] and [8], but a complete description for general parameters d, e remains unknown.

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Projectivity criteria for algebraic spaces

János Kollár
(joint work with D. Villalobos-Paz)

The talk discussed 2 results on the projectivity of algebraic spaces.

It was observed about 40 years ago that, as a consequence of Mori's program, proper but non-projective algebraic spaces should contain rational curves. In varying generality, results of this type were conjectured and proved by Campana, Kollár, Peternell, Shokurov. A theorem of Villalobos-Paz turns this into a necessary and sufficient condition.

Theorem 1. [2] *Let X be a compact Moishezon manifold. Then X is non-projective iff there is a rational curve $C \subset X$ such that $-[C] \in \overline{\mathrm{NE}}(X)$, where $\overline{\mathrm{NE}}(X) \subset H_2(X, \mathbb{R})$ denotes the closed cone of curves.*

More generally, this holds if X is $\mathbb{Q}$-factorial with log terminal singularities. There is also a variant for proper morphisms of algebraic spaces over $\mathbb{C}$.

The main technical result shows that one can run the relative Minimal Model Program for projective morphisms of algebraic spaces. Then one needs to apply this to a projective modification of X, and study the first MMP step that results in a non-projective variety.

The second result shows that, among algebraic spaces, flat deformations of a projective scheme are projective. We state the results for deformations over the disc $\mathbb{D}$, though similar results hold over arbitrary bases.

Theorem 2. [1] *Let $g : X \to \mathbb{D}$ be a proper, flat morphism of complex analytic spaces. Assume that*

(1) X_0 is projective,
(2) the fibers X_s have rational singularities for $s \neq 0$, and
(3) g is bimeromorphic to a projective morphism $g^{\mathrm{p}} : X^{\mathrm{p}} \to \mathbb{D}$.

Then g is projective over a smaller punctured disc $\mathbb{D}^{\circ}_{\epsilon} \subset \mathbb{D}$.

Note that g need not be projective over any smaller disc $\mathbb{D}_{\epsilon} \subset \mathbb{D}$.

To prove this, we may assume that X retracts to X_0. Now take an ample line bundle L_0 on X_0 and lift its Chern class to a cohomology class $\Theta \in H^2(X, \mathbb{Z})$. Usually Θ is not an algebraic cohomology class, but, a Chow-variety argument shows that it satisfies Seshadri's ampleness criterion on all but countably many fibers X_s. One then proves that these fibers are projective. A Baire category argument then shows that the non-projective fibers form a discrete set.

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Noether-Lefschetz theory of hyper-Kähler varieties via Gromov-Witten invariants

Georg Oberdieck

1. Hassett-Looijenga-Shah divisors

Let $\mathcal{M}_2$ be the moduli space of quasipolarized K3 surfaces of genus 2. It is birational to the GIT quotient $\mathbb{P}(H^0(\mathbb{P}^2, \mathcal{O}(6)))//\mathrm{SL}_6$ parametrizing double covers of $\mathbb{P}^2$ branched along a sextic curve. Hwoever, there are differences in moduli. For once, the Noether-Lefschetz divisor in $\mathcal{M}_2$ of elliptic K3 surfaces with section,

$$\mathsf{NL}_{\left(\begin{smallmatrix} 2 & 1 \\ 1 & 0 \end{smallmatrix}\right)} \in \mathrm{Div}(\mathcal{M}_2),$$

can *not* correspond to any divisor in the GIT moduli space.[1] The divisor $\mathsf{NL}_{\left(\begin{smallmatrix} 2 & 1 \\ 1 & 0 \end{smallmatrix}\right)}$ is the simplest example of an *Hassett-Looijenga-Shah (HLS) divisor*.

Let V_{10} be a 10-dimensional vector space and $\sigma \in \wedge^3 V_{10}^*$ a non-zero tri-linear form. The Debarre-Voisin variety [4] associated to σ is defined by

$$X_\sigma = \{V \in \mathrm{Gr}(6, 10) \,|\, \sigma|_V = 0\}$$

and is hyperkähler if it is smooth of dimension 4. The period map gives rise to a birational [7] morphism from the GIT moduli space to the moduli space of hyperkähler fourfolds of $K3^{[2]}$-type of degree 22 and divisibility 2:

$$\mathcal{P} \colon \mathbb{P}(\wedge^3 V_{10}^*)//\mathrm{SL}_{10} \dashrightarrow \mathcal{M}_{22}^{(2)}$$

[1] If such K3 were realized as double cover of $\mathbb{P}^2$, the algebraic class of square zero and degree 1 against the quasi-polarization would give rise to an elliptic curve mapping with degree 1 to $\mathbb{P}^2$, which is absurd.

The rational map $\mathcal{P}$ restricts to a regular morphism $\mathcal{P}: U \to \mathcal{M}_{22}^{(2)}$ on the open subset parametrizing 4-dimensional Debarre-Voisin varieties which are smooth or at most nodal along a smooth K3 surface. The complement of U has codimension ≥ 2, see [4, 1, 6].

Definition 1. *An irreducible divisor $D \subset \mathcal{M}_{22}^{(2)}$ is HLS if $\mathcal{P}^*(D) = 0$ in $\mathrm{Div}(U)$.*

Remark 1. *This definition was first given in* [2] *although in slightly different but equivalent form (see* [6, Sec.6.2] *why they are equivalent).*

Following [3], let $\mathcal{C}_{2e} \in \mathrm{Div}(\mathcal{M}_{22}^{(2)})$ be the Noether-Lefschetz divisor of first kind of discriminant e. We have that e is a square mod 11.

Theorem 1 ([2, 6])**.** *$\mathcal{C}_2, \mathcal{C}_6, \mathcal{C}_8, \mathcal{C}_{10}, \mathcal{C}_{18}$ are HLS divisors on $\mathcal{M}_{22}^{(2)}$.*

Remark 2. *The HLS-ness of $\mathcal{C}_2, \mathcal{C}_6, \mathcal{C}_{10}, \mathcal{C}_{18}$ was first proven geometrically in* [2]*. An independent proof of these cases plus an extension to $\mathcal{C}_8$ was later given in* [6] *using Gromov-Witten theory (the intersection theory on the moduli space of stable maps).*

2. Proof

The approach of [6] to the theorem is as follows: Let $\widetilde{U} \subset \mathbb{P}(\wedge^3 V_{10}^*)$ be the preimage of U under the quotient map. The complement of $\widetilde{U}$ is of codimension 2. Since $\widetilde{U}$ is contained in the stable locus, the quotient map $\pi : \widetilde{U} \to U$ is a SL_{10}-torsor. Let $\ell \subset \widetilde{U}$ be a fixed line, and $\iota_\ell \to \mathcal{M}_{22}^{(2)}$ the classifying map of the associated pencil of Debarre-Voisin varieties. Then we have:

$$\begin{aligned} \mathcal{P}^*(\mathcal{C}_{2e}) = 0 \quad &\iff \quad \pi^*\mathcal{P}^*(\mathcal{C}_{2e}) = 0 \\ &\iff \quad (\pi^*\mathcal{P}^*\mathcal{C}_{2e}) \cdot \ell = 0 \\ &\iff \quad \int_\ell \iota_\ell^* \mathcal{C}_{2e} = 0. \end{aligned}$$

The numbers $\int_\ell \iota_\ell^* \mathcal{C}_{2e}$ are the Noether-Lefschetz numbers of first kind of the pencil ℓ. They are related by an explicit invertible upper-triangular relation to the Noether-Lefschetz numbers of second kind $\mathsf{NL}^\ell(e)$ [6, Prop.5]. By a result of Borcherds the generating series of Noether-Lefschetz numbers of second kind

$$\varphi(q) = \sum_e \mathsf{NL}^\ell(e) q^{e/11}$$

is a modular form of weight 11 for a specific congruence subgroup. In particular, it only depends on finitely many data. In fact, when working with the corresponding vector-valued modular form, one finds that 6 Fourier coefficients are enough to fix the modular form completely. These coefficients are obtained by from the following mostly formal input:

(i) the Gromov-Witten/Noether-Lefschetz relation [5],
(ii) known cases of a new multiple-cover conjecture for $K3^{[n]}$-type hyperkähler varieties [6] (the original motivation for this work),

(iii) the abelian/non-abelian correspondence [8] (which allows to compute genus 0 Gromov-Witten invariants for zero loci of homogeneous vector bundles in GIT quotients by passing to the abelian quotient).

This yields a closed evaluation of the Noether-Lefschetz series which we state next.

3. Generating series

Define the weight $1, 2, 3$ modular forms

$$E_1(\tau) = 1 + 2\sum_{n\geq 1} q^n \sum_{d|n} \chi_p\left(\frac{n}{d}\right), \qquad \Delta_{11}(\tau) = \eta(\tau)^2\eta(11\tau)^2$$
$$E_3(\tau) = \sum_{n\geq 1} q^n \sum_{d|n} d^2\chi_p\left(\frac{n}{d}\right)$$

where $\eta(\tau) = q^{1/24}\prod_{n\geq 1}(1-q^n)$ is the Dirichlet eta function, $q = e^{2\pi i\tau}$ and χ_{11} is the Dirichlet character given by the Legendre symbol $\left(\frac{\cdot}{11}\right)$. Consider the following weight 11 modular forms for $\Gamma_0(11)$ and character χ_{11}:

$$\begin{aligned}\varphi_0(q) = &-5\,E_1^{11} + 430\,E_1^8E_3 + \frac{5199920}{9}\,\Delta_{11}^3E_1^5 - \frac{35407490}{27}\,\Delta_{11}^4E_1^3\\ &+ \frac{49194440}{9}\,\Delta_{11}^2E_1^4E_3 + 248350\,E_1^5E_3^2 - \frac{596661440}{27}\,\Delta_{11}^3E_1^2E_3\\ &- \frac{306631760}{9}\,\Delta_{11}E_1^3E_3^2 + \frac{51243500}{3}\,\Delta_{11}^4E_3 + \frac{1331452540}{27}\,\Delta_{11}^2E_1E_3^2\\ &+ \frac{349019440}{9}\,E_1^2E_3^3\\ \varphi_1(q) = &-5\,E_1^{11} + 110\,E_1^8E_3 + \frac{722740}{3993}\,\Delta_{11}^3E_1^5 - \frac{1805750}{3993}\,\Delta_{11}^4E_1^3\\ &- \frac{12660620}{11979}\Delta_{11}^2E_1^4E_3 - 990E_1^5E_3^2 + \frac{118940}{363}\Delta_{11}^5E_1 + \frac{5609180}{3993}\Delta_{11}^3E_1^2E_3\\ &+ \frac{29208460}{11979}\,\Delta_{11}E_1^3E_3^2 + \frac{3500}{33}\,\Delta_{11}^4E_3 + \frac{2610980}{1089}\,E_1^2E_3^3\end{aligned}$$

Theorem 2. *The generating series of Noether-Lefschetz numbers of a generic pencil of Debarre-Voisin varieties is:*

$$\begin{aligned}\varphi(q^{11}) = \varphi_0(q^{11}) + \varphi_1(q) = &-10 + 640q^{11} + 990q^{12} + 5500q^{14} + 11440q^{15}\\ &+21450q^{16} + 198770q^{20} + 510840q^{22} + \dots\end{aligned}$$

The vanishing of the coefficients q^1, q^3, q^4, q^5, q^9 implies that the corresponding divisors are HLS. The number 640 is the number of singular fibers of the pencil. The numbers 990 and 5500 have been independently computed by J. Song, see [1] for a study of the geometry of the associated divisors.

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On the problem of generating Chow groups by smooth subvarieties

Olivier Benoist

Fix two integers $c, d \geq 1$. In this talk, we will consider the following classical question [5], as well as variants of it in real algebraic geometry.

Question 1 (Borel, Haefliger)**.** *Let X be a smooth projective variety of dimension $c+d$ over $\mathbb{C}$. Is the Chow group $\mathrm{CH}_d(X) = \mathrm{CH}^c(X)$ generated by classes of d-dimensional smooth closed subvarieties $Y \subset X$?*

It will be useful to keep the following heuristic in mind. In differential geometry, Whitney has shown that a generic $\mathcal{C}^\infty$ map $f : Y^d \to X^{c+d}$ of differentiable manifolds is an embedding if $d < c$. If $d \geq c$, this is not the case anymore, as Y will typically self-intersect in X. In particular, if $d = c$ one expects finitely many transverse self-intersections. It is thus easy to construct submanifolds of X if $d < c$, but hard if $d \geq c$ in general. By analogy, one may expect that Question 1 has a positive answer if $d < c$, but not necessarily so if $d \geq c$.

Let us now review what is known about the question of Borel and Haefliger. Positive results are due to Hironaka [8] and Kleiman [10].

Theorem 1 (Hironaka)**.** *Question 1 has a positive answer if $d < c$ and $d \leq 3$.*

Theorem 2 (Kleiman)**.** *Question 1 has a positive answer if $c = 2$ and $d \in \{2, 3\}$.*

Of these two theorems, the most important for us is Theorem 1. The bound stemming from the Whitney heuristic appears clearly in its statement. The principle of its proof is the following. Given a possibly singular subvariety $Y \subset X$, Hironaka embeds a resolution of singularities $\widetilde{Y}$ of Y in a relative projective space $\pi : \mathbb{P}^N \times X \to X$. Using linkage, he then moves $\widetilde{Y}$ in $\mathbb{P}^N \times X$ to put it in general position. That $d \leq 3$ ensures that the linkage process does not create singularities. Once $\widetilde{Y}$ has been put in general position, the Whitney-type bound $d \leq c$ guarantees that the projection $\pi|_{\widetilde{Y}}$ is an embedding, hence that $\pi(\widetilde{Y})$ is smooth.

The first counterexamples to Question 1 were found by Harthorne, Rees and Thomas [7] for $c = 2$ and $d \geq 7$. A different construction due to Debarre [6] works for $c = 2$ and $d \geq 5$.

Theorem 3 (Hartshorne, Rees, Thomas, Debarre)**.** *Question 1 has a negative answer in general if $c = 2$ and $d \geq 5$.*

The argument of Harthorne, Rees and Thomas would give counterexamples for higher values of c as well. Debarre's method yields the lowest-dimensional known counterexamples. Their works do not allow to reach the Whitney threshold $d = c$. Our first theorem remediates this situation. To formulate it, we let $\alpha(n)$ denote the number of 1's in the dyadic expansion of n.

Theorem 4 ([2])**.** *Question 1 has a negative answer in general if $d \geq c$ and $\alpha(c+1) \geq 3$.*

In the crucial case where $d = c$, our proof is inspired by the Whitney heuristic. We construct a smooth projective variety X of dimension $2d$ over $\mathbb{C}$ (a well-chosen fixed point free quotient of an abelian variety) and a class $\alpha \in \mathrm{CH}_d(X)$. To show, for instance, that α cannot be the class of a smooth subvariety $Y \subset X$, we compute the number of self-intersections of Y in X. By means of a double point formula due to Fulton, and of divisibility results for Chern numbers due to Rees and Thomas, we show that the number of these self-intersections is odd, hence nonzero. This contradicts the smoothness of Y.

We will now successively consider three questions in real algebraic geometry that are related to Question 1, and we will state positive and negative results about these questions that are inspired by Theorems 1 and 4.

The first question is motivated by the fact that, in real algebraic geometry, many questions concerning real loci are insensitive to singularities at non-real points.

Question 2. *Let X be a smooth projective variety of dimension $c + d$ over $\mathbb{R}$. Is the Chow group $\mathrm{CH}_d(X)$ generated by classes of subvarieties $Y \subset X$ that are smooth along their real locus $Y(\mathbb{R})$?*

Theorem 5 ([2])**.** *Question 2 has a positive answer if $d < c$.*

Theorem 6 ([2])**.** *Question 2 has a negative answer in general if $d \geq c$ and $\alpha(c+1) \geq 3$.*

The proof of Theorem 5 builds on Hironaka's smoothing by linkage technique. A notable feature is the removal of the hypothesis $d \leq 3$ appearing in the statement of Theorem 1. As the linkage process is bound to create singularities if $d > 3$, proving Theorem 5 requires to control these singularities, and to ensure that they are not real. Our main tool to do so is a study of linkage in families, for which we rely on results of Peskine and Szpiro and of Huneke and Ulrich.

The proof of Theorem 6 follows the same method of counting self-intersections as that of Theorem 4.

The second question investigates the subgroup of the Chow group generated by subvarieties with no real points. To state it, we recall that for any smooth variety X

over $\mathbb{R}$, the real cycle class map $\mathrm{cl}_{\mathbb{R}} : \mathrm{CH}_*(X) \to H_*(X(\mathbb{R}), \mathbb{Z}/2)$ associates with a subvariety $Y \subset X$ the homology class $[Y(\mathbb{R})]$ of its real locus.

Question 3. *Let X be a smooth projective variety of dimension $c + d$ over $\mathbb{R}$. Is* $\mathrm{Ker}\big(\mathrm{cl}_{\mathbb{R}} : \mathrm{CH}_d(X) \to H_d(X(\mathbb{R}), \mathbb{Z}/2)\big)$ *generated by classes of subvarieties $Y \subset X$ whose real locus $Y(\mathbb{R})$ is empty?*

Theorem 7 ([2])**.** *Question 3 has a positive answer if $d < c$.*

Theorem 8 ([2])**.** *Question 3 has a negative answer in general if $d \geq c$ and $\alpha(c+1) \geq 2$.*

Kucharz [11] had already proved Theorem 7 if $d = 1$ and $c = 2$, and Theorem 8 if c is even. The proofs of Theorems 7 and 8 follow again, respectively, Hironaka's smoothing by linkage technique (combined with Ischebeck and Schülting's description of $\mathrm{Ker}(\mathrm{cl}_{\mathbb{R}})$ [9]), and the method of counting self-intersections.

The last question that we examine concerns algebraic approximation of $\mathcal{C}^\infty$ submanifolds. If X is a smooth projective variety over $\mathbb{R}$, and if $\iota : M \hookrightarrow X(\mathbb{R})$ is a $\mathcal{C}^\infty$ submanifold of its real locus, we consider the following property:

(i) For all neighbourhoods $\mathcal{U} \subset \mathcal{C}^\infty(M, X(\mathbb{R}))$ of ι, there exist $j \in \mathcal{U}$ and a subvariety $Y \subset X$ smooth along $Y(\mathbb{R})$ such that $j(M) = Y(\mathbb{R})$.

There are homological obstructions to the validity of (i). For instance, assertion (i) obviously implies that $\iota_*[M] \in \mathrm{Im}(\mathrm{cl}_{\mathbb{R}})$. More generally, (i) implies the algebraicity in $X(\mathbb{R})$ of products of Stiefel–Whitney classes of M:

(ii) For all integers $i_1, \ldots, i_r$, one has $\iota_*(w_{i_1}(M) \ldots w_{i_r}(M)) \in \mathrm{Im}(\mathrm{cl}_{\mathbb{R}})$.

It is then natural to ask:

Question 4. *Let X be a smooth projective variety of dimension $c + d$ over $\mathbb{R}$ and let $\iota : M \hookrightarrow X(\mathbb{R})$ be a d-dimensional $\mathcal{C}^\infty$ submanifold of its real locus. Are properties (i) and (ii) equivalent?*

Theorem 9 ([2])**.** *Question 4 has a positive answer if $d < c$.*

Theorem 10 ([2])**.** *Question 4 has a negative answer in general if $d \geq c$ and $\alpha(c+1) = 2$.*

Theorem 9 for $d = 1$ had already been proven by Bochnak and Kucharz [4] when $c = 2$ and by Wittenberg and myself [3] in general. The proofs of Theorems 9 and 10 are further applications of Hironaka's smoothing by linkage technique (combined with the relative Nash–Tognoli theorem of Akbulut and King [1]), and of the method of counting self-intersections.

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Varieties of general type with small volume

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(joint work with Chengxi Wang)

For smooth complex projective varieties X of general type, there is a constant r_n depending only on the dimension n of X such that the pluricanonical linear system $|mK_X|$ gives a birational embedding of X into projective space for all $m \geq r_n$. This is a result of Hacon-M$^{\text{c}}$Kernan, Takayama, and Tsuji [9, 13, 15]. It follows that there is a positive lower bound v_n for the volume of all smooth n-folds of general type; namely, $1/(r_n)^n$ is a lower bound. (By definition, the volume of X measures the asymptotic growth of the plurigenera: $\text{vol}(X) = \limsup_{m\to\infty} h^0(X, mK_X)/(m^n/n!)$. This is equal to the intersection number K_X^n if the canonical bundle K_X is ample.) It is a fundamental problem in the classification of algebraic varieties to find the optimal values of these constants. Our interest here is the asymptotics in high dimensions.

In low dimensions, strong results are known. In dimension 1, the optimal bounds are $r_1 = 3$ and $v_1 = 2$, with equality for a curve of genus 2. In dimension 2, we have $r_2 = 5$ (by Bombieri) and $v_2 = 1$, the extreme case being a general weighted projective hypersurface of degree 10 in $P(1,1,2,5)$. Finally, in dimension 3, we have $r_3 \leq 57$ and $v_3 \geq 1/1680$ by J. Chen and M. Chen [5, Theorem 1.6], [7], [6]. (It is an important feature of dimensions at least 3 that the volume of a smooth variety of general type need not be an integer, because the canonical divisor of the canonical model need not be Cartier.) The smallest known volume for a smooth 3-fold of general type is $1/420$, by Iano-Fletcher: take a resolution of the weighted projective hypersurface $X_{46} \subset P(4,5,6,7,23)$ [12, section 15]. In that example, $|mK_X|$ is birational if and only if $m = 23$ or $m \geq 27$. The smooth 4-fold of general type with smallest known volume is a resolution of $X_{165} \subset P(10,12,17,33,37,55)$, with volume $1/830280$, found by Brown and Kasprzyk [4].

In dimensions at least 4, no explicit lower bound for the volume of a smooth variety of general type is known. We at least give a bound in the other direction, by finding varieties of general type with exotic behavior (Theorem 1) [14]. As in previous examples, we do this by resolving suitable weighted projective hypersurfaces with canonical singularities. These exhibit a huge range of behavior, and our examples are certainly not optimal. (In contrast, Theorem 2 below should be nearly optimal.)

Theorem 1. *(1) For every sufficiently large positive integer n, there is a smooth complex projective n-fold of general type with volume less than $1/n^{(n \log n)/3}$.*

(2) For every sufficiently large positive integer n, there is a smooth complex projective n-fold X of general type such that the linear system $|mK_X|$ does not give a birational embedding for any $m \leq n^{(\log n)/3}$.

Theorem 1 builds on ideas of Ballico, Pignatelli, and Tasin. They found n-folds of general type with volume about $1/n^n$, and such that $|mK_X|$ does not give a birational embedding for m at most a constant times n^2 [1, Theorems 1 and 2]. Here we improve the latter bound from n^2 to about $n^{\log n}$, which grows faster than any polynomial.

Our methods also give super-exponentially small examples for several other problems: (1) the constant a_n in Noether-type inequalities $\mathrm{vol}(X) \geq a_n p_g(X) - b_n$ for smooth projective n-folds X of general type, (2) the volume of terminal Fano varieties, and (3) the volume of ample Weil divisors on terminal Calabi-Yau varieties. The existence of some positive lower bound in each dimension was proved by M. Chen and Z. Jiang for problem (1), and by C. Birkar for problems (2) and (3) [8, Corollary 5.1], [2, Theorem 1.1], [3, Corollary 1.4].

If we weaken the assumptions on singularities, we can produce varieties with even more extreme behavior [14]:

Theorem 2. *For every integer $n \geq 2$, there is a complex projective klt n-fold X with ample canonical class such that* $\mathrm{vol}(K_X) < 1/2^{2^n}$.

We know that there is some positive lower bound in each dimension for the volumes of klt varieties with ample canonical class, by Hacon-M^{c}Kernan-Xu's theorem that these volumes satisfy DCC [11, Theorem 1.3].

In this direction, Kollár constructed what may be the klt pair (Y, Δ) of general type with standard coefficients that has minimum volume in each dimension [10, Introduction]. The example X described in Theorem 2 should be nearly optimal, since it is in the more special class of klt varieties rather than klt pairs, and yet it has the property that $\log(\mathrm{vol}(K_X))$ is asymptotic to $\log(\mathrm{vol}(K_Y + \Delta))$ in Kollár's example, as the dimension goes to infinity. The details of the construction are intricate, combining Sylvester's sequence $2, 3, 7, 43, 1807, \ldots$ with several sequences of polynomials defined by recurrence relations.

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