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Dynamische Systeme (hybrid meeting)

Organized by
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Abstract. This workshop continued a biannual series of workshops at Oberwolfach on dynamical systems that started with a meeting organized by Moser and Zehnder in 1981. Workshops in this series focus on new results and developments in dynamical systems and related areas of mathematics, with symplectic geometry playing an important role in recent years in connection with Hamiltonian dynamics. In this year special emphasis was placed on various kinds of spectra (in contact geometry, in Riemannian geometry, in dynamical systems and in symplectic topology) and their applications to dynamics.

Mathematics Subject Classification (2010): 37-XX, 53Dxx.

Introduction by the Organizers

The workshop *Dynamische Systeme*, organised by M.-C. Arnaud (Paris), H. Hofer (Princeton), M. Hutchings (Berkeley) and V. Kaloshin (Vienna), was well attended with over 60 participants with broad geographic representation from 15 countries. The workshop covered a broad range of topics in dynamical systems and related areas, with a special emphasis on various kinds of spectra and their applications to dynamics.

Several striking results related to the existence of homoclinic or heteroclinic tangencies were presented. They concern the local generic existence of wild behaviours related to the so-called Newhouse phenomenon, for some rigid dynamics where it is hard to obtain such results. In particular, S. Biebler showed wild behaviour for polynomial Hénon maps in complex dimension equal to 2, and K. Callis showed similar results for Birkhoff billiards.

Others works were connected with the existence of transveral homoclinic intersections: I. Baldoma and M. Guardia presented works related to the existence of transversal homoclinic intersections and the existence of chaos for the 3-body problem. Using an approach coming from finite dimensional dynamical systems, namely singular perturbations to study certain homoclinic orbits, T. Seara proved the non-existence of small breathers of non-linear Klein-Gordon PDE.

Other results were presented concerning billiards. S. Zelditch presented a survey on several open problems for the Laplace spectrum of convex billiards. E. Koudjinan proved that non-trival deformations of circular billiards cannot preserve 2 and $(2m+1)$-rational caustics. P. Berk explained that translation flows that are disjoint with their inverse are abundant.

Renormalization techniques were used by several lecturers: C. Ulcigrai presented the proof of a conjecture on generalized interval exchange transformations by Marmi-Moussa and Yoccoz in genus two: under a full measure condition C^0 linearization is C^1. F. Trujillo proved the existence of KAM tori of lower dimension in a non-convex setting. For flat tangencies of area preserving maps, a non generic case, Krikorian gave conditions for the existence of KAM curves accumulating a separatrix. Yi Pan presented reducibility results for quasi-periodic cocycles.

B. Fayad dealt with KAM rigidity of parabolic actions on the torus.

Different speakers discussed Birkhoff sections and their generalizations for contact flows. O. Von Koert proved a generalization of the Poincaré-Birkhoff theorem for Hamiltonian twist maps of Liouville domains in all dimensions and applied this to find periodic orbits for the restricted 3-body problem. For Reeb flows in dimension 3, V. Colin introduced the notion of "broken book decomposition" and used this to derive general dynamical properties.

Several speakers discussed additional new relations between dynamics and symplectic geometry. A. Abbondandolo explained how Zoll contact forms maximize the systolic ratio (relating minimal period of Reeb orbits to volume). U. Hryniewicz presented a characterization of the special three-dimensional contact forms whose Reeb flow has exactly two periodic orbits. D. Cristofaro-Gardiner discussed how spectral invariants in periodic Floer homology can be used to resolve the simplicity conjecture for area-preserving homeomorphisms of the disk, and related questions for area-preserving homeomorphisms of the two-sphere. S. Seyfaddini showed how new spectral invariants in Heegaard Floer homology can be used to extend these results to higher genus surfaces. V. Ramos described how integrability of billiard dynamics on certain domains leads to new results on the existence or nonexistence of symplectic embeddings involving Lagrangian products. E. Shelukhin used new Lagrangian spectral invariants to obtain applications to Poincaré recuurence for Lagrangian submanifolds of symplectic manifolds. S. Tanny presented new results on how Hamiltonian spectral invariants behave under composition of maps with disjoint support. C. Viterbo discussed new results on the completion of the set of exact Lagrangians in a cotangent bundle with respect to the spectral norm.

The meeting was held in an informal and stimulating atmosphere, bringing together the in-person and remote participants. Unfortunately the traditional Wednesday afternoon walk had to be cancelled due to inclement weather.

Workshop (hybrid meeting): Dynamische Systeme

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Abstracts

A generalization of the Poincaré-Birkhoff fixed point theorem and the spatial, restricted three-body problem

Otto van Koert
(joint work with Augustin Moreno)

In joint work with Agustin Moreno, we propose a generalization of the Poincaré-Birkhoff fixed point theorem. This generalization is motivated by a construction of global hypersurfaces of section in the spatial restricted three-body problem. We define RTBP as the study of the dynamics of the Hamiltonian

$$H = \frac{1}{2}\|p\|^2 + q_1p_2 - q_2p_1 - \frac{1-\mu}{\|q-e\|} - \frac{\mu}{\|q-m\|}$$

on $(T^*\mathbb{R}^3, dp\wedge dq)$, where $e = (-\mu, 0, 0)$, and $m = (1-\mu, 0, 0)$ for some fixed energy c. This Hamiltonian has five critical points corresponding to five critical points of the effective potential

$$U = -\frac{1-\mu}{\|q-e\|} - \frac{\mu}{\|q-m\|} - \frac{1}{2}(q_1^2 + q_2^2).$$

We call these points Lagrange points: $L_1, \dots, L_5 \in T^*\mathbb{R}^3$ and order their values as $H(L_1) < H(L_2) < H(L_3) < H(L_4) = H(L_5)$ (which can be done for most mass ratios μ). After regularization of two body collisions (classically done with Levi-Civita, but Moser regularization generalizes better), one may wonder whether global (hyper)surfaces of section exist. In the planar problem, global surfaces of section have been constructed using classical means by Poincaré, Birkhoff, Conley, and McGehee. More recently, modern symplectic techniques have been applied. In the classical constructions, the boundary of the global surface of section consists of the so-called retrograde or prograde (approximately circular) planar periodic orbits.

To construct a global hypersurface of section in the regularized spatial problem S_c, we need first of all an invariant set of codimension 2. There is of course an easy candidate, namely the planar problem, say $P_c \subset S_c$. The complement of P_c in S_c is foliated by copies of T^*S^2 for $c < H(L_1)$, and by copies of $T^*S^2 \natural T^*S^2$ for $c \in (H(L_1), H(L_2))$. Using the geodesic flow on a sphere as inspiration, we can embed these leaves transverse to the flow, and with an estimate, one can show that orbits always return. This construction works for a wider class of problems enjoying similar symmetry properties as RTBP.

This setup motivates a generalization of the Poincaré-Birkhoff fixed point theorem. Instead of an annulus, we consider a general Liouville domain (W, λ) (such as the disk bundle D^*S^2 or $D^*S^2 \natural D^*S^2$, but the annulus is also an example). We define the following version of the twist condition.

Definition 1. *A map* $\tau : W \to W$ *is a* **Hamiltonian twist map** *with respect to* $\alpha = \lambda|_{\partial W}$ *if*

- *τ is generated by a smooth Hamiltonian $H : \mathbb{R} \times W \to \mathbb{R}$*
- *$X_{H_t}|_{\partial W} = h_t R_\alpha$ for some positive, smooth function $h : \mathbb{R} \times \partial W \to \mathbb{R}^+$.*

We then have the following theorem.

Theorem 1 (A generalized Poincaré–Birkhoff theorem). *Suppose that τ is an exact symplectomorphism of a connected Liouville domain (W, λ), and let $\alpha = \lambda|_B$. Assume the following:*

- *all fixed points of τ are isolated;*
- ***(Hamiltonian twist map)** τ is a Hamiltonian twist map, where the generating Hamiltonian is at least C^2;*
- ***(index-definiteness)** If $\dim W \geq 4$, then assume $c_1(W)|_{\pi_2(W)} = 0$, and $(\partial W, \alpha)$ is strongly index-definite. In addition,*
- ***(Symplectic homology)** $SH_\bullet(W)$ is infinite dimensional.*

Then τ has simple interior periodic points of arbitrarily large (integer) period.

Here, strongly index-definite means that there is a global symplectic trivialization, and $c > 0$ such that

$$|\mu_{RS}(\gamma|_{[0,T]}, \epsilon)| \geq cT + d$$

In dimension 2, the condition that $SH_\bullet(W)$ is infinite dimensional just means that W is not a disk. The Hamiltonian twist condition is used to extend the Hamiltonians generating τ to admissible Hamiltonian for symplectic homology with some control on the orbits of the extension.

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Quantitative conditions for right-handedness of dynamically convex Reeb flows

Anna Florio

(joint work with Umberto L. Hryniewicz)

In a joint work with Umberto L. Hryniewicz, we establish a quantitative condition to guarantee that a dynamically convex Reeb flow in S^3 is right-handed. In particular, this enables us to provide quite a large class of examples of the so-called right-handed flows. The notion of right-handedness was introduced by Étienne Ghys in [2]. Roughly speaking, a flow ϕ_t on S^3 without rest points is right-handed if almost all pairs of trajectories link positively. More precisely, let μ_1, μ_2 be ergodic ϕ_t-invariant probability measures. There can be two cases.

(i) If $\text{supp}(\mu_1) = \text{supp}(\mu_2) = \gamma$ for some periodic orbit γ, then one says that μ_1, μ_2 are *positively linked* if the transverse rotation number of γ computed in a Seifert framing is strictly positive.

(*ii*) Otherwise, fix an auxiliary Riemannian metric and consider two recurrent points p, q, generic for μ_1, μ_2 respectively, whose orbits are disjoint. Let $\{T_n\}_n, \{S_n\}_n$ be increasing sequences such that $\phi_{T_n}(p) \to p, \phi_{S_n}(q) \to q$. Let α_n, respectively β_n, be the (unique) shortest geodesic arc connecting $\phi_{T_n}(p)$ to p, respectively $\phi_{S_n}(q)$ to q. Up to C^1 perturbations of the paths α_n and β_n, we can assume that the loops $k(T_n, p)$ and $k(S_n, q)$, obtained by concatenating α_n with $\phi_{[0,T_n]}(p)$ and β_n with $\phi_{[0,S_n]}(q)$, are disjoint. Defining then

$$\ell(p, q) = \inf_{\{T_n\}_n, \{S_n\}_n} \liminf_{n \to +\infty} \frac{\text{linking}(k(T_n, p), k(S_n, q))}{T_n S_n},$$

one says that μ_1, μ_2 are positively linked if for $\mu_1 \times \mu_2$-almost all points (p, q) it holds $\ell(p, q) > 0$.

Definition 1. *A flow ϕ_t without rest points in S^3 is right-handed if every pair of ergodic ϕ_t-invariant probability measures links positively.*

Right-handedness has interesting dynamical consequences, in terms of global surfaces of section and open book decompositions. A global surface of section (GSS) is a compact, embedded surface Σ such that: (*i*) $\partial\Sigma$ is a finite union of periodic orbits, (*ii*) the interior of Σ is transverse to the vector field generating ϕ_t, (*iii*) the trajetory of every point in $S^3 \setminus \partial\Sigma$ meets infinitely many times in the future and in the past the interior of Σ. An open book decomposition is then the given of a pair $(\partial\Sigma, \pi)$ where $\partial\Sigma$ is an oriented link, called *binding*, while $\pi : S^3 \setminus \partial\Sigma \to S^1$ is a fibration of the complement of $\partial\Sigma$ such that for each $\theta \in S^1$, the preimage $\pi^{-1}(\theta)$ is a global surface of section.

In [2], Ghys proved the following result for right-handed flows.

Theorem 1. *For a right-handed flow ϕ_t, every finite collection of periodic orbits is the binding of an open book decomposition.*

In particular, Ghys's theorem imposes some restrictions on the periodic orbits that can be realised by right-handed flows: knots or links that are not fibered, i.e. whose complement cannot be fibered over S^1, cannot be realised.

In our work we look for a numerical condition for right-handedness within the class of dynamically convex Reeb flows on S^3. Such flows were first introduced by Hofer, Wysocki and Zehnder in [4]. Different interesting examples and applications lie in this class. Indeed, a Hamiltonian flow restricted to a strictly convex energy level of $\mathbb{R}^4$ and the geodesic flow on S^2 with respect to a δ-pinched Riemannian metric for $\delta > \frac{1}{4}$ are both dynamically convex Reeb flows (see [4, Theorem 3.4] and [3]). This notion of dynamically convex Reeb flows is of special interest due to the following result by Hofer, Wysocki and Zehnder in [4].

Theorem 2. *Any dynamically convex Reeb flow admits a disk-like global surface of section.*

In [5], Hryniewicz provides a characterisation of the orbits binding a GSS: any periodic Reeb orbit γ for a dynamically convex Reeb flow on S^3 binds a disk-like global surface of section if and only if γ is unknotted and has self-linking number -1. Our condition is a lower bound on the asymptotic ratio between the amount of rotation of the linearised flow and the linking number of trajectories with a periodic orbit that spans a disk-like GSS. More precisely, let γ_0 be a periodic Reeb orbit that binds a disk-like global surface of section. Observe that, since we are in S^3, the contact structure $(\xi, d\lambda)$ associated to our Reeb flow is trivial and we find a global symplectic trivialisation σ of $(\xi, d\lambda)$ with respect to which we can consider the angular coordinate $\Theta_\sigma(u)$ of a vector $u \in \xi$. For any $x \in S^3, u \in \xi_x \setminus \{0\}$ we consider then a lift $t \mapsto \tilde{\Theta}_\sigma(t; x, u)$ of the angle coordinate function $t \mapsto \Theta_\sigma(D\phi_t(x)u)$. For $x \in S^3 \setminus \gamma_0$ and for $t > 0$ we define a canonical way to obtain a loop $k(t; x, \Sigma)$ from the piece of trajectory $\phi_{[0,t]}(x)$, where Σ is a disk-like global surface of section spanned by γ_0. Thus, the quantity linking$(k(t; x, \Sigma), \gamma_0)$ can be seen almost as the number of intersections of $\phi_{[0,t]}(x)$ with Σ. We can now introduce the following quantity

$$\kappa(\gamma_0) := \liminf_{T\to+\infty} \inf_{\substack{x\in S^3\setminus\gamma_0 \\ u\in\xi_x\setminus\{0\}}} \frac{\tilde{\Theta}_\sigma(T; x, u) - \tilde{\Theta}_\sigma(0; x, u)}{\text{linking}(k(T; x, \Sigma), \gamma_0)} .$$

Our quantitative condition is then the following one.

Theorem 3. *Let ϕ_t be a dynamically convex Reeb flow on S^3. Let γ_0 be an unknotted periodic orbit with self-linkinf number -1. If $\kappa(\gamma_0) > 2\pi$, then the flow is right-handed.*

In the framework of geodesic flows on a Riemannian 2-sphere (S^2, g), we can obtain an explicit interval for the pinching constant where the right-handedness condition is satisfied.

Theorem 4. *Let g be a Riemannian metric on S^2 such that $\delta \geq 0.7225$, where $\delta := \dfrac{\min K_g}{\max K_g}$ is the pinching constant, with K_g being the Gaussian curvature. Then the geodesic flow on (S^2, g) lifts to a right-handed flow on S^3.*

In order to prove Theorem 4, we need to verify the quantitative condition given by Theorem 3 using comparison theorems coming from Riemannian geometry.

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On the algebraic structure of groups of area-preserving homeomorphisms

Sobhan Seyfaddini
(joint work with Daniel Cristofaro-Gardiner, Vincent Humilière, Cheuk Yu Mak, Ivan Smith)

The simplicity question. Let (Σ, ω) denote a compact and connected surface, possibly with boundary, equipped with an area-form. Let $\mathrm{Homeo}_0(\Sigma, \omega)$ denote the identity component in the group of homeomorphisms of Σ which preserve the measure induced by ω and coincide with the identity near the boundary of Σ, if the boundary is non-empty. We say $\varphi \in \mathrm{Homeo}_0(\Sigma, \omega)$ is a **Hamiltonian homeomorphism** if it can be written as a uniform limit of Hamiltonian diffeomorphisms. The set of all such homeomorphisms is denoted by $\overline{\mathrm{Ham}}(\Sigma, \omega)$; this is a normal subgroup of $\mathrm{Homeo}_0(\Sigma, \omega)$. Hamiltonian homeomorphisms have been studied extensively in the surface dynamics community; see, for example, [8, 7].[1]

There exists a homomorphism out of $\mathrm{Homeo}_0(\Sigma, \omega)$, called the **mass-flow** homomorphism, introduced by Fathi [3], whose kernel is $\overline{\mathrm{Ham}}(\Sigma, \omega)$. The normal subgroup $\overline{\mathrm{Ham}}(\Sigma, \omega)$ is proper when Σ is different from the disc or the sphere. In the 1970s, Fathi asked in [3, Section 7] if $\overline{\mathrm{Ham}}(\Sigma, \omega)$ is a simple group; in higher dimensions, one can still define mass-flow and Fathi showed [3, Thm. 7.6] that its kernel is always simple, under a technical assumption on the manifold which always holds when the manifold is smooth. When Σ is a surface with genus 0, Fathi's question was answered in [1, 2]; however, the higher genus case has remained open.

By using our new spectral invariants, we can answer Fathi's question in full generality:

Theorem 1. $\overline{\mathrm{Ham}}(\Sigma, \omega)$ *is not simple.*

Theorem 1 generalizes the aforementioned results of [1, 2] proving this result in the genus zero case. Our proof is logically independent of these works. To prove the theorem, following [1, 2] we construct a normal subgroup $\mathrm{FHomeo}(\Sigma, \omega)$, called the group of **finite energy homeomorphisms**, and we prove that it is proper. The group FHomeo is inspired by Hofer geometry, and one can define Hofer's metric on it, see [2, Sec. 5.3]. For another proof in the genus 0 case, see [11].

The group $\mathrm{FHomeo}(\Sigma, \omega)$ contains the commutator subgroup of $\overline{\mathrm{Ham}}(\Sigma, \omega)$, hence we learn from our main result that $\overline{\mathrm{Ham}}(\Sigma, \omega)$ is not perfect, either.

Extending the Calabi invariant. One would like to understand more about the algebraic structure of $\overline{\mathrm{Ham}}(\Sigma, \omega)$ beyond the simplicity question. Recall that $\mathrm{Ham}(\Sigma, \omega)$ denotes the subgroup of Hamiltonian diffeomorphisms and suppose

[1] We remark that when $\Sigma = S^2$, $\overline{\mathrm{Ham}}$ is the group of area and orientation preserving homeomorphisms, and when $\Sigma = D^2$, it is the group of area preserving homeomorphisms that are the identity near the boundary.

now that the boundary of Σ is non-empty. Then, the group of Hamiltonian diffeomorphisms admits a homomorphism

$$\mathrm{Cal} : \mathrm{Ham}(\Sigma, \omega) \to R,$$

called the **Calabi invariant**, defined as follows. Let $\theta \in \mathrm{Ham}(\Sigma, \omega)$. Pick a Hamiltonian $H : [0,1] \times \Sigma \to R$, supported in the interior of Σ, such that $\theta = \phi^1_H$. Then,

$$\mathrm{Cal}(\theta) := \int_0^1 \int_\Sigma H\, \omega\, dt.$$

The above integral does not depend on the choice of H and so $\mathrm{Cal}(\theta)$ is well-defined. Moreover, it defines a non-trivial group homomorphism.

A question of Fathi from the 1970s [3, Section 7] asks if Cal admits an extension to $\overline{\mathrm{Ham}}(D, \omega)$. An illuminating discussion by Ghys of this question appears in [6, Section 2]; it follows from results of Gambaudo-Ghys [5] and Fathi [4] that Calabi is a topological invariant of Hamiltonian diffeomorphisms, i.e. if $f, g \in \mathrm{Ham}(\Sigma, \omega)$ are conjugate by some $h \in \mathrm{Homeo}_0(\Sigma, \omega)$, then $\mathrm{Cal}(f) = \mathrm{Cal}(g)$. Hence, it seems natural to try and extend Calabi to $\overline{\mathrm{Ham}}(\Sigma, \omega)$, or at least to a proper normal subgroup.[2] Our proof of Theorem 1 involves constructing an "infinite twist" Hamiltonian homeomorphism which, heuristically, has infinite Calabi invariant, so our interest in what follows will be extending the Calabi homomorphism to a proper normal subgroup rather than the full group.

There is a later conjecture of Fathi about what an appropriate normal subgroup for the purpose of extending Calabi might be. In the article [9], Oh and Müller introduced a normal subgroup $\mathrm{Hameo}(\Sigma, \omega)$, called the group of **Hameomorphisms** of Σ; the idea of the definition is that these are elements of $\overline{\mathrm{Ham}}(\Sigma, \omega)$ that have naturally associated Hamiltonians. The group $\mathrm{Hameo}(\Sigma, \omega)$ is contained in $\mathrm{FHomeo}(\Sigma, \omega)$, and so our proof of Theorem 1 shows that it is proper. The aforementioned conjecture of Fathi is that the Calabi invariant admits an extension to $\mathrm{Hameo}(\Sigma, \omega)$ when Σ is the disc; see [10, Conj. 6.1]. We prove this for any Σ with non-empty boundary.

Theorem 2. *The Calabi homomorphism admits an extension to a homomorphism from the group* $\mathrm{Hameo}(\Sigma, \omega)$ *to the real line.*

Theorem 2 implies that $\mathrm{Hameo}(\Sigma, \omega)$ is neither simple nor perfect, when $\partial\Sigma \neq \emptyset$; we do not know whether or not the kernel of Calabi on Hameo is simple.

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Normal forms and sharp systolic inequalities

Alberto Abbondandolo
(joint work with Gabriele Benedetti)

Let α be a contact form on the closed $(2n-1)$-dimensional manifold M, i.e. a 1-form such that $\alpha \wedge d\alpha^{n-1}$ never vanishes. The Reeb vector field of α is the unique vector field R_α such that $\imath_{R_\alpha} d\alpha = 0$ and $\imath_{R_\alpha}\alpha = 1$. We denote by $T_{\min}(\alpha)$ the minimum of all periods of closed orbits of R_α. The systolic ratio of α is the number

$$\rho(\alpha) := \frac{T_{\min}(\alpha)^n}{\mathrm{vol}(M,\alpha)},$$

where the volume $\mathrm{vol}(M,\alpha)$ is defined by integrating $\alpha \wedge d\alpha^{n-1}$ over M. The choice of the n-th power here makes this quantity scaling invariant: $\rho(c\alpha) = \rho(\alpha)$ for every $c \neq 0$. The systolic ratio of a contact form is a natural generalization of the systolic ratio in metric geometry, i.e. the ratio between the n-th power of the length of the shortest closed geodesic on a closed n-dimensional Riemannian manifold (Q, g) and the Riemannian volume of (Q, g). Indeed, the geodesic flow of (Q, g) is a Reeb flow on the unit tangent bundle of Q, the length of any closed geodesic is its period as closed orbit of this flow, and the contact volume of the unit tangent bundle of Q is, up to a multiplicative constant depending only on n, the Riemannian volume of (Q, g).

The systolic ratio is a dynamical invariant: Contact forms inducing smoothly conjugate Reeb flows have the same systolic ratio.

Still borrowing terminology from metric geometry, the contact form α is said to be Zoll if all its Reeb orbits are closed and have the same minimal period. A standard example is the contact form defining the Hopf fibration on $S^{2n-1} \subset \mathbb{C}^n$. The systolic ratio of a Zoll contact form is the inverse of a positive integer N, which is the Euler number of the circle bundle associated to the free circle action induced by the Zoll Reeb flow. In my talk, I discussed the proof of the following result obtained in collaboration with Gabriele Benedetti, see [1]:

Theorem. *Every Zoll contact form α_0 on the closed manifold M has a C^3 neighborhood $\mathcal{U}$ in the space of contact forms such that*

$$\rho(\alpha) \leq \rho(\alpha_0) \qquad \forall \alpha \in \mathcal{U},$$

with equality holding if and only if α is Zoll. In the latter case, $\alpha = c\,\varphi^* \alpha_0$ for some positive number c and some diffeomorphism $\varphi : M \to M$.

Previous results are: An argument based on non-converging normal forms was used in [4] in order to prove a weaker statement about paths of contact forms emanating from a Zoll one. The above theorem was proven in the special case $M = S^3$ in [2] and for arbitrary closed 3-manifolds in [5]. The proofs of the latter two papers use global surfaces of section and do not extend to higher dimension. It should also be remarked that the systolic ratio is always unbounded from above on the space of all contact forms inducing any given contact structure: See [3] for the case of 3-manifolds and [8] for the general case.

Consequences of the above theorem are:

(i) Sharp local systolic inequalities in metric geometry, see [6]: Zoll Riemannian and Finsler metrics are local maximizers of the metric systolic ratio.
(ii) Local version of a conjecture of Viterbo, see [9]: Among all convex bodies of fixed volume in $\mathbb{R}^{2n}$, balls are local maximizers of the symplectic EHZ-capacity.
(iii) Local non-squeezing in the intermediate dimensions: The volume of any $2k$-dimensional symplectic projection of the image of the unit ball in $\mathbb{R}^{2n}$ by a symplectomorphism that is close to a linear one is not smaller than the volume of the unit ball in $\mathbb{R}^{2k}$.

The proof of the above theorem is based on a new normal form for contact forms that are close to Zoll ones:

Normal form. *Let α_0 be a Zoll contact form on the closed manifold M and let $\pi : M \to B$ be the circle bundle induced by the corresponding free circle action. For every contact form α on M which is C^2-close to α_0 there exists a diffeomorphism $\varphi : M \to M$ such that*

$$\varphi^* \alpha = S \circ \pi\, \alpha_0 + \eta + df,$$

where S is a smooth real function on B, η is a 1-form such that $\imath_{R_{\alpha_0}} \eta = 0$, f is a smooth real function on M, and

$$\imath_{R_{\alpha_0}} d\eta = F\big(d(S \circ \pi)\big), \tag{1}$$

for some linear endomorphism $F : T^*M \to T^*M$. Moreover, φ is close to the identity and $S - 1$, η, f, F are suitably small when α is suitably close to α_0.

The proof of this normal form uses old ideas of Moser and Weinstein and a result about normal forms for flows that are close to periodic ones which is due to Bottkol [7].

Thanks to (1), S is a variational principle detecting the closed orbits of R_α bifurcating from the set B of periodic orbits of R_{α_0}: If $b \in B$ is a critical point of S, then $\varphi(\pi^{-1}(b))$ is a closed orbit of R_α of minimal period $T_0 \cdot S(b)$, where T_0 denotes the common minimal period of the orbits of R_{α_0}. In particular,

$$T_{\min}(\alpha) \leq T_0 \min_B S.$$

The next ingredient is this identity, which follows from the normal form:

$$\operatorname{vol}(M,\alpha) = \int_M p(x, S\circ\pi(x))\, \alpha_0 \wedge d\alpha_0^{n-1},$$

where

$$p(x,s) = s^n + \sum_{j=0}^{n-1} p_j(x)\, s^j, \qquad \text{with} \quad \int_M p_j\, \alpha_0 \wedge d\alpha_0^{n-1} = 0,$$

and $\|p_j\|_{C^0}$ is small when $\|\alpha - \alpha_0\|_{C^3}$ is small. In particular, the function $s \mapsto p(x,s)$ is strictly monotonically increasing for s close to 1 and we obtain

$$\begin{aligned}\operatorname{vol}(M,\alpha) &= \int_M p(x, S\circ\pi(x))\, \alpha_0 \wedge d\alpha_0^{n-1} \geq \int_M p(x, \min_B S)\, \alpha_0 \wedge d\alpha_0^{n-1} \\ &= \Big(\min_B S\Big)^n \operatorname{vol}(M,\alpha_0) \geq \frac{T_{\min}(\alpha)^n}{T_0^n} \operatorname{vol}(M,\alpha_0) = \frac{T_{\min}(\alpha)^n}{\rho(\alpha_0)},\end{aligned}$$

proving the inequality of the Theorem above. In the equality case, S must be constant and the characterization of α follows.

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Problems in billiard dynamics arising in the inverse spectral problem

Steven Zelditch
(joint work with Hamid Hezari)

The billiard problems of this talk pertain to billiards in bounded smooth plane domain $\Omega \subset \mathbb{R}^2$. The spectral problem is the classical eigenvalue problem with Dirichlet or Neumann boundary conditions. The Dirichlet eigenvalue problem is,

$$\begin{cases} \Delta\phi_j = -\lambda_j^2\phi_j, \\ \\ \phi_j|_{\partial\Omega} = 0. \end{cases}$$

The Dirichlet spectrum is denoted by $Sp(\Omega) = \{\lambda_j^2\}$. The dynamical problems are essentially the same for Neumann boundary conditions.

The inverse spectral problem is to recover information about Ω from $Sp(\Omega)$. In 1965, M. Kac proved that one can 'hear' the area $|\Omega|$ and perimeter $|\partial\Omega|$. What other geometric invariants of Ω can be deduced from $Sp(\Omega)$? Are they sufficient to determine Ω up to isometry?

Since the $1970's$, the principal tool for recovering geometric information on Ω from $Sp(\Omega)$ has been the trace of the wave group. Let Δ_Ω^B be the (positive) Laplacian on Ω with boundary condition B on $\partial\Omega$. The trace of the even wave operator is defined by

$$w_\Omega^B(t) := \mathrm{Tr}\cos\left(t\sqrt{\Delta_\Omega^B}\right) = \sum_{j=1}^{\infty}\cos(t\lambda_j).$$

The sum converges in the sense of tempered distributions. The trace is singular only at

$$t \in \mathrm{Lsp}(\Omega) = \{\mathrm{L}(\gamma) = \text{Length of } \gamma\}$$

where γ is a closed geodesic (closed billiard trajectory). As this suggests, the singularities of the wave trace are invariants of periodic orbits of the billiard dynamics of Ω.

In general, the strategy for determining Ω from $Sp(\Omega)$ is to first obtain dynamical invariants of the billiard map (lengths of closed orbits, Birkhoff normal form coefficients at periodic orbits, etc.) and then to reduce the problem to an inverse problem in dynamical systems. Recently, there has been dramatic progress in inverse billiard dynamics due to Avila, de -Simoi- Kaloshin [ADK16], Kaloshin-Sorrentino [KS18] and others, mainly on the Birkhoff problem to determine ellipses from the unique dynamical properties of their billiard maps. The work [ADK16] was then used by the authors in [HeZe19] to prove that ellipses of small eccentricity are uniquely determined by $Sp(\Delta)$.

Probably the most natural problem is to prove the 'quantum analogue' of Birkhoff's conjecture that ellipses E_e of any eccentricity e are the unique domains with integrable billiards.

Conjecture. *Ellipses E_e are uniquely determined by $Sp(E_e)$.*

Local versions of the classical dynamical Birkhoff conjecture are proved in [ADK16, KS18]. But the dynamical and Laplace inverse spectral problems are not the same. One of the main obstructions is that the wave trace does not determine useful information when there exists multiplicity in $Lsp(\Omega)$, the length spectrum. That is, when there exist distinct orbits of the same length. In that case, the wave trace singularity at $t = L$ is a sum over contributions from the different orbits of length L. Connected 1-parameter families are not a problem but distinct components are, since they can lead to cancellations in the wave trace. Moreover, to get explicit formulae, one must assume that the periodic orbits are Bott-Morse non-degenerate, which is generically true among domains but might possible not be true within an isospectral class of domains. These problems on cancellations and degeneracy are unavoidable in inverse Laplace-spectral theory. It is rather clear that a non-isometric but isospectral domain to an ellipse would have to be pathological: its wave trace would have large singularities at each length in the length spectrum of an ellipse, the kind of singularity produced by a smooth 1-parameter family of orbits, yet it could not arise from a one-parameter family (else it would be rationally integrable, hence an ellipse). But its periodic orbits could not be isolated if they produce these large singularities. So, what kind of sets would periodic orbits of a given length be be? Cantor sets of positive measure? Could those ever produce the wave trace expansion of an ellipse? In view of the results of [KS18] it seems that the following conjecture could be accessible:

Conjecture. *Ellipses E_e are Δ-spectrally rigid.*

In other words, there do not exist non-trivial deformations of E_e by domains with the same Dirichlet spectra as E_e. Trivial means isometric. In [HZ12], it is proved that non-trivial isospectral deformations preserving the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry of the ellipse do not exist. Hence the above Conjecture is to remove the symmetry assumptions. It is proved in [KS18] that iso-length-spectral deformations do not exist but the possible multiplicities in the length spectra of the deformed domains apriori obstruct the application of this result.

It is a reasonably accurate simplification to say that in Δ-inverse spectral theory we try to determine properties of Ω from oscillatory integrals. For large q, the contribution to $\mathrm{Tr}\cos t\sqrt{-\Delta}$ of q-bounce orbits is essentially the Fourier transform $\lambda \to t$ of the (Marvizi-Melrose) integral [MM82],

$$I_q(\lambda) := \int_{\partial\Omega} e^{i\lambda\psi_q(x)} A_q(\lambda, x) dx. \tag{1}$$

One hopes to reduce the inverse spectral problem to recovering ψ_q from $Sp(\Delta)$. Here, $\psi_q(x) : \partial\Omega \to \mathbb{R}$ is the 'loop length function' for $\Gamma(1, q)$ orbits (winding number 1, q bounces). Namely it is the length of 'the' geodesic loop of winding number 1 with q bounces that starts and ends at x. The critical points $\{x \in \partial\Omega : d_x\psi_q(x) = 0\}$ are precisely the bounce points of a periodic orbit in $\Gamma(1, q)$. What we can determine from the spectrum is – at best– the set of critical values of ψ_q and integrals (or sums) over the critical points.

When a domain is 'nearly circular', it turns out that for all $q \geq 3$, the asymptotic expansion of $I_q(\lambda)$ as $\lambda \to \infty$ is a spectral invariant. In the jargon of inverse spectral theory, one can 'hear' each integral $I_q(\lambda)$. This was a crucial ingredient in the inverse result of [HeZe19]. In the case of an ellipse, ψ_q is a constant function for each q and that is sufficient to deploy the results of [ADK16]. But the argument that $I_q(\lambda)$ is a spectral invariant breaks down. In fact, it is not even clear that ψ_q is well-defined where we need it to be: I.e. is there is a unique geodesic loop at x with winding number 1 and q bounces?

There of course exist domains with 'messy' length spectra and ψ_q functions. It is a smooth function of one real variable but its critical set could be any closed set. One of the main problems is to derive information on the critical point structure of ψ_q from asymptotics of the integrals $I_q(\lambda)$. But in bad cases, the contribution to the wave trace from $\Gamma(1, q)$ orbits might involve a finite (or, worse) sum of integrals like $I(q, \lambda)$ and it is difficult to extract information from the sum.

A conjecture that seems reasonable and not inaccessible is:

Conjecture. *One can tell from $Sp(\Omega)$ if Ω is convex, at least if Ω is assumed to be real analytic.*

If Ω is convex, the length of its perimeter $|\partial\Omega|$ is an accumulation point in $Lsp(\Omega)$ from below: there exist sequences of $(1, q)$ orbits whose Lengths approach $|\partial\Omega|$ as $q \to \infty$. When Ω is convex, are multiples of $|\partial\Omega|$ the only possible accumulation points? What if Ω is analytic?

J. De Simoi has recently proved a number of results on accumulation points in the length spectrum for convex domains. It is plausible that the asymptotic expansions of Mather and Melrose-Marvizi of the maximal length L_q of $\Gamma(1, q)$ orbits can be used to distinguish these accumulation points from the one at $|\partial\Omega|$.

The main dynamical invariants one can deduce from $Sp(\Delta)$ when $Lsp(\Omega)$ is simple is the sequence of Birkhoff normal form invariants around (non-degnerate) periodic orbits. The question then arises whether one can have two non-isometric domains with the same length spectrum and the same Birkhoff normal form invariants. A local version of this problem is whether there exist two non-isometric (germs of) domains for which the Birkhoff normal forms around periodic 2-link orbits (bouncing ball orbits) are the same.

Conjecture. *There exist two non-isometric domain (germs) around bouncing ball orbits for which the Birkhoff normal form of the billiard map around the orbits are the same.*

Hezari and I have proven that such domains exist 'formally' , i.e. we construct the Taylor expansions of the defining functions of the domains near the endpoints of the bouncing ball orbit [HeZe21]. It remains to prove that the Taylor series converge when the domains are real analytic or can be completed to make smooth domains.

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On the accumulation of separatrices by invariant curves

Raphaël Krikorian
(joint work with Anatole Katok)

Let f be a smooth symplectic diffeomorphism of the plane admitting a (non-split) separatrix associated to a hyperbolic fixed point. We prove that if f is a perturbation of the time-1 map of a symplectic autonomous vector field, this separatrix is accumulated by a positive measure set of invariant circles. On the other hand, we provide examples of smooth symplectic diffeomorphisms with a Lyapunov unstable non-split separatrix that are not accumulated by invariant circles.

A theorem by M.R. Herman,"Herman's last geometric theorem", (cf. [3], [4]), asserts that if a smooth orientation and area preserving diffeomorphism f of the 2-plane admits a KAM circle Σ (by definition, a smooth invariant curve on which the dynamics of f is conjugated to a Diophantine translation) then this KAM circle is accumulated by other KAM circles the union of which has positive 2-dimensional Lebesgue measure in any neighborhood of Σ. In this report we investigate whether such a phenomenon holds if, instead of being a KAM circle, the invariant set Σ is a *separatrix* of a hyperbolic fixed (or periodic) point of f.

The situation we consider is the following. Let f_0 be the time-1 map of a smooth autonomous symplectic vector field X_0 of the 2-plane admitting 0 as a hyperbolic fixed point. We assume that the stable and unstable manifolds of 0 coincide and form a compact set Σ. We now consider a hamiltonian diffeomorphism f which is a smooth perturbation of f_0 and still admits 0 as a hyperbolic fixed point. It is not true in general that the stable and unstable manifold of 0 coincide but we assume that this is the case. For example we can construct $f := f_\epsilon$ as the time-1 map of a 1-periodic hamiltonian vector field $X_\epsilon(t,\cdot) = X_0(\cdot) + O(\epsilon)$ which is tangent to Σ.

Theorem A. *For any $r \in \mathbf{N}^*$ there exists $\epsilon_r > 0$, such that, for any $\epsilon \in]-\epsilon_r, \epsilon_r[$, there exists a set of f_ϵ-invariant C^r KAM-circles accumulating the separatix Σ and which covers a set of positive Lebesgue measure of $\mathbf{R}^2$ in any neighborhood of Σ.*

On the other hand, in the non perturbative case the situation is different.

Let Δ_Σ be the bounded connected component of $\mathbf{R}^2 \setminus \Sigma$.

Theorem B. *There exists a smooth symplectic diffeomorphism $f : \mathbf{R}^2 \to \mathbf{R}^2$ admitting a separatrix Σ which is included in an open set W of $\Sigma \cup \Delta_\Sigma$ that contains no f-invariant circle.*

The proofs of Theorems A and B rely on renormalization techniques.

Let us sketch the proof of Theorem A. After performing some Birkhoff Normal Form and applying a symplectic Sternberg Theorem ([1], [2]), we define a convenient fundamental domain for the Hamiltonian diffeomorphism f_ϵ. This fundamental domain is an abstract annulus and we consider the first return map of f_ϵ in this annulus. After uniformization, this abstract annulus becomes the standard annulus, and the first return map a diffeomorphism on this annulus (the renormalization of f_ϵ) that has the intersection property. In the perturbative situation we are dealing with, one can then apply Moser-Rüssmann's invariant curve theorem ([5], [6]) and prove the existence of a set of positive measure of KAM circle for the renormalization of f_ϵ. One then shows that these curves correspond to KAM curves for f_ϵ accumulating the separatrix Σ.

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Chaotic phenomena around L_3 in the restricted 3-body problem

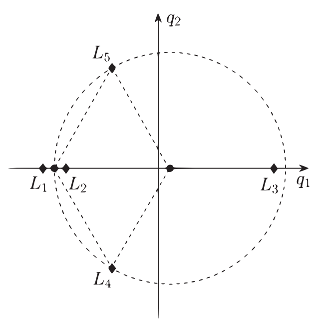
Inmaculada Baldomá Barraca
(joint work with Mar Giralt, Marcel Guardia)

The so-called Restricted Planar Circular 3-Body Problem (RPC3BP) is a simplified configuration of the 3-Body Problem, the motion of three bodies under their mutual gravitational attraction. We assume that a) one of the bodies (say the third) has mass zero (Restricted), b) the two first bodies, the primaries, are not influenced by the massless one and they perform circular motions (Circular), c) the third body is coplanar with the primaries (Planar).

Normalizing the primaries masses, we set them to $1-\mu$ and μ, with $\mu \in \left(0, \frac{1}{2}\right]$. Since the primaries follow circular orbits, in rotating coordinates, their positions can be fixed at $(\mu, 0)$ and $(\mu - 1, 0)$. Then, the RPC3BP is a 2-degrees of freedom Hamiltonian system with respect to the autonomous hamiltonian

$$(1) \qquad H(q,p;\mu) = \frac{||p||^2}{2} - q^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} p - \frac{(1-\mu)}{||q-(\mu,0)||} - \frac{\mu}{||q-(\mu-1,0)||}.$$

where $(q,p) \in \mathbb{R}^2 \times \mathbb{R}^2$ are the position and momenta of the third body.



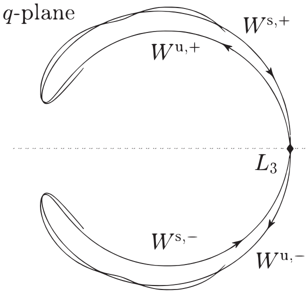
For $\mu > 0$, it is a well known fact [7] that (1) has five critical points, usually called Lagrange points. The three collinear Lagrange points, L_1, L_2 and L_3, are of center-saddle type whereas, for small μ, the triangular ones, L_4 and L_5, are of center-center type.

On an inertial (non-rotating) system of coordinates, the Lagrange points correspond to periodic orbits with the same period as the two primaries, i.e on a 1:1 mean motion resonance.

We focus on the invariant manifolds of L_3, the Lagrangian point "at the other side" of the massive primary, in the perturbative setting $0 < \mu \ll 1$. The eigenvalues of L_3 are of the form

$$(2) \qquad \left\{ \pm\sqrt{\mu}\sqrt{\frac{21}{8}} + \mathcal{O}(\mu^{3/2}),\ \pm i(1+\mathcal{O}(\mu)) \right\},$$

as $\mu \to 0$ and therefore, due to the different size in the eigenvalues, the system posseses two time scales, which translates to rapidly rotating dynamics coupled with a slow hyperbolic behavior around the critical point L_3.

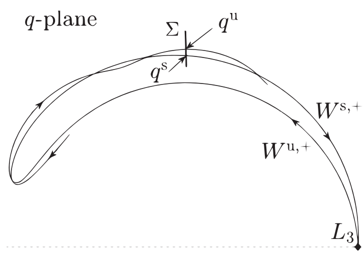


The one dimensional unstable and stable invariant manifolds have two branches each which are symetric with respect to $\Phi(q,p) = (q_1, -q_2, -p_1, p_2)$. Then we only need to focus in (for instance) the branches $W^{u,+}, W^{s,+}$. In the figure, the projection of the stable and unstable manifolds of L_3 on the q-plane.

The invariant manifolds associated to L_3 (more precisely its center-stable and center-unstable invariant manifolds) play an important role in the dynamics of the RPC3BP since they act as boundaries of *effective stability* of the stability domains around L_4 and L_5 (see [6]). Moreover, being far from collision, the dynamics close to the Lagrange point L_3 and its invariant manifolds for small μ are rather similar to that of other mean motion resonances which play an important role in creating instabilities in the Solar system, see [4].

Over the past years, one of the main focus of study of the dynamics "close" to L_3 and its invariant manifolds has been the so called "horseshoe-shaped orbits" which are quasi-periodic orbits that encompass the critical points L_4, L_3 and L_5. The interest on these types of orbits arise when modeling the motion of co-orbital satellites, the most famous being Saturn's satellites Janus and Epimetheus, and near Earth asteroids. Recently, in [5], the authors have proved the existence of 2-dimensional elliptic invariant tori on which the trajectories mimic the motions followed by Janus and Epimetheus

Rather than looking at stable motions "close to" L_3 as [5], our works ([1, 2]) are a first step towards a rigourous proof of the existence of instabilities in some instance of the 3-Body Problem.



Let us be more explicit. Consider the transversal tridimensional section Σ defined by $\Sigma = \{q_1 = 0, q_2 > 0 \ \|q\| > 1\}$. We define (q^u, p^u) and (q^s, p^s) being the first crossing of $W^{u,+}$ and $W^{s,+}$ with Σ respectively.

Then

$$\|q^s - q^u\| + \|p^s - p^u\| = |\Theta| \mu^{\frac{1}{3}} e^{-\frac{A}{\sqrt{\mu}}} \left(1 + \mathcal{O}(|\log \mu|^{-1})\right)$$

where $|\Theta| \sim 1.63$ is a Stokes constant which comes from an associated *inner equation* and can only be numerically computed whereas A is explicit:

$$A = \int_0^{\frac{\sqrt{2}-1}{2}} \frac{2}{1-x} \sqrt{\frac{x}{3(x+1)(1-4x-4x^2)}} dx \sim 0.177744.$$

To finish let us to make some comments about our result. This work rules out the existence of primary homoclinic connections to L_3 in the RPC3BP for $0 < \mu \ll 1$. However, it does not prevent the existence of multiround homoclinic orbits, that is homoclinic orbits which pass close to L_3 multiple times. Numerical evidences, [3], indicate that multi-round homoclinic connections to L_3 should exist for $\{\mu_k\}_{k\in\mathbb{R}}$ satisfying $\mu_k \to 0$ as $k \to \infty$.

From our result, one should expect that there exist Lyapunov periodic orbits exponentially close to L_3 whose (two dimensional) stable and unstable invariant manifolds intersect transversally. This would create chaotic motions "exponentially close" to L_3 and its invariant manifolds.

Finally, the analysis performed in this paper can be seen as a humble first step towards constructing Arnold diffusion in the 1 : 1 mean motion resonance if one considers either the Restricted Spatial Circular 3-Body Problem with small $\mu > 0$, the Restricted Planar Elliptic 3-Body Problem with small $\mu > 0$ and eccentricity of the primaries $e_0 > 0$, or the planar 3-Body Problem (i.e. all three masses positive, two small). One should be able to construct orbits with a drastic change in angular momentum (or inclination in the spatial setting).

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KAM rigidity of commuting parabolic automorphism of the torus

Bassam Fayad
(joint work with Danijela Damjanovic, Maria Saprykina)

This is a report on a joint work with Danijela Damjanovic and Maria Saprykina. The question we address is whether an affine ergodic $\mathbb{Z}^2$-action is locally rigid.

Two famous manifestations of local rigidity are KAM rigidity of Diophantine torus translations and smooth local rigidity of hyperbolic or partially hyperbolic higher rank actions.

The paper [DF] extended Damjanovic and Katok's local rigidity for partially hyperbolic higher rank affine actions on tori [DK], by showing KAM rigidity of affine $\mathbb{Z}^2$-actions that have a higher rank linear part except for a rank one factor that is the Identity.

To complete the study of local rigidity of $\mathbb{Z}^k$-actions on the torus one needs to address the case of affine actions with parabolic generators.

Definition 1. *We say that a linear map $A \in SL(d,\mathbb{Z})$ is (parabolic) of step n if $(A - Id)^n = 0$, and $(A - Id)^{n-1} = 0$. An affine map $a = A + \alpha$, $A \in SL(d,\mathbb{Z})$ and $\alpha \in \mathbb{R}^d$, is said to be of step n if A is of step n.*

We say that a $\mathbb{Z}^2$ affine action by parabolic elements is of step n if all of its elements are of step at most n.

As shown by A. Katok in [K], the cohomological equation above an individual parabolic affine action on the torus is stable, under a Diophantine condition on its translation part. Namely, there is a countable number of invariant distributions that can be computed as sums of Fourier coefficients (along the dual orbits) that constitute the only obstructions for the existence of a tame solution with a finite loss of regularity to the cohomological equation.

However, in the case of step-n, $n \geq 3$, the loss of differentiability is comparable to the order of smoothness considered. This shows that the local rigidity theory of step-2 parabolic actions should be very special, with much stronger rigidity features than the general case.

Given an affine map $a = A + \alpha$, a diffeomorphism of the torus f that is close to a is of the form $A + \alpha + \Delta f$ with Δf a $\mathbb{Z}^d$ periodic vector function.

Definition 2 (KAM rigidity of affine actions under volume preserving perturbations). *We say that an affine $\mathbb{Z}^2$-action $(a,b) = (A+\alpha, B+\beta)$ is KAM rigid under volume preserving perturbations, if there exists $u \in \mathbb{N}$ and $r_0 \in \mathbb{N}$ and $\varepsilon > 0$ that satisfy the following :*

If $r \geq r_0$ and $(f,g) = (a+\Delta f, b+\Delta g)$ is a smooth volume preserving $\mathbb{Z}^2$-action such that

$$\|\Delta f\|_r \leq \varepsilon, \quad \|\Delta g\|_r \leq \varepsilon, \quad \int_{\mathbb{T}^d} \Delta f d\lambda = 0, \quad \int_{\mathbb{T}^d} \Delta g d\lambda = 0, \tag{1}$$

then there exists $h \in Diff^{\infty}_{\lambda}(\mathbb{T}^d)$ such that $\|h - Id\|_{r-u} \leq C(a,b)\varepsilon$ and

$$h \circ f \circ h^{-1} = a, \quad h \circ g \circ h^{-1} = b.$$

Given an affine $\mathbb{Z}^2$-action $(a,b) = (A+\alpha, B+\beta)$, every element of the action is of the form $a^k b^l = A^k B^l + \alpha_{k,l}$. The commutation relation may force some of the coordinates of $\alpha_{k,l}$ to be 0. For some pairs (A,B) this may result in the existence of an Identity factor for any corresponding affine action (a,b). In this case the affine action is never ergodic and KAM rigidity fails. An example is given by the pair $A = \mathrm{Id} + E_{21}$, $B = \mathrm{Id} + E_{31}$ ($E_{i,j}$ is the matrix with a single nonzero entry 1 at row i and column j).

We show that besides this situation KAM rigidity holds for step-2 actions.

Theorem 1. *Given a commuting pair (A,B) of step-2 parabolic matrices, we have the following dichotomy*

(i) *For any choice of* (α, β) *such that* $a = A + \alpha$ *and* $b = B + \beta$ *commute, the action of* (a, b) *has Identity for factor and is thus not ergodic and not locally rigid.*
(ii) *For almost every choice of* (α, β) *such that* $a = A + \alpha$ *and* $b = B + \beta$ *commute (relative to the Haar measure on the coordinates that do not vanish because of the commutation relation), the action of* (a, b) *is ergodic and KAM-rigid under volume preserving perturbations.*

We can give a more precise version of *(ii)* in which we precise the full measure Diophantine conditions required on the pair (α, β) to guarantee KAM-rigidity.

The proof is based on a KAM scheme of successive conjugacies that converge quadratically to the linear model. An important ingredient of the proof is the cohomological stability above a step-2 parabolic automorphism that was proved by Katok in [K].

Problem 2. *In fact, for Diophantine step-2 maps a, it is very well possible that if a smooth perturbation of a is* C^1 *conjugated to a, then it is smoothly conjugated to a.*

We do not address this question in this work.

For individual step-n transformations, $n \geq 3$, we can expect an absence of local rigidity even when a C^1 conjugacy is supposed to exist with the affine map. Fix an arbitrary $\alpha \in \mathbb{R}$, $\bar{\alpha} = (\alpha, 0, 0,)$, consider the affine map $a = A + \bar{\alpha}$ where

$$A = \mathrm{Id} + E_{2,1} + E_{3,2} \in SL(3, \mathbb{Z}). \tag{2}$$

We show that

Proposition 1. *For any* $\alpha, r \in \mathbb{N}$ *and any* $\varepsilon > 0$*, there exists a function* $f \in C^{2r}(\mathbb{T}^3, \mathbb{R})$ *such that* $\|f\|_{2r} < \varepsilon$ *and for which*

$$H \circ A \circ H^{-1} = A + (0, 0, f)$$

for some diffeomorphism H *of class* C^{r-3} *such that* $\|H - Id\|_{r-3} < \varepsilon$ *while* H *is not of class* C^{r+1}.

This proposition indicates that because of the loss of regularity in the cohomological equation above a step-3 (or higher) transformation, we cannot expect the same rigidity features for general step-n actions as those of step-2 actions.

However, there are examples of higher step affine actions that are KAM-rigid. This is the case for some of the actions where one element is step-2. Define for an example

$$a(x, y, z, t) = (x + \alpha, y + x, z + y, t + \beta)$$
$$b(x, y, z, t) = (x, y + \alpha, z + t, t).$$

We show that

Theorem 3. *If* α *and* β *are Diophantine, the step-3 action* (a, b) *is KAM-rigid among volume preserving perturbations.*

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Ruelle zeta at zero for nearly hyperbolic 3-manifolds

Semyon Dyatlov

(joint work with Mihajlo Cekić, Benjamin Küster, Gabriel Paternain)

This talk reports on the recent paper [1]. We consider a negatively curved compact connected oriented Riemannian 3-manifold (Σ, g) and study the order of vanishing at 0 of the Ruelle zeta function

$$\zeta(\lambda) = \prod_{\ell \in \mathscr{L}_M} (1 - e^{-\lambda\ell}), \quad \operatorname{Re}\lambda \gg 1 \tag{1}$$

where $\mathscr{L}_M$ is the set of lengths of primitive closed geodesics on Σ, taken with multiplicity. The product (1) converges when $\operatorname{Re}\lambda$ is sufficiently large and it is known that it continues meromorphically to $\lambda \in \mathbb{C}$.

Denote by $m(0)$ the order of vanishing of ζ at 0, i.e.

$$\lambda^{-m(0)}\zeta(\lambda) \quad \text{is holomorphic and nonvanishing at } \lambda = 0.$$

Then we show the following

Theorem 1. *1. If g is the hyperbolic metric on Σ, then $m(0) = 4 - 2b_1(\Sigma)$ where $b_1(\Sigma)$ is the first Betti number of Σ.*

2. If g is a small generic conformal perturbation of the hyperbolic metric, then $m(0) = 4 - b_1(\Sigma)$.

The first part of Theorem 1 is not new – it is due to Fried [3], using the Selberg trace formula. The second part is where the novelty lies. The proof uses the microlocal approach to dynamical zeta functions. In this approach the order of vanishing of ζ at 0 is expressed as

$$m(0) = \sum_{k=0}^{4} (-1)^k \dim \operatorname{Res}_0^{k,\infty}$$

where $\operatorname{Res}_0^{k,\infty}$ are the spaces of currents (i.e. distributional differential forms) defined as follows:

$$\operatorname{Res}_0^{k,\infty} := \{u \in \mathcal{D}'(S\Sigma; \Omega^k) \mid \operatorname{WF}(u) \subset E_u^*,\ \iota_X u = 0,\ \exists \ell : \mathcal{L}_X^\ell u = 0\}.$$

Here $S\Sigma$ is the sphere bundle of (Σ, g), $X \in C^\infty(S\Sigma; T(S\Sigma))$ is the generator of the geodesic flow, $\Omega^k = \wedge^k T^*(S\Sigma)$ is the bundle of k-forms, $E_u^* = \{(x,\xi) \in T^*(S\Sigma) \mid \xi \in E_u^*(x)\}$ is the subset of $T^*(S\Sigma)$ induced by the dual unstable space, and $\operatorname{WF}(u) \subset T^*M \setminus 0$ is the *wavefront set* of a current u, which is a standard object in microlocal analysis giving the location of the singularities of u in the position-frequency space.

The proof of Theorem 1 then proceeds by first obtaining a detailed understanding of the spaces $\mathrm{Res}_0^{k,\infty}$ and the action of the operator d on them in the hyperbolic case. The case of perturbations is handled using first order perturbation theory, where the key ingredient is to show that pushforwards to Σ of certain currents determined by elements of $d\,\mathrm{Res}_0^{1,\infty}$ are nonzero.

Theorem 1 is in contrast with previous result of the speaker and Zworski [2] which showed that if Σ is a negatively curved surface, then the order of vanishing $m(0)$ is determined by just the topology of Σ, specifically $m(0) = b_1(\Sigma) - 2$. In [1] we conjecture that $m(0) = 4 - b_1(\Sigma)$ for a generic negatively curved metric.

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Reeb dynamics in dimension 3 and broken book decompositions

Vincent Colin

(joint work with Pierre Dehornoy, Ana Rechtman)

On a closed 3-manifold M, the *Giroux correspondence* asserts that every contact structure ξ is carried by *some* open book decomposition of M: there exists a Reeb vector field for ξ transverse to the interior of the pages and tangent to the binding [Gir]. The dynamics of this specific Reeb vector field is then captured by its first-return map on a page, which is a flux zero area preserving diffeomorphism of a compact surface, a much simplified data. When one is interested in the dynamics of a *given* Reeb vector field this Giroux correspondence is quite unsatisfactory – though there are ways to transfer some properties of an adapted Reeb vector field to every other one through contact homology techniques [CH, ACH] – and the question one can ask is: Is every Reeb vector field adapted to some (rational) open book decomposition? Equivalently, does every Reeb vector field admit a Birkhoff section?

We give a positive answer to these questions for the generic class of nondegenerate Reeb vector fields and the extended class of *broken book decompositions.*

Theorem 1. *Every nondegenerate Reeb vector field on a 3-manifold is carried by a broken book decomposition.*

A contact form and the corresponding Reeb vector field are *nondegenerate* if all the periodic orbits of the Reeb vector field are nondegenerate, namely the eigenvalues of a Poincaré map along a periodic orbit are all different from one (even when the orbit is travelled several times). The nondegeneracy condition is generic for Reeb vector fields, see for example [CH, Lemma 7.1].

A *Birkhoff section* of a vector field R on a 3-manifold is a surface with boundary whose interior is embedded and transverse to R and whose boundary is immersed and composed of periodic orbits. A Birkhoff section must also intersect all orbits of R within bounded time, so that there is a well-defined return map in the interior of the surface. The boundary will be called the *binding*. These surfaces are also known as global surfaces of section. A Birkhoff section induces a rational open book decomposition of the manifold.

Broken book decompositions are generalisations of Birkhoff sections and rational open book decompositions, reminiscent of finite energy foliations constructed by Hofer-Wyszocki-Zehnder for nondegenerate Reeb vector fields on $\mathbb{S}^3$ [HWZ]. In a broken book decomposition we allow the binding to have *broken* components, in addition to *radial* ones modelled on the classical open book case. The complement of the binding is foliated by surfaces. A radial component of the binding has a tubular neighborhood in which the pages of the broken book induce a radial foliation by annuli. The foliation in a tubular neighborhood of a broken component has sectors that are radially foliated by annuli and sectors that are transversally foliated by hyperbolas.

A broken book decomposition *carries*, or *supports*, a Reeb vector field if the binding is composed of periodic orbits, while the other orbits are transverse to the foliation given on the complement of the binding by the interior of the pages (this foliation by relatively compact leaves is usually non trivial, as opposed to the genuine open book case). In the proof of Theorem 1, we construct a supporting broken book decomposition for any fixed nondegenerate Reeb vector field on a 3-manifold M from a cover of M by pseudo-holomorphic curves, given by the non-triviality of the U-map in embedded contact homology. The projected pseudo-holomorphic curves are then converted into surfaces with boundary whose interiors are embedded and transverse to the Reeb vector field using a construction of Fried [Fri]. These surfaces give a complete system of transverse surfaces to the Reeb vector field, meaning that their union intersects every orbit. The novelty in our approach is to combine these two known techniques.

We believe that the notion of a (degenerate) broken book decomposition is interesting in its own right. Near the binding, the broken book foliation looks like the mapping torus of a transverse invariant foliation of Le Calvez [L] and our study could also be seen as a first step towards generalising Le Calvez theory to vector fields in three dimensions.

Weinstein conjectured in 1979 that a Reeb vector field on a closed 3-manifold always has at least one periodic orbit [Wei]. The conjecture was proved in full generality by Taubes using Seiberg-Witten Floer homology [Tau]. It is also a consequence of the U-map property we use here, and it is no surprise that our result indeed implies the existence of the binding periodic orbits. Taubes' result was then improved by Cristofaro-Gardiner and Hutchings [C-GH], who proved that every Reeb vector field on a closed 3-manifold has at least two periodic orbits, following a work of Ginzburg, Hein, Hryniewicz and Macarini on $\mathbb{S}^3$ [GHHM]. It is now moreover conjectured that a Reeb vector field has either two or infinitely many periodic orbits. The existence of infinitely many periodic orbits has been established under some hypothesis (see the survey [GG]) and it is known to be generic [Iri]. We extend a recent result of Cristofaro-Gardiner, Hutchings and Pomerleano, originally obtained for *torsion contact structures* ξ (with $c_1(\xi) \in \mathrm{Tor}(H^2(M,\mathbb{Z}))$) [C-GHP] and prove the conjecture for nondegenerate Reeb vector fields.

Theorem 2. *If M is a closed oriented 3-manifold that is not the sphere or a lens space, then every nondegenerate Reeb vector fields on M has infinitely many simple periodic orbits. In the case of the sphere or a lens space, there are either two or infinitely many periodic orbits.*

We point out that the cases where Reeb vector fields have exactly two nondegenerate periodic orbits are well-understood: they exist only on the sphere or on lens spaces, both periodic orbits are elliptic and are the core circles of a genus one Heegaard splitting of the manifold [HuT]. Also the contact structure has to be tight, since a nondegenerate Reeb vector field of an overtwisted contact structure always has a hyperbolic periodic orbit (see for example Theorem 8.9 in [HK]). A recent work of Cristofaro-Gardiner, Hryniewicz, Hutchings and Liu shows that when there are exactly two periodic orbits they are in fact nondegenerate, see [C-GHHL].

Beyond the number of periodic orbits, the study of the topological entropy of Reeb vector fields started with the works of Macarini and Schlenk [MS] and has been continued by Alves [ACH, Alv]. We recall that topological entropy measures the complexity of a flow by computing the growth of the number of "different" orbits. If this number grows exponentially then the entropy is positive. For flows in dimension 3, if the topological entropy is positive then the number of periodic orbits is infinite.

As an application of Theorem 1 we get a result on topological entropy

Theorem 3. *If M is a closed oriented 3-manifold that is not a graph manifold, then every nondegenerate Reeb vector field on M has positive topological entropy.*

Theorems 2 and 3 are obtained by analysing the broken binding components of the broken book decomposition. Indeed, a broken component of the binding is a hyperbolic periodic orbit and we can prove that there are heteroclinic cycles between these periodic orbits. If there are no such broken components, then we have a rational open book decomposition and the results come from an analysis of its monodromy. In particular, we obtain

Theorem 4. *If M is a closed oriented 3-manifold, then every strongly nondegenerate Reeb vector fields on M without homoclinic orbits has a Birkhoff section.*

A *homoclinic* orbit is an orbit that is contained in a stable and an unstable manifold of the same hyperbolic periodic orbit. Equivalently, it is an orbit that is forward and backward asymptotic to the same hyperbolic periodic orbit.

A vector field is *strongly nondegenerate* if it is nondegenerate and the intersections of the stable and unstable manifolds of the hyperbolic orbits are transverse. A strongly nondegenerate vector field with a homoclinic orbit has positive topological entropy, thus Theorem 4 implies that a strongly nondegenerate Reeb vector field whose topological entropy is zero is carried by a rational open book decomposition.

Our techniques, combined with Fried's construction [Fri] and Arnaud-Bonatti-Crovisier's results [BC, ABC], also allow to establish the existence of a Birkhoff section for a C^1-dense subset of Reeb vector fields.

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PFH spectral invariants, the simplicity conjecture, and beyond

Daniel Cristofaro-Gardiner
(joint work with Vincent Humiliére, Sobhan Seyfaddini)

In the 60s and 70s, there was a flurry of activity concerning the question of whether or not various subgroups of homeomorphism groups of manifolds are simple, with beautiful contributions by Fathi, Kirby, Mather, Thurston, and many others[1]. A funnily stubborn case that remained open was the case of area-preserving homeomorphisms of surfaces. For example, the following question, Question A below, had remained unsolved. Let $Homeo_c(D^2, \omega)$ denote the group of area-preserving homemorphisms of the two-disk, with its standard area form, that are the identity near the boundary; this is a normal subgroup of the group of area-preserving homeomorphisms of the two-disk.

Question A: Is $Homeo_c(D^2, \omega)$ a simple group?

In contrast, for balls of higher dimensions the analogous group is known to be simple by work of Fathi from the 70s [4]. Over time, it has generally been thought that the symplectic structure, which is a unique feature of the two-dimensional case, forces the group to be not simple in contrast to the situation in higher dimensions. This has been called the "symplectic conjecture", and it has attracted considerable interest.

Our main result confirms it.

Theorem A [1]**:** The group $Homeo_c(D^2, \omega)$ is not simple.

To prove Theorem A, we significantly develop the theory of what are called **PFH spectral invariants**. The PFH spectral invariants are a sequence of numbers $c_d(\varphi) \in \mathbb{R}$ associated (after a choice of "reference cycle") to area-preserving homeomorphisms or diffeomorphisms of the two-sphere. They were originally defined for diffeomorphisms by Michael Hutchings and one of the results of our work extends their definition to homeomorphisms. Their definition makes use of a kind of Floer homology for three-dimensional mapping torii, defined by Hutchings, called periodic Floer homology. We show that they satisfy various useful properties. For example, they are continuous with respect to the C^0-topology, which is the key property that we use to show that they can be defined for homeomorphisms. They also are Lipschitz with respect to Hofer's metric, with Lipschitz constant d.

To state another important property that they satisfy, recall that a compactly supported area-preserving diffeomorphism φ of the two-disc has a well-defined **Calabi invariant** defined by writing φ as a time-1-map of a Hamiltonian H,

[1] For a summary of some of this history, see [1, Sec. 1.1.1].

normalized to be 0 near the boundary of the disc, and then defining $Cal(\varphi) = \int H\omega$. This measures the "average rotation" of φ and it turns out that it does not depend on the choice of H. Hutchings has conjectured that the PFH spectral invariants recover Calabi in their asymptotic limit:

Conjecture A: $lim_{d\to\infty}\frac{c_d(\varphi)}{d} = Cal(\varphi)$.

We prove this for "monotone twist" maps and it is an interesting question whether it holds more generally. We believe this to be the case.

We can now explain the basic idea behind the proof of Theorem A. We define a subgroup $FHomeo \subset Homeo_c(D^2, \omega)$ of "finite Hofer energy" homeomorphisms. These are maps which can be uniformly approximated by diffeomorphisms while keeping the distance from the identity in Hofer's metric bounded. It is not too difficult to show that this is a non-trivial normal subgroup, and the challenge is to show that it is proper. To do this, we show that a particular kind of "infinite twist" map, defined by preserving the radius but twisting more and more as one approaches the origin, is not in $FHomeo$. The idea is that an infinite twist map morally has infinite Calabi invariant, and so the PFH spectral invariants grow super linearly for such a map by Conjecture A. On the other hand, a finite Hofer energy homeomorphism must have no more than linear growth of its PFH spectral invariants by the Hofer Lipschitz property mentioned above. The idea of defining a normal subgroup and showing properness via an infinite twist map was inspired by the article [7].

Moving forward, PFH spectral invariants have had further uses. In particular, we used them in [2] to resolve the Kapovich-Polterovich question about the large-scale geometry of the group of Hamiltonian diffeomorphisms of the two-sphere, equipped with Hofer's metric: this question, which was featured as Problem 21 in the McDuff-Salamon problem list [6, Sec. 14.2], asked whether or not this group was quasi-isometric to $\mathbb{R}$ and we showed that it was not. Simultaneously, this question was resolved by Polterovich-Shelukhin [8] using a different family of "link spectral invariants" that built on work of Mak and Smith [5]. We also showed using PFH spectral invariants that the group of area and orientation preserving homeomorphisms of the two-sphere is not simple; the two sphere was the last closed manifold for which the simplicity question for the group of volume preserving homeomorphisms was not known. The main technical challenges of these works as far as PFH spectral invariants are concerned is defining invariants that depend only on the time 1-map and not on the choice of reference cycle.

Returning now to the algebraic structure of $Homeo_c(D^2, \omega)$, the following seems very important.

Question B: Can we understand the quotient of $Homeo_c(D^2, \omega)$ by $FHomeo$?

We know by [1] that the quotient is abelian, and our infinite twist generates a copy of $\mathbb{R}$ in it. Polterovich and Shelukhin have shown [8] that this copy of $\mathbb{R}$ is not the entire group, but not much is known beyond that. In a different direction, it is also natural to try to understand the structure of FHomeo. It contains a normal subgroup Hameo whose definition we omit here for brevity. We showed in [3] that Calabi extends to Hameo[2] and in particular FHomeo is not simple.

Question C: Is the kernel of Calabi on Hameo simple?

It would also be interesting to understand the relationship between PFH spectral invariants and the link spectral invariants. It was observed by Polterovich and Shelukhin [8] that these turn out to agree in certain cases where they can both be computed.

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Disjointness of translation flows with their inverses

Prsemyslaw Berk

(joint work with Krzysztof Fraczek, Thierry de la Rue)

The problem of existence of isomorphism of a flow with its inverse is a very classical issue in the theory of dynamical systems. Many classical examples of flows have the property that they are isomorphic with their inverses. More precisely, if $(X, \mathcal{B}, \mu, \mathcal{T} = \{T_t\}_{t\in\mathbb{R}})$ is a measure preserving flow then it is isomorphic with its inverse iff there exists a measurable map $S : X \to X$ such that $S_*\mu = \mu$ and

$$S \circ T_t = T_{-t} \circ S \quad \text{for every } t \in \mathbb{R}.$$

One of those classical examples is a linear flow on the torus, where S can be seen as a rotation by π. This can be generalized to the broader setting of translation surface. Namely, (M, η) is a translation surface, where M is a compact topological surface of genus $g \geq 1$ and η is a maximal atlas of charts defined everywhere minus finite number of points, where each transition map is a translation. Such translation surfaces can be obtained by glueing parallel sides in a polygon obtained from $d \geq 2$ pairs of identical segments. Then such surface can be parametrized by

[2]Polterovich-Shelukhin have communicated that they have a proof of this as well.

the parameters of the sides and the ordering of those sides on the circumference of the polygon.

On the translation surface we consider a translation flow T_t associated to a constant unit vector field. If there exists a Poincaré section such that the first return map is an interval exchange transformation given by a symmetric permutation, then such flow is isomorphic to its inverse ant the isomorphism can be seen as a rotation by π like in the toral case. We say that this is a *hyperelliptic* case. Our main result in [1] shows that this is not the case in non-hyperelliptic case.

Theorem 1. *In non-hyperellptic case, for G_δ-dense set of translation surfaces, the vertical flow is disjoint with its inverse. On the other hand for a dense set of translation surfaces, the vertical flow is isomorphic with its inverse.*

We say that two flows $(X, \mathcal{B}, \mu, \{T_t\}_{t\in\mathbb{R}})$ and $(Y, \mathcal{C}, \nu, \{S_t\}_{t\in\mathbb{R}})$ are disjoint if for every measure λ on $X \times Y$ such that

- λ projects as μ on X and as ν on Y;
- λ is invariant under action of the product flow $\{T_t \times S_t\}_{t\in\mathbb{R}}$

we have $\lambda = \mu \otimes \nu$. In particular, it is easy to see that disjointness implies non-isomorphism.

To prove Theorem 1, we consider G_δ and density condition separately. First we construct locally defined continuous embedding from the space of translation surfaces into the space of measure preserving flows $Flow$. Then we use the result of Danilenko and Rhyzhikov in [2] that the property of being disjoint with its inverse is typical in $Flow$. This yields G_δ condition. To prove density, we consider surfaces with short vertical saddle connections. Then we pass to special representation the vertical flow and we obtain a special flow over rotation with two extra discontinuity points. Finally, we prove a criterion on disjointness for special flows which can be applied in our case.

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Can one-parameter families of 2– and $(2m+1)$–periodic billiard trajectories be co–preserved?

Comlan Edmond Koudjiann

(joint work with Vadim Kaloshin)

Let $\Omega \subset \mathbb{R}^2$ be a convex domain with sufficiently smooth boundary $\partial\Omega$ parametrized by an arclength s, with corresponding parametrization $\gamma\colon \mathbb{T} := \mathbb{R}/\ell\mathbb{Z} \ni s \mapsto \mathbb{R}^2$, ℓ being the length of the boundary $\partial\Omega$. The billiard map, denoted by B_Ω, is a map $B_\Omega\colon \mathbb{T} \times (0, \pi) \ni (s_0, \theta_0) \longmapsto (s_1, \theta_1) \in \mathbb{T} \times (0, \pi)$ defined as follows: s_1 is the arclength parameter of the (second) intersection point of $\partial\Omega$ with the line passing through $\gamma(s_0)$ and making the angle θ_0 with the tangent vector $\dot{\gamma}(s_0)$; θ_1 is the angle made by the reflection of the vector $\overrightarrow{\gamma(s_1)\gamma(s_0)}$ *w.r.t* the inner–normal to $\partial\Omega$ at $\gamma(s_1)$, and the tangent vector $\dot{\gamma}(s_1)$. Moreover, B_Ω can be extended smoothly to the boundaries $\mathbb{T} \times \{0\}$ and $\mathbb{T} \times \{\pi\}$ as the identity.

In this talk, I am interested in convex billiard tables with convex rational caustics.

Definition 1:

- A caustic for B_Ω is a curve $\Gamma \subset \Omega$ *s.t.* if a B_Ω–trajectory is once tangent to Γ, then this trajectory stays tangent to Γ after each reflection.
- A caustic Γ is called convex if Γ is a closed curve which bounds a strictly convex domain.
- A caustic Γ in Ω is called (m, n)–rational integrable convex caustic ((m, n)–rational caustic for short) if all its B_Ω–tangent trajectories are (m, n)–periodic. An orbit[1] $\{(s_k, \theta_k)\}_{k\in\mathbb{N}_0}$ of the billiard map is called (m, n)–periodic if its lift $\{(\tilde{s}_k, \theta_k)\}_{k\in\mathbb{N}_0}$ to the universal cover $\mathbb{R} \times [0, \pi]$ satisfies: $\tilde{s}_n = \tilde{s}_0 + n\ell$, $\theta_n = \theta_0$.

Example 2:

- Any inner–circle in a disc $\mathscr{D}$ is a convex caustic for $B_{\mathscr{D}}$ and therein exists a (m, n)–rational caustic for any $m \geq 1$, $n \geq 2$.
- Any inner–confocal ellipse of an ellipse $\mathscr{E}$ is a convex caustic for $B_{\mathscr{E}}$ and therein exists a (m, n)–rational caustic for any $m \geq 1$, $n \geq 3$.

J. Mather proved that if a convex domain admits a convex caustic then this domain must be strictly convex. Moreover, Lazutkin showed that, near the boundary of a sufficiently smooth, strictly convex billiard table Ω, there exists uncountably many convex caustics accumulating at the boundary and whose union is a set of positive measure; in addition, B_Ω is conjugated to a rigid motion with irrational rotation number on each of those convex caustics. One can then naturally wonder about the rational caustic: Do they exist in general? How many of them should one expects?

Unlike irrational caustics which tend to be robust under perturbations, rational caustics tend to be very rigid and can easily be destroyed by perturbations. As pointed out in **Example 2**, ellipses (including circles) admits rational caustics for all (m, n), $m \geq 1$ and $n \geq 2$, except for $n = 2$ for ellipses with positive eccentricities. Thus, one may ask

Question 3 ([2]): Given two such pairs (m_1, n_1) and (m_2, n_2), are they smooth strictly convex domains other than ellipses which admit (simultaneously) a (m_1, n_1)– and a (m_2, n_2)–rational caustics?

In this context, S. Tabachnikov conjectured the following.

Conjecture 4: In a C^r $(n = 2, \cdots, \infty, w)$ neighborhood of a circle, there are no other billiard tables that admit a $(1, 2)$– and a $(1, 3)$–rational caustic than circles.

[1] $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

In this talk, I will answer affirmatively a deformative version of this conjecture. To this end, we will need the following definitions.

Definition 5:

- We call a deformation of a disc, any 1–parameter family $\{\Omega_\varepsilon\}_{0\le\varepsilon<\varepsilon_0}$ of strictly convex planar domains such that Ω_0 is a disc.
- We shall say that a deformation $\{\Omega_\varepsilon\}_{0\le\varepsilon<\varepsilon_0}$ of a disc preserves the (m,n)–rational caustic if, for any $0\le\varepsilon<\varepsilon_0$, B_{Ω_ε} admits a (m,n)–rational caustic $\mathscr{C}_\varepsilon$ and $\mathscr{C}_\varepsilon=\mathscr{C}_0+O(\varepsilon)$.

Then, we proved the following

Theorem 6 (V. Kaloshin & C.E. Koudjinan[1]): Let $m\in\mathbb{N}$ and $\{\Omega_\varepsilon\}_{0\le\varepsilon<\varepsilon_0}$ be a deformation of a disc. Assume that:

(i) the map $[0,\varepsilon_0)\ni\varepsilon\longmapsto\Omega_\varepsilon$ is C^3;

(ii) the boundary $\partial\Omega_\varepsilon$ is real–analytic for each $\varepsilon\in[0,\varepsilon_0)$;

(iii) the deformation $\{\Omega_\varepsilon\}_{0\le\varepsilon<\varepsilon_0}$ preserves simultaneously the $(1,2)$– and the $(1,2m+1)$–rational caustics.

Then, the deformation $\{\Omega_\varepsilon\}_{0\le\varepsilon<\varepsilon_0}$ is trivial, *i.e.* Ω_ε is a disc for each $\varepsilon\in[0,\varepsilon_0)$.

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Oscillatory motions and symbolic dynamics in the three body problem

Marcel Guardia

(joint work with Pau Martín, Tere M. Seara)

The three body problem models the motion of three bodies with masses m_0, m_1, $m_2>0$ under the Newton gravitational force. It is given by the equations

$$\begin{aligned}\ddot q_0 &= m_1\frac{q_1-q_0}{\|q_1-q_0\|^3}+m_2\frac{q_2-q_0}{\|q_2-q_0\|^3}\\ \ddot q_1 &= m_0\frac{q_0-q_1}{\|q_0-q_1\|^3}+m_2\frac{q_2-q_1}{\|q_2-q_1\|^3}\\ \ddot q_2 &= m_0\frac{q_0-q_2}{\|q_0-q_2\|^3}+m_1\frac{q_1-q_2}{\|q_1-q_2\|^3}.\end{aligned}\tag{1}$$

We consider two fundamental questions for this classical model: The analysis of the possible *final motions* and the existence of *chaotic motions (symbolic dynamics)*. These questions go back to the first half of the 20th century.

We call final motions to the possible qualitative behaviors that the complete trajectories of the 3 body problem may possess as time tends to infinity (forward or backward). The analysis of final motions was proposed by Chazy [2], who proved that the final motions of the three body problem should fall into one of the following categories. To describe them, we denote by r_k the vector from the point mass m_i to the point mass m_j for $i \neq k$, $j \neq k$, $i < j$.

Theorem 1 (Chazy (1922))**.** *Every solution of the three body problem defined for all (future) time belongs to one of the following seven classes.*

- *Hyperbolic (H):* $|r_i| \to \infty$, $|\dot{r}_i| \to c_k > 0$ *as* $t \to \infty$.
- *Hyperbolic–Parabolic (HP$_k$):* $|r_i| \to \infty$, $i = 1, 2, 3$, $|\dot{r}_k| \to 0$, $|\dot{r}_i| \to c_i > 0$, $i \neq k$, *as* $t \to \infty$.
- *Hyperbolic–Elliptic, (HE$_k$):* $|r_i| \to \infty$, $|\dot{r}_i| \to c_i > 0$, $i \neq k$, *as* $t \to \infty$, $\sup_{t \geq t_0} |r_k| < \infty$.
- *Parabolic-Elliptic (PE$_k$):* $|r_i| \to \infty$, $|\dot{r}_i| \to 0$, $i \neq k$, *as* $t \to \infty$, $\sup_{t \geq t_0} |r_k| < \infty$.
- *Parabolic (P):* $|r_i| \to \infty$, $|\dot{r}_i| \to 0$ *as* $t \to \infty$.
- *Bounded (B):* $\sup_{t \geq t_0} |r_i| < \infty$.
- *Oscillatory (OS):* $\limsup_{t \to \infty} \sup_i |r_i| = \infty$ *and* $\liminf_{t \to \infty} \sup_i |r_i| < \infty$.

Note that this classification applies both when $t \to +\infty$ or $t \to -\infty$. To distinguish both cases we add a superindex $+$ or $-$ to each of the cases, e.g H^+ and H^-.

At the time of Chazy all types of motions were known to exist except the oscillatory ones. Their existence was proven later by Sitnikov [3] for the Restricted 3 Body Problem and by Alekseev [1] for the (full) 3 body problem for some choices of the masses.

We have proven the following result.

Theorem 2. *Consider the three body problem with masses* $m_0, m_1, m_2 > 0$ *such that* $m_0 \neq m_1$. *Then,*

$$X^- \cap Y^+ \neq \emptyset \qquad \text{with} \qquad X, Y = OS, B, PE_3, HE_3.$$

Note that this theorem gives the existence of orbits which are oscillatory in the past and in the future. It also gives different combinations of past and future final motions. Indeed, the orbits that we construct have negative energy and are such that:

- The bodies of masses m_0 and m_1 perform (approximately) circular motions. That is, $|q_0 - q_1|$ is aproximately constant.
- The third body may have radically different behaviors: oscillatory, bounded, hyperbolic or parabolic.

This theorem is a consequence of the following one, which proves the existence of chaotic dynamics for the 3 Body Problem.

Theorem 3. *Consider the three body problem with masses $m_0, m_1, m_2 > 0$ such that $m_0 \neq m_1$ and denote by Φ_t its flow. Then, there exists a section Π transverse to Φ_t such that the induced Poincaré map*

$$\mathcal{P} : \mathcal{U} = \mathring{\mathcal{U}} \subset \Pi \to \Pi$$

has an invariant set $\mathcal{X}$ which is homeomorphic to $\mathbb{N}^{\mathbb{Z}}$, the set of sequences of natural numbers. Moreover, the dynamics $\mathcal{P} : \mathcal{X} \to \mathcal{X}$ is topologically conjugated to the shift

$$\sigma : \mathbb{N}^{\mathbb{Z}} \to \mathbb{N}^{\mathbb{Z}}, \qquad (\sigma\omega)_k = \omega_{k+1}$$

The set $\mathcal{X}$ is a hyperbolic set once the 3 body problem is reduced by its classical first integrals and it leads to positive topological entropy. The oscillatory motions given by Theorem 2 belong also to this invariant set $\mathcal{X}$.

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Absolutely Periodic Orbits in Smooth Convex Billiards

Keagan Callis

We consider a bounded domain $\Omega \subset \mathbb{R}^2$, and the eigenvalue problem for the laplacian, i.e. solutions ϕ to $\Delta\phi = \lambda\phi$ for some eigenvalue λ. Then if one considers all the eigenvalues which admit solutions, $\{\lambda_1 < \lambda_2 \leq \lambda_2 \leq ... \leq \lambda_n \leq ...\}$ and the counting function $N(\lambda) = |\{\lambda_i \leq \lambda\}|$, Weyl was able to prove that the first term in the asymptotic expansion of N depends only on the area of Ω. For the second term in the expansion, Weyl conjectured the following:

Conjecture. *(Weyl's Conjecture,* [10]*) Let $\Omega \in C^\infty(\mathbb{R}^2)$. Then,*

$$N(\lambda) = \frac{1}{4\pi}\text{Area}(\Omega)\lambda + C\ell(\partial\Omega)\lambda^{1/2} + o(\lambda^{1/2}), \tag{1}$$

where C is a constant and $\ell(\partial\Omega)$ is the length of the boundary of Ω.

This was proven by Ivrii in [3], provided Ivrii's conjecture holds, which states:

Conjecture. *(Ivrii's Conjecture) The set of periodic billiards has measure zero for all $\Omega \in C^\infty(\mathbb{R}^2)$.*

Thus, it makes sense to study the prevalence of periodic orbits in smooth billiard systems. We note that if one were to find an open set of periodic points in a billiard system, say of period q, then the differential at a point in this set must be the identity, i.e. $df^q(x_0) = Id$ for each x_0 in this open set. Following the definition given in [4], a periodic orbit is called **absolutely periodic** if it has such a point. Thus, in [4] there is the following conjecture:

Conjecture. *There are no absolutely periodic orbits for euclidean billiards.*

This has been proven in the case of convex domains with analytic boundary [9], and has been proven in the case of convex domains with smooth boundary up to periods of period 4 [8]. However, if one were to find such an orbit, it would be significant progress in a disproof of Ivrii's conjecture.

In this paper, we detail a proof on the existence of a billiard system exhibiting an **absolutely periodic orbit of order** n, or in other words an orbit such that $df^q(x_0) = Id + F(x_0)$, where $F(x_0) = 0$ up to order n. In fact, we prove that such domains can always be found close to billiard maps with strictly convex boundaries in $C^r(\mathbb{R}^2)$ that exhibit homoclinic tangencies. Our main theorem then is

Theorem 1. *For any $n \in \mathbb{N}$ and any strictly convex $\Omega \in C^r(\mathbb{R}^2)$, there exists a strictly convex boundary $\tilde{\Omega} \in C^{r-1}(\mathbb{R}^2)$ arbitrarily close to Ω in the C^{r-1} topology such that $\tilde{\Omega}$ has an absolutely periodic orbit of order n.*

To prove this we seek to use the methods from [1] to obatin a similar result for billiard maps. In their setting, something stronger is proven that involves what are called Newhouse domains. The notion of Newhouse domains comes from another area of study - that of homoclinic orbits and their bifurcations. Newhouse proved in [5], [6], [7], that there exist open sets (Newhouse domains) in which maps that exhibit homoclinic tangencies are dense. Moreover it was shown that these domains exist around any map that exhibits a homoclinic tangency.

It was shown in [1] (Theorem 3) that in the Newhouse domains in the space of area preserving $C^\infty(\mathbb{R}^2)$ maps, maps with infinitely many homoclinic tangencies of all orders are dense. We seek to prove an analogue of their proof by extending the result to the case of billiard maps, though here we are only interested in obtaining a map with a single periodic orbit of high order close to our original.

To do this, we follow essentially the same proof as in [1], *only in the case of billiard maps.*

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Non existence of small breathers of non-linear Klein-Gordon equations

Tere Seara

(joint work with Otàvio M.L. Gomide, Marcel Guardia, Chongchun Zeng)

Breathers are nontrivial time-periodic and spatially localized solutions of nonlinear dispersive partial differential equations (PDEs). In this work (see full details and proofs in [1], as well as results about the so-called *generalized breathers*) through a bifurcation approach in a singular perturbation framework, we study the existence/non-existence of *small* breathers of a class of nonlinear Klein-Gordon equations:

$$\partial_t^2 u - \partial_x^2 u + u - \frac{1}{3}u^3 - f(u) = 0, \tag{1}$$

where the nonlinearity f is a real-analytic odd function satisfying $f(u) = \mathcal{O}(u^5)$.

Let $\omega > 0$. A solution $u(x,t)$ of (1) is a breather of temporal frequency ω if $u(x,t)$ is $\frac{2\pi}{\omega}$-periodic in t and in some appropriate metric

$$\lim_{x\to\pm\infty} u(x,\cdot) = 0.$$

Given $\sigma \in (0,1)$ we call the breather σ-multi-bump in x in the ℓ_1 norm if there exist $x_1 < x_2 < x_3 < x_4 < x_5$ such that

$$\max\{\|u(x_{j_1},\cdot)\|_{\ell_1} \mid j_1 \in 1,3,5\} \le \sigma \min\{\|u(x_{j_2},\cdot)\|_{\ell_1} \mid j_2 \in 2,4\}.$$

We call the breather σ-single-bump if it is not σ-multi-bump.

Theorem 1. *Assume $f(u) = \mathcal{O}(u^5)$ is odd and analytic. Then, there exists $C_{\rm in} \in \mathbb{C}$, which depends on $f(\cdot)$ analytically, such that if $C_{\rm in} \neq 0$, then for any $\sigma \in (0,1)$, there exists $\rho^* > 0$ such that there does not exist any solution $u(x,t)$ to (1) which:*

(1) is $\frac{2\pi}{\omega}$-periodic in t for some $\omega > 0$,
(2) is σ-single-bump in the ℓ_1 norm.
(3) satisfies that, as $|x| \to +\infty$,

$$\|u(x,\cdot)\|_{H_t^1\left((-\frac{\pi}{\omega},\frac{\pi}{\omega})\right)} + \|\partial_x u(x,\cdot)\|_{L_t^2\left((-\frac{\pi}{\omega},\frac{\pi}{\omega})\right)} \to 0, \tag{2}$$

(4) satisfies

$$\sup_{x\in\mathbb{R}} \|u(x,\cdot)\|_{\ell_1} < \min\{1, \rho^*\omega^{\frac{1}{2}}\}. \tag{3}$$

The constant $C_{\rm in}$, known as the *Stokes constant* in the literature, depends analytically on f, and we can prove that $C_{\rm in} \neq 0$ for generic f. Moreover, $C_{\rm in}$ does not depend on ω, therefore just *one condition* ($C_{\rm in} \neq 0$) rules out the existence of single-bump small amplitude breathers *of any frequency*.

Theorem 1 is proved using the spatial dynamics method [2, 3]. Fix $\omega > 0$. Considering x as the evolutionary variable, given any $\frac{2\pi}{\omega}$-periodic-in-t initial values $\big(u(0,\cdot), \partial_x u(0,\cdot)\big)$, the nonlinear Klein-Gordon equation (1) defines a well-posed Hamiltonian dynamical system depending on the paramenter ω in appropriate spaces of $\frac{2\pi}{\omega}$-periodic-in-t functions. The trivial state 0 is stationary and breathers correspond to orbits which converge to 0 as both $x \to \pm\infty$, that is, homoclinic orbits to 0 which belong to the intersection between their stable and unstable manifolds. In the spatial dynamics framework of (1), the dimension of the hyperbolic eigenspace of 0 is finite and increases by 1 as the frequency ω decreases through $\frac{1}{k}$. Therefore, small homoclinics can only apperar when ω crosses these values. We take $0 < \varepsilon_0 \le 1/2$ and consider the intervals:
(4)

$$I_k(\varepsilon_0) = \left[\sqrt{\frac{1}{k(k+\varepsilon_0^2)}}, \frac{1}{k}\right), \ k \in \mathbb{N} \ J_k(\varepsilon_0) = \left[\frac{1}{k+1}, \sqrt{\frac{1}{k(k+\varepsilon_0^2)}}\right), \ k \in \mathbb{N} \cup \{0\}$$

Roughly speaking, when $\omega \in J_k(\varepsilon_0)$, the hyperbolicity of the linearized (1) at 0 is strong enough to prevent the existence of small homoclinic orbits. In contrast, when ω decreases through $\frac{1}{k}$ and enters $I_k(\varepsilon_0)$, the linearized (1) is weakly hyperbolic in the newly generated hyperbolic directions and small homoclinic orbits may appear through a homoclinic bifurcation.

The following theorem rephrases (and implies) Theorem 1 in terms of invariant manifolds and is obtained through a careful analysis of the spatial dynamics of (1) near 0. In the intervals $I_k(\varepsilon_0)$ it requires the study of the exponentially small splitting between the stable and unstable manifolds of 0.

Theorem 2. *Assume $f(u)$ satisfies te same hypothesis as in Theorem 1. Then the following statements hold.*

(1) There exists $\rho_1^* > 0$ such that for any $\varepsilon_0 \in (0, 1/2]$, $\omega \in J_k(\varepsilon_0)$, $k \in \mathbb{N} \cup \{0\}$, if $u(x,t)$ is a $\frac{2\pi}{\omega}$-periodic-in-t solution to (1) satisfying (2) as $x \to +\infty$ or $-\infty$, then

$$\sup_{x\in\mathbb{R}} \|u(x,\cdot)\|_{\ell_1} \ge \rho_1^* \min\{1, \varepsilon_0 \omega^{\frac{1}{2}}\}.$$

(2) There exist $\varepsilon_0, M > 0$ such that for $\omega \in I_k(\varepsilon_0)$, there exist $\frac{2\pi}{\omega}$-periodic and odd in t solutions $u^\star_{\mathrm{wk}}(x,t)$, $\star = s, u$, to (1) such that

(a) For $x \ge 0$ if $\star = s$ and $x \le 0$ if $\star = u$, they can be approximated as

$$\left\| \left(1 - \frac{1}{(k\omega)^2}\partial_t^2\right) \left(\begin{pmatrix} u^\star_{\mathrm{wk}}(x,t) \\ \frac{\partial_x u^\star_{\mathrm{wk}}(x,t)}{\sqrt{k}\varepsilon\omega} \end{pmatrix} - \sqrt{k}\varepsilon\omega \begin{pmatrix} v^h(\varepsilon\sqrt{k}\omega x) \\ (v^h)'(\varepsilon\sqrt{k}\omega x) \end{pmatrix} \sin k\omega t \right) \right\|_{\ell_1}$$

$$\le M k^{-\frac{3}{2}} \varepsilon^3 v^h(\varepsilon\sqrt{k}\omega x), \ \text{where } v^h(y) = \frac{2\sqrt{2}}{\cosh y}$$

(b) They also satisfy:

$$\left\| \left(|-\partial_t^2 - 1|^{\frac{1}{2}} (u^u_{\mathrm{wk}} - u^s_{\mathrm{wk}}) + i\partial_x (u^u_{\mathrm{wk}} - u^s_{\mathrm{wk}}) \right)(0,t) - 4\sqrt{2} C_{\mathrm{in}} e^{-\frac{\sqrt{2k}\pi}{\varepsilon}} \sin 3k\omega t \right\|_{\ell_1}$$

$$\leq \frac{M e^{-\frac{\sqrt{2k}\pi}{\varepsilon}}}{\frac{1}{2}\log k - \log \varepsilon},$$

where C_{in} is the Stokes constant given in Theorem 1.

(3) Suppose $C_{\mathrm{in}} \neq 0$, then for any $\sigma \in (0,1)$, if $\omega \in I_k(\varepsilon_0)$ then if $u(x,t)$ is a $\frac{2\pi}{\omega}$-periodic-in-t solution to (1) satisfying (2) as $|x| \to \infty$ and

$$\sup_{x\in\mathbb{R}} \|u(x,\cdot)\|_{\ell_1} \leq \rho_2^* \sqrt{\omega},$$

then $u(x,t)$ is σ-multi-bump in the ℓ_1 norm.

For $\omega \in J_k(\varepsilon_0)$, statement (1) implies that all orbits on both the stable and unstable manifolds of 0 leave a small neighborhood of 0 and therefore there are no small orbits homoclinic-in-x to 0. For $\omega \in I_k(\varepsilon_0)$, statement (2a) indicates that there two solutions on the $(2k+1)$-dimensional stable/unstable manifolds of 0, both of which are well approximated by the exponentially localized $v^h(x)$.

The most significant result is statement (2b), which gives the precise leading $\mathcal{O}(e^{-\frac{\sqrt{2k}\pi}{\varepsilon}})$ order term of the splitting between u^u_{wk} and u^s_{wk}.

To obtain the leading order term of the splitting requires to extend suitable parameterizations of the stable and unstable manifolds of 0 to the complex plane and use different approximations for these manifolds using a non-linear equation, known as the *inner equation* which provides the Stokes coonstant C_{in}. Here we stress that the precise exponentially small leading order approximation is obtained for this problem which has *infinitely many oscillatory directions*.

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A global rigidity result for Poincaré sections of higher genus flows

Corinna Ulcigrai

(joint work with Selim Ghazouani)

It is well known that (minimal) *circle diffeomorphisms* $T : S^1 \to S^1$ arise a Poincaré first return maps of (minimal) flows $(\varphi_t)_{t\in\mathbb{R}}$ on a torus S, i.e. on a compact, orientable surface of genus one. Consider now a (smooth) flow $(\varphi_t)_{t\in\mathbb{R}}$ on a compact, orientable surface S of higher genus $g \geq 2$. Notice that in this case the flow always has *fixed points*. We will assume that it is *minimal* in the sense that

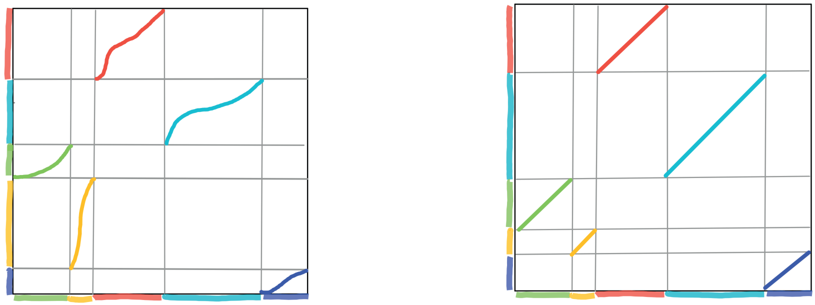

FIGURE 1. A generalized IET (GIET) and a (standard) IET with $d = 4$.

all orbits different than fixed points are dense (this notion is also sometimes called *quasi-minimality*). This implies in particular that all fixed points are of saddle-type. Poincaré first return maps $T : I \to I$ of $(\varphi_t)_{t\in\mathbb{R}}$ on a transverse segment $I \subset S$ are one-to-one *piecewise diffeomorphisms* known as *generalized interval exchange transformations*[1]. More precisely, a map $T : I \to I$ is a generalized interval exchange transformations or, for short, a GIET, if one can partition I into finitely many intervals $I_1, \ldots, I_d$ so that the restriction T_i of T to I_i, for each $1 \le i \le d$, is a diffeomorphism onto its image which extends to a diffeo of the closure $\overline{I}_i$. We say that T is *of class* $\mathcal{C}^r$ if each T_i is a diffeo of class $\mathcal{C}^r$ from $\overline{I}_i$ onto its image, see Figure 1 (left).

Linear models for circle diffeomorphisms and for generalized interval exchange transformations are provided, respectively, by (rigid) circle rotations (i.e. by the map $T_0(x) = x + \alpha \mod 1$ on $I = [0, 1]$) and by (standard or classical) *interval exchange transformations* T_0 (or for short, IETs), namely GIETs such that the derivative T_i' of each branch is constant and equal to one (so that T_0 is a piecewise isometry, see Figure 1, right). Notice that rigid rotations are IETs with $d = 2$ branches.

A classical problem in dynamics, which is at the heart of the *theory of circle diffeomorphisms*, is to understand when a circle diffeomorphism T is *linearizable*, i.e. conjugate to a rigid rotation T_0, i.e. when there exists a homeomorphism $h : I \to I$ (called the *conjugacy*) such that $h \circ T = T_0 \circ h$ and, if it is linearizable, what is the *regularity* of the conjugacy h. The *local theory*, which treats the case of circle diffeos T which are $\mathcal{C}^\infty$*-close* (or analytically, or $\mathcal{C}^r$ close) to a rigid rotation T_0, is the realm of KAM theory. Among the few *global* results (which do not assume that T is close to T_0), we recall that Denjoy showed that as soon as a circle diffeo T is sufficiently smooth, for example $\mathcal{C}^2$ (but $\mathcal{C}^1$ with bounded variation derivative suffices) and the *rotation number* $\gamma(T)$ of T is *irrational*, T is

[1]The adjective *generalized* is used to distingush them from the more commonly studied (standard) interval exchange transformations, which are one-to-one piecewise *isometries*.

minimal and linearizable (in particular conjugate to the rotation R_α with $\alpha = \gamma(T)$ equal to the rotation number of T). A celebrated theorem by Michael Herman [4] and Jean-Christophe Yoccoz [10] shows furthermore that if T is $\mathcal{C}^\infty$, under a full measure condition on the rotation number $\gamma(T)$ (which J.C. Yoccoz showed to coincide with the class of *Diophantine numbers*), h is $\mathcal{C}^\infty$.

In analogy with the case of circle diffeos, we say that a GIET T is *linearizable* if it is topologically conjugate to a linear model, namely to a (standard) IET T_0. A *local theory* in higher genus was initiated by the semimal works by Giovanni Forni first and Stefano Marmi, Pierre Moussa and Jean-Christophe Yoccoz later. Forni in particular showed that there are *obstructions* to solve the *cohomlological equation*[2] (which can be seen as a *linearized* conjugacy problem and a crucial first step towards starting a KAM scheme), but that under a *finite number* of them (depending on the regularity) it can be solved for almost every T_0. A refinement of this result by Marmi, Moussa and Yoccoz (with an explicit full measure arithmetic condition on T_0 called *Roth-type*) then led to the proof, by the same authors, that, for any $r \geq 2$, the $\mathcal{C}^r$ local conjugacy class of (Roth-type) IETs is a *finite codimension* submanifold. They also conjectured that for $r = 1$ it is a submanifold of codimension $(d-1)+(g-1)$, where d is the number of exchanged intervals and g the genus of the surface of which T is a Poincaré section. For special rotation numbers (namely for IETs of *hyperbolic periodic type*) this has recently been proved by Selim Ghazouani [2].

Marmi, Moussa and Yoccoz proposed (also in [9]) a *rigidity conjecture* for GIETs, which states that under a full measure condition, the existence of a topological conjugacy implies automatically that the conjugacy is differentiable (systems with this property are sometimes known in the one-dimensional dynamics literature as *geometrically rigid*), as long as the *boundary* is the same. Here the *boundary* $B(T)$ of a GIET T is a $\mathcal{C}^1$-conjugacy invariant which takes values in $\mathbb{R}^\kappa$ (where κ is the number of saddles of the flow of which T is a Poincaré section), which encodes the *holomomy* of the leaves of the flow foliation *around* each saddle.

In joint work with Selim Ghazouani, we very recently proved this rigidity conjecture by Marmi, Moussa-Yoccoz for minimal GIETs of $d = 4, 5$ intervals, which correspond to Poincaré sections of flows on surfaces in genus two:

[2] Let us recall that the cohomological equation is the equation $\varphi \circ T_0 - \varphi = \psi$, where ψ is a smooth (or $\mathcal{C}^r$) observable $\psi : I \to \mathbb{R}$, T_0 an IET and one looks for a smooth (or $\mathcal{C}^r$) solution $\varphi : I \to \mathbb{R}$. When T_0 is a rigid rotation R_α, this equation can be solved for almost every α as long as $\int \psi \, \mathrm{d}x = 0$, namely a non-zero mean is the only *obstruction* to solvability. Forni in [1] showed on the other hand that if T_0 is a Poincaré section of a genus g flow, there are finitely many *obstructions* to solve it, but as long as the linear functionals which describe these obstructions (called *invariant distributions*) are zero, then one can find a solution (continuous, or finite smoothness if one adds more distributions) for almost every IET T_0, i.e. for Lebesgue almost every choice of the lengths.

Theorem 1 (Ghaozuani-U', [3]). *For a full measure set*[3] *of IETs T_0, if T is a GIET of class $\mathcal{C}^3$ with zero boundary $B(T) = 0$ which is topologically conjugate to T_0 via a conjugacy h, then it is differentiably conjugate to T_0, namely h is a diffeomorphism of class $\mathcal{C}^1$.*

We remark that this is a *global result* that shows in particular that, under a full measure condition, the $\mathcal{C}^1$ conjugacy class coincides with the $\mathcal{C}^0$ conjugacy class. We expect the optimal regularity of h to be $\mathcal{C}^{1+\alpha}$, i.e. the derivative h' to be Hölder with some exponent $0 \leq \alpha \leq 1$ (this is in accordance with results on circle diffeomorphisms with break points and suggested by results on the regularity of solutions to the cohomological equation).

As a corollary, we show also that, if S is a surface of genus two and $(\varphi_t)_{t\in\mathbb{R}}$ has (two) *Morse saddles* (i.e. the trajectories are level sets of the Morse function $xy = c$), then if the foliation given by trajectories of $(\varphi_t)_{t\in\mathbb{R}}$ is *linearizable*, i.e. topologically conjugate, in the sense of foliations, to a *linear* foliation, then it is also *differentiably* conjugate. In this setting, indeed, the *zero boundary* assumption is automatically satisfied by the GIETs which arise as Poincaré sections (since a Morse saddle has trivial holonomy).

The proof of the theorem is based on *renormalization*, in the spirit of Sinai-Khanin [5] and Khanin-Teplisky [6] revisitations of Herman's theory. The *renormalization operator* $\mathcal{R}$ which maps the space of GIETs of d intervals on $I = [0, 1]$ into itself, is given here by Rauzy-Veech induction, a well-known algorithm which plays a key role in the ergodic theory of IETs. The iterates $\mathcal{R}^n(T), n \in \mathbb{N}$ are obtained by inducing T on a sequence of nested intervals $I^{(n+1)} \subset I^{(n)} \subset I$ and normalizing the induced map (which is a GIET of the same number of intervals) linearly so that it acts again on $[0, 1]$.

At the heart of our rigidity result, there is the following *dynamical dichotomy* for renormalization, which we prove in any genus $g \geq 1$. Let us say that a GIET is *infinitely renormalizable* if the iterates $\mathcal{R}^n(T)$ of T can be defined for any $n \in \mathbb{N}$. Notice that if T is conjugate to a (minimal, or at least Keane) IET, it is infinitely renormalizable.

Theorem 2 (Ghazouani-U', dynamical dichotomy [3]). *Let T be a GIET of any number $d \geq 2$ of intervals which is infinitely renormalizable. Under a full measure condition, there exists a sequence $(n_k)_{k\in\mathbb{N}}$ of return times (depending on T) such that the orbit of $(\mathcal{R}^n(T))_{n\in\mathbb{N}}$ of T under renormalization satisfies the following dichotomy:*

(1) either $(\mathcal{R}^n(T))_{n\in\mathbb{N}}$ is recurrent *in the $\mathcal{C}^1$-topology along the subsequence $(n_k)_{k\in\mathbb{N}}$, namely there exists a constant $C > 0$ such that*

$$\frac{1}{C} \leq D\mathcal{R}^{n_k}(T) \leq C, \qquad \text{for all } k \in \mathbb{N}, \tag{1}$$

[3]The notion of *full measure* (or *almost every* IET) refers here to the Lebesgue measure on the space of d-IETs: we say that a result holds for almost every IET if it holds for IETs with *irreducible* permutations for Lebesgue almost every choice of the continuity intervals lengths.

(2) *or, the* $(\mathcal{R}^n(T))_{n\in\mathbb{N}}$ diverges *and there exists an* affine *IET* T_1 *(i.e. a GIET whose branches have constant slope) so that the derivatives are approximated in the leading order (see equation* (2) *below for details) by the slopes of the iterates of* $(\mathcal{R}^n(T_1))_{n\in\mathbb{N}}$.

The first case is what we call *recurrent case* of the dichotomy; the bounds given by (1) are a form of *a priori bounds*, that guarantee that the geometry does not degenerate under renormalization. The second case is what we call *affine shadowing.* The key quantity that encodes the evolution is the *log-average vector* $\omega(T)$, i.e. a vector in $\mathbb{R}^d$ whose entries ω_i, for $1 \le i \le d$, are $\omega_i = \log \rho_i$, where ρ_i is the *average slope* of the i^{th} branch T_i of T, given by $\rho_i = |T(I_i)|/|I_i|$ (here $|I|$ denotes the length of the interval I). Notice that if T_1 is an *affine* IET, the evolution of the log-slopes vectors $\omega_1^{(n)} := \omega(\mathcal{R}^n(T_1))$ is linear and governed by a linear cocycle (the Rauzy-Veech cocycle). In the affine shadowing case, the norm of the vectors $\omega_1^{(n)}$ grows exponentially and

$$\left\|\omega(\mathcal{R}^n(T)) - \omega_1^{(n)}\right\| \le C \left\|\omega_1^{(n)}\right\|^{\epsilon}, \qquad \text{for all } \epsilon > 0. \tag{2}$$

In the *recurrent case*, the presence of a priori bounds (which play the role of *Denjoy-Koksma inequality* for circle diffeomorphisms) and the boundary zero assumption allow to prove that, in this case, $(\mathcal{R}^n(T))_{n\ge 0}$ converge, in the $\mathcal{C}^1$ norm, to the subspace of IETs exponentially fast (a phenomenon known as *exponential convergence of renormalization*) and then conclude, as in the classical Herman's theory, that T is $\mathcal{C}^1$-conjugate to its linear model.

In the *affine shadowing* case, we exploit previous work by Marmi, Moussa and Yoccoz on affine IETs, which in particular shows that in genus two the affine IET T_1 which appears as affine shadow always has wandering intervals[4] and conclude, thanks to the shadowing, that also T has wandering intervals. Under the assumption that T is topologically conjugate to an IET (and hence has no wandering intervals), this case cannot hence occur. This shows that in genus two, the existence of a topological conjugacy provides a priori bounds and implies the desired rigidity.

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Integrable systems and symplectic embeddings

Vinicius Ramos
(joint work with Jean Gutt, Yaron Ostrover, Daniele Sepe, Brayan Ferreira)

Symplectic embeddings have been a central subject in symplectic topology with a strong link to Reeb dynamics. Sharp embeddings of very simple domains turn out to be quite difficult to determine. Symplectic capacities are quantitative invariants providing obstructions to the existence of such embeddings. Calculating them and determining when they are sharp has been an important topic for over two decades.

In 1989 Ekeland and Hofer [2] defined a sequence of capacities and they computed their values for ellipsoids and polydisks. Their definition uses analytical methods and a Fadell-Rabinowitz type index, which is quite difficult to compute. Over the years it was thought that they might be expressed as spectral invariants coming from Floer theory. But it was not until 2018 that Jean Gutt and Michael Hutchings were able to define a similar sequence coming from positive S^1 equivariant symplectic homology, see [4]. They checked that their capacities coincided with the Ekeland-Hofer capacities for ellipsoids and polydiscs and they were able to compute them for a large class of toric domains. Using the Morse homology defined by Abbondandolo and Majer, Jean Gutt and I have been able to prove the following result.

Theorem 1 (Gutt–R. [5])**.** *For any star-shaped domain $X \subset \mathbb{R}^{2n}$ the Ekeland–Hofer and the Gutt–Hutchings capacities coincide.*

Toric domains are generalizations of ellipsoids and polydiscs and many symplectic embedding problems for these domains have been understood, particularly in dimension 4. A different class of domains for which not much was known are Lagrangian products. Given open sets $K, L \subset \mathbb{R}^n$, their Lagrangian product is the cartesian product $K \times L \subset \mathbb{R}^n_{\mathbf{x}} \times \mathbb{R}^n_{\mathbf{y}}$, where we endow $\mathbb{R}^{2n}$ with the symplectic form $\omega_0 = \sum_i dx_i \wedge dy_i$. These domains are related to billiard dynamics and convex geometry as explained in [1]. The integrability of some billiard systems implies that many lagrangian products are symplectomorphic to toric domains.

Theorem 2 (R. [8])**.** *The Lagrangian bidisc $D^2 \times D^2$ is symplectomorphic to a concave toric domain X_Ω where $\Omega \subset \mathbb{R}^2_{\geq 0}$ is the relatively open set bounded by the coordinate axes and the curve*

$$2\left(\sqrt{1-v^2} + v(\pi - \arccos v), \sqrt{1-v^2} - v\arccos v\right), \quad \textit{for } v \in [-1,1]. \tag{1}$$

Using the same integrable system, we were able to deform $D^2 \times D^2$ and prove the following result.

Theorem 3 (Ostrover–R. [6])**.** *Denote the lagrangian ℓ^p-sum of two discs by*

$$D^2 \oplus_p D^2 = \{(q_1, p_1, q_2, p_2) \in \mathbb{R}^4 \mid (q_1^2 + q_2^2)^{p/2} + (p_1^2 + p_2)^{q/2} < 1\}.$$

Then $D^2 \oplus_p D^2$ is symplectomorphic to a convex or a concave toric domain, if $p \in [1,2]$ or $p \in [2,\infty)$, respectively.

Using the one-dimensional billiard system, in joint work with Sepe, we found a large family of lagrangian products which are also symplectomophic to toric domains.

Theorem 4 (R.–Sepe [9])**.** *Let $\Omega \subset \mathbb{R}^n$ be a star-shaped open set such that*

$$(x, y) \in \Omega \Rightarrow [-|x|, |x|] \times [-|y|, |y|] \subset \Omega.$$

Then the lagrangian product $[0,1]^n \times \Omega$ is symplectomorphic to $X_{2\Omega \cap \mathbb{R}_{\geq 0}}$.

The idea of using integrable systems to find hidden toric domains has proven very fruitful, although the details vary widely from a situation to another. Recently we have been able to show the following results.

Theorem 5 (Ostrover–R.–Sepe [7])**.** *Let T be a triangle with angles $(\pi/3, \pi/3, \pi/3)$, $(\pi/4, \pi/4, \pi/2)$ or $(\pi/6, \pi/3, \pi/2)$. Then the lagrangian product of T and a subset $\Omega \subset \mathbb{R}^2$ with a similar symmetry is symplectomorphic to a toric domain. In particular, the lagrangian product of an equilateral triangle and its related regular hexagon is a symplectic ball.*

Theorem 6 (Ferreira–R. [3])**.** *The disc bundles of the punctured sphere $D^*(S^2 \setminus \{p\})$ and the hemisphere $D^*(S^2_+)$ are symplectomorphic to the symplectic polydisc $P(2\pi, 2\pi)$ and the ball $B^4(2\pi)$, respectively. Moreover, the Gromov widths of D^*S^2 and $D^*\mathbb{R}P^2$ are 2π.*

The results above show that the integrability of a billiard system on a certain table with the standard reflection law gives rise to a family of symplectomorphisms of lagrangian products of this table and a sufficiently symmetric subset. That implies that many lagrangian products which correspond to non-integrable Minkowski billiards are still toric domains. One is then led to ask the following question.

Question 7. *For which domains $K, L \subset \mathbb{R}^n$ is the lagrangian product $K \times L$ a toric domain and how is the integrability of the corresponding Minkowski billiard system related to that?*

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Asymptotic Hofer geometry and Lagrangian Poincaré recurrence

Egor Shelukhin

(joint work with Leonid Polterovich)

Poincaré recurrence is a classical phenomenon of measure-preserving dynamics in a finite-measure space introduced in [12]. It was inspired by the work of Poincaré on the three-body problem and implies the following weaker statement on the return times of a positive-measure subset to itself. A stronger version of this kind of statement appears in [1].

Theorem 1. *Let (X, μ) be a finite-measure space and let $A \subset X$ be a positive measure subspace. Set $a = \mu(A)/\mu(X) > 0$. Let $\phi : X \to X$ be an invertible measure-preserving transformation. Then there exists an increasing sequence $\overline{k} = \{k_i\}_{i \geq 1}$ of natural numbers of density $d(\overline{k}) = \lim_{m\to\infty} \frac{1}{m}\#\{k_i \mid k_i \leq m\}$ at least a such that for all $i \geq 1$,*

$$\phi^{k_i}(A) \cap A \neq \emptyset.$$

Inspired by the fact that Poincaré's work pertained to classical mechanics, and by the paradigm of symplectic packing obstructions [6, 11, 2], which are stronger than simply volume-preserving ones, it is natural to ask whether a similar type of result could hold for symplectically rigid subsets of measure zero. A prototypical example of such a rigid set is a Lagrangian submanifold of a symplectic manifold, especially a Lagrangian torus. This leads to the symplectic packing question and the symplectic Poincaré recurrence question [5]. We formulate both for Lagrangian submanifolds. Suppose that (M, ω) is a closed symplectic manifold and $L \subset M$ is a displaceable closed Lagrangian submanifold. That is, there is a Hamiltonian diffeomorphism $\phi \in \mathrm{Ham}(M, \omega)$ of M such that $\phi(L) \cap L = \emptyset$. Note that without this assumption both questions below have trivial answers.

Question 2 (Lagrangian packing)**.** *Let the Lagrangian packing number $k(L) \in \mathbb{N} \cup \{\infty\}$ be the supremum of natural numbers k such that there exist Hamiltonian diffeomorphisms $\phi_1, \ldots, \phi_k \in \mathrm{Ham}(M, \omega)$ such that $L_1 = \phi_1(L), \ldots, L_k = \phi_k(L)$ are pair-wise disjoint. Is $k(L) < \infty$?*

Question 3 (Lagrangian Poincaré recurrence)**.** *Let $\phi \in \mathrm{Ham}(M, \omega)$ be a Hamiltonian diffeomorphism. Does there exist a symplectic invariant $a(L) > 0$ of L such that there exists a sequence $\overline{k} = \{k_i\}_{i \geq 1}$ of natural numbers of (lower) asymptotic density $\underline{d}(\overline{k}) = \liminf_{m \to \infty} \frac{1}{m} \#\{k_i \mid k_i \leq m\}$ at least $a(L)$ satisfying for all $i \geq 1$,*

$$\phi^{k_i}(L) \cap L \neq \emptyset?$$

I have presented results on these questions obtained jointly with Leonid Polterovich in [14]. I have first explained how a positive answer to the first question yields a positive answer to the second question with $a(L) = 1/k(L)$. Both questions, as well as this implication, apply for arbitrary subsets of M of course. However, they are most interesting for measure-zero subsets. Subsequently, I introduced a symplectic invariant $\alpha(A) \in [0, 1]$ of a compact subset $A \subset M$ based on asymptotic Hofer geometry, lower bounds on which allow us to prove upper bounds on $k(A)$.

The invariant α is defined as follows. Let $\nu : \widetilde{\mathrm{Ham}}(M, \omega) \to \mathbb{R}_{\geq 0}$ be the Hofer pseudo-norm [7, 13, 9] on the universal cover of the Hamiltonian group $\mathrm{Ham}(M, \omega)$ of (M, ω). The asymptotic Hofer norm $\overline{\nu} : \widetilde{\mathrm{Ham}}(M, \omega) \to \mathbb{R}_{\geq 0}$ is defined as $\overline{\nu}(\psi) = \lim_{m \to \infty} \frac{1}{m} \nu(\psi^m)$ for $\psi \in \widetilde{\mathrm{Ham}}(M, \omega)$. Set $\psi_f = [\{\phi_f^t\}_{t \in [0,1]}] \in \widetilde{\mathrm{Ham}}(M, \omega)$ for the class of the Hamiltonian isotopy generated by a function $f \in C^\infty(M; \mathbb{R})$. Ordering open neighborhoods U of A by inverse inclusion, set

$$\alpha(A) = \lim_{U \supset A} \sup\{\overline{\nu}(\psi_f) \mid f \in C^\infty(M; \mathbb{R}),\ |f|_{C^0} = 1,\ \mathrm{supp}(f) \subset U,\ f|_A \equiv 1\}.$$

In view of a classical argument of Sikorav [16, 8] we obtain the following.

Proposition 2. *The packing number of A satisfies*

$$k(A) \leq \left\lfloor \frac{1}{\alpha(A)} \right\rfloor,$$

where the right hand side is by convention ∞ if $\alpha(A) = 0$.

Finally, we prove by introducing and using Lagrangian spectral invariants for (symmetric) product Lagrangian submanifolds in symmetric product orbifolds (inspired by [10]), the following result.

Theorem 4. *Let $L_0 \subset S^2$ be the boundary of a disk of area $B \in (1/k, 1/(k+1))$, $k \geq 2$, where the total area of S^2 is 1. Let S be the equator in a sphere $S^2(2b)$ of total area $2b$. Suppose that $b \in (0, \frac{(k+1)B-1}{k-1})$. Then $L = L_0 \times S$ in $M = S^2 \times S^2(2b)$ satisfies $\alpha(L) = 1/k$ and $k(L) = k$.*

This is the first example known to date of a positive answer to either one of the questions above, as stated. In the case of certain compact symplectic four-manifolds with non-empty boundary, a similar result via different methods is work in progress by Dimitroglou-Rizell and Opshtein. A positive answer to a version of

Question 3 with the lower bound depending also on ϕ was proved by Ginzburg and Gürel [4] for a special class of Hamiltonian diffeomorphisms of $\mathbb{C}P^n$, the so-called pseudo-rotations. Finally, the same result with volume-preserving diffeomorphisms instead of symplectomorphisms is false: there exists a volume-preserving diffeomorphism ϕ of M such that $\{\phi^i(L)\}_{i\in\mathbb{N}}$ are all disjoint.

Further applications of the methods.

The method of [14] (Lagrangian spectral invariants in symmetric products) used for Lagrangian Poincare recurrence, has numerous additional applications to geometry and dynamics of Hamiltonian diffeomorphisms on surfaces (see [14], as well as [3] and forthcoming paper [15]). Our current impression is that the merit of the orbifold setting (as in [10, 14]) is that it extends to dimension 4 and it would be very interesting to extend it to higher dimensions.

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Contact three-manifolds with exactly two simple Reeb orbits

Umberto Hryniewicz
(joint work with Daniel Cristofaro-Gardiner, Michael Hutchings, Hui Liu)

The goal of this talk is to present a description of Reeb flows on closed 3-manifolds with precisely two periodic orbits, which is as complete as possible in a very precise sense to be described. This is in parallel to what happens for area-preserving pseudo-rotations of the 2-disk: there are a number of common features shared by all such disk-maps, but their dynamics can be quite different, ranging from integrable situations to cases where the Lebesgue measure is ergodic [1].

Our characterization theorem is particularly motivated by the conjecture that every Reeb flow on a closed 3-manifold must have two or infinitely many periodic orbits. One may not necessarily learn too much about the structure of a specific Reeb flow in dimension three from knowing that it has infinitely many periodic orbits, but our results tell that one does get lots of detailed information about such a flow in the case of two periodic orbits. In any case, if this conjecture is true then we are compelled to understand the very special case of two periodic orbits. The main step is to show that a contact form with exactly two periodic Reeb orbits has the property that both closed orbits are irrationally elliptic.

The proof combines the powerful ECH volume formula from [3] with a study of the behavior of the ECH index under non-degenerate perturbations of the contact form which is based on [2]. As a consequence, the ambient contact 3-manifold is a standard lens space, the contact form is dynamically convex, the Reeb flow admits rational disk-like global surfaces of section, and the dynamics are described by a pseudo-rotation of the 2-disk. Hence, the analogy to pseudo-rotations is not only heuristic: we establish it in a mathematically rigorous way. Moreover, the periods and rotation numbers of the closed orbits satisfy the same relations as in the case of (quotients of) irrational ellipsoids. In the case of S^3 the transverse knot-type of the periodic orbits is fully determined, in the case of lens spaces there is a weaker characterization of the transverse knot type.

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Surviving lower dimensional tori from an invariant resonant torus

Frank Trujillo

The classical KAM theory establishes the persistence, under sufficiently small perturbations, of most of the n-dimensional invariant tori for non-degenerate integrable Hamiltonians with n degrees of freedom. The surviving tori are those carrying a quasi-periodic motion by a Diophantine vector and, in particular, their restricted dynamics is minimal. On the other hand, such systems also admit n-dimensional invariant tori whose restricted dynamics is not minimal. These tori, which we call *resonant*, are foliated by *lower dimensional invariant tori*, that is, by invariant tori whose dimension is smaller than the number of degrees of freedom of the system. The codimension of these tori is called the *number of resonances* of the resonant torus.

In the resonant setting not only the hypotheses of the classical KAM theorem are not satisfied but, in general, the resonant invariant torus tends to disappear under small perturbations. Nevertheless, invariant lower dimensional tori, with the same dynamics as that of the ones in the invariant foliation of the resonant torus, might still be found in the perturbed system.

Most of the existing results in this direction deal with *generic* perturbations and hold for resonant vectors with any number of resonances [7], [5] [3]. However, similar results for *arbitrary* perturbations are only available when the resonant vector has exactly 1 or $m-1$ resonances [1], [2], [3], [6], [4].

In this work we present a criterion for the existence of at least one lower dimensional invariant torus, associated to a resonant invariant torus (with any number of resonances) in the unperturbed system, for a class of near-integrable non-convex Hamiltonians. Typically, this invariant torus will be of *hyperbolic type*. As a particular case of our main result, we obtain the following.

Theorem 1. *Let H be a real analytic Hamiltonian over $\mathbb{T}_r^d \times \mathbb{T}_r^l \times B_s^d \times B_s^l$, where $\mathbb{T}_r^m = (\{z \in \mathbb{C} \mid |\mathrm{Im}(z)| < r\}/\mathbb{Z})^m$ and $B_s^m = \{z \in \mathbb{C}^m \mid |z| < s\}$, of the form*

$$H(q,x,p,y) = \langle \omega, p \rangle - \frac{1}{2}|p|^2 + \frac{1}{2}|y|^2 + f(q,x,p,y),$$

with $\omega \in \mathbb{R}^d$ Diophantine. There exists $\epsilon_0 > 0$, depending only on r, s, d, l and ω, such that if

$$\|f\|_\infty < \epsilon_0,$$

then H admits an invariant d-dimensional invariant torus, whose restricted dynamics is analytically conjugated to a continuous translation by ω.

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Germ-typicality of the Newhouse phenomenon

Pierre Berger
(joint work with Sylvain Crovisier, Enrique Pujals)

A long standing problem is that of ergodicity of a typical dynamical system. This goes back to the Boltzmann ergodic hypothesis, which has been reformulated in modern terms by Birkhoff-Koopman (1932) as follows: a typical proper Hamiltonian systems is ergodic on a.e. [component of] energy level. Here ergodicity means that Liouville a.e. point has its orbit which is equidistributed over the Liouville probability measure.

This was disproved by Kolmogorov in 1954, with the so-called KAM's theorem.

A topological and weaker version of the Bolzmann ergodic hypothesis, is the quasi-ergodic hypothesis. It was stated by Poincaré (1892) as the existence of a dense set of proper Hamiltonian systems which displays a dense orbit at a dense subset of [components of] energy levels.

This was disproved by Hermann in 1992 [for non-exact symplectic form]

A great idea of Smale was to remove and simplify the structure left invariant by the system. Namely, he proposed to focus on differentiable systems on low-dimensional and closed manifolds which does not need to preserve the volume. In the early 60's he conjectured the open-density of diffeomorphisms satisfying Axiom A. One of the numerous properties of Axiom A is the finiteness of topological attractors (and even the finite quasi-ergodicity). Moreover by the works of Sinai and Bowen-Ruelle, they are *finitely ergodic*: There is a finite set of probability measures which model the statistical behavior of the orbits of Leb. a.e. point. Hence a corollary of Smale's program would be that any system in an open and dense set should be finitely ergodic and displays finitely many topological attractors.

The density of Axiom A conjecture was disproved by Abraham-Smale in 1970.
The openness and density of finite ergodicity or quasi-ergodicity was disproved by Newhouse in 1974.

More precisely Newhouse proved the existence of a *locally topologically generic set* (an open set intersected with countably many open-dense sets) formed by smooth surface diffeomorphisms with the following property. The dynamics displays infinitely attracting periodic orbits which accumulate (in law) onto the set of all invariant measures of a Smale's horseshoe (a stably embedded Bernoulli shift). This dynamical property is called the *Newhouse phenomenon*.

Newhouse phenomenon is perhaps the most complex and rich phenomenon known in differentiable dynamical systems. Indeed from the topological or statistical viewpoints, these dynamics are presently extremely far from being understood;

it is not clear that the current dynamical paradigms would even allow one to state a description of such dynamics.

Since the early 70's, the problem of the typicality of the Newhouse phenomenon has been fundamental, see for instance [9]. But the notion of topological genericity is not completely satisfactory as a notion of typicality. For instance, a surface conservative maps displaying countably many elliptic points with Liouville rotation number is topologically generic, but this property sounds negligible (the set of real Liouville numbers has Lebesgue measure zero). That is why many important works and programs [10, 8, 7, 4] wondered if the complement of the Newhouse phenomenon could be typical in some stronger sense, such as the one of Kolmogorov (ICM 1964) which involves parameter families.

In 2016, it has been shown that finite ergodicity and quasi-ergodicity is not typical in the sense of Kolmogorov.

More precisely, I showed in [1, 2] the existence of an open set of finite differentiable families of dynamics in which a topologically generic family displays the Newhouse phenomenon at every parameter.

Now a natural question is whether Newhouse phenomenon is locally typical.

Typicality's notions	Open	Kolmogorov typ.	Topo. gen.	Density
Finite ergocity	No [5]	No [1, 2]	No [5]	?
Newhouse phen.	locally ?	locally ?	locally Yes [5]	locally Yes [5]

In the presented work, with Crovisier and Pujals we showed that the Newhouse phenomenon is typical according to the following notion inspired by Kolmogorov idea and subsequent developments:

Definition 1 (Germ-typicality)**.** *A behavior $\mathcal{B}$ is* C^r-germ-typical *in an open set $\mathcal{U}$ of C^r-self-maps of a manifold M, if there exist a topologically generic set $\mathcal{R}$ in the space of C^r-families $\hat{f} = (f_a)_{a\in\mathbb{R}}$ of maps in $\mathcal{U}$ and a locally constant function $\delta\colon \mathcal{R} \to (0, +\infty)$ such that for every $\hat{f} \in \mathcal{R}$ and for all $|a| < \delta(\hat{f})$, the map f_a presents the behavior $\mathcal{B}$.*

Interestingly, the two known obstructions to robust finite ergodicity, KAM and Newhouse phenomena, appear nearby (very simple) *configurations*. KAM phenomenon appears robustly when the dynamics displays a twisted invariant torus. Newhouse's phenomenon is topologically generic nearby dynamics which displays an area contracting homoclinic tangency [6]. In a similar way, we show that the germ typicality of the Newhouse phenomenon occurs nearby any system displaying a simple configuration that we call a *bicycle*:

Definition 2. *A local diffeomorphism displays a* bicycle *if one of its saddle points has a homoclinic tangency and a heterocycle. A saddle point P displays a* heterocycle *if $W^u(P)$ contains a projectively hyperbolic source S and if the strong unstable manifold $W^{uu}(S)$ intersects $W^s(P)$. The bicycle is* dissipative *if the dynamics contracts area along the orbit of P.*

The main theorem of the presented work is the following:

Theorem 1 ([3]). *For every $2 \leq r < \infty$ and for every local C^r-diffeomorphism of a surface $f \in \mathrm{Diff}^r_{loc}(M)$ which displays a dissipative bicycle, there exists a (non empty) open set $\mathcal{U}^r \subset \mathrm{Diff}^r_{loc}(U, M)$ whose closure contains f and where the Newhouse phenomenon is C^r-germ-typical.*

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Poisson brackets of partitions of unity and Floer theory

Shira Tanny

In [2], Entov and Polterovich discovered a surprising relation between the Poisson bracket and the notion of *displaceability*, which can be thought of as a symplectic "small scale". Recall that the Poisson bracket of a pair of Hamiltonians $F, G : M \times [0,1] \to \mathbb{R}$ is defined by $\{F, G\}(x) := \frac{d}{dt}\Big|_{t=0} G \circ \varphi^t_F(x)$. A subset U of M is called displaceable if there exists a Hamiltonian H such that $\varphi^1_H(\bar{U}) \cap \bar{U} = \emptyset$. For a displaceable subset $U \subset M$, its *displacement energy* $e(U)$ is the infimum of the Hofer norm over Hamiltonians whose time-1 flow map displaces U.

On a closed connected symplectic manifold (M, ω), consider a finite open cover $\mathcal{U} := \{U_i\}_{i \in I}$ by displaceable sets. The non-displaceable fiber theorem [2] implies that any subordinate partition of unity $\mathcal{F} = \{f_i\}_{i \in I}$ cannot be Poisson commuting, namely there exist i, j such that $\{f_i, f_j\} \not\equiv 0$. The Poisson bracket invariant, which was introduced by Polterovich in [6], measures this non-commutativity:

$$pb(\mathcal{U}) := \inf_{\mathcal{F}} \max_{|x_i|, |y_j| \leq 1} \Big\| \Big\{ \sum_{i \in I} x_i f_i, \sum_{j \in I} y_j f_j \Big\} \Big\|. \tag{1}$$

Here the infimum is taken over all partitions of unity $\mathcal{F}$ subordinate to $\mathcal{U}$. In [7], Polterovich explained the relation of this invariant to operational quantum mechanics and conjectured a lower bound for $pb(\mathcal{U})$ in terms of the maximal displacement energy of a set from $\mathcal{U}$:

Conjecture (Polterovich)**.** *Let (M, ω) be a closed symplectic manifold, and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of M by displaceable sets. Then, there exists a constant $C = C(M, \omega) > 0$ depending only on the symplectic manifold, such that*

$$pb(\mathcal{U}) \cdot e(\mathcal{U}) \geq C, \tag{2}$$

where $e(\mathcal{U}) := \max_{i \in I} e(U_i)$ and $e(U_i)$ is the displacement energy of U_i.

This conjecture was proved in dimension 2 by Buhovsky-Logunov-T. [1], using geometric and combinatorial arguments. In higher dimensions Polterovich's onjecture is still open, and the current approoce towards it relies on Floer theory and spectral invariants. The spectral invariant $c(F)$ of a Hamiltonian F measures the smallest action level in which the fundamental class appears in Floer homology constructed with respect to F. Entov-Polterovich-Zapolsky [3] and Polterovich [7] provided a method to produce lower bounds for $pb(\mathcal{U})$ out of upper bounds for spectral invariants of Hamiltonians supported in disjoint unions of sets from the cover $\mathcal{U}$. Motivated by this approach, Seyfaddini [8] and Ishikawa [5] proved upper bounds for spectral invariants of Hamiltonians supported in certain disjoint domains. More recently, Humilière-Le Roux-Seyfaddini [4] proved that if $\pi_2(M) = 0$ and F, G are Hamiltonians supported in certain disjoint domains, then $c(F+G) = \max\{c(F), c(G)\}$. They also gave a counter example for this statement on the sphere S^2. It turns out that an inequality, which is sufficient from a Poisson bracket point of view, holds in a more general setting. For example, an inequality holds if the domains are "far enough" from each other:

Definition 1. *Let $U \subset M$ be the image of an embedding ψ of a nice star shaped domain in $\mathbb{R}^{2n}$. We say that U is σ-extendable if ψ extends to $\sqrt{1+\sigma} \cdot \psi^{-1} U$. Denote by $(1+\sigma) \cdot U := \psi(\sqrt{1+\sigma} \cdot \psi^{-1} U)$ the extension of U in M.*

Theorem 1. *Assume that (M, ω) is rational, namely $\omega(\pi_2(M)) = \kappa\mathbb{Z}$, and*

- *U_i are σ_i-extendable embeddings of nice star shaped domains such that the extensions $\{(1+\sigma_i) \cdot U_i\}$ are disjoint,*
- *$supp(F_i) \subset U_i$ and $c(F_i) < \min\{\kappa, \sigma_i \cdot T_{\min}(\partial U_i)\}$.*

Then, $c(\sum_i F_i) \leq \max_i c(F_i)$.

Another example is if the boundary of U is Zoll of certain period:

Theorem 2. *Assume that (M,ω) is rational, namely $\omega(\pi_2(M)) = \kappa\mathbb{Z}$, and*

- *U_i are disjoint embeddings of star shaped domains whose bounaries are Zoll: all periodic orbits have period T_i, and $T_i|\kappa$,*
- *$supp(F_i) \subset U_i$ and $c(F_i) < T_i$.*

Then, $c(\sum_i F_i) \leq \max_i c(F_i)$.

Lastly, on monotone manifolds an inequality holds for convex domains that are "not too big". The size of the domain is measured by the maximal action-index ration of a closed Reeb orbit:

Definition 2. *Let U be the image of a nice star shaped domain. We define*

$$C(U) := \sup\left\{\frac{2T(\gamma)}{CZ(\gamma)-n+1} \ : \ \gamma \in \mathcal{P}(\partial U)\right\},$$

where $T(\gamma)$ is the period and $CZ(\gamma)$ is the Conley-Zehnder index.

Theorem 3. *Assume that (M,ω) is monotone, namely $\omega = \kappa c_1$ on $\pi_2(M)$, $\dim M > 2$, and*

- *U_i are embeddings of strictly convex domains,*
- *if $\kappa > 0$ then U_i satisfy $C(U_i) \leq \kappa$.*

Then, for every F_i supported in U_i respectively, $c(\sum_i F_i) \leq \max_i c(F_i)$.

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Wild holomorphic dynamics

Sébastien Biebler
(joint work with Pierre Berger)

In the 60s, in a mathematical optimistic movement aiming to describe a typical dynamical system, Smale conjectured the density of uniform hyperbolicity in the space of C^r-diffeomorphisms f of a compact manifold M. In the 70s, Newhouse discovered an extremely complicated new phenomenon, resulting in an obstruction to Smale's conjecture. Specifically, he proved the following result:

Theorem (Newhouse [N, N2]**).** *For any surface $\mathcal{S}$, there exists a nonempty open set $\mathcal{U}$ of C^2-diffeomorphisms of $\mathcal{S}$ such that any f in a Baire generic subset of $\mathcal{U}$ has infinitely many attracting periodic points.*

Dynamics with infinitely many attracting periodic points are particularly interesting since one can not hope to describe them satisfactorily from an ergodic point of view with finitely many invariant probability measures. It is therefore natural to try to generalize the Newhouse phenomenon to other settings.

In holomorphic dynamics, polynomial automorphisms form a natural class of maps to investigate the existence of the Newhouse phenomenon. A polynomial automorphism of $\mathbb{C}^k$, $k \geq 2$, is a bijection of $\mathbb{C}^k$ such that both f and f^{-1} have polynomial coordinates. In the 90s, Buzzard [Bu] showed that there exist Newhouse domains in the family of polynomial automorphisms of $\mathbb{C}^2$ of degree d when d is large enough. In higher dimension, we have the following result:

Theorem (Biebler [Bi]**).** *For any integer $d \geq 2$, there exists a nonempty open set $\mathcal{U}$ of polynomial automorphisms of $\mathbb{C}^3$ of degree d such that any f in a Baire generic subset of $\mathcal{U}$ has infinitely many attracting periodic points.*

Examples of applications of the Newhouse phenomenon include the existence of generic sets of diffeomorphisms displaying a superexponential growth of the number of isolated periodic points by Kaloshin [K], the density of universal maps in any Newhouse domain in the family of area-preserving two-dimensional maps by Gonshenko, Shilnikov and Turaev [GST] and the density of finitely differentiable maps having a wandering domain in any Newhouse domain by Kiriki and Soma [KS].

A new application deals with wandering Fatou components in complex dynamics. It is classical that for any holomorphic map f acting on a complex manifold M, one can write M as the disjoint union of the Fatou set, where the dynamics is locally normal, and its complement, the Julia set, where it is chaotic. A Fatou component is a connected component of the Fatou set, and it is wandering if it is not preperiodic. The dynamics on wandering components can be very complicated, and thus the question of their existence is crucial.

A celebrated theorem of Sullivan [S] states that any rational map of the Riemann sphere does not have a wandering Fatou component. On the contrary, in higher dimension, examples of wandering Fatou components have been found for transcendental maps of $\mathbb{C}^2$ by Fornæss and Sibony [FS] and for holomorphic endomorphisms of $\mathbb{P}^2(\mathbb{C})$ by Astorg, Buff, Dujardin, Peters and Raissy [ABDPR]. Recently, for polynomial automorphisms of $\mathbb{C}^2$, it has been shown that:

Theorem (Berger, Biebler [BB]). *There exists a locally dense set of parameters* $(p_i, b) \in \mathbb{R}^6$ *such that the following polynomial automorphism has a wandering Fatou component* $\mathcal{C}$*:*

$$f_p : (z, w) \in \mathbb{C}^2 \mapsto (z^6 + \sum_{i=0}^{4} p_i \cdot z^i - w, b \cdot z) \in \mathbb{C}^2 \, .$$

Moreover for every $z \in \mathcal{C}$*, the sequence of empirical measures*

$$\mathsf{e}_n := \frac{1}{n} \sum_{i=1}^{n} \delta_{f_p^i(z)}$$

does not converge. The set of accumulation points of $(\mathsf{e}_n)_n$ *has a covering number* $\mathcal{N}(\eta)$ *at scale* η *for the Wasserstein distance which is superpolynomial (and even stretched exponential):*

$$\liminf_{\eta \to 0} \frac{\log \log \mathcal{N}(\eta)}{- \log \eta} > 0 \, .$$

The statistical behavior of the wandering component is therefore very difficult to describe. The proof of this theorem is based on the Newhouse phenomenon and uses in particular renormalization techniques introduced in [Be]. This also generalizes the work of Kiriki and Soma to the C^∞ and analytic cases.

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Support of elements in the Humilière completion, γ-coisotropic subsets and inverse reduction inequalities

Claude Viterbo

Except when explicitly mentioned all results here will appear in [9].

1. Support of elements in the Humilière completion and γ-coisotropic subsets

Let $(M,\omega = d\lambda)$ be an exact symplectic manifold. In [1], V. Humilière defined the completion for the spectral norm γ of the set of compact supported Hamiltonian maps and the set of exact Lagrangians. We denote these completions by $\widehat{\mathrm{Ham}_c}(M,\omega)$ and $\widehat{\mathcal{L}}(M,\omega)$, and shall only deal with the case $M = T^*N$. Our first goal is to define the support of an element $L \in \widehat{\mathcal{L}}(M,\omega)$. We first set the following

Definition 1. *We shall say that $x \notin \mathrm{supp}(L)$ if for all neighbourhoods U of x, and all Hamiltonian maps supported in U we have $\gamma(\varphi(L),L) = 0$.*

The first question is to try to understand what are the properties of the support. For this we need another definition, that coincides with the one in [6] except that we replace the Hofer distance by the γ-distance.

Definition 2. *We shall say that a set V is not γ-coisotropic at $z \in V$ if for all neighbourhoods U of z, and ball $B(z,\eta) \subset U$ there is a sequence φ_j of Hamiltonian maps supported in U, such that $\gamma - \lim\varphi_j = \mathrm{Id}$ and $\varphi_j(V) \cap B(z,\eta) = 0$.*

Our first result is

Proposition 3. *Let $L \in \widehat{\mathcal{L}}(T^*M)$. Then $\mathrm{supp}(L)$ is γ-coisotropic.*

It is not hard to check that a smooth submanifold is γ-coisotropic if and only if it is coisotropic in the usual sense (i.e. $(T_zV)^\omega \subset T_zV$). It is also not hard to check that being γ-coisotropic is a property invariant by symplectic homeomorphism (or more generally a homeomorphism preserving γ). Finally we prove in [3]

Proposition 4. *The singular support of a sheaf $\mathcal{F} \in D^b(N)$ is γ-coisotropic in T^*N.*

This is more precise than the definition of involutivity in [4], which is moreover only invariant by C^1 symplectic maps.

Returning to supports of Lagrangians, we see that such a support cannot have topological dimension less than $n = \dim(N)$. However it can be arbitrarily large:

Proposition 5 (Existence of Peano Lagrangians)**.** *There exists* $L \in \widehat{\mathcal{L}}(T^*M)$ *such that* $\mathrm{supp}(L)$ *contains a symplectic image of*

$$[0,1]^n \times [0,1]^r = \{(q_1, ..., q_n, p_1, ...p_r, 0, ...0) \mid \forall j |q_j| \le 1, |p_j| \le 1\}.$$

However when $\mathrm{supp}(L)$ is minimal, we expect that in some sense L coincides with $\mathrm{supp}(L)$. For example we can define topological Lagrangians, as elements in $\widehat{\mathcal{L}}(T^*M)$ such that $\mathrm{supp}(L)$ is a topological n-dimensional submanifold. This makes sense in particular when $\mathrm{supp}(L)$ is a smooth submanifold. Being coisotropic it must be Lagrangian, and then we may ask whether $L = \mathrm{supp}(L)$?

We cannot prove this in general, even though we expect it to be true, but it holds for example if $\mathrm{supp}(L) = G$ is a Lie group. A crucial ingredient for the proof is explained in the next section

2. Inverse reduction inequalities

Let $L \in T^*(X \times Y)$ be an exact Lagrangian, and for $x \in X$, set $L_x = (L \cap (T^*_x X \times T^*Y))/T^*_x X \subset T^*Y$. For each x we obtain the reduction L_x of L at $x \in X$.We then have (see [7])

Proposition 6. *We have for all* $x \in X$ *that*

$$\gamma((L_1)_x, (L_2)_x) \le \gamma(L_1, L_2).$$

Now the spectral norm γ is the difference between the two spectral invariants $c_+(L_1, L_2)$ and $c_-(L_1, L_2)$. Our new result here is

Proposition 7. *Assume all* L_x *are embedded. Then setting* $d = \dim(X)$ *we have*

$$c_+(L_1, L_2) \le \sup_x c_+((L_1)_x, (L_2)_x) + d \sup_x \gamma(L_1)_x, (L_2)_x).$$

A consequence of this inverse reduction inequality, related to the previous section is

Proposition 8. *Let* G/H *be a homogeneous space, and for* $g \in G$ *let* τ_g *the symplectic map induced by* $g \in G$ *on* $T^*(G/H)$*. Then there is a constant* C *such that for all* L *exact Lagrangian in* $T^*(G/H)$ *we have*

$$\gamma(L) \le \sup_{g \in G} \gamma(\tau_g L, L).$$

We also get, as a byproduct of the above Proposition, the following result, conjectured in [8] and proved first by Shelukhin, and later independently by Guillermou-Vichery and the author (see [5, 2, 9])

Proposition 9. *Let* N *be a homogeneous space, and* DT^*N *be the unit disc bundle of* T^*N*. Then there is a constant* C *such that for all* L*, exact Lagrangian in* DT^*N*, we have*

$$\gamma(L) \le C.$$

3. Singularities of Hamiltonians

Let $H \in C^0(M \setminus V)$ where V is a compact subset of M. We may ask whether H defines an element in $\widehat{\mathrm{Ham}_c}(M,\omega)$. In other words, let $(H_k)_{k\geq 1}$ be a sequence in $C^\infty(M)$ converging uniformly on compact subsets in $M \setminus V$ to H. Let φ_k be the time one map of H_k. Is it true that φ_k converges for the γ-topology to an element $\varphi \in \widehat{\mathrm{Ham}_c}(M,\omega)$? Humilière proved in [1] that if $\dim(V) < n$ then this holds. We extend this as

Proposition 10. *Let H and H_k as above. If V is nowhere γ-coisotropic, then $(\varphi_k)_{k\geq 1}$ converges for the γ-topology to an element $\varphi \in \widehat{\mathrm{Ham}_c}(M,\omega)$.*

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