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MATRIX-MFO Tandem Workshop: Invariants and Structures in Low-Dimensional Topology (hybrid meeting)

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Abstract. The first ever MATRIX-MFO tandem workshop addressed several research questions in low-dimensional topology and related areas.

Mathematics Subject Classification (2020): 57K10, 57K20, 57K30, 57K40.

Introduction by the Organizers

The MATRIX-MFO Tandem Workshop *Invariants and Structures in Low-Dimensional Topology* was organized by Stefan Friedl (Regensburg) and Arunima Ray (Bonn) on the MFO side and by Jessica Purcell (Melbourne) and Stephan Tillmann (Sydney) on the MATRIX side. We had initially planned that 20 mathematicians would meet at MFO and that another 20 would congregate at the MATRIX institute in Creswick. In the end 19 mathematicians did indeed make it to MFO. Unfortunately the ongoing Covid-19 pandemic prevented the Australian mathematicians from coming to the MATRIX institute. Instead the Australian participants joined the workshop virtually from their respective homes. An unexpected positive outcome of this was that more Australian mathematicians were able to participate than initially planned.

To accommodate the hybrid setting and the time difference, the workshop had an unusual format, consisting primarily of discussion sessions and with very few formal talks. Our first goal was to give young mathematicians an opportunity to introduce themselves and their work. The second goal was to get conversations started among all the participants and to try to make the workshop as interactive as possible. To achieve our second goal we wrote to all participants before the meeting, soliciting questions or topics to be discussed at the meeting. Each participant also got a chance to indicate which problems they would be interested in discussing.

Among the suggestions from the participants, we picked four problems which had enough traction at MFO as well as on the Australian side:

(1) Twisted intersection forms of spin 4-manifolds.
(2) Knots in 3-manifolds.
(3) Profinite rigidity of 3-manifold groups.
(4) Cobordisms up to stable diffeomorphism.

Each topic had two discussion leaders, one at MFO and one in Australia.

The setup resulted in an a rather unusual schedule. We had a common time which worked for most participants between 9am and 12pm German time. On Monday every participant got one minute to introduce themselves. Afterwards the discussion leaders gave short introductions to the four problems. Finally the various working groups met in different rooms at MFO and they were joined via zoom by the Australian mathematicians to get started.

On the remaining days of the workshop, the tandem time had the same structure. Each day began with short (10 minute) talks by young researchers. There were four such talks from from MFO on Tuesday, and ten from Australia from Wednesday to Friday. The talks were followed by brief summaries from the working groups, and occasionally an indication of the plans for the coming day. The bulk of the shared time was left to the working groups to discuss and collaborate, with more detailed reports from the groups on either continent. Over the course of the week the problems being discussed also evolved – for example the twisted intersection forms group merged with the cobordism group and the knot group split into smaller groups. Participants were encouraged to switch between groups during the week as they wished, but most remained with their original choices.

The afternoons at MFO had only had one scheduled event per day. Peter Feller, Holger Kammeyer, Markus Land and Delphine Moussard each gave a 45 minute talk on recent results in their respective research areas. These talks took place at 2:15pm in the main lecture hall. A few Australian mathematicians joined despite the late hour. The talks were also recorded and made available to the Australian participants. Each working group also meet from 4pm to 6pm to work on their problem.

In summary, the 19 mathematicians at MFO relished the opportunity to talk in person about mathematics again. The discussions on mathematical and academic issues were extremely lively and a welcome relief after spending many months in front of a laptop screen. The collaboration across time zones and continents worked out better than expected. Indeed, the seven hour time difference turned out to be a blessing since it gave us enough time to meet together, but it also gave each continent to work on the problem separately and in smaller groups.

The extended abstracts are organized as follows. First we provide three summaries of what happened in the working groups. These are followed by the extended abstracts of the four invited talks by Feller, Kammeyer, Land and Moussard. We conclude with some extended abstracts of young researchers.

The organizers of this workshop hope that interactive meetings, which are more focussed on problem solving instead of a long list of talks, will become more common. The organizers also wish to thank the staff at MFO for making this meeting possible and for providing us with all the technical support which was required.

MATRIX-MFO Tandem Workshop (hybrid meeting): Invariants and Structures in Low-Dimensional Topology

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Abstracts

Profinite rigidity of 3-manifold groups

Holger Kammeyer
(joint work with Carol Badre, Grace Garden, Boris Okun, Jessica Purcell, Marcy Robertson, Benjamin Ruppik)

The purpose of our problem group was to understand the methods of the recent proofs by Bridson–McReynolds-Reid-Spitler [1] that certain Kleinian groups are profinitely rigid in the absolute sense: they are distinguished from all other finitely generated residually finite groups by the set of finite quotient groups. In addition, we wanted to assess if these methods can be adapted to establish profinite rigidity for further examples of Kleinian groups, possibly using computer assistance for some explicit computations.

After working through [1] and the preprint [2], we have come up with the following seven step template to prove that a certain Kleinian group $\Gamma \leq \mathrm{PSL}_2(\mathbb{C})$ is profinitely rigid, meaning that if a finitely generated residually finite group Λ has the property that the profinite completion $\widehat{\Lambda}$ is isomorphic to $\widehat{\Gamma}$, then actually Λ must be isomorphic to Γ.

(1) One constructs a field isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}_p}$ with the property that $\Gamma \leq \mathrm{PSL}_2(\mathbb{C}) \cong \mathrm{PSL}_2(\overline{\mathbb{Q}_p})$ has precompact image, so that the closure of the image is a profinite group. Consequently, the homomorphism $\Gamma \to \mathrm{PSL}_2(\overline{\mathbb{Q}_p})$ extends to a homomorphism $\widehat{\Gamma} \to \mathrm{PSL}_2(\overline{\mathbb{Q}_p})$ and using $\widehat{\Gamma} \cong \widehat{\Lambda} \geq \Lambda$, we obtain a homomorphism $\phi\colon \Lambda \to \mathrm{PSL}_2(\mathbb{C})$. In doing so, it is assured that if $\Gamma \to \mathrm{PSL}_2(\mathbb{C})$ has Zariski dense image, then so does $\phi\colon \Lambda \to \mathrm{PSL}_2(\mathbb{C})$. So the upshot is that a reasonably large quotient of Λ also maps to $\mathrm{PSL}_2(\mathbb{C})$.

(2) Assuming that Γ is arithmetic, it is essentially given by the reduced norm one elements in an order $\mathcal{O}$ of a quaternion algebra A over a number field k with exactly one complex place. We consider the trace field $k' = \mathbb{Q}(\mathrm{tr}(\phi(\Lambda))$ and the quaternion algebra $A' = k'(\phi(\Lambda))$ associated with $\phi(\Lambda)$. It follows from Galois rigidity that k' is likewise a number field and there exists a maximal order $\mathcal{O}' \subset A'$ such that ϕ has image in the reduced norm one elements of $\mathcal{O}'$.

(3) The goal is now to show that $k \cong k'$ and $A \cong A'$. As a technical core step, the assumption $\widehat{\Gamma} \cong \widehat{\Lambda}$ implies that the number fields have isomorphic finite adele rings $\mathbb{A}_k^f \cong \mathbb{A}_{k'}^f$ and A is isomorphic to A' over this ring isomorphism.

(4) Since k has precisely one complex place, it is arithmetically solitary meaning that the last step implies $k \cong k'$ and also $A \cong A'$.

(5) Now for certain Kleinian groups Γ, the associated order in the quaternion algebra is maximal and all maximal orders in the quaternion algebra are conjugate. In these cases, we thus obtain a homomorphism $\phi\colon \Lambda \to \Gamma$.

(6) Using specific properties of Γ, such as the precise group structure of the abelianization, one shows that ϕ is an epimorphism.

(7) The profinite completion $\widehat{\Gamma}$ of a Kleinian group is cofinite Hopfian. An improved argument for this step is given in [2]. It follows that $\widehat{\phi}$, hence ϕ is an isomorphism.

A key property in these proofs, which is actually used more than once, is *Galois rigidity*. It can be described as the phenomenon that a Kleinian group Γ has no more Zariski dense representations to $\mathrm{PSL}_2(\mathbb{C})$ than the inclusion and its Galois conjugates coming from other embeddings of the trace field to $\mathbb{C}$. For certain groups Γ, it can be possible to establish Galois rigidity by a brute force method. The relations in a suitable finite presentation of Γ restrict the subset of $\mathrm{PSL}_2(\mathbb{C})$ to which generators of Γ can map and by computer assistance, it can be possible to see that not more representations are possible than the degree of the trace field. We discussed in how far this method might also be feasible for the Whitehead link group. Additionally, the first few steps of the proof sketeched above are not limited to the group $\mathrm{PSL}_2(\mathbb{C})$. We pondered in how far these steps could be adapted to improve profinite rigidity results relative to some class of groups to absolute profinite rigidity. In particular, we see chances to do this for lattices in the complex exceptional groups of type E_8, F_4, and G_2 as considered in [3].

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Discussions on knot theory in general 3-manifolds

Marc Kegel, Joan Licata, and Arunima Ray

Our discussion group was formed around the straightforward, but rather general, question: which aspects of classical knot theory in S^3 can be transported to knot theory in general 3-manifolds? Broadly speaking, there are two ways to approach this question:

(1) Which *results* from classical knot theory are still true for knots in general 3-manifolds?
(2) Which *invariants* of classical knots can be generalised to give invariants of knots in arbitrary 3-manifolds?

Since the scope of the question was quite large, our first tandem meeting was a problem session, where we brainstormed as a group which specific questions might be interesting to pursue. Some of the questions that we discussed were:

(1) Is there a Jones/HOMFLYPT polynomial for knots in 3-manifolds?
(2) What are knot projections for knots in 3-manifolds? What are the desirable properties for such projections? Is there a Reidemeister theorem for knots in 3-manifolds?

(3) What is the right analogue of the Gordon-Luecke theorem [5] for knots in general 3-manifolds (and is it true)?
(4) There is a prime decomposition for 3-manifolds. There is a prime decomposition for knots in S^3. Is there a prime decomposition for knots in 3-manifolds?

On Monday and Tuesday the group stayed together, switching between problems. On Wednesday morning we split into three groups, focused on considering (1) general knot projections, (2) prime decompositions of knots in 3-manifolds, and (3) the Boileau volumes of 3-manifolds. Prior to this we also discussed the analogue of the Gordon–Luecke theorem, and there was an impromptu group discussing skein modules for general 3-manifolds, a possible way to generalise knot polynomials to other 3-manifolds.

Are knots determined by their complements? First we discussed what the correct analogue of the Gordon–Luecke theorem would be in general 3-manifolds. There are two options:

(Strong GL) Let K_1 and K_2 be knots in a closed, oriented 3-manifold Y. Then $Y \setminus K_1$ is orientation preserving homeomorphic to $Y \setminus K_2$ if and only if K_1 and K_2 are (ambient) isotopic.

(Weak GL) Let K_1 and K_2 be knots in a closed, oriented 3-manifold Y. Then $Y \setminus K_1$ is orientation preserving homeomorphic to $Y \setminus K_2$ if and only if K_1 and K_2 are *equivalent*, i.e. there is an orientation preserving homeomorphism $f\colon Y \to Y$ satisfying $f(K_1) = K_2$.

Strong GL is false in its full generality: Consider $K \subseteq S^1 \times S^2$ and $G(K)$ the image of K under the Gluck twist $G\colon S^1 \times S^2 \to S^1 \times S^2$, which in particular is an orientation preserving diffeomorphism. It was shown in [1] that the Gluck twist does not, in general, preserve isotopy classes, e.g. whenever K represents an odd element $w \neq \pm 1 \in \mathbb{Z} \cong H_1(S^1 \times S^2; \mathbb{Z})$. It seems possible that Weak GL might be true for knots in closed, oriented, irreducible 3-manifolds, whenever the knots are not rational unknots in some lens spaces. Weak GL appears on Kirby's list of problems, where it is called the oriented knot complement conjecture [6, Problem 1.81D].

Weak GL is related to the cosmetic surgery conjecture [6, Problem 1.81A]: Suppose K is a knot in a closed oriented 3-manifold Y such that $Y \setminus K$ is irreducible and not homeomorphic to the solid torus. If two different Dehn surgeries on K are orientation preserving homeomorphic, then there is a homeomorphism of $Y \setminus K$ which takes one slope to the other. This conjecture is still open.

The cosmetic surgery conjecture implies Weak GL (except for rational unknots in some lens spaces), as follows. Suppose there are knots K and J in some 3-manifold M and an orientation preserving homeomorphism $\varphi\colon M \setminus K \xrightarrow{\cong} M \setminus J$. Assume further that $M \setminus K$ is irreducible and not homeomorphic to the solid torus. Then the Dehn filling of $M \setminus K$ along the meridian μ_K of K, i.e. the manifold M, and the Dehn filling on $M \setminus J$ along $\varphi(\mu_J)$ are orientation preserving homeomorphic. By the cosmetic surgery conjecture, the two slopes must be equal, i.e. φ maps meridian to meridian. But this means we can extend φ to an orientation preserving homeomorphism of M to itself, with $\varphi(K) = J$.

Knot projections. Part of the discussion focused on the desirable properties of knot projections for knots in general 3-manifolds. In particular, we think that a representation of knots in general 3-manifolds is good if it-

- is useful for hand-computations of invariants,
- is useful for defining invariants,
- can be drawn on a piece of paper,
- can be fed to a computer,
- has good compression properties,
- is easy to see and fix errors,
- can represent a wide variety of knots/manifolds, and
- admits a Reidemeister-type theorem.

The discussion evolved to focus on projections of knots to special spines in arbitrary 3-manifolds. A number of the workshop participants have continued working on this problem and expect to write a short paper on the topic.

Prime decompositions of 3-manifolds. The goal was to give an existence and uniqueness theorem for knots in general 3-manifolds. We came up with a notion of complexity which decreases when a nullhomologous knot has a decomposition as a connected sum, which is necessary for a decomposition into summands to eventually terminate. After some discussion, a google search revealed that the question was completely answered by Miyazaki in 1989 [8]. Notably, not every knot in a general 3-manifold has a terminating prime decomposition. For example, by the lightbulb trick, the knot given by $S^1 \times \{*\} \subseteq S^1 \times S^2$ can be infinitely decomposed.

Skein modules. David Reutter gave a talk about the skein module $Sk(M)$ of an arbitrary 3-manifold M defined using the Kauffman skein relations. He explained how one can prove that $Sk(M)$ is always finite-dimensional (the idea is divide-and-conquer). We expect that the skein module is always non-zero, but without an explicit alternative formula for evaluating links (as for $M = S^3$), this seems to be a hard question. We then discussed $Sk(S^1 \times S^2)$ and wondered how $Sk(S^1 \times S^2)$ is related to the decategorification of the Khovanov homology of links in $S^1 \times S^2$.

Boileau volumes. Every closed, oriented 3-manifold contains hyperbolic knots and links [9]. Boileau defines the *volume* of a closed, oriented 3-manifold M, denoted $\mathrm{vol}_B(M)$, as the minimum possible volume of hyperbolic link complements in M [2]. Jorgensen–Thurston theory implies that a minimum is achieved [10]. If M is hyperbolic, then $\mathrm{vol}_B(M)$ is the hyperbolic volume of M, since we can use the empty link. It is a deep theorem that $\mathrm{vol}_B(S^3)$ is the volume of the figure eight knot complement [3].

Using work of Dunfield [4] and Milley [7], we computed the Boileau volume of some simple non-hyperbolic 3-manifolds like $\mathbb{RP}^3$. On the other hand, we performed computer experiments using SnapPy and Regina to develop upper bounds on the Boileau volume and formulate conjectures how the Boileau volume should behave on lens spaces, on more general Seifert fibered spaces, under connected sums and under finite-sheeted covers. We went on to prove the limit cases of some of our conjectures. For example, we showed that $\mathrm{vol}_B(L(p,1))$ converges, as $p \to \infty$, strictly monotonically to the volume of the Whitehead link complement and that for $M_i \ncong S^3$, the $\mathrm{vol}_B(M_1 \# \cdots \# M_n) \to \infty$ as $n \to \infty$.

Participants

A majority of the workshop attendees participated in this discussion group, both at MFO and in Australia, including at least the following people.

At MFO

- Jonathan Bowden
- Rachael Boyd
- Rima Chatterjee
- Marc Kegel
- Delphine Moussard
- José Quintanilha
- Katherine Raoux
- Arunima Ray
- David Reutter
- Saul Schleimer
- Paula Truöl
- Claudius Zibrowius

In Australia

- Jack Brand
- Ben Burton
- Zsuzsanna Dancso
- Alex He
- Tamara Hogan
- Adele Jackson
- Joan Licata
- Thiago de Paiva
- Jessica Purcell
- Jonathan Spreer
- Emily Thompson
- Stephan Tillmann

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Twisted intersection forms of spin 4-manifolds

Stefan Friedl

(joint work with Diarmuid Crowley, Peter Feller, Urs Fuchs, Daniel Kasprowski, Markus Land, Delphine Moussard, Csaba Nagy, Mark Powell, Katherine Raoux, David Reutter)

In this working group, organized jointly by Diarmuid Crowley (Melbourne) and Stefan Friedl (Regensburg), we discussed twisted intersection forms of spin 4-manifolds. Recall that one of the most important invariants of a (closed oriented smooth) 4-manifold M is the intersection form $I_M\colon H_2(M;\mathbb{Z}) \times H_2(M;\mathbb{Z}) \to \mathbb{Z}$, which can be defined via the cup product and Poincaré duality. It is a classical result that if M admits a spin structure, then the intersection form is even, i.e. for any $a \in H_2(M;\mathbb{Z})$ the integer $I_M(a,a)$ is even.

Stefan Friedl raised the question whether the analogous statement also holds for twisted intersection forms on spin 4-manifolds. More precisely, let $\lambda\colon P \times P \to \mathbb{Z}$ be a non-singular symmetric form over a finitely generated free abelian group P, let M be a 4-manifold and let $\alpha\colon \pi_1(M) \to \mathrm{Aut}(P,\lambda)$ be a representation. Using the cup product, Poincaré duality and the form λ one can define the twisted intersection form

$$I_{M,\alpha}\colon H_2(M;P_\alpha) \times H_2(M;P_\alpha) \to \mathbb{Z}.$$

The question is, whether this form is even, if M admits a spin structure. We note that for the trivial representation $\alpha\colon \pi_1(M) \to \{1\} \to \mathrm{Aut}(P,\lambda)$, the twisted intersection form is simply the tensor product of the intersection form of M and (P,λ), and a tensor product of two symmetric forms is even if one of them is even. Note furthermore note that the signature of a non-singular even symmetric form is necessarily divisible by eight. Thus a related question is, whether the signature of such a twisted intersection form is always divisible by eight.

Throughout the week we discussed this question and related issues. Furthermore several talks were given on related topics:

(1) On the first day Markus Land gave an impromptu talk, recalling the proof of the fact that the classical intersection form of a spin 4-manifold (in fact, a 4-dimensional spin PD complex) is even.

(2) Next we discussed at length the relationship between several alternative definitions of spin structures. For example, let M be a manifold that is equipped with a CW-structure. In [6] it is stated, without proof, that spin structures on M are in bijection with the set of orientations on the 1-skeleton M^1 that extend to the 2-skeleton M^2. We found out that this statement is not correct for 2-manifolds. Indeed, S^2 is spin, but the tangent bundle is non-trivial. In particular if we equip S^2 with the standard CW-structure, then the tangent bundle is trivial on the 1-skeleton, which is just a point, but this trivialization does not extend over the 2-skeleton, which is S^2, since the tangent bundle is non-trivial. Nevertheless, we convinced ourselves that the claimed statement is correct for manifolds of dimension ≥ 3.

(3) Tuesday morning Mark Powell reported on a result of Kasprowski, Powell, Teichner [5] that there exists a spin 4-manifold M such that the equivariant intersection form $H_2(M;\mathbb{Z}[\pi]) \times H_2(M;\mathbb{Z}[\pi]) \to \mathbb{Z}[\pi]$ is weakly even but not strongly even.
(4) Tuesday afternoon and Wednesday morning we discussed quadratic refinements of the linking form on rational homology 3-spheres, that are equipped with a spin structure. In particular Wednesday morning Diarmuid Crowley gave a talk on aspects of his work with Nordström [4, §2.8]. He reviewed Wall's definition of the quadratic refinement of the linking form of an $(s-1)$-connected $(2s+1)$-manifold, $s \neq 1,3,7$ [7, §12A] and explained how to define a quadratic refinement for linking forms of $(2s+1)$-manifolds $(M,\overline{\nu})$ with a v_s-structure $\overline{\nu}$, at least when the Hurewicz map $\pi_s(M) \to H_s(M;\mathbb{Z})$ is onto. For oriented 3-manifolds $\overline{\nu}$ is equivalent to the choice of a spin structure.
(5) To address the original question Diarmuid Crowley proposed to study intersection forms of 4-manifold bundles over spin 4-manifolds. These intersection forms are special cases of twisted intersection forms. Friday morning Diarmuid Crowley sketched a proof that in some circumstances such twisted intersection forms are indeed even.
(6) On Thursday and Friday Markus Land sketched a proof, building on his recent work [1, 2, 3] that the signatures of twisted intersection forms of spin 4-manifolds are divisible by eight. Since signatures of even non-singular forms are divisible by eight this can be seen as evidence for an affirmative answer to the original question.

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Recent highlights in low-dimensional topology

Peter Feller

We present three highlights, one from each of the years 2019, 2020, and 2021.

2020: Topology input yields Euclidean geometry result. An n-gone in the Euclidean plane $\mathbb{R}^2$ is said to *inscribe* in a Jordan curve $\Gamma \subset \mathbb{R}^2$ if there exists an orientation-preserving similarity of $\mathbb{R}^2$ that maps the vertices of the n-gone into Γ. A quadrilateral (i.e. a four-gone) in $\mathbb{R}^2$ is called *cyclic* if its vertices lie on a circle. The following result characterizes the quadrilaterals that inscribe in all smooth Jordan curves.

Theorem 1 (Greene-Lobb, 2020 [GL20]). *Every cyclic quadrilateral inscribes in every smooth Jordan curve in the Euclidean plane.*

The surprising (symplectic) topology input in Greene and Lobb's proof of Theorem 1 is the fact that there do not exist embedded Lagrangian tori in $(\mathbb{R}^4, \omega_{\text{std}})$ with minimum Maslov number 4 [Vit90, Pol91], where ω_{std} denotes the symplectic form $dx \wedge dy + dz \wedge dw$. We describe three stepping stones towards Theorem 1.

Firstly, in 2018, Hugelmeyer [Hug21] discovered a strategy of proof that allows to recover Schnirelman's result that squares inscribe in all smooth Jordan curves using the following knot theory input. The $T(4,5)$ torus knot in the three-sphere $\mathbb{S}^3 = \partial\mathbb{D}^4$ is not the boundary of an embedded smooth Möbius band in the four-ball $\mathbb{D}^4$. In fact, this is only implicit in Hugelmeyer's work (see [FG20] for details); instead, Hugelmeyer proved the following new result: rectangles with aspect ratio $\sqrt{3}$ inscribe in every smooth Jordan curve. For this he used that another knot, the $T(5,6)$ torus knot, is not the boundary of an embedded smooth Möbius band in $\mathbb{D}^4$ [Hug18]. Secondly, in 2019, Hugelmeyer followed up by showing that for every smooth Jordan curve "a third" of all rectangles inscribe [Hug21]. Thirdly, building on ideas from Hugelmeyer's follow-up, but crucially employing a symplectic topology perspective, Greene and Lobb showed that all rectangles inscribe in all smooth Jordan curves [GL21]. For this they employ that there do not exist embedded Lagrangian Klein bottles in $(\mathbb{R}^4, \omega_{\text{std}})$. The proof of Theorem 1 can be understood as an improvement on their argument for this result.

2019: Porting $\text{Diff}^+(S_g)/\text{Diff}_0(S_g)$ technology to $\text{Diff}_0(S_g)$. The identity component of the group of C^∞-diffeomorphisms of a compact smooth manifold M, denoted by $\text{Diff}_0(M)$, is perfect [Mat71, Mat74, Thu74]. In fact, results from [BIP08, Tsu08, Tsu12] imply that, for every closed and oriented manifold M that is diffeomorphic to a sphere or has dimension two or four, the group $\text{Diff}_0(M)$ is uniformly perfect: there exist an $N \in \mathbb{N}$ such that every element can be written as a product of at most N commutators. In contrast, for the smooth, oriented, and closed surfaces S_g of genus $g \geq 1$ one has the following striking result.

Theorem 2 (Bowden-Hensel-Webb, 2019 [BHW19]). *For $g \geq 1$, the space of homogeneous quasimorphisms on $\text{Diff}_0(S_g)$ is (uncountably) infinite dimensional.*

Theorem 2 relates to uniform perfectness as follows. A short calculation shows, that, if G is a group for which there exists of a homogeneous quasimorphism $f\colon G \to \mathbb{R}$ that is not constantly 0, then G is not uniformly perfect. Hence, Theorem 2 implies that $\mathrm{Diff}_0(S_g)$ is not uniformly perfect.

For the proof of Theorem 2, the authors proceed in analogy to an idea that can be used to show that the mapping class group $\mathrm{MCG} := \mathrm{Diff}^+(S_g)/\mathrm{Diff}_0(S_g)$ for $g \geq 3$ has many homogeneous quasimorphisms and hence, while being perfect, is not uniformly perfect (originally proven in [EK01]). Here is a terse account of this idea for MCG. Set $\mathcal{C} := \{[K] \mid K \text{ is an essential simple closed curve in } S_g\}$, where $[K]$ denotes the isotopy class of K. The group MCG acts on $\mathcal{C}$ via $\mathrm{MCG} \times \mathcal{C} \to \mathcal{C}, ([\phi],[K]) \mapsto [\phi(K)]$. This action allows to construct many homogeneous quasimorphism on MCG, using the following celebrated fact. Equipped with the curve graph metric[1], $\mathcal{C}$ is a Gromov-hyperbolic metric space [MM99].

Bowden, Hensel, and Webb fearlessly consider the following "large" analogue of $\mathcal{C}$: the set $\mathcal{C}^\dagger := \{K \mid K \text{ is an essential simple closed curve in } S_g\}$ with a similarly defined metric (simply dropping equivalence classes in the definition). Guided by analogy to the MCG setup, they show that $\mathcal{C}^\dagger$ is Gromov-hyperbolic and they construct many homogeneous quasimorphisms on $\mathrm{Diff}_0(S_g)$ using the action $\mathrm{Diff}_0(S_g) \times \mathcal{C}^\dagger \to \mathcal{C}^\dagger, (\phi, K) \mapsto \phi(K)$.

2021: A space version of the light bulb theorem for all dimensions. In this section results are only described in vague terms. In particular, information about orientations and framings is suppressed.

The (folklore) light bulb theorem says that all smooth embeddings of the interval $\mathbb{D}^1$ in $\mathbb{S}^2 \times \mathbb{D}^1$ with boundary $\{p\} \times (\partial\mathbb{D}^1)$ are isotopic rel boundary. Recent developments are Gabai's 4D light bulb theorem (same statement with $\mathbb{D}^1$ replaced by $\mathbb{D}^2$) and further results concerning the fourth dimension [Gab20, Sch20, ST19].

An elegant perspective allows to put all of this in a "spacified" context. Informally, the following result says that for $1 \leq k \leq d$ the space of embeddings of the k-disk $\mathbb{D}^k$ into a smooth oriented d-dimensional manifold M with prescribed boundary $s\colon \mathbb{S}^{k-1} \hookrightarrow \partial M$, denoted by $\mathrm{Emb}_\partial(\mathbb{D}^k, M)$, is homotopy equivalent to a certain path space of embeddings of the $(k-1)$-disk into a d-dimensional manifold, if s has a *geometrically dual sphere* G, i.e. $\mathbb{S}^{d-k} \cong G \subseteq \partial M$ and $|s(\mathbb{S}^{k-1}) \pitchfork G| = 1$.

Theorem 3 (Kosanović-Teichner, 2021 [KT21]). *Let $G \subseteq \partial M$ be a geometrically dual sphere for s, and set M_G to be the result of attaching a $(d-k+1)$-handle to M along G. Then* $\mathrm{Emb}_\partial(\mathbb{D}^k, M) \simeq \Omega\mathrm{Emb}_\partial(\mathbb{D}^{k-1}, M_G)$.

Without further describing the path space $\Omega\mathrm{Emb}_\partial(\mathbb{D}^{k-1}, M_G)$, we note that in the case of $k = 1$ and $d \geq 3$, one finds $\pi_0(\Omega\mathrm{Emb}_\partial(\mathbb{D}^{k-1}, M_G)) \cong \pi_1(M_G) \cong \pi_1(M)$. This recovers the light bulb theorem, since $\pi_0(\mathrm{Emb}_\partial(\mathbb{D}^1, M)) \cong \pi_1(M) = \{1\}$ for $M = \mathbb{S}^2 \times \mathbb{D}^1$. In case of $k = 2$ and $d = 4$, Kosanović and Teichner explicitly describe $\pi_0(\Omega\mathrm{Emb}_\partial(\mathbb{D}^1, M_G))$ using a so-called Dax invariant. This π_0-calculation amounts to a generalization of all prior light bulb theorems in 4D due

[1]The metric is the one induced from the graph with vertices $\mathcal{C}$ and one edge (of length 1) between $[K]$ and $[L]$ for all disjoint, non-isotopic, and essential simple closed curves K and L.

to the bijection between $\pi_0(\mathrm{Emb}_\partial(\mathbb{D}^2, M))$ and $\pi_0(\Omega\mathrm{Emb}_\partial(\mathbb{D}^1, M_G))$ provided by Theorem 3. In general, the homotopy type of embedding spaces (and loop spaces thereof) are easier to understand the larger the codimension $d-k$ is. The striking point of Theorem 3 is that, in the presence of dual spheres, the homotopy type of the embedding space of interest can be understood via the homotopy type of an embedding space with larger codimension.

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On and around profinite rigidity

Holger Kammeyer
(joint work with Steffen Kionke)

Recall that the *profinite completion* $\widehat{\Gamma}$ of a group Γ is defined as the projective limit of the system of all finite quotient groups of Γ. We have a canonical homomorphism $\Gamma \to \widehat{\Gamma}$ and we say that Γ is *residually finite* if this homomorphism is injective. The profinite completion satisfies the universal property that every homomorphism $\Gamma \to P$ to a profinite (i.e. compact totally disconnected) group factors uniquely through $\Gamma \to \widehat{\Gamma}$. The universal property shows in particular that "$\wedge$" defines a functor from the category of groups to the category of profinite groups. Arguably, the birth of the field of *profinite rigidity* is the following question asked by A. Grothendieck [5] in 1970.

Question 1 (Grothendieck 1970)**.** Let $u\colon \Gamma \to \Lambda$ be a homomorphism of finitely presented and residually finite groups such that $\widehat{u}\colon \widehat{\Gamma} \to \widehat{\Lambda}$ is an isomorphism. Does it follow that $u\colon \Gamma \to \Lambda$ is an isomorphism?

Note that by residual finiteness of Γ, the homomorphism $\widehat{u}$ just restricts to u on $\Gamma \leq \widehat{\Gamma}$, hence it comes with no loss of generality to assume in this question that u is injective. The timeline of the results regarding this question is as follows:

- **1990:** *Platonov–Tavgen'* [11]*:* Asking the question for finitely generated instead of finitely presented groups, the answer is negative. Counterexamples arise as fibered products $F_n \times_Q F_n \leq F_n \times F_n$ for an epimorpism $F_n \to Q$ defininig a presentation of an infinite group Q with n generators, n relations, and no finite quotients. Examples of such groups Q were found by Higman [6].
- **2004:** *Bridson–Grunewald* [3]*:* A careful improvement of the above construction yields counterexamples to Grothendieck's original question (which they found in Oberwolfach!) Instead of using free groups, the counterexamples (called *Grothendieck pairs*) arise as fibered products $H \times_Q H \leq H \times H$ defined by an epimorphism $H \to Q$ where H is a residually finite Gromov hyperbolic group with two dimensional classifying space and Q is a Higman type group whose presentation complex is aspherical.
- **2011**: *Long–Reid* [9]*:* In the positive direction, no Grothendieck pairs exist among fundamental groups of closed 3-manifolds.
- **2019:** *Boileau–Friedl* [2]*:* There are no Grothendieck pairs among fundamental groups of irreducible 3-manifolds with toral boundary, either.
- **2021:** *Sun* [12]*:* There are no Grothendieck pairs among finitely generated 3-manifold groups at all.

Viewing that 3-manifold groups are in this sense *Grothendieck rigid*, we now move on to a related but more subtle concept.

Definition 2. A finitely generated residually finite group Γ is called *profinitely rigid* if for every other such group Λ with $\widehat{\Gamma} \cong \widehat{\Lambda}$, we have $\Gamma \cong \Lambda$.

Note that in this definition, it does not matter whether $\widehat{\Gamma} \cong \widehat{\Lambda}$ means abstract isomorphism or isomorphism of topological groups because an abstract isomorphism of finitely generated profinite groups is automatically a homeomorphism as follows from a deep theorem by Nikolov–Segal [10]. It is not hard to see that finitely generated abelian groups are profinitely rigid. However, stepping only an inch away from abelian groups, counterexamples emerge: the virtually cyclic groups $\mathbb{Z}/25\mathbb{Z} \rtimes_{(\cdot 6)} \mathbb{Z}$ and $\mathbb{Z}/25\mathbb{Z} \rtimes_{(\cdot 11)} \mathbb{Z}$ have isomorphic profinite completions but are not isomorphic [1], the intuitve reason being that $\widehat{\mathbb{Z}}$ has a wealth of nontrivial automorphisms while $\mathbb{Z}$ has only one. More substantial counterexamples can be found among lattices in higher rank Lie groups as we showed in joint work with S. Kionke [8, Theorem 1.3].

Theorem 3. *Let G be a connected simple Lie group of higher rank with trivial center which is neither isomorphic to $\mathrm{PSL}_m(\mathbb{H})$ nor to any real or complex form of type E_6. Then G contains arbitrarily many pairwise non-isomorphic uniform lattices with isomorphic profinite completions.*

The key property of G as in the theorem is that arithmetic lattices $\Gamma \leq G$ are known to satisfy the *congruence subgroup property*: essentially all finite quotient groups arise from reducing matrix coefficients modulo some ideal in the ring of integers of the number field over which Γ is defined. The examples are constructed by carefully choosing different congruence subgroups in a fixed arithmetic group. We should stress, however, that while the resulting lattices are not isomorphic, they are commensurable. This has led us to the question whether lattices in G are *profinitely solitary*, meaning if two of them have commensurable profinite completions, must the lattices be commensurable themselves? We show in loc. cit. that lattices in most higher rank Lie groups are not profinitely solitary. But interestingly, there are three exceptions: when G is the complex form of exceptional type E_8, F_4, or G_2. These observations have led us to the problem of classifying higher rank lattices up to profinite commensurability, a task which we have undertaken in [7].

In view of Theorem 3, it might come as a surprise that lattices in $\mathrm{PSL}_2(\mathbb{C})$, and more generally *Kleinian groups*, meaning discrete subgroups of $\mathrm{PSL}_2(\mathbb{C})$, are conjectured to be profinitely rigid. The first examples of profinitely rigid Kleinian groups have only been found recently by Bridson–McReynolds–Reid–Spitler [4]. It is intriguing to pin down what properties of these particular Kleinian group are relevant for concluding profinite rigidity. For an outline of the proof methods, we refer to the report of the problem group "Profinite rigidity of 3-manifold groups".

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Mapping class groups, symplectic groups, and their stable cohomology

Markus Land

(joint work with Fabian Hebestreit, Thomas Nikolaus)

Let Σ_g be an oriented closed surface of genus g. In this talk I explained how to access the cohomology (in a stable range) of its mapping class group, and that a similar method can be employed to study the cohomology of the symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$ – the group of automorphisms of the intersection form of Σ_g. Let $\mathrm{MCG}_g^+ = \pi_0(\mathrm{Diff}^+(\Sigma_g))$ be the group of isotopy classes of orientation preserving diffeomorphisms of Σ, called the (oriented) mapping class group of Σ_g.

Theorem 1 (Eells–Earle). *For $g \geq 2$, the map $\mathrm{Diff}^+(\Sigma_g) \to \mathrm{MCG}_g^+$ is a homotopy equivalence.*

In other words, the theorem says that the group $\mathrm{Diff}_0(\Sigma_g)$ of diffeomorphisms isotopic to the identity is contractible once $g \geq 2$, i.e. once Σ_g admits a hyperbolic metric. This implies that elements of $H^*(\mathrm{MCG}_g^+; R)$ are precisely the universal characteristic classes of oriented bundles of genus g surfaces.

Letting a diffeomorphism act on the intersection form, one obtains a group homomorphism $\mathrm{MCG}_g^+ \to \mathrm{Sp}_{2g}(\mathbb{Z})$. This map turns out to be surjective, and its kernel is called the Torelli group. It follows that the cohomology of the Torelli group is a representation of $\mathrm{Sp}_{2g}(\mathbb{Z})$ and it is interesting to study this representation. The Serre spectral sequence associated to the map $\mathrm{MCG}_g^+ \to \mathrm{Sp}_{2g}(\mathbb{Z})$ allows to access the fixed points of the $\mathrm{Sp}_{2g}(\mathbb{Z})$-action on the cohomology of the Torelli group. As a first step, one would therefore like to determine the cohomology of $\mathrm{Sp}_{2g}(\mathbb{Z})$ and the effect of the map $\mathrm{MCG}_g^+ \to \mathrm{Sp}_{2g}(\mathbb{Z})$ on cohomology.

With rational coefficients, much is known already, and the purpose of this talk is to indicate how the new developments in hermitian K-theory [1, 2, 3] can be used to obtain information about $\mathbb{F}_2$-coefficients – something which was to the best of my knowledge previously inaccessible.

In then indicated what the d-dimensional cobordism category Cob_d^+ is: Roughly speaking, objects are closed oriented $(d-1)$-manifolds, morphisms are oriented d-dimensional cobordisms, and composition in the category is given by glueing cobordisms. This category is naturally a topologically enriched (or an $(\infty,1)$-category in modern language) having the following property: For oriented $(d-1)$-manifolds M and N, one has an equivalence

$$\mathrm{Map}_{\mathrm{Cob}_d^+}(M,N) = \coprod_{[W]} \mathrm{BDiff}(M,N)$$

where $[W]$ runs through the diffeomorphism classes of cobordisms between M and N. The following theorem (which combes various results) is a way one can access the cohomology of MCG_g^+, as exploited in [4]. It uses in particular homological stability, the group completion theorem, and the Galatius–Madsen–Tillmann–Weiss theorem.

Theorem 2. *There are canonical maps*

$$\mathrm{BDiff}^+(\Sigma_g, D^2) \longrightarrow \Omega_{\emptyset,S^1}|\mathrm{Cob}_2^+| \longrightarrow \Omega_0^\infty \mathrm{MTSO}(2)$$

where the middle term is the space of paths from $\emptyset$ to S^1 in the geometric realisation of the cobordism category, and the latter term is the Thom spectrum of $-\gamma_2$, where γ_2 is the universal oriented rank 2 vectorbundle over BSO(2). *The first map is a homology equivalence in a range depending on g (roughly for degrees $\leq \frac{2g}{3}$), and the latter map, a parametrised Pontryagin–Thom map, is a homotopy equivalence.*

Therefore, the stable homology of $\mathrm{Diff}^+(\Sigma_g, D^2)$, and therefore of $\mathrm{MCG}_{g,1}^+$ is described by the homology of the unit component of the infinite loop space associated to the spectrum MTSO(2) – something accessible by completely different means that mapping class group itself. For instance, the rational cohomology of $\Omega_0^\infty\mathrm{MTSO}(2)$ can be deduced immediately. The cohomology with finite coefficients is more complicated, but Galatius was able to describe it as a kernel of a certain map of (very concrete) Hopf algebras, [4].

The following is the algebraic analog of the above theorem offer using the results of [1, 2, 3] and a result of Hebestreit–Steimle [5]. It makes use of an algebraic version $\mathrm{Cob}^{\mathrm{alg}}$ of a cobordism category.

Theorem 3. *There are maps*

$$\mathrm{BSp}_{2g}(\mathbb{Z}) \longrightarrow \Omega_0|\mathrm{Cob}_2^{\mathrm{alg}}| \longrightarrow \Omega_0^\infty \mathrm{GW}(\mathbb{Z})$$

of which the first one is a homology equivalence in a range depending on g and the latter map is a homotopy equivalence.

Here, $\mathrm{GW}(\mathbb{Z})$ is a Grothendieck–Witt spectrum associated to a particular Poincaré ∞-category in the sense of [1, 2, 3]. Its homotopy type away from 2 has been known for a long time, the novelty the papers [1, 2, 3] bring to the table is (among others) the following theorem.

Theorem 4. *There is a canonical map* $\mathrm{GW}(\mathbb{Z}) \to K(\mathbb{Z})^{hC_2}$ *which induces an isomorphism on 2-localised homotopy groups in positive degrees.*

One can ues this theorem to give a concrete description of the 2-localised homotopy type of $\mathrm{GW}(\mathbb{Z})$. This can in turn be exploited to deduce the following calculation, which part of a joint project with Hebestreit and Nikolaus where we calculate the stable mod 2 cohomology of various isometry groups of forms over $\mathbb{Z}$.

Theorem 5. *There is an isomorphism*

$$H^*(\mathrm{Sp}(\mathbb{Z}); \mathbb{F}_2) \cong \mathbb{F}_2[c_1, c_2, \dots] \otimes_{\mathbb{F}_2} \Lambda_{\mathbb{F}_2}(e_1, e_2, \dots)$$

where $|c_i| = 2i$ *and* $|e_i| = 4i - 1$.

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A glimpse of trisections

Delphine Moussard

Trisections are decompositions of 4–manifolds into three 1–handlebodies, introduced by Gay and Kirby in [GK16], and analogous to Heegaard splittings of 3–manifolds. Here, by 4–manifold, we mean a closed, smooth, connected and oriented 4–dimensional manifold. A *trisection* of a 4–manifold is a decomposition $X = X_1 \cup X_2 \cup X_3$ such that:

- $X_i \cong \natural^{k_i}(S^1 \times B^3)$,
- $H_{ij} = X_i \cap X_j \cong \natural^g(S^1 \times B^2)$,
- $\Sigma = X_1 \cap X_2 \cap X_3 \cong \sharp^g(S^1 \times S^1)$.

Theorem 1 (Gay–Kirby)**.** *Any* 4*–manifold admits a trisection.*

A trisection can be represented by a trisection diagram, namely the central surface Σ with three systems of curves, each being a cut system for one of the three dimensional 1–handlebodies H_{ij}. Note that ∂X_i is diffeomorphic to $\sharp^{k_i}(S^1 \times S^2)$ and comes with a Heegaard splitting $\partial X_i = H_{ij} \cup_\Sigma H_{i\ell}$. This provides the following constraint on trisection diagrams: any pair of cut system is handleslide-diffeomorphic to the one in Figure 2. Thanks to the result of Laudenbach and

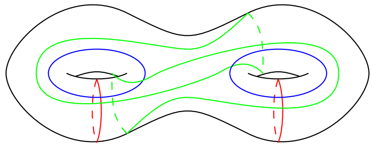

Figure 1. A trisection diagram

Poénaru [LP72] asserting that any diffeomorphism of the boundary of a 4–dimensional 1–handlebody extend to the whole handlebody, one can conclude that a trisection diagram determines a unique 4–manifold up to diffeomorphism.

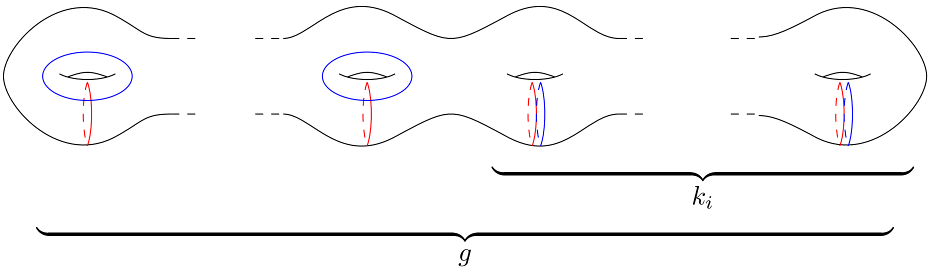

Figure 2. Genus–g Heegaard diagram for $\sharp^{k_i} S^1 \times S^2$

The simplest example of a trisection is $S^4 = B^4 \cup B^4 \cup B^4$, with trisection diagram a 2–sphere. Writing $S^1 \times S^3 = S^1 \times (B^3 \cup B^3 \cup B^3)$, one easily gets a genus one trisection for $S^1 \times S^3$, represented by the left-hand side diagram in Figure 3. A genus one trisection of $\mathbb{C}P^2$ is obtained by setting $X_i = \{|z_j| \leq |z_i|\}$, using homogenous coordinates $[z_1 : z_2 : z_3]$; the corresponding diagram is represented in Figure 3, right-hand side. A genus two trisection of $S^2 \times S^2$ can be worked out using the projection on one factor, whose diagram is that of Figure 1, see [GK16] for details.

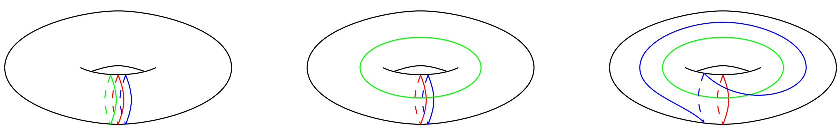

Figure 3. Genus one trisection diagrams

Given a trisection of the 4–manifold X, one can produce another by a stabilization move. Take a boundary parallel arc in one of the H_{ij} and add a tubular neighborhood of it to X_ℓ, removing it from X_i and X_j. This adds 1 to k_ℓ and g

and leaves k_i and k_j unchanged. The effect on a diagram is a connected sum with the genus one diagram in the middle of Figure 3.

Theorem 2 (Gay–Kirby)**.** *Any two trisections of a given* 4*–manifold can be made isotopic after a number of stabilizations.*

The diagrammatic counterpart of this result says that any two trisection diagrams of a given 4–manifold are related by diffeomorphisms, handleslides within each cut system and stabilizations.

Trisections naturally provide an invariant of 4–manifolds called the *trisection genus*, namely the smallest possible genus of a trisection of the given manifold. It is obvious that the only 4–manifold with trisection genus 0 is S^4. One easily checks that only $S^1 \times S^3$, $\mathbb{C}P^2$ and $\overline{\mathbb{C}P^2}$ have trisection genus 1. From genus 2, handleslides come into play, making the classification problem harder. Manifolds with trisection genus 2 have been classified by Meier and Zupan in [MZ17].

Theorem 3 (Meier–Zupan)**.** *If X admits a genus two trisection, then X is either $S^2 \times S^2$ or a connected sum of $S^1 \times S^3$, $\mathbb{C}P^2$ and $\overline{\mathbb{C}P^2}$ with two summands. Moreover, each of these* 4*–manifolds has a unique genus two trisection up to diffeomorphism.*

From genus three, it turns out there are infinitely many distinct manifolds, as established in [Mei18].

Theorem 4 (Meier)**.** *Spun lens spaces form an infinite family of* 4*–manifolds with trisection genus* 3*.*

Meier conjectured that every irreducible 4–manifold with trisection genus three is either the spin of a lens space, or a Gluck twist on a specific 2–knot in the spin of a lens space.

It is immediate that the trisection genus g_t is subadditive: for any 4–manifolds X and X', $g_t(X \sharp X') \leq g_t(X) + g_t(X')$. The additivity conjecture asserts that the equality always holds. It was pointed out by Lambert-Cole and Meier in [LCM20] that this conjecture would imply that the trisection genus is invariant under homeomorphism. As a consequence it would imply that there is no exotic S^4, nor exotic X for X any 4–manifold of trisection genus at most 2. Lambert-Cole and Meier exhibited infinitely many exotic pairs of 4–manifolds with the same trisection genus, what can be considered as a supporting evidence for the additivity conjecture.

In [MZ18], Meier and Zupan showed that embedded surfaces in a trisected 4–manifold can be put in a "good position" with respect to the trisection.

Theorem 5 (Meier–Zupan)**.** *Any smoothly embedded surface S in a trisected* 4*–manifold $X = X_1 \cup X_2 \cup X_3$ can be isotoped into a bridge position,* ie*:*

- *S meets Σ transversely in an even number of points,*
- *$S \cap X_i \cap X_j$ is a boundary parallel tangle,*
- *$S \cap X_i$ is a boundary parallel disk tangle.*

Bridge trisections of surfaces in $\mathbb{C}P^2$ are a key tool in the recent proof by Lambert-Cole of the Thom conjecture [LC20].

Theorem 6 (Thom conjecture)**.** *If S is a smoothly embedded oriented connected surface in $\mathbb{C}P^2$, of degree $d > 0$, then $g(S) \geq \frac{1}{2}(d-1)(d-2)$.*

Since equality is realized by complex curves in $\mathbb{C}P^2$, this gives the minimal genus for a representative of a given homology class in $H_2(\mathbb{C}P^2)$. The Thom conjecture was first proved by Kronheimer and Mrowka in [KM94] using gauge theory. The novelty in Lambert-Cole's proof is that it avoids gauge theory.

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Knots in overtwisted contact manifolds

Rima Chatterjee

Classification and structure problems are very important and difficult problems in knot theory. We say two knots are isotopic if one can continuously deform one knot into another. If we put extra geometric structure on knots, this classification problem becomes harder. A contact structure ξ on a 3-manifold M is a no-where integrable 2-plane field. A contact 3-manifold (M, ξ) is a 3-manifold equipped with a contact structure ξ. There are two types of contact structures– tight and overtwisted. If one can find an overtwisted disc (a disc where the contact planes are tangential along the boundary) embedded in a contact manifold, we call it overtwisted. There are two types of knots in a contact manifold– Legendrian and transverse. The classification problem of Legendrian and transverse knots are lot more finer that its topological counterparts as there are knots which are smoothly isotopic but not isotopic as Legendrian and transverse knots. The classical invariants of Legendrian knots are the Thurston–Bennequin number and the rotation number. On the other hand, transverse knots have only one classical invariant– the self-linking number. These invariants are not complete as one can find Legendrian (resp.transverse) knots with same classical invariants which are not Legendrian (resp. transverse) isotopic [8], [2].

This talk was focused on knots in overtwisted manifolds. We can find two types of knots in an overtwisted manifold–non-loose (also known as exceptional) and loose (also known as non-exceptional). Non-loose knots have tight complements, where loose knots are knots with overtwisted complements. While using Eliashberg's celebrated result on the classification of overtwisted manifolds [3], we understand the loose knots and links fairly well [5], [1], non-loose knots still remain a mystery. In this talk, we discuss a brief history of these knots and how the classification and structure problems are lot harder for this class of knots. The only known knot types whose non-loose representatives are understood, are the unknot [4] and the torus knots [9], though the classication of torus knots is a partial result. The only known non-loose link type which is completely classified is the Hopf link [10]. It is still an open question if every knot type in an overtwisted S^3 has a non-loose representative.

Next we discuss the structure problems of the non-looose knots. Topological satellite operations have always been an interesting area of study in contact geometry. While the satellite operations of Legendrian and transverse knots are well understood in the tight manifolds [7], [6], not many things are known for knots in overtwisted manifolds. We would like to know how these operations affect the geometric features of the knots. In other words, we are interested to know when these topological operations preserve non-looseness. We have examples which tell us that the non-loose knots do not behave well with cabling operation, that is cabling of two non-loose knots can produce a loose knot. The next candicate to look for is the cable of a non-loose knot. In a joint work with Etnyre, Min and Mukherjee, we could show that when the cabling slope is greater than the Thurston-Bennequin invariant of the knot, it remains non-loose. For positive cables, this is true for every non-loose knot when we ignore the Giroux torsion in the complement . For negative cables, one needs to work a little more. This is a work in progress. We still have no idea what happens to the Whitehead double of a non-loose knot.

To conclude, there are lots of open questions in this area which we do not know an answer of and thus makes this area a very interesting field of study.

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Earthquakes on the once-punctured torus

Grace Garden

Earthquakes originated in work by Thurston (see [1] for an exposition) and have proved to be useful in the study of hyperbolic manifolds and hyperbolic structures. Most notably, a hyperbolic structure on a surface can be related to any other hyperbolic structure on a surface by a unique earthquake, which was used in the proof of the Nielsen realisation problem [6].

Given a hyperbolic orientable surface S and a simple closed curve γ, the earthquake about γ is achieved by cutting the surface open along γ, twisting around γ by some distance r, and regluing. In this way, earthquakes about simple closed curves can be viewed as "fractional" Dehn twists. These were actually defined for objects called geodesic laminations (of which simple closed geodesics are an example), but for our purposes it suffices to only consider simple closed curves. Earthquakes can be considered in three different ways:

(1) as deformations of S,
(2) as maps from the Teichmüller space of S to itself,
(3) as paths in the Teichmüller space of S as r varies.

In this note, I present some results for earthquakes on the once-punctured torus $S_{1,1}$. In particular, I describe two new methods to get an explicit form of the earthquake for any simple closed geodesic. Both methods start by considering Dehn twists on $S_{1,1}$ and extending to earthquakes in the sense of (3).

The first method studies the repeated action of Dehn twists on the $\mathrm{SL}_2(\mathbb{C})$-character variety of $S_{1,1}$. By the work of Fricke and Klein (see [3, 4]) the $\mathrm{SL}_2(\mathbb{C})$-character variety is isomorphic to $\mathbb{C}^3$ by taking an element ρ of the character variety to

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \mathrm{tr}(\rho(\alpha)) \\ \mathrm{tr}(\rho(\beta)) \\ \mathrm{tr}(\rho(\alpha\beta)) \end{pmatrix}$$

where $\pi_1(S_{1,1}) = \langle \alpha, \beta \rangle$.

A component X of the character variety can be identified with Teichmüller space of $S_{1,1}$ and we focus on the action here (see [5]).

Theorem 1. *For a fixed starting point $\mathbf{v}_0 = (x_0, y_0, z_0) \in X$ the earthquake about α in trace coordinates is given by*

$$\mathcal{E}^{tr}_\alpha(\mathbf{v}_0) : \mathbb{R} \to \mathbb{C}^3$$
$$r \mapsto \begin{pmatrix} x_0 \\ y_\alpha(\mathbf{v}_0)(r) \\ y_\alpha(\mathbf{v}_0)(r-1) \end{pmatrix}$$

where

$$y_\alpha(\mathbf{v}_0) : \mathbb{R} \to \mathbb{C}$$
$$r \mapsto \left(\frac{y_0}{2} + \frac{(x_0y_0 - 2z_0)\sqrt{x_0^2-4}}{2(x_0^2-4)}\right)\left(\frac{x_0 + \sqrt{x_0^2-4}}{2}\right)^r + \left(\frac{y_0}{2} - \frac{(x_0y_0 - 2z_0)\sqrt{x_0^2-4}}{2(x_0^2-4)}\right)\left(\frac{x_0 - \sqrt{x_0^2-4}}{2}\right)^r.$$

The second method studies the action of Dehn twists on the Teichmüller space of $S_{1,1}$ using maximal collar neighbourhoods. Cooper and Pfaff introduce coordinates called triangle lengths [2], which measure the half lengths of the unique geodesics associated with $\alpha, \beta, \alpha\beta$,

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \frac{\ell(\alpha)}{2} \\ \frac{\ell(\beta)}{2} \\ \frac{\ell(\alpha\beta)}{2} \end{pmatrix}. \tag{1}$$

Theorem 2. *For a fixed starting point $\mathbf{w}_0 = (a_0, b_0, c_0) \in \mathbb{R}^3_+$, the left earthquake deformation about α in triangle lengths is given by*

$$\mathcal{E}^{\ell}_\alpha(\mathbf{w}_0) : \mathbb{R} \to \mathbb{R}^3_+$$
$$r \mapsto \begin{pmatrix} a_0 \\ b_\alpha(\mathbf{w}_0)(r) \\ b_\alpha(\mathbf{w}_0)(r-1)) \end{pmatrix}$$

where

$$b_\alpha(\mathbf{w}_0) : \mathbb{R} \to \mathbb{R}_+$$
$$r \mapsto \cosh^{-1}\left(\cosh(b_0)\cosh(ra_0) - (\cosh(c_0) - \cosh(a_0)\cosh(b_0))\frac{\sinh(ra_0)}{\sinh(a_0)}\right).$$

The two expressions for the earthquake about α align and similar results can be derived and extended to any simple closed curve γ. This provides both an algebraic and geometric interpretation of earthquakes on the once-punctured torus. I have some pictorial results for a few examples in trace coordinates, triangle lengths, in Fenchel-Nielsen length-twist coordinates, and coordinates on the simplex. This aligns with previous research on earthquakes by Waterman and Wolpert [7].

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The link between Drinfeld associators and Kashiwara-Vergne solutions

Tamara Hogan
(joint work with Zsuzsanna Dancso and Marcy Robertson)

In algebraic situations where $xy = yx$, it is well-known that $e^x e^y = e^{x+y}$. However, in the non-commutative setting, we have

$$e^x e^y = e^{x+y+\frac{1}{2}[x,y]+\frac{1}{12}[x,[x,y]]+\cdots}, \tag{1}$$

where $[x, y] = xy-yx$. The series of Lie brackets in the exponent on the right-hand side of (1) is called the Baker-Campbell-Hausdorff series of x and y and denoted $\mathrm{BCH}(x, y)$.

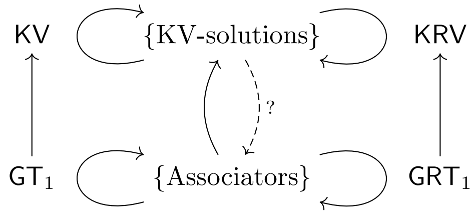
The Kashiwara-Vergne Conjecture, posed in [9] and proven in all generality in [2], posited that there always exists a pair of convergent power series $A(x, y), B(x, y)$ such that $\mathrm{BCH}(x, y)$ can be expressed as a combination of A, B and the morphisms $\mathrm{Ad}(x)$ and $\mathrm{Ad}(y)$. Solution pairs (A, B) are called Kashiwara-Vergne solutions (hereafter called KV-solutions). The collection of all KV-solutions form a bi-torsor with left (respectively right) symmetry group denoted by KV (respectively KRV).

In the quest for explicit KV-solutions, Alekseev, Enriquez and Torossian had the key observation that a unique KV-solution can be constructed from any Drinfeld associator [1] [3], of which there are known explicit examples (notably, the Knizhnik-Zamolodchikov associator).

Drinfeld associators come from Drinfeld's study of quasi-Hopf algebras and arise from a specific kind of weakening of the (co)-associativity conditions on a Hopf algebra [8]. The collection of Drinfeld associators also form a bi-torsor with left (respectively, right) symmetry group given by the Grothendieck-Teichmuller group GT_1 (respectively, the graded Grothendieck-Teichmuller group GRT_1).

Thus, the relation described by [1] and [3] gives a map of bi-torsors between Drinfeld associators and KV-solutions. This algebraic relationship is summarised by the diagram in (2).

(2)



Since KV-solutions and Drinfeld associators are both bi-torsors, being able to understand their symmetry groups would elucidate a lot about their structure. The following is an open conjecture of Alekseev and Torossian, which is of interest to myself and my co-authors.

Conjecture 1. [3, Remark 9.14] *The symmetry groups of KV-solutions are such that* $\mathsf{KV} \cong \mathsf{GT}_1 \times \mathbb{K}$ *and* $\mathsf{KRV} \cong \mathsf{GRT}_1 \times \mathbb{K}$.

An alternate perspective on all these objects is given by homomorphic expansions (alternately called universal finite type invariants).

Definition 2. [6] *Let* $\mathcal{A}$ *be an algebra with augmentation map* $\epsilon : \mathcal{A} \to \mathbb{K}$ *and augmentation ideal* $I := ker(\epsilon)$. *A homomorphic expansion of* $\mathcal{A}$ *is a map*

(3) $$f : \mathcal{A} \to gr(\mathcal{A}) := \bigoplus_{n=1}^{\infty} I^n/I^{n+1}$$

such that:

- f *respects the algebraic operations on* $\mathcal{A}$*;*
- $f|_{I^m}$ *vanishes in degrees less than* m*;*
- *the degree* m *part of* $f|_{I^m}$ *is the projection map* $I^m \to I^m/I^{m+1}$.

It was a result of Bar-Natan in [5] that Drinfeld associators are in one-to-one correspondence with specific homomorphic expansions of the category of parenthesised braids, PaB. The category of parenthesised braids consists of the union of groupoids $\bigcup_{n\in\mathbb{N}} PaB_n$. PaB_n is the groupoid with objects given by parenthesised permutations of n, PaP_n and morphisms given by the pure braid group on n strands, $\mathbb{PB}_n$. The associated graded structure of PaB, $gr(PaB)$, is isomorphic to the category of parenthesised chord diagrams, $PaCD$. This result implies that $Aut(PaB) \cong \mathsf{GT}_1$ and $Aut(PaCD) \cong \mathsf{GRT}_1$ [4].

Similarly, Bar-Natan-Dancso proved in [6] that KV-solutions are in one-to-one correspondence with specific homomorphic expansions of the circuit algebra of welded tangled foams, wF. Welded tangled foams are knotted ribbon tubes in $\mathbb{R}^4$, with the addition of 'foamed' vertices where two tubes fuse together into one (forming a Y-shaped vertex). The associated graded structure of wF, $gr(wF)$, is isomorphic to the circuit algebra of welded arrow diagrams, A^w. This result implies that $Aut(wF) \cong \mathsf{KV}$ and $Aut(A^w) \cong \mathsf{KRV}$ [7].

Given these two results and the results of [1] and [3], we can then also define a map between expansions of PaB and expansions of wF. However, the problem in fully linking together the algebraic and topological perspectives given above together is that the algebraic structures of parenthesised braids and welded tangled foams don't agree with each other. Ergo, we can't define a homomorphism from $PaB \to wF$, and it is not easy to understand the maps $\mathsf{GT}_1 \to \mathsf{KV}$ and $\mathsf{GRT}_1 \to \mathsf{KRV}$ as automorphisms of all of these algebras. There are however, natural set maps $PaB \to wF$ (given by the 'tube' map [10]) and $PaCD \to A^w$ which were explored in [6] and [7]. It is hoped we can reach an understanding of this topological side of the problem which agrees with these naturally arising set maps.

In a forthcoming paper from myself, Dancso and Robertson we provide a diagrammatic interpretation of the map $\rho : \mathsf{GRT}_1 \to \mathsf{KRV}$ from [3, Theorem 4.6], $\tilde{\rho}$, and use it to classify GRT_1 as a specific subgroup of automorphisms of arrow diagrams.

Theorem 1. *There is an automorphism of groups* $\mathsf{GRT}_1 \cong Aut_{\tilde{\rho}}(A^w)$.

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Q-forms & 4-manifolds

Csaba Nagy

We consider the problem of classifying manifolds up to diffeomorphism / h-cobordism in a given stable diffeomorphism class, in the setting of manifolds equipped with B-structures. We define the "Q-form" of a B-manifold, and present an example which shows that this invariant can distinguish stably diffeomorphic manifolds from each other. Finally we state the Q-form conjecture, which claims that the Q-form is a complete invariant of such manifolds under certain conditions.

Recall that for a fibration $B \to BO$ a B*-structure* on a manifold M is a lift $M \to B$ of the classifying map of the stable normal bundle of M (up to homotopy). We say that the $2q$-dimensional B-manifolds M and M' are *stably diffeomorphic*, if there is a (B-structure preserving) diffeomorphism between $M\#k(S^q \times S^q)$ and $M'\#k(S^q \times S^q)$ for some k (where $S^q \times S^q$ is equipped with a fixed B-structure). If B is the normal $(q-1)$-type of M, then the stable diffeomorphism type of M is determined by the bordism class of the normal $(q-1)$-smoothing $M \to B$:

Theorem (Kreck [3, Corollary 3])**.** *Let M and M' be $2q$-manifolds with the same Euler-characteristic and normal $(q-1)$-smoothings $\bar{\nu} : M \to B$ and $\bar{\nu}' : M' \to B$. If $\bar{\nu}$ and $\bar{\nu}'$ are bordant, then M and M' are stably diffeomorphic.*

We define the Q-form, a diffeomorphism invariant of B-manifolds, by adding some extra data to the intersection form:

Definition. Let M be a $2q$-manifold with a B-structure $\bar{\nu} : M \to B$. Its *Q-form* is the triple

$$E_q(M, \bar{\nu}) = (H_q(M), \lambda_M, \bar{\nu}_*)$$

where $\lambda_M : H_q(M) \times H_q(M) \to \mathbb{Z}$ is the intersection form of M and $\bar{\nu}_* : H_q(M) \to H_q(B)$ is the induced homomorphism.

For example, if M is a 4-manifold with a spin^C-structure $s : M \to BSpin^C$, then $s_* : H_2(M) \to H_2(BSpin^C) \cong \mathbb{Z}$ is the homomorphism determined by the first Chern-class $c_1(s) \in H^2(M)$ of the spin^C-structure.

Theorem (Conway-Crowley-Powell-Sixt [2])**.** *For every $k > 0$ there exist k spin^C-structures on $S^2 \times S^2$, $s_1, \dots, s_k$, such that $(S^2 \times S^2, s_i)$ and $(S^2 \times S^2, s_j)$ are stably diffeomorphic but not diffeomorphic for every $i \neq j$.*

This is proved by constructing a list of k non-isomorphic triples which are realized as Q-forms of spin^C-structures on $S^2 \times S^2$ which are all bordant to each other over their common normal 1-type (hence stably diffeomorphic). This shows that the Q-form is capable of distinguishing stably diffeomorphic manifolds.

In fact, Crowley's Q-form conjecture [1, Problem 11] claims that in the setting of Kreck's theorem, if the Q-forms of the manifolds are isomorphic, then the manifolds are h-cobordant (and hence diffeomorphic if $q > 2$):

Conjecture (Q-form conjecture for simply-connected manifolds with q even)**.** *Let M and M' be simply-connected $2q$-manifolds with normal $(q-1)$-smoothings $\bar{\nu} : M \to B$ and $\bar{\nu}' : M' \to B$ for some even q. If $\bar{\nu}$ and $\bar{\nu}'$ are bordant and $E_q(M, \bar{\nu}) \cong E_q(M', \bar{\nu}')$, then M and M' are h-cobordant (over B).*

Theorem (Nagy [4])**.** *The Q-form conjecture holds if $H_q(B)$ is torsion-free.*

Remark. In this special case a stronger statement holds: any normal bordism between $\bar{\nu}$ and $\bar{\nu}'$ is bordant (rel boundary) to an h-cobordism.

This is proved by defining a new surgery obstruction, associated to a normal bordism between $\bar{\nu}$ and $\bar{\nu}'$, and showing that if $E_q(M, \bar{\nu}) \cong E_q(M', \bar{\nu}')$, then the obstruction is elementary.

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Concordances in (non-orientable 3-manifold) $\times$ [0, 1]

Benjamin Matthias Ruppik

(joint work with Michael Klug)

Let RHT $= 3_1$ denote the right-handed trefoil knot, and RHT#RHT the connected sum of a right-handed trefoil with itself. The trefoil knot is chiral, which means that it is not isotopic to its mirror image, the left handed trefoil LHT. We will watch a "movie" (Figure 1) taking place in a non-orientable 3-manifold N^3, which describes an interesting bounded punctured torus in the 4-manifold $N^3 \times [0, 1]$.

In the movie, reading the slices from top to bottom, the first frame describes the connected sum RHT#RHT sitting as a small local knot in the top boundary $N^3 \times \{0\}$. We see a saddle splitting the two summands apart. Now one of the summands stays put while the other goes on a journey around an orientation reversing loop in the non-orientable 3-manifold N^3. On returning, this summand has changed into the mirror image, a left-handed trefoil, which joins the right-handed trefoil at a fusion saddle to form the connected sum RHT#LHT. After another saddle, we are left with an unlink of two components, which vanish at two minima at the very bottom.

From lower bounds on the 4-genus, for example the Levine-Tristram signatures, we know that any smooth[1] properly embedded surface in $\mathbb{S}^3 \times [0, 1]$ with boundary RHT#RHT $\subset \mathbb{S}^3 \times \{0\}$ needs to be of genus at least 2. We will say that the $(\mathbb{S}^3 \times [0, 1])$-genus of the connected sum of a right-handed trefoil with itself is

$$g^{\text{smooth}}_{\mathbb{S}^3 \times [0,1]}(\text{RHT\#RHT}) = 2.$$

The movie figure proves the following theorem, which states that there exist more genus-efficient surfaces if our ambient 3-manifold has interesting topology.

Theorem 1. *For any* ***non-orientable*** *3-manifold N^3, there is an example of a local knot $K \subset \mathbb{D}^3 \subset N^3$ for which the $(N^3 \times [0, 1])$-genus is strictly less than the $(\mathbb{S}^3 \times [0, 1])$-genus. We can take $K =$ RHT#RHT:*

$$g^{\text{smooth}}_{N^3 \times [0,1]}(\text{RHT\#RHT}) \leq 1$$

[1] This lower bound is also true if 'smooth' is replaced by 'locally flat', but we will not discuss the difference between the Smooth and Topological category in this abstract and stay in the smooth world.

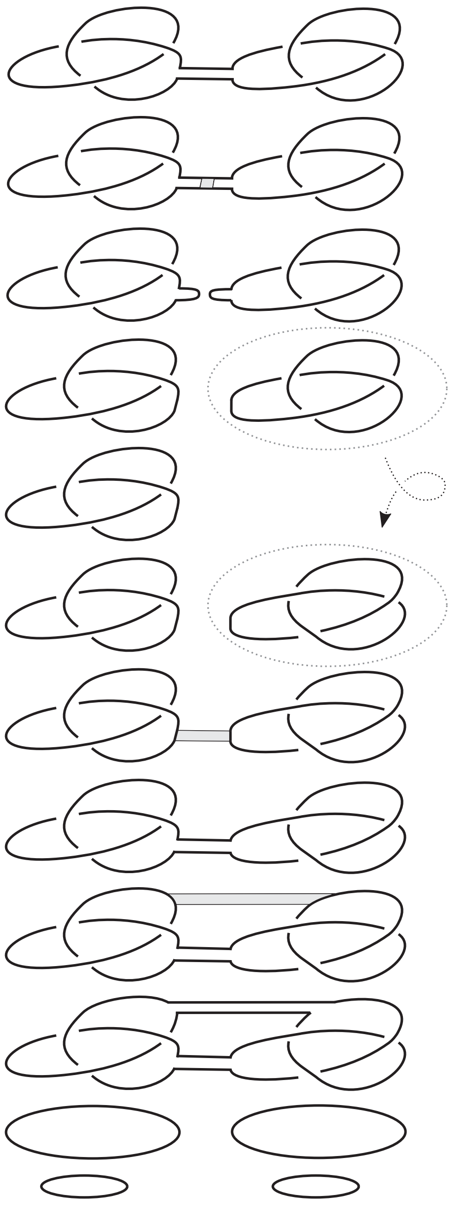

FIGURE 1. Punctured genus 1 surface with boundary RHT#RHT properly embedded in $N^3 \times [0,1]$, where N^3 is a non-orientable 3-manifold. The loopy arrow indicates that one of the trefoil summands travels around an orientation reversing loop in N^3.

In joint work with Michael Klug [4], we investigate the sliceness of knots in collars $M^3 \times [0,1]$ of boundaries $M^3 = \partial X^4$ of 4-manifolds. In particular, we study the difference between *shallow slice* knots (those knots which are slice in a collar of the boundary) and *deep slice* knots (those where every slice disk need to go deep into the 4-manifold X^4). Our main result is that every 4-manifold build from a 4-ball by attaching a non-zero number of 2-handles contains null-homotopic deep slice knots in its boundary.

Question 1. *Is there an example of the phenomenon of the theorem in an* ***orientable*** *3-manifold? In other words, can we use the "additional topology" of an orientable 3-manifold for finding more efficient cobordisms between local knots?*

The question is interesting in the context of the following theorem of Boden-Nagel, which depends on embedding the universal cover of a punctured 3-manifold into the 3-sphere and a lifting argument.

Theorem 2 (Boden-Nagel, [1])**.** *For a local knot $K \subset \mathbb{D}^3 \subset M^3$ in an orientable 3-manifold M^3, sliceness in $M^3 \times [0,1]$ implies sliceness in $\mathbb{S}^3 \times [0,1]$.*

Further work investigating concordances in general 3-manifolds are [2] and [3].

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Concordance of positive braid knots

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This talk is concerned with the following question.

Question 1. *Are concordant positive braid knots isotopic?*

We study *knots* in the 3-sphere S^3, i.e. non-empty, connected, oriented, closed, smooth 1-dimensional submanifolds of S^3, considered up to ambient isotopy. Two knots K and J are called *concordant* if there exists an annulus $A \cong S^1 \times [0,1]$ smoothly and properly embedded in $S^3 \times [0,1]$ such that $\partial A = K \times \{0\} \cup J \times \{1\}$ and such that the induced orientation on the boundary of the annulus agrees with the orientation of K, but is the opposite one on J. Knots up to concordance form a group, the *concordance group* $\mathcal{C}$, with the group operation induced by connected sum. A knot is concordant to the unknot if and only if it is *slice*, i.e. if it bounds a smoothly embedded 2-dimensional disk D^2 in B^4, the 4-ball bounded by S^3.

Clearly, isotopic knots are concordant. The converse is in general not true as any nontrivial slice knot shows. For example, for any nontrivial knot K the knot

$K\# -K$ is slice. Here $-K$ denotes the *inverse* of K in $\mathcal{C}$, the image of K under an orientation-reversing diffeomorphism of S^3 with the opposite orientation.

However, it was shown by Litherland [6] that *algebraic knots*, which are knots of isolated singularities of complex algebraic plane curves, are isotopic if they are concordant. This naturally leads to Question 1 when looking at the following set of inclusions. We have

$$\{\text{positive torus knots}\} \subset \{\text{algebraic knots}\} \subset \{\text{positive braid knots}\}.$$

Note that the torus knots $T_{p,q}$ for coprime positive integers p and q are obtained as knots associated to the singularity $z^p - w^q = 0$ for $z, w \in \mathbb{C}$. Algebraic knots are certain iterated cables of torus knots and they are known to be *positive braid knots*, i.e. they can be obtained as closures of positive braids.

By a fundamental theorem of Alexander [1], every knot in S^3 can be represented as the closure of an n-braid for some positive integer n. Here, an *n-braid* is an element of the *braid group on n strands*, denoted B_n, whose classical presentation with $n-1$ generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2, \text{ and } \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$$

was introduced by Artin [2]. A *positive braid* is an element of the braid group B_n for some n that can be written as a positive braid word $\sigma_{s_1}\sigma_{s_2}\cdots\sigma_{s_l}$ with $s_i \in \{1, \ldots, n-1\}$. The set of positive braid knots is a subset of the set of *positive knots*, i.e. the knots that admit a diagram with only positive crossings.

Let $\mathrm{wr}(\gamma)$ denote the *writhe* of a braid $\gamma \in B_n$ for some $n > 0$, i.e. the exponent sum of the word γ in the generators $\sigma_1, \ldots, \sigma_{n-1}$. If γ is a positive braid such that its closure $K = \widehat{\gamma}$ is a knot, then, by work of Bennequin [4] and Rudolph [7] — the latter building on Kronheimer and Mrowka's proof of the local Thom conjecture [5] — we have

$$g_4(K) = g(K) = \frac{\mathrm{wr}(\gamma) - n + 1}{2}. \tag{1}$$

Here $g(K)$ denotes the 3-*genus of* K, the minimal genus of a compact, connected, oriented smooth surface in S^3 with oriented boundary the knot K, and $g_4(K)$ denotes the 4-*genus* of K, the minimal genus of a compact, connected, oriented surface smoothly embedded in B^4 with oriented boundary the knot K in $S^3 = \partial B^4$. A corollary of Equation (1) is that there can be only finitely many positive braid knots in each concordance class in $\mathcal{C}$. In fact, by a result of Baader, Dehornoy and Liechti [3] this is true in more generality: every concordance class in $\mathcal{C}$ contains at most finitely many isotopy classes of positive knots. For positive braid knots, this follows by combining Equation (1) with the facts that the 4-genus is a concordance invariant for positive braid knots and that the writhe of a positive braid γ equals the number of generators in the corresponding braid word and is linearly bounded from below by twice the *positive braid index* of $\widehat{\gamma}$ — the minimal number of strands among the positive braid representatives of $\widehat{\gamma}$. The question whether there is indeed only one isotopy class of positive braid knots of fixed braid index in each concordance class remains open. We are particularly interested in this question when the braid index is fixed to be 3, the first interesting case.

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