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Convex Geometry and its Applications (hybrid meeting)

Organized by
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Abstract. The geometry of convex domains in Euclidean space plays a central role in several branches of mathematics: functional and harmonic analysis, the theory of PDE, linear programming and, increasingly, in the study of algorithms in computer science. The purpose of this meeting was to bring together researchers from the analytic, geometric and probabilistic groups who have contributed to these developments.

Mathematics Subject Classification (2010): 52Axx (68Q25, 60D05).

Introduction by the Organizers

The meeting *Convex Geometry and its Applications*, organized by Shiri Artstein-Avidan, Franck Barthe, and Monika Ludwig, was held from December 12 to December 18, 2021, in hybrid format. It was attended by 63 participants working in all areas of convex geometry (24 of them were present in Oberwolfach, and 39 attended online). Of these 15% were female and more than 20% were younger participants. There were 10 plenary lectures of 45 minutes duration and 21 shorter lectures of 20 minutes; 4 of the longer lectures and 9 short lectures were given by online participants. Two of the participants, Sophie Huiberts and Chiara Meroni, acted as Video Conference Assistants.

The lectures illustrated the diversity of research activities in the field, from theoretical aspects to applications. Among the main topics, we can list the study of geometric inequalities (including Brunn–Minkowski theory, isoperimetric inequalities), classification of valuations, stochastic geometry, high dimensional convex geometry and its probabilistic approaches, differential geometry, convex algebraic geometry, combinatorial geometry, algorithmic problems but also applications to harmonic analysis, optimal transport or statistical physics. Some highlights of the program were as follows.

The opening lecture, by Artem Zvavitch, was devoted to new relationships between volumes of sumsets. The topic is at the heart of the Brunn–Minkowski theory, and offers many challenging open questions. One of them asks about the monotonicity in m of the volume of the Minkowski average of m copies of a compact set. This property is obvious for convex sets, but may fail for general sets in high dimensions. Zvavitch provided more examples where the property holds. Following the analogy between volume of sets and entropy of random variables, he addressed and proved several new inequalities for sumsets which correspond to inequalities in information theory.

Yuansi Chen presented an improved bound on the spectral gap of convex sets and log-concave probability measures, which almost matches the famous conjecture of Kannan–Lovász–Simonovits (Chen's bound still involves a constant depending on the dimension, but its growth is slower than any power). This result has very important consequences, and improves the known results about the hyperplane conjecture and the variance conjecture. He gave a very pedagogic presentation of the strategy of proof, which is based on Eldan's stochastic localization scheme. Nevertheless the limitations and the actual reach of the methods are still not well understood, so that further improvements are still plausible.

The topic of classification of valuations, although a very classical one, is still moving fast. Andrea Colesanti presented a complete classification of continuous, epi-translation and rotation invariant valuations on the set of super-coercive convex functions, which involves singular Hessian measures. This analogue of Hadwiger's classification for valuations on sets was proved with Monika Ludwig and Fabian Mussnig; it was motivated by the search of a natural extension of the notion of mixed volumes from sets to functions. In a different direction, Thomas Wannerer presented the recent progress in the study of translation-invariant valuations from the viewpoint of representation theory, as initiated by Semyon Alesker. This includes in particular a detailed study of highest-weight vectors. As an application, analogues of the Hodge–Riemann relations were conjectured and proved in special cases by Jan Kotrbatý. In his lecture, Semyon Alesker showed that these statements imply a whole new family of inequalities involving mixed volumes.

An active direction of research investigates improved Brunn–Minkowski type inequalities when restricting attention to origin-symmetric convex sets. In his lecture, Liran Rotem presented a joint result with Dario Cordero-Erausquin, which establishes an improved log-concavity property of log-concave rotation invariant measures, for dilates of a symmetric convex sets (known as the B-inequality). The proof relies on Hörmander's L^2 method and uses, in a tricky way, several Poincaré type inequalities (one of them being an infinitesimal version of the Brunn–Minkowski inequality with weights, put forward by Emanuel Milman and Alexander Kolesnikov). In his lecture Emanuel Milman announced an isomorphic local solution to the main conjecture in this direction which is due to Böröczky–Lutwak–Yang–Zhang and known as the log-Brunn–Minkowski inequality: close to every symmetric convex body, one can find another one, satisfying an appropriate improved Poincaré inequality for even functions.

Many models of stochastic geometry consider the approximation of a convex set by the convex hull of many random points (for instance, the lecture of Pierre Calka gave very precise asymptotics for its facets of maximal area). The behaviour of random simplices inside a convex set was also thoroughly investigated. Somewhat surprisingly, the convex hull of less random points than the dimension was not well studied. In his talk, Christoph Thäle considered the average length of the segment formed by two independent random points, uniformly distributed on a convex body $K \subset \mathbb{R}^d$. He showed optimal bounds on this expectation, for bodies K satisfying an appropriate size condition.

Elisabeth Werner gave a lecture about multiset or multifunctional versions of the classical Blaschke–Santaló inequality, which asserts that the product of the volume of an origin symmetric convex body with the volume of its polar body is maximal when the body is an ellipsoid. Various extensions were proved in her joint work with Alexander Kolesnikov in the case of several unconditional functions. One instance of the inequality turns out to be dual to a new fundamental inequality relating transportation cost and Gaussian entropy, in the case of Wasserstein barycenters of several measures. In the case of just two measures, this recovers a recent inequality of Max Fathi, refining the famous transport-entropy inequality of Talagrand. Katarzyna Wyczesany discussed measure transportation with respect to non standard costs, and proved a new Rockafellar-type results for such costs. Tomasz Tkocz illustrated in his talk the close connection of Khinchine type inequalities to volumes of hyperplane sections of the cube. He proved new sharp inequalities for negative moments, which recover optimal cube sections, and he showed how to derive strong stability results when optimal sections of ℓ_p balls are known.

In Problem 19 of the Scottish Book, Ulam asked whether a convex body, that floats in equilibrium in any orientation in a liquid of twice its density, has to be a Euclidean ball. A decade later, Rolf Schneider gave a positive answer for bodies having a center of symmetry. In the closing lecture, Dmitry Ryabogin constructed a non-symmetric body of revolution (actually a perturbation of the ball) that does float in equilibrium in any direction, thereby giving a negative solution to Ulam's problem.

Many more exciting results were presented in other talks, for instance Ronen Eldan showed how ideas from convex geometry yield a simpler and more general proof of chaos for p-spins models. On the computational side, Mark Rudelson proved good approximation bounds for a deterministic algorithm for computing the volume of polytopes given by inequalities. Chiara Meroni gave a sample of results and open question about semialgebraic convex sets, and Stephanie Mui presented her results on the solution of the L_p Alexandrov problem. We refer to the following research reports for more details.

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Workshop (hybrid meeting): Convex Geometry and its Applications

Table of Contents

Abstracts

Sumset estimates in convex geometry

Artem Zvavitch
(joint work with Matthieu Fradelizi, Mokshay Madiman)

Sumset estimates, which provide bounds on the cardinality of sumsets of finite sets in a group, form an essential part of the toolkit of additive combinatorics. In this talk we presented a number of inequalities in Convex Geometry inspired by classical sumsets estimates. We also discussed the connections of those inequalities to tools used in information theory. In particular, we explored sharp constants in the convex geometry analogues of Plünnecke-Ruzsa inequalities. This analog was proposed by S. Bobkov and M. Madiman [2] who developed a technique for going from entropy to volume estimates and back, by using certain reverse Hölder inequalities and, in particular, proved that

$$|A+B+C||A| \le 3^n |A+B||A+C|,$$

where A, B, C are convex, compact sets in $\mathbb{R}^n$ and by $|A|$ we denote the volume of a set A. To study the best constant in the above inequality we define

$$c(A,B,C) = \frac{|A||A+B+C|}{|A+B||A+C|} \text{ and } c_n = \sup_{A,B,C\subset\mathbb{R}^n} c(A,B,C).$$

We proved that

(1) $c_2 = 1$, $c_3 = 4/3$ and $c(3/2)^{n/4} \le c_n \le 2^{n-2}$.

(2) $c(B_2^n, B, C) \le 1$, B is a zonoid (a limit of zonotopes which are finite Minkowski sum of segments) and C is any convex compact set. Actually, it is the same as $c(\mathcal{E}, B, C) \le 1$, $\mathcal{E}$ is an ellipsoid, B is a zonoid and C is any convex set. This gives a partial answer to T. Courtade's question, who asked if $c(B_2^n, B, C) \le 1$, for any (convex) B, C and Euclidean ball $B_2^n = \{x \in \mathbb{R}^n : |x| \le 1\}$.

(3) $c_n(\Delta, B, C) \le 1$, where Δ is a simplex and $n = 2, 3, 4$.

We should note that T. Nayar and P. Tkocz also independently obtained upper and lower bounds on c_n. To prove the above results we have connected convex geometry analogues of Plünnecke-Ruzsa inequalities to the local versions of Alexandrov's inequality. Those inequalities can be traced back to the classical work of W. Fenchel [3] who proved that

$$|A|V(A[n-2], B, C) \le 2V(A[n-1], B)V(A[n-1], C),$$

for any convex compact sets A, B, C in $\mathbb{R}^n$. The inequality was further generalized in [1, 4, 5]. The inequality turned out to be a part of reach class of Bezout inequalities proposed in [5]. In particular, J. Xiao [6] proved that

$$|A|V(A[n-j-m], B[j], C[m]) \le \min\left(\binom{n}{j}, \binom{n}{m}\right) V(A[n-j], B[j])V(A[n-m], C[m]).$$

We also discussed a connection of Plünnecke-Ruzsa inequalities to the inequalities of the volume of orthogonal projections of a convex bodies, in particular to a local Loomis-Whitney type inequality, which allowed us to show the lower bound for the constant c_n.

We presented a weaker version of Plünnecke-Ruzsa inequalities which has an optimal constant: If A, B, C are convex bodies in $\mathbb{R}^n$, then

$$|A||B+C| \leq |A+B||A+C|.$$

This inequality was previously proved in [2] with additional multiplicative constant 2^n. Finally, we discussed an application of the above inequality to a convex geometry analog of the Ruzsa triangle inequality: For convex bodies A, B, C in $\mathbb{R}^n$,

$$|A||B-C| \leq \frac{1}{2^n}\binom{2n}{n}|A-B||A-C|.$$

Moreover in planar case, we were able to find the optimal constant in the above inequality by proving that for any planar, convex bodies A, B, C we have

$$|A||B-C| \leq |A-B||A-C|.$$

One of the main steps in the proof of the above inequality is the following inequality of the Rogers-Shephard type which seems to have interest on its own: Consider two convex sets A, C in $\mathbb{R}^2$, then

$$V(A,-C) \leq V(A,C) + \sqrt{|A||C|},$$

the equality in above inequality is only possible in the following cases

- One of the sets A or C is a singleton or a segment and another is any convex body.
- A is a triangle and $C = tA + b$, for some $t > 0$ and $b \in \mathbb{R}^2$.

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Fluctuations of random convex hulls

Pierre Calka
(joint work with J. E. Yukich)

We consider the convex hull K_λ of the point set constituted as the intersection of a homogeneous Poisson point process of intensity $\lambda > 0$ in $\mathbb{R}^d$ with a smooth convex body K of $\mathbb{R}^d$. We assume that K has a $\mathcal{C}^2$ boundary with positive Gaussian curvature. We are interested in estimating the so-called maximal radial and longitudinal fluctuations of the random polytope. In other words, we investigate the deviation of the convex envelop from the boundary of the mother body K, i.e. the Hausdorff distance between the two, and the maximal area of the facets of K_λ. The problem is inspired by recent works on the fluctuations of interfaces of several two-dimensional random growth models used in statistical physics such as the random cluster model, the polynuclear growth model or several constrained random walks, see e.g. the survey [3]. In most cases, the radial and longitudinal fluctuations are proved to grow like $\ell^{\frac{1}{3}}$ and $\ell^{\frac{2}{3}}$ respectively when the area ℓ^2 inside the random interface goes to infinity, a property which is shared by the random polytope λK_λ in dimension two. Additionally, our model makes it possible to derive an explicit expansion of the maximal fluctuations up to a term which converges in distribution to a Gumbel distribution. To the best of our knowledge, this kind of property has been unreachable up to now for the classical random growth models cited above.

More precisely, we denote by $d_H(K, K_\lambda)$ the Hausdorff distance between K and K_λ and by $MFA(K_\lambda)$ the maximal facet area of K_λ. The quantity $d_H(K, K_\lambda)$ has been studied notably by Bárány [1] and Bräker, Hsing and Bingham [2]. In [1], the mean Hausdorff distance is proved to be $\Theta\left(\frac{\log(\lambda)}{\lambda})^{\frac{2}{d+1}}\right)$ where $f(\lambda) = \Theta(g(\lambda))$ means that the ratio $\frac{f}{g}$ is bounded from below and from above. In [2], the convergence to the Gumbel distribution is obtained in the planar case. To the best of our knowledge, the variable $MFA(K_\lambda)$ has not been considered before in the literature. Our main results are described below.

(i) When λ goes to infinity, we obtain in the cases ($d = 2$, K is any smooth convex body) and ($d \geq 3$, K is any ball of $\mathbb{R}^d$) that

$$d_H(K, K_\lambda) = \lambda^{-\frac{2}{d+1}}(a_0(a_1 \log \lambda + a_2 \log(\log \lambda) + a_3 + \xi_\lambda))^{\frac{2}{d+1}}$$

where ξ_λ converges in distribution to a Gumbel distribution, i.e.

$$\lim_{\lambda\to\infty} \mathbb{P}(\xi_\lambda \leq t) = e^{-e^{-t}}, \quad t \in \mathbb{R}.$$

(ii) When λ goes to infinity, we obtain for any $d \geq 2$ and any smooth convex body K that

$$MFA(K_\lambda) = \lambda^{-\frac{d-1}{d+1}}(b_0(b_1 \log \lambda + b_2 \log(\log \lambda) + b_3 + \psi_\lambda))^{\frac{d-1}{d+1}}$$

where ψ_λ converges in distribution to a Gumbel distribution.

Here a_i and b_i, $0 \leq i \leq 3$, are explicit constants which depend on dimension d and also on the convex body K in the cases $i = 0$ and $i = 3$.

Additional results concern the location and shape of the facet which reaches the maximal area (resp. the maximal distance to the boundary): we prove that the Gauss curvature at the point on the boundary of K which is the closest to the facet with the maximal area (resp. maximal distance to the boundary) converges to the minimum (resp. maximum) of the Gauss curvature along the boundary of K. Moreover, we obtain an explicit limit distribution for the location of that closest boundary point. Finally, in the case of the facet with the maximal area, we prove that up to affine transformation, its shape converges to the shape of a regular simplex.

Our method relies notably on the introduction and study of a so-called typical facet of K_λ, i.e. a facet *chosen uniformly at random*. In particular, explicit asymptotics for the distribution of the distance to the boundary, the area and the diameter of the typical facet are some of our intermediary results which constitute the basis for deriving the extreme value convergences listed above.

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Funk geometry and projective invariance

Dmitry Faifman

The Funk metric is the lesser-known cousin of the Hilbert metric in the interior of a convex body. Unlike the Hilbert metric, the Funk metric is not a projective invariant, but as it turns out it comes close, in particular, its Holmes-Thompson volume is a projective invariant. I will discuss several questions in Funk geometry which generalize some well-known theorems and conjectures in convex geometry, such as the Blaschke-Santalo inequality, the Mahler conjecture, and the Santalo point of a body, where projective invariance plays a role. Then, motivated by projective invariance, we will attempt to define the regularized total Funk volume of a convex body, obtaining a quantity reminiscent of the O'Hara conformal energy of a knot.

1. Introduction

1.1. **Background.** Let $K \subset \mathbb{R}^n$ be a convex body with non-empty interior. The *Funk metric* is a non-symmetric distance on the interior of K, given by $d_K^F(x, y) = \log \frac{|xz|}{|yz|}$, where z is the intersection point with ∂K of the ray emanating from x towards y.

Equivalently, and more naturally, one can define the Funk metric as the non-reversible Finsler metric on $\mathrm{int}(K)$ whose unit tangent ball at $x \in \mathrm{int}(K)$ is simply K with x at its origin.

The more well-studied *Hilbert metric* is the symmetrization of the Funk metric, yielding the distance function $d_K^H(x,y) = \frac{1}{2}\log\frac{|xz||wy|}{|wx||yz|}$, where w is the other intersection point of the line through x, y with ∂K.

Both metrics are projective, that is straight segments are geodesic. The Funk metric is clearly affinely invariant by construction, while the explicit formula for the Hilbert distance reveals it is projectively invariant.

We will consider the outward metric balls in the Funk metric, namely

$$B_K^F(q,r) := \{x \in \mathrm{int}(K) : d_K^F(q,x) \le r\} = q + (1-e^{-r})(K-q).$$

For a thorough introduction to Funk and Hilbert geometries, see [11]. Some of the results presented below appeared in a preprint by the author [4]. Others are part of a joint ongoing collaboration with C. Vernicos and C. Walsh.

1.2. **Volume in Funk geometry.** We will be making use of the Holmes-Thompson definition of volume. For the Funk metric, the volume density is $\frac{1}{\omega_n}|K^x|dx$, where $\omega_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$, and K^x the polar body of K with respect to x.

We examine the volume of balls, $\mathrm{vol}_K^F(B_K^F(q,r))$. Consider first its asymptotics.

As $r \to 0$, $\mathrm{vol}_K^F(B_K^F(q,r)) \sim \omega_n^{-1}|K \times K^q|r^n$, which is essentially the volume product, or the Mahler volume, with respect to q.

When $r \to \infty$, it is a result of Berck-Bernig-Vernicos [1] (which they prove in the setting of the Hilbert metric) that if K is sufficiently regular, e.g. C^2 and strictly convex, then $\mathrm{vol}_K^F(B_K^F(q,r)) \sim c_n\Omega_n(K,q)e^{\frac{n-1}{2}r}$ for a certain numerical constant c_n, where $\Omega_n(K,q)$ is the centro-affine surface area of K with respect to q.

2. Projective invariance

Surprisingly, it turns out that the Funk metric is almost projectively-invariant.

Theorem 1. *(F.) Let $g : \mathbb{RP}^n \to \mathbb{RP}^n$ be a projective map, and assume $g(K) \subset \mathbb{R}^n$. Denoting by ϕ_K^F the Funk Finsler norm on* $\mathrm{int}(K)$, *it holds that $g^*\phi_{gK}^F - \phi_K^F \in C(TK)$ is an exact 1-form.*

This readily implies that the volume in Funk geometry is projectively invariant.

Further evidence of the projective nature of the Funk metric, which is not shared by the Hilbert metric, is manifested through projective duality. Consider $K \subset \mathbb{RP}^n$, and let $K^\vee$ denote the polar body of K, which lies in the dual projective space.

Theorem 2. *(F.) If $K \subset L$ are convex bodies in $\mathbb{RP}^n$, then* $\mathrm{vol}_L^F(K) = \mathrm{vol}_{K^\vee}^F(L^\vee)$.

3. Santalo point and volume extremals

The Santalo point of a convex body is the unique point x in its interior which minimizes the volume $|K^x|$. This statement admits the following extension.

Theorem 3. *(F.-Vernicos-Walsh, in progress) For any convex body K and any $0 < r < \infty$ there is a unique point $q = S(K, r) \in \mathrm{int}(K)$ minimizing $\mathrm{vol}_K^F(B_K^F(q, r))$.*

The classical Santalo point corresponds to infinitesimal radius. The proof in the general case relies on the projective invariance of the Funk volume.

The Funk volume $M_r(K)$ of the r-ball centered at $S(K, r)$ is now an affine invariant of K for every r. It is then natural to look for its extremals.

3.1. The upper bound.

Examining the endpoints $r \to 0, \infty$, one is led to conjecture that the maximum of $M_r(K)$ is attained by ellipsoids, which is the Blaschke-Santalo inequality for $r \to 0$, and the centro-affine isoperimetric inequality [9] for $r \to \infty$. We prove this under an additional symmetry assumption.

Theorem 4. *(F.) $M_r(K)$ is uniquely maximized by ellipsoids among unconditional convex bodies.*

Curiously, this leads to a new proof of the Colbois-Verovic volume entropy conjecture in Hilbert geometry [3, 13, 14], albeit only in the unconditional case.

3.2. The lower bound.

A more difficult question is the minimization of $M_r(K)$. A natural conjecture is that the minimum is attained by the simplex in general, and by Hanner polytopes when $K = -K$. Indeed one can easily verify that all Hanner polytopes have the same value of $M_r(K)$.

When $r \to 0$ this is the famous Mahler conjecture, which remains open in general, although several special cases are known [6, 7, 12]. Curiously, the $r \to \infty$ case is resolved in the general K case, as Vernicos and Walsh have shown in [14] that $M_r(K) \sim c_n \mathrm{Flag}(K) r^n + o(r^n)$ as $r \to \infty$ when K is a polytope (otherwise, the growth rate is super-polynomial). Here $\mathrm{Flag}(K)$ is the full flag number of K, that is the number of complete flags formed by its faces, which is clearly minimized by the simplex. In the symmetric case, the minimum of $\mathrm{Flag}(K)$ is conjectured by Kalai [8] to be attained by Hanner polytopes.

Theorem 5 (F.-Vernicos-Walsh, in progress). *Among unconditional bodies K, $M_r(K)$ is minimized by Hanner polytopes for all $0 < r < \infty$.*

4. The total volume

The centro-affine area of K can be viewed as a regularization of the total Funk volume of K as it is exhausted by metric balls of increasing radius. However, this regularization has two unpleasant properties: it depends on the centerpoint, and it is not projectively invariant. Let us propose a different regularization.

We identify $K \subset \mathbb{RP}^n$ with a geodesically convex set on the Euclidean sphere $S^n \subset \mathbb{R}^{n+1}$ in the obvious way, and $K^\vee \subset S^n$ is the polar set.

For $A \subset \mathrm{int}(K)$, one can check that $\int_{A \times K^\vee} \langle x, \xi \rangle^{-(n+1)} dx d\xi$ is proportional to the Funk volume of A. This suggests the following definition.

Definition 1. The Beta function of K is $B_K(z) = \int_{K\times K^\vee} \langle x, \xi\rangle^z dx d\xi$, which is well-defined for $z \in \mathbb{C}$ with sufficiently large real part.

It is reminiscent of the Beta function of a knot introduced by Brylinski [2], which is related to the O'Hara conformal energy of a knot [5, 10]. We are interested in the behavior of $B_K(z)$ near $z = -(n+1)$.

Theorem 6. *(F.) Assume K is C^∞-smooth and strictly convex. Then*

(1) *$B_K(z)$ admits a meromorphic extension to $\mathbb{C}$, with possible simple poles at $z = -\frac{n+3+2k}{2}$, $k \geq 0$.*
(2) *For even n, $B_K(-n-1)$ is a projective invariant of K. For odd n, the residue $\mathrm{Res}_{-n-1} B_K(z)$ is a projective invariant.*

It would be interesting to understand the extremals of those quantities.

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Asymptotic Bounds on the Combinatorial Diameter of Random Polytopes

Sophie Huiberts
(joint work with Gilles Bonnet, Daniel Dadush, Uri Grupel, Galyna Livshyts)

The combinatorial diameter diam(P) of a polytope $P \subset \mathbb{R}^n$ is the minimum number k such that any pair of vertices can be connected by k edges. We provide upper and lower bounds on the combinatorial diameter of a random "spherical" polytope, which is tight to within one factor of dimension when the number of inequalities is large compared to the dimension. More precisely, for an n-dimensional polytope P defined by the intersection of m i.i.d. half-spaces whose supporting normals are chosen uniformly from the sphere, we show that diam(P) is $\Omega(nm^{\frac{1}{n-1}})$ and $O(n^2 m^{\frac{1}{n-1}} + n^5 4^n)$ with high probability when $m \geq 2^{\Omega(n)}$.

For the upper bound, we first prove that the number of vertices in any fixed two-dimensional projection sharply concentrates around its expectation when m is large, where we rely on the $\Theta(n^2 m^{\frac{1}{n-1}})$ bound on the expectation due to Borgwardt [Math. Oper. Res., 1999]. To obtain the diameter upper bound, we stitch these "shadows paths" together over a suitable net using worst-case diameter bounds to connect vertices to the nearest shadow. For the lower bound, we first reduce to lower bounding the diameter of the dual polytope P°, corresponding to a random convex hull, by showing the relation $\text{diam}(P) \geq (n-1)(\text{diam}(P^\circ) - 2)$. We then prove that the shortest path between any "nearly" antipodal pair vertices of P° has length $\Omega(m^{\frac{1}{n-1}})$. This obtains the following result.

Theorem 1. *Suppose that $n, m \in \mathbb{N}$ satisfy $n \geq 2$ and $m \geq 2^{\Omega(n)}$. Let $A^\mathsf{T} := (a_1, \ldots, a_M) \in \mathbb{R}^{n \times M}$, where M is Poisson distributed with $\mathbb{E}[M] = m$, and $a_1, \ldots, a_M$ are sampled independently and uniformly from $\mathbb{S}^{n-1}$. Then, letting $P(A) := \{x \in \mathbb{R}^n : Ax \leq 1\}$, with probability at least $1 - m^{-n}$, we have that*

$$\Omega(nm^{\frac{1}{n-1}}) \leq \text{diam}(P(A)) \leq O(n^2 m^{\frac{1}{n-1}} + n^5 4^n).$$

1. Approximate Lower Bound

We leave the details on the result to the full paper. Here we sketch a simpler lower bound. We first reduce to lower bounding the diameter of the polar polytope $P(A)^\circ = Q(A)$, where we show that $\text{diam}(P(A)) \geq (n-1)(\text{diam}(Q(A)) - 2)$. This relation holds as long as $P(A)$ is a simple polytope containing the origin in its interior (which holds with probability $1 - 2^{-\Omega(m)}$). To prove it, we show that given any path between vertices v_1, v_2 of $P(A)$ of length D, respectively incident to distinct facets F_1, F_2 of $P(A)$, one can extract a facet path, where adjacent facets share an $n-2$-dimensional intersection (i.e., a ridge), of length at most $D/(n-1) + 2$. Such facet paths exactly correspond to paths between vertices in $Q(A)$, yielding the desired lower bound.

For $m \geq 2^{\Omega(n)}$, proving that $\text{diam}(P(A)) \geq \Omega(nm^{1/(n-1)})$ reduces to showing that $\text{diam}(Q(A)) \geq m^{1/(n-1)}$ with high probability. For the $Q(A)$ lower bound, we examine the length of paths between vertices of $Q(A)$ maximizing antipodal objectives, e.g., $-e_1$ and e_1. From here, one can one easily derive an $\Omega((m/\log m)^{\frac{1}{n-1}})$ lower bound on the length of such a path, by showing that every edge of $Q(A)$ has length $\epsilon := \Theta((\log m/m)^{\frac{1}{n-1}})$ and that the vertices in consideration are at distance $\Omega(1)$. This is a straightforward consequence of $Q(A)$ being tightly sandwiched by a Euclidean ball, namely $(1-\epsilon^2/2)B_2^n \subseteq Q(A) \subseteq B_2^n$ with high probability. This sandwiching property is itself a consequence of the rows of A being ϵ-dense on $\mathbb{S}^{n-1}$, as mentioned in the previous section.

Removing the extraneous logarithmic factor (which makes the multiplicative gap between our lower and upper bound go to infinity as $m \to \infty$), requires a much more involved argument as we cannot rely on a worst-case upper bound on the length of edges. Instead, we first associate any antipodal path above to a continuous curve on the sphere from $-e_1$ to e_1, corresponding to objectives maximized by vertices along the path. From here, we decompose any such curve into $\Omega(m^{\frac{1}{n-1}})$ segments whose endpoints are at distance $\Theta(m^{-1/(n-1)})$ on the sphere. Finally, by appropriately bucketing the breakpoints and applying a careful union bound, we show that for any such curve, an $\Omega(1)$ fraction of the segments induce at least 1 edge on the corresponding path with overwhelming probability. For further details on the upper and lower bound, we refer the reader to the paper.

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A simple convex-geometric approach to proving that spin glasses exhibit chaotic behavior

Ronen Eldan

A Gaussian field exhibits a chaotic behaviour if the location of its maximizer is highly unstable under perturbations. More formally, given two independent copies of a Gaussian field, $(g_i)_i$ and $(g_i')_i$ we define

$$g_i^\epsilon = g_i + \epsilon g_i'.$$

A family of Gaussian fields depending on a parameter n exhibits chaos if the correlation between the maximizer of (g_i) and that of (g_i^ϵ) goes to zero for some ϵ_n going to zero with n, where the correlation between two Gaussians is defined as

$$R(i,j) := \frac{\mathbb{E}[g_i g_j]}{\sqrt{\mathrm{Var}[g_i]\mathrm{Var}[g_j]}}.$$

This phenomenon was considered in the physics literature as early as the 70's in context of models in statistical mechanics, and in the general context of Gaussian fields the definition was put forth by Chatterjee in 2008 [1]. Chatterjee observed that there is a connection between the phenomenon of chaos and other phenomena such as superoconcentration, which refers to the variance of the supremum being small, and "multiple peaks" which refers to the abundance of almost-uncorrelated near-maximizers.

In this talk we discuss these phenomena in the context of mixed p-spin glasses, or in other words the family of Gaussian fields indexed by the discrete hypercube $\{-1,1\}^n$ whose distributions are invariant under its symmetry group. Perhaps the most canonical example is the Sherrington-Kirkpatrick model defines as follows: Let G be an n by n matrix of i.i.d standard Gaussians, and consider the maximizer $v(G)$ of the expression $v^T Gv$ among all sign vectors $v \in \{-1,1\}^n$. We, therefore, discuss the question of stable $v(G)$ is under small perturbations of G? In 2018, Chen, Handschy and Lerman [2] showed that this model, as well as all even p-spin models, indeed exhibit chaos. Their proof relies heavily on the Parisi-Guerra-Talagrand framework which stems from the cavity method. We will explain an (arguably) much simpler proof which mostly uses classical results in convexity. We take advantage of the convexity of the "cells" which correspond to different maximizers in the Gaussian field and show that the application of a classical result by Hargé can make the stability of the maximizer tractable. Our proof also generalizes to all mixed p-spin models of spin glasses.

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Smoothing rearrangements and the Pólya-Szegő inequality

Gabriele Bianchi

(joint work with Richard J. Gardner, Paolo Gronchi, Markus Kiderlen)

A familiar version of the Pólya-Szegő inequality states that if $\Phi : [0,\infty) \to [0,\infty)$ is convex and $\Phi(0) = 0$ (i.e. a Young function) and $f \in \mathcal{V}(\mathbb{R}^n)$ is Lipschitz, then $f^\#$ is Lipschitz and

$$\int_{\mathbb{R}^n} \Phi\left(\|\nabla f^\#(x)\|\right)\, dx \leq \int_{\mathbb{R}^n} \Phi(\|\nabla f(x)\|)\, dx; \tag{1}$$

see, e.g., [1]. Here $\mathcal{V}(\mathbb{R}^n)$ is the class of nonnegative measurable functions on $\mathbb{R}^n$ that vanish at infinity, (i.e. such that the measure $\mathcal{H}^n\{x \in \mathbb{R}^n : f(x) > t\}$ is finite for each $t > 0$) and $f^\#$ denotes the *symmetric decreasing rearrangement* of f, the function whose superlevel sets have the same $\mathcal{H}^n$-measure of those of f and such that, for $t > 0$, $\{x : f^\#(x) > t\}$ is a ball centered at the origin o of $\mathbb{R}^n$.

The map that takes f to $f^\#$ is the primary example of a rearrangement on $\mathcal{V}(\mathbb{R}^n)$. Other examples are the (k,n)-*Steiner* rearrangement with respect to a k-dimensional subspace in $\mathbb{R}^n$, *polarization* with respect to a hyperplane, and the *Solynin rearrangement* associated to the continuous Steiner symmetrization. The Pólya-Szegő inequality (1) holds for each of the just-mentioned rearrangements.

Diverse variants and applications of the Pólya-Szegő inequality have generated a very substantial literature, surveyed by Talenti who in [7, p. 126] provides over fifty references. The main themes are: Pólya-Szegő inequalities on spheres, hyperbolic, or other spaces, and for other functionals of the gradient; weighted versions involving other measures; versions invariant under affine transformations (see, for instance, [6] and all the papers citing it); anisotropic inequalities; the examination of equality cases; connections with capacitary inequalities; and applications to mathematical physics, PDEs, and function spaces.

This research has arisen from earlier work on symmetrization and rearrangement, including our previous investigations [2–4]. As in those articles, the attention is less on particular symmetrizations or rearrangements than on general properties that allow results for those special cases to be extended and unified.

At the heart of the definition of the rearrangement Tf of a function f there is the formula

$$\{x : Tf(x) > t\} = \diamondsuit_T\{x : f(x) > t\} \tag{2}$$

where $\diamondsuit_T$ denotes a map from $\mathcal{L}^n$, the class of $\mathcal{H}^n$-measurable sets with finite measure, to itself. The superlevel set $\{x : Tf(x) > t\}$ depends only on $\{x : f(x) > t\}$ and this relation, the map $\diamondsuit_T$, is the same for each t. A rearrangement is a map from function spaces and one may wonder which properties of this map make (2) valid. An answer to this has been given in [4], where the following result is proved (together with results for more general function spaces).

Theorem 1 ([4])**.** *Let $T : \mathcal{V}(\mathbb{R}^n) \to \mathcal{V}(\mathbb{R}^n)$ be equimeasurable (i.e. $\mathcal{H}^n\{x : Tf(x) > t\} = \mathcal{H}^n\{x : f(x) > t\}$ for $t > 0$) and monotonic (i.e. $f \le g$ up to sets of measure zero implies $Tf \le Tg$ up to sets of measure zero). Then there exists a map $\diamondsuit_T : \mathcal{L}^n \to \mathcal{L}^n$ for which (2) is valid. This map is defined for $A \in \mathcal{L}^n$ by*

$$\diamondsuit_T A = \{x : T1_A(x) = 1\},$$

and it is measure preserving and monotonic with respect to inclusion. Moreover T is defined, up to sets of measure zero, by $\diamondsuit_T$.

Thus a rearrangement $T : \mathcal{V}(\mathbb{R}^n) \to \mathcal{V}(\mathbb{R}^n)$ is any map which is equimeasurable and monotonic in the sense defined above.

The main purpose of this research is to find conditions on a rearrangement that make the Pólya-Szegő inequality hold. We say that a rearrangement T is *smoothing* if

$$(\diamondsuit_T A)^* + dB^n \subset \diamondsuit_T(A + dB^n)^*,$$

up to sets of measure zero, for each $d > 0$ and bounded measurable set A. Here E^* denotes the set of density points of E. It turns out that this notion is equivalent to the rearrangement reducing the modulus of continuity.

Theorem 2 ([5])**.** *A rearrangement $T : \mathcal{V}(\mathbb{R}^n) \to \mathcal{V}(\mathbb{R}^n)$ is smoothing if and only if T reduces the modulus of continuity, that is, T is such that $\omega_d(Tf) \le \omega_d(f)$ for $d > 0$ and $f \in \mathcal{V}(\mathbb{R}^n)$, where*

$$\omega_d(f) = \operatorname{ess\,sup}_{\|x-y\| \le d} |f(x) - f(y)|.$$

All special rearrangements mentioned in the lines following (1) are smoothing. We are able to prove the validity of the Pólya-Szegő inequality for all smoothing rearrangements on $\mathcal{V}(\mathbb{R}^n)$.

Theorem 3 ([5])**.** *If $T : \mathcal{V}(\mathbb{R}^n) \to \mathcal{V}(\mathbb{R}^n)$ reduces the modulus of continuity, Φ is a Young function and $f \in \mathcal{V}(\mathbb{R}^n)$ is Lipschitz, then Tf is Lipschitz and*

$$\int_{\mathbb{R}^n} \Phi\left(\|\nabla Tf(x)\|\right) dx \leq \int_{\mathbb{R}^n} \Phi\left(\|\nabla f(x)\|\right) dx. \tag{3}$$

We state the result for Lipschitz functions, for simplicity of exposition, but we have proved it in a much larger function space, and we believe that we can extend it to the Orlicz space $W^{1,\Phi}(\mathbb{R}^n)$, the same function space where (1) is valid. The method of proof is new and we sketch it briefly.

Step 1. Assume, for simplicity, that f has compact support. Let K_f and K_{Tf} denote the subgraphs of f and Tf, respectively. Let $C \subset \mathbb{R}^{n+1}$ denote a convex body which is rotationally symmetric about the x_{n+1}-axis and contains o in its interior. This body in a later step is chosen to represent Φ. We prove, for $\varepsilon > 0$,

$$\mathcal{H}^{n+1}\big(K_{Tf} + \varepsilon C\big) \leq \mathcal{H}^{n+1}\big(K_f + \varepsilon C\big). \tag{4}$$

We prove this slice by slice: more precisely we prove, for $t > 0$,

$$\Big(K_{Tf} + \varepsilon C\Big) \cap \{x_{n+1} = t\} \subset \diamondsuit_T\Big[\Big(K_f + \varepsilon C\Big) \cap \{x_{n+1} = t\}\Big]. \tag{5}$$

Since $\diamondsuit_T$ is measure preserving, the validity of (5) for each $t > 0$ implies (4).

Step 2. Since $\mathcal{H}^{n+1}\big(K_f\big) = \mathcal{H}^{n+1}\big(K_{Tf}\big)$, because T is a rearrangement, (4) gives

$$\frac{\mathcal{H}^{n+1}\big(K_{Tf} + \varepsilon C\big) - \mathcal{H}^{n+1}\big(K_{Tf}\big)}{\varepsilon} \leq \frac{\mathcal{H}^{n+1}\big(K_f + \varepsilon C\big) - \mathcal{H}^{n+1}\big(K_f\big)}{\varepsilon},$$

and, passing to the limit, it gives an inequality between the C-Minkowski contents of K_{Tf} and of K_f. This can be expressed in terms of integrals over the graphs of Tf and f, by results proved by G. Zhang and by L. Lussardi and E. Villa in the case of C^1 functions and of Lipschitz functions, respectively. The inequality becomes

$$\int_{\text{graph of } Tf} h_C(\nu(x))\, d\mathcal{H}^n(x) \leq \int_{\text{graph of } f} h_C(\nu(x))\, d\mathcal{H}^n(x),$$

where $\nu(x)$ denotes the outer normal to the graph, i.e.

$$\int_{\text{support of } Tf} h_C\left(-\nabla Tf(x), 1\right) dx \leq \int_{\text{support of } f} h_C(-\nabla f(x), 1)\, dx.$$

Step 3. It can be proved that, given any $M > 0$, it is possible to choose C so that

$$h_C(v, 1) = 1 + \alpha\, \Phi(\|v\|), \quad \forall v \in \mathbb{R}^n : \|v\| < M,$$

for some $\alpha > 0$. Thus, choosing M larger than the Lipschitz constant of f, we have

$$\int_{\text{support of } Tf} 1+\alpha\Phi(\|\nabla Tf(x)\|)\,dx \le \int_{\text{support of } f} 1+\alpha\Phi(\|\nabla f(x)\|)\,dx$$

and, since $\mathcal{H}^n(\text{support of } Tf) = \mathcal{H}^n(\text{support of } f)$, we get (3). □

The same method also yields more general Pólya-Szegő type inequalities, like Klimov and anisotropic inequalities.

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Isoperimetric inequalities for polar L_p-centroid bodies

Peter Pivovarov

(joint work with Radosław Adamczak, Grigoris Paouris, Paul Simanjuntak)

I discussed isoperimetric inequalities for the volume of L_p-centroid bodies. The focus was on the interplay between geometric and probabilistic features.

Geometrically, centroid bodies are defined as follows: given an origin-symmetric convex body K in $\mathbb{R}^n$, the centroids of halves of K cut by hyperplanes through the origin form the surface of its centroid body $Z(K)$. The Busemann-Petty centroid body inequality [2, 14] says that the volume of $Z(K)$ is minimized when K is an origin-symmetric ellipsoid of the same volume as K. It is a fundamental isoperimetric principle known to imply affine invariant versions of the standard isoperimetric inequality [7].

Two key papers in the development of L_p-Brunn-Minkowski theory are by Lutwak-Zhang [9] and Lutwak-Yang-Zhang [8]. They introduce L_p-centroid bodies and establish fundamental isoperimetric inequalities. For $1 \le p < \infty$, the L_p-centroid body of a star-shaped body K, denoted $Z_p(K)$, is defined (up to normalization) by its support function as

$$h(Z_p(K),u) = \left(\frac{1}{|K|}\int_K |\langle x,u\rangle|^p\,dx\right)^{\frac{1}{p}} \quad (u\in\mathbb{R}^n). \tag{1}$$

In [9], it is shown that

$$|Z_p^\circ(K)| \le |Z_p^\circ(K^*)|, \tag{2}$$

where K^* is the dilate of the unit ball centered at the origin of the same volume as K. A stronger, non-polar version was established in [8]. Campi and Gronchi [3] provided an alternate approach using shadow systems, which opened the path to various generalizations and extensions via this method.

Alongside their geometric aspects, L_p-centroid bodies have compelling probabilistic features. One can associate L_p-centroid bodies to probability densities rather than sets, as put forth by the second-named author [12]. In [13], the second and third-named authors introduced an empirical approach to isoperimetric inequalities for convex sets, including centroid bodies. For $1 \leq p < \infty$, the empirical L_p-centroid body has support function

$$h(Z_{p,N}(f), u) = \left(\frac{1}{N} \sum_{i=1}^{N} |\langle X_i, u \rangle|^p \right)^{\frac{1}{p}} \quad (u \in \mathbb{R}^n), \tag{3}$$

where $X_1, \ldots, X_N$ are independent random vectors drawn from a continuous probability distribution with density f. An empirical approach for their duals was developed in [4]; in particular, it was shown that

$$\mathbb{E}|Z_{p,N}^{\circ}(f)| \leq \mathbb{E}|Z_{p,N}^{\circ}(I_{B_f})|,$$

where I_{B_f} is the indicator of an origin-symmetric Euclidean ball with the same height and integral as f. The method relies on random linear operators acting in L_p spaces, a viewpoint from the asymptotic theory of normed spaces [10].

All of the isoperimetric inequalities for L_p-centroid bodies for $p \geq 1$ and their extensions have concerned convex objects. There is broad interest in the cases when $p < 1$. Such bodies have been studied from several perspectives, including that of intersection bodies, which played a crucial role in the resolution of the Busemann-Petty problem [5]. For $p < 1$, Yaskin and Yaskina defined polar L_p-centroid bodies, for $p \in (-1, 1)$, via their radial function:

$$\rho(Z_p^{\diamondsuit}(K), u) = \left(\int_K |\langle x, u \rangle|^p dx \right)^{-1/p} \quad (u \in \mathbb{R}^n), \tag{4}$$

and studied volume comparison problems [15]. Characterizations of such operators were treated by Haberl and Ludwig in [6]. Convexity of $Z_p(K)$ for $p \geq 1$ is immediate from the definition in (1) (regardless of K, provided the integrals in (1) exist). When K is an origin-symmetric convex body and $p \in (-1, 1)$, a result of Berck [1] shows that $Z_p(K)$ is also convex. For $p = -1$, $Z_p(K)$ corresponds to the intersection body of K and the analogue of (2) holds, known as the Busemann intersection inequality [7]. In this way, the bodies $Z_p^{\diamondsuit}(K)$, $p \in [-1, 1]$ interpolate between intersection bodies and polar L_p-centroid bodies. However, isoperimetric inequalities for $Z_p^{\diamondsuit}(K)$, $p < 1$, up to now, had remained open. Proofs of (2) and its empirical forms have relied on methods based on convexity of the L_p-norm, which does not apply to the case $p < 1$.

We establish a sharp isoperimetric inequality that extends the Lutwak-Zhang theorem (2) to the case $p \in (0, 1)$. For this range of p, we define empirical L_p

centroid bodies via their radial function:

$$\rho(Z_{p,N}^{\diamondsuit}(f), u) = \left(\frac{1}{N}\sum_{i=1}^{N} |\langle X_i, u\rangle|^p\right)^{-\frac{1}{p}} \qquad (u \in \mathbb{R}^n), \tag{5}$$

where the X_i's are as above. With this notation, for $p \in (0,1)$, we have

$$\mathbb{E}|Z_{p,N}^{\diamondsuit}(f)| \leq \mathbb{E}|Z_{p,N}^{\diamondsuit}(I_{B_f})|. \tag{6}$$

Consequently,

$$|Z_{p}^{\diamondsuit}(f)| \leq Z_{p}^{\diamondsuit}(I_{B_f})|. \tag{7}$$

Our main inspiration and technical ingredient is a volume formula for sections of L_p balls by Nayar-Tkocz [11]. This allows for a reduction from star-shaped sets to convex sets and the use of dual inequalities for random convex bodies from [4].

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Stability of the Prékopa–Leindler inequality

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(joint work with Alessio Figalli, João P. G. Ramos)

We prove that if a triplet of functions satisfies almost equality in the Prékopa–Leindler inequality, then these functions are close to a common log-concave function, up to multiplication and rescaling. Our result holds for general measurable functions in all dimensions, and provides a quantitative stability estimate with computable constants.

The Prékopa-Leindler inequality, due to Prékopa [17] and Leindler [16] in dimension one, was generalized in Prékopa [18] and Borell [3] to any dimension. The case of equality is characterized by Dubuc [8]. In order to state it precisely, we recall that a function $f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be log-concave if $f((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} f(y)^\lambda$ for all $x, y \in \mathbb{R}^n$ and $\lambda \in (0,1)$; in other words, f is log-concave if it can be written as $f = e^{-\varphi}$ for some convex function $\varphi : \mathbb{R}^n \to (-\infty, \infty]$.

Theorem 1 (Prékopa, Leindler; Dubuc)**.** *Let $\lambda \in (0,1)$ and $f, g, h : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be measurable functions such that*

$$h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^\lambda \qquad \forall\, x, y \in \mathbb{R}^n. \tag{1}$$

Then

$$\int_{\mathbb{R}} h \geq \left(\int_{\mathbb{R}} f\right)^{1-\lambda} \left(\int_{\mathbb{R}} g\right)^{\lambda}. \tag{2}$$

Also, equality implies that there exist $a > 0$, $w \in \mathbb{R}^n$, and a log-concave function $\tilde{h}$, such that $h = \tilde{h}$, $f = a^{-\lambda}\tilde{h}(\cdot - \lambda w)$, $g = a^{1-\lambda}\tilde{h}(\cdot + (1-\lambda)w)$ almost everywhere.

Here, we announce the first quantitative stability result for the Prékopa-Leindler inequality on arbitrary functions in any dimension, including $n = 1$.

Theorem 2. *Given $\tau \in (0, \frac{1}{2}]$ and $\lambda \in [\tau, 1-\tau]$, let $f, g, h : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $n \geq 1$, be measurable functions such that $h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^\lambda$ for all $x, y \in \mathbb{R}^n$, and*

$$\int_{\mathbb{R}^n} h < (1+\varepsilon) \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g\right)^{\lambda} \qquad \textit{for some } \varepsilon > 0. \tag{3}$$

There are a computable dimensional constant Θ_n and computable constants $Q_n(\tau)$ and $M_n(\tau)$ depending only on n and τ, such that the following holds: If $0 < \varepsilon < e^{-M_n(\tau)}$, then there exist $\tilde{h}$ log-concave and $w \in \mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} |h - \tilde{h}| + \int_{\mathbb{R}^n} |a^\lambda f - \tilde{h}(\cdot + \lambda w)| + \int_{\mathbb{R}^n} |a^{\lambda-1} g - \tilde{h}(\cdot + (\lambda-1)w)| < \frac{\varepsilon^{Q_n(\tau)}}{\tau^{\Theta_n}} \int_{\mathbb{R}^n} h,$$

where $a = \int_{\mathbb{R}^n} g / \int_{\mathbb{R}^n} f$.

Remark 3. If f, g, h are *a priori* assumed to be log-concave, then Theorem 2 was established by Ball, Böröczky [1] and Böröczky, De [4] in the case $n = 1$ (in this case, $\varepsilon^{Q_n(\tau)}/\tau^{\Theta_n}$ in Theorem 2 can be essentially replaced by $(\varepsilon/\tau)^{\frac{1}{3}}$), and by Böröczky, De [4] in the case $n \geq 2$ (in that case, $\varepsilon^{Q_n(\tau)}/\tau^{\Theta_n}$ in Theorem 2 can be replaced by $(\varepsilon/\tau)^{\frac{1}{19}}$). Further, we note that Bucur, Fragalà [5] proved another interesting stability version of the Prékopa-Leindler inequality for log-concave functions, bounding the distance of all one dimensional projections.

Theorem 2 is probably quite far from the optimal version, that one could conjecture to provide a bound of the form $C(n,\tau)\varepsilon^{1/2}$. In this direction, already for $n = 1$, we prove that the error term in Theorem 2 is at least $c\varepsilon^{1/2}$ in any dimension even if the functions are assumed to be log-concave.

Note that, if f, g, h are the indicator functions of some sets A, B, C, then Theorem 1 corresponds exactly to the Brunn-Minkowski inequality. Let us list some result for particular cases of the stability of the Brunn-Minkowski inequality.

- When $n = 1$, then the optimal stability result is due to Freiman (see Christ [7] for the first published argument), and the error term is of order $c\tau^{-1}\varepsilon$ for an absolute contant $c > 0$.
- When $n = 2$, then van Hintum, Spink, Tiba [14] obtained the optimal stability version, where the error term is of the from $c_n\tau^{-\frac{1}{2}}\varepsilon^{\frac{1}{2}}$ with $c_n > 0$ depending only on n.
- When $A = B$ and $n \geq 3$, then van Hintum, Spink, Tiba [13] obtained the optimal stability version, where the error term is of the from $c_n\tau^{-\frac{1}{2}}\varepsilon^{\frac{1}{2}}$ with $c_n > 0$ depending only on n.
- For any A, B and $n \geq 3$, the stability of the Brunn-Minkowski inequality is proved by Figalli, Jerison [9].

When at least one of the sets A or B is convex in the Brunn-Minkowski inequality, then several results have been obtained with optimal error $\varepsilon^{\frac{1}{2}}$ in ε, as described below.

- When either A or B is convex, an optimal stability estimate has been proved by Barchiesi, Julin [2]. This extends earlier results about the case where both A and B are convex by Figalli, Maggi, Pratelli [11,12], or when either A or B is the unit ball by Figalli, Maggi, Mooney [10].
- If A and B are convex and n is large, then Kolesnikov, Milman [15] provided an error term of the form $c\,n^{2.75}\tau^{-\frac{1}{2}}\varepsilon^{\frac{1}{2}}$, for some absolute constant c. Actually, we note that the term $n^{2.75}$ can be improved to $n^{2.5+o(1)}$ by combining the general estimates of Kolesnikov, Milman [15, Section 12] with the bound $n^{o(1)}$ on the Cheeger constant of a convex body in isotropic position, that follows from Yuansi Chen's work [6] on the Kannan-Lovasz-Simonovits conjecture.

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Convex Algebraic Geometry

Chiara Meroni

What is convex algebraic geometry? By this name we refer to the study of convex geometry from the point of view of algebraic geometry and real algebra. This approach has its origin in the theory of polytopes, connected to linear algebra and combinatorics. From there, it is natural to go beyond linear algebra and enter the world of *nonlinear algebra* [9].

In this setting, we study the family of semialgebraic convex bodies. A convex body $K \subset \mathbb{R}^n$ is called semialgebraic if it is a semialgebraic set, i.e. it is defined by a boolean combination of polynomial inequalities. Informally, it is a finite union of finite intersections of polynomial inequalities. For a background in semialgebraic geometry and real algebraic geometry, we refer to [3]. This is a notion that behaves well with respect to convexity. Indeed, given a convex body $K \subset \mathbb{R}^n$, the following are equivalent:

- K is semialgebraic;
- the support function of K is a semialgebraic function;

- the radial function of K is a semialgebraic funciton;
- the dual/polar body of K is a semialgebraic convex body.

Moving towards algebraic geometry and convex geometry, the object that better encodes this interaction is the *algebraic boundary*.

Definition 1. Let $K \subset \mathbb{R}^n$ be a convex body. Its *algebraic boundary* is the smallest variety that contains the topological boundary of K. In other words, it is the closure of the topological boundary with respect to the Zariski topology.

In this way, we associate a variety to a convex body. We can study such a variety using tools from algebraic geometry (see [5] for an introduction) in order to get information about K. For instance, the algebraic boundary detects the semialgebraicity of a convex body: K is semialgebraic if and only if its algebraic boundary is an algebraic hypersurface. Polytopes are an example of semialgebraic convex bodies, and their algebraic boundary is the hyperplane arrangement defined by the facets. One hopes to extend notions and techniques from the theory of polytopes to semialgebraic convex bodies. Some examples are [10], where the authors develop a broad definition of an f-vector, and [14], that discusses a generalization of the neighborliness for non-polyhedral convex cones.

One can generalize polytopes in many ways, in order to obtain classes of semialgebraic convex bodies. The family of *spectrahedra* is one option [11]. They arise as the intersection of the cone of positive semidefinite matrices with a linear subspace. Spectrahedra are relevant in optimization because they are the feasible regions of semidefinite programming. Their study is intimately related to the study of matrices and determinantal varieties. Another direction is that of the *convex hull of a variety* [12, 13]. Understanding the boundary of a convex hull is a difficult task in general. However, algebraic geometry gives the answer in the case of convex hulls of varieties, as stated in [12, Theorem 1.1]. Such a formula describes the components of the algebraic boundary of K, where K is the convex hull of a smooth compact real algebraic variety in $\mathbb{R}^n$.

Not all convex bodies are semialgebraic. For instance, *zonoids* have a non-empty intersection with the set of semialgebraic convex bodies, but are not contained in it. Hence, the immediate question is: which zonoids are semialgebraic? This lies in the context of the Zonoid Problem [2, 4]. This problem is very hard to tackle [15, 16]: restricting to the subclass of semialgebraic convex bodies would potentially make it easier. In [6], we investigate a class of semialgebraic zonoids called *discotopes*. They are Minkowski sums of finitely many discs. We study their algebraic boundary, in order to be able to characterize them. The beauty and the strength of this problem is that it can be approached from many points of view: algebraic geometry, as in our work, measure theory, random geometry.

Many areas of convex geometry investigate objects that are defined starting from a convex body. A goal is to understand how semialgebraic geometry behaves with respect to these constructions. In [8], we show that the *fiber body* of a semialgebraic convex body is not in general semialgebraic. On the other hand, the main result of [1] states that the *intersection body* of a polytope is always a semialgebraic starshaped set. We also provide an algorithm to compute intersection bodies of polytopes and their algebraic boundary, available at `https://mathrepo.mis.mpg.de/intersection-bodies`.

A final remark is that semialgebraic convex bodies also appear in applications; an example is [7]. In this work, we meticulously analyse the *correlation body*, a fundamental object for quantum physics. It is a self-dual semialgebraic convex body which is sandwiched between two polytopes. In [7, Section 6] we exhibit a number of open problems and research directions.

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Legendre transforms, Laplace transforms, and function-valued valuations

Jin Li

The theory of valuations originates from Dehn's solution to Hilbert's third problem. Various important functionals on convex bodies or on functions were characterized in the theory of valuations with their natural geometric invariances; for example, volumes [8, 10, 11, 16], projection bodies (LYZ-bodies) and moment bodies [7, 13, 15, 17]. However, it is somehow surprising that not too much study on transforms of functions in the theory of valuations, although many of them are valuations. Besides the results stated here, the author only found two works on such theory: the Laplace transform and the Fourier transform on Lebesgue space; see [12, 18].

It should be remarked that there is a series of beautiful characterizations of transforms of functions including the Legendre transform and the Fourier transform by Artstein-Avidan, V. Milman and many others, for example [2, 3, 5].

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and we always assume $n \geq 2$ if there are no further remarks. Denote by $\mathrm{Cvx}(\mathbb{R}^n)$ the set of all lower semi-continuous, convex functions $u : \mathbb{R}^n \to (-\infty, \infty]$ and by $\mathrm{Conv}_{sc}(\mathbb{R}^n)$ the set of proper, super-coercive $u \in \mathrm{Cvx}(\mathbb{R}^n)$. Here a convex function u is proper if $u \neq \infty$; and it is super-coercive if $\lim\limits_{|x|\to\infty} \frac{u(x)}{|x|} = \infty$.

For $u, v \in \mathrm{Cvx}(\mathbb{R}^n)$, we define

$$(u \vee v)(x) = \max\{u(x), v(x)\},\ (u \wedge v)(x) = \min\{u(x), v(x)\},\ x \in \mathbb{R}^n.$$

Let $\langle \mathbb{A}, + \rangle$ be an abelian semigroup. A map $z : \mathrm{Conv}_{sc}(\mathbb{R}^n) \to \langle \mathbb{A}, + \rangle$ is a valuation if

$$z(u \vee v) + z(u \wedge v) = zu + zv$$

whenever all four function $u, v, u \vee v, u \wedge v \in \mathrm{Conv}_{sc}(\mathbb{R}^n)$. If we restrict z to indicator functions of convex bodies, we get valuations on convex bodies.

The Legendre transform of $u \in \mathrm{Cvx}(\mathbb{R}^n)$ is

$$\mathcal{L}e\, u(x) = \sup_{y\in\mathbb{R}^n} \langle x, y\rangle - u(y),\ x \in \mathbb{R}^n.$$

Artstein-Avidan and V. Milman [5] showed that the Legendre transform is essentially the only bijection $z : \mathrm{Cvx}(\mathbb{R}^n) \to \mathrm{Cvx}(\mathbb{R}^n)$ satisfying the following monotonicity conjugate:

$$u \leq v \Longleftrightarrow zu \geq zv.$$

Since $z : \mathrm{Cvx}(\mathbb{R}^n) \to \mathrm{Cvx}(\mathbb{R}^n)$ is a bijection, the monotonicity conjugate is equivalent to the following maximum and "minimum" conjugate:

$$z(u \vee v) = (zu)\tilde{\wedge}(zv),\ z(u\tilde{\wedge}v) = (zu) \vee (zv).$$

(the equivalence is also pointed out in [5] and is a key step in their proof). Here $(u\tilde{\wedge}v)(x) = \max\{w(x) : w \in \mathrm{Cvx}(\mathbb{R}^n), w \leq u \wedge v\}$. It is easy to see that the maximum and "minimum" conjugate is much stronger than the valuation property. A natural question arises: can we characterize the Legendre transform in the theory of valuations? (the question is also asked by V. Milman in the online AGA seminar).

The Legendre transform is a duality on convex functions. Its analog on convex bodies is the polar body. A characterization of polar bodies analog to [5] was established by Böröczky and Schneider [6]; see also, Gruber [9], Artstein-Avidan and V. Milman [4]. Meanwhile, characterizations of the polar bodies in the theory of valuations were established by Ludwig [14, 15]. Let $\mathcal{K}^n$ be the set of convex bodies, and $\mathcal{K}^n_{(o)}$ be the set of convex bodies containing the origin in their interiors.

Theorem 1 (Ludwig [14, 15])**.** *A map* $Z : \mathcal{K}^n_{(o)} \to \langle \mathcal{K}^n, + \rangle$ *is a continuous valuation satisfying* $Z(\phi K) = \phi^{-t} ZK$ *for any* $K \in \mathcal{K}^n_{(o)}$ *and* $\phi \in \mathrm{GL}(n)$*, if and only if there are* $c_1, c_2 \geq 0$ *such that*

$$ZK = c_1 K^* + c_2(-K^*)$$

for every $K \in \mathcal{K}^n_{(o)}$*. Here "+" can be the Minkowski addition or the radial addition.*

However, the following example shows that we can not characterize the Legendre transform analog to Theorem 1 without additional assumptions.

Example: $zu(x) = \int_0^\infty h(\{e^{-u} \geq t\}, x) dt,\ u \in \mathrm{Conv}_{sc}(\mathbb{R}^n)$.

Here $h(\{e^{-u} \geq t\}, \cdot)$ is the support function of $\{e^{-u} \geq t\} := \{y \in \mathbb{R}^n : e^{-u(x)} \geq t\}$. It is easy to see that both the valuation z and the Legendre transform have the following properties: $z(u \circ \phi^{-1}) = z(u)\phi^t$ for any $u \in \mathrm{Conv}_{sc}(\mathbb{R}^n)$ and $\phi \in \mathrm{GL}(n)$; and $z(\tau_y u)(x) = zu + x \cdot y$ for any $y \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$, where $\tau_y u(x) := u(x - y)$.

The function $\tau_y u$ is a (usual) translation of the function $u \in \mathrm{Conv}_{sc}(\mathbb{R}^n)$, and the function $u + \ell_y$ is an additional translation, which is called dually translation, of $u \in \mathrm{Conv}_{sc}(\mathbb{R}^n)$; see e.g. [1, 8]. Here $\ell_y(x) = y \cdot x$ for any $x \in \mathbb{R}^n$, where $y \cdot x$ is the inner product of $y, x \in \mathbb{R}^n$. Further assuming $z(u + \ell_y) = \tau_y z(u)$, we characterize the Legendre transform.

Let $F(\mathbb{R}^n : \mathbb{R})$ be the set of all finite functions on $\mathbb{R}^n$.

Theorem 2. *A map* $z : \mathrm{Conv}_{sc}(\mathbb{R}^n) \to F(\mathbb{R}^n : \mathbb{R})$ *is a continuous and* $\mathrm{SL}(n)$ *contravariant valuation which is a translation conjugate if and only if there is a constant* $c \in \mathbb{R}$ *such that*

$$zu = \mathcal{L}e\, u + c$$

for every $f \in \mathrm{Conv}_{sc}(\mathbb{R}^n)$*.*

Here z is continuous if zu_i converges pointwise to zu whenever u_i epi-converges to u (the epigraph converges in the sense of Kuratowski convergence); it is $\mathrm{SL}(n)$ contravariant if

$$z(u \circ \phi^{-1}) = z(u) \circ \phi^t$$

for any $\phi \in \mathrm{SL}(n)$; and it is a translation conjugate if

$$z(u + \ell_y) = \tau_y z(u),\ z(\tau_y u) = z(u) + \ell_y$$

for every $u \in \mathrm{Conv}_{sc}(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$.

We characterize not only the Legendre transform but also the Laplace transform $\mathcal{L}a$:

$$\mathcal{L}a\, (\exp\{-u\})\, (x) := \int_{\mathbb{R}^n} \exp\{\langle x, y \rangle - u(y)\}\, dy,\ x \in \mathbb{R}^n,$$

by changing the translation conjugate to the log-translation conjugate:

$$z(u + \ell_y) = \tau_y z(u),\ z(\tau_y u) = z(u) \exp\{\ell_y\}.$$

Theorem 3. *A map* $z : \mathrm{Conv}_{sc}(\mathbb{R}^n) \to F(\mathbb{R}^n : \mathbb{R})$ *is a continuous and* $\mathrm{SL}(n)$ *contravariant valuation which is a log-translation conjugate if and only if there are constants* $c_1, c_2 \in \mathbb{R}$ *such that*

$$zu = c_1 \exp\{\mathcal{L}e\, u\} + c_2 \mathcal{L}a\, (\exp\{-u\})$$

for every $u \in \mathrm{Conv}_{sc}(\mathbb{R}^n)$*.*

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Functional versions of intrinsic and mixed volumes

Andrea Colesanti

Intrinsic and mixed volumes are two central notions in Convex Geometry. Intrinsic volumes arise naturally as coefficients of the Steiner formula; the latter is a special case of a more general phenomenon: the polynomiality of volume with respect to Minkowski addition. This phenomenon is at the basis of the definition of mixed volumes.

The notions of intrinsic and mixed volumes have been extended to various functional spaces; we focus on *quasi-concave functions* and convex functions.

A function $u\colon \mathbb{R}^n \to [0,+\infty)$ is said to be quasi-concave, if for every $t>0$, the set

$$\{x\colon u(x) \geq t\}$$

is compact and convex. We denote by $\mathrm{QC}(\mathbb{R}^n)$ the family of quasi concave functions. In [10], Milman and Rotem defined the following operations on the space of quasi-concave functions. Given u, w quasi-concave, $\alpha, \beta \geq 0$,

$$(\alpha \cdot u \oplus \beta \cdot w)(z) = \sup\{\min\{u(x), w(y)\}\colon \alpha x + \beta y = z\}. \tag{1}$$

They also introduced a volume type functional

$$u \ \to \ I_n(u) := \int_{\mathbb{R}^n} u \ dx \ \in [0,+\infty].$$

In [10], it is proved that the functional I_n is *polarized* by the operations defined in (1). Namely, there exists a mapping $I\colon (\mathrm{QC}(\mathbb{R}^n))^n \to [0,\infty]$ such that:

$$I_n(t_1 \cdot u_1 \oplus \cdots \oplus t_m \cdot u_m) = \sum_{i_1,\dots,i_n=1}^{m} t_{i_1} \dots t_{i_n} I(u_{i_1}, \dots, u_{i_n})$$

for every $u_1, \dots, u_m \in \mathrm{QC}(\mathbb{R}^n)$, $t_1, \dots, t_m \geq 0$. Then, $I(u_1, \dots, u_n)$ is a natural candidate as mixed volume of $u_1, \dots, u_n \in \mathrm{QC}(\mathbb{R}^n)$. By choosing the characteristic function of the unit ball B^n of $\mathbb{R}^n$, denoted by χ_{B^n}, as the unit ball of $\mathrm{QC}(\mathbb{R}^n)$, intrinsic volumes can now be defined as:

$$V_i(u) = c(n,i) I(\underbrace{u, \dots, u}_{i \text{ times}}, \underbrace{\chi_{B^n}, \dots, \chi_{B^n}}_{n-i \text{ times}})$$

($c(n,i)$ is a constant depending on i and n).

A different, but equivalent, characterization was found in [2], where the authors observed that intrinsic volumes of quasi-concave functions can be obtained integrating (classical) intrinsic volumes of their level sets:

$$V_i(u) = \int_0^{+\infty} V_i(\{u \geq t\}) \ dt, \tag{2}$$

V_i in the last integral denotes the standard intrinsic volume of convex bodies.

Making one step back, we now observe that numerous and significant connections have been established between intrinsic and mixed volumes, and *valuations*, in the realm of Convex Geometry. The Hadwiger theorem asserts that a real-valued functional defined on convex bodies is a continuous and rigid motion invariant valuation if and only if it is the linear combination of intrinsic volumes. The McMullen decomposition theorem states that every continuous and translation invariant valuation Z can be written as

$$\mathrm{Z} = \sum_{i=0}^{n} \mathrm{Z}_i,$$

where Z_i is i-homogeneous. The mixed volume operator V naturally provides a class of homogeneous valuations. Indeed, for fixed $i \in \{0, \dots, n\}$ and convex bodies $K_{n-i}, \dots, K_n$, the functional Z defined by

$$\mathrm{Z}(K) = V(\underbrace{K, \dots, K}_{i \text{ times}}, K_{n-i}, \dots, K_n)$$

is a i-homogeneous and translation invariant valuation. Finally, the solution by Alesker of the McMullen conjecture asserts that linear combinations of valuations of this form are dense in the space of continuous and translation invariant valuations.

The notion of valuation can be transferred to a functional setting. Let $\mathfrak{F}$ be a function space; a functional $\mathrm{Z}\colon \mathfrak{F} \to \mathbb{R}$ is a valuation if

$$\mathrm{Z}(u \vee v) + \mathrm{Z}(u \wedge v) = \mathrm{Z}(u) + \mathrm{Z}(v),$$

for every $u, v \in \mathfrak{F}$ such that $u \vee v$, $u \wedge v \in \mathfrak{F}$ (here $\vee$ and $\wedge$ denote the point-wise minimum and maximum, respectively). We refer to the papers [5–8] for more details about the theory of valuations on spaces of functions.

In [3, 4], valuations on the space of quasi-concave functions are studied. In particular, in [3] some classification results of Hadwiger type are established. According to one of them, every continuous (w.r.t. to a suitable topology), increasing and rigid motion invariant valuation Z on $\mathrm{QC}(\mathbb{R}^n)$ is the sum of terms of the form

$$\int_0^{+\infty} V_i(\{u \geq t\})\, d\nu(t)$$

where $i \in \{1, \dots, n\}$ and ν is a Radon measure on $[0, +\infty)$. Comparing with (2), we see that the valuation point of view gives us a larger class of functionals, as intrinsic volumes of quasi-concave functions.

We now turn to the case of convex functions. In [5–8], we consider the space $\mathrm{Conv}_{sc}(\mathbb{R}^n)$ of functions $u\colon \mathbb{R}^n \to (-\infty, +\infty]$, which are convex, lower semi-continuous and super coercive:

$$\lim_{|x| \to \infty} \frac{u(x)}{|x|} = +\infty.$$

This space can be endowed with an addition and a multiplication by non-negative reals, and by the topology induced by epi-convergence (or Γ-convergence). Note this space is *dual* to the space $\mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$ of finite convex functions on $\mathbb{R}^n$; i.e. $u \in \mathrm{Conv}_{sc}(\mathbb{R}^n)$ if and only if $u^* \in \mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$, where u^* is the conjugate (or Legendre transform) of u. The bridge between $\mathrm{Conv}_{sc}(\mathbb{R}^n)$, the primal setting, and $\mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$, the dual setting, provided by the * transform, permits to transfer notions and results from one setting to the other, almost automatically.

In [1], the author defines a class of functionals, which admits an integral representation on smooth functions, in the dual setting $\mathrm{Conv}(\mathbb{R}^n; \mathbb{R})$. The primal version of these functionals turn out to be continuous, translation invariant valuations, Z, which are moreover *vertically invariant*: $\mathrm{Z}(u + c) = \mathrm{Z}(u)$, for every u and every constant c. In [5] we establish a homogeneous decomposition theorem for this type of valuations.

Finally, in [6] (see also [7, 8]) we proved a Hadwiger type result, which classifies all continuous, rigid motion and vertically invariant valuations on $\mathrm{Conv}_{sc}(\mathbb{R}^n)$. In order to give an idea of how the functionals involved in this classification look like, we mention that their behaviour on smooth functions u is of the form:

$$\sum_{i=0}^{n} \int_{\mathbb{R}^n} \zeta_i(|\nabla u(x)|)[\mathrm{D}^2 u(x)]_{n-i}\, dx, \tag{3}$$

where $i \in \{0, \dots, n\}$, $[\mathrm{D}^2 u]_j$ = j-th elementary symmetric function of the eigenvalues of $\mathrm{D}^2 u$, and $\zeta_i \in C((0, \infty))$ has bounded support, and has a possible, controlled, singularity at 0.

In view of this result, the summands of (3) represent functional analog of intrinsic volumes for convex functions.

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Singular Hessian valuations and mixed Monge–Ampère measures

Fabian Mussnig

(joint work with Andrea Colesanti, Monika Ludwig)

For $j \in \{0, \dots, n\}$ and suitable $\zeta \in C((0, \infty))$ with bounded support, we consider the functional

$$\mathrm{V}^*_{j,\zeta}(v) := \int_{\mathbb{R}^n} \zeta(|x|)[\mathrm{D}^2 v(x)]_j \,\mathrm{d}x$$

for convex $v \in C^2(\mathbb{R}^n)$. Here $[\mathrm{D}^2 v(x)]_j$ is the jth elementary symmetric function of the eigenvalues of the Hessian matrix of v at $x \in \mathbb{R}^n$. It was shown that these operators continuously extend to all convex functions $v : \mathbb{R}^n \to \mathbb{R}$ [2, Theorem

1.4]. However, the densities ζ are allowed to have certain singularities at 0^+, which seems counterintuitive at first. On the other hand, the operators $\mathrm{V}^*_{j,\zeta}$ were characterized in a Hadwiger-type theorem and play the role of functional intrinsic volumes [2, Theorem 1.5]. In that sense, these singularities are canonical.

For a convex function $v : \mathbb{R}^n \to \mathbb{R}$, let $\mathrm{MA}(v; \cdot)$ denote its *Monge–Ampère measure*, that is,

$$\mathrm{MA}(v; B) := V_n \left(\bigcup\nolimits_{x \in B} \partial v(x) \right)$$

for every Borel set $B \subseteq \mathbb{R}^n$, where $\partial v(x)$ is the subdifferential of v at $x \in \mathbb{R}^n$. For convex functions $v_1, \dots, v_n : \mathbb{R}^n \to \mathbb{R}$, we write $\mathrm{MA}(v_1, \dots, v_n; \cdot)$ for the polarization of $\mathrm{MA}(v; \cdot)$ and we call $\mathrm{MA}(v_1, \dots, v_n; \cdot)$ the *mixed Monge–Ampère measure* of $v_1, \dots, v_n$. We show the following result.

Theorem 1 ([3], Theorem 2.5)**.** *If $j \in \{0, \dots, n\}$ and $\zeta \in D^n_j$, then*

$$\mathrm{V}^*_{j,\zeta}(v) = \int_{\mathbb{R}^n} \alpha(|x|) \,\mathrm{dMA}(v[j], h_{B^n}[n-j]; x) \tag{1}$$

for every convex $v : \mathbb{R}^n \to \mathbb{R}$, where $\alpha \in C_c([0, \infty))$ is given by

$$\alpha(s) := \binom{n}{j} \left(s^{n-j} \zeta(s) + (n-j) \int_s^\infty t^{n-j-1} \zeta(t) \,\mathrm{d}t \right)$$

for $s > 0$. Moreover, for $j \in \{1, \dots, n\}$,

$$\mathrm{V}^*_{j,\zeta}(v) = \int_{\mathbb{R}^n} \alpha(|x|) \det(\mathrm{D}^2 v(x)[j], \mathrm{D}^2 h_{B^n}(x)[n-j]) \,\mathrm{d}x \tag{2}$$

if in addition $v \in C^2(\mathbb{R}^n)$.

Here the classes $D^n_j \subset C((0, \infty))$ describe the possible singularities of the densities ζ. In addition, $h_{B^n}(x) = |x|$, $x \in \mathbb{R}^n$, is the support function of the unit ball in $\mathbb{R}^n$ and in the mixed Monge–Ampère measure in (1) the function v is repeated j times and h_{B^n} is repeated $(n-j)$ times. Furthermore, det in (2) denotes the mixed discriminant of symmetric $n \times n$ matrices. Since $\mathrm{D}^2 h_{B^n}(x)$ exists for every $x \neq 0$, the right side of (2) is well-defined as a Lebesgue integral.

Note that the function α which appears in Theorem 1 is continuous at 0^+. Thus, by the properties of mixed Monge–Ampère measures, representation (1) also provides a new proof of the existence of the operators $\mathrm{V}^*_{j,\zeta}$.

Theorem 1 is directly connected to the following new Steiner formulas.

Theorem 2 ([3], Theorem 2.3)**.** *If $\zeta \in D^n_n$, then*

$$\mathrm{V}^*_{n,\zeta}(v + r h_{B^n}) = \sum_{j=0}^{n} r^{n-j} \kappa_{n-j} \,\mathrm{V}^*_{j,\zeta_j}(v) \tag{3}$$

for every convex $v : \mathbb{R}^n \to \mathbb{R}$ and $r > 0$, where $\zeta_j \in D^n_j$ is given by

$$\zeta_j(s) := \frac{1}{\kappa_{n-j}} \left(\frac{\zeta(s)}{s^{n-j}} - (n-j) \int_s^\infty \frac{\zeta(t)}{t^{n-j+1}} \,\mathrm{d}t \right)$$

for $s > 0$ and $j \in \{0, \dots, n\}$.

Here κ_{n-j} is the $(n-j)$-dimensional volume of the unit ball in $\mathbb{R}^{n-j}$. Furthermore, the densities ζ_j are obtained from ζ by a bijective integral transform and thus, every functional intrinsic volume V^*_{j,ζ_j} for $j \in \{1,\ldots,n\}$ and $\zeta_j \in D^n_j$ will appear exactly once on the right side of (3) as ζ ranges in D^n_n (for $j=0$ multiple densities give the same functional intrinsic volume). In addition, the classical Steiner formula is retrieved from (3) if v is chosen to be the support function of a convex body in $\mathbb{R}^n$.

By duality, equivalent results are obtained on the space of proper, lower semicontinuous, super-coercive, convex functions $u : \mathbb{R}^n \to (-\infty, \infty]$. In particular, under additional smoothness assumptions, an explicit representation of the measure appearing in (1) in terms of the convex conjugate of v is given, where the elementary symmetric functions of the principal curvatures of the sublevel sets appear.

Finally, let us remark that mixed operators on convex functions, similar to (1), were considered by Alesker [1] and Knoerr [4].

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An Almost Constant Lower Bound of the Isoperimetric Coefficient in the KLS Conjecture

Yuansi Chen

Abstract: Kannan, Lovász, and Simonovits (KLS) conjectured in 1995 that the Cheeger isoperimetric coefficient of any log-concave density or any convex body is achieved by half-spaces up to a universal constant factor. This conjecture now plays a central role in the field of convex geometry, unifying or implying older conjectures. In particular, it implies Bourgain's slicing conjecture (1986) and the thin-shell conjecture (2003). The previous best bound with dimension dependency $d^{1/4}$ was established by Lee and Vempala in 2017, which also matches the best dimension dependency Klartag obtained in 2006 for Bourgain's slicing conjecture. In recent work, we improve the Eldan's stochastic localization proof technique deployed in Lee and Vempala (2007) to prove an almost constant Cheeger isoperimetric coefficient in the KLS conjecture with dimension dependency $d^{o(1)}$. In this talk, first we briefly survey the origin and the main consequences of these conjectures. Then we present the development and the refinement of Eldan's stochastic localization scheme. Finally, we explain a few proof details which result in the current best bound of the Cheeger isoperimetric coefficient in the KLS conjecture.

Main text: Given a distribution, the isoperimetric coefficient of a subset is the ratio of the measure of the subset boundary to the minimum of the measures of the subset and its complement. Taking the minimum of such ratios over all subsets defines the isoperimetric coefficient of the distribution, also called the Cheeger isoperimetric coefficient of the distribution.

Kannan, Lovász and Simonovits (KLS) [1] conjecture that for any distribution that is log-concave, the Cheeger isoperimetric coefficient equals to that achieved by half-spaces up to a universal constant factor. If the conjecture is true, the Cheeger isoperimetric coefficient can be determined by going through all the half-spaces instead of all subsets. For this reason, the KLS conjecture is also called the KLS hyperplane conjecture. To make it precise, we start by formally defining log-concave distributions and then we state the conjecture.

A probability density function $p : \mathbb{R}^d \to \mathbb{R}$ is *log-concave* if its logarithm is concave, i.e., for any $x, y \in \mathbb{R}^d \times \mathbb{R}^d$ and for any $\lambda \in [0, 1]$,

$$p(\lambda x + (1 - \lambda)y) \geq p(x)^{\lambda} p(y)^{1-\lambda}. \tag{1}$$

Common probability distributions such as Gaussian, exponential and logistic are log-concave. This definition also includes any uniform distribution over a *convex set* defined as follows. A subset $K \subset \mathbb{R}^d$ is *convex* if $\forall x, y \in K \times K, z \in [x, y] \implies z \in K$. The *isoperimetric coefficient* $\psi(p)$ of a density p in $\mathbb{R}^d$ is defined as

$$\psi(p) := \inf_{S \subset \mathbb{R}^d} \frac{p^+(\partial S)}{\min(p(S), p(S^c))} \tag{2}$$

where $p(S) = \int_{x \in S} p(x)dx$ and the boundary measure of the subset is

$$p^+(\partial S) := \liminf_{\epsilon \to 0^+} \frac{p(\{x : \mathbf{d}(x, S) \leq \epsilon\}) - p(S)}{\epsilon},$$

where $\mathbf{d}(x, S)$ is the Euclidean distance between x and the subset S.

The KLS conjecture is stated by Kannan, Lovász and Simonovits [1] as follows.

Conjecture 1. *There exists a universal constant c, such that for any log-concave density p in $\mathbb{R}^d$, we have*

$$\psi(p) \geq \frac{c}{\sqrt{\rho(p)}},$$

where $\rho(p)$ is the spectral norm of the covariance matrix of p. In other words, $\rho(p) = \|A\|_2$, where $A = Cov_{X \sim p}(X)$ is the covariance matrix.

An upper bound of $\psi(p)$ of the same form is relatively easy and it was shown to be achieved by half-spaces [1]. Proving the lower bound on $\psi(p)$ up to some small factors in Conjecture 1 is the main goal of this paper. We say a log-concave density is *isotropic* if its mean $\mathbb{E}_{X \sim p}[X]$ equals to 0 and its covariance $\mathrm{Cov}_{X \sim p}(X)$ equals to $\mathbb{I}_d$. In the case of isotropic log-concave densities, the KLS conjecture states that any isotropic log-concave density has its isoperimetric coefficient lower bounded by a universal constant.

In this talk, we make use of the stochastic localization proof technique introduced by Eldan [2] to prove a lower bound of the isoperimetric coefficient with the state-of-the-art dimension dependency $d^{-o_d(1)}$.

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Optimal transport with respect to costs attaining infinite values

Katarzyna Wyczesany
(joint work with Shiri Artstein-Avidan, Shay Sadovsky)

In collaboration with S. Artstein-Avidan and S. Sadovsky, we are working on a project that builds on the work of S. Artstein-Avidan and V. Milman on *order reversing functional dualities* [1, 2] where it was shown that on the class of lower semi-continuous convex function (Cvx) the famous Legendre transform, defined as $\mathcal{L}\varphi(y) = \sup_x(\langle x, y\rangle - \varphi(x))$, is essentially the unique order-reversing involution. Somewhat surprisingly, on the subclass of Cvx of non-negative convex functions that take value zero at the origin (Cvx_0) there are *two* essentially different order-reversing involutions: $\mathcal{L}$ and the Polarity transform $\mathcal{A}$, defined as $\mathcal{A}\varphi(y) = \sup_{\{x:\langle x,y\rangle>1\}} \frac{\langle x,y\rangle-1}{\varphi(x)}$.

Both Legendre and Polarity transforms can be re-written as *cost transforms*: given a cost function $c : X \times Y \to \mathbb{R} \cup \{+\infty\}$ and a function $\varphi : X \to [-\infty, +\infty]$, its c-transform $\varphi^c : Y \to [-\infty, \infty]$ is given by

$$\varphi^c(y) = \inf_x \left(c(x, y) - \varphi(x)\right).$$

It is folklore that the cost function related to the Legendre transform is the *quadratic cost* $\|x - y\|^2/2$ (or equivalently $-\langle x, y\rangle$). The *polar* cost associated with $\mathcal{A}$ is of the form $-\ln(\langle x, y\rangle - 1)_+$. This cost function assumes infinite values and does not fall to any well-understood family of costs considered in transport theory, hence it became the motivating example for our research.

Given a cost function $c : X \times Y \to (-\infty, \infty]$, the classical mass transport problem of Monge is to find the transport map $T : X \to Y$ of infimal total cost, that is

$$\inf\left\{\int c(x, Tx)d\mu : T_{\#}\mu = \nu\right\}. \tag{1}$$

Here $T_{\#}\mu$ stands for a measure on Y such that for every measurable set $A \subset Y$ we have $(T_{\#}\mu)(A) = \mu(T^{-1}(A))$. Since a transport map does not always exist one considers a generalization of the problem to finding a transport plan, i.e. a probability measure $\gamma \in \mathcal{P}(X \times Y)$ whose marginals are μ and ν.

In the classical case, one usually considers finite-valued cost functions and the theory is well-developed. However, despite the interest in cost functions attaining value $+\infty$, which corresponds to prohibiting certain pairs of points to be mapped to one another, there was no unified approach and only special families of costs were considered (see for example [3–5]). In our work [11, 12] we were able to find new methods and presented a unified tool to address these cases.

The celebrated theorem of Brenier [6] (generalised by McCann [7]) which states that for $X = Y = \mathbb{R}^n$ and the quadratic cost $c(x, y) = \|x - y\|^2/2$ one may find T which attains the infimum (1), and moreover, this optimal map (which is called the Brenier map) is given as the gradient of some convex function φ. The function φ for which $\nabla\varphi = T$ is called a *potential* for the map T. This elegant result, which has a multitude of applications is proven using an important geometric interpretation of optimality called *cyclic monotonicity* of the support of the optimal plan. Rockafellar's Theorem [8] states that a set is cyclically monotone if and only if it lies in the subgradient of some convex function. The generalization to other finite-valued costs was established by Rochet and Rüschendorf [9, 10] (for a general cost one considers c-subgradient of a function φ defined as $\partial^c\varphi = \{(x, y) : \varphi(x)+\varphi^c(y) = c(x, y)\}$). It was known, however, that one cannot expect this result to hold if the cost admits the value $+\infty$ (see [13, Example 3.1.]). In [11] we show that the potential exists if and only if the support of the plan is *"path-bounded"*, which is a new notion that we introduce, and which implies cyclic monotonicity (if the cost is finite-valued these notions coincide). The result relies on our new method which reduces the problem of finding a potential to proving solvability of a special (possibly infinite) family of linear inequalities. As a consequence, we obtain a new and elementary proof of the Rockafellar-Rochet-Rüschendorf theorem.

Furthermore, in [12], we establish sufficient 'compatibility conditions' between two measures, which together with the assumption that there exists a plan with finite total cost, imply the existence of a potential. The discussion of compatibility becomes necessary when, as in our case, some transportation schemes are prohibited. In the simplest case of discrete probability measures with an equal number of atoms, each with equal weight, the condition is that of Hall's marriage theorem (a transport map gives a matching). In the case when one of the measures is discrete, compatibility gives rise to a new notion of *'Hall polytopes'*, which we introduced in [12] and studied in depth. We also present a second approach for general measures (under a continuity assumption on the cost), which utilizes our results from [11]: using known results we establish the existence of the optimal transport plan and associate a directed graph with its support. We show that this graph is strongly connected, which in turn guarantees path-boundedness.

In [12], we define *cost duality for sets.* Let $c : X \times Y \to (-\infty, \infty]$ be a cost function and fix $t \in (-\infty, \infty]$, then for any $K \subset X$ the c-dual of K (we suppress t in the notation as it is a fixed parameter) is given by

$$K^c = \{y : c(x, y) \geq t, \forall x \in K\}.$$

For simplicity assume that $X = Y$ and that the cost is symmetric. Note, that taking the polar set $^\circ$ can be realised, for example, using the classical cost $-\langle x, y\rangle$ and $t = -1$, or the polar cost $-\ln(\langle x, y\rangle - 1)_+$ and $t = \infty$. In [14] we show that any order reversing quasi-involution on subsets of space X corresponds to a cost duality. This characterization offers a unifying point of view that deepens the understanding of the underlying principles and structures. One can also easily create a multitude of examples that, of course, include convex sets containing the origin, but also reciprocal bodies and flowers. Further, we characterize when an order reversing quasi involution on a subclass can be extended to the whole space, discuss the uniqueness of the extension and study invariant sets of such transforms.

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Spectral monotonicity under Gaussian convolution

Bo'az Klartag

(joint work with Eli Putterman)

The Poincaré constant $C_P(\mu)$ of a Borel probability measure μ on $\mathbb{R}^n$ is the smallest constant $C \geq 0$ such that for any locally-Lipschitz function $f \in L^2(\mu)$,

$$\mathrm{Var}_\mu(f) \leq C \cdot \int_{\mathbb{R}^n} |\nabla f|^2 d\mu$$

where $\mathrm{Var}_\mu(f) = \int_{\mathbb{R}^n} f^2 d\mu - \left(\int_{\mathbb{R}^n} f d\mu\right)^2$ and $|\cdot|$ is the Euclidean norm. A non-negative function ρ on $\mathbb{R}^n$ is log-concave if $K = \{x \in \mathbb{R}^n\, ;\, \rho(x) > 0\}$ is convex, and $\log \rho$ is concave in K. An absolutely continuous probability measure on $\mathbb{R}^n$ is called log-concave if it has a log-concave density. An arbitrary probability measure on $\mathbb{R}^n$ is called log-concave if it is the pushforward of some absolutely continuous log-concave probability measure on $\mathbb{R}^k$ under an injective affine map.

We show that the Poincaré constant of a log-concave measure is monotone increasing along the heat flow. Write γ for the standard Gaussian measure in $\mathbb{R}^n$. Slightly improving upon a result of Cattiaux and Guillin [2, Theorem 9.4.3], we show the following:

Theorem 1. Let μ be a log-concave probability measure on $\mathbb{R}^n$. Then,

$$C_p(\mu) \leq C_P(\mu * \gamma).$$

In fact, the entire spectrum of the associated Laplace operator is monotone decreasing. Two proofs of Theorem 1 are given. The first proof analyzes a curvature term of a certain time-dependent diffusion, and the second proof constructs a contracting transport map following the approach of Kim and Milman. Recall that a map $T : \mathbb{R}^n \to \mathbb{R}^n$ is a contraction if $|T(x) - T(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}^n$.

Theorem 2. Let μ be a log-concave probability measure on $\mathbb{R}^n$. Then there exists a contraction $T : \mathbb{R}^n \to \mathbb{R}^n$ that pushes forward $\mu * \gamma$ to μ.

Theorem 2 is reminiscent of Caffarelli's theorem [1], which states that there is a contraction pushing forward γ to μ in the case where the density of μ with respect to the measure γ is log-concave. Theorem 2 implies Theorem 1, as well as similar inequalities between the log-Sobolev constants and isoperimetric constants of $\mu * \gamma$ and μ.

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Kohler-Jobin meets Ehrhard

Galyna Livshyts

(joint work with Orli Herscovici)

Consider a log-concave measure μ on $\mathbb{R}^n$ with density e^{-V}, for some convex function V, and its associated Laplacian

$$L\cdot = \Delta \cdot - \langle \nabla \cdot, \nabla V \rangle.$$

Let K be a convex domain in $\mathbb{R}^n$. By $W^{1,2}(K,\mu)$ denote the weighted Sobolev space and by ∇ the weak gradient. Letting $W_0^{1,2}(K,\mu) = W^{1,2}(K,\mu) \cap \{u|_{\partial K} = 0\}$ (where the boundary value is understood in the sense of the trace), define the μ-torsional rigidity of K as

$$T_\mu(K) = \sup_{u \in W_0^{1,2}(K,\mu)} \frac{(\int_K u\, d\mu)^2}{\int_K |\nabla u|^2 d\mu}.$$

See Pólya and Szegö [16] for more details.

The μ-principal frequency of a domain K is defined to be

$$\Lambda_\mu(K) := \inf_{u \in W_0^{1,2}(K,\mu)} \frac{\int_K |\nabla u|^2 d\mu}{\int_K u^2 d\mu}.$$

Note that the torsional rigidity is monotone increasing while the principle frequency is monotone decreasing, i.e. whenever $K \subset L$, we have $T_\mu(K) \leq T_\mu(L)$ and $\Lambda_\mu(K) \geq \Lambda_\mu(L)$.

In the case when μ is the Lebesgue measure and $L = \Delta$, these quantities have been studied extensively, and are intimately tied with the subject of isoperimetric inequalities. See, e.g. Kawohl [8], Pólya and Szegö [16], Burchard [2], Lieb and Loss [14], Kesavan [9], or Vazquez [18]. In particular, the Faber-Krahn inequality [7], [12], [13] states that the Lebesgue principal frequency of a domain K of a fixed Lebesgue measure is *minimized* when K is a euclidean ball. The result of Saint-Venant (see e.g. [16]) states that, conversely, the torsional rigidity of a domain K of a fixed Lebesgue measure is *maximized* when K is a euclidean ball.

The easiest way to prove these results is via rearrangements. For a set K in $\mathbb{R}^n$, denote by K^* the euclidean ball of the same Lebesgue measure as K. Recall that the Schwartz rearrangement of a non-negative function $u : K \to \mathbb{R}$ is the function $u^* : K^* \to \mathbb{R}$ whose level sets $\{u^* \geq t\}$ are all euclidean balls, and such that $|\{u \geq t\}| = |\{u^* \geq t\}|$ (where $|\cdot|$ stands for the Lebesgue measure.) The Pólya-Szegö principle [16] (which is a consequence of the isoperimetric inequality) implies that $\int_K |\nabla u|^2 dx \geq \int_{K^*} |\nabla u^*|^2 dx$, while the definition of the symmetrization yields that $\int_K u^2 dx = \int_{K^*} (u^*)^2 dx$, and, for a non-sign-changing function u, $\int_K u\, dx = \int_{K^*} u^* dx$. Therefore, the Faber-Krahn and the Saint-Venant results follow (together with some additional information that the extremal function for the torsional rigidity is non-negative).

In the case of the Gaussian measure γ (which is the measure with density $\frac{1}{(\sqrt{2\pi})^n} e^{-\frac{x^2}{2}}$), the analogue of the Schwartz rearrangement was developed by Ehrhard [5, 6]. The euclidean balls are replaced with the Gaussian isoperimetric regions, which are half-spaces (see Sudakov and Tsirelson [17], Borel [3]). In the Gaussian world, K^* is the half-space of the same Gaussian measure as the domain K, and u^* is the function whose level sets are half-spaces, and such that $\gamma(\{u \geq t\}) = \gamma(\{u^* \geq t\})$. The Pólya-Szegö principle is replaced with the analogous Ehrhard principle, which yields $\int_K |\nabla u|^2 d\gamma \geq \int_{K^*} |\nabla u^*|^2 d\gamma$. As a result, whenever the Gaussian measure of the domain K is fixed, the Gaussian principal frequency $\Lambda_\gamma(K)$ is minimized when K is a half-space (see Carlen and Kerce [4]), and the Gaussian torsional rigidity $T_\gamma(K)$ is maximized when K is a half-space (see e.g. Livshyts [12]).

In the Lebesgue world, Pólya and Szegö asked another natural question: if for a set K, not its measure, but its torsional rigidity is fixed, then is the principal frequency still minimized on the euclidean ball? This question was answered in the affirmative by Kohler-Jobin [10, 11] back in the 1970s. The main tool which she developed was the so-called modified torsional rigidity with respect to a function $w \in W_0^{1,2}(K)$. For a general measure μ, one may extend this notion as follows:

$$T_\mu^{mod}(K; w) = \sup_{u \in Cl(w)} \frac{(\int_K u\, d\mu)^2}{\int_K |\nabla u|^2 d\mu},$$

where $Cl(w) \subset W_0^{1,2}(K)$ consists of functions obtained as a composition of some non-decreasing one-dimensional function (vanishing at zero) with the function w. In other words, $Cl(w)$ is the collection of functions which have the same level sets as w, and which vanish at the boundary. As follows from the definition of torsional rigidity, $T_\mu(K) \geq T_\mu^{mod}(K; w)$ for any w. Kohler-Jobin analyzed the concept of modified torsional rigidity, and found an explicit way of describing the maximizing function for $T_\mu^{mod}(K; w)$, given an arbitrary w. This analysis is at the core of the Kohler-Jobin rearrangement technique. For further generalizations and applications, as well as a nice exposition of the topic, see Brasco [1].

We develop the Gaussian analogue of the Kohler-Jobin rearrangement, and show

Theorem 1 (Herscovici, Livshyts). *For any convex domain $K \subset \mathbb{R}^n$, letting H be the half-space such that $T_\gamma(K) = T_\gamma(H)$, we have $\Lambda_\gamma(K) \geq \Lambda_\gamma(H)$.*

We would like to emphasize that the analogy of our work and the works of Kohler-Jobin [10, 11] and Brasco [1] is not to be expected, like in the case of the inequalities of Faber-Krahn and Saint-Venant! Our proof relies heavily on the particular properties of certain special functions, and not just on soft properties of rearrangements. It is somewhat miraculous that Theorem 1 is true at all!

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From harmonic analysis of translation invariant valuations to geometric inequalities

Thomas Wannerer

(joint work with Jan Kotrbatý)

A valuation is a function $\phi\colon \mathcal{K}(\mathbb{R}^n) \to \mathbb{C}$ on the space of convex bodies in $\mathbb{R}^n$ which is additive in the sense that

$$\phi(K \cup L) = \phi(K) + \phi(L) - \phi(K \cap L)$$

holds whenever $K \cup L$ is convex. Here we are interested in the infinite dimensional vector space $\mathrm{Val}(\mathbb{R}^n)$ of translation invariant and continuous valuations. By a theorem of McMullen this space is naturally graded

$$\mathrm{Val}(\mathbb{R}^n) = \bigoplus_{r=0}^{n} \mathrm{Val}_r(\mathbb{R}^n)$$

by the degree of homogeneity of a valuation. Here ϕ is said to be homogeneous of degree r if $\phi(tK) = t^r \phi(K)$ for all convex bodies K and all $t > 0$.

The Alesker-Bernig-Schuster theorem [1] asserts that each irreducible representation of the special orthogonal group appears with multiplicity at most one as a subrepresentation of the space of continuous translation invariant valuations with a fixed degree of homogeneity. Moreover, the theorem describes in terms of highest weights which irreducible representations appear with multiplicity one.

We present a refinement of this result, namely the explicit construction of the highest weight vector in each irreducible subrepresentation. We then describe how important operations on valuations (pullback, pushforward, Fourier transform, Lefschetz operators, Alesker-Poincaré pairing) act on these highest weight vectors.

Let $n \geq 2$ be fixed throughout and let

$$\lambda_{k,m} = (m, \underbrace{2, \ldots, 2}_{k-1}, 0, \ldots, 0).$$

Let $*$ denote the convolution of valuations, let $\iota\colon \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \oplus \mathbb{R}$ and $\pi\colon \mathbb{R}^{n-1} \oplus \mathbb{R} \to \mathbb{R}^{n-1}$, let $\mathbb{F}$ denote the Fourier transform and let Λ denote the Lefschetz operator $\Lambda\phi(K) = \frac{d}{dt}\big|_{t=0} \phi(K+tD^n)$. Let v_n denote the volume of the n-dimensional euclidean unit ball and let s_n denote the surface area of the n-dimensional unit sphere. The following theorem summarizes our results.

Theorem 1. *For any $r, k, m \in \mathbb{N}$ with $r \leq n-1$, $k \leq \min\{r, n-r\}$, and $m \geq 2$, let $\omega^{(n)}_{r,k,m} \in \Omega^{n-1}(S\mathbb{R}^n)$ be given by formula* (6) *below. The smooth valuation*

$$\phi^{(n)}_{r,k,m}(K) = \frac{(\sqrt{-1})^{\lfloor \frac{n}{2} \rfloor} (\sqrt{2})^{m-2}}{s_{n+m-r-3}} \int_{N(K)} \omega^{(n)}_{r,k,m}, \quad K \in \mathcal{K}(\mathbb{R}^n),$$

is a highest weight vector of weight $\lambda_{k,m}$ in the $SO(n)$-representation $\mathrm{Val}_r(\mathbb{R}^n)$. Moreover, these valuations satisfy the following properties:

(1) $$\overline{\phi}^{(n)}_{r,k,m} * \phi^{(n)}_{n-r,k,m} = (-1)^k \frac{(m+k-1)(n+m-k)\binom{n-2k}{r-k} v_{n+2m-2}}{v_{n+m-r-2} v_{r+m-2} s_{2m-3}};$$

(2) $$\iota^* \phi^{(n)}_{r,k,m} = \begin{cases} \phi^{(n-1)}_{r,k,m} & \text{if } r < n-1 \text{ and } k < n-r, \\ 0 & \text{otherwise;} \end{cases}$$

(3) $$\pi_* \phi^{(n)}_{r,k,m} = \begin{cases} \phi^{(n-1)}_{r-1,k,m} & \text{if } k < r, \\ 0 & \text{if } k = r; \end{cases}$$

(4) $$\mathbb{F}\phi^{(n)}_{r,k,m} = (-1)^{k-1} (\sqrt{-1})^m \phi^{(n)}_{n-r,k,m};$$

(5) $$\Lambda \phi^{(n)}_{r,k,m} = \begin{cases} (n-r-k+1) \frac{v_{n+m-r-1}}{v_{n+m-r-2}} \phi^{(n)}_{r-1,k,m} & \text{if } k < r, \\ 0 & \text{if } k = r. \end{cases}$$

Let us give the definition of the valuations $\phi^{(n)}_{r,k,m}$. Let the standard coordinates on $\mathbb{R}^n \times \mathbb{R}^n$ be denoted by $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$ and let $S\mathbb{R}^n = \mathbb{R}^n \times S^{n-1}$ denote the sphere bundle of $\mathbb{R}^n$. Put $l = \lfloor \frac{n}{2} \rfloor$ and define for $j \in \{1, \dots, l\}$

$$z_j = \frac{1}{\sqrt{2}}(x_{2j-1} + \sqrt{-1}x_{2j}),$$
$$z_{\overline{j}} = \frac{1}{\sqrt{2}}(x_{2j-1} - \sqrt{-1}x_{2j}),$$
$$\zeta_j = \frac{1}{\sqrt{2}}(\xi_{2j-1} + \sqrt{-1}\xi_{2j}),$$
$$\zeta_{\overline{j}} = \frac{1}{\sqrt{2}}(\xi_{2j-1} - \sqrt{-1}\xi_{2j}).$$

Let us assume for the sake of simplicity from now on that n is even and put

$$\mathcal{I} = \{1, \overline{1}, \dots, l, \overline{l}\}.$$

For any subset $I \subset \mathcal{I}$ define 0-forms on $\mathbb{R}^n \times \mathbb{R}^n$ with values in the exterior algebra $\Lambda(\mathbb{R}^n)^* \otimes \mathbb{C}$ by

$$dz_I = \sum_{i \in I} z_i \otimes dz_i,$$
$$d\zeta_I = \sum_{i \in I} \zeta_i \otimes dz_i$$

and put $\Theta = dz_1 \wedge dz_{\overline{1}} \wedge \cdots \wedge dz_l \wedge dz_{\overline{l}}$.

We define translation invariant differential $(n-1)$-forms on $S\mathbb{R}^n$ by

$$\omega_{r,k,m} = \zeta_{\overline{1}}^{m-2}\omega_{r,k} \in \Omega^{n-1}(S\mathbb{R}^n)^{\mathrm{tr}} \tag{6}$$

where

$$\omega_{r,k} \otimes \Theta = \frac{1}{(n-r-1)!(r-k)!k!}\zeta_J \wedge (d\zeta_J)^{n-r-1} \wedge (dz_J)^{r-k} \wedge (\overline{dz_K})^k$$

and $K = \{1, \dots, k\}$, $J = \mathcal{I} \setminus K$.

We explain how the information of Theorem 1 yields a proof of the Hodge-Riemann relations for valuations in the case of euclidean balls as reference bodies, generalizing previous results of Kotrbatý [3] for even valuations and for valuations of degree 1. Finally, we discuss how the Hodge-Riemann relations imply new geometric inequalities for convex bodies and explain how our results together with [4] extend the scope of the inequalities for mixed volumes recently discovered by Alesker [2].

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Hodge-Riemann relations in valuations and inequalities for mixed volumes

Semyon Alesker

Very recently Kotrbatý [5] has discovered a new fundamental property of valuations - the Hodge-Riemann (HR) type relations. He proved them for even valuations, and most recently Kotrbatý and Wannerer [6] proved HR for odd valuations. These results, in combination with previously developed theory, opened a way for applications of valuations to geometric inequalities.

Furthermore, Kotrbatý has formulated a conjectural more general (mixed) version of HR relations (MHR). We are going to formulate this conjecture as well as two new geometric inequalities which follow from Kotrbatý's theorem.

Let us introduce some background. Let $Val^\infty(V)$ denote the space of smooth translation invariant valuations on an n-dimensional vector space V equipped with the Garding topology. We have McMullen's decomposition with respect to degrees of homogeneity

$$Val^\infty(V) = \oplus_{i=0}^n Val_i^\infty(V).$$

Set $Val^{\infty,i}(V) := Val_{n-i}^\infty(V)$.

Theorem 1 (Bernig-Fu [3]). *Fix a Lebesgue measure vol on V.*
(1) There exists a unique continuous (in the Garding topology) map called convolution

$$*\colon Val^\infty \times Val^\infty \to Val^\infty$$

such that if $\phi(\bullet) = vol(\bullet + A)$, $\psi(\bullet) = vol(\bullet + B)$ *then*

$$(\psi * \psi)(\bullet) = vol(\bullet + A + B).$$

(2) $(Val^\infty, *)$ *is a commutative associative algebra with a unit (= vol).*
(3) $Val^{i,\infty} * Val^{j,\infty} \subset Val^{i+j,\infty}$.

Moreover $(Val^\infty, *)$ satisfies the Poincaré duality:

$$Val^{i,\infty} \times Val^{n-i,\infty} \xrightarrow{*} Val^{n,\infty} = \mathbb{C}\cdot\chi$$

is a perfect paring, i.e. for any non-zero valuation $\phi \in Val^{i,\infty}$ there exists $\psi \in Val^{\infty,n-i}$ such that $\phi * \psi \neq 0$.

In the conjecture below all bodies A_i are assumed to have smooth positively curved boundary. Denote by V_A the mixed volume $V_A(\bullet) := V(\bullet[n-1], A)$.

Conjecture 2 (Kotrbatý [5]). *(1) (MHL) Let* $i < n/2$ *then the map* $Val^{i,\infty} \to Val^{n-i,\infty}$ *given by*

$$\phi \mapsto \phi * V_{A_1} * \cdots * V_{A_{n-2i}}$$

is an isomorphism.

(2) (MHR) Let $i \leq n/2$. *Define primitive subspace*

$$P^i = \{\phi \in Val^{i,\infty} | \, \phi * V_{A_1} * \cdots * V_{A_{n-2i}} * V_{A_{n-2i+1}} = 0\}.$$

Then the Hermitian form on P^i

$$\phi \mapsto (-1)^i \phi * \bar{\phi} * V_{A_1} * \cdots * V_{A_{n-2i}} \geq 0$$

with equality iff $\phi = 0$.

Remark 3. (1) In the special case when all $A_i = B$ are the Euclidean balls part (1) was proved by me [1] (even valuations) and Bernig-Bröcker [4] (general case).

(2) In the special case when all $A_i = B$ part (2) was proved by Kotrbatý [5] for even valuations, and for odd ones the result was announced in 2021 by Kotrbatý-Wannerer.

(3) The case $i = 1$ of part (2) implies easily the Alexandrov-Fenchel inequalities. This case was proved by Kotrbatý-Wannerer [6] using the method of the proof of the Alexandrov-Fenchel inequality. It implies a few new inequalities as well, in the known case when all $A_i = B$ I proved one of them.

As we have explained, in the case when all $A_i = B$ are Euclidean balls the above conjecture is proven. In this case, it has an equivalent version on the language of the product on valuations which is obtained by applying the Fourier type transform (this also was observed by Kotrbatý). From the latter version I obtained two new inequalities for mixed volumes as follows.

Theorem 4 (Alesker [2])**.** *Let* $n \geq 2$. *Let* $A_1, \ldots, A_{n-1} \subset \mathbb{R}^n$ *be convex compact sets. Then*

$$V_{2n}(\iota_1(A_1), \ldots, \iota_1(A_{n-1}); \iota_2(A_1), \ldots, \iota_2(A_{n-1}); \Delta(B)[2]) \geq$$
$$V_{2n}(\iota_1(A_1), \ldots, \iota_1(A_{n-1}); -\iota_2(A_1), \ldots, -\iota_2(A_{n-1}); \Delta(B)[2]).$$

Theorem 5 (Alesker [2])**.** *Let* $n \geq 2$. *Let* $A_1, \ldots, A_{n-1} \subset \mathbb{R}^n$ *be convex compact sets. One has*

$$V_{2n}(\iota_1(A_1), \ldots, \iota_1(A_{n-1}); \iota_2(A_1), \ldots, \iota_2(A_{n-1}); \Delta(B)[2]) +$$
$$V_{2n}(\iota_1(A_1), \ldots, \iota_1(A_{n-1}); -\iota_2(A_1), \ldots, -\iota_2(A_{n-1}); \Delta(B)[2]) \leq$$
$$\gamma_n V_n(A_1, \ldots A_{n-1}, B)^2,$$

where γ_n *is such a constant that the equality is achieved for* $A_1 = \cdots = A_{n-1} = B$.

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Intrinsic volumes on Kähler manifolds

Andreas Bernig
(joint work with Joe Fu, Gil Solanes, Thomas Wannerer)

The classical Steiner formula for the volume of parallel bodies is the easiest way to define intrinsic volumes of convex bodies. It admits a differential-geometric analogue, Weyl's tube formula, which applies to submanifolds in euclidean spaces. Surprisingly, the coefficients in Weyl's tube formula only depend on the intrinsic geometry of the submanifold, and not on the embedding. They are also called "intrinsic volumes".

In Alesker's modern framework of "valuations on manifolds", both notions of intrinsic volume really are the same. Weyl's theorem can then be rephrased by saying that for every Riemannian manifold, there is a canonical subalgebra in the algebra $\mathcal{V}(M)$ of smooth valuations, the Lipschitz-Killing algebra, and this assignment is compatible with isometric embeddings. If the dimension of M is n, then the Lipschitz-Killing algebra is isomorphic to the algebra $\mathrm{Val}^{\mathrm{SO}(n)}$ of continuous, translation invariant and $\mathrm{SO}(n)$-invariant valuations on $\mathbb{R}^n$. By Hadwiger's theorem and results by Alesker, $\mathrm{Val}^{\mathrm{SO}(n)} \cong \mathbb{R}[t]/(t^{n+1})$.

In an ongoing work with Joe Fu, Thomas Wannerer, and Gil Solanes, we give a complex version of this theorem. The algebra $\mathrm{Val}^{\mathrm{U}(n)}$ of continuous, translation invariant and $\mathrm{U}(n)$-invariant valuations on $\mathbb{C}^n$ was described by Fu [3] as

$$\mathrm{Val}^{\mathrm{U}(n)} \cong \mathbb{R}[t,s]/(f_{n+1}, f_{n+2}), \quad \log(1+tx+sx^2) = \sum_i f_i(t,s)x^i.$$

This algebra was studied in detail in [1].

A Kähler manifold is a complex manifold with a compatible Riemannian metric such that the fundamental form is closed. Kähler geometry thus combines Riemannian, complex and symplectic geometry.

Theorem 1. *For any Kähler manifold M of complex dimension n, there is a canonical subalgebra $\mathrm{KLK}(M) \subset \mathcal{V}(M)$, isomorphic to $\mathrm{Val}^{\mathrm{U}(n)}$, with the property that if $M' \hookrightarrow M$ is a Kähler embedding then the natural restriction map $\mathcal{V}(M) \to \mathcal{V}(M')$ restricts to a natural surjection $\mathrm{KLK}(M) \to \mathrm{KLK}(M')$.*

As a corollary, we find a canonical isomorphism between the algebra of isometry invariant valuations on complex projective space and $\mathrm{Val}^{\mathrm{U}(n)}$, which gives a satisfactory explanation for phenomena observed in [2]. In particular, the global or local kinematic formulas on complex projective space are formally identical to the kinematic formulas on $\mathbb{C}^n$.

In the Riemannian setting, there are two different ways to prove the Weyl principle. The first is by using Nash' theorem and restricting valuations from an ambient Euclidean space. This does not work in the Kähler case, since not every Kähler manifold may be embedded in flat space. The second proof uses a description of all invariant differential forms on the sphere bundle of a Riemannian manifold that can be built within the Cartan apparatus. Although this could work in the Kähler case as well, we were not able to overcome the combinatorial difficulties in this approach. Fortunately, a mixture of both approaches works well: first use restrictions for embedded Kähler manifolds to construct a sequence of differential forms, and then extend these forms to all Kähler manifolds.

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Improved log-concavity for rotationally invariant measure

Liran Rotem

(joint work with Dario Cordero-Erausquin)

Let $W : \mathbb{R}^n \to (-\infty, \infty]$ be a measurable function, and let μ denote the Borel measure on $\mathbb{R}^n$ with density $\frac{\mathrm{d}\mu}{\mathrm{d}x} = e^{-W}$. It is well known that if W is convex then μ is log-concave, i.e. it satisfies the Brunn-Minkowski type inequality

$$\text{(BM)} \qquad \mu\left((1-\lambda)A + \lambda B\right) \geq \mu(A)^{1-\lambda}\mu(B)^{\lambda}$$

for all Borel sets $A, B \subseteq \mathbb{R}^n$ and $0 < \lambda < 1$ (see e.g. [1]). In fact the converse is also true: if (BM) holds then W is equal almost everywhere to a convex function. By taking in (BM) $\lambda = \frac{1}{2}$ and $A = B = K$, a convex body in $\mathbb{R}^n$, we deduce that

$$\text{(BM}^*\text{)} \qquad \mu\left(\frac{a+b}{2}K\right) \geq \sqrt{\mu(aK)\mu(bK)}.$$

We will now improve the inequalities (BM) and (BM*), under the additional assumption that the sets involved are convex and symmetric. We say that $K \subseteq \mathbb{R}^n$ is a *convex body* if K is convex and compact. We say that K is *symmetric* if $K = -K$.

Definition. (1) μ satisfies the (B) property if for every symmetric convex body $K \subseteq \mathbb{R}^n$ and every $a, b > 0$,

$$\text{(B)} \qquad \mu(\sqrt{ab}K) \geq \sqrt{\mu(aK)\,\mu(bK)}.$$

(2) μ satisfies the Gardner-Zvavitch property if for every symmetric convex bodies $K, L \subseteq \mathbb{R}^n$ and every $0 < \lambda < 1$

$$\text{(GZ)} \qquad \mu\left((1-\lambda)K + \lambda L\right)^{\frac{1}{n}} \geq \lambda\mu(K)^{\frac{1}{n}} + (1-\lambda)\mu(L)^{\frac{1}{n}}.$$

Clearly (B) implies (BM*) and (GZ) implies (BM) when the sets involved are convex and symmetric. There are non-trivial examples of measures that satisfy (B) and (GZ):

Theorem. The Gaussian measure γ with density $\frac{\mathrm{d}\gamma}{\mathrm{d}x} = \frac{1}{(2\pi)^{\frac{n}{2}}}e^{-|x|^2/2}$ satisfies both (B) and (GZ).

The fact that γ satisfies (B) was conjectured by Banaszczyk and first published by Latała ([11]). The conjecture was proved by Cordero-Erausquin, Fradelizi and Maurey ([3]). The fact that γ satisfies(GZ) was first conjectured by Gardner and Zvavitch ([7]). It was proved by Eskenazis and Moschidis ([5]). Their result relied on a previous paper of Kolesnikov and Livshyts ([8]), which proved the same result with exponent $\frac{1}{2n}$ instead of $\frac{1}{n}$. The tools used in [8] were previously developed by Kolesnikov and Milman in [10] and [9] to attack other Brunn-Minkowski type inequalities such as the L^p-Brunn-Minkowski conjecture.

Before this work we did not have many examples of measures which satisfy either (B) or (GZ) except γ. For (GZ) one can trivially take the Lebesgue measure restricted to a convex body T, but no other examples were known. In [12] Livshyts did show that (GZ) holds for every even log-concave measure, but with the optimal exponent $\frac{1}{n}$ replaced by $\frac{1}{n^{4+o(1)}}$. Eskenazis, Nayar and Tkocz showed that the (B) property holds for certain Gaussian mixtures, which include the measure $\mathrm{d}\mu = e^{-\|x\|_p^p}\mathrm{d}x$ for $0 < p \leq 1$ ([6]). It is also known that *if* the log-Brunn-Minkowski conjecture holds in dimension n, then every even log-concave measure μ on $\mathbb{R}^n$ satisfies (B) and (GZ) ([14], [13]). In particular this is true in dimension $n = 2$ ([2]).

Our work ([4]) concerns measures with the following property:

Definition. We say μ is of the form $(\star)$ if μ has density $e^{-w(|x|)}$, where $w : (0,\infty) \to (-\infty,\infty)$ is an increasing function such that $t \mapsto w(e^t)$.

Theorem 1. *Every measure of the form $(\star)$ satisfies (B) and (GZ).*

This theorem creates for the first time a large family of examples of measures which satisfy (B) and (GZ). This includes all rotation invariant log-concave measures, and in particular the measures μ_p with density $e^{-|x|^p/p}$ for all $p > 0$. In particular see that μ_∞, the uniform measure on the unit ball B_2^n, satisfies both properties. But there are also many examples which are not log-concave. For example, for every $\beta > 0$ the measure ν_β with density $\frac{1}{\left(1+|x|^2\right)^\beta}$ also satisfies (B) and (GZ).

The main ingredient in the proof that every measure of the form $(\star)$ satisfies (B) is the following result which is of independent interest:

Theorem 2. *Assume μ satisfies $(\star)$, and let ν be an even measure which is log-concave with respect to μ (i.e. $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} = e^{-V}$ for $V : \mathbb{R}^n \to (-\infty,\infty]$ convex). Let $f : \mathbb{R}^n \to \mathbb{R}$ be an* even *function. Then*

$$\mathrm{Var}_\nu f \leq \int \left\langle \left(\nabla^2 W + \frac{w'(|x|)}{|x|} Id\right)^{-1} \nabla f, \nabla f \right\rangle \mathrm{d}\nu.$$

Equality holds when $\nu = \mu$ and $f = \langle \nabla W, x\rangle$.

The inverse of $\nabla^2 W + \frac{w'(|x|)}{|x|}\,\mathrm{Id}$ can by computed explicitly. For example for μ_p we obtain

$$\begin{aligned}\mathrm{Var}_{\mu_p} f &\leq \left(\frac{1}{2}|x|^{2-p}|\nabla f|^2 - \frac{p-2}{2p}\cdot\frac{\langle \nabla f, x\rangle^2}{|x|^p}\right)\mathrm{d}\mu_p \\ &\leq \max\left\{\frac{1}{p},\frac{1}{2}\right\}\cdot\int |x|^{2-p}|\nabla f|^2\,\mathrm{d}\mu_p.\end{aligned}$$

Theorem 2 is proved using L^2 methods similar to the ones used in [3]. There, the theorem reduced to the Gaussian Poincaré inequality, which is well known. In our case, Theorem 2 reduces to the far less trivial:

Theorem 3. *Assume μ satisfies ($\star$), and let ν be an even measure which is log-concave with respect to μ. Let $h : \mathbb{R}^n \to \mathbb{R}$ be an* odd *function. Then*

$$\int \frac{w'(|x|)}{|x|}h^2\mathrm{d}\nu \leq \int |\nabla h|^2\,\mathrm{d}\nu.$$

Theorem 3 is proved using polar coordinates. One of the main ingredients is an infinitesimal version of the Brunn-Minkowski inequality (BM) which follows from a result of Kolesnikov and Milman [10]. This concludes our sketch of the proof of the (B) property.

Finally, in order to show that every measure of the form ($\star$) satisfies (GZ) one also needs to use Theorem 3. In fact, one simply repeats the proof of [8] and [5]: The proof there only needs the fact that $\mu = \gamma$ in order to use the Gaussian Poincaré inequality. Replacing this inequality with our Theorem 3, the prove transfers with minor changes.

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Sharp inequalities for the mean distance of random points in convex bodies

Christoph Thäle

Let $K \subset \mathbb{R}^d$ be a convex body and X_1, X_2 be two independent random points selected according to the uniform distribution on K. By

$$\Delta(K) := \mathbb{E}\|X_1 - X_2\|$$

we denote the expected length of the line segment connecting X_1 with X_2, the so-called *mean distance of* K. Explicit values for $\Delta(K)$ are rare and known only for a few particular special cases. In the plane, these are a regular triangle, a rectangle, a regular hexagon, a circle or an ellipse. In higher space dimensions only the values for the ball B^d and an ellipsoid with semi-axes $a_1, \dots, a_d > 0$ are available from papers of Miles and Heinrich, respectively:

$$\Delta(B^d) = \frac{2^{2d+2} d \cdot \left[\Gamma\left(\frac{d}{2}+1\right)\right]^2}{(2d+1)!!(d+1)\pi},$$

and

$$\Delta(K) = \frac{2^{d+1}\left[\Gamma\left(\frac{d}{2}+1\right)\right]^3}{(d+1)\pi^{(d+1)/2}\Gamma\left(d+\frac{3}{2}\right)} \int_{\mathbb{S}^{d-1}} \sqrt{a_1^2 u_1^2 + \ldots + a_d^2 u_d^2}\, \sigma(\mathrm{d}u).$$

Moreover, it is known from the work of Blaschke (in the plane) and Pfiefer (in all space dimensions) that the ratio $\Delta(K)/V_d(K)^{1/d}$ between the mean distance and the dth root of the volume of K is minimized precisely if K is a ball. In fact,

$$\Delta(K) \geq \frac{2^{2d+2} d \cdot \left[\Gamma\left(\frac{d}{2}+1\right)\right]^{2+1/d}}{(2d+1)!!(d+1)\pi^{3/2}} V_d(K)^{1/d}.$$

Motivated in particular by the explicit value for the mean distance of an ellipsoid, in which the integral term can be identified, up to a dimension-dependent constant, with the first intrinsic volume, our approach to inequalities for $\Delta(K)$ uses as a different normalization the first intrinsic volume $V_1(K)$ of K. We show that

$$\frac{3d+1}{2(d+1)(2d+1)} < \frac{\Delta(K)}{V_1(K)} < \frac{1}{3}.$$

We also argue that this inequality is best possible in the sense that there are two sequences K_δ and K'_δ of d-dimensional convex bodies such that

$$\lim_{\delta\to 0}\frac{\Delta(K_\delta)}{V_1(K_\delta)}=\frac{3d+1}{2(d+1)(2d+1)}\qquad\text{and}\qquad\lim_{\delta\to 0}\frac{\Delta(K'_\delta)}{V_1(K'_\delta)}=\frac{1}{3}.$$

More explicitly, K_δ and K'_δ our examples given by

$$K_\delta := \operatorname{conv}(e_1,-e_1,\delta e_2,\delta e_3,\dots,\delta e_d),$$
$$K'_\delta := [-1,1]\times[0,\delta]^{d-1}.$$

The material presented in this talk is based on a joint work [1] with G. Bonnet, A. Gusakova and D. Zaporozhets.

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Explicit representations of isotropic measures in extremal positions

Julián Haddad
(joint work with Fernanda M. Baêta)

Let $K \subset \mathbb{R}^n$ be a convex body. In 1948 Fritz John [3] studied the problem of determining the ellipsoid $\mathcal{E}_J \subseteq K$ of maximal volume inside K (known today as *John's Ellipsoid*) and showed a set of necessary conditions for $\mathcal{E}_J$ to be the unit Euclidean ball $\mathbb{B}$ (Theorem 1 below). A position of a convex body K is a set of the form $A(K)$ where A is an invertible affine transformation. We say that K is in John position if $\mathcal{E}_J = \mathbb{B}$. A construction that is dual to John's ellipsoid is the *Löwner ellipsoid* $\mathcal{E}_L \supseteq K$ which is the unique ellipsoid of minimal volume containing K. The set K is in Löwner position if $\mathcal{E}_L = \mathbb{B}$. John's Theorem can be stated as follows.

Theorem 1. [3, Application 4, pag. 199 - 200] *Assume K is in John (resp. Löwner) position, then there exists a finite set of points $\{\xi_1,\dots,\xi_m\} \in S^{n-1}\cap\partial K$, positive numbers $\{c_1,\dots,c_m\}$ and $\lambda \neq 0$, for which*

$$\sum_i c_i\xi_i\otimes\xi_i=\lambda\ \mathrm{Id}\quad\text{and}\quad\sum_i c_i\xi_i=0. \tag{1}$$

Here $v\otimes w$ is the rank-one matrix $(v\otimes w)_{i,j}=v_iw_j$.

A measure μ on the sphere S^{n-1} is said to be *isotropic* if

$$\int_{S^{n-1}}(\xi\otimes\xi)d\mu=\lambda\ \mathrm{Id},$$

for some $\lambda \neq 0$, and *centered* if

$$\int_{S^{n-1}}\xi d\mu=0.$$

Then we see that equation (1) can be expressed as the fact that the atomic measure $\mu_K = \sum_i c_i \delta_{\xi_i}$ is centered and isotropic.

Even though Theorem 1 is a central result about the geometry of convex sets, and the literature around John position and the decomposition of the identity is vast, the existence of the contact points and weights ξ_i, c_i are generally proved in a non-constructive way. In this work we propose the following minimization procedure aimed at computing a measure μ as above.

Theorem 2. *Let K be a convex body in Löwner position. Choose any finite positive and non-zero measure ν in $\partial K \cap S^{n-1}$, and any C^1 function $F : \mathbb{R} \to \mathbb{R}$ that is non-negative, non-decreasing, convex, strictly convex in $[0, \infty)$, and assume $F'(0) > 0$. Let S_0 be the linear space of $n \times n$ symmetric matrices of zero trace, and consider the convex functional $I_\nu : S_0 \times \mathbb{R}^n \to \mathbb{R}$ defined by*

$$I_\nu(M, w) = \int_{S^{n-1}} F(\langle \xi, M\xi + w\rangle) d\nu(\xi).$$

If I_ν has a unique global minimum (M_0, w_0), then the measure

$$F'(\langle \xi, M_0\xi + w_0\rangle) d\nu(\xi)$$

is non-negative, non-zero, centered and isotropic.

Let us consider the situation where $\partial K \cap S^{n-1}$ is finite. In this case, a natural choice of ν is the counting measure c. We obtain the following:

Corollary 3. *Let K be a convex body in Löwner position and assume*

$$\partial K \cap S^{n-1} = \{\xi_1, \dots, \xi_m\}.$$

Choose any C^1 function $F : \mathbb{R} \to \mathbb{R}$ that is non-negative, non-decreasing, convex, strictly convex in $[0, \infty)$, and assume $F'(0) > 0$. Consider the convex functional $I_c : S_0 \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$I_c(M, w) = \sum_{i=1}^{m} F(\langle \xi_i, M\xi_i + w\rangle).$$

If I_c has a unique global minimum (M_0, w_0), then the numbers

$$c_i = F'(\langle \xi_i, M_0\xi_i + w_0\rangle), \; i = 1, \dots, m$$

together with the vectors ξ_i, $i = 1, \dots, m$, satisfy equation (1).

Depending on the set $\partial K \cap S^{n-1}$ and the measure ν, the function I_ν might or might not have a minimum. This can be a consequence of a "bad choice" of ν, or of the fact that $\partial K \cap S^{n-1}$ is in a "degenerate" position. To make this precise we recall the following properties about John/Löwner position.

Theorem 4. *Let L be any convex body. The following statements are equivalent*

(1) *L is in Löwner position.*
(2) *$L \subseteq \mathbb{B}$ and for every $(M, w) \in (S_0 \times \mathbb{R}^n) \setminus (0, 0)$ there exists $\xi \in \partial K \cap S^{n-1}$ for which $\langle \xi, M\xi + w\rangle \geq 0$.*
(3) *$L \subseteq \mathbb{B}$ and $(\frac{1}{n} \mathrm{Id}, 0) \in \mathrm{co}\left(\{(\xi \otimes \xi, \xi) / \xi \in \partial K \cap S^{n-1}\}\right)$.*

Recalling that K is always assumed to be in Löwner position, we can prove:

Theorem 5. *The following statements are equivalent*

(1) *I_ν has a unique global minimum (M_0, w_0).*
(2) *For every $(M, w) \in (S_0 \times \mathbb{R}^n) \setminus (0, 0)$*

$$\nu(\{\xi \in \partial K \cap S^{n-1} / \langle \xi, M\xi + w\rangle > 0\}) > 0.$$

If $\partial K \cap S^{n-1}$ is finite and $\nu = c$, the statements above are also equivalent to the following:

(3) *$(\frac{1}{n}\operatorname{Id}, 0)$ lies in the relative interior of $\operatorname{co}\left(\{(\xi \otimes \xi, \xi)/\xi \in \partial K \cap S^{n-1}\}\right) \subseteq S_1 \times \mathbb{R}^n$, where S_1 is the set of matrices of trace 1.*

Although the proof of Theorem 2 is straightforward, the geometric interpretation of the minimizer (M_0, w_0) is not. In [2], Artstein and Katzin introduced a new one-parameter family of positions: A convex body K is said to be in *maximal intersection position of radius r* if $r\mathbb{B}$ is the ellipsoid maximizing $\operatorname{vol}(r\mathbb{B} \cap K)$ among all ellipsoids of same volume as $r\mathbb{B}$. It is also shown that every centrally symmetric convex body K admits at least one of such positions $T_r K$ with $T_r \in \operatorname{SL}_n(\mathbb{R})$, and in this case the uniform measure in $S^{n-1} \cap r^{-1}T_r K$ is isotropic (modulo some technical assumptions). Then the following constructive proof of the existence of the measure μ is given.

Theorem 6 (Theorem 1.6, [2])**.** *Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body in Löwner position. For every $r < 1$, denote by ν_r the uniform probability measure on $S^{n-1} \cap r^{-1}T_r K$, where $T_r K$ is in maximal intersection position of radius r. Then there exists a sequence $r_j \nearrow 1$ such that the sequence of measures ν_{r_j} weakly converges to an isotropic measure whose support is contained in $\partial K \cap S^{n-1}$.*

In this work we show that Theorems 6 and 2 are intimately related, in the sense that Theorem 2 can be thought as an "infinitesimal version" of Theorem 6. Also, a geometric interpretation of the minimizer (M_0, w_0) in Theorem 2 is given.

This work was published in a recent preprint [1].

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Sharp affine isoperimetric inequalities for the volume decomposition functionals of polytopes

Ge Xiong
(joint work with Yude Liu and Qiang Sun)

A *polytope* in $\mathbb{R}^n$ is the convex hull of a finite set of points in $\mathbb{R}^n$ provided it has positive *volume* V_n (i.e., n-dimensional Lebesgue measure). The convex hull of a subset of these points is called a *face* of the polytope if it lies entirely on the boundary of the polytope and if it has positive $(n-1)$-dimensional Lebesgue measure. Write $\mathcal{P}_o^n$ for the set of polytopes in $\mathbb{R}^n$ with the origin in their interiors.

Let $P \in \mathcal{P}_o^n$ and u be a unit outer normal vector to a face F of P. The *cone-volume* $V_P(\{u\})$ of P associated with u is the volume of the convex hull of the origin o and face F. The simplest form of cone-volume is reduced to the area formula of triangles in ancient geometry. However, it is striking that combining the notions of cone-volume of polytopes, linear independence of vectors and dimension of spaces, a *new* geometry of polytopes has emerged: The nth power of volume of polytopes in $\mathbb{R}^n$ is naturally decomposed into n terms, and each term is a homogeneous polynomial of degree n.

In [3], we introduce the so-called *volume decomposition functional* of polytopes.

Definition 1. Suppose $P \in \mathcal{P}_o^n$, and $u_1, u_2, \ldots, u_N$ are the unit outer normal vectors of its faces. We define the *kth volume decomposition functional* $X_k(P)$, $k = 1, 2, \ldots, n-1, n$, by

$$X_k(P)^n = \sum_{\dim(\operatorname{span}\{u_{i_1},\ldots,u_{i_n}\})=k} V_P(\{u_{i_1}\})V_P(\{u_{i_2}\})\cdots V_P(\{u_{i_n}\}).$$

As usual, $\operatorname{span}\{u_{i_1}, \ldots, u_{i_n}\}$ in the above definition denotes the linear subspace spanned by n normal vectors $u_{i_1}, \ldots, u_{i_n}$ of the polytope P. Obviously $X_k(P)^n$ is a homogeneous polynomial in cone-volumes of degree n, $k = 1, 2, \ldots, n$; $X_k(P)$ is *centro-affine* invariant, i.e., $X_k(TP) = X_k(P)$ for $T \in \mathrm{SL(n)}$; and $X_k(\lambda P) = \lambda^n X_k(P)$ for $\lambda > 0$.

It is interesting that if $k = n$, then

$$X_n(P) = \Big(\sum_{u_{i_1}\wedge\cdots\wedge u_{i_n}\neq 0} V_P(\{u_{i_1}\})V_P(\{u_{i_2}\})\cdots V_P(\{u_{i_n}\})\Big)^{\frac{1}{n}},$$

which is *identical* to the functional U introduced by E. Lutwak, D. Yang and G. Zhang (LYZ) [4] to attack the longstanding unsolved *Schneider projection problem* in convex geometry; For $k \neq n$, all the functionals $X_k(P)$ are totally *new*.

It is interesting that the nth power of the volume functional $V_n(P)$ satisfies the following tidy *identity*

$$V_n(P)^n = X_1(P)^n + X_2(P)^n + \cdots + X_n(P)^n, \tag{1}$$

which says that the nth power of volume $V_n(P)$ of a polytope P is decomposed into n homogeneous polynomials $X_k(P)^n$, $k = 1, 2, \ldots, n$. Moreover, the identity (1) suggests that these new functionals X_k, $k = 1, 2, \ldots, n-1$, are *complementary* to the Lutwak-Yang-Zhang U functional. So, in some sense, we trace the origin of LYZ's U functional for the first time.

In 2001, LYZ [4] conjectured that if P is a polytope in $\mathbb{R}^n$ with its centroid at the origin, then

$$\frac{X_n(P)}{V_n(P)} \geqslant \frac{(n!)^{\frac{1}{n}}}{n} \tag{2}$$

with equality if and only if P is a *parallelotope*.

It took more than one dozen years to completely settle this conjecture [1, 2, 5].

Back to the new volume decomposition functionals X_k, $k = 1, 2, \ldots, n-1$, in light of the identity (1), we are tempted to raise the following problem.

Problem X. Let P be a polytope in $\mathbb{R}^n$ with its centroid at the origin. Does there exist a constant $c(n, k)$ depending on n and $k, k \in \{1, 2, \ldots, n-1\}$, such that

$$\frac{X_k(P)}{V_n(P)} \leq c(n, k)?$$

The Problem X in $\mathbb{R}^3$ is satisfactorily solved by Liu-Sun-Xiong [3].

Theorem 1. *If P is a polytope in $\mathbb{R}^3$ with its centroid at the origin, then*

$$\frac{X_1(P)}{V_3(P)} \leq \left(\frac{1}{3}\right)^{\frac{2}{3}}, \quad \frac{X_2(P)}{V_3(P)} \leq \left(\frac{2}{3}\right)^{\frac{1}{3}}, \quad \textit{and} \quad \frac{X_3(P)}{V_3(P)} \geq \frac{2^{\frac{1}{3}}}{3^{\frac{2}{3}}},$$

and equality holds in each inequality if and only if P is a parallelepiped.

Theorem 2. *If P is a polytope in $\mathbb{R}^n$ with its centroid at the origin, then*

$$\frac{X_1(P)}{V_n(P)} \leq n^{\frac{1}{n}-1}$$

with equality if and only if P a parallelotope.

Restricted to $\mathcal{P}_3^n$, i.e., the set of polytopes in $\mathbb{R}^n$ whose any *three* outer normal vectors (up to their antipodal normal vectors) are linear independent, we prove the following.

Theorem 3. *If P is a polytope in $\mathcal{P}_3^n$ with its centroid at the origin and $n \geq 3$, then*

$$\frac{X_2(P)}{V_n(P)} \leq n^{\frac{1}{n}-1}[(2^{n-1}-1)(n-1)]^{\frac{1}{n}}$$

with equality if and only if P a parallelotope.

In the appendix of [3], we show that the set $\mathcal{P}_k^n$, $k = 1, 2, \ldots, n$, is dense in $\mathcal{K}_o^n$ in the sense of Hausdorff metric.

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Blaschke–Santaló inequality for many functions and geodesic barycenters of measures

Elisabeth M. Werner
(joint work with Alexander V. Kolesnikov)

Motivated by the geodesic barycenter problem from optimal transportation theory, we prove a natural generalization of the Blaschke-Santalo inequality and the affine isoperimetric inequalities for many sets and many functions. We derive from it an entropy bound for the total Kantorovich cost appearing in the barycenter problem.

We list next two of our results. The complete work can be found in [1]

Theorem. *Let* $f_i\colon \mathbb{R}^n \to \mathbb{R}_+$, $1 \le i \le k$, *be unconditional integrable functions satisfying*

$$\prod_{i=1}^{k} f_i(x_i) \le \rho\left(\sum_{\substack{i,j=1\\ i<j}}^{k} \langle x_i, x_j\rangle\right) \quad \textit{for every } \; x_i, x_j \in \mathbb{R}^n_+,$$

where ρ *is a positive non-increasing function on* $[0,\infty)$ *such that* $\int_{\mathbb{R}} \rho^{\frac{1}{k}}(t^2)dt < \infty$. *Then*

$$\prod_{i=1}^{k} \int_{\mathbb{R}^n} f_i(x_i)dx_i \le \left(\int_{\mathbb{R}^n} \rho^{\frac{1}{k}}\left(\frac{k(k-1)}{2}|u|^2\right) du\right)^k.$$

For $k > 2$, *equality holds in this inequality if and only if there exist positive constants* c_i, $1 \le i \le k$, *such that* $\prod_{i=1}^{k} c_i = 1$, *and such that for all* $1 \le i \le k$, *almost everywhere on* $\mathbb{R}^n$,

(1)
$$f_i(x) = c_i\, \rho^{\frac{1}{k}}\left(\frac{k(k-1)}{2}|x|^2\right).$$

(2) *The function* ρ *satisfies the inequality*

$$\prod_{i=1}^{k} \rho^{\frac{1}{k}}\left(\frac{k(k-1)}{2}\,|x_i|^2\right) \le \rho\left(\sum_{i,j=1,i<j}^{k} \langle x_i, x_j\rangle\right).$$

Our proof uses the Prékopa–Leindler inequality for many functions and an exponential change of variables as an intermediate step.

We study equality cases for unconditional functions and prove the above-stated equality characterizations. To do so, we need equality characterizations in the Prékopa–Leindler inequality. We could not find such characterizations in the literature and therefore give a proof of those.

We prove a generalization of the Blaschke–Santaló inequality which involves more than two convex bodies. There, $\|\cdot\|_K$ denotes the norm with the convex body K as unit ball.

Theorem. *Let K_i, $1 \le i \le k$, be unconditional convex bodies in $\mathbb{R}^n$ such that*

$$\prod_{i=1}^{k} e^{-\frac{1}{2}\|x_i\|_{K_i}^2} \le \rho\left(\sum_{i,j=1,i<j}^{k} \langle x_i, x_j\rangle\right) \quad \text{for every } x_i, x_j \in \mathbb{R}^n_+,$$

where ρ is a positive non-increasing function $[0,\infty)$ such that $\int_{\mathbb{R}} \rho^{\frac{1}{k}}(t^2)dt < \infty$. Then

$$\prod_{i=1}^{k} \mathrm{vol}_n(K_i) \le \left(\frac{\mathrm{vol}_n(B_2^n)}{(2\pi)^{\frac{n}{2}}}\right)^k \left(\int_{\mathbb{R}^n} \rho^{\frac{1}{k}}\left(\frac{k(k-1)}{2}\,|x|^2\right)dx\right)^k.$$

For $k > 2$, equality holds if and only if $K_i = r\,B_2^n$ and $\rho(t) = e^{-\frac{t}{(k-1)r^2}}$ for some $r > 0$.
In particular, if $\rho(t) = e^{-\frac{t}{k-1}}$, then, if $\sum_{i=1,i<j}^{k}\langle x_i, x_j\rangle \le \frac{k-1}{2}\sum_{i=1}^{k}\|x_i\|_{K_i}^2$, we have that

$$\prod_{i=1}^{k} \mathrm{vol}_n(K_i) \le (\mathrm{vol}_n(B_2^n))^k$$

and for $k > 2$ equality holds if and only if $K_i = B_2^n$ for all $1 \le i \le k$.

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On the L^p Aleksandrov problem for negative p

Stephanie Mui

The integral curvature measure, also known as the first curvature measure introduced by Federer [2] for sets of positive reach, was first defined by Aleksandrov [1]. The corresponding classical Aleksandrov problem is a type of Minkowski problem and asks about the existence of a convex body with prescribed integral curvature. When the given measure has density $f : S^{n-1} \to (0,\infty)$, the Aleksandrov problem amounts to solving the following Monge-Ampère-type partial differential equation

$$\det\left(\nabla_{ij}^2 h + h\delta_{ij}\right) = \frac{f \cdot \left(|\nabla h|^2 + h^2\right)^{\frac{n}{2}}}{h},$$

where ∇h is the gradient of h, $\nabla_{ij}^2 h$ is the Hessian of h, and δ_{ij} is the identity matrix with respect to an orthonormal frame on S^{n-1}. The classical Aleksandrov problem was solved by Aleksandrov [1] first for polytopes and then generalized via an approximation argument. Oliker [5] gave an alternate solution to the existence question using mass transport for the polytope case and then extended it to more general shapes with the same approximation approach. Huang-Lutwak-Yang-Zhang [4] provided another solution to the existence problem for even measures with a direct variational proof.

The L^p Aleksandrov problem arose in work by Huang-Lutwak-Yang-Zhang [3], where the concept of dual curvature measures $\widetilde{C}_q(K,\cdot)$ and related variational formulas were discovered. The dual Minkowski problem, which analogously asks about the existence and uniqueness of a convex body with predetermined dual curvature measure, interpolates between some previously disconnected questions. In particular, the $q = 0$ case of the dual Minkowski problem becomes the classical Aleksandrov problem, and the $q = n$ case is the logarithmic Minkowski problem.

The L^p integral curvature comes from a variational formula in Huang-Lutwak-Yang-Zhang [4] for a certain entropy integral. For each $K \in \mathcal{K}_o^n$, define its entropy $\mathcal{E}$ by

$$\mathcal{E}(K) = -\int_{S^{n-1}} \log h_K(v)\ dv.$$

Then for each $p \neq 0$ and $K \in \mathcal{K}_o^n$, we define (see Huang-Lutwak-Yang-Zhang [4]) the L^p integral curvature measure, $J_p(K,\cdot)$, of K as the Borel measure on S^{n-1} that satisfies

$$\frac{d}{dt}\mathcal{E}(K \hat{+}_p t\cdot L)\bigg|_{t=0} = \frac{1}{p}\int_{S^{n-1}} \rho_L(u)^{-p}\ dJ_p(K,u)$$

for all $L \in \mathcal{K}_o^n$, where the L^p harmonic combination is defined as $K \hat{+}_p t\cdot L = (K^* +_p t{\cdot}L^*)^*$, and K^* is the polar of K. It turns out that the L^p integral curvature measure is related to the classical integral curvature measure in the following way

$$dJ_p(K,\cdot) = \rho_K^p\ dJ(K,\cdot).$$

Observe that when $p = 0$, $J_0(K,\cdot) = J(K,\cdot)$, the classical case.

The L^p Aleksandrov problem asks about the existence of a convex body with predetermined L^p integral curvature. More specifically:

Problem. Fix $p \in \mathbb{R}$. What are the necessary and sufficient conditions on a given Borel measure μ on S^{n-1} so that there exists a convex body $K \in \mathcal{K}_o^n$ with $\mu = J_p(K,\cdot)$?

It was shown that if μ has density f, this problem amounts to solving the Monge-Ampère-type partial differential equation

$$\det\left(\nabla_{ij}^2 h + h\delta_{ij}\right) = \frac{f\cdot\left(|\nabla h|^2 + h^2\right)^{\frac{n}{2}}}{h^{1-p}},$$

where ∇h is the gradient of h (unknown function), $\nabla^2_{ij}h$ is the Hessian of h, and δ_{ij} is the identity matrix with respect to an orthonormal frame on S^{n-1}. Huang-Lutwak-Yang-Zhang [4] completely solved existence for when $p > 0$.

Theorem (Huang-Lutwak-Yang-Zhang 2018)**.** Suppose $p \in (0, \infty)$ and μ is a finite Borel measure on S^{n-1}. Then there exists $K \in \mathcal{K}^n_o$ such that μ is the L^p integral curvature measure of K if and only if μ is not concentrated on any great subsphere.

Furthermore, Huang-Lutwak-Yang-Zhang [4] solved existence under some strong conditions for the origin symmetric case and when $p < 0$. More specifically,

Theorem (Huang-Lutwak-Yang-Zhang 2018)**.** Suppose $p \in (-\infty, 0)$ and μ is a finite, even, nonzero Borel measure on S^{n-1} that vanishes on all great subspheres of S^{n-1}. Then there exists $K \in \mathcal{K}^n_o$ such that μ is the L^p integral curvature measure of K.

Note that this result excludes many shapes, including polytopes. Zhao [6] addressed part of this gap by proving existence for origin symmetric polytopes and $p \in (-1, 0)$.

Theorem (Zhao 2019)**.** Suppose $p \in (-1, 0)$ and μ is a finite, even, discrete, nonzero Borel measure on S^{n-1} . Then there exists an origin symmetric polytope $K \in \mathcal{K}^n_o$ such that μ is the L^p integral curvature measure of K if and only if μ is not concentrated on any great subsphere of S^{n-1}.

We will extend the result by Zhao [6] by completely proving existence for the origin-symmetric case of the L^p Aleksandrov problem, for $p \in (-1, 0)$.

Theorem 1. *Let $-1 < p < 0$ and μ be a nonzero even finite Borel measure on S^{n-1}. Then there exists an origin symmetric convex body $K \in \mathbb{R}^n$ such that $\mu = J_p(K, \cdot)$ if and only if μ is not completely concentrated on any lower-dimensional subspace.*

For the remaining negative index cases ($p \leq -1$), we will weaken the assumptions on the $p < 0$ existence result by Huang-Lutwak-Yang-Zhang [4] from completely no concentration to requiring some measure concentration condition. More specifically, we will show the following:

Theorem 2. *Let $p \leq -1$ and μ be a nonzero even finite Borel measure on S^{n-1}. Suppose, on all great subspheres $\xi \subset S^{n-1}$, that*

$$\frac{\mu(\xi)}{\mu(S^{n-1})} \leq C(n)^p,$$

where $C(n) = \exp\left[\frac{1}{2}\left(\psi\left(\frac{n}{2}\right) - \psi\left(\frac{1}{2}\right)\right)\right]$ is a constant depending only on n, and ψ is the digamma function. Then there exists a $K \in \mathcal{K}^n_e$ such that $\mu = J_p(K, \cdot)$.

The approach for both of these results is variational. We first convert the existence question into an optimization problem and then prove the existence of an optimizer. Whether the measure concentration bound in Theorem 2 is optimal is an open problem. Furthermore, the non-origin symmetric case of the L^p Aleksandrov problem for negative p remains unsolved.

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A quick estimate for the volume of a polyhedron

Mark Rudelson
(joint work with Alexander Barvinok)

The problem of efficient computation (approximation) of the volume of a polytope, and, more generally, of a given convex body has attracted a lot of attention. The most successful approach is via Markov Chain Monte Carlo randomized algorithms. In particular, randomized algorithms allow one to approximate the volume of a polytope in $\mathbb{R}^n$ within relative error $\epsilon > 0$ in time polynomial in n and ϵ^{-1}. The polytope can be defined as the convex hull of a finite set of points or as the intersection of halfspaces, or by a membership oracle, in which case the algorithms extend to the class of all "well-conditioned" convex bodies.

Deterministic algorithms enjoyed less success. For a general convex body $B \subset \mathbb{R}^n$, the only available polynomial time algorithm approximates volume within a factor of $n^{O(n)}$, using an approximation of B by an ellipsoid, see [2]. For B defined by a membership oracle, this approximation factor is basically the best possible (up to some logarithmic terms) that can be achieved in deterministic polynomial time [1]. If P is a polytope defined as the convex hull of a set of points or as the intersections of halfspaces, deterministic algorithms in principle may turn out to be as powerful as randomized ones, but so far the approximation ratio achieved in deterministic polynomial time is the same as for general convex bodies. We remark that if $P \subset \mathbb{R}^n$ is a polytope defined as the convex hull of $n + O(1)$ points or as the intersection of $n + O(1)$ halfspaces, then $\text{vol}(P)$ can be computed exactly in polynomial time, in the former case by a triangulation into $n^{O(1)}$ simplices and in the latter case by a dual procedure of expressing P as a signed linear combination of $n^{O(1)}$ simplices.

We consider the class of polyhedra P defined as the intersection of the non-negative orthant $\mathbb{R}^n_+$ and an affine subspace in $\mathbb{R}^n$. In coordinates, P is defined by a system of linear equations $Ax = b$, where A is an $m \times n$ matrix, x is an n-vector of variables and b is an m-vector, and inequalities $x \geq 0$, meaning that the coordinates of x are non-negative. We assume that $m < n$, that $\operatorname{rank}(A) = m$ and that P has a non-empty relative interior, that is, contains a point x with all coordinates positive. Hence $\dim P = n - m$ and we measure the $(n-m)$-dimensional volume of P in its affine span with respect to the Euclidean structure inherited from $\mathbb{R}^n$. We also assume that P is bounded, that is, a polytope. Generally, any $(n-m)$-dimensional polyhedron with n facets can be represented as the intersection of $\mathbb{R}^n_+$ and an affine subspace of codimension m. Furthermore, many interesting polyhedra, such as transportation polytopes are naturally defined in this way.

We present a deterministic polynomial time algorithm which approximates the volume of such a polytope P within a factor of γ^m, where $\gamma > 0$ is an absolute constant (for m large enough, one can choose $\gamma = 4.89$). In fact, our algorithm is basically a *formula*. The only "non-formulaic" part of our algorithm consists of solving some standard convex optimization problem on P, namely finding its *analytic center* $z = (\zeta_1, \ldots, \zeta_n) \in P$ defined as the unique point of maximum of the function

$$f(x) = n + \sum_{j=1}^{n} \ln \xi_j$$

over $x = (\xi_1, \ldots, \xi_n) \in P$. The function f is strictly concave and hence the maximum point z can be found in polynomial time. With this notation,

$$\alpha^n \mathcal{F}(A, b) \leq \operatorname{vol}(P) \leq \beta^n \mathcal{F}(A, b),$$

where

$$\mathcal{F}(A, b) = e^{f(z)} \sqrt{\frac{\det(AA^\top)}{\det\Big(A\big(\operatorname{diag}(\zeta_1, \ldots, \zeta_n)\big)^2 A^\top\Big)}}$$

and $\alpha < 1 < \beta$ are some explicit absolute constants such that $\beta/\alpha < \gamma$. To evaluate the estimator $\mathcal{F}(A, b)$, we only need to compute two $m \times m$ determinants, which, as is well-known, can be accomplished in $O(m^3)$ time. While the approximation factor γ^m looks big compared to $1 + \epsilon$ achieved by randomized algorithms, it appears to be the best achieved to date by a deterministic polynomial time algorithm for many interesting classes of polytopes, such as transportation polytopes. Since the algorithm is basically a formula, it allows one to analyze how the volume changes as P evolves inside its class, which turns out to be important for studying some statistical phenomena related to contingency tables. The approximation factor looks more impressive when $n \gg m$, which is indeed the case for many interesting classes of polytopes. Note that if we dilate a d-dimensional polytope by a factor of $(1 + \epsilon)$, its volume gets multiplied by $(1 + \epsilon)^d$. Hence sometimes one considers the volume ratio

$$\operatorname{vr}(P) = \inf_{\substack{\mathcal{E} \subset P \\ \mathcal{E} - \text{ ellipsoid}}} \left(\frac{\operatorname{vol}(P)}{\operatorname{vol}(\mathcal{E})}\right)^{1/d}$$

which has an advantage of being invariant under linear transformations. Since the optimal ellipsoid $\mathcal{E}$ can be approximated in polynomial time, our algorithm (formula) approximates the volume ratio within a factor of $1 + o(1)$ whenever $n \gg m$.

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The Bochner Formula in Convex Geometry

Emanuel Milman

Bochner's formula is a classical tool in geometry for relating between a differential operator, usually the Laplacian, with curvature of the underlying manifold, usually Ricci curvature. On a weighted Riemannian manifold $(K, g, \mu = \exp(-V)d\text{Vol}_K)$, Bochner's identity reads as follows:

$$\frac{1}{2}\Delta_\mu|\nabla_g u|^2 = \langle \nabla_g \Delta_\mu u, \nabla_g u\rangle + \|\text{Hess}_g u\|^2 + \langle \text{Ric}_{g,\mu} \nabla_g u, \nabla_g u\rangle\ ,$$

where $\Delta_\mu = \Delta - \langle \nabla_g V, \nabla_g\rangle$ is the weighted Laplacian and $\text{Ric}_{g,\mu} = \text{Ric}_g + \text{Hess}_g V$ is the weighted Ricci curvature. When the manifold in question has a boundary ∂K, an integration of Bochner's identity results in additional boundary terms, and the resulting integrated form is called Reilly's formula. As observed in [6], it is actually useful to perform a second integration by parts on ∂K, yielding:

$$\int_K (\Delta_\mu u)^2 d\mu = \int_K \left(\langle Ric_{g,\mu} \nabla_g u, \nabla_g u\rangle + \|\text{Hess}_g u\|^2\right) d\mu$$
$$+ \int_{\partial K} H_{\partial K,\mu} u_\nu^2 d\mu^{\partial K} - 2\int_{\partial K} \langle \nabla_{\partial K} u_\nu, \nabla_{\partial K} u\rangle\, d\mu^{\partial K} + \int_{\partial K} \langle \text{II}_{\partial K} \nabla_{\partial K} u, \nabla_{\partial K} u\rangle\, d\mu^{\partial K}.$$

Here $d\mu^{\partial K} = \exp(-V)d\text{Vol}_{\partial K}$, $u_\nu = \nabla_\nu u$ where ν is the outer unit-normal to ∂K, $\text{II}_{\partial K}$ is the second fundamental form and $H_{\partial K,\mu} = tr(\text{II}_{\partial K}) - V_\nu$ is the weighted mean-curvature.

The Bochner and Reilly formulas are powerful tools for obtaining Poincaré-type inequalities on a manifold K (with possible boundary ∂K), when combined with the so-called L^2-method, going back (at least) to the work of Lichnerowicz and Hörmander from the 1950-60's. Given a test function f on K, the idea is to solve:

$$\Delta_\mu u = f \text{ on } K\ ,\ u/u_\nu \equiv C \text{ on } \partial K,$$

where C is a constant chosen to make the above PDE solvable (in the case of Neumann boundary conditions), and apply the Bochner / Reilly formula to the "dual function" u. This idea has been used in the context of Brunn-Minkowski theory to obtain Brascamp–Lieb-type inequalities, Prékopa–Leindler inequalities, Thin-Shell estimates, Blaschke–Santaló inequalities, Busemann inequalities, and more.

In our joint work with Sasha Kolesnikov [5] (which was eventually split into two halves for publication purposes [6, 7]), we introduced a twist on the above template, which may be called a "boundary L^2-method". Given a test function f on the *boundary* ∂K, the idea is to exchange the role played by f and C as the right-hand-side of the PDE on the interior and on the boundary, and solve:

$$\Delta_\mu u \equiv C \text{ on } K \ , \ u/u_\nu = f \text{ on } \partial K.$$

Applying the Reilly formula to u on K, one will obtain a Poincaré-type inequality on the boundary ∂K.

Using this idea, we obtained in [5] a direct proof of Colesanti's Poincaré-type inequality [4] on the boundary of a smooth, strongly convex body K in $\mathbb{R}^n$ of volume 1, which Colesanti derived as an equivalent local form of the classical Brunn-Minkowski inequality:

$$\int_{\partial K} H_{\partial K} f^2 dx - \frac{n-1}{n}\left(\int_{\partial K} f dx\right)^2 \leq \int_{\partial K} \left\langle \mathrm{II}_{\partial K}^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \right\rangle dx \ \ \forall f \in C^1(\partial K).$$

In fact, our proof applies to any weighted-Riemannian manifold (K, g, μ) satisfying the Bakry–Émery Curvature-Dimension $\mathsf{CD}(0, N)$ condition and convex boundary ∂K, yielding a novel Brunn–Minkowski inequality on such manifolds and a new proof of the Brunn–Minkowski inequality in $\mathbb{R}^N$.

In our joint follow-up work [8], we extended the above proof to incorporate a trace Poincaré inequality for lower-bounding the term $\int \|\mathrm{Hess}_g u\|^2 \, d\mu$ appearing in the Bochner formula (where u is harmonic), which we had previously discarded. As a consequence, we were able to obtain a strengthened version of the Brunn–Minkowski inequality for origin-symmetric convex bodies K, confirming the local form of the even L^p-Brunn–Minkowski conjecture of Böroczaky–Lutwak–Yang–Zhang (BLYZ) [1, 2] for $p = 1 - \frac{c}{n^{3/2}}$. Thanks to local-to-global results of Chen–Huang–Li–Liu [3] and Putterman [11], our results can be globalized.

In [8], we interpreted the (local form of the) BLYZ conjecture as an even spectral-gap estimate for the Hilbert–Brunn–Mikowski (HBM) operator Δ_K, which we introduced. Under different normalization, this operator was used by Hilbert himself in his spectral proof of the Brunn–Minkowski inequality. With our normalization, Δ_K is the weighted-Laplacian associated to the weighted Riemannian manifold $(\mathbb{S}^{n-1}, g_K, V_K)$, where V_K is the cone volume measure of K and g_K is seemingly mysterious Riemannian metric which pops up from the computations. We showed that the (local form of the) BLYZ conjecture is equivalent to showing that the even spectral-gap $\lambda_{1,e}(-\Delta_K)$ is minimized for (an appropriate approximation of) the cube. We also conjectured that it should be maximized for Euclidean balls. In [9], we were finally able to resolve the latter conjecture: for every C^2-smooth strongly convex body K, one has $\lambda_{1,e}(-\Delta_K) \leq 2n$ with equality if and only if K is a centered ellipsoid.

Finally, we presented a novel connection between the Brunn–Minkowski theory to centro-affine differential geometry which we developed in [10]. It turns out that when equipping ∂K with the centro-affine normalization, one naturally obtains from the corresponding Gauss equation the Riemannian metric g_K as the induced centro-affine second fundamental form, and the cone volume measure V_K as the induced volume form. This normalization turns every ∂K into a centro-affine sphere having curvature equal to 1, and in particular, the induced centro-affine connection ∇_K has Ricci curvature equal to $n-2$. As a consequence, the HBM operator coincides with the centro-affine Laplacian, yielding a natural explanation for its centro-affine equivariance. One may then derive a centro-affine Bochner formula of the form:

$$\int (\Delta_K f)^2 dV_K = \int \left\| \mathrm{Hess}^*_K f \right\|^2_{g_K} dV_K + (n-2) \int |\mathrm{grad}_{g_K} f|^2 dV_K,$$

where Hess^*_K is the Hessian with respect to the conjugate connection ∇^*_K. The Brunn–Minkowski inequality is then an immediate consequence of the Lichnerowicz classical spectral-gap estimate under positive Ricci curvature.

Using the above centro-affine machinery, we were able in [10] to resolve the isomorphic version of the BLYZ even log-Brunn–Minkowski conjecture: for any origin-symmetric convex body K in $\mathbb{R}^n$, there exists another origin-symmetric convex body $\tilde{K}$ at a (geometric, or Banach-Mazur) distance of at most 8 from it, so that $\tilde{K}$ satisfies the conjectured even spectral-gap lower-bound, satisfies the log-Minkowski inequality, and is uniquely determined by its cone volume measure $V_{\tilde{K}}$. The constant 8 can be improved to $1+\epsilon$ if $d_{BM}(K, \mathrm{Ball}) \ll \sqrt{n}$ to begin with.

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Khinchin inequalities and sections of convex bodies

Tomasz Tkocz

(joint work with Giorgos Chasapis, Keerthana Gurushankar, Hermann König, Piotr Nayar)

We discuss a natural connection between Khinchin-type inequalities for negative moments and extremal-volume sections of convex bodies. This has recently led to a certain probabilistic extension of Ball's cube slicing result, as well as new stability results for hyperplane sections of ℓ_p-balls. We highlight several results from recent works [2–4], emphasising our initial motivation as well as the natural probabilistic context behind the geometric results. For comprehensive references and a historical account, we refer to the aforementioned papers.

Let $Q_n = [-\frac{1}{2}, \frac{1}{2}]^n$ be the centred cube of unit volume in $\mathbb{R}^n$. Throughout, $a = (a_1, \dots, a_n)$ denotes a *unit* vector in $\mathbb{R}^n$, that is with $|a| = \sqrt{a_1^2 + \cdots + a_n^2} = 1$. Motivated by questions in geometric number theory, Good conjectured that the minimal volume section of the cube Q_n by a linear subspace is 1 (plainly attained by coordinate subspaces). This was proved by Hensley and earlier, independently by Hadwiger, in the case of hyperplane, that is co-dimenision one sections (see [He, 5]). A quite suprising, deep and influential reversal was established by Ball in [1]. Together, these results state that for every $n \geq 2$ and every unit vector a in $\mathbb{R}^n$, we have

$$\begin{aligned}(1)\qquad \mathrm{vol}_{n-1}\left(Q_n \cap (1,0,\dots,0)^\perp\right) &\leq \mathrm{vol}_{n-1}\left(Q_n \cap a^\perp\right) \\ &\leq \mathrm{vol}_{n-1}\left(Q_n \cap (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \dots, 0)^\perp\right).\end{aligned}$$

It is rather unexpected that the maximising direction is not the diagonal one $\left(\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}\right)$, in contrast with the analogous results about sections of ℓ_p-balls when $0 < p < 2$.

Let us recall a classical result in probability going back to early works of Khinchin on the law of the iterated logarithm for independent random signs $\varepsilon_1, \varepsilon_2, \dots$, with a sharp version due to Haagerup (see [6, 8]): for every $n \geq 2$ and every unit vector a in $\mathbb{R}^n$, we have

$$(2)\qquad \mathbb{E}\left|\sum_{j=1}^n a_j\varepsilon_j\right|^q \geq \begin{cases} \lim_{m\to\infty} \mathbb{E}\left|\sum_{j=1}^m \frac{1}{\sqrt{m}}\varepsilon_j\right|^q, & q_0 \leq q \leq 2, \\ \mathbb{E}\left|\frac{1}{\sqrt{2}}\varepsilon_1 + \frac{1}{\sqrt{2}}\varepsilon_2\right|^q, & 0 < q \leq q_0, \end{cases}$$

where $q_0 = 1.84..$ is the numerical constant such that for $q = q_0$ the two expressions on the right hand side are equal. Here we see a phase transition of the extremising (minimising) direction from the (asymptotic) diagonal one to the one featured in Ball's result (1). (The reverse bound $\mathbb{E}\left|\sum_{j=1}^n a_j\varepsilon_j\right|^q \leq \mathbb{E}|\varepsilon_1|^q = 1$, $0 < q < 2$ is a direct consequence of the monotonicity of the moments, as $\mathbb{E}\left|\sum_{j=1}^n a_j\varepsilon_j\right|^2 = \sum_{j=1}^n a_j^2 = 1$.)

Our main result from [3] provides an extension of (1) to *negative* moments, akin to (2): for every $n \geq 2$ and every unit vector a in $\mathbb{R}^n$, we have

$$\mathbb{E}\left|\sum_{j=1}^n a_j U_j\right|^q \leq \begin{cases} \lim_{m\to\infty} \mathbb{E}\left|\sum_{j=1}^m \frac{1}{\sqrt{m}} U_j\right|^q, & q_1 \leq q < 0, \\ \mathbb{E}\left|\frac{1}{\sqrt{2}}U_1 + \frac{1}{\sqrt{2}}U_2\right|^q, & -1 < q \leq q_1. \end{cases} \tag{3}$$

Here $U_1, U_2, \ldots$ are independent random variables uniform on $[-\frac{1}{2}, \frac{1}{2}]$ and $q_1 = -0.79..$ is the numerical constant such that for $q = q_1$ the two expressions on the right hand side coincide. To see that this result is indeed an extension of the upper bound from (1), note that

$$\lim_{q \downarrow -1} \frac{1+q}{2} \mathbb{E}\left|\sum_{j=1}^n a_j U_j\right|^q = \mathrm{vol}_{n-1}\left(Q_n \cap a^\perp\right). \tag{4}$$

Such sharp inequalities for $q \geq 1$ were established by Latała and Oleszkiewicz in [9]. Out main result from [2] closes the picture by addressing the case $0 < q < 1$: for every $n \geq 2$ and every unit vector a in $\mathbb{R}^n$, we have

$$\mathbb{E}\left|\sum_{j=1}^n a_j U_j\right|^q \geq \lim_{m\to\infty} \mathbb{E}\left|\sum_{j=1}^m \frac{1}{\sqrt{m}} U_j\right|^q, \qquad 0 < q < 1. \tag{5}$$

The methods of [2] and [3] build up on Ball's, Haagerup's as well as Nazarov and Podkorytov's works (see [1, 6, 11]). One highlight is that to overcome certain technicalities, we employ an inductive argument on the number of summands n using random vectors uniformly distributed on Euclidean spheres in $\mathbb{R}^3$ instead of the initial uniform distribution on $[-\frac{1}{2}, \frac{1}{2}]$. We only briefly mention that it has been elusive how to extend this Archimedes' Hat-Box *trick* to address for instance lower-dimensional sections.

We finish with indicating that (4) generalises to subspace sections of arbitrary dimension of arbitrary symmetric convex bodies and can be efficiently used to establish stability results for extremal-volume hyperplane sections of B_p^n, the n-dimensional ℓ_p-ball, in all regimes of p where the extremisers are known (see [4], as well as [10] for an independent and different approach when $p = \infty$). What remains open is the case of the maximum-volume section when $2 < p < \infty$. We offer the following conjecture: for every $n \geq 2$, there is a unique $2 < p_n^* < \infty$ with

$$\begin{aligned} &\max_{a \in \mathbb{R}^n,\ |a|=1} \mathrm{vol}_{n-1}\left(B_p^n \cap a^\perp\right) \\ &\quad = \begin{cases} \mathrm{vol}_{n-1}\left(Q_n \cap (\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})^\perp\right), & 2 < p \leq p_n^* \\ \mathrm{vol}_{n-1}\left(Q_n \cap (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \ldots, 0)^\perp\right), & p_n^* \leq p < \infty, \end{cases} \end{aligned} \tag{6}$$

where the two values on the right hand side coincide for $p = p_n^*$.

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p-Alexandrov type inequalities

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(joint work with A. Carbery, F. MacIntyre)

We will study quantities which can be associated to a collection of surfaces in Euclidean space, and which take the form

$$Q_j^p(\mathbb{S}_1, \ldots, \mathbb{S}_j) := \left(\int_{S_j} \cdots \int_{S_1} |v_1(x_1) \wedge \cdots \wedge v_j(x_j)|^p \mathrm{d}\sigma_1(x_1) \ldots \mathrm{d}\sigma_j(x_j) \right)^{1/jp}.$$

Each S_i is a surface in Euclidean space $\mathbb{R}^d$ which is equipped with a suitable surface measure σ_i, each $v_i : S_i \to \mathbb{R}^d$ is a measurable vector field,= the wedge product $|v_1 \wedge \cdots \wedge v_j|$ is interpreted as the j-dimensional volume of the parallelotope with edges $v_1, \ldots, v_j \in \mathbb{R}^d$, and p is a parameter.

A general upper bound is given by the following theorem.

Theorem 1. *Let $\mathbb{S}_1, \ldots, \mathbb{S}_j$ be generalised d-hypersurfaces with $d \geq j$ and let $0 < p < \infty$. Then*

$$Q_j^p(\mathbb{S}_1, \ldots, \mathbb{S}_j) \leq \prod_{i=1}^{j} Q_{j-1}^p(\mathbb{S}_1, \ldots, \widehat{\mathbb{S}_i}, \ldots, \mathbb{S}_j)^{1/j}.$$

The above theorem is sharp. Let $\mathbb{S}_i$ be the unit cube in the coordinate hyperplane in $\mathbb{R}^j$ which is perpendicular to e_i, together with Lebesgue measure and unit normal e_i. In this case $Q_j^p(\mathbb{S}_1, \ldots, \mathbb{S}_j)$ and each term on the RHS are 1 and so we see that we cannot improve the constant in the above theorem to anything smaller than 1. On the other hand, in the diagonal case when each $\mathbb{S}_i = \mathbb{S}$, on a set of positive measure the vectors $\{v(x_1), \dots, v(x_j)\}$ fail to be mutually orthogonal, and so we shall have strict inequality. Indeed, in the diagonal case and $p = 1$, we have that:

Theorem 2. *Let $\mathbb{S}$ be a generalised d-hypersurface as above. For $1 \le j \le d-1$ we have*

$$Q^1_{j+1}(\mathbb{S}) \le \frac{Q^1_{j+1}(\mathbb{S}^{d-1})}{Q^1_j(\mathbb{S}^{d-1})}\, Q^1_j(\mathbb{S}),$$

where, for $1 \le j \le d$,

$$Q^1_j(\mathbb{S}^{d-1}) = \omega_{d-1} \left(\frac{\omega_d d!}{\omega_{d-j}(d-j)!} \right)^{1/j}.$$

To prove this we need a connection with convex geometry through mixed volumes. Then, the result comes after using Alexandrov-Fenchel inequality.

For $p = 2$ we can prove a similar theorem using the Alexandrov inequality for mixed discriminants.

For other values of p we have to study the quantity

$$W_{k,p} = \left(\frac{1}{\omega_k^p} \int_{G_{n,k}} |P_E K|^p \, d\nu_{n,k}(E) \right)^{1/kp}.$$

Our goal is to compare $W_{k,p}$ with $W_{m,p}$ and especially, $W_{k,p}$ with $W_{k-1,p}$.

- In [1] the authors proved Alexandrov inequalities for $W_{k,p}$ for $p \in [-n, 0]$, whenever $\max\{k, m\} \ge -p$.
- It fails for $p < -n$.
- We also found examples where, for $2 < p < \infty$, the log-concavity condition

  $$W^2_{k,p} \ge W_{k-1,p} W_{k+1,p},$$

  doesn't hold.

1.1. **"Discrete case".** A calculation shows that when $\mathbb{S}$ is the surface of a box in $\mathbb{R}^d$ with side-lengths $s_1, s_2, \dots, s_d$, then

$$Q^1_k(\mathbb{S}) = 2(k!)^{1/k}(s_1 \dots s_d) \left(\sum_{1 \le i_1 < \dots < i_k \le d} \frac{1}{s_{i_1} \dots s_{i_k}} \right),$$

therefore the comparison between between $Q^1_k(\mathbb{S})$ and $Q^1_{k+1}(\mathbb{S})$, can be done using the classical Newton-Maclaurin inequality for symmetric sums. Now, when $\mathbb{S}$ is the surface of a box in $\mathbb{R}^d$, the comparison becomes the Vector-Valued Maclaurin inequality.

Conjecture 3 (Vector-Valued Maclaurin inequality)**.**

$$\left(\frac{\sum_{1\le i_1<\cdots<i_k\le d}|v_{i_1}\wedge\cdots\wedge v_{i_k}|^p}{\binom{d}{k}}\right)^{\frac{1}{kp}}\le\left(\frac{\sum_{1\le i_1<\cdots<i_{k-1}\le d}|v_{i_1}\wedge\cdots\wedge v_{i_{k-1}}|^p}{\binom{d}{k-1}}\right)^{\frac{1}{(k-1)p}},$$

with $p\in[0,\infty]$ *and* $2\le k\le d$*.*

In [2] the conjecture confirmed in the following cases.

Theorem 4. *When*

- *p=2*
- *p=0*
- $p=\infty$
- $p=1$ *and* $k=2,3,d$*.*
- $p=1$ *but with a constant* $1<C<2$*.*

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Face numbers of high-dimensional Poisson polytopes

Zakhar Kabluchko

Let $\mathcal{Z}_d$ be the zero cell of the d-dimensional, isotropic and stationary Poisson hyperplane tessellation. An explicit formula for the expected number of k-dimensional faces of $\mathcal{Z}_d$ is known [1]. We review several results on the *asymptotic* behavior of the expected f-vector of $\mathcal{Z}_d$, as $d\to\infty$. For example [2], the expected number of hyperfaces of $\mathcal{Z}_d$ is asymptotically equivalent to $\sqrt{2\pi/3}\,d^{3/2}$, as $d\to\infty$. Based on these formulas, we investigate the question of whether the dual polytope of $\mathcal{Z}_d$ is k-neighborly, i.e. whether every k vertices of $\mathcal{Z}_d^\circ$ span a $(k-1)$-dimensional face of $\mathcal{Z}_d^\circ$, with probability approaching 1 as $d\to\infty$. Most results of the present talk are stated in terms of expectations. Whether they hold with high probability, remains open.

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A negative answer to Ulam's Problem 19 from the Scottish Book

Dmitry Ryabogin

The following intriguing problem was proposed by Ulam [U, Problem 19]: *If a convex body $K \subset \mathbb{R}^3$ made of material of uniform density $\mathcal{D} \in (0,1)$ floats in equilibrium in any orientation (in water, of density* 1 *), must K be a Euclidean ball?*

Schneider [Sch1] and Falconer [Fa] showed that this is true, provided K is centrally symmetric and $\mathcal{D} = \frac{1}{2}$. No results are known for other densities $\mathcal{D} \in (0,1)$ and no counterexamples have been found so far.

The "two-dimensional version" of the problem is also very interesting. In this case, we consider floating logs of uniform cross-section and seek the ones that will float in every orientation with the axis horizontal. If $\mathcal{D} = \frac{1}{2}$, Auerbach [A] has exhibited logs with non-circular cross-section, both convex and non-convex, whose boundaries are so-called Zindler curves [Zi]. More recently, Bracho, Montejano and Oliveros [BMO] showed that for densities $\mathcal{D} = \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ and $\frac{2}{5}$ the answer is affirmative, while Wegner proved that for some other values of $\mathcal{D} \neq \frac{1}{2}$ the answer is negative, [Weg1], [Weg2]; see also related results of Várkonyi [V1], [V2]. Overall, the case of general $\mathcal{D} \in (0,1)$ is notably involved and widely open.

In this talk we discuss the following result.

Theorem 1. *Let $d \geq 3$. There exists a strictly convex* non-centrally-symmetric *body of revolution $K \subset \mathbb{R}^d$ which floats in equilibrium in every orientation at the level $\frac{\mathrm{vol}_d(K)}{2}$.*

This gives

Theorem 2. *The answer to Ulam's Problem 19 is negative, i.e., there exists a convex body $K \subset \mathbb{R}^3$ of density $\mathcal{D} = \frac{1}{2}$, which is not a Euclidean ball, yet floats in equilibrium in every orientation.*

Our bodies are *small perturbations of the Euclidean ball.* We combine our recent results from [R] together with work of Olovjanischnikoff [O], and then use the machinery developed together with Nazarov and Zvavitch in [NRZ]. The proofs of Theorem 1 for even and odd d are different. For even d we solve a finite moment problem to obtain our body as a *local* perturbation of the Euclidean ball. The case of odd d is more involved. To control the perturbation, we use the properties of the spherical Radon transform, [Ga, pp. 427-436], [He, Chapter III, pp. 93-99].

We refer the reader to [CFG, pp. 19-20], [Ga, pp. 376-377], [G], [M, pp. 90-93] and [U] for an exposition of known results related to the problem.

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