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Describing distance: from the plane to spectral triples

Francesca Arici • Bram Mesland

Geometry draws its power from the abstract structures that govern the shapes found in the real world. These abstractions often provide deeper insights into the underlying mathematical objects. In this snapshot, we give a glimpse into how certain "curved spaces" called manifolds can be better understood by looking at the (complex) differentiable functions they admit.

1 Introduction

The word "geometry" brings to our mind pictures and shapes. From the objects that we see in the world around us, our minds create ideal mathematical figures such as balls, cones, cubes, and polyhedra in 3-dimensional space. In the physical world, we can build approximations of these objects, and with them, we create more complicated structures.

It should therefore not come as a surprise that, from a historical perspective, the development of geometry and mathematics in ancient times finds its roots in practical problems related to construction, but also stemming from astronomy and surveying. The word *geometry* itself comes from the two Greek words $\gamma\eta$, meaning "Earth" and $\mu\varepsilon\tau\varrho\iota\alpha$, meaning "measurement". An expert in geometry is someone who occupies themselves with measurements, dealing with notions like length, distance, area, volume, angles, and dimension.

Figure 1: Measuring the world and finding abstractions.

Philosophers and mathematicians soon took the study of physical space and geometry a step further, treating forms and shapes as abstract mathematical objects. A more modern abstraction is offered by the concept of “manifold”, which builds on our ideas of geometry. To get an idea of what a manifold is, the reader can start thinking about lines and curves, such as circles, ellipses, parabolas. When we zoom in on a very small part of a curve, we observe that the curve itself actually looks almost like a straight line. A surface of a sphere has a similar property. We know from our experience that, from close by, a sphere looks like a flat plane: it is tempting to think that the Earth is flat because we only ever see a small piece of it.

Roughly speaking, manifolds are shapes that can be described “locally” by copies of flat space, together with rules for how the different flat patches fit together – this is what mathematicians call *charts*. Not surprisingly, such a collection of charts is called an *atlas*. For the surface of the Earth, we can use a minimum of two charts that overlap around the equator.

The notion of a manifold is not confined to dimension 2 or 3, but actually makes sense in every dimension. While the study of abstract curves and surfaces dates back to the work of Carl Friedrich Gauss (1777–1855), the *Princeps mathematicorum*, we owe the more general concept of manifold to his doctoral student Bernhard Riemann (1826–1866), who developed the field that currently takes his name, namely Riemannian geometry.

Riemann’s ideas found a very important application in the early 20th century, in Einstein’s theory of general relativity. There, gravity is understood as an intrinsic feature of a 4-dimensional curved space-time, and space is a 3-dimensional curved manifold in the sense of Riemann. The theory still stands as one of the most experimentally successful models in physics and shows how abstract mathematical thinking can lead to new insights about the physical world.

After having taken the reader through the classical world of geometry in Section 2, we then show in Section 3 how the concept of distance can be characterized in completely algebraic terms. We finally give a glimpse into a modern approach to the study of manifolds (and other geometric objects) that goes under the name of “noncommutative geometry”.

Noncommutative geometry is an active field of modern mathematics initiated by the 1982 Fields medalist Alain Connes [2]. We present one of the fundamental ideas of Connes and his reconstruction theorem, which roughly states that the (geo)metric information about a space can be recovered by looking at properties of generalized coordinates on that space.

2 Geometry: measuring distances

Familiar notions like distance, length, and angles can be rigorously defined in mathematical terms. Here we will give an overview of how this can be achieved, starting from Cartesian geometry and then building up towards Riemannian geometry.

2.1 The Cartesian plane

Cartesian geometry lends its name from the French mathematician and philosopher René Descartes (1596–1650). It deals with a description of geometric objects using numbers and relies on the notion of coordinate system.

The crucial assumption in this approach is the fact that any point in the plane is uniquely determined by a pair of coordinates: x_1 and x_2. As a result, we denote the plane with $\mathbb{R}^2$, in line with the description of its points by a pair of real numbers.

Notions like angles, distance, and area, as well as parallelism, incidence, and tangency, are the subject of *Euclidean* geometry. It is named after Euclid, the Greek mathematician and philosopher who formalized and put into writing our geometric intuitions around 300 B.C. In *Cartesian* geometry, geometric quantities are defined using formulas (see Figure 3). These formulas describe the underlying Euclidean geometry in terms of algebraic features.

For a point $P = (x_1, x_2)$ in the plane $\mathbb{R}^2$, we define its distance from the origin – or equivalently, the length of line segment connecting it to the origin of the plane – as its *Euclidean norm*

$$\|P\| := \sqrt{x_1^2 + x_2^2}. \tag{1}$$

The norm allows us to talk about the *distance* between P and Q as the length of the line segment that connects them: for two points P and Q with coordinates (x_1, x_2) and (y_1, y_2), their distance $d(P, Q)$ is

$$d(P, Q) := \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}. \tag{2}$$

The Euclidean norm in Formula (1) is related to the notion of angle in Euclidean geometry since it comes from an *inner product*, a fundamental operation in algebra and geometry which is defined in the following way. Identify two points $P = (x_1, x_2)$ and $Q = (y_1, y_2)$ in the Cartesian plane with the vectors x and y that connect them to the origin. Then their inner product is defined as

$$\langle x, y \rangle := x_1 y_1 + x_2 y_2. \tag{3}$$

The inner product is related to the angle θ between the two vectors x and y by the formula

$$\langle x, y \rangle = \|x\| \|y\| \cos(\theta),$$

where $\| \cdot \|$ is the Euclidean length of the vector defined in (1). Note that it relates back to the inner product via the formula

$$\|x\|^2 = \langle x, x \rangle.$$

This fact has the important consequence that whenever we have such an inner product, we automatically have a norm, and hence a distance.

2.1.1 The circle in the plane

We now consider one of the simplest examples of a closed curve, namely the circle of radius 1 in the plane. This is the set of points of distance 1 from the origin. Using the Euclidean norm (1), we see that such points are exactly those with coordinates (x_1, x_2) satisfying the algebraic equation $x_1^2 + x_2^2 = 1$. Mathematicians denote the circle with S^1 and write

$$S^1 := \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1 \right\}.$$

The distance between two points $P = (x_1, x_2)$ and $Q = (y_1, y_2)$ on S^1 is given by the shortest path from P to Q along the circumference of the circle. This gives a new notion of distance; the *radial distance*.

2.2 Higher dimensions: the Euclidean space

We need not restrict our attention to the 2-dimensional world that we can draw on paper! Elaborating on the previous example, we can consider the n-dimensional *Euclidean spaces*. Points in an n-dimensional Euclidean space are in one-to-one correspondence with n-tuples of real numbers, which we denote by $\mathbb{R}^n$. Given two points P and Q with coordinates $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$, we can define an inner product, generalizing Definition (3), as

$$\langle x, y \rangle := x_1 y_1 + \cdots + x_n y_n. \tag{4}$$

For a vector x, the resulting norm induced by the inner product (4) is

$$\|x\| := \sqrt{\langle x, x \rangle} = \sqrt{x_1^2 + \cdots + x_n^2}, \tag{5}$$

and the distance between two points is defined again as the norm of the vector that connects them:

$$d(P, Q) = \|x - y\| := \sqrt{\langle x - y, x - y \rangle} = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}. \tag{6}$$

If we insert $n = 2$ in the above formulas, we recover the Formulas (2), (1), and (3) that we encountered earlier.

2.2.1 Curves in the Euclidean space

Mathematically, a *curve* is a continuous function γ from a closed interval taking values in the Euclidean space $\mathbb{R}^n$, in symbols

$$\gamma : [a, b] \to \mathbb{R}^n.$$

Here “continuous” means that sufficiently small changes in the function’s input yield arbitrarily small changes in the output; the function does not have any “jumps”.

In view of applications to physics, where curves are used to describe the trajectory of a body over time, we would like to restrict ourselves to curves for which velocity makes no abrupt changes.[1] Geometrically speaking, we expect that a tangent vector – which in physics represents the velocity – can be defined at each point of the curve, and that this tangent vector varies “nicely” from point to point. In mathematical terms, we are assuming that the derivative of the function γ is again a continuous function – that γ is a continuously

[1] We do, however, allow for abrupt changes in acceleration, like those produced by sudden forceful interactions with the system.

differentiable curve or C^1-curve. We say that γ *joins* P *and* Q if $\gamma(a) = P$, $\gamma(b) = Q$.

For a curve γ of class C^1, the derivative of γ and the Euclidean norm (5) come together in a formula that computes the length $L(\gamma)$[2] of the curve γ as an integral:

$$L(\gamma) = \int_a^b \|\gamma'(t)\| \mathrm{d}t. \tag{7}$$

Given two points P and Q in $\mathbb{R}^n$, the distance (2) admits an alternative characterization using the language of curves in the plane. Indeed, one can compute the distance between P and Q as the "minimal length of paths joining P and Q". In a formula:

$$d(P, Q) = \min \left\{ L(\gamma) : \gamma : [a, b] \to \mathbb{R}^n \text{ a } C^1\text{- curve joining } P \text{ and } Q \right\}. \tag{8}$$

In $\mathbb{R}^n$ the shortest path that connects two points is a straight line, which is a continuously differentiable curve – so that we recover the distance formula 6 we introduced earlier.[3]

This is the first example of a reformulation of the concept of distance. The advantage of working with such a formula is that it allows us to make sense of the distance between points in different contexts, such as on non-flat spaces like spheres and other surfaces.

2.3 Distances in Riemannian geometry

As explained in the introduction, manifolds are made out of local patches that look like flat Euclidean space. We can use this local structure to make sense of the notions of tangent vector and differentiability for *any* manifold.

To describe the continuity of our collection of tangent vectors, we use the metric structure coming from the Euclidean inner product on each patch. We do so by prescribing a way of gluing together the inner products on $\mathbb{R}^n$ in a way that is compatible with the structure of the manifold. The mathematical object that does the trick is the "Riemannian metric tensor".

The Riemannian metric tensor allows us to assign a length to tangent vectors. For any connected Riemannian manifold, we can then define the distance between two points as the infimum of the lengths of all paths connecting them, exactly as in Formula (8). As our experience with flights suggests, in a curved space, the shortest path between two points may not look like a straight line on the charts in the atlas anymore.

[2] The requirement of differentiability is important here; non-differentiable curves can not always be assigned a length.

[3] Showing that a shortest path between P and Q *exists* may be in some cases a difficult, if not impossible task, but we avoid going into such details here.

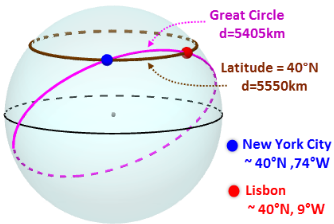

Figure 2: The great circle between the cities of Lisbon and New York.

Indeed, on a 2-dimensional sphere, which is a good approximation of the shape of the terrestrial globe, the shortest paths are those that go along great circles, that is, circles centered at the center of the sphere. Given two points on a sphere that are not directly opposite each other, there exist a unique great circle.[4]

Using the Riemannian metric tensor, it is possible to define various other geometric notions on a Riemannian manifold as well, such as angles, areas, volumes, and curvature. In physics, these geometrically defined quantities describe notions like force, work, field strength, and flux. As Galileo Galilei (1564–1642) wrote centuries before the advent of Riemannian geometry, the book of nature "is written in the language of mathematics, and its characters are triangles, circles and other geometric figures" [4].

3 An algebraic look at distances

3.1 Cartesian coordinates revisited

Going back to the example of Cartesian geometry in the plane, we see that the use of a coordinate system allows for a deeper study of geometrical objects: not only can we describe the position of points in terms of coordinates, but also other features of geometrical shapes.

Indeed, lines, curves, polygons, and areas of space are described using equations, inequalities, or systems involving them. Notions like incidence, parallelism, and tangency are understood by analyzing the corresponding equations. In this way, problems arising in geometry and physics are translated into problems in algebra or analysis. This interplay between algebra and geometry is referred to as "duality". An instance of this can be found in Subsection 2.1.1, where

[4] On the contrary, given two antipodal points, there are infinitely many great circles passing through them.

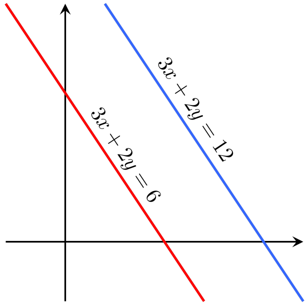

Figure 3: Two parallel lines and their equations. The property of being parallel is translated in the algebraic condition of having the same *slope*, which is defined as minus the ratio between the y and x coefficient.

we described points on the circle using an algebraic equation. It dates back to the work of Descartes, but it has influenced the development of mathematics ever since. As the French mathematician Sophie Germain (1776–1831) sharply concluded,

Algebra is but written geometry and geometry is but figured algebra.

In the 19th and 20th century, the notion of duality was taken to a more abstract level by mathematicians of the caliber of David Hilbert (1862–1943) and Emmy Noether (1882–1935), and later Alexander Grothendieck (1928–2014), who set the stage for algebraic and arithmetic geometry.

A similar type of duality, more analytical in nature, was discovered by the Russian mathematicians Israel Gelfand (1913–2009) and Mark Naimark (1909–1978), and it is now referred to as "Gelfand duality".

3.2 Complex numbers and the algebra of complex-valued functions

We will now discuss Gelfand duality for the manifolds we have encountered before, that is, for those geometrical objects that locally look like flat space. We will also assume that our manifolds are *compact*, meaning that every atlas can be reduced to one that contains only a finite number of charts.

To be able to introduce our algebraic machinery, we need the complex numbers $\mathbb{C}$. These are an extension of the real numbers obtained by introducing an additional element i with $\mathrm{i}^2 = -1$.

A complex number can be written as $z = x + \mathrm{i}y$, with x and y real numbers. Every such complex number determines a point P with Euclidean coordinates (x, y).

Consider the collection of all continuous functions on a compact manifold X that take values in the complex numbers $\mathbb{C}$. We denote them by

$$C(X) := \{f : X \to \mathbb{C} : f \text{ is continuous}\}.$$

The set $C(X)$ comes with two natural operations, addition and multiplication, which are defined point-wise: for $f, g \in C(X)$, their *sum function* is given by

$$(f + g)(x) := f(x) + g(x)$$

for every $x \in X$, and their *product function* by

$$(fg)(x) := f(x)g(x)$$

for every $x \in X$.

There is a natural norm [5] on the space $C(X)$, given by

$$\|f\| = \max_{x \in X} |f(x)|. \tag{9}$$

There is even more structure on $C(X)$, turning it into what is called a C^*-algebra. This goes beyond the scope of this note, but we refer to the recent snapshot [5] for an introduction to C^*-algebras and their classification, and to [1, 6] for more on the role of C^*-algebras in the study of quantum symmetries.

The operation of passing from a space to the collection of functions thereon is not a mere mathematical abstraction, but rather a rigorous reformulation of the concept of *coordinate* on a space:

Example 3.1. *Let us go back to the example of the Euclidean space $\mathbb{R}^n$: the coordinate functions*

$$f_k : (x_1, \cdots, x_n) \mapsto x_k, \quad k = 1, \cdots n,$$

are continuous, and they contain all the information about the Euclidean space.

Bringing the idea of "generalized coordinates" further, one can prove that any manifold is in a sense "the same" as its algebra of coordinates – the manifold can be recovered from the coordinates.

[5] Note that this is a norm in the function space $C(X)$, not on the space itself. In fact, there need not be a distance function on X.

3.3 The distance formula in function spaces

We are now ready to present another reformulation of the notion of *distance between points*, which involves function algebras.

To understand how this works, let us consider again the circle S^1.

We can define a distance between two points P and Q on S^1 by considering all functions having derivative at most 1, and then by evaluating them at the two points:

$$d(P,Q) = \max_{f \in C^1(S^1)} \{|f(P) - f(Q)| \ : \ |f'(x)| \leq 1 \ \text{ for all } x \in S^1\}. \qquad (10)$$

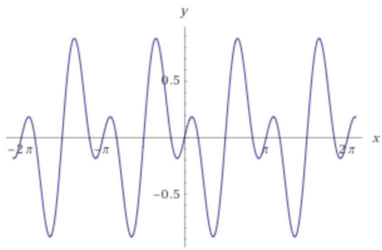

Figure 4: An example of a periodic, continuously differentiable function.

To simplify matters, one can identify functions on the circle with functions on $\mathbb{R}$ with are 2π-periodic. Those are determined by their values on the interval $[0, 2\pi]$. We identify $C(S^1)$ with the set of continuous 2π-periodic functions, which we denote by $C_{\text{per}}([0, 2\pi])$. Then the collection of continuously differentiable functions on the circle $C^1(S^1)$ corresponds to the algebra

$$C^1_{\text{per}}([0, 2\pi]) = \{f \in C_{\text{per}}([0, 2\pi]) : f' \in C_{\text{per}}([0, 2\pi])\}$$

of continuous 2π-periodic functions on the interval $[0, 2\pi]$ whose derivative is also continuous and of period 2π.

We now introduce a third space of functions:

$$L^2_{\text{per}}([0, 2\pi]) := \left\{ f : [0, 2\pi] \to \mathbb{C} : \int_0^{2\pi} |f(x)|^2 dx < \infty \right\},$$

known as the space of *square integrable periodic functions*; it consists of all functions on the interval $[0, 2\pi]$ for which the integral $\int_0^{2\pi} |f(x)|^2 dx$ is finite. This finite integral is important here because it gives us a way to measure distances between functions via a norm defined by

$$\left(\int_0^{2\pi} |f(x)|^2 dx\right)^{\frac{1}{2}}.$$

This norm is called the L^2-norm of the periodic function f, and we often denote it by $\|f\|_{L^2}$. Spaces like this play an important role in physics and they are known as Hilbert spaces, in honour of the German mathematician David Hilbert, who studied them at the beginning of the 20th century.

The process of "taking the derivative" can be thought of as an operator (that is, a function that acts on a function space)

$$D\!\!\!/ : C^1_{\mathrm{per},L^2}([0,2\pi]) \to L^2_{\mathrm{per}}([0,2\pi]) \quad (D\!\!\!/ f)(x) = if'(x).$$

Point-wise multiplication by a function also gives an operator on $L^2_{\mathrm{per}}([0,2\pi])$. Using the formula for the derivative of a product, the condition $|f'(x)| \leq 1$ for all $x \in \mathbb{R}$ can be shown to be equivalent to

$$\|D\!\!\!/(fg)(x) - f(D\!\!\!/ g)(x)\|_{L^2} \leq \|g(x)\|_{L^2} \tag{11}$$

for any function $g \in C^1_{\mathrm{per}}([0,2\pi])$.

The expression $(D\!\!\!/ f - fD\!\!\!/)$ is called a *commutator*, as it measures the failure of the multiplication operator f to commute with the differentiation operator $D\!\!\!/$. The commutator $(D\!\!\!/ f - fD\!\!\!/)$ is written shortly as $[D\!\!\!/, f]$. Formula (11) can then be written

$$\|[D\!\!\!/, f]g(x)\|_{L^2} \leq \|g(x)\|_{L^2}.$$

We can now write the distance formula (10) as

$$d(P,Q) = \max_{f \in C^1(S^1)} \left\{ |f(P) - f(Q)| : \|[D\!\!\!/, f]g(x)\| \leq \|g(x)\| \text{ for all } g \in C^1_{\mathrm{per}}([0,2\pi]) \right\} \tag{12}$$

This formula, which expresses a relation between distance, points, and differentiation, is the backbone of the theory of spectral triples and noncommutative geometry.

4 A glimpse into the world of spectral triples

Using the material of the previous section as a model, we can generalize and write analogously the distance formula for a Riemannian manifold

$$d_D(p,q) = \sup\{|f(p) - f(q)| \ : \ f \in \mathcal{A}, \ \|\,[D,f]\,\| \leq 1\}.$$

The derivative operator from the previous example, $\not{D}$, is now replaced by D, a certain operator acting on a suitable space of functions H associated to the manifold X. $\mathcal{A}$ is an algebra of functions contained in $C(X)$ on which D is defined (think for example of $C^1(X)$), and which *acts* on a Hilbert space H by multiplication. We thus see that the space and the metric distance can be *recovered* from the data $(\mathcal{A}, H, D)$. The main idea here is that the triple $(\mathcal{A}, H, D)$ contains a lot of information about the Riemannian manifold X.

This can be made mathematically precise, and the discussion of the previous section leads to a generalization of metric geometry to the case of noncommutative algebras and yields Connes' notion of *spectral triple* [3] $(\mathcal{A}, H, D)$, consisting of the following data:

An algebra $\mathcal{A}$ acting on a Hilbert space H, together with an operator D that relates to $\mathcal{A}$ via a number of axioms expressing compactness and differential properties. Although this may seem very abstract, the above discussion shows that Riemannian geometry can be captured with this framework.

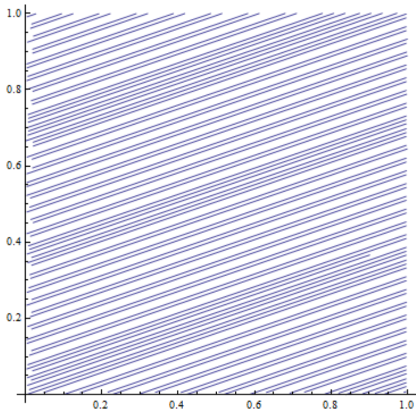

Figure 5: Irrational rotations on the square with identified edges.

We think of the interaction of D and $\mathcal{A}$, acting on the Hilbert space H as describing the geometry of A. The theory of spectral triples has seen a steady development since the early 1990s, and the spectral triples have been investigated in situations arising across the mathematical spectrum, as well as in connection with applications to physics. The Standard Model of particle physics, for example, can be conveniently encoded in a spectral triple on the product of a classical space with an algebra encoding the fundamental interactions.

Another important area of noncommutative geometry is the study of *foliations*. Roughly speaking, a foliation prescribes a way of slicing a manifold into submanifolds (called *leaves*) of smaller dimension. To illustrate this concept, we will provide an example.

Given a square with the opposite sides glued together, we draw a straight line in the square whose slope is irrational. Since we identify the sides of the square, the line will come back in at the other side of the square. We obtain a path that never intersects itself and continues indefinitely. Any starting point on a given horizontal line in the square gives a different such path, and together these fill up the entire square.

The different lines get infinitely close together, and the resulting foliation, which consists of one single leaf winding up indefinitely, can be conveniently described with the aid of a non-commutative algebra. The advantage of this approach is that the *noncommutative* geometry of this algebra contains information about the foliation which cannot be described in ordinary geometric terms.

The paradigm of spectral triples as describing the geometry of noncommutative algebras greatly widens the scope of classical Riemannian geometry. Whether one stays close to Riemannian geometry, such as in the above foliation example, or considers a situation where there is no apparent Riemannian geometry at all, the same set of general tools can be applied to study these systems. The original motivation for the study of spectral triples came from the study of the Standard Model of particle physics [3], where it had considerable success. It led to a general mathematical theory that can be studied for its own sake, and then can be applied in widely different contexts. More recently, the theory of spectral triples has found applications in the study of topological phases of matter. This showcases the fact that a lot of the power of modern mathematics comes from linking apparently unrelated contexts by recognizing an underlying common pattern through abstraction.

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References

[1] M. Caspers, *Quantum symmetry*, Snapshots of modern mathematics from Oberwolfach $\mathcal{N}^o$09/2020, 2020.

[2] A. Connes, *Noncommutative geometry*, Academic Press, 1994.

[3] ______, *Geometry from the spectral point of view*, Letters in Mathematical Physics **34** (1995), no. 3, 203–238, http://dx.doi.org/10.1007/BF01872777.

[4] Galileo Galilei, *The assayer: Translated from the Italian by Stillman Drake*, pp. 151–336, University of Pennsylvania Press, 2016, https://doi.org/10.9783/9781512801453-006.

[5] D. Kerr, *C*-algebras: structure and classification*, Snapshots of modern mathematics from Oberwolfach $\mathcal{N}^o$02/2021, 2021.

[6] M. Weber, *Quantum symmetry*, Snapshots of modern mathematics from Oberwolfach $\mathcal{N}^o$05/2020, 2020.

Francesca Arici *is an Assistant Professor and NWO Veni fellow at Leiden University, the Netherlands.*

Bram Mesland *is an Assistant Professor at Leiden University, the Netherlands.*

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Mathematisches Forschungsinstitut
Oberwolfach gGmbH
Schwarzwaldstr. 9 – 11
77709 Oberwolfach
Germany

Director
Gerhard Huisken

Mathematisches Forschungsinstitut Oberwolfach

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