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Set Theory

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Abstract. While set theory continues reaching out into various other fields of mathematics but also becomes more and more specialized, recent times have seen important results around holy grails of set theory which gave a new momentum to the whole field as a unit.

Mathematics Subject Classification (2020): 03Exx.

Introduction by the Organizers

After having had a meeting being cancelled in April 2020 due to Covid-19, we were very happy to now be able to realize a hybrid workshop with 17 participants being physically present. We explored topics in all areas of set theory, and the group of 17 were filled with joy from discussing mathematics again face to face.

At the pure end of set theory, there have been breakthroughs e.g. in the theory of forcing over determinacy models as well as in the study of large cardinal axioms in the absence of the axiom of choice.

Sargsyan reported on a $\mathbb{P}_{\max}$ type of construction of a model satisfying a consequence of the Proper Forcing Axiom (PFA) from surprisingly weak large cardinal hypotheses. This might open the door for building such models for even stronger such consequences or maybe even all of PFA. Goldberg reported on large cardinals which cannot exist in ZFC (a topic which recently saw a strong revival). A large cardinal concept that was introduced quite a while ago but that now gains a lot of attention again is "Berkeley cardinals." Schlutzenberg analyzed (in joint work with Steel) the exact extent of determinacy in ω-small mice.

Viale's talk was on a topic that was inspired by the result of Asperó-Schindler according to which Woodin's $\mathbb{P}_{\max}$ axiom $(*)$ follows from Martin's Maximum^{++} (MM^{++}). He collects sophisticated arguments which support the view that the Continuum Hypothesis (CH) is false.

Dobrinen reported on important progress on the infinite dimensional Ramsey theory for Fraïssé structures. Džamonja gave an account of her joint work with Buhagiar on sufficient conditions on the paracompactness of box products.

Still in the realm of pure set theory, Hamkins gave insights into the bi-interpretability of weak set theories. Jackson, in joint work with Chan and Trang, lifted results concerning the continuity and monotonicity of certain functions to bigger cardinals by replacing determinacy arguments by arguments using the strong partition property. Mildenberger addressed the exciting problem of perfect tree forcings at singular (rather than regular) cardinals.

Prikry forcing is mostly relevant in the theory of cardinal arithmetic. Poveda (in joint work with Rinot and Sinapova) presents a general framework of versions of Prikry forcings with an eye on applications on compactness and incompactness results. Sakai studies an extension of Jensen's Subcomplete Forcing Axiom which produces a variant of Jensen's diamond principle. Wilson further explores the exciting area of virtual large cardinals. Zapletal presents methods to be able to obtain challenging independence results about chromatic numbers of graphs on Euclidean spaces. Zeman gave a proof related to the distributivity of iterations of club shooting posets.

Bridging set theory with abstract topology, Rinot addressed the (still open) question in in ZFC there is a Dowker space of size $\aleph_1$. He obtained an amazing variety of combinatorial insights related to this problem.

At the applied end of set theory, there were talks in the very active areas of descriptive set theory as well as group theory.

Gao gave a survey of known results about omnigenous groups, and prove that there are continuum many pairwise non-isomorphic, omnigenous, universal countable locally finite groups. Foreman presented an anti-classification result for a program initiated by Smale. He showed that no Borel map from the C^k-diffeomorphisms of compact manifolds to a Polish space gives complete invariants for the equivalence relation of conjugacy-by-homeomorphims. Kwiatkowsa demonstrated that finite connected graphs with confluent epimorphism form a projective Fraïssé class and she investigated the continuum obtained as the topological realization of the projective Fraïssé limit. Sabok characterizes hyperfinite bipartite graphings that admit measurable perfect matchings.

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Workshop: Set Theory

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Abstracts

Infinite dimensional Ramsey theory of homogeneous structures: a progress report

Natasha Dobrinen

In their seminal 2005 paper, Kechris, Pestov and Todorcevic [3] asked for the development of infinite dimensional Ramsey theory for Fraïssé structures. A step in this direction was begun in Dobrinen's 2019 [2] paper proving a Galvin-Prikry style theorem for a topological space of subcopies of the Rado graph. We have recently extended this line of work to a class of homogeneous structures with relations of arity at most two satisfying a certain amalgamation property.

Let $\mathcal{K}$ be a Fraïssé class and let $\mathbf{K}$ denote its Fraïssé limit. Then $\mathbf{K}$ is *homogeneous*, meaning that any isomorphism between two finite induced substructures of $\mathbf{K}$ extends to an automorphism of $\mathbf{K}$. The Galvin-Prikry theorem states that Borel subsets of the Baire space are Ramsey, meaning that if $\mathcal{X} \subseteq [\mathbb{N}]^\omega$ is Borel, then there is some infinite subset $M \subseteq \mathbb{N}$ such that either $[M]^\omega \subseteq \mathcal{X}$ or else $[M]^\omega \cap \mathcal{X} = \emptyset$. To develop infinite dimensional Ramsey theory on homogeneous structures, we start by letting $\mathbb{N}$ be the universe of $\mathbf{K}$ and identifying the collection of subcopies of $\mathbf{K}$ with the subspace of the Baire space determined by their universes.

The Substructure Free Amalgamation Property (SFAP) and its disjoint amalgamation version (SDAP) were developed recently in work of Coulson, Dobrinen, and Patel [1] and shown to guarantee big Ramsey degrees which have a simple characterization in terms of diagonal antichains in coding trees of 1-types. That work gave a uniform approach to prior results of Devlin, Laflamme–Sauer–Vuksanovic, Laflamme-Nguyen Van Thé–Sauer, and others, as well as exact big Ramsey degrees for new homogeneous structures. Known results on big Ramsey degrees imply that any hope for infinite dimensional structural Ramsey theory must restrict to subcopies of $\mathbf{K}$ with the same similarity type.

We prove that for a homogeneous structure $\mathbf{K}$ with finitely many relations of arity at most two and satisfying SFAP, given a diagonal antichain $\mathbb{D}$ in the coding tree of 1-types for $\mathbf{K}$, the Baire space of all subcopies of $\mathbf{K}$ induced by similarity copies of $\mathbb{D}$ have the property that all Borel subsets have the Ramsey property. Furthermore, we answer a question of Todorcevic asked at the 2019 Luminy meeting, showing that a Nash-Williams style corollary recovers the exact big Ramsey degrees. We also present some preliminary results for the wider class of SDAP$^+$ structures.

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On middle box products and paracompact cardinals

Mirna Džamonja
(joint work with David Buhagiar)

We discuss several sufficient conditions on the paracompactness of box products with an arbitrary number of many factors and boxes of arbitrary size. The former include results on generalised metrisability and Sikorski spaces. Of particular interest are products of the type ${}^{<\kappa}2^{\lambda}$, where we prove that for a regular uncountable cardinal κ, if ${}^{<\kappa}2^{\lambda}$ is paracompact for every $\lambda \geq \kappa$, then κ is at least inaccessible. The case of the products of the type ${}^{<\kappa}X^{\lambda}$ for κ singular has not been studied much in the literature and we offer various results. The question if ${}^{<\kappa}2^{\lambda}$ can be paracompact for all λ when κ is singular has been partially answered but remains open in general.

Descriptive Set Theory and Dynamical Systems

Matthew Foreman
(joint work with Anton Gorodetski)

The talk surveyed known impossibility results on the quantitative aspects of dynamical systems (ergodic theory) and the qualitative aspects of dynamical systems (smooth dynamics).

Relatively recent results in ergodic theory have shown that measure isomorphism between ergodic measure preserving transformations is not Borel (joint work with D. Rudolphs and B. Weiss, [1]) and is not classifiable by countable algebraic invariants (joint work with B. Weiss [5]). This was improved in joint work with Weiss ([4], [2], [3]) to show:

Theorem. *Let $\mathcal{ED}$ be the collection of C^{∞}, ergodic, measure preserving diffeomorphisms of the 2-torus. Then measure isomorphism is a complete analytic subset of $\mathcal{ED} \times \mathcal{ED}$. In particular it is not a Borel equivalence relation.*

This result gives a negative solution to von Neumann's 1932 proposal to classify the statistical behavior of differentiable systems ([6])

The new results are similar anti-classification results for a program of Smale ([7] and [8]). Smale proposed classifying diffeomorphisms of compact manifolds up to the equivalence relation of conjugacy-by-homeomorphisms. This equivalence relation preserves the *qualitative* behavior of the diffeomorphisms, such as stable points, attractors, and so forth.

Considerable work has been done in this area, such as the classification of Anosov and Morse-Smale diffeomorphisms. Any collection of structurally stable diffeomorphisms have continuously computable complete numerical invariants. (Continuity is with respect to the C^1-topology.)

However, the general problem is not Borel. In the statements that follow the topology on the space of diffeomorphisms is the C^k-topology, where M is a C^k-manifold. The theorems are joint work with A. Gorodetski.

The main new results of the talk are two-fold:

Theorem. *Let M be a C^k-manifold of dimension at least 2. The equivalence relation E_0 is continuously reducible to the equivalence relation of conjugacy-by-homeomorphisms on C^k-diffeomorphisms.*

The point is the following corollary:

Corollary. *There is no Borel function from the C^k-diffeomorphisms to any Polish space that gives complete invariants for the equivalence relation of conjugacy-by-homeomorphisms.*

On dimension at least 5, the talk contained a stronger result:

Theorem. *Let M be a C^k manifold of dimension at least 5. Then the equivalence relation of conjugacy-by-homeomorphism is complete analytic. In particular it is not a Borel subset of the space of pairs of C^k-diffeomorphisms.*

Remark. After the conference, in March 2022, Foreman and Gorodetski were able to improve the two theorems to apply to dimensions one and above.

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Omigenous Groups

Su Gao
(joint work with Mahmood Etedadialiabadi, François Le Maître and Julien Melleray)

Omnigeous groups are countable locally finite groups with certain ultrahomogeneity-like property. All of them are embeddable into the isometry group of the Urysohn space (and other similar "large groups") as dense subgroups. The Fraïssé limit of finite groups, known as Hall's group, is an example of an omnigenous group that is universal among all countable locally finite groups. In this talk I will give a survey of known results about omnigenous groups, and prove that there are continuum many pairwise non-isomorphic, omnigenous, universal countable locally finite groups.

Measurable cardinals and choiceless axioms

Gabriel Goldberg

One of the most influential ideas in the history of large cardinals is Scott's reformulation of measurability in terms of elementary embeddings [5]: the existence of a measurable cardinal is equivalent to the existence of a nontrivial elementary embedding from the universe of sets V into a transitive submodel M. In the late 1960s, Solovay and Reinhardt realized that by imposing stronger and stronger closure constraints on the model M, one obtains stronger and stronger large cardinal axioms, an insight which rapidly led to the discovery of most of the modern large cardinal hierarchy. Around this time, Reinhardt formulated the ultimate large cardinal principle of this kind: there is an elementary embedding from the universe of sets to itself.[1] Soon after, however, Kunen [4] showed that this principle is inconsistent:

Theorem (Kunen)**.** *There is no elementary embedding from the universe of sets to itself.*

Kunen's proof relies heavily on the Axiom of Choice, however, and the question of whether this is necessary immediate arose.[2] Decades later, Woodin returned to this question and discovered that although the traditional large cardinal hierarchy stops short at Kunen's bound, there lies beyond it a further realm of large cardinal axioms incompatible with the Axiom of Choice, axioms so absurdly strong that Reinhardt's so-called ultimate axiom appears tame by comparison. Yet since this discovery, despite significant efforts of many researchers, no one has managed to prove the inconsistency of a single one of these choiceless large cardinal axioms.

"The difficulty," according to Woodin [7], "is that without the Axiom of Choice it is extraordinarily difficult to prove anything about sets." One remedy to this

[1] Of course, the identity is such an elementary embedding. Whenever we write "elementary embedding," we will really mean "nontrivial elementary embedding."

[2] The question was first raised by the anonymous referee of Kunen's paper.

difficulty, proposed by Woodin himself [6, Theorem 227], is to *simulate* the Axiom of Choice using auxiliary large cardinal hypotheses, especially extendible cardinals. Cutolo [2] expanded on this idea to establish the striking result that the successor of a singular Berkeley limit of extendible cardinals is measurable. While Reinhardt's axiom does not imply the existence of an extendible cardinal, Asperó [1] showed that it does imply the existence of elementary embeddings reminiscent of those associated with extendible cardinals.

This talk concerns a technique that combines Woodin, Cutolo, and Asperó's ideas to show that one can simulate the Axiom of Choice using large cardinal-like notions that follow from Reinhardt's principle.

A cardinal λ is *rank Berkeley* if for all $\xi < \lambda \leq \alpha$, there is an elementary embedding from V_α to itself whose critical point lies between ξ and λ. The existence of a rank Berkeley cardinal is a first-order principle that seems to capture all the set-theoretic consequences of the existence of an elementary embedding from the universe of sets to itself; the latter principle implies the former by an argument due independently to Woodin and Schlutzenberg. A cardinal κ is *rank reflecting* if for all ordinals $\xi < \kappa$ and all formulas φ in the language of set theory, if there is an ordinal α such that $V_\alpha \vDash \varphi(\xi)$, then there is such an ordinal less than κ. This reflection property can be seen as a very weak version of supercompactness — so weak, in fact, that the existence of a proper class of rank reflecting cardinals is a consequence of ZF.

Theorem. *If κ is a rank reflecting cardinal above the least rank Berkeley cardinal, then either κ or κ^+ is a regular cardinal.*

Rank Berkeley cardinals yield the following analysis of the closed unbounded filter:

Theorem. *Suppose λ is rank Berkeley, $\kappa \geq \lambda$ is rank reflecting, and $\delta \geq \kappa$ is regular. Let F denote the closed unbounded filter on δ. Then the following hold:*

- *F is κ-complete.*
- *Every stationary subset of δ contains an atom of F.*
- *The atoms of F are almost linearly ordered by Jech's reflection order.*

Recall here that a set S is an *atom* of the filter F if $\{A \cap S : A \in F\}$ is an ultrafilter on S. If $S_0, S_1 \subseteq \delta$ are stationary, then $S_0 < S_1$ in Jech's *reflection order* if $S_0 \cap \alpha$ is stationary in α for almost all $\alpha \in S_1$; that is, there is a closed unbounded set $C \subseteq \delta$ such that $S_0 \cap \alpha$ is stationary in α for almost all $\alpha \in S_1 \cap C$. In the context of the theorem above, Jech's order is almost linear in the sense that every antichain of atoms in this order has cardinality less than or equal to λ.

Corollary. *Suppose λ is rank Berkeley and $\kappa \geq \lambda$ is rank reflecting. If κ is regular, then κ is measurable, and if κ is singular, then κ^+ is measurable.*

One also obtains an analysis of ultrafilters on ordinals reminiscent of the Ultrapower Axiom [3]:

Theorem. *If λ is rank Berkeley and $\kappa \geq \lambda$ is rank reflecting, then the following hold:*

- *The κ-complete ultrafilters on ordinals are almost well ordered by the Ketonen order.*
- *Every κ-complete filter on an ordinal extends to a κ-complete ultrafilter.*

An ultrafilter U is *well founded* if $\text{Ult}(\text{Ord}, U) \cong \text{Ord}$. (In the context of DC, this is equivalent to U being countably complete.) The *Ketonen order* is defined on the well-founded ultrafilters on an ordinal δ by setting $W \leq U$ if there is a sequence $\langle W_\alpha : \alpha < \delta \rangle$ of well-founded ultrafilters such that

$$W = \{A \subseteq \delta : \{\alpha < \delta : A \cap \alpha \in W_\alpha\} \in U\}$$

ZF alone implies that the Ketonen order is well founded. In the context of the theorem above, one can show that κ-complete ultrafilters on ordinals are well founded and almost linearly ordered by the Ketonen order in the sense that each antichain has cardinality less than or equal to λ. This is arguably a weak version of the Ultrapower Axiom, which assuming AC is equivalent to the statement that the Ketonen order is linear.

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Bi-interpretation of weak set theories

Joel D. Hamkins

(joint work with Alfredo Roque Freire)

Set theory exhibits a truly robust mutual interpretability phenomenon: in any model of one set theory we can define models of diverse other set theories and vice versa. In any model of ZFC, we can define models of ZFC + GCH and also of ZFC + $\neg$CH and so on in hundreds of cases. And yet, it turns out, in no instance do these mutual interpretations rise to the level of bi-interpretation. Ali Enayat [1] proved that distinct theories extending ZF are never bi-interpretable, and models of ZF are bi-interpretable only when they are isomorphic. So there is no nontrivial bi-interpretation phenomenon in set theory at the level of ZF or above. Nevertheless, for natural weaker set theories, we prove, including ZFC$^-$ without power set and Zermelo set theory Z, there are nontrivial instances of bi-interpretation. Specifically, there are well-founded models of ZFC$^-$ that are bi-interpretable, but not isomorphic–even H_{ω_1} and H_{ω_2} can be bi-interpretable–and there are distinct bi-interpretable theories extending ZFC^-. Similarly, using a construction of Mathias, we prove that every model of ZF is bi-interpretable with a model of Zermelo set theory in which the replacement axiom fails.

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Continuity and monotonicity results at partition cardinals

Stephen Jackson
(joint work with William Chan and Nam Trang)

We prove several results concerning the continuity and monotonicity of functions Φ from a function space κ^ϵ to the ordinals, where $\epsilon \leq \kappa$ and κ has the weak (if $\epsilon < \kappa$) or strong (if $\epsilon = \kappa$) partition property. For example, we show that if ϵ has cofinality ω, then f almost everywhere depends only on $\sup(f)$ and f restricted to δ for some fixed δ less than κ. As a consequence of this result we obtain the following result about cardinalities: if κ has the weak partition property, then there does not exist an injection from $\kappa^{<\kappa}$ into ON^λ for any $\lambda < \kappa$.

This cardinality result was previously known for $\kappa = \omega_1$ by a determinacy argument. We also show that for κ having the strong partition property, any Φ is monotonically increasing on a measure one set.

The projective Fraïssé limit of graphs with confluent epimorphisms

Aleksandra Kwiatkowska
(joint work with Włodzimierz J. Charatonik and Robert P. Roe)

In [5] Irwin and Solecki introduced the concept of a projective Fraïssé limit analogous to injective Fraïssé limit from model theory. They considered finite linear (combinatorial) graphs together with epimorphisms, and showed that the topological realization of the Fraïssé limit is the pseudo-arc. Bartošová and Kwiatkowska continued this study in [1], where they considered finite (combinatorial) rooted trees and found that in that case the topological realization of the Fraïssé limit is the Lelek fan. Panagiotopoulos and Solecki [6] introduced appropriate definitions for connectedness and monotone maps on combinatorial graphs. They considered finite connected graphs with monotone epimorphisms and showed that the topological realization of the Fraïssé limit is the Menger curve. In [4], Charatonik and Roe considered finite trees with epimorphisms which are respectively monotone, confluent, order-preserving, retractions, light, etc. As topological realizations of the various Fraïssé limits they obtained known continua, such as the Cantor fan and the generalized Ważewski dendrite D_3, as well as previously unknown ones.

Confluent mappings between continua were introduced by Charatonik [2] as a generalization of monotone maps. Those notions were adapted to topological graphs by Charatonik-Roe [4]. Given two topological graphs G and H, an epimorphism $f\colon G \to H$ is called *monotone* if preimages of connected sets are connected. It is called *confluent* if for every connected subset Q of vertices of H and every connected component C of $f^{-1}(Q)$ we have $f(C) = Q$.

In a joint work with Charatonik and Roe [3], we show that finite connected graphs with confluent epimorphism form a projective Fraïssé class and we investigate the continuum obtained as the topological realization of the projective Fraïssé limit. We show that this continuum is indecomposable, but not hereditarily indecomposable, as arc-components are dense. It is one-dimensional, pointwise self-homeomorphic, and each point is the top of the Cantor fan. Moreover, it is hereditarily unicoherent, in particular, it does not embed a circle; however, it embeds a solenoid.

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Distributivity and Minimality in Perfect Tree Forcings for Singular Cardinals

Heike Mildenberger

(joint work with Maxwell Levine)

Dobrinen, Hathaway and Prikry studied a forcing $\mathbb{P}_\kappa$ consisting of perfect trees of height λ and width κ where κ is a singular λ-strong limit of cofinality λ. They showed that if κ is singular of countable cofinality, then $\mathbb{P}_\kappa$ is minimal for ω-sequences assuming that κ is a supremum of a sequence of measurable cardinals. We obtain this result without the measurability assumption.

Prikry proved that $\mathbb{P}_\kappa$ is (ω, ν)-distributive for all $\nu < \kappa$ given a singular ω-strong limit cardinal κ of countable cofinality, and Dobrinen et al. asked whether this result generalizes if κ has uncountable cofinality. We answer their question in the negative by showing that $\mathbb{P}_\kappa$ is not $(\lambda, 2)$-distributive if κ is a λ-strong limit of uncountable cofinality λ and we obtain the same result for a range of similar forcings, including one that Dobrinen et al. consider that consists of pre-perfect trees. We show that $\mathbb{P}_\kappa$ collapses κ to λ.

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Iterations, stationary reflection and Prikry-type forcings

Alejandro Poveda
(joint work with Assaf Rinot and Dima Sinapova)

In this talk we give an overview of the theory of $(\Sigma, \vec{\mathbb{S}})$-Prikry forcings and their iterations, recently introduced in a series of papers [1, 2, 3]. We will begin presenting the simpler class of Σ-Prikry forcings and demonstrating that many Prikry-type posets that center on countable cofinalities fall into this framework. Among these examples one finds Prikry forcing, Gitik-Sharon forcing, the Extender-Based Prikry forcing and its supercompact version, and the recent Gitik's *collapsing generators* Extender-Based forcing. Afterwards, we shall discuss how these forcings can be iterated in a successful and abstract fashion.

The very first applications of the $(\Sigma, \vec{\mathbb{S}})$-Prikry framework concern the study of *Compactness* and *Incompactness Principles* at the level of successors of singulars. More particularly, these latter touch upon the tension between *Stationary Reflection* and the failure of the *Singular Cardinal Hypothesis* (SCH). In his *Annals of Mathematics* paper from 1977, Magidor [4] proved that the SCH can fail at the first singular cardinal, $\aleph_\omega$. Some years later, in 1982, Magidor [5] obtained a result of an opposite nature, asserting that stationary reflection may hold at the level of the successor of the first singular cardinal, $\aleph_{\omega+1}$. Ever since, it remained open whether Magidor's compactness and incompactness results may co-exist. In Part III of our project [3] we improve the machinery developed in [1, 2] aiming to iterate Prikry-type forcings that enable to bring down the cardinal structure to small cardinals. As an application of this general framework we settle, in the affirmative, the above-mentioned long-standing problem by Magidor:

Theorem (P.-Rinot-Sinapova)**.** *Suppose that there are infinitely many supercompact cardinals. Then, there exists a forcing extension of the set-theoretic universe where the following properties hold:*

(1) $2^{\aleph_n} = \aleph_{n+1}$ *for all* $n < \omega$*;*
(2) $2^{\aleph_\omega} = \aleph_{\omega+2}$*, hence* $\mathrm{SCH}_{\aleph_\omega}$ *fails;*
(3) $\mathrm{Refl}(\aleph_{\omega+1})$ *holds.*

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A dual of Juhász' question

Assaf Rinot
(joint work with Roy Shalev)

A *Dowker space* is a normal topological space whose product with the unit interval is not normal. Whether such a space exists was asked by Dowker in a paper from 1951 [3]. The first consistent example was soon given by Rudin in 1955 [10], who constructed a Dowker space of size $\aleph_1$, assuming the existence of a Souslin tree. Curiously, the existence of a Souslin tree was shown to be consistent only at the late 1960's [5, 13, 6].

By now, there are a few constructions of Dowker spaces in ZFC; a space of size $(\aleph_\omega)^{\aleph_0}$ [11], of size continuum [1], and of size $\aleph_{\omega+1}$ [8]. The following problem is still standing:

Question 1. *Is there a Dowker space of size $\aleph_1$?*

The list of known sufficient conditions include CH [7], $\clubsuit$ [2], a Luzin set [14], and a certain tailored instance of a strong club-guessing principle [4]. Recall:

- Jensen: $\diamondsuit$ implies the existence of a Souslin tree;
- Devlin: $\diamondsuit$ is equivalent to $\mathsf{CH} + \clubsuit$;
- Jensen: CH does not imply the existence of a Souslin tree;
- Juhász: Does $\clubsuit$ imply the existence of a Souslin tree?

Juhász' question remains open for 35 years now. Here, we propose to look at its dual.

The literal dual would ask whether the existence of a Souslin tree imply $\clubsuit$, but this is easily refuted. So, we ask:

Question 2. *Does the existence of a Souslin tree imply a weak form of $\clubsuit$, strong enough to entail the existence of a Dowker space of size $\aleph_1$?*

In a joint work with Shalev [9], we gave an affirmative answer. Hereafter, κ denotes a regular uncountable cardinal.

Definition 3 ([9]). Let $\mathcal{S}$ be a nonempty collection of stationary subsets of κ. The principle $\clubsuit_{\mathsf{AD}}(\mathcal{S}, \mu, {<}\theta)$ asserts the existence of a sequence $\langle \mathcal{A}_\alpha \mid \alpha \in \bigcup \mathcal{S}\rangle$ such that:

(1) $\mathcal{A}_\alpha$ is a pairwise disjoint family of μ many cofinal subsets of α;
(2) For every $\mathcal{B} \subseteq [\kappa]^\kappa$ of size $< \theta$ and every $S \in \mathcal{S}$, there are stationarily many $\alpha \in S$ with $\sup(A \cap B) = \alpha$ for all $A \in \mathcal{A}_\alpha$ and $B \in \mathcal{B}$;
(3) For all $A \neq A'$ from $\bigcup_{S\in\mathcal{S}} \bigcup_{\alpha\in S} \mathcal{A}_\alpha$, $\sup(A \cap A') < \sup(A)$.

Theorem 4 ([9]). *Assume any of the following:*

(1) $\clubsuit_{\mathsf{AD}}(\mathcal{S},1,2)$ *holds for some infinite partition* $\mathcal{S}$ *of a nonreflecting stationary subset of* κ*;*

(2) $\clubsuit_{\mathsf{AD}}(\{E^\kappa_\lambda\},\lambda,1)$ *holds, where* $\kappa=\lambda^+$ *for an infinite regular cardinal* λ*.*

Then there exists a Dowker space of size κ*.*

Theorem 5 ([9]). *If there exists a Souslin tree, then* $\clubsuit_{\mathsf{AD}}(\mathcal{S},\omega,{<}\omega)$ *holds for any partition* $\mathcal{S}$ *of* ω_1 *into stationary sets.*

A generalization of the preceding theorem involves the concept of *vanishing branches* of Souslin trees. For a κ-Souslin tree $\mathcal{T}$, let $V(\mathcal{T})$ stand for the set of limit ordinals $\alpha<\kappa$ such that, for every node x in $\mathcal{T}$ of height $<\alpha$, there exists an α-branch containing x that has no upper bound in $\mathcal{T}$. The general form of Theorem 5 asserts that for every κ-Souslin tree $\mathcal{T}$, there exists a club $C\subseteq\kappa$ such that $\clubsuit_{\mathsf{AD}}(\mathcal{S},\mu,{<}\theta)$ holds, provided that $\mu<\kappa=\kappa^{<\theta}$, and $\mathcal{S}$ is a partition of $V(\mathcal{T})\cap C\cap E^\kappa_{\geq\theta}$ into stationary sets.

Motivated by this finding, in the last part of our talk, we turned to study $\mathrm{Vspec}(\kappa)=\{V(\mathcal{T})\mid\mathcal{T}\text{ is a }\kappa\text{-Souslin tree}\}$.

A λ-complete λ^+-Souslin tree $\mathcal{T}$ satisfies $V(\mathcal{T})=E^{\lambda^+}_\lambda$, and a uniformly coherent κ-Souslin tree $\mathcal{T}$ satisfies $V(\mathcal{T})=E^\kappa_\omega$. In [12], Shelah gave a forcing construction of a *full* κ-Souslin tree, which is a tree $\mathcal{T}$ that in particular satisfies $V(\mathcal{T})=\emptyset$. In an upcoming joint paper with Greenstein we prove that the following two propositions hold in L:

(1) If κ is an inaccessible cardinal that is not weakly compact, then $\mathrm{Vspec}(\kappa)\cap(\mathrm{NS}_\kappa)^+$ is dense in $(\mathrm{NS}_\kappa)^+$;

(2) If κ is a Mahlo cardinal that is not weakly compact, then there exists a family of 2^κ-many full κ-Souslin trees such that the product of any finitely many of them is again κ-Souslin.

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Extension of Subcomplete Forcing Axiom which implies $\diamondsuit^+_{\omega_1}$

Hiroshi Sakai

Jensen [1] introduced the class of subcomplete forcings, which includes all σ-closed forcings, Namba forcing (under CH), Prikry forcing and club shootings through stationary subsets of $\mathrm{Cof}(\omega)$. Subcomplete forcings add no reals and preserve stationary subsets of ω_1. Also, revised countable support iterations of subcomplete forcings are subcomplete.

Jensen [2] studied the forcing axiom for subcomplete forcings, which is called the Subcomplete Forcing Axiom and abbreviated as SCFA. SCFA is consistent with $\diamondsuit_{\omega_1}$. Also, SCFA implies several important consequences of Martin's Maximum (MM), such as the Singular Cardinal Hypothesis and the reflection of stationary subsets of $\kappa \cap \mathrm{Cof}(\omega)$ for a regular cardinal $\kappa > 2^\omega$.

MM also has the following interesting consequences, which are consistent with $\diamondsuit_{\omega_1}$: $\mathrm{MA}^+(\sigma\text{-closed})$, The Weak Reflection Principle, Chang's Conjecture, the failure of Kurepa Hypothesis. It is natural to ask whether SCFA implies them.

In this talk, we show that SCFA implies none of the above mentioned consequences of MM. Note that all of them fails under $\diamondsuit^+_{\omega_1}$. In fact, we prove that SCFA is consistent with $\diamondsuit^+_{\omega_1}$. Very rough idea of our proof is as follows.

For some kind of a $\diamondsuit^-_{\omega_1}$-sequence $\vec{K} = \langle K_\xi \mid \xi < \omega_1 \rangle$, we introduce the notion of $\vec{K}$-subcomplete forcings and consider its forcing axiom, which is denoted as $\vec{K}$-SCFA. Then, we have the following, where a nice iteration in (2) is a variation of a revised countable support iteration, which was introduced by Miyamoto [3].

(1) For any $\vec{K}$, all $\vec{K}$-subcomplete forcings are subcomplete. So $\vec{K}$-SCFA implies SCFA.
(2) All nice iterations of $\vec{K}$-subcomplete forcings are $\vec{K}$-subcomplete. So $\vec{K}$-SCFA for some $\vec{K}$ is consistent.
(3) For any $\vec{K}$, $\vec{K}$-SCFA implies $\diamondsuit^+_{\omega_1}$.

Then, it follows that SCFA is consistent with $\diamondsuit^+_{\omega_1}$.

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The universe below Woodin limit of Woodins

Grigor Sargsyan

Since 2000s many interesting statements were shown to be consistency wise weaker than a Woodin cardinal that is a limit of Woodin cardinals. Examples include:

(1) $\mathrm{MM}^{++}(c)$ [3].
(2) CH + dense ideal on ω_1.
(3) Sealing (generic absoluteness for universally Baire sets) [2].
(4) $2^\omega = \omega_2$ + failure of $\square_{\omega_3}$ and $\square(\omega_3)$ [1].

All of what is mentioned above were believed to be at least as strong as superstrong cardinals. Clause 4 above implies that one cannot prove in ZFC alone that countable submodels of a K^c construction are iterable. In this talk, we will outline the above progress and the future research directions.

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Perfect matchings in hyperfinite graphings

Marcin Sabok
(joint work with Matthew Bowen and Gábor Kun)

We characterize hyperfinite bipartite graphings that admit measurable perfect matchings. In particular, we prove that every regular hyperfinite one-ended bipartite graphing admits a measurable perfect matching.

We give several applications of this result. We extend the Lyons-Nazarov theorem by showing that a bipartite Cayley graph admits a factor of iid perfect matching if and only if the group is not isomorphic to the semidirect product of $\mathbb{Z}$ and a finite group of odd order, answering a question of Lyons and Nazarov and Kechris and Marks in the bipartite case. We also answer a question of Bencs, Hrušková and Tóth arising in the study of balanced orientations in graphings. Finally, we show how our results generalize and lead to a simple approach to recent results on the measurable circle squaring by Grabowski, Máthé and Pikhurko.

The extent of determinacy in ω-small mice

Farmer Schlutzenberg
(joint work with John Steel)

Assume ZFC + infinitely many Woodin cardinals and a measurable above. The minimal iterable proper class mouse M_ω with infinitely many Woodin cardinals satisfies "the reals $\mathbb{R}$ are wellordered in $L(\mathbb{R})$". However, determinacy holds essentially as far as possible in $L(\mathbb{R}\cap M_\omega)$, in that M_ω also satisfies "there is an ordinal δ such that

(i) $L_\delta(\mathbb{R}) \models \mathrm{AD}$ and
(ii) there is a wellorder of $\mathbb{R}$ in $L_{\delta+1}(\mathbb{R})$"

In fact, (i) holds in the strong sense that there is a Σ_1-elementary embedding

$$j : L_\delta(\mathbb{R}\cap M_\omega) \to L(\mathbb{R})$$

so δ measures more generally the extent to which $L(\mathbb{R})^{M_\omega}$ is correct, as in, agrees with the true $L(\mathbb{R})$. (Note that the failure of determinacy in $L_{\delta+1}(\mathbb{R}\cap M_\omega)$ prevents a Σ_1-elementary embedding $j : L_{\delta+1}(\mathbb{R}\cap M_\omega) \to L(\mathbb{R})$.) See [1, §7].

Rudominer and Steel conjectured in 1999 [2] that a similar phenomenon should arise in all ω-small mice which model ZF^- + "$\mathbb{R}$ exists" (these include M_ω and all mice below it in the mouse-order which model ZF^- + "$\mathbb{R}$ exists"). They confirmed the conjecture or a weakening thereof in certain cases, but other cases have remained open.

The case distinction is determined by the degree of closure of the reals of M with respect to definability over segments $\mathcal{J}_\beta(\mathbb{R})$ (of the true $L(\mathbb{R})$; the $\mathcal{J}$ here refers to Jensen's $\mathcal{J}$-hiearchy over $\mathbb{R}$). Let β_0 be the least β such that for some $x \in \mathbb{R}\cap M$ and $n < \omega$, the set X of all reals which are Σ_n-definable from x and (codes for) ordinal parameters over $\mathcal{J}_\beta(\mathbb{R})$ is such that either $X \not\subseteq M$ or X is uncountable in M. Then a Σ_1-gap ends at β_0. (Recall that a Σ_1-gap of $L(\mathbb{R})$ (see [3]) is a maximal interval $[\alpha,\beta]$ such that $\mathcal{J}_\alpha(\mathbb{R}) \preccurlyeq_{1,\mathbb{R}} \mathcal{J}_\beta(\mathbb{R})$. Here $\preccurlyeq_{1,\mathbb{R}}$ denotes Σ_1-elementarity with respect to parameters in $\mathbb{R}\cup\{\mathbb{R}\}$, but actually full Σ_1-elementarity $\mathcal{J}_\alpha(\mathbb{R}) \preccurlyeq_1 \mathcal{J}_\beta(\mathbb{R})$ holds). The case distinction in the Rudominer-Steel proof depends on the nature of the gap ending at β_0. In particular, the conjecture has remained open in the case that β_0 ends a weak gap. We report on some further progress toward a positive resolution of the conjecture, in particular establishing a slightly weakened variant of the conjecture in the weak gap case.

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Continuous logic and Borel equivalence relations

Todor Tsankov
(joint work with Andreas Hallbäck and Maciej Malicki)

The theory of Borel reducibility of definable equivalence relations was initiated by Friedman and Stanley who were specifically interested in the equivalence relation of isomorphism of countable structures. Since then, the scope of the theory has considerably expanded but isomorphism of countable structures remains one of the situations where the most detailed results are available and where both methods of infinitary model theory and descriptive set theory can be applied. In particular, Hjorth, Kechris and Louveau [2] have developed a rich theory for Borel isomorphism equivalence relations in this setting.

In our work [1], we use infinitary continuous logic to extend parts of this theory to metric structures. Our main result is a model-theoretic characterization of when isomorphism of locally compact metric structures is an essentially countable equivalence relation. It is a common generalization of theorems of Hjorth (for pseudo-connected locally compact metric spaces) and Hjorth and Kechris (for countable structures).

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Absolute model companionship, and the continuum problem

Matteo Viale

Absolute model companionship (AMC) is a strict strengthening of model companionship defined as follows: For a theory T, $T_{\exists\vee\forall}$ denotes the logical consequences of T which are boolean combinations of universal sentences. S is the AMC of T if it is model complete and $T_{\exists\vee\forall} = S_{\exists\vee\forall}$. The $\{+,\cdot,0,1\}$-theory ACF of algebraically closed field is the model companion of the theory of Fields but not its AMC as $\exists x(x^2+1=0) \in \mathsf{ACF}_{\exists\vee\forall} \setminus \mathsf{Fields}_{\exists\vee\forall}$. Any model complete theory T is the AMC of $T_{\exists\vee\forall}$.

We use AMC to study the continuum problem and to gauge the expressive power of forcing. We show that (a definable version of) $2^{\aleph_0} = \aleph_2$ is the unique solution to the continuum problem which can be in the AMC of a *partial Morleyization* of the $\in$-theory ZFC+*there are class many supercompact cardinals*. We also show that (assuming large cardinals) forcibility overlaps with the apparently weaker notion of consistency for any mathematical problem ψ expressible as a Π_2-sentence of a (very large fragment of) third order arithmetic (CH, the Suslin hypothesis, the Whitehead conjecture for free groups are a small sample of such problems ψ).

Partial Morleyizations can be described as follows: let Form_τ be the set of first order τ-formulae; for $A \subseteq \mathsf{Form}_\tau$, τ_A is the expansion of τ adding atomic relation symbols R_ϕ for all formulae ϕ in A and $T_{\tau,A}$ is the τ_A-theory asserting that each τ-formula $\phi(\vec{x}) \in A$ is logically equivalent to the corresponding atomic formula $R_\phi(\vec{x})$. For a τ-theory T $T + T_{\tau,A}$ is the *partial Morleyization* of T induced by $A \subseteq \mathsf{Form}_\tau$.

We refer the reader to [1, 2] for more details.[1]

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On the additivity of strong homology for locally compact separable metric spaces

Justin T. Moore

(joint work with Nathaniel Bannister, Jeff Bergfalk)

We show that it is consistent relative to a weakly compact cardinal that strong homology is additive and compactly supported within the class of locally compact separable metric spaces.

This complements work of Mardešić and Prasolov [2] showing that the Continuum Hypothesis implies that a countable sum of Hawaiian earrings witnesses the failure of strong homology to possess either of these properties.

Our results build directly on work of Bergfalk and Lambie-Hanson [1] which establishes the consistency, relative to a weakly compact cardinal, of $\lim^s \mathbf{A} = 0$ for all $s \geq 1$ for a certain pro-abelian group $\mathbf{A}$; we show that that work's arguments carry implications for the vanishing and additivity of the $\lim^s$-functors over a substantially more general class of pro-abelian groups indexed by ω^ω.

Note: the speaker needed to cancel his talk at the last minute and the results mentioned in this abstract were not presented at this Oberwolfach meeting after all.

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Rigidity conjectures in continuous quotients

Alessandro Vignati

The focus of this work is the following question: How does a change of an ideal change the structure of a quotient?

Some context: Suppose $\mathcal{M}$ and $\mathcal{N}$ are Borel spaces with a compatible algebraic structure (such as groups, Boolean algebras, C*-algebras), and E and F are Borel ideals in $\mathcal{M}$ and $\mathcal{N}$. We call $\mathcal{M}/E$ and $\mathcal{N}/E$ *Borel quotient structures*, and denote by π_E and π_F the canonical quotient maps. Suppose further that $\Phi\colon \mathcal{M}/E \to \mathcal{N}/F$ is a homomorphism. Can we find a *lifting* of Φ, i.e., a map making the following diagram commute, which has some desirable topological or algebraic properties?

$$
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\quad\Phi_*\quad} & \mathcal{N} \\
{\scriptstyle \pi_E}\big\downarrow & & \big\downarrow{\scriptstyle \pi_F} \\
\mathcal{M}/E & \xrightarrow{\quad\Phi\quad} & \mathcal{N}/F
\end{array}
$$

We focus in particular on the case when Φ is an isomorphism. Our study revolves around the following definition:

Definition 1. Let Φ be an isomorphism between Borel quotient structures $\mathcal{M}/E$ and $\mathcal{N}/E$. Φ is *topologically trivial* if it has a Borel-measurable lifting.

Question 2. *Under what assumptions is it true that every isomorphism between $\mathcal{M}/E$ and $\mathcal{N}/F$ is topologically trivial?*

The assumptions referred to in this question come in two varieties:

(1) The assumptions on the structures $\mathcal{M}$ and $\mathcal{N}$ and on E and F.
(2) The additional set-theoretic assumptions.

We focus on the second case.

An example: In 1956 W. Rudin proved that the Continuum Hypothesis (CH) implies that the Čech–Stone remainder of $\mathbb{N}$ (with the discrete topology), $\beta\mathbb{N}\setminus\mathbb{N}$, has $2^{\mathfrak{c}}$ homeomorphisms. By the Stone Duality, autohomeomorphisms of $\beta\mathbb{N}\setminus\mathbb{N}$ correspond to automorphisms of the Boolean algebra $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$, and therefore Rudin's result provides a topologically nontrivial automorphism of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$. In 1979, Shelah described a forcing extension of the universe in which every autohomeomorphism of $\beta\mathbb{N}\setminus\mathbb{N}$ is the restriction of a continuous map of $\beta\mathbb{N}$ into itself. This gives that in Shelah's model, there are only topologically trivial automorphisms of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$.

Extensions of Shelah's argument (nowadays facilitated by Forcing Axioms) show that this rigidity of $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$ is shared by other similar quotient structures. In general, we work on the following pattern:

Conjecture 3. For Borel quotient structures $\mathcal{M}/E$ and $\mathcal{N}/F$, consider the following statements.

(1) The Continuum Hypothesis (CH) implies that $\mathcal{M}/E$ has $2^{\mathfrak{c}}$ automorphisms (and therefore $2^{\mathfrak{c}}$ topologically nontrivial automorphisms).

(2) Forcing Axioms imply that every isomorphism between $\mathcal{M}/E$ and $\mathcal{N}/F$ is topologically trivial.

We work mainly on quotients of Boolean algebras, and certain quotients of C*-algebras known as corona C*-algebras, noncommutative generalisation of Čech-Stone remainders of locally compact noncompact topological spaces. We prove instances of Conjecture 3 for large classes of such quotient structures.

This work is contained in the survey [1].

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Virtually strong cardinals and virtually Woodin cardinals

Trevor M. Wilson

A large cardinal property defined in terms of elementary embeddings can be weakened by allowing the elementary embeddings to exist in a generic extension of the universe, to obtain what is known as a virtual large cardinal property. For example, Magidor's characterization of supercompactness can be weakened in this way to virtual supercompactness, a.k.a. remarkability as defined by Schindler. We may similarly weaken the definition of strong cardinal to obtain a definition of virtually strong cardinal. (We don't require the codomain of the elementary embedding to be well-founded above the point of its agreement with V, because if we did, the definition would be equivalent to virtual supercompactness.)

We outline a proof that virtual strongness of a cardinal κ is equivalent to a Löwenheim–Skolem property for a certain fragment of infinitary second-order logic, namely the one obtained from atomic formulas and their negations by finitary quantification, arbitrary disjunctions, and $<\kappa$-length conjunctions.

We then define virtually Woodin from virtually strong just as Woodin is defined from strong, and we discuss some equivalent characterizations and equiconsistency results involving virtually Woodin cardinals. First, we characterize the smallest virtually Woodin cardinal as the smallest cardinal such that every rayless graph of that cardinality is isomorphic to a proper subgraph of itself, where "rayless" means having no infinite path. (This characterization also holds for rayless trees in place of rayless graphs.) Second, we give a combinatorial characterization of virtual Woodinness of a cardinal κ as a transfinite generalization of n-subtlety in which regressive functions on $[\kappa]^{n+1}$ are replaced by regressive functions on fronts on κ of rank κ, where "front" is defined as in the Nash-Williams bqo theory.

Finally, we define an algebraic property of small structures that is equiconsistent via Mitchell forcing with the existence of a virtually Woodin cardinal: it is the virtual weak Vopěnka property at ω_2, which says there is no ω_2-sequence of structures M_α, each of cardinality ω_1 in a common signature of cardinality ω_1, such that in every generic extension of the universe the number of homomorphisms from M_α to M_β is 0 if $\alpha < \beta$ and 1 otherwise. The non-virtual version of this equiconsistency remains open in the inner model direction.

Coloring distance graphs in Euclidean spaces

Jindrich Zapletal

The talk presents a challenging independence results about chromatic numbers of graphs on Euclidean spaces.

Definition 1. Let $n \geq 1$ be a number. The graph Γ_n on $\mathbb{R}^n$ connects points of rational Euclidean distance.

In ZFC, chromatic numbers of the graphs Γ_n have been completely determined. After Komjáth, Erdős, and Hajnal showed in ZFC that the graphs Γ_2 and Γ_3 are countably chromatic, Komjáth showed that the same holds for an arbitrary value of n. The method of proof relies on the fact that the graphs Γ_n are σ-algebraic, i.e. a countable union of algebraic sets.

In the choiceless theory $\mathrm{ZF}+\mathrm{DC}$ the situation is more interesting. It is impossible to show that any of these graphs are countably chromatic. The main result of this talk is that in $\mathrm{ZF}+\mathrm{DC}$, one can even obtain a consistency result separating different dimensions.

Theorem 2. *Let $n \geq 1$ be a number. It is consistent relative to an inaccessible cardinal that* $\mathrm{ZF}+\mathrm{DC}$ *holds, the chromatic number of Γ_n is countable, yet the chromatic number of Γ_{n+1} is not.*

In fact, I obtained a much stronger result than indicated in the theorem. If Δ is an arbitrary σ-algebraic graph on $\mathbb{R}^n$ which contains no perfect clique, then it is consistent that $\mathrm{ZF}+\mathrm{DC}$ holds, the chromatic number of Δ is countable, and in every non-meager subset of $\mathbb{R}^{n+1}$, every small enough distance can be found. This clearly implies that the chromatic number of Γ_{n+1} is uncountable, because in every partition of $\mathbb{R}^n$ into countably many pieces one of them would have to be non-meager, and that piece could not be a Γ_{n+1}-anticlique.

The theorem is proved using the methodology of balanced forcing, developed jointly with Paul Larson in the book Geometric Set Theory, AMS Surveys and Monographs 248. The model for the theorem is obtained by forcing over the standard choiceless Solovay model using a suitable analytic coloring poset. In fact, it is easy to define the coloring poset in a few lines:

Definition 3. Let $n \geq 1$ be a number. The coloring poset P_n consists of conditions p such that there is a countable real closed subfield $\mathrm{supp}(p) \subset \mathbb{R}$ such that p is a function with domain $\mathrm{supp}(p)^n$, it is a Γ_n-coloring, and for every $x \in \mathrm{dom}(p)$, $p(x)$ is a basic open subset of $\mathbb{R}^n$ containing x as an element. The ordering is defined by $q \leq p$ if $p \subseteq q$ and for every $x \in \mathrm{dom}(q \setminus p)$, the set $q(x)$ contains no elements of $\mathrm{dom}(p)$ Γ_n-connected to x.

The challenge resides in the high degree of control one has to exercise over the P_n-extension of the Solovay model.

Distributivity of iterations of club shooting posets

Martin Zeman

This is a continuation of the joint work of M. Foreman, M. Magidor and M. Zeman on games with filters [1]. The main result concerns the distributivity of iterations of club shooting posets, which is also of independent interest, and very likely has broader applications. In our situation, this kind of result can be used to gain more control over winning strategies constructed for Player II in the Welch's variant of Holy-Schlicht games with filters.

The active stages in the iterations in question are typically, but not necessarily inaccessible cardinals, at each active stage α a closed unbounded set is added through the complement of a carefully chosen non-reflecting stationary subset of α^+, and the supports are sufficiently large. For instance, Easton supports would be suitable here (but the result seems to hold for larger supports as well).

The conclusion is that if the first active stage is δ then the entire iteration is (δ^+, ∞)-distributive. The main point in the argument is passing through inverse limits. Whereas passing through inverse limits of small cofinalities can be done in ZFC using methods known for a long time (and most likely the result has been known for a long time), passing through inverse limits of large cofinalities seems to be less clear, and the only way we know how to do it at this point is using fine structure of extender models.

In this talk a simple instance of such an argument is presented which nevertheless features all essential combinatorial aspects of the construction. The presentation will be self-contained and accessible to a broad set-theoretic audience. The model used will be the constructible universe L and no background on fine structure of L will be assumed.

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