Snapshots of modern mathematics
from Oberwolfach

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Jewellery from tessellations of hyperbolic space

Herbert Gangl

In this snapshot, we will first give an introduction to hyperbolic geometry and we will then show how certain matrix groups of a number-theoretic origin give rise to a large variety of interesting tessellations of 3-dimensional hyperbolic space. Many of the building blocks of these tessellations exhibit beautiful symmetry and have inspired the design of 3D printed jewellery.

1 The axiom of parallels

Some of the most influential books on mathematics in history were written by Euclid around 300 B.C., and are called the “Elements”. For an online translation of these see [1].

With the Elements, Euclid set out to put geometry on a solid footing. This amounted to a huge endeavour. The first task was to find as few *axioms* upon which to base the theory of geometry as possible, where an axiom is a statement or proposition that is regarded as being self-evidently true. The second task was to deduce all the known geometric theorems from those axioms.

In modern parlance, we can state the axioms to which he reduced the whole theory as follows ([1], p.7):

1. Each pair of points can be joined by one and only one straight line segment.
2. Any straight line segment can be indefinitely extended in either direction.
3. There is exactly one circle of any given radius with any given center.

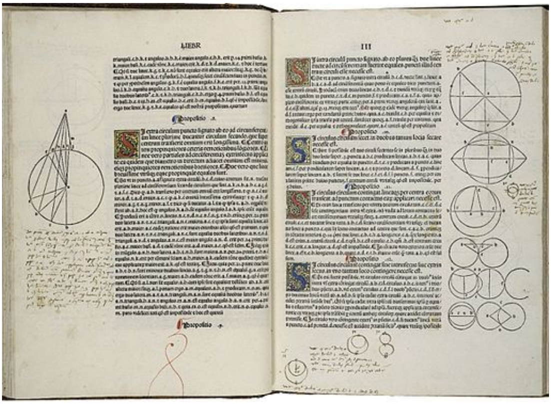


Figure 1: From an Italian translation of Euclid’s Elements.

4. All right angles are congruent to one another.
5. Through any point not lying on a straight line there passes one and only one straight line that does not intersect the given line.

Of such a set of axioms one needs to make sure in particular that they do not lead to a contradiction, that is, they are *consistent*, and that none of the axioms can be deduced from one or more of the others, in other words, that they are *independent*. The fifth axiom, the axiom of parallel lines, was even in Euclid’s time considered something of an enigma, as it seemed superfluous, yet nobody was able to deduce it from the other axioms. The following question naturally arose: Is the fifth axiom independent of the others? The importance of Euclid’s Elements and this somewhat nagging issue for the foundations of mathematics, in that the possible redundance of the fifth axiom was presumably felt to be a blot on the perceived elegance of the axiomatic setting, spurred many attempts over the centuries to deduce the fifth axiom from the other four.

Surprisingly, the solution to this longstanding conundrum was found when (presumably independent) flashes of genius struck at least five people at around the start of the nineteenth century. Most famously, these mathematicians include János Bolyai (1802–1860), Nikolai Lobachevsky (1792–1856) and Carl Friedrich Gauss (1777–1855), but also Ferdinand Karl Schweikart (1780-1859) and his nephew Franz Taurinus (1794–1874) corresponded with Gauss about the problem. Each of them showed, in their own way, that one can replace the fifth axiom in two fundamentally different ways and still obtain a consistent geometry. Roughly speaking, we can replace the parallel postulate with either of the following versions:

5a. Through any point not lying on a straight line there are **no** straight lines that do not intersect the given line.
5b. Through any point not lying on a straight line there are **at least two** straight lines that do not intersect the given line.

The first of these options gives rise to what is called *spherical* geometry, which we can think of as geometry on the surface of a sphere. Here the straight lines are *great circles*, which are the intersections of the sphere with planes that pass through the centre of the sphere. In this snapshot we are interested in the second option, having more than one parallel line through a given point, which gives rise to *hyperbolic* geometry. Fortunately one can picture this kind of geometry using intuition from the spaces we are used to, that is, the *Euclidean* spaces, albeit with some of the "rules" changed. Here we list some of the consequences of allowing more than one parallel line:

- The angle sum in a triangle is strictly smaller than π (as opposed to the equality we are used to from Euclidean geometry).
- If two triangles are similar (that is, have the same angles), then they also have the same side lengths (that is, they are *congruent*), as opposed to the Euclidean case where one has infinitely many non-congruent similar triangles.
- The area of a triangle with angles α, β and γ is equal to $\pi - (\alpha + \beta + \gamma)$, in other words, the area of a hyperbolic triangle can be read off directly from its angles.
- There are (non-empty) triangles with all angles being zero! These triangles, which are called *ideal*, have all of their vertices on the boundary of hyperbolic space, and they have maximal area.
- The "hyperbolic Pythagoras" rule—in a right-angled triangle (which still means with one angle equal to $\pi/2$) with sides of length a, b and hypotenuse c one has
$$\cosh(a)\cosh(b) = \cosh(c)\,.$$

2 First glance at hyperbolic geometry

How can we picture such a strange geometry? There are several rather different models in which one can view it, here we will use the *upper half-plane* model $\mathbb{H}^2 = \{x + iy \in \mathbb{C} \mid x \in \mathbb{R}, y > 0\}$, the upper half of the complex plane. One can view the real line (embedded in $\mathbb{C}$) as part of the boundary of $\mathbb{H}^2$, which we denote by $\partial\mathbb{H}^2$. Apart from the real line, there is one more point ("the point at infinity") that is considered to be part of the boundary. Altogether, $\partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$, and it can be identified with the real projective line in projective geometry. Think stereographic projection[1] from the north pole of a circle to the line which is tangent to the south pole—in our picture the north pole plays the role of the point at infinity and the tangent to the south pole is identified with $\mathbb{R}$.

This model ought to be familiar for those readers who have seen *Möbius transformations* in complex analysis. Recall that these are maps of the form

$$z \mapsto \frac{az+b}{cz+d}$$

with $a, b, c, d \in \mathbb{C}$, and most of them map the complex plane, together with the point at infinity, into itself. We can also use matrix notation to denote such maps, where we simply collect the coefficients into a 2×2 matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

We can consider in particular those Möbius transformations which preserve $\mathbb{H}^2$ (that is, which map the upper half-plane onto itself). These transformations are the ones with coefficients a, b, c, d that are real numbers with $ad - bc > 0$. We are even more interested in the transformations that not only map the upper half plane onto itself, but also preserve the underlying "geometry" of the space. In other words, they do not change the distance between points or the angles in a geometric shape. Maps of a metric space[2] with this property are called *isometries*. We have the following fact:

Fact: Any isometry of $\mathbb{H}^2$ is captured by some matrix A as above with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. The set of all such matrices is called $\mathrm{SL}_2(\mathbb{R})$.

Let us see some examples:

1. The matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ encodes the map $z \mapsto \frac{1 \cdot z + 2}{0 \cdot z + 1} = z + 2$. This is a translation of the complex plane, it simply shifts each point to the right by two units.
2. The matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ encodes the map $z \mapsto -\frac{1}{z}$. A simple calculation shows that this sends the number $a + ib$ to the number $\frac{-a+ib}{a^2+b^2}$, and so it again preserves $\mathbb{H}^2$. It can be seen geometrically as first a reflection in the unit circle (by which we mean each point $a + ib$ outside the unit circle gets mapped to the point $\frac{a+ib}{a^2+b^2}$ inside the circle on the same (Euclidean) straight

[1] See https://en.wikipedia.org/wiki/Stereographic_projection

[2] A metric space is a set of points along with a function that determines a notion of distance between the points. For more details see https://en.wikipedia.org/wiki/Metric_space.

line from the origin which is at the same hyperbolic distance from the circle), and then a reflection in the imaginary axis, sending a to $-a$.

To proceed further we need to introduce the notion of *group*. A group G is a set together with a binary operation (often denoted by multiplication $\cdot$) that satisfies the following rules:

- Closure: for all $g, h \in G$, we have $g \cdot h \in G$.
- Associativity: for all $f, g, h \in G$, we have $f \cdot (g \cdot h) = (f \cdot g) \cdot h$.
- Identity: there is a particular element $e \in G$ which satisfies $g \cdot e = e \cdot g = g$ for all $g \in G$.
- Inverses: for each $g \in G$, there exists an element g^{-1} which satisfies the property that $g \cdot g^{-1} = g^{-1} \cdot g = e$.

One obvious example is the set of non-zero real numbers with ordinary multiplication. Then the identity is 1, and the inverse of every number a is $1/a$. The matrices $\mathrm{SL}_2(\mathbb{R})$ also form a group, with the operation of matrix multiplication. It is easy to check that the closure and associativity properties are satisfied. The identity in $\mathrm{SL}_2(\mathbb{R})$ is given by the 2×2 matrix

$$\mathrm{Id}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and for the inverses, we have that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

A *subgroup* of a group is a subset of the set G which is also a group under the same operation. For example, the set of matrices in $\mathrm{SL}_2(\mathbb{R})$ with entries a, b, c, d that are all integers is a subgroup, which we call $\mathrm{SL}_2(\mathbb{Z})$. A (sub)group is said to be *generated* by a set of elements in G if every element of the group can be written as a finite product of those elements.

The two matrices given as examples 1 and 2 above generate a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ which we shall call Γ. We shall use this subgroup Γ to illustrate various of the notions that we will need later. First of all, we want to have an idea of the "size" of a subgroup inside a group. This is done using the *cosets* with respect to Γ which are defined to be the sets

$$g\Gamma = \{g \cdot h : h \in \Gamma\}, \quad \text{for each } g \in G.$$

For the group Γ as a subgroup of $\mathrm{SL}_2(\mathbb{Z})$, it is an interesting exercise to compute these cosets, and show that there are precisely three of them. We say that Γ is of *index* 3 in $\mathrm{SL}_2(\mathbb{Z})$. That means that three copies of Γ are enough to "cover" the group $\mathrm{SL}_2(\mathbb{Z})$. In that sense we think of Γ as being a *large* subgroup. Let us also note that Γ is an example of an "arithmetic" group, as is $\mathrm{SL}_2(\mathbb{Z})$ itself; such groups play an important role in number theory.

Fix now a number $z \in \mathbb{H}^2$. The set of all image points $\{g(z) : g \in \Gamma\}$ is called the Γ-*orbit* of the point z. We would like to find a subset of $\mathbb{H}^2$ that contains exactly one element of the Γ-orbit of each point in $\mathbb{H}^2$. A one-dimensional analogue would be the set of all translates of a point $x \in \mathbb{R}$ under the group of integers with addition: $\{\dots, x-2, x-1, x, x+1, x+2, \dots\}$. Then a set containing precisely one of these translates for each $x \in \mathbb{R}$ would be the half-open interval $[0, 1)$. Note that this is not the only possible choice, but it is (in some sense) a natural one.

If we can find such a set for a subgroup, such as the subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$, and the set is also "connected" (that is, it is in one piece, like the interval example above), we call it a *fundamental domain for the action of* Γ *on* $\mathbb{H}^2$.

Now each point in $\mathbb{H}^2$ has a Γ-translate in the half-strip $\{x + iy \mid -1 \leq x < 1,\ y > 0\}$: for each $z_0 = x_0 + iy_0$ simply add or subtract integer multiples of 2 from z_0 so that it lands between -1 and 1 (in terms of the maps, this means applying the map $z \mapsto z + 2$ or its inverse $z \mapsto z - 2$ repeatedly until the point is moved to the strip).

Furthermore, since the second matrix, corresponding to the map $z \mapsto -1/z$, maps elements from inside the unit circle to the outside, it seems plausible that a fundamental domain is the region shown in Figure 2 (this is to be thought as extended to the "point at infinity" where the two vertical boundary lines "meet").

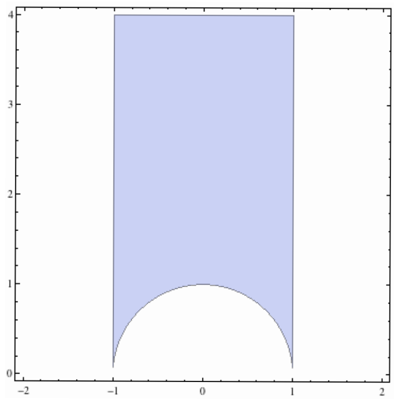


Figure 2: Fundamental domain for the group Γ (it is unbounded in the y-direction).

Indeed, it turns out that this is essentially the correct picture, except that one needs to be a bit more careful at the boundary (only "half" the points are to be counted in, so that there is no overlap).

Note that this fundamental domain is a (hyperbolic) triangle in $\mathbb{H}^2$ with all vertices at the boundary: $(-1,0)$, $(1,0)$ and the point at infinity. In particular, the angles in the triangle, which are calculated by finding the angles between the straight lines tangent to the sides, are all equal to zero. Therefore, this is an example of an ideal triangle, as mentioned above. Let us also mention here that the *geodesics*, or straight lines, in $\mathbb{H}^2$ are the half-circles (and as a limiting case also the straight lines, which can be imagined as circles with infinite radius) orthogonal to the real line, thought of as the boundary $\partial\mathbb{H}^2$.

Once a fundamental domain for a given arithmetic group is found, its translates will determine a "tessellation" of the original space. Simply take all its translates under the group (typically one allows for overlaps along the boundaries, so one is more casual about the boundary of the fundamental domain). For the group Γ, this tessellation is shown in Figure 3, where the black and white colours are only to make the triangles easier to see.

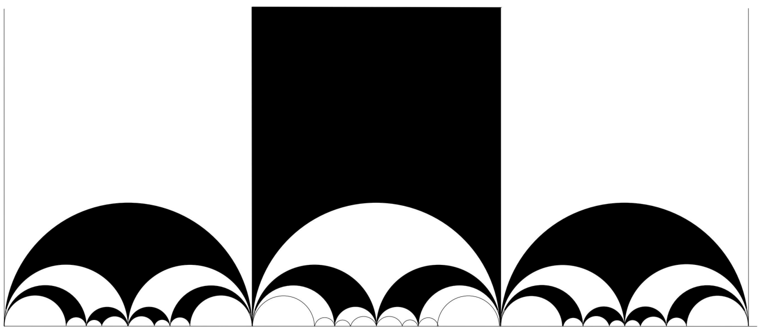


Figure 3: Tessellation of the hyperbolic plane.

Moreover, one can view this picture in a different model of the hyperbolic plane, the *disc model* $\mathbb{D}^2 := \{z \in \mathbb{C} : |z| < 1\}$, the inside of the unit disc in the complex plane. We can send one model to the other by using, for example, the map

$$\begin{array}{ccc} \mathbb{H}^2 & \longrightarrow & \mathbb{D}^2 \\ z & \mapsto & \dfrac{z-i}{z+i} \end{array}$$

and the tessellation in this case is shown in Figure 4. In this model it is perhaps easier to see that each of the fundamental domains is indeed a triangle with vertices on the boundary, which here is the *unit circle* $\mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\}$.

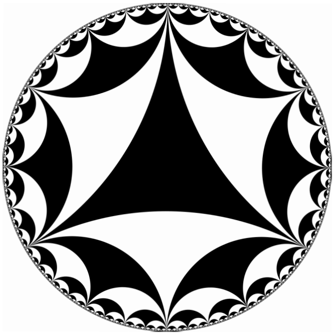


Figure 4: A hyperbolic tessellation of the unit disc model. Each (curved) triangle has the same hyperbolic area.

3 From 2D to 3D

We obtain an analogous picture when we pass from 2D to 3D, that is, from the hyperbolic plane to the hyperbolic *space*. It is often depicted as the "upper half" of the usual (Euclidean) space $\mathbb{R}^3$, consisting of those points $(x, y, z) \in \mathbb{R}^3$ for which $z > 0$, and it is denoted $\mathbb{H}^3$. A beautiful introductory text, including historical information on this topic, was written by Milnor [4]; here we briefly describe some of the features of $\mathbb{H}^3$.

- Its boundary $\partial\mathbb{H}^3$ is given by the xy-plane (consisting of the points $(x, y, z) \in \mathbb{R}^3$ for which $z = 0$) together with a point at infinity. This gives topologically a "1-point compactification" of the plane, and geometrically we obtain a sphere.
- Its geodesics are again certain half-circles; in this 3-dimensional case they must be orthogonal to the boundary plane. As a limiting case, if a half-circle passes through the point at infinity, it becomes a straight line in $\mathbb{H}^3$. Again, think stereographic projection from the north pole, this time of a sphere to the plane which is tangent to the south pole—in our picture the north pole plays the role of the point at infinity and the tangent plane to the south pole is identified with the plane "underneath" the upper half-space.

- The hyperplanes in $\mathbb{H}^3$ are half-spheres, again orthogonal to the boundary plane (a limiting case being half-spheres through ∞ which are planes that intersect this boundary plane at a right angle).
- Isometries now are encoded by elements in $\mathrm{SL}_2(\mathbb{C})$, rather than the group $\mathrm{SL}_2(\mathbb{R})$ of isometries of the hyperbolic plane. Hence we are looking for interesting subgroups of the matrix group $\mathrm{SL}_2(\mathbb{C})$.

4 Tessellations in hyperbolic space

Let us consider a couple of examples. Perhaps the simplest 3-dimensional example of a tessellation in hyperbolic space arises from the group $\mathrm{SL}_2(\mathbb{Z}[i])$ where $\mathbb{Z}[i] \subset \mathbb{C}$ denote the *Gaussian integers*, that is, the set of complex numbers $\{a + ib : a, b \in \mathbb{Z}\}$. We illustrate it in Figure 5. The actual fundamental domain is slightly more complicated, but if one passes to a subgroup (of index 4) of $\mathrm{SL}_2(\mathbb{Z}[i])$, which corresponds to gluing 4 copies of the fundamental domain together, then one obtains a nice octahedron with all six vertices at the boundary. You can think of the octahedron as two square pyramids glued together along the base, and a fundamental domain for the full group $\mathrm{SL}_2(\mathbb{Z}[i])$ is given as half of such a square pyramid.

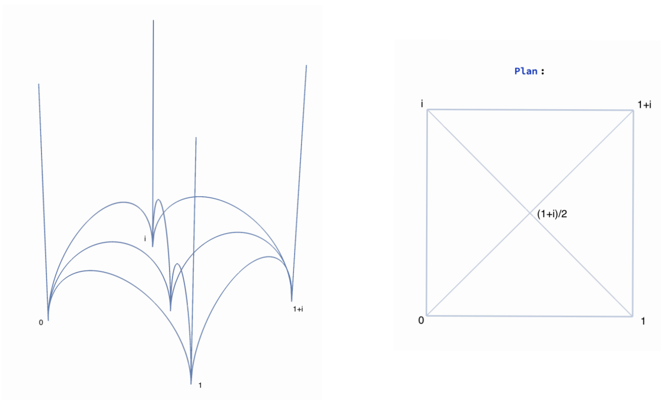
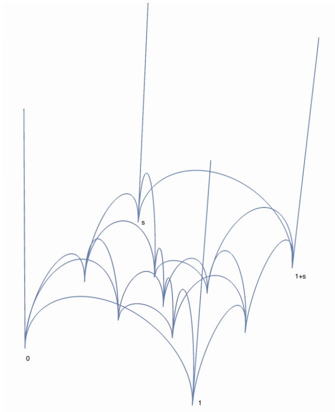


Figure 5: Tessellation of hyperbolic 3-space using $\mathrm{SL}_2(\mathbb{Z}[i])$.

A second example is given by $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$, where $\mathbb{Z}[\sqrt{-2}]$ refers to set of elements $a + b\sqrt{-2}$ where a and b are integers, and the picture is as in Figure 6. We note also that similar projection pictures of a number of cases can be found in [3].



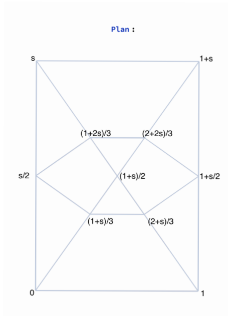


Figure 6: Tessellation of the hyperbolic 3-space using $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$.

How should we interpret these pictures? In both Figures 5 and 6, the image on the left shows (parts of) hyperplanes in $\mathbb{H}^3$. In Figure 6, four of them are vertical, whilst the other 10 of them are half-spheres. The interior of this figure gives a polyhedron which is essentially a fundamental domain arising from the action of $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$, or, more precisely, a suitably large subgroup (recall that this means a subgroup of small index). The right hand picture shows a projection of this polyhedron from the point "at infinity" to the plane below.

It is reasonably straightforward to picture parts of the tessellation of $\mathbb{H}^3$ arising from this fundamental domain via simple translates using the matrices $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ with a an integer or, say, an integer multiple of $\sqrt{-2}$. However, it is considerably harder to picture the image under the "inversion" $z \mapsto -1/z$, and moreover some of the faces become quite small. In any case, from this point on we are more interested in the fundamental domains themselves and what can be done with them.

A schematic 3D-picture, produced using the computing software Mathematica, of the half-spheres bounding the polyhedron is shown in Figure 7 (the polyhedron itself consists of the points above those half-spheres).

If we push the vertex "at infinity" down to a finite point, we can see a compact approximation of the polyhedron (with the same combinatorial data). This is shown on the left-hand side of Figure 8. Then we can try to recognise it as a more familiar polyhedron, at least after straightening out the faces. Indeed we find its Euclidean counterpart to be a cuboctahedron, as shown on the right-hand side of Figure 8.

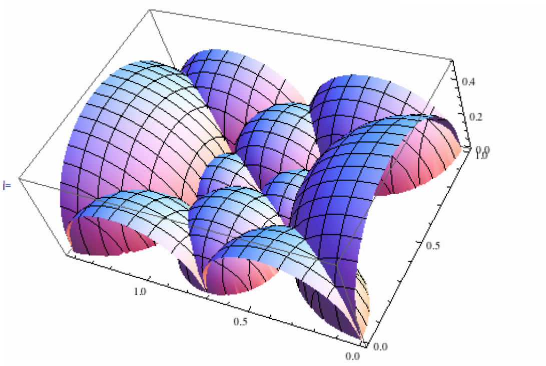


Figure 7: Ideal fundamental domain arising from $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$.

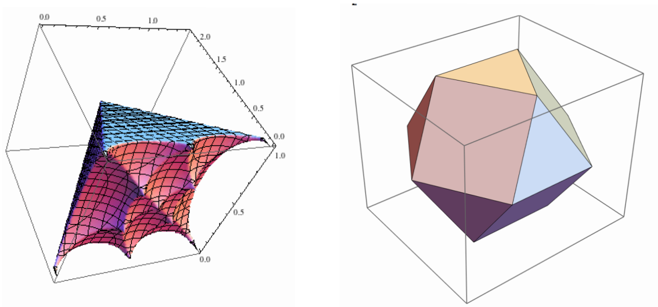


Figure 8: Approximate ideal fundamental domain arising from $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$, and, on the right, a "straightened" version.

5 Further arithmetic examples

There are a dozen further arithmetic examples known, elaborated upon in [2], all of which arise from $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-d}])$ (or at least a closely related such matrix group; the mathematically precise notion is for it to be "commensurable"[3]) for some small positive integer d.

[3] Two groups are called commensurable if their intersection is a subgroup of finite index in each of the two groups (that is, if their intersection constitutes a "large" part of each group).

5.1 The case $d = 6$.

It turns out that the case $d = 6$ is one of the rare cases where we can find a fundamental domain which is a single convex polyhedron in its own right (rather than a union of such polyhedra). More precisely, one obtains a rhombicuboctahedron, which we depict as a 2-dimensional projection from the point at infinity on the left and in its Euclidean avatar on the right, as shown in Figure 9. We suggest that the reader try the following visualisation challenge: Can you "see" that the left hand image is combinatorially, that is, not taking into account distances, a projection of the right-hand image from one of its vertices?

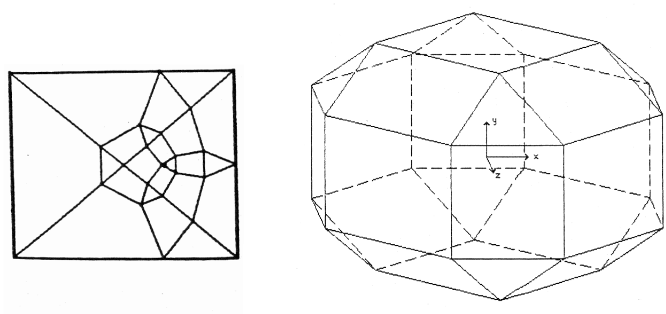


Figure 9: A polyhedron tessellating hyperbolic 3-space, projected to the Euclidean plane from one of its vertices (left), and a "straightened" version of that polyhedron in Euclidean space (right).

6 Hyperbolic polyhedra for decoration and jewellery

We can apply the same procedure to groups closely related to $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-d}])$ for many integers $d > 0$, and it turns out that in a good number of cases one finds interesting looking yet rather skewed polyhedra. Several students in Durham working on summer research projects have been toying around with these over the years and found ways to depict them (M. Spencer) and to exhibit their symmetry better by "spherifying" them with suitable affine transformations (J. Inoue).

These pictures triggered a desire to realise the polyhedra as models, and other students were able to produce the first such models via 3D-printing (E. Woodhouse, J. Inoue). There are plenty of computer aided design (CAD) programs like OpenSCAD and Rhino3D which allow the manipulation of the

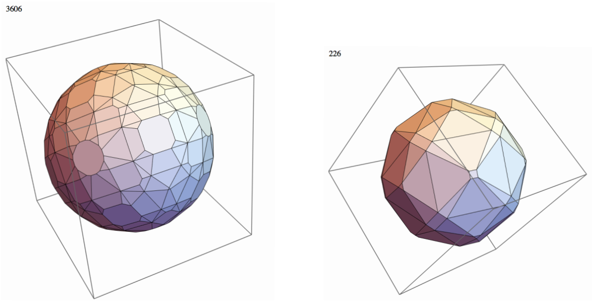


Figure 10: Pictures of polyhedra arising from an ideal tessellation of hyperbolic 3-space from a group commensurable with $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-d}])$ with $d = 3606$ and $d = 226$, respectively.

data which encode the vertices, edges and faces of the models. Moreover, one has the option to produce files from the ensuing models that can be uploaded to the web page of a 3D printing service who in turn print and ship the results—in a good variety of materials—to their customers. Our first such trial runs produced wireframe models of those polyhedra in plastics, and in steel materials like Bronze Steel or Gold Steel; examples of the latter are the following rather decorative models (arising from $d = 3606$ and $d = 226$, respectively), shown in Figure 10. In Figures 11 and 12, we show some of the 3D-printed results, adapted in some cases to produce jewellery.

Figure 11: Pictures of 3D-printed wireframe models of the polyhedra shown in Figure 10 in Polished Bronze Steel and Polished Gold Steel, respectively.

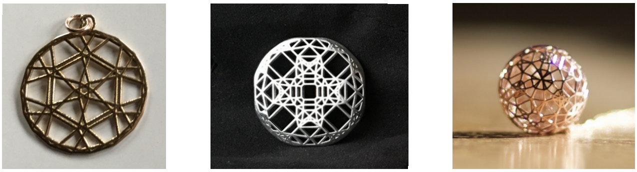


Figure 12: Pictures of polyhedra in Polished Bronze, Rhodium plated and Rose Gold plated, respectively, arising from an ideal tessellation of hyperbolic 3-space for $d = 34$, $d = 11782$ and $d = 1409$, respectively.

It is surprising how nicely poised many of the ensuing polyhedra emerge, as there does not seem to be a compelling reason *a priori* that the vertices of such a polyhedron (given as the simultaneous integer solutions of a linear and a quadratic Diophantine equation[4]) should have such a rich (hidden) symmetry at all. It is a pleasing empirical observation that many polyhedra arising in this way turn out to be combinatorially different—this is in notable contrast to a different tessellation procedure for closely related groups given by Yasaki [5] where only nine combinatorially different polyhedra appear to occur. In fact, the maximal number of vertices for our polyhedra might even grow indefinitely with increasing d (the current record: for $d = 20009$ we find a polyhedron with 2496 vertices).

[4] A Diophantine equation is a polynomial equation in two or more variables in which only integer solutions are allowed.

Image credits

Figure 1 From the Folger Shakespeare Library, under license CC BY-SA 4.0.

Figure 3 Created by the editor.

Figure 4 Licensed under Creative Commons Attribution-Share Alike 3.0 via Wikimedia Commons, https://en.wikipedia.org/wiki/File:Ideal-triangle_hyperbolic_tiling.svg, visited on February 4th, 2022.

All other images created by the author.

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Connections to other fields
Fine Arts

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IMAGINARY
open mathematics