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Topologie

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Abstract. The lectures in the workshop covered various topics in modern topology, including algebraic and geometric topology, homotopy theory, geometric group theory, and manifold topology, as well as connections to neighboring areas, most prominently symplectic topology/geometry. The following current research topics received more attention during the workshop: manifolds and K-theory, symplectic topology and Floer homology, generalizations of hyperbolic techniques in geometric group theory, and equivariant and motivic homotopy theory. The aim of the various topics was to foster communication and provide chances for participants to see and experience driving questions and important methods in nearby fields within the realm of topology.

Mathematics Subject Classification (2020): 55-xx, 57-xx.

Introduction by the Organizers

In the final program we had 5 lectures on Monday and Tuesday, 3 on Wednesday and 4 on Friday (due to the traditional hike on Wednesday and some early departures on Friday). On Thursday we had 3 hour talks in the morning and 4 shorter talks in the afternoon. The format seemed to work very well. The workshop featured three lectures by Mohammed Abouzaid (Columbia University). Professor Abouzaid explained how novel techniques allowed him (and his collaborators) to develop a method of building spaces (more precisely, spectra) which recover Floer homologies through their usual homologies. These spaces have been already used in particular situations (eg in Khovanov homology) to find invariants beyond the known ones in knot theory. To put this advance in perspective, we need to step back and recall some of the ideas of symplectic topology and geometry.

Just as a smooth structure on a manifold allows us to use methods and techniques of calculus in topology, the existence of symplectic and contact structures (through compatible almost-complex structures) provides a possibility to apply complex analysis in solving topological questions. In addition, these structures (having their roots in physics) also provide the framework for connections to theoretical physics, mirror symmetry being the most well-known and most fundamental.

Since their introduction by Gromov 35 years ago, pseudo-holomorphic curves have been an essential tool for studying symplectic and contact manifolds. The main questions revolve around how to organize the information one gets from the spaces of these holomorphic maps. Important developments include the works of Floer, Fukaya, and many others, leading to Floer homology of Lagrangians, the Fukaya category of a symplectic manifold, and contact and symplectic (co)homology. These ideas and methods have now widespread applications, and led to solutions of purely topological questions.

Purely topological results also follow from symplectic/contact theories; for example, it has been shown that the symplectomorphism type of the cotangent bundles of lens spaces are complete diffeomorphism invariants, and (a specific version of) the contact homology of the unit conormal bundle of a knot provides a complete knot invariant. A further example of this relationship is given by Abouzaid's result about exotic spheres and Lagrangian embeddings.

In addition to the lecture series of Professor Abouzaid, there were several lectures in low dimensional topology. The talk of Jen Hom centered on a version of knot Floer homology (a variant of the general symplectic Floer homology package, which turned out to be of utmost importance in contemporary 3-manifold topology and knot theory), with an application for the algebraic structure of the homology cobordism group of integral homology 3-spheres. The talk of Lisa Piccirillo provided a state-of-the-art overview regarding the construction of interesting smooth four-manifolds, at the same time discussing the situation with topological four-manifolds with fixed fundamental groups. Maggie Miller's short presentation verified a new phenomenon for Seifert surfaces of knots: they can stay distinct even when we allow isotopies through the fourth dimension. Marco Marengon showed a natural link between the smooth topology of four-manifolds and knot theory, and outlined some intriguing conjectures regarding knot Floer homology and Khovanov homology groups of knots which bound smooth embedded disks in the standard four-space. Andy Putman's talk focused on the moduli space of curves with level structures and its stable cohomology.

In the area of homotopy theory, we heard reports of many interesting new developments. Jeremy Hahn discussed a revolutionary new approach (based on complex cobordism) he has developed with his collaborators to compute prismatic cohomology of rings which generalizes to define and compute prismatic cohomology of ring spectra. This gives a new streamlined approach to algebraic K-theory computations. This line of research was augmented by the talk of Thomas Nikolaus, who described how prisms can be used to understand the algebraic K-theory of the rings $\mathbb{Z}/p^k$. Tomer Schlank gave a new spectrum-level generalization of the classical Nullstellensatz to chromatic homotopy theory, and used this to deduce new enhanced red-shift results in algebraic K-theory. Lennart Meier reported on his joint work with Gepner to produce and compute integral genuine equivariant elliptic cohomology. Another advance in equivariant homotopy theory was discussed by Mike Hill, who detailed an approach to the computation of the equivariant dual Steenrod algebra for the cyclic groups C_{2^k}.

Several talks featured current work in geometric group theory. Geometric group theory has developed a large body of techniques for studying groups that act geometrically on spaces with negative or non-positive curvature, known as hyperbolic and CAT(0) spaces. The talks in this workshop focused on extending the theory of hyperbolic and CAT(0) groups to broader classes of groups. Two talks, by Indira Chatterji and Elia Fioravanti concerned groups acting on coarse versions of median spaces, which serve as natural generalizations of hyperbolic spaces. A third talk, by Davide Spriano, discussed new ideas for generalizing techniques from CAT(0) cube complexes to broader classes of CAT(0) spaces by introducing an analogue of hyperplanes and curve graphs for these spaces.

We also had several talks relating to high dimensional manifolds, with some overlap with both homotopy theory and symplectic topology. Alexander Kupers spoke about his joint work with Randal-Williams about Torelli Lie algebras, in which they resolved a conjecture of Hain from 1997 using high dimensional manifolds among other tools. Nathalie Wahl gave an enlightening survey of invariance and non-invariance results in string topology, and presented compelling evidence for a non-invariance conjecture. Thomas Kragh talked about spaces of Legendrian knots, and Oscar Randal-Williams talked about algebraic independence of the topological Pontryagin classes.

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Workshop: Topologie

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Abstracts

Towards quantum Morava K-theory

Ivan Smith
(joint work with Mohammed Abouzaid, Mark McLean)

This talk discussed a new interaction between symplectic topology and chromatic homotopy theory, in the vein of Abouzaid and Blumberg's [1].

Symplectic manifolds admit no local invariants: if (X,ω) is connected then the group of Hamiltonian symplectomorphisms $\mathrm{Ham}(X)$ acts transitively. One is therefore led to ask global 'topological' questions. Any symplectic manifold (X,ω) admits a contractible space of 'taming' almost complex structures J, ones for which $\omega(v,Jv) > 0$ whenever $v \neq 0$. Ever since the seminal works of Gromov and Floer, arguably the key tool in studying the global topology of symplectic manifolds has been the package of enumerative invariants defined by counting J-holomorphic curves for some chosen taming J (the taming condition ensures compactness of the moduli spaces).

The moduli spaces $\mathcal{M}(J,\beta)$ of J-curves in some fixed class $\beta \in H_2(X;\mathbb{Z})$ are often highly singular, and have usually been studied via *local Kuranishi charts*: this writes a neighbourhood of a point of the moduli space as the zero-set of a section of an orbibundle over an orbifold. Such local charts are difficult to patch together since the dimension of the base and the fibre vary, only their difference is well-defined. Recently [2], Abouzaid, McLean and the author proved that moduli spaces of genus zero curves admit *global* Kuranishi charts. A global chart for a Hausdorff topological space $\mathcal{M}$ is a quintet $(G,\mathcal{T},E,s,\phi)$ where G is a compact Lie group, $\mathcal{T}$ is a G-manifold for which the action has finite stabilisers, $E \to \mathcal{T}$ is a G-bundle, and s is a G-equivariant section for which there is a homeomorphism $\phi : s^{-1}(0)/G \xrightarrow{\simeq} \mathcal{M}$. Then we show:

Theorem. (Abouzaid-McLean-S.) If (X,ω) is closed, J tames ω and $\beta \in H_2(X;\mathbb{Z})$, the space of stable genus zero curves with n marked points $\mathcal{M}_{0,n}(X,J,\beta)$ admits a global chart.

One can assume that $\mathcal{T}$ is smooth, and that $T\mathcal{T} - \mathfrak{g}$ and E admit stable almost complex structures. The chart is unique up to a reasonable set of 'moves', which preserve the 'virtual dimension' $\dim(\mathcal{T}) - \dim(G) - \mathrm{rk}(E)$. (Work in preparation gives the same result for spaces of closed higher genus stable maps.)

The Morava K-theories are generalised cohomology theories $K_p(n)$ indexed by a prime p and integer $n > 0$, with coefficients $\mathbb{F}_p[v,v^{-1}]$ for a variable v of degree $|v| = 2(p^n - 1)$. When the 'height' $n = 1$ the theory is basically a summand of mod p complex K-theory, but for larger height the theories are constructed algebraically. They are the 'primes' in the stable homotopy category, and intrinsic to it; just as one studies rings one prime at a time in classical commutative algebra, so in chromatic homotopy theory one studies spectra one $K_p(n)$ at a time.

Greenlees and Sadofsky [4] showed that the Morava K-theories satisfy a Poincaré duality theorem for classifying spaces of finite groups. The connection between symplectic topology and chromatic homotopy theory comes from a striking generalisation of this result due to Cheng [3]:

Theorem (Cheng) If X is a smooth manifold admitting an action of G with finite stabilisers, then there is an isomorphism $\tilde{H}^{-*}(X^{-TX\oplus\mathfrak{g}};\mathbb{K}) \cong H_*(X;\mathbb{K})$ whenever $\mathbb{K}$ is a Morava K-theory.

This leads to a definition of a Morava K-theoretic virtual fundamental class for a space $\mathcal{M}$ of J-holomorphic rational curves. Fix a Morava-local theory $\mathbb{K}$; take a global chart $(G,\mathcal{T},E,s)$ for $\mathcal{M}$. Being stably almost complex, the virtual bundles $T\mathcal{T}-\mathfrak{g}$ and E are $\mathbb{K}$-oriented. Start with the equivariant Euler class $e_G(E)\in H^{\mathrm{rk}(E)}_{G,c}(\mathcal{T};\mathbb{K})$, which lives in compactly supported cohomology since $\mathcal{M}$ is compact, and then consider its image under the maps

$$H^*_{G,c}(\mathcal{T};\mathbb{K}) \longrightarrow H^{*-\dim(\mathcal{T})+\dim(G)}_{G,c}(\mathcal{T}^{-T\mathcal{T}\oplus\mathfrak{g}};\mathbb{K}) \longrightarrow H^G_{\dim(\mathcal{T})-\dim(G)-*}(\mathcal{T};\mathbb{K})$$

given by Thom isomorphism and Cheng's duality. Although the class is supported on $\mathcal{T}$, it serves as well as the usual virtual class, since for instance evaluation and stabilisation maps to X or the moduli space $\mathcal{M}_{0,n}$ of stable domains are well-defined on the total space $\mathcal{T}$ of the global chart.

As an application of these ideas, we discussed the proof of the following, from [2]. A map $\gamma: S^1\to \mathrm{Ham}(X)$ defines by clutching a fibration $P_\gamma\to S^2$. The fact that γ lands in the Hamiltonian group ensures that P_γ admits a symplectic structure extending the obvious fibrewise symplectic structure. Lalonde, McDuff and Polterovich [6], building on work of Seidel, proved that the rational cohomology of P_γ splits additively.

Theorem (Abouzaid-McLean-S.) If $\gamma: S^1\to\mathrm{Ham}(X)$ is a Hamiltonian loop, then $H^*(P_\gamma;\mathbb{Z})\cong H^*(X;\mathbb{Z})\otimes H^*(S^2;\mathbb{Z})$ additively.

The additive splitting holds for any complex-oriented theory, for instance for complex K-theory, but simple examples show it doesn't hold for KO-theory. Working over $\mathbb{Q}$, [6] studied moduli spaces of J-holomorphic sections of $P_\gamma\to S^2$; the integral splitting uses the existence of Morava virtual classes for such spaces of sections. The proof in fact shows that a certain stable sweepout map

$$S^1\wedge X_+\to X_+$$

becomes null after smashing with complex cobordism MU. That vanishing is lifted from the vanishing of the corresponding sweepout over all $K_p(n)$-local theories, using a deep result of Hovey [5] from chromatic homotopy theory, namely that the map from the p-completion of BP to the product of its localisations at all $K_p(n)$'s actually splits. The circuitous route via the Morava theories is because the virtual class construction (and Cheng's theorem) does not directly apply over MU.

Quantum cohomology $QH^*(X)$ is an associative deformation of the usual cohomology ring $H^*(X)$ built out of moduli spaces $\mathcal{M}_{0,3}(X,J,\beta)$ of genus zero curves with three marked points. Work in progress uses the new virtual classes to build quantum ordinary K-theory and quantum Morava K-theory for a general closed symplectic manifold. As for the quantum K-theory of smooth algebraic varieties, the treatment of associativity is more involved, and the formal group of the underlying cohomology theory makes an appearance.

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Algebraic independence of topological Pontrjagin classes

Oscar Randal-Williams
(joint work with Søren Galatius)

The spaces $B\mathrm{O}(d)$ classifying d-dimensional vector bundles carry two well-known kinds of rational characteristic classes. For $d = 2n$ there is the Euler class $e \in H^{2n}(B\mathrm{O}(2n);\mathbb{Q}^{w_1})$, in cohomology with twisted coefficients corresponding to the determinant line: its square yields an untwisted cohomology class $e^2 \in H^{4n}(B\mathrm{O}(2n);\mathbb{Q})$. On the other hand, for all d there are the Pontrjagin classes $p_i \in H^{4i}(B\mathrm{O}(d);\mathbb{Q})$. By their construction these satisfy some elementary relations:

$$(\dagger)\qquad \begin{aligned} p_i &= 0 \text{ for } 2i > d,\\ p_n &= e^2 \text{ for } d = 2n. \end{aligned}$$

Furthermore, the Euler and Pontrjagin classes give a complete description of the cohomology of $B\mathrm{O}(d)$, and these relations are the only ones satisfied: we have

$$H^*(B\mathrm{O}(d);\mathbb{Q}) = \begin{cases} \mathbb{Q}[p_1,p_2,\ldots,p_n] & d = 2n+1\\ \mathbb{Q}[p_1,p_2,\ldots,p_n,e^2]/(p_n - e^2) & d = 2n. \end{cases}$$

There are analogous spaces $B\mathrm{Top}(d)$ classifying fibre bundles with fibre the euclidean space $\mathbb{R}^d$, and structure group $\mathrm{Top}(d)$, the group of homeomorphisms of $\mathbb{R}^d$ fixing the origin. Neglecting the fibrewise linear structure of a vector bundle gives maps $B\mathrm{O}(d) \to B\mathrm{Top}(d)$, and it follows from work of Sullivan and Kirby–Siebenmann that the map of stabilisations $B\mathrm{O} \to B\mathrm{Top}$ is a rational homotopy equivalence. We therefore have

$$H^*(BTop;\mathbb{Q}) \xrightarrow{\sim} H^*(B\mathrm{O};\mathbb{Q}) = \mathbb{Q}[p_1, p_2, p_3, \ldots].$$

This means that there are uniquely defined rational cohomology classes on $B\mathrm{Top}$ which restrict to the usual Pontrjagin classes on $B\mathrm{O}$: these are the topological Pontrjagin classes. These can be pulled back to $B\mathrm{Top}(d)$ for any d, and it is also easy to construct the (squared) Euler class on $B\mathrm{Top}(2n)$.

As their construction is rather indirect it is by no means clear whether the relations (†) should be expected to hold on $B\mathrm{Top}(d)$. Reis and Weiss [1, 2, 3] developed a refined strategy to show that these relations do hold, but in breakthrough recent work Weiss [4] has shown that in fact they do not! For example, his results imply that $p_{n+i} \neq 0 \in H^{4(n+i)}(B\mathrm{Top}(2n);\mathbb{Q})$ for fixed i and all large enough n.

In my talk I presented the following result.

Theorem A. *For all $2n \geq 6$ the map*

$$\mathbb{Q}[e^2, p_1, p_2, p_3, \ldots] \longrightarrow H^*(B\mathrm{Top}(2n);\mathbb{Q})$$

is injective.

It is easy to deduce an analogous statement for $B\mathrm{Top}(2n+1)$. What this means is that not only do the relations (†) not hold, but no universal relations hold among the Euler and Pontrjagin classes for euclidean bundles of dimension $d \geq 6$.

In the form I have presented it this result relies on the theorem of Kupers [5] that $B\mathrm{Top}(2n)$ has finitely-generated cohomology groups for $n \geq 6$ (which in turn relies on the ideas of Weiss' [4]). However, if one is happy to work with rationalised integral cohomology, rather than rational cohomology, then this ingredient can be avoided, and in fact the argument then goes through equally well in dimension $2n = 4$, giving:

Theorem B. *The map*

$$\mathbb{Q}[e^2, p_1, p_2, p_3, \ldots] \longrightarrow H^*(B\mathrm{Top}(4);\mathbb{Z}) \otimes \mathbb{Q}$$

is injective.

I spent most of the talk outlining the proof of these results. The general idea is as follows, in the case $2n \geq 6$. If the maps in question were not injective then there would be non-zero rational polynomial Ξ in the Euler and Pontrjagin classes (or better, Hirzebruch L-classes) which vanishes. Such a counterexample would already exist over $\mathbb{Z}[\frac{1}{S}]$ for some large S, because Ξ has finitely-many rational coefficients and each Hirzebruch L-class is defined after inverting finitely-many primes. One can then try to obtain a contradiction by constructing non-linear representations $\phi : \mathbb{Z}/p \to \mathrm{Top}(2n)$ for infinitely-many primes $p > S$ satisfying $\phi^*\Xi \neq 0 \in H^{4i}(B\mathbb{Z}/p;\mathbb{Z}[\frac{1}{S}]) = \mathbb{Z}/p$. This what we do. We produce such ϕ's by constructing fake $(2n-1)$-dimensional lens spaces using Wall realisation and the calculation of the surgery obstruction groups $L^s_{2n}(\mathbb{Z}[\mathbb{Z}/p])$, then taking the non-linear $\mathbb{Z}/p$-representations given by the open cone on their universal covers. The crucial step is then to determine the Hirzebruch L-classes of these non-linear representations, which makes use of localisation in equivariant cohomology, as well as the Family Signature Theorem [6].

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On δ-median spaces and groups

Indira Chatterji
(joint work with Cornelia Drutu)

To say something meaningful about groups, we often restrict ourselves to a class of groups. Here we will focus on groups acting geometrically (that is, properly discontinuously, by isometries and co-compactly) on a metric space with nice geometric properties. Our focus today are δ-median spaces, a class of spaces that includes δ-hyperbolic groups, as well as CAT(0) cubical complexes.

Intervals and hyperbolicity. A metric space (X, d) is called *geodesic* if for any $x, y \in X$ there is an isometry $c : [0, d] \to X$ with $c(0) = x$ and $c(d) = y$, called *geodesic*. On any metric space, for $\delta \geq 0$ and two points x, y, we define the δ*-interval* by

$$I_\delta(x, y) = \{t \in X \mid d(x, t) + d(t, y) \leq d(x, y) + \delta\}$$

Notice that, for $\delta = 0$, this is the set of points on a geodesic between x and y.

Definition 1. A space is called *hyperbolic* (also, *Gromov hyperbolic*, or δ*-hyperbolic*) if there is $\delta \geq 0$ such that, for any $x, y, z \in X$

$$I(x, y) \subseteq I_\delta(x, z) \cup I_\delta(z, y).$$

Equivalently a metric space is hyperbolic if, for any $\delta > 0$, there is $K \geq 0$ such that

$$d_H(I_\delta(x, y), [0, d(x, y)]) < K.$$

Here $[0, d]$ denotes the interval of length d in $\mathbf{R}$ and d_H the Hausdorff distance. Trees, classical hyperbolic spaces, rank-one Lie groups, Cayley graphs of fundamental groups of surfaces of genus greater than 2 are instances of hyperbolic metric spaces. We say that a group is *hyperbolic* if it admits a geometric action of a hyperbolic metric space. Equivalently, any of its Cayley graphs will be a hyperbolic metric space. Having a $\mathbf{Z}^2$ as a subgroup is an obstruction to hyperbolicity, and hyperbolicity is not preserved under taking direct products.

Knowing that a group is hyperbolic gives several useful properties, like finite presentation, many free subgroups, a nice boundary, a contractible Rips complex, so a finite $K(G,1)$, and the Baum-Connes conjecture, a conjectural way to compute the K-theory of the reduced C*-algebra of a group. However, we still don't know if any hyperbolic group is residually finite or acts geometrically on a CAT(0) space, namely a space with a metric notion of non-positive curvature.

Medians. We now turn to another condition on triples of points, that turns out quite different from hyperbolicity. Here again, intervals play an important role.

Definition 2. A metric space (X, d) is called *median* if, for any $x, y, z \in X$ then

$$I(x,y) \cap I(y,z) \cap I(z,x) = \{m\}.$$

Namely, given any 3 points, there is a single point lying simultaneously on 3 geodesics between those 3 points. Examples include real and simplicial trees, the plane with the L^1-metric, products of trees with the sum of the tree metrics and CAT(0) cubical complexes where cubes are endowed with the L^1-metric. In fact median graphs are graphs whose vertex set is a median space and according to Chepoi [4], those are exactly 1-skeletas of CAT(0) cubical complexes. In fact median spaces are to CAT(0) cubical complex what $\mathbf{R}$-trees are to simplicial trees.

Sageev's construction of CAT(0) cubical complexes were instrumental in the 2012 solution of the virtually Haken conjecture by Agol. An important piece was Bergeron-Wise cocompact action of the fundamental group of a hyperbolic 3-manifold on a finite dimensional CAT(0) cubical complex.

We showed in 2007 with Drutu and Haglund [2] that actions (or lack of) on a median space allows to characterize property (T) and the Haagerup property, a property that implies the Baum-Connes conjecture.

Approximate medians. In a recent paper with Cornelia Drutu [1], we look at a minimal notion including both hyperbolicity, and median spaces. According to a recent result of Petyt [7], this class of groups include mapping class groups of closed surfaces.

Definition 3. A metric space (X, d) is called δ*-median* if there is $\delta \geq 0$ such that for any $x, y, z \in X$

$$I_\delta(x,y) \cup I_\delta(y,z) \cup I_\delta(z,x) \sim_d \{m\}$$

where by $\sim_d$ we mean "non empty and at uniformly bounded distance to". A group is called δ-median if it acts geometrically on a δ-median space.

Groups that are δ-median include cocompact lattices in products of rank one Lie groups, as well as non-cocompact lattices in $SO(n,1)$, and lattices in $\widetilde{SL(2,\mathbf{R})}$. This notion is close to Bowditch's notion of coarse median. According to Niblo, Wright and Zhang [6], δ-median spaces are in particular coarse median, but it is not clear if any coarse median space admits a δ-median metric. Notice that just requiring non-empty for the triple intersection of those coarse intervals is very different from δ-median, and is a notion called δ*-tripodal*. For $\delta = 0$, one can show that the building associated to $Sp(4, \mathbf{Q}_p)$ admits a 0-tripodal metric, but definitely not median since it has property (T).

Walls. We are trying to understand how far are δ-median spaces and groups from being median. For instance, a uniform lattice in $Sp(n, 1)$ is δ-hyperbolic, hence δ-median, but because it has property (T), any action on a median space has a bounded orbit. Median spaces all come from and have a structure of spaces with walls, that we now explain.

Definition 4. A space with walls is a set X, with a set of partitions, described by $\mathcal{H} \subset \mathcal{P}(X)$ closed under taking complements and endowed with a measure μ satisfying that

$$\mu\{w(x|y)\} = \mu\{h \in \mathcal{H} | x \in h, y \in h^c\} < \infty$$

The pseudo-distance defined by $pd(x, y) = \mu\{w(x|y)\}$ induces a distance called wall-metric.

Wall metrics on graphs are called *cut metrics* by computer scientists and their embeddings in L^1-spaces allows certain fast computations on those graphs. In the 2007 joint work with Drutu and Haglund [2] we showed that any wall space isometrically embeds in a median space, and in a recent joint work [1] with Drutu, we show the following.

Theorem A. *Let X be a wall space. If the wall metric is δ-median, then the image of the embedding of X in a median space $M(X)$ is at finite Hausdorff distance from $M(X)$. Moreover, for $X = \mathbb{H}$ the real hyperbolic n-space, $M(\mathbb{H})$ is locally compact.*

This result is interesting because according to Caprace's appendix to a joint work with Iozzi and Fernos [3], uniform lattices cannot act on a finite dimensional CAT(0) cubical complex without a fix point, and this result has been extended by Fioravanti [5] to actions on finite dimensional median spaces. Hence, the class of cocompact median groups is different from the one of CAT(0) cubical groups but it is still unknown if the class of finite dimensional median groups is different from the one of finite dimensional CAT(0) cubical groups.

Open questions. The interest of δ-median spaces resides in trying to use both hyperbolicity and cubical techniques at the same time. Many questions for δ-median spaces and groups remain open, and we list here a few of interest.

(1) Does any action on a uniformly locally finite δ-median space translate into a proper action on an L^p-space?
(2) Does any uniformly locally finite δ-median space admit a strongly bolic metric?
(3) Are hierarchically hyperbolic spaces δ-median?
(4) Are δ-median spaces hierarchically hyperbolic?

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Floer Homotopy Theory

Mohammed Abouzaid

I explain joint work with I. Smith and M. Mclean on the construction of Kuranishi bordism groups, as well as joint work with A. Blumberg on a model for spectra built from flow categories.

Curve Graphs and Surface Homeomorphisms

Jonathan Bowden
(joint work with Sebastian Hensel, Richard Webb)

Curve Graphs. In the mid 70's Harvey [9] introduced the curve complex as a combinatorial model for Teichmüller space of a closed, oriented, connected surface that we denote by S. This complex has 0-simplices given by isotopy classes of essential, simple closed curves on S so that a collection of distinct vertices span a simplex if they have (simultaneously) disjoint representatives. This complex has a natural simplicial action by the mapping class group $MCG(S)$ of isotopy classes of orientation preserving diffeomorphisms of the surface.

The curve complex has taken on prominence in many parts of low dimensional topology as well as in the algebraic study of mapping class groups. For example the proof of Homological stability for mapping class groups of surfaces due to Harer [5]. The curve complex, or more specifically its 1-skeleton - the *curve graph* $\mathcal{C}(S)$ - took on a new significance sue to the following fundamental result, which endowed it with a geometry of negative curvature (in the coarse sense):

Theorem A (Masur-Minsky [6])**.** *The curve graph $\mathcal{C}(S)$ is δ-hyperbolic for all hyperbolic surfaces S.*

Furthermore, the action of the mapping class group on this graph is isometric. This action then has many important applications to the dynamics of surface homeomorphisms as well as to the geometry of the mapping class groups. One such example of this is the following

Theorem B (Bestvina-Fujiwara [1]). *The space of unbounded quasimorphisms*[1] *on the mapping class group $MCG(S)$ of a hyperbolic surface is infinite dimensional.*

This result is often phrased as a conference of the fact that the action of the mapping class group on the curve graph satisfies a weak from of properness called WPD.

Geometry of Diffeomorphism Groups. It is a classical result of Thurston that for any smooth, closed manifold M the identity component of the group of self-diffeomorphisms $\mathrm{Diff}_0(M)$ is simple. This group has several natural conjugate invariant *norms* in the sense of Burago-Ivanov-Polterovich [4]. These a defined by considering the word norm of any conjugate-invariant, symmetric generating set $\mathcal{S}$. Namely set

$$\|g\|_{\mathcal{S}} = \min\{N \ |g = s_1 \cdots s_N \ , \ s_i \in \mathcal{S}\}.$$

Important examples are $\mathcal{S}_{com}$ the set of commutators and $\mathcal{S}_{frag}$ the set of diffeomorphism supported on embedded (open) balls. This is then gives a natural way of endowing the group with a (metric) geometry. A fundamental question is then whether these norms contain large scale information, that is whether they are bounded or not. Somewhat surprisingly, it was shown by Burago-Ivanov-Polterovich [4] for $\dim(M) = 3$ and by Tsuboi [7], [8] for $\dim(M) \geq 5$ that these norms are always *bounded* (this also holds for S^1, which is an exercise). However, in the case of surfaces the situation is very different

Theorem C (B-Hensel-Webb [2]). *The commutator norm and the fragmentation norms are unbounded on* $\mathrm{Diff}_0(S)$ *if and only if* $S \neq S^2$.

In order to prove the above theorem it suffices to construct unbounded quasi-morphisms which can be done using the philosophy of Bestvina-Fujiwara.

A new curve graph. In order to construct quasi-morphisms we consider a "rigidified" or "discretized" version of the classical curve graph – the *fine curve graph* $\mathcal{C}^\dagger(S)$. The vertices of this graph are *actual* essential, simple, closed curves in S, as opposed to isotopy classes, and edges are given by disjoint pairs. The diffeomorphism group then acts isometrically on this graph and most importantly we have:

Theorem D (B-Hensel-Webb [2]). *The fine curve graph $\mathcal{C}^\dagger(S)$ is δ-hyperbolic for all hyperbolic surfaces S.*

[1] Recall that a *quasi-morphism* on a group G is a map $\varphi : G \to \mathbb{R}$ so that

$$\sup_{g,h \in G} |\varphi(gh) - (\varphi(g) + \varphi(h))| < \infty.$$

A similar result also holds for the torus T^2 and various surfaces with boundary. The key point now is that the identity component of the diffeomorphism group now acts on $\mathcal{C}^\dagger(S)$ and one can then use the methods of Bestina-Fujiwara to construct unbounded quasimorphisms.

Dynamics of Surface Homemorphisms. The fine curve graph provides a new tool to study the dynamics of surface homemorphisms as on the one hand the diffeomorphism group acts naturally on the underlying surface, but it also acts on the fine curve graph. It is important to understand the dictionary between these actions (in analogy to that for Mapping Class Groups) and as a first step we have the following result the characterises when elements act loxodromically in the case of the torus.

Theorem E (B-Hensel-Mann-Militon-Webb [3]). *Let $f \in \mathrm{Diff}_0(T)$. The following are equivalent*

(1) f acts loxodromically on $\mathcal{C}(T)$,
(2) The rotations set $\mathrm{rot}(f)$ has non-empty interior, and
(3) there is a finite, f-invariant set $P \subset T$ such that the restriction of f to $T - P$ represents a pseudo-Anosov mapping class.

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Homology cobordism and Heegaard Floer homology

Jennifer Hom
(joint work with Irving Dai, Kristen Hendricks, Matthew Stoffregen,
Linh Truong, and Ian Zemke)

Two closed, oriented 3-manifolds Y_0, Y_1 are *homology cobordant* if there exists a smooth, compact, oriented 4-manifold W such that $\partial W = -Y_0 \sqcup Y_1$ and the inclusions $\iota\colon Y_i \to W$ induce isomorphisms

$$\iota_*\colon H_*(Y_i;\mathbb{Z}) \to H_*(W;\mathbb{Z})$$

for $i = 0, 1$. The key point is that, on the level of homology, W looks like a product. The *3-dimensional homology cobordism group* $\Theta^3_{\mathbb{Z}}$ consists of integer homology 3-spheres modulo homology cobordism, under the operation induced by connected sum.

Fintushel-Stern used gauge theory to show that $\Theta^3_{\mathbb{Z}}$ is infinite, and Furuta and Fintushel-Stern improved this result to show that $\Theta^3_{\mathbb{Z}}$ contains a subgroup isomorphic to $\mathbb{Z}^\infty$. Frøyshov used Yang-Mills theory to define a surjective homomorphism $\Theta^3_{\mathbb{Z}} \to \mathbb{Z}$, showing that $\Theta^3_{\mathbb{Z}}$ has a direct summand isomorphic to $\mathbb{Z}$. (This is stronger than having a $\mathbb{Z}$ subgroup, since, for example, $\mathbb{Z}$ is a subgroup of $\mathbb{Q}$ but not a summand.) In joint work with Dai, Stoffregen, and Truong, we use Hendricks-Manolescu's involutive Heegaard Floer homology to prove the following:

Theorem A. *The homology cobordism group $\Theta^3_{\mathbb{Z}}$ contains a direct summand isomorphic to $\mathbb{Z}^\infty$.*

Fundamental questions about the structure of $\Theta^3_{\mathbb{Z}}$ remain open:

Question. *Does $\Theta^3_{\mathbb{Z}}$ contain any torsion? Modulo torsion, is $\Theta^3_{\mathbb{Z}}$ free abelian?*

In a different direction, it is natural to ask which types of manifolds can represent a given class $[Y] \in \Theta^3_{\mathbb{Z}}$. The first answers to this question were in the positive. Livingston showed that every class in $\Theta^3_{\mathbb{Z}}$ can be represented by an irreducible integer homology sphere and Myers improved this to show that every class has a hyperbolic representative.

In the negative direction, Frøyshov, F. Lin, and Stoffregen independently showed that there are classes in $\Theta^3_{\mathbb{Z}}$ that do not admit Seifert fibered representatives. Nozaki-Sato-Taniguchi improved this result to show that there are classes that do not admit a Seifert fibered representative nor a representative that is surgery on a knot in S^3. However, none of these results were sufficient to obstruct $\Theta^3_{\mathbb{Z}}$ from being generated by Seifert fibered spaces. In joint work with Hendricks, Stoffregen, and Zemke, we prove the following:

Theorem B. *The homology cobordism group $\Theta^3_{\mathbb{Z}}$ is not generated by Seifert fibered spaces. More specifically, let Θ_{SF} denote the subgroup generated by Seifert fibered spaces. The quotient $\Theta^3_{\mathbb{Z}}/\Theta_{SF}$ is infinitely generated.*

The proofs of both of these theorems rely on involutive Heegaard Floer homology, which associates an algebraic object called an iota-complex to a homology sphere Y; the chain homotopy type of the iota-complex is an invariant of the diffeomorphism type of Y. There exists a weaker notion of equivalence, called local equivalence, and the local equivalence type of an iota-complex is in fact invariant under homology cobordism. We characterize such complexes up to (almost) local equivalence, and use this characterization to

(1) define an infinite family of surjective linearly independent integer-valued homology cobordism invariants, proving Theorem A, and
(2) show that (the almost local equivalence type of) an iota-complex of any element in Θ_{SF} has a particular form, and there exist homology spheres whose iota-complexes are not of this form, proving Theorem B.

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Coarse-median preserving automorphisms

Elia Fioravanti

Let G be a finitely generated group. A fundamental problem in the study of automorphisms of G is the structure of their *fixed subgroups*.

In complete generality, it is not uncommon for these subgroups to be completely wild: they might be distorted in G, lack a finite classifying space, or not even be finitely generated.

Remarkably, in many important cases related to low-dimensional topology, the structure of $\mathrm{Fix}(\varphi)$ turns out to be much better-behaved.

Example.

(1) Let $G = \pi_1(S)$ for a closed, orientable surface S of genus $g \geq 2$. For every $\varphi \in \mathrm{Aut}(G)$, the subgroup $\mathrm{Fix}(\varphi)$ is either trivial, a maximal cyclic subgroup of G, or the fundamental group of an embedded subsurface of S. This can be easily deduced from the Nielsen–Thurston classification of homeomorphisms of S.
(2) Consider $G = F_n$, a free group of rank $n \geq 2$. For every $\varphi \in \mathrm{Aut}(G)$, the subgroup $\mathrm{Fix}(\varphi)$ is isomorphic to F_m for $m \leq n$. This was known as the P. Scott Conjecture in the 70s and 80s, before being solved by Bestvina and Handel in [2].
(3) More generally, let G be a negatively-curved group (a.k.a. a Gromov-hyperbolic group). W. Neumann showed that, for every $\varphi \in \mathrm{Aut}(G)$, the

subgroup $\mathrm{Fix}(\varphi)$ is finitely generated and undistorted in G (i.e. quasi-isometrically embedded) [5]. In particular, $\mathrm{Fix}(\varphi)$ is itself negatively-curved and admits a finite (rational) classifying space.

It should not be too surprising that, in the above cases, fixed subgroups are tame. Automorphisms of negatively-curved groups fit in a rich and complicated theory that originated from major breakthroughs of Rips and Sela (e.g. [6]) and now forms one of the most important branches of Geometric Group Theory. Unfortunately, the above results all rely — in one way or the other — on some form of Nielsen–Thurston theory, so they are not susceptible to generalisations.

In fact, already when G belongs to the important class of *non-positively curved* groups, almost nothing is known on the automorphisms of G in full generality.

Disappointingly, fixed subgroups can actually be completely wild in this context. This is evident already when G is a harmless-looking right-angled Artin group: there can be automorphisms $\varphi \in \mathrm{Aut}(G)$ for which $\mathrm{Fix}(\varphi)$ is one of the pathological groups constructed by Bestvina and Brady [1].

There is a way, however, of recovering some structure in automorphisms of non-positively curved groups, if we fix a *coarse median* structure on the group. This is a coarse notion of barycentre recently introduced by Bowditch [3].

For this we restrict to the case when G is *cocompactly cubulated*, i.e. acts properly and cocompactly on a simply connected, non-positively curved cube complex. Note that these groups include all known non-positively curved groups with interesting automorphisms, in particular all right-angled Artin/Coxeter groups.

Theorem ([4]). Let G be a cocompactly cubulated group and let $\varphi \in \mathrm{Aut}(G)$ be a coarse-median preserving automorphism. Then $\mathrm{Fix}(\varphi)$ is finitely generated, undistorted in G, and itself cocompactly cubulated. In particular, $\mathrm{Fix}(\varphi)$ admits a rational classifying space with finitely many cells.

A similar argument can be used to re-prove Neumann's result mentioned above. Indeed, all automorphisms of hyperbolic groups are coarse-median preserving.

Also all automorphisms of right-angled Coxeter groups are coarse-median preserving. Instead, for right-angled Artin groups, automorphisms turn out to be coarse-median preserving exactly when they lie in the well-known class of "untwisted" automorphisms.

Corollary ([4]). Consider one of the following two settings:

(1) G is a right-angled Coxeter group and $\varphi \in \mathrm{Aut}(G)$;
(2) G is a right-angled Artin group and $\varphi \in \mathrm{Aut}(G)$ is untwisted.

Then, for some $n \geq 1$, the subgroup $\mathrm{Fix}(\varphi^n)$ is quasi-convex in the standard word metric on G. In particular, it is separable (closed in the profinite topology on G).

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Prismatic cohomology of ring spectra

Jeremy Hahn
(joint work with Arpon Raksit, Dylan Wilson)

In homotopy theory, one is interested in a variety of multiplicative cohomology theories, such as

- Ordinary cohomology with coefficients in a commutative ring,
- Complex K-theory ku,
- Real K-theory ko,
- Complex bordism MU, and
- Framed bordism $\mathbb{S}$.

All of the above examples enjoy a great deal of functoriality–they are $\mathbb{E}_\infty$-ring spectra.

Definition 1. A multiplicative cohomology theory is said to be *even* if the cohomology of a point is concentrated in even degrees.

Of the above examples, ordinary cohomology, complex K-theory, and complex bordism are all even $\mathbb{E}_\infty$-ring spectra. On the other hand, real K-theory ko and framed bordism $\mathbb{S}$ are not even.

In this talk, following [1], we discussed a universal method of approximating $\mathbb{E}_\infty$-rings via even $\mathbb{E}_\infty$-rings.

Definition 2 (The even filtration). If R is an $\mathbb{E}_\infty$-ring and n is an integer, we define

$$\mathrm{fil}^n_{\mathrm{ev}}(R)$$

to be the limit, over all $\mathbb{E}_\infty$-ring maps $R \to B$ with B even, of $\tau_{\geq 2n}B$. As n varies, the $\mathrm{fil}^n_{\mathrm{ev}}(R)$ assemble to form a filtered $\mathbb{E}_\infty$-ring $\mathrm{fil}^\star_{\mathrm{ev}}(R)$.

If R is even, then $\mathrm{fil}^\star_{\mathrm{ev}}(R)$ is the double-speed Postnikov tower $\tau_{\geq 2\star}R$. In more complicated examples, the even filtration is computable by the following pair of definition and theorem:

Definition 3. A map $A \to B$ of $\mathbb{E}_\infty$-ring spectra is said to be *evenly free* if, for all $\mathbb{E}_\infty$-ring maps $A \to C$ with C even, the natural map

$$C \to C \otimes_A B$$

presents $\pi_*(C \otimes_A B)$ as a non-zero, free π_*C-module concentrated in even degrees.

Theorem A. *If $A \to B$ is an evenly free map of $\mathbb{E}_\infty$-ring spectra, then*

$$\mathrm{fil}^\star_{\mathrm{ev}}(A) = \lim_\Delta \mathrm{fil}^\star_{\mathrm{ev}}(B^{\otimes_A \bullet+1}),$$

where the limit is taken over a cosimplicial diagram in filtered $\mathbb{E}_\infty$-ring spectra.

Example 1. The unit map $\mathbb{S} \to \mathrm{MU}$ is evenly free. To see this, note that if C is any even $\mathbb{E}_\infty$-ring then

$$\pi_*(C \otimes_{\mathbb{S}} \mathrm{MU}) \cong C_*(\mathrm{BU}) \cong \pi_*(C)[b_1, b_2, \cdots],$$

by the Thom isomorphism and Atiyah–Hirzebruch spectral sequence. It follows that

$$\mathrm{fil}^\star_{\mathrm{ev}}(\mathbb{S}) = \lim_\Delta \left(\tau_{\geq 2*}(\mathrm{MU}^{\otimes_{\mathbb{S}} \bullet+1})\right),$$

which is the Adams–Novikov filtration of the sphere spectrum.

A similar argument shows that, if R is any $\mathbb{E}_\infty$-ring with $\pi_*(\mathrm{MU} \otimes_{\mathbb{S}} R)$ concentrated in even degrees, then the even filtration on R is the Adams–Novikov filtration on R. To give additional examples, we recall the definition of topological Hochschild homology:

Definition 4. If A is an $\mathbb{E}_\infty$-ring, then

$$\mathrm{THH}(A) = A \otimes_{A \otimes_{\mathbb{S}} A} A.$$

Example 2. THH(MU) has homotopy groups $\Lambda_{\pi_*(\mathrm{MU})}(\sigma b_1, \sigma b_2, \cdots)$, where the degree of σb_i is $2i+1$. As it turns out, the augmentation map

$$\mathrm{THH}(\mathrm{MU}) \to \mathrm{MU}$$

is evenly free. Indeed, if $\mathrm{THH}(\mathrm{MU}) \to C$ is an $\mathbb{E}_\infty$-ring map then one can calculate $\pi_*(C \otimes_{\mathrm{THH}(\mathrm{MU})} \mathrm{MU})$ by means of a Tor spectral sequence with E_2-page given by

$$\mathrm{Tor}_{\Lambda_{\pi_*(\mathrm{MU})}(\sigma b_1, \sigma b_2, \cdots)}(\pi_* C, \pi_* \mathrm{MU}).$$

If C is even, then the σb_i must act by zero on $\pi_* C$, for degree reasons. It follows that the Tor spectral sequence collapses to give a free $\pi_*(C)$-module on generators in even degrees.

The techniques illustrated by the above example can be used to compute the even filtration on THH quite generally. For example, we are able to prove the following result:

Theorem B. *Let R be a discrete commutative ring with bounded p-power torsion for all primes p. If the algebraic cotangent complex of R is concentrated in Tor amplitude* $[0, 1]$*, then*

$$\mathrm{fil}^\star_{\mathrm{ev}}\mathrm{THH}(R) \simeq \mathrm{fil}^\star_{\mathrm{mot}}\mathrm{THH}(R),$$

where $\mathrm{fil}^\star_{\mathrm{mot}}\mathrm{THH}(R)$ *is the* global motivic filtration *of Morin and Bhatt–Lurie, building on the local motivic filtration of Bhatt–Morrow–Scholze.*

In contrast to the Morin [2], Bhatt–Lurie [3], and Bhatt–Morrow–Scholze [4] definitions of motivic filtrations, the even filtration makes no reference to perfectoid or qrsp rings.

The key to proving Theorem B is the construction of an evenly free map $\mathrm{THH}(R) \to B$ such that B is even. We are able to make such a construction in the following more general set-up:

Definition 5. A connective $\mathbb{E}_\infty$-ring R is said to be *chromatically quasisyntomic* if $\mathrm{MU}_* R$ is concentrated in even degrees, has bounded p-power torsion for all primes p, and (when considered as an ungraded commutative ring) has algebraic cotangent complex with Tor amplitude in $[0, 1]$.

Theorem C. *If R is chromatically quasisyntomic, then there exists an evenly free map $\mathrm{THH}(R) \to B$ such that B is even. Furthermore, it is possible to make this map compatible with all of the cyclotomic structure present on $\mathrm{THH}(R)$ (e.g., one can choose B to have a circle action and the map to be an S^1-equivariant $\mathbb{E}_\infty$-ring map).*

In the situation of the above theorem, we define the (Nygaard-completed, Breuil–Kisin twisted) absolute prismatic cohomology of R to be the associated graded of the filtered $\mathbb{E}_\infty$-ring

$$\lim_\Delta \left(\tau_{\geq 2*}\left(\left(B^{\otimes_{\mathrm{THH}(A)}\bullet+1}\right)^{tS^1}\right)\right).$$

This agrees with previous definitions when R is discrete, and is independent of the choice of B and the choice of S^1-equivariant $\mathbb{E}_\infty$-ring map $\mathrm{THH}(R) \to B$.

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Nearby Legendrians and K-theory

Thomas Kragh

(joint work with Yasha Eliashberg)

Let Q be a smooth compact manifold possibly with boundary of dimension d. Let $\pi : T^*Q \to Q$ denote the cotangent bundle. On T^*Q (considered as a $2d$ dimensional smooth manifold) we have a canonical 1-form called the Liouville form

$$\lambda_z(v) = z(\pi_*(v)).$$

It has the property that for any 1-form $\beta \in \Omega^1(Q)$ corresponding to a section $Q \to T^*Q$ we have $\beta^*\lambda = \beta$. Thus closed (resp. exact) 1-forms corresponds to graphical submanifolds on which the restriction of λ is closed (resp. exact). A *Lagrangian* in T^*Q is *any* d-dimensional submanifolds on which λ is closed. An *exact* Lagrangian is one which has λ exact.

The Jet-1-bundle J^1Q is defined as the product $T^*Q\times\mathbb{R}$ of the cotangent bundle with $\mathbb{R}$. A section of this corresponds to both a 1 form and a function. It also has a canonical one form defined by

$$\alpha = \lambda - dz$$

where z is the coordinate in $\mathbb{R}$. Given a section (β, f) the pull back is identically 0 iff $df = \beta$. Hence, such sections are exact Lagrangians with a choice of primitive for the restriction of λ. A *Legendrian* is a d dimensional submanifold in J^1Q such α restricted to it vanishes.

The *front* of a Legendrian Λ is the image of Λ in the projection to $N \times \mathbb{R}$. One may reconstruct the Legendrian from its front projection. This is essentially because the "1-form" part is given by d of the function part.

The zero section in $J^1(Q)$ is of course a Legendrian, and we define $\mathcal{L}\mathrm{eg}(Q)$ as the space of Legendrians that equals the zero-section outside a compact set in the interior of Q. For the purpose of simplifying notation we restrict to the connected component of the zero section. So we assume that $\mathcal{L}\mathrm{eg}(Q)$ is connected.

The goal of the talk: To sketch how to define and detect "exotic" homotopy groups of the space $\mathcal{L}\mathrm{eg}(Q)$ using algebraic K-theory of spaces.

The word exotic here indicates that if one forgets the Legendrian condition and consider these as smooth embedded submanifolds the homotopy groups become trivial, *and* if one forgets the embedded condition and considers these as immersed Legendrians they also become trivial.

For a function $F : Q \times \mathbb{R}^{2k} \to \mathbb{R}$ the fiber-wise gradient $\nabla^f F : Q \times \mathbb{R}^{2k} \to \mathbb{R}^{2k}$ is defined by taking the usual gradient of $F_q : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ for each $q \in Q$. We say that F is *generic* if the equation $\{\nabla^f F = 0\}$ is regular - hence the subspace of fiber-wise critical points, denoted $\Sigma_F \subset Q \times \mathbb{R}^{2k}$, is a manifold of dimension d. For a generic function we get an immersion

$$\iota : \Sigma_F \to J^1Q \tag{1}$$

given by taking the horizontal derivative of F (derivative w.r. to Q directions) and the function value of F (as the $\mathbb{R}$ component). By construction the 1-form factor is the derivative of the function part - hence this is a Legendrian immersion. We say that this Legendrian is *generated* by F, and F is called a generating function for the Legendrian. Locally such always exists, and hence one could use this as the local definition of a Legendrian.

One may *stabilize* a generating function F as above to a function

$$s(F) : Q \times \mathbb{R}^{2k+2} \to \mathbb{R}$$

by $s(F)(q, z, x, y) = F(q, z) + x^2 - y^2$. As the fiber-wise critical points and values are unchanged we have identifications $\Sigma_F \cong \Sigma_{s(F)}$ compatible with the map to J^1Q.

We define a space $\mathcal{G}\mathrm{en}(Q)$ of stable (repeatedly using s above) generating functions quadratic at infinity that generates a Legendrian in $\mathcal{L}\mathrm{eg}(Q)$. We will not spell out the quadratic condition, but it is equivalent to being equal to a fixed quadratic form at infinity. We will need the following result (see [Fer97] for a survey and references)

Theorem(Chekanov, Chaperon/Théret): $\mathcal{G}\mathrm{en}(Q) \to \mathcal{L}\mathrm{eg}(Q)$ is a Serre fibration.

We define the h-cobordism space $\mathcal{H}(D^{d-1})$ to be the space of functions

$$f : D^{d-1} \times [-1,1] \to [-1,1]$$

such that

- f is the obvious projection outside a compact set in the interior,
- 0 is a regular value and
- the set $f^{-1}([-1,0])$ deformation retracts onto $D^{d-1} \times \{-1\}$.

The set in the last point is an actual h-cobordism rel S^{d-2}.

Now pick a path from the zero section in $\mathcal{L}\mathrm{eg}(Q)$ to a Legendrian Λ such that Λ has a fold (always possible). We get by the theorem above that Λ has a generating function in $\mathcal{G}\mathrm{en}(Q)$. Let $F : Q \times \mathbb{R}^{2k} \to \mathbb{R}$ be a choice of such. We may parametrize a piece of the fold by D^{d-1} (see figure 1 part a). In fact we may choose a map

$$D^{d-1} \times [-1,1] \to Q$$

together with a lift $D^{d-1} \times [-1,1] \times D^{2k} \to Q \times \mathbb{R}^{2k}$ such that in this local

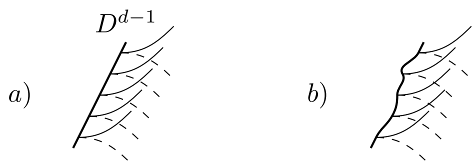

FIGURE 1. Front projection of a fold and the modified fold.

parametrization we have

$$F(d,t,z) = z_1^3 - tz_1 + q(z_2,\ldots,z_{2k})$$

where q is a non-degenerate quadratic form. This is a standard birth-death singularity at $t = 0$ stabilized by a quadratic form. We may now define a map $\mathcal{H}(D^{d-1}) \to \mathcal{G}\mathrm{en}(Q)$ by implanting a function $f \in \mathcal{H}(D^{d-1})$ into this and change it in the local parametrization to be

$$F^f(d,t,x) = z_1^3 - f(d,t)z_1 + q(z_2,\ldots,z_{2k})$$

As the birth-death is now at $f = 0$ (the outgoing boundary of the associated h-cobordism) this has the effect of moving the fold as illustrated in Figure 1 b).

For each $G \in \mathcal{G}\mathrm{en}(Q)$ we define $DG : Q \times \mathbb{R}^{2k} \times \mathbb{R}^{2k} \to \mathbb{R}$ by

$$DG(q,z_1,z_2) = G(q,z_1) - G(q,z_2).$$

The cobordism $(DG)^{-1}([\epsilon, \epsilon^{-1}])$ is in fact an h-cobordism for small $\epsilon > 0$, and by using so-called stabilizing and extensions (for h-cobordisms) we construct a map

$$\mathcal{G}\text{en}(Q) \to \mathcal{H}_\infty(*)$$

where the target is the stable h-cobordism space of a point essentially defined by stabilizing with quadratic forms like we did for generating functions. The composition

$$\mathcal{H}(D^{d-1}) \to \mathcal{G}\text{en}(Q) \to \mathcal{H}_\infty(*)$$

can be computed to be such a stabilization. Hence non-trivial in a stable range using results from algebraic K-theory of spaces in [Igu88], [Wal82] and [WJR13].

It is also relatively easy to check that the composition

$$F \to \mathcal{G}\text{en}(Q) \to \mathcal{H}_\infty(pt),$$

where F is the fiber of the Serre fibration in the theorem above, is null homotopic. Hence the map $\mathcal{H}(D^{d-1}) \to \mathcal{L}\text{eg}(G)$ is injective on homotopy groups.

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Non-triviality and non-invariance in string topology

Nathalie Wahl

This talk is based on joint work with Nancy Hingston and Florian Naef, as well extended conversations with Kai Cieliebak and Alexandru Oancea.

Let M be a closed oriented manifold, on which we have chosen a Riemannian metric, and let $LM = \text{Maps}(S^1, M)$ denote its free loop space. Morse theory on the energy functional tells us that the homology of the free loop space is build out of geodesics in M. Yet, as a graded vector space, it only depends on the homotopy type of M. This leads to the following question: is there a collection of operations on $H_*(LM)$ that would make $H_*(LM)$ equipped with these operations a finer invariant?

An answer proposed by Chas and Sullivan 20 years ago in [1, 8] is *string topology*. We will here consider in particular the following two operations: the string product

$$\wedge\colon H_p(LM) \otimes H_q(LM) \longrightarrow H_{p+q-n}(LM)$$

that "intersects the chains of basepoints in M and concatenates the loops" and the string coproduct

$$\vee\colon H_p(LM, M) \longrightarrow H_{p+1-n}(LM \times LM, M \times LM \cup LM \times M)$$

in homology relative to the constant loops, that "looks for a self-intersection of the form $\gamma(0) = \gamma(t)$ and cuts". (We can consider the coproduct as a non-relative operation by extending it to be zero on the constant loops, see [3].) These operations live in a large chain complex of operations known to act non-trivially at least on rational homology of LM for simply-connected manifolds via a Hochschild complex model, e.g. by combining [4, 7, 9, 10].

The string product was shown by Cohen-Klein-Sullivan to be a homotopy invariant:

Theorem A. [2] *A homotopy equivalence $f : M_1 \to M_2$ induces an isomorphim $Lf_* : H_*(LM_1) \to H_*(LM_2)$ of algebras over the string product.*

Even though the string coproduct is a priori a very similar operation, it was shown more recently by Naef that the coproduct in fact is not homotopy invariant:

Theorem B. [5] *Consider the homotopy equivalent lens spaces $\mathcal{L}_{1,7} \simeq \mathcal{L}_{2,7}$. The induced isomorphism $H_*(L\mathcal{L}_{1,7}) \cong H_*(L\mathcal{L}_{1,7})$ does not respect the coalgebra structure defined by the string coproduct.*

This result was extended in [6, Thm 2.11] to show that for any degree 1 homotopy equivalence $f : \mathcal{L}_{p,q_1} \to \mathcal{L}_{p,q_2}$ between 3-dimensional lens spaces, the map f is homotopic to a homeomorphism if and only if the map Lf_* respects the coproduct of degree 3 classes $H_3(L\mathcal{L}_{p,q_i}) \xrightarrow{\vee} H_1(L\mathcal{L}_{p,q_i} \times L\mathcal{L}_{p,q_i})$.

In this talk, we explain how the failure of homotopy invariance of the coproduct can be understood from considering the string coproduct as a relative version of the so-called trivial coproduct (see [3, Thm 2.13] or [6, Sec 2.6]), giving an explicit description of the (non-relative!) homotopy invariance of the trivial product. This leads to a formula for the failure of invariance, essentially of the form conjectured by Naef in [5]: Let $f : M_1 \to M_2$ be a degree 1 homotopy equivalence between closed manifolds, and let $\vee_1$ and $\vee_2$ denote the string coproduct of $H_*(LM_1)$ and $H_*(LM_2)$ and $\wedge_2$ the string product of $H_*(LM_2)$. Suppose that $\alpha \in H_*(LM_1)$ is a homology class. Then

$$(f \circ \vee_1 - \vee_2 \circ f)\,\alpha \;=\; E'_f \otimes E''_f \wedge_2 f_*\alpha + f_*\alpha \wedge_2 E'_f \otimes E''_f$$

for $E'_f \otimes E''_f = \vee_0(H[\Delta_1]) \in H_1(LM_2 \times LM_2)$ with $\vee_0$ the trivial string coproduct, H the above mentioned homotopy witnessing the invariance of $\vee_0$, and $[\Delta_1]$ the diagonal in M_1 considered as a class in constant loops.

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The stable cohomology of the moduli space of curves with level structures

Andrew Putman

Let $\mathcal{M}_{g,p}$ be the moduli space of smooth genus g algebraic curves over $\mathbb{C}$ equipped with p distinct ordered marked points. The (orbifold) fundamental group of $\mathcal{M}_{g,p}$ is the mapping class group $\mathrm{Mod}_{g,p}$ of an oriented genus g surface $\Sigma_{g,p}$ with p punctures, i.e., the group of isotopy classes of orientation-preserving diffeomorphisms of $\Sigma_{g,p}$ that fix each puncture. In fact, $\mathcal{M}_{g,p}$ is an (orbifold) classifying space for $\mathrm{Mod}_{g,p}$, so

$$\mathrm{H}_*(\mathcal{M}_{g,p};\mathbb{Q}) \cong \mathrm{H}_*(\mathrm{Mod}_{g,p};\mathbb{Q}).$$

There is a rich interplay between the topology of $\mathrm{Mod}_{g,p}$ and the algebraic geometry of $\mathcal{M}_{g,p}$. In this talk, we study the cohomology of certain finite covers of $\mathcal{M}_{g,p}$, or equivalently finite-index subgroups of $\mathrm{Mod}_{g,p}$.

1.1. **Analogy.** More generally, let $\Sigma_{g,p}^b$ be an oriented genus g surface with p punctures and b boundary components[1] and let $\mathrm{Mod}_{g,p}^b$ be its mapping class group, i.e., the group of isotopy classes of orientation-preserving diffeomorphisms of $\Sigma_{g,p}^b$ that fix each puncture and boundary component pointwise. We will omit p or b if it vanishes. There is a fruitful analogy between $\mathrm{Mod}_{g,p}^b$ and arithmetic groups like $\mathrm{SL}_n(\mathbb{Z})$. The following table lists some parallel structures and results:

[1] There are various ways to define $\mathcal{M}_{g,p}^b$ when $b \geq 1$, e.g., by identifying smooth algebraic curves over $\mathbb{C}$ with hyperbolic metrics on the associated surfaces and letting $\mathcal{M}_{g,p}^b$ be the moduli space of complete hyperbolic metrics on $\Sigma_{g,p}^b$ with geodesic boundary. However, these moduli spaces are not varieties.

	$\mathrm{SL}_n(\mathbb{Z})$	$\mathrm{Mod}_{g,p}^b$
natural action	vector in $\mathbb{Z}^n$	curve on $\Sigma_{g,p}^b$
associated space	locally symmetric space	$\mathcal{M}_{g,p}$
normal form	Jordan normal form	Thurston normal form (see [9])
Bieri–Eckmann duality	Borel–Serre [3]	Harer [14]
homological stability	Charney [6], Maazen [18]	Harer [13]
calculation of stable H_*	Borel [2]	Madsen–Weiss [19]

See [5] for more details.

1.2. **Stable cohomology.** Our main theorem provides another entry in this table. To motivate it, we first discuss homological stability and introduce the stable cohomology of the mapping class group, focusing for simplicity on surfaces without punctures. If $\Sigma_g^b \hookrightarrow \Sigma_{g'}^{b'}$ is an embedding, then there is an induced map $\mathrm{Mod}_g^b \to \mathrm{Mod}_{g'}^{b'}$ that extends mapping classes by the identity. Harer [13] proved that the resulting map

$$\mathrm{H}^k(\mathrm{Mod}_{g'}^{b'}) \to \mathrm{H}^k(\mathrm{Mod}_g^b)$$

is an isomorphism if $g \gg k$. The cohomology in this regime is known as the *stable cohomology* of the mapping class group. At least rationally, it was calculated by Madsen–Weiss [19], who showed that it is a polynomial algebra in classes $\kappa_n \in \mathrm{H}^{2n}$ called the Miller–Morita–Mumford classes. See [11, 15, 28] for expository accounts of this circle of ideas.[2]

1.3. **Borel stability.** Borel's stability theorem [2] concerns another kind of stability where instead of embedding the group into a larger one, we pass to a finite-index subgroup. Roughly speaking, it says that in a stable range, the rational cohomology of a lattice Γ in a semisimple Lie group is independent of the lattice Γ. In particular, it is unchanged when you replace Γ by a subgroup of finite index. For instance, for $\ell \geq 2$ define $\mathrm{SL}_n(\mathbb{Z}, \ell)$ be the level-ℓ subgroup of $\mathrm{SL}_n(\mathbb{Z})$, i.e., the kernel of the action of $\mathrm{SL}_n(\mathbb{Z})$ on $(\mathbb{Z}/\ell)^n$. We thus have a short exact sequence

$$1 \longrightarrow \mathrm{SL}_n(\mathbb{Z}, \ell) \longrightarrow \mathrm{SL}_n(\mathbb{Z}) \longrightarrow \mathrm{SL}_n(\mathbb{Z}/\ell) \longrightarrow 1. \tag{1}$$

By the congruence subgroup property [1, 20], for $n \geq 3$ every finite-index subgroup of $\mathrm{SL}_n(\mathbb{Z})$ contains $\mathrm{SL}_n(\mathbb{Z}, \ell)$ for some $\ell \geq 2$. Borel's theorem implies that the inclusion $\mathrm{SL}_n(\mathbb{Z}, \ell) \hookrightarrow \mathrm{SL}_n(\mathbb{Z})$ induces an isomorphism[3] $\mathrm{H}_k(\mathrm{SL}_n(\mathbb{Z}, \ell); \mathbb{Q}) \cong \mathrm{H}_k(\mathrm{SL}_n(\mathbb{Z}); \mathbb{Q})$ for $n \gg k$. See [7] for a direct proof of this fact.[4]

[2] Adding punctures does change the cohomology, but in a controlled way. See [17, Proposition 2.1].

[3] We have switched to homology since that is more natural for our subsequent work.

[4] The paper [7] does not state its main result in this way, but the above is implicit in it. See [26, Theorem C] for a explicit proof along the same lines of a more general result allowing twisted coefficients.

1.4. **Level-ℓ subgroup.** For $\ell \geq 2$, the *level-ℓ subgroup* of $\mathrm{Mod}_{g,p}^b$, denoted $\mathrm{Mod}_{g,p}^b(\ell)$, is the kernel of the action of $\mathrm{Mod}_{g,p}^b$ on $\mathrm{H}_1(\Sigma_{g,p}^b;\mathbb{Z}/\ell)$. This action preserves the algebraic intersection form, which is a symplectic form if $p+b \leq 1$. In that case, we have a short exact sequence

$$1 \longrightarrow \mathrm{Mod}_{g,p}^b(\ell) \longrightarrow \mathrm{Mod}_{g,p}^b \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}/\ell) \longrightarrow 1$$

that is analogous to (1). For $p+b \geq 2$, we get a similar exact sequence, but with a more complicated cokernel. For $b=0$ and $p \leq 1$, the associated finite cover of $\mathcal{M}_{g,p}$ is the moduli space $\mathcal{M}_{g,p}(\ell)$ of smooth genus-g curves over $\mathbb{C}$ with p marked points equipped with a full level-ℓ structure, i.e., a basis for the ℓ-torsion in their Jacobian.[5]

1.5. **Main theorem.** The following is our analogue for $\mathrm{Mod}_{g,p}^b$ of the Borel stability theorem.

Theorem 1. *Let $g,p,b \geq 0$ and $\ell \geq 2$. Then the map $\mathrm{H}_k(\mathrm{Mod}_{g,p}^b(\ell);\mathbb{Q}) \to \mathrm{H}_k(\mathrm{Mod}_{g,p}^b;\mathbb{Q})$ induced by the inclusion $\mathrm{Mod}_{g,p}^b(\ell) \to \mathrm{Mod}_{g,p}^b$ is an isomorphism if $g \gg k$.*

1.6. **Prior work.** Two special cases of Theorem 1 were already known. The case $k=1$ was proved by Hain [12] using work of Johnson [16] on H_1 of the Torelli subgroup of $\mathrm{Mod}(\Sigma_g)$. Little is known about the higher homology groups of the Torelli group, so this approach does not generalize. The case $k=2$ was proved[6] by the author in [23].

1.7. **Necessity of hypotheses.** The hypotheses in Theorem 1 are necessary:

- No result like Theorem 1 can hold for integral cohomology. Indeed, Perron [22], Sato [27], and Putman [24] identified exotic torsion elements of $\mathrm{H}_1(\mathrm{Mod}_{g,p}^b(\ell);\mathbb{Z})$ that do not come from $\mathrm{H}_1(\mathrm{Mod}_{g,p}^b;\mathbb{Z})$. Presumably similar torsion phenomena also occur for higher integral homology groups.
- Theorem 1's conclusion is false outside a stable range. Indeed, Church–Farb–Putman [8] and Morita–Sakasai–Suzuki [21] independently proved that $\mathrm{H}^{4g-5}(\mathrm{Mod}(\Sigma_g);\mathbb{Q}) = 0$, but Fullarton–Putman [10] proved that $\mathrm{H}^{4g-5}(\mathrm{Mod}(\Sigma_g,\ell);\mathbb{Q})$ is enormous.[7] The significance of $4g-5$ here is that it is the rational cohomological dimension of $\mathrm{Mod}(\Sigma_g)$; see [14].

[5] For $p \geq 2$, the cover of $\mathcal{M}_{g,p}$ associated to $\mathrm{Mod}_{g,p}(\ell)$ covers $\mathcal{M}_{g,p}(\ell)$. The subgroup corresponding to $\mathcal{M}_{g,p}(\ell)$ is the kernel $\mathrm{Mod}_{g,p}[\ell]$ of the action of $\mathrm{Mod}_{g,p}$ not on $\mathrm{H}_1(\Sigma_{g,p};\mathbb{Z}/\ell)$ but on $\mathrm{H}_1(\Sigma_g;\mathbb{Z}/\ell)$, which $\mathrm{Mod}_{g,p}$ acts on via the map $\mathrm{Mod}_{g,p} \to \mathrm{Mod}_g$ that fills in the p punctures. Our main theorem does imply a corresponding theorem for $\mathcal{M}_{g,p}(\ell)$.

[6] Actually, this paper only deals with the kernel $\mathrm{Mod}_{g,p}^b[\ell]$ of the action of $\mathrm{Mod}_{g,p}^b$ on $\mathrm{H}_1(\Sigma_g;\mathbb{Z}/\ell)$ coming from the map $\mathrm{Mod}_{g,p}^b \to \mathrm{Mod}_g$ that fills in the punctures, glues discs to the boundary components, and extends mapping classes over these discs by the identity. Thus even for $k=2$ our theorem is stronger than the one in the literature. A similar thing is true for the case $k=1$ proved by Hain [12], though his proof can be generalized to this more general setting using the generalization of Johnson's work in [25].

[7] Brendle–Broaddus–Putman [4] generalized [10] to $\mathrm{Mod}_{g,p}^b$; however, it is still unknown whether the cohomology of $\mathrm{Mod}_{g,p}^b$ vanishes in its virtual cohomological dimension in general.

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Topological and smooth 4-manifolds and knots with fundamental group 1 or $\mathbb{Z}$

Lisa Piccirillo

One of the major areas of study in 4-manifold topology is the distinction between the topological and smooth categories. In this abstract I will survey what is known about the classification of 4-manifolds and the classification of surfaces inside them under the drastic hypothesis that the manifold or surface complement has $\pi_1 \in \{1, \mathbb{Z}\}$[1]. In the topological category, classifications are known. In the smooth category, for closed manifolds or surfaces, the problem of determining whether the topological classification fails in all settings is the domain of two very difficult conjectures; the Poincare conjecture and the unknotting conjecture. However, for smooth manifolds with boundary, we know the topological classifications always fail. I will discuss briefly what is known in each of these categories for 4-manifolds and then for surfaces embedded in them.

I will begin with a discussion of 4-manifolds. In the topological category there are known classification theorems for 4-manifolds with $\pi_1 = 1$ or $\mathbb{Z}$ both with and without boundary [12, 11, 18, 4, 6]. I will not attempt to state these classification here, but I will comment on some of the invariants which show up in the classification. The richest algebraic information comes from the *equivariant intersection form.* For a 4-manifold X this is defined to be the intersection form on the integer homology of the universal cover. Notice that if X is simply connected this is just the intersection form on the second homology of X. In the setting where the manifold has boundary, the homeomorphism type of the boundary is an invariant of X, and there are two additional invariants coming from the relationship between the algebraic topology of X and the algebraic topology of ∂X.

In the setting of closed smooth 4-manifolds, classification theorems are well out of reach. We say that a smooth 4-manifold X is *exotic* if there exists another smooth 4-manifold X' which is homeomorphic but not diffeomorphic to X. Notice,

[1] There is some literature for other groups; in the topological category some classifications are known for finite cyclic groups [13] and solvable Baumslag-Solitar groups [14], see also [5]. In the smooth category, there are some exotica results for arbitrary fundamental group, eg. [17]. I restrict attention to 1 and $\mathbb{Z}$ because these are the groups with the most complete understanding across categories.

in particular, that if a 4-manifold is exotic then the invariants involved in the topological classification fail to determine the smooth structure. It is reasonable to conjecture that all smooth 4-manifolds are exotic; there are already many exotic smooth 4-manifolds known, the simplest of which is $\mathbb{C}P^2\#_2\overline{\mathbb{C}P^2}$ [2, 3]. The smooth 4-dimensional Poincare conjecture posits that S^4 is not exotic; this remains open, and it is also unknown whether, for example, $S^1 \times S^3$ is exotic, or whether there are exotic definite 4-manifolds.

In the setting of smooth 4-manifolds with boundary, much more is known. For example, it is known that there are exotic contractible 4-manifolds and exotic 4-manifolds homotopy equivalent to S^1 [1]. In fact, for any fixed equivariant intersection form it is known that there are exotic 4-manifolds with boundary with that intersection form [6]. Future study of exotica for 4-manifolds with boundary should attempt to fix the boundary 3-manifold[2], or to fix multiple invariants from the topological classification and demonstrate exotica with those invariants. Notice that, if one can give exotica with boundary S^3 and a given equivariant intersection form, one obtains closed exotica with that form.

Moving now to the study of surfaces in 4-manifolds, the literature takes a parallel structure. For this exposition, we will assume that all surfaces are properly embedded in a simply connected ambient 4-manifold X with boundary S^3, and that the surface complement has $\pi_1 = 1$ or $\mathbb{Z}$. Abusively, we will say the surface has $\pi_1 = 1$ or $\mathbb{Z}$. Notice that this mimics the setting of classical knot theory, where knots live in S^3 (which is simply connected) and the interesting knot invariant is π_1 of the complement. We will say that surfaces Σ and Σ' are equivalent if there exists a homeomorphism or diffeomorphism F of pairs $(X, \Sigma) \cong (X, \Sigma')$ such that F induces the trivial isometry on the intersection form of X.

In the topological category, the ambient manifold is topological and the surface is assumed to be locally flat. Here, classifications are known for closed surfaces with $\pi_1 = 1$ or $\mathbb{Z}$ [4, 6] and for surfaces with boundary and $\pi_1 = \mathbb{Z}$ [6]. There is no classification in the literature for surfaces with boundary and $\pi_1 = 1$, but this should follow readily using the techniques of [4, 6]. Again I will not state these classifications explicitly; invariants include the equivariant intersection form of the surface complement, and the genus of the surface, and in the case that the surface has boundary, the boundary knot in S^3. One particularly striking consequence of the machinery which goes into these theorems is that in S^4, for g not equal to 1 or 2, there is a unique locally flat surface with genus g and $\pi_1 = \mathbb{Z}$ [7, 10, 7].

Much less is known for smooth closed surfaces. Here the governing conjecture is the *unknotting conjecture* which posits that for any $g \in \mathbb{N}$ there is a unique smooth surface in S^4 with genus g and $\pi_1 = \mathbb{Z}$. We will say that a smooth surface Σ is *exotic* if there exists a smooth surface Σ' which is topologically but not smoothly equivalent to Σ. It is known that there exist 4-manifolds X which contain exotic closed surfaces [9] and that these surfaces can be taken to be nullhomologous [16] spheres [19]. It is open to find exotic nullhomologous surfaces with $\pi_1 = 1$ or $\mathbb{Z}$ in particularly simple X, such as $\mathbb{C}P^2$ or S^4.

[2]This study was begun in [8].

For surfaces with boundary, the geography of exotic surfaces is better known. For example, it is known that there are exotic disks in B^4 with $\pi_1 = \mathbb{Z}$ and exotic disks in $\mathbb{C}P^2 \# B^4$ with $\pi_1 = \mathbb{Z}$ [15]. In fact, for any 2-handlebody X with S^3 boundary, and any equivariant intersection form λ there is an exotic disk in X with $\pi_1 = \mathbb{Z}$ and such that the equivariant intersection form of the complement is isometric to λ [6]. Future study of exotic surfaces should focus on controlling the boundary knot, as well as the other invariants from the topological classifications. Notice that an exotic surface with unknot boundary can be capped to give an exotic closed surface.

A parallel story can be told for the study of mapping class groups of topological and smooth 4-manifolds with $\pi_1 = 1$ or $\mathbb{Z}$, although less is known. I will not tell it here, but I will mention that the story diverges from that of manifolds or surfaces in that presently there doesn't seem to be additional leverage on the study of exotic diffeomorphisms by examining 4-manifolds with boundary.

Future work in this area will undoubtedly focus on the problems about exotica with boundary mentioned here, as well as showing that closed manifolds such as $S^1 \times S^3$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ and closed surfaces in S^4 with $\pi_1 = \mathbb{Z}$. I mention these closed problems in particular because the gauge theoretic invariants which have already been used to produce (more complicated) exotica could also detect exotica in each of these settings. Thus, work that will push this field forward must be, in large part, constructive.

To conclude, I will remark that while these problems about detecting closed exotica in the simplest settings seem difficult, these are in fact just the first problems in a fledgling theory. For one, the study of these questions for more complicated fundamental groups should be systematized and explored. But even with the fundamental group restriction, we should be working towards classification theorems for smooth manifolds; demonstrating that the topological classifications fail is just the 0^{th} step towards this goal.

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On the Torelli Lie algebra

Alexander Kupers
(joint work with Oscar Randal-Williams)

We let $\Sigma_{g,1}$ be a surface of genus g with one boundary component and define its *mapping class group* $\Gamma_{g,1} = \pi_0 \operatorname{Diff}_\partial(\Sigma_{g,1})$ as the group of isotopy classes of diffeomorphisms of $\Sigma_{g,1}$ fixing the boundary pointwise. This group acts on the first homology group $H_1(\Sigma_{g,1};\mathbb{Z}) \cong \mathbb{Z}^{2g}$ of $\Sigma_{g,1}$ preserving the intersection form, and hence yields a homomorphism $\Gamma_{g,1} \to \mathrm{Sp}_{2g}(\mathbb{Z})$ whose kernel is the *Torelli group* $T_{g,1}$. Taking the limit of the Lie algebras of the Mal'cev completions of its lower central series quotients, we obtain the *Torelli Lie algebra* $\mathfrak{t}_{g,1}$, a pro-unipotent Lie algebra over $\mathbb{Q}$ with $\mathrm{Sp}_{2g}(\mathbb{Z})$-action.

By construction $\mathfrak{t}_{g,1}$ comes with a filtration, whose associated graded $\mathrm{Gr}^\bullet\, \mathfrak{t}_{g,1}$ is a Lie algebra with additional grading, which we refer to as *weight*. Its Lie algebra homology groups—for example computed by the Chevalley–Eilenberg complex—are consequently bigraded; we denote homological degree p and weight w by $H_p^{\mathrm{Lie}}(\mathrm{Gr}^\bullet\, \mathfrak{t}_{g,1})_w$. If these groups were to vanish when $p \neq w$, we would say $\mathrm{Gr}^\bullet\, \mathfrak{t}_{g,1}$ is *Koszul*, but this is not true. However, a slightly weaker statement *is* true: if we say that $\mathrm{Gr}^\bullet\, \mathfrak{t}_{g,1}$ is *Koszul in weights* $\leq W$ if $H_p^{\mathrm{Lie}}(\mathrm{Gr}^\bullet\, \mathfrak{t}_{g,1})_w$ vanishes when $p \neq w$ and $w \leq W$, then the first main result of [1] is as follows.

Theorem A (Kupers–Randal-Williams)**.** *The Lie algebra* $\mathrm{Gr}^\bullet\, \mathfrak{t}_{g,1}$ *is Koszul in weights* $\leq \frac{g}{3}$.

Remark. We prove the same result for several variants of $\mathfrak{t}_{g,1}$, and for the variant $\mathfrak{u}_g$ this result was independently obtained by Felder–Naef–Willwacher [2].

Theorem A, or rather its proof, also sheds light on the *geometric Johnson homomorphism.* This is the map of Lie algebras with additional grading

$$\tau_{g,1}\colon \mathrm{Gr}^\bullet\, \mathfrak{t}_{g,1} \longrightarrow \mathrm{Der}(\mathrm{Lie}(H_1(\Sigma_{g,1};\mathbb{Q})))$$

induced by the action of the Torelli group on the fundamental group of the surface $\Sigma_{g,1}$; here $\mathrm{Der}(-)$ denotes the Lie algebra with additional grading of (not necessarily degree-preserving) derivations, and $\mathrm{Lie}(-)$ denotes the free Lie algebra. The group $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts on the domain and target of $\tau_{g,1}$, and with respect to these actions, the map $\tau_{g,1}$ is $\mathrm{Sp}_{2g}(\mathbb{Z})$-equivariant. The second main result of [1] concerns its kernel.

Theorem B (Kupers–Randal-Williams)**.** *In weights $\leq \frac{g}{3}$, the kernel of $\tau_{g,1}$ is given by trivial $\mathrm{Sp}_{2g}(\mathbb{Z})$-representations that lie in the centre of $\mathrm{Gr}^\bullet\, \mathfrak{t}_{g,1}$.*

Remark. It is known that the kernel contains at least one copy of $\mathbb{Q}$, and this may in fact be the entire kernel.

The proof of both theorems makes use of results on moduli spaces of high-dimensional manifolds obtained in [3]. The proof of Theorem A may be summarised as follows. The Lie algebra $\mathrm{Gr}^\bullet\, \mathfrak{t}_{g,1}$ is quadratically presented for $g \geq 4$ by work of Hain, so we may instead prove that its quadratic dual commutative algebra is Koszul in the same range of weights. This commutative algebra may be described as the algebra of twisted Miller–Morita–Mumford classes on the moduli space of $\Sigma_{g,1}$-bundles with trivial homological monodromy, and may—up to scaling its grading by odd $n \geq 3$—be replicated in high dimensions on the moduli space of $W_{g,1} = D^{2n}\#(S^n \times S^n)^{\#g}$-bundles with trivial homological monodromy and a "Euler" tangential structure. An unstable rational Adams spectral sequence relates the homotopy groups of the latter moduli space to the commutative algebra homology groups—for example computed by the Harrison complex—of the algebra of twisted Miller–Morita–Mumford classes. These homotopy groups were studied independently through Goodwillie–Klein–Weiss embedding calculus in [3], and the results in that paper are strong enough to yield the desired vanishing *except for trivial* $\mathrm{Sp}_{2g}(\mathbb{Z})$*-representations.* However, by translating the entire problem to the setting of graph complexes via Schur–Weyl duality, and using a transfer argument in that setting, the desired vanishing for trivial $\mathrm{Sp}_{2g}(\mathbb{Z})$-representations may be deduced from that for the non-trivial $\mathrm{Sp}_{2g}(\mathbb{Z})$-representations. Theorem B is obtained by similar methods, upon identifying the high-dimensional analogue of the geometric Johnson homomorphism $\tau_{g,1}$.

Remark. As the above summary may suggest, Theorems A and B have applications to the study of high-dimensional manifolds and graph complexes. These can be found in [1].

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The Chromatic Nullstellensatz

Tomer Schlank
(joint work with Robert Burklund and Allen Yuan)

Stable homotopy theory has greatly benefited from insights offered by three fundamental perspectives. First, spectra should be considered as ∞-categorical analogues of abelian groups. Second, it is fruitful to generalize notions from algebra and algebraic geometry to the world of spectra. Third, these generalizations should avoid element-based formulae and be given in terms of categorical properties. In this way, for example, the Zariski spectrum of a ring is replaced by the notion of the Balmer spectrum, which presents chromatic homotopy theory as the analog for spectra of the primary decomposition for abelian groups. Some aspects of homotopy theory thus became akin to a game of Taboo, where classical notions from algebra are redefined without using the words, *element*, *equation* or *subset*. In this talk I present a joint work with R. Burklund and A. Yuan where we suggest a redefinition for algebraically closed fields. The idea is that algebraically closed fields are precisely those commutative rings that satisfy a form of Hilbert's Nullstellensatz.

Definition 1. Let $\mathcal{C}$ be a presentable ∞-category. We say that $\mathcal{C}$ is *Nullstellensatzian* if every compact and non-terminal object in $\mathcal{C}$ admits a map to the initial object of $\mathcal{C}$. Similarly, we say that an object $A \in \mathcal{C}$ is *Nullstellensatzian* if A is non-terminal and $\mathcal{C}_{A/-}$ is Nullstellensatzian. $\triangleleft$

Hilbert's Nullstellensatz is essentially the statement that an object in the category of commutative rings satisfies the Nullstellensatz if and only if it is an algebraically closed field.

Our first result is the classification of Nullstellensatzian $\mathbb{E}_\infty$-algebras in the monochromatic world. Through the connection between being Nullstellensatzian and being algebraically closed (in the sense of Galois theory) work of Baker and Richter [BR08] on Galois extensions suggests that the natural candidates for Nullstellensatzian $T(n)$-local $\mathbb{E}_\infty$-algebras are the Lubin-Tate theories attached to algebraically closed fields. Indeed, we show that these are exactly the Nullstellensatzian $T(n)$-local $\mathbb{E}_\infty$-algebras.

Theorem A (Chromatic Nullstellensatz)**.** *A $T(n)$-local $\mathbb{E}_\infty$-algebra R is Nullstellensatzian in* $\mathrm{CAlg}(\mathrm{Sp}_{T(n)})$ *if and only if $R \cong E(L)$, where $E(L)$ is the Lubin-Tate spectrum attached to an algebraically closed field L.*[1]

1.1. **The constructible spectrum.**
Given an arbitrary $T(n)$-local $\mathbb{E}_\infty$-algebra R, Theorem A supports the idea of considering $\mathbb{E}_\infty$ maps $R \to E(L)$ out to Lubin–Tate theories of algebraically closed fields as "geometric points of $\mathrm{Spec}(R)$". In classical algebra, the geometric points of A are usually organized into a topological space—the Zariski spectrum of A. The utility of the spectrum comes from the fact that often algebraic questions over a base ring A can be studied locally or even point-wise over $\mathrm{Spec}(A)$. Our understanding of the algebraically closed fields in $\mathrm{CAlg}(\mathrm{Sp}_{T(n)})$ allows us to make an analogous construction in the chromatic setting.

Definition 2. Let $R \in \mathrm{CAlg}(\mathrm{Sp}_{T(n)})$. A *geometric point* of R is an equivalence class of maps $f\colon R \to E(L)$ for L algebraically closed, under the equivalence relation identifying two maps $f_1\colon R \to E(L_1)$ and $f_2\colon R \to E(L_2)$ whenever $E(L_1) \otimes_R E(L_2) \neq 0$.[2] ◁

The set[3] of geometric points can be endowed with the so-called **constructible topology**, which gives it the structure of a compact Hausdorff topological space.

Theorem B. *There is a functor*

$$\mathrm{Spec}^{\mathrm{cons}}_{T(n)}\colon \mathrm{CAlg}(\mathrm{Sp}_{T(n)})^{\mathrm{op}} \to \mathrm{CHaus}$$

which sends $R \in \mathrm{CAlg}(\mathrm{Sp}_{T(n)})$ to the set of geometric points endowed with the topology in which a subset $U \subset \mathrm{Spec}^{\mathrm{cons}}_{T(n)}(R)$ is closed if and only if it is the image a map $\mathrm{Spec}^{\mathrm{cons}}_{T(n)}(S) \to \mathrm{Spec}^{\mathrm{cons}}_{T(n)}(R)$ induced by some map of algebras $R \to S$.

In the classical case, a not-often-mentioned property is that is that one can check whether an element $a \in A$ is nilpotent by checking whether it is nilpotent at every geometric point of A. In fact, it is this property which guarantees that the Zariski spectrum of A has enough points. The analogous result in the $T(n)$-local setting is the following theorem:

Definition 3. Let $R \in \mathrm{CAlg}(\mathrm{Sp}_{T(n)})$ and let $g\colon M \to N$ be a map of compact $T(n)$-local R-modules. We say that g is *nilpotent* if there exists some $k \gg 0$ such that $g^{\otimes k}\colon M^{\otimes k} \to N^{\otimes k}$ is null. A map $f\colon R \to S$ in $\mathrm{CAlg}(\mathrm{Sp}_{T(n)})$ *detects nilpotence* if a map $M \to N$ of compact $T(n)$-local R-modules is nilpotent if and only if the induced map $M \otimes_R S \to N \otimes_R S$ is nilpotent in $\mathrm{Mod}_S(\mathrm{Sp}_{T(n)})^\omega$. ◁

[1] Here and throughout the paper, by "$E(L)$ for an algebraically closed field L" we mean for height $n > 0$ that $\mathrm{char}\, L = p$ and $E(L)$ is as in [GH04, Lur18], and for height $n = 0$ that $\mathrm{char}\, L = 0$ and $E(L) \coloneqq L[u^{\pm 1}]$ where the generator u is placed in degree 2, that is, $E(L) \cong L \otimes KU$.

[2] Note that for a discrete ring A, taking equivalence classes of maps $A \to L$ to algebraically closed fields under the analogous equivalence relation gives rise to the set of points of $\mathrm{Spec}(A)$.

[3] Although it is not immediate, the collection of geometric points of R turns out to be a set.

Theorem C. *Let $f\colon R \to S$ be a map in $\mathrm{CAlg}(\mathrm{Sp}_{T(n)})$. Then f detects nilpotence if and only if the induced map $\mathrm{Spec}^{\mathrm{cons}}_{T(n)}(S) \to \mathrm{Spec}^{\mathrm{cons}}_{T(n)}(R)$ is surjective.*

Theorem C tells us that, in this theory, we have "enough points". In particular, since the map $R \to 0$ detects nilpotence only if $R = 0$, we deduce that any nonzero R has at least one geometric point. In other words:

Theorem D. *Let R be a non-zero $T(n)$-local $\mathbb{E}_\infty$-algebra. Then there exists some algebraically closed field L and a map of $\mathbb{E}_\infty$-algebras $R \to E(L)$.*

1.2. **Applications.**
Theorem D hascan be applied to the study of $T(n)$-local $\mathbb{E}_\infty$-algebras. As an immediate consequence we obtain an alternative proof for Hahn's celebrated result on the chromatic support of $\mathbb{E}_\infty$-algebras.

Theorem E (Hahn [Hah16])**.** *Let $R \in \mathrm{CAlg}(\mathrm{Sp})$. Then for every $n \geq 0$, we have that $R \otimes T(n) = 0$ implies $R \otimes T(n+1) = 0$.*

Indeed, since algebras over the zero algebra are zero, Theorem D allows us to reduce the statement to the case of $E(L)$, where it is a straightforward computation.

In view of Theorem E, it is natural to define the *height* of a nonzero $\mathbb{E}_\infty$-algebra $R \in \mathrm{CAlg}(\mathrm{Sp})$ by

$$\mathrm{height}(R) := \max\{n \geq -1 | T(n) \otimes R \neq 0\}.$$[4]

Based on computations at small heights, Ausoni and Rognes suggested a far-reaching conjectural organizing principle for the interaction between algebraic K-theory and chromatic height. This phenomena, known as *redshift*, can be summarized by the slogan "algebraic K-theory raises the chromatic height by one." Theorem D allows us to prove this conjecture for arbitrary $\mathbb{E}_\infty$-algebras. Note that if R is an $\mathbb{E}_\infty$-algebra, then $K(R)$ also admits the natural structure of an $\mathbb{E}_\infty$-algebra. We get:

Theorem F (Redshift for $\mathbb{E}_\infty$-algebras)**.** *Let $0 \neq R \in \mathrm{CAlg}(\mathrm{Sp})$ be such that $\mathrm{height}(R) \geq 0$. Then*

$$\mathrm{height}(K(R)) = \mathrm{height}(R) + 1.$$

The inequality $\mathrm{height}(K(R)) \leq \mathrm{height}(R) + 1$ has been recently proved in the groundbreaking papers [LMMT20, CMNN20], so we are reduced to proving that the height always increases. Once again, Theorem D allows us to reduce the claim to the case of $E(L)$, where it was proven by the third author in [Yua21].

Many of the best studied $\mathbb{E}_\infty$-algebras, including cobordism rings, occur as Thom spectra. As a consequence of Theorem D we find that $\mathbb{E}_\infty$-algebra maps from Thom spectra to algebraically closed Lubin–Tate theories, known as orientations, are particularly well-behaved.

[4]Here we set $T(-1) = \mathbb{S}$.

Theorem G. *Let L be an algebraically closed field and let*

$$f\colon X \to \mathrm{pic}(\mathrm{Mod}_{E(L)}(\mathrm{Sp}_{T(n)}))$$

be a map in $\mathrm{Sp}_{\geq 0}$ with $T(n)$-local Thom spectrum Mf. Then the following are equivalent:

(1) $Mf \neq 0$.
(2) There is map $Mf \to E(L)$ *in* $\mathrm{CAlg}_{E(L)}(\mathrm{Sp}_{T(n)})$.
(3) f *is null-homotopic.*
(4) $Mf \cong E(L)[X] \in \mathrm{CAlg}_{E(L)}(\mathrm{Sp}_{T(n)})$.

For a map $g\colon X \to \mathrm{pic}(\mathrm{Sp})$ in $\mathrm{Sp}_{\geq 0}$, it follows from Theorem G that there exists an $\mathbb{E}_\infty$-algebra map $Mg \to E(\overline{\mathbb{F}}_p)$ if and only if $K(n) \otimes Mg \neq 0$.

Theorem H. *Taking g to be the complex J-homomorphism* $\mathrm{ku} \to \mathrm{pic}(\mathrm{Sp})$, *we obtain an equivalence of spaces*

$$\mathrm{Map}_{\mathrm{CAlg}(\mathrm{Sp})}(\mathrm{MUP}, E(\overline{\mathbb{F}}_p)) \cong \mathrm{Map}_{\mathrm{Sp}_{\geq 0}}(\mathrm{ku}, \mathrm{gl}_1(E(\overline{\mathbb{F}}_p))).$$

In particular, there exists an $\mathbb{E}_\infty$-algebra map

$$\mathrm{MU} \to E(\overline{\mathbb{F}}_p).$$

The question of whether such $\mathbb{E}_\infty$ complex orientations of Lubin–Tate theories exist has a long history. In [And95], Ando gave a "norm-coherence" condition based on power operations for when a Lubin–Tate theory admits an $\mathbb{H}_\infty$ map from MU. Building on work by Ando in the case of the Honda formal group, Zhu [Zhu20] checked this condition for all Lubin–Tate theories. A general obstruction theory for constructing $\mathbb{E}_\infty$ complex orientations was described by Hopkins–Lawson [HL18], which recovered previous results of Walker and Möllers [Wal08, Möl10] at height 1. The more general case of MUP-orientations was demonstrated in a height 1 example by [HY20], and then proven for all Lubin–Tate theories of height $n \leq 2$ by Balderrama [Bal21].

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Hyperbolic models for CAT(0) spaces

Davide Spriano
(joint work with Harry Petyt and Abdul Zalloum)

This talk is based on joint work with Harry Petyt and Abdul Zalloum developed in [7]

Two of the most well-studied topics in geometric group theory are CAT(0) cube complexes and mapping class groups. This is in part because they both admit powerful combinatorial-like structures that encode interesting aspects of their geometry: hyperplanes for the former and curve graphs for the latter. In recent years, analogies between the two theories have become more and more apparent. In this talk, we bring CAT(0) spaces into the picture by developing versions of hyperplanes and curve graphs for them.

The main new notion introduced is that of *curtains*, which are CAT(0) analogues of hyperplanes.

Definition: Let X be a CAT(0) space. A *curtain* is $\pi_\alpha^{-1}(P)$, where α is a geodesic, π_α the closest-point projection, and P a subinterval of α of length one not containing the endpoints.

Given an arbitrary CAT(0) space X, we use curtains to define a new family of metrics d_L on X and write $X_L = (X, d_L)$. This construction is inspired by work of Genevois and Hagen on CAT(0) cube complexes [4, 3]. It will be seen from the results described below that these spaces share many fundamental properties with curve graphs. In the first place, we prove the following.

Theorem A: For any natural number L there is a constant δ_L such that for every CAT(0) space X, the space X_L is δ_L–hyperbolic, and $\mathrm{Isom}X \leq \mathrm{Isom}X_L$. Furthermore, geodesics of X descend to unparametrised quasigeodesics of X_L.

Both surfaces and CAT(0) spaces have automorphisms that can naturally be considered hyperbolic-like, namely pseudo-Anosov homeomorphisms and rank-one isometries. Pseudo-Anosovs are precisely those mapping classes that act loxodromically on the curve graph [6], and, in the cubical setting, rank-one isometries are those that skewer a pair of *separated* hyperplanes [1, 2]. For CAT(0) spaces we have the following.

Theorem B: Let g be a semisimple isometry of a proper CAT(0) space X. The following are equivalent.

- g is rank-one.
- g skewers a pair of separated curtains.
- g acts loxodromically on some X_L.

As a consequence, if G acts properly coboundedly on X and some X_L is unbounded, then Gromov's classification of actions on hyperbolic spaces yields a loxodromic isometry of X_L, which is rank-one by the above theorem.

Aside from hyperbolicity, one of the most important results about the curve graph is *Ivanov's theorem* [5], which states that every automorphism of the curve graph is induced by some mapping class. Recall that a CAT(0) space has the geodesic extension property if every geodesic segment appears in some biinfinite geodesic.

Theorem C: Let X be a proper CAT(0) space with the geodesic extension property. If any one of the following holds, then $\mathrm{Isom}X = \mathrm{Isom}X_L$ for all L.

- X admits a proper cocompact action by a group that is not virtually free, or
- X is a tree that does not embed in $\mathbf{R}$, or
- X is one-ended.

Right now, most of the geometric result relied on rank-one elements or unboundedness of the spaces X_L. It is therefore natural to ask whether it is possible to draw some conclusions for the case where the X_L are all bounded. We obtain the following dichotomy.

Theorem D: Let G be a group acting properly cocompactly on a CAT(0) space X. Exactly one of the following holds.

- Every X_L has diameter at most two, in which case G is wide.
- Some X_L is unbounded, in which case G has a rank-one element, and if G is not virtually cyclic then it is *acylindrically hyperbolic*.

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Non-isotopic Seifert surfaces

Maggie Miller
(joint work with Kyle Hayden, Seungwon Kim, JungHwan Park, and Isaac Sundberg)

Livingston [2] previously showed that any two connected, same-genus Seifert surfaces for an unlink become isotopic rel. boundary in B^4 once their interiors are pushed slightly into B^4. This is a common phenomenon; many Seifert surfaces (i.e. essentially all examples from the literature except for those of [3], which I explain in the talk) become isotopic once their interiors are pushed into B^4.

In the figure below, I illustrate two Seifert surfaces Σ_0, Σ_1 for a knot K. These surfaces come from recent joint work with Hayden, Kim, Park and Sundberg [1] and are adaptations of a construction of Lyon [3]. In this talk, I will show that Σ_0 and Σ_1 are not even topologically isotopic in B^4 by proving that the 2-fold branched covers of B^4 branched along Σ_0 or Σ_1 (respectively) are not homeomorphic (they are distinguished by intersection form on $H_2(\,-\,;\mathbb{Z})$).

While I will not prove it in this talk, I will also discuss our further work showing that Whitehead doubles of Σ_0 and Σ_1 (defined in the talk as certain genus-2 Seifert surfaces) are topologically but not smoothly isotopic rel. boundary in B^4. The surfaces are distinguished smoothly by showing they induce distinct maps on Khovanov homology.

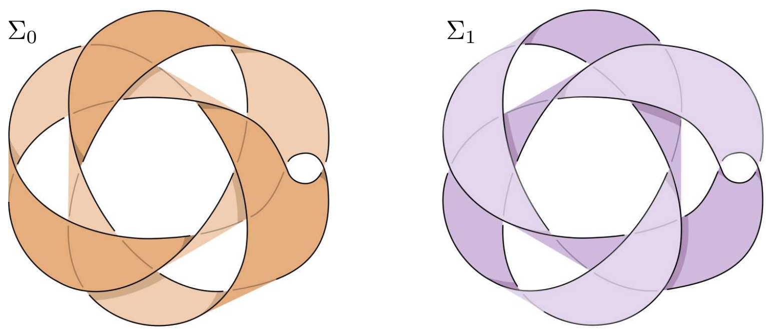

Figure 1. Two genus-1 Seifert surfaces Σ_0 (left) and Σ_1 (right) for the same knot K that are not isotopic even when their interiors are pushed into B^4. Image from [1].

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Locally flat embedded surfaces in 4-manifolds

Anthony Conway
(joint work with Lisa Piccirillo and Mark Powell)

In what follows, manifolds are assumed to be compact, connected and oriented. We work in the topological category: a 4-manifold will always refer to a topological 4-manifold and embeddings are assumed to be locally flat.

1. Brief history

The history of locally flat surfaces in 4-manifolds took off with the work of Freedman [Fre82]. Indeed, in his 1982 paper on the classification of closed simply-connected 4-manifolds, Freedman proved the following two striking results:

- If an embedded 2-sphere $\Sigma \subset S^4$ has $\pi_1(S^4 \setminus \Sigma) = \mathbb{Z}$, then it is unknotted.
- A knot K with Alexander polynomial 1 bounds a disc $D \subset D^4$ with $\pi_1(D^4 \setminus D) = \mathbb{Z}$.

Here a surface $\Sigma \subset S^4$ is *unknotted* if it bounds a solid handlebody in S^4 or, equivalently, if it is isotopic to the standard embedding of the genus g surface in S^4. Returning to the history of locally flat surfaces, here is a (non-exhaustive) list of noteworthy results:

- Boyer classified surfaces in simply-connected 4-manifolds whose complements are simply-connected [Boy93].
- Lee and Wilczynski obtained partial results towards the classification of surfaces whose complements have abelian fundamental group [LW90, LW93, LW97, LW00].
- Friedl and Teichner gave conditions on a knot $K \subset S^3$ for it to bound a disc $D \subset D^4$ whose complement has fundamental group $BS(1,2) = \langle a, b \mid aba^{-1} = b^2 \rangle$ [FT05].

Many authors have obtained criteria for pairs of surfaces in a 4-manifold to be topologically isotopic. We will not attempt a survey of the vast litterature on this topic but instead point the reader to [Sun15, FL19, FMN$^+$21, KPRT22] for a couple of recent interesting results on locally flat surfaces in 4-manifolds.

2. Results

In order to state our results, it is convenient to introduce some terminology. A locally flat surface Σ in a 4-manifold X is called a $\mathbb{Z}$-*surface* if $\pi_1(X \setminus \Sigma) = \mathbb{Z}$. Using this language, Freedman proved that $\mathbb{Z}$-spheres in S^4 are unknotted. Our first result, obtained in joint work with Mark Powell, is the higher genus analogue of Freedman's unknotting theorem.

Theorem A ([CP20])**.** *If $\Sigma \subset S^4$ is a $\mathbb{Z}$-surface of genus $g > 2$, then it is unknotted.*

Combining Freedman's result with Theorem A, genus g $\mathbb{Z}$-surfaces in S^4 are known to be unknotted for all $g \neq 1, 2$. The statement is still open for $g = 1, 2$

although, as we will see in Section 3, the question has now been reduced to an algebraic problem.

In fact, the methods from [CP21] apply in more general ambient 4-manifolds than S^4. The price to pay is the appearance of an additional invariant: given a $\mathbb{Z}$-surface $\Sigma \subset X$ in a simply-connected 4-manifold, the *equivariant intersection form* of the exterior $X_\Sigma := X \setminus \nu\Sigma$ refers to the non-degenerate Hermitian form

$$\begin{aligned} \lambda_\Sigma \colon H_2(X_\Sigma^\infty) \times H_2(X_\Sigma^\infty) &\to \mathbb{Z}[t^{\pm 1}] \\ (x,y) &\mapsto \sum_k (x \cdot t^k y) t^{-k}. \end{aligned}$$

Here X_Σ^∞ denotes the universal cover of X_Σ with deck transformation group $\mathbb{Z} = \langle t \rangle$ and $x \cdot t^k y$ refers to the algebraic intersection of x with $t^k y$. The equivariant intersection form admits a more algebraic description in terms of Poincaré duality but we will not pursue it here.

The classification of $\mathbb{Z}$-surfaces in simply-connnected 4-manifolds, obtained in collaboration with Lisa Piccirillo and Mark Powell, now reads as follows.

Theorem B ([CP20, CPP22]). *Let X be a closed simply-connected 4-manifold.*

1. *For $\mathbb{Z}$-surfaces $\Sigma_0, \Sigma_1 \subset X$, there is an equivalence $(X, \Sigma_0) \cong (X, \Sigma_1)$ if and only if there is an isometry $\lambda_{\Sigma_0} \cong \lambda_{\Sigma_1}$.*
2. *There are necessary and sufficient conditions for a non-degenerate Hermitian form $\lambda \colon \mathbb{Z}[t^{\pm 1}]^n \times \mathbb{Z}[t^{\pm 1}]^n \to \mathbb{Z}[t^{\pm 1}]$ to arise as the equivariant intersection form λ_Σ of the exterior of a $\mathbb{Z}$-surface $\Sigma \subset X$.*

We record a couple of remarks concerning this theorem and adjacent results.

- The first statement of Theorem B can be upgraded from equivalence to isotopy using an additional algebraic condition on the isometry $\lambda_{\Sigma_0} \cong \lambda_{\Sigma_1}$.
- We focused on closed surfaces in closed 4-manifolds but the articles [CP20, CPP22] also contain results for properly embedded surfaces in simply-connected 4-manifolds with S^3 boundary. For example, for every $g \neq 1, 2$, an Alexander polynomial one knot bounds a unique $\mathbb{Z}$-surface of genus g in D^4 up to isotopy rel. boundary [CP20].
- Concerning other groups, [CP21] builds on the work of Friedl-Teichner [FT05] to give a full classification of $BS(1,2)$-ribbon discs in D^4. The reason for the focus on the groups $\mathbb{Z}$ and $BS(1,2)$ is that these are currently the only known groups that are both good (in the sense of Freedman) and arise as fundamental groups of ribbon disc complements in the 4-ball. In theory all ribbon groups could be good and [Con22] describes the resulting classification of G-ribbon discs in the 4-ball under this optimistic hypothesis.

3. Proof sketches

We sketch how Theorem B implies Theorem A and then outline the proof of the existence statement of Theorem B. A word of warning: these sketches are scarce on details.

Proof sketch of Theorem A assuming item (1) *of Theorem B..* Use $\mathcal{U}_g \subset S^4$ to denote the unknotted surface of genus g. It is not difficult to verify that $\pi_1(S^4 \setminus \mathcal{U}_g) = \mathbb{Z}$. Given a $\mathbb{Z}$-surface $\Sigma \subset S^4$, there are three main steps to prove that Σ is ambiently isotopic to $\mathcal{U}_g$.

(1) Show that the unknot $\mathcal{U}_g \subset S^4$ satisfies $\lambda_{\mathcal{U}_g} \cong \mathcal{H}_2^{\oplus g}$, where $\mathcal{H}_2 = \begin{pmatrix} 0 & t-1 \\ t^{-1}-1 & 0 \end{pmatrix}$. This is an explicit calculation that can be performed using a handle decomposition for the surface exterior $X_{\mathcal{U}_g} = S^4 \setminus \mathcal{U}_g$.
(2) Apply a result of Baykur and Sunukjian [BS16] to prove that Σ and $\mathcal{U}_g$ become isotopic after adding a large enough number $n > 0$ of tubes to each surface. Prove that adding a tube to an embedded surface adds an $\mathcal{H}_2$-connected summand to the equivariant intersection form and deduce that $\lambda_\Sigma \oplus \mathcal{H}_2^{\oplus n} \cong \lambda_{\mathcal{U}_g} \oplus \mathcal{H}_2^{\oplus n}$.
(3) Use a cancellation result of Bass [Bas73] to "cancel off" the $\mathcal{H}_2^{\oplus n}$-summands leading to $\lambda_\Sigma \cong \lambda_{\mathcal{U}_g}$. Some work is needed to pass from our setting to one where Bass's result applies. This step is the most technical one and requires the hypothesis that $g \geq 3$.

Since we have now proved that $\lambda_\Sigma \cong \lambda_{\mathcal{U}_g}$, we can now apply item (1) of Theorem B to obtain an equivalence $(S^4, \Sigma) \cong (S^4, \mathcal{U}_g)$. The conclusion that Σ and $\mathcal{U}_g$ are ambiently isotopic follows from the fact that orientation-preserving self-homeomorphisms of S^4 are isotopic to the identity [Qui86]. □

Proof sketch of item (2) *of Theorem B.* Fix a simply-connected 4-manifold X and a non-degenerate Hermitian form $\lambda\colon \mathbb{Z}[t^{\pm 1}]^n \times \mathbb{Z}[t^{\pm 1}]^n \to \mathbb{Z}[t^{\pm 1}]$ that satisfies some additional algebraic conditions. We wish to build a $\mathbb{Z}$-surface $\Sigma \subset X$ with $\lambda_\Sigma \cong \lambda$.

The first step of the proof is to build a 4-manifold M with the algebraic topology of a $\mathbb{Z}$-surface exterior. We achieve this in three substeps.

(1) Add 2-handles to $(\Sigma_g \times S^1) \times [0,1]$ to obtain a cobordism $(W, \Sigma_g \times S^1, Y)$ between $\Sigma_g \times S^1$ and a 3-manifold Y with $H_1(Y^\infty) = 0$. Roughly speaking, this step is carried out by attaching the 2-handles according to the framing and linking data specified by the Hermitian form λ.
(2) Use surgery theory to prove that Y bounds a 4-manifold B that is homotopy equivalent to a circle. This should be thought of as a $\pi_1 = \mathbb{Z}$ analogue of Freedman's theorem that every integer homology 3-sphere bounds a contractible 4-manifold.
(3) The required 4-manifold M is now defined as $M := W \cup_Y B$. One verifies that M has the appropriate algebraic topology, namely fundamental group $\pi_1(M) = \mathbb{Z}$, equivariant intersection form $\lambda_M \cong \lambda$ and boundary $\partial M = \Sigma_g \times S^1$.

The second step of the proof is to show that M is a $\mathbb{Z}$-surface exterior. A calculation shows that $X' := M \cup_\partial (\Sigma_g \times D^2)$ is a closed simply-connected 4-manifold with $\mathbb{Z}$-intersection form $Q_{X'} \cong Q_X$ (here we use one of the conditions on the form λ). Freedman's classification of closed simply-connected 4-manifolds then implies that X' is homeomorphic to X. We have now obtained the required $\mathbb{Z}$-surface in X, namely $\Sigma_g \times \{0\} \hookrightarrow X' \cong X$. □

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Integral equivariant elliptic cohomology

Lennart Meier

(joint work with David Gepner)

1. Motivation

The most classical equivariant cohomology theory is *Borel equivariant cohomology*, sending a space X with a G-action to $H^*(EG \times_G X; \mathbb{Z})$. For us, G will always be a compact Lie group. For $X = \mathrm{pt}$, we obtain $H^*(BG; \mathbb{Z})$. As shown by Swan [6],

$H^*(BG;\mathbb{Z})$ is nonzero in infinitely many degrees for every non-trivial G and thus $H^*(BG;\mathbb{Z})$ will never be finitely generated as an abelian group.

Equivariant K-theory enjoys better finiteness properties. Assuming from now on X to be a finite G-CW-complex (e.g. any smooth compact G-manifold), $K^0_G(X)$ is defined as the Grothendieck group of G-equivariant complex vector bundles on X. Thus, $K^0_G(\mathrm{pt})$ is the representation ring $R(G)$ of G, which coincides as an abelian group with the free abelian group on the set of irreducible complex representations of G. Thus, $K^0_G(\mathrm{pt})$ is finitely generated for every finite G. But for $G = S^1$, we obtain $K^0_{S^1}(\mathrm{pt}) \cong \mathbb{Z}[t^{\pm 1}]$, being of infinite rank.

Question. *What are examples of equivariant cohomology theories having good finiteness properties for all compact Lie groups?*

Our answer will be *equivariant elliptic cohomology.* To motivate it, let us first reinterpret equivariant K-theory in algebro-geometric terms. For every G, we can consider the scheme $\operatorname{Spec} R(G)$. The $R(G)$-algebra structure on $K^0_G(X)$ corresponds to considering $K^0_G(X)$ as a sheaf of quasi-coherent algebras on $\operatorname{Spec} R(G)$. For $G = S^1$, we obtain $\operatorname{Spec} R(S^1) \cong \operatorname{Spec} \mathbb{Z}[t^{\pm 1}] = \mathbb{G}_m$, the multiplicative group – this represents the functor sending a commutative ring to its group of units. The group structure is actually induced by the multiplication map $S^1 \times S^1 \to S^1$. We could also paint a similar picture for Borel equivariant cohomology, where we see $\operatorname{Spec} H^*(BS^1) \cong \operatorname{Spec} \mathbb{Z}[x] \cong \mathbb{G}_a$, the additive group. The lack of finiteness in these examples corresponds to the fact that $\mathbb{G}_a$ and $\mathbb{G}_m$ are not *proper.* But there is a third family of one-dimensional proper group schemes, namely elliptic curves.

2. Equivariant Elliptic Cohomology

The aim of equivariant elliptic cohomology is repeat the above story for equivariant K-theory, replacing $\mathbb{G}_m$ by a fixed elliptic curve E. In particular, the S^1-equivariant theory takes values in quasi-coherent sheaves on E. The original motivations for constructions such theories came both from the theory of elliptic genera (Miller, Rosu) and from geometric representation theory. Motivated by the latter, Grojnowksi gave in [3] the first construction of equivariant elliptic cohomology for elliptic curves *over the complex numbers* and for *connected* compact Lie groups of equivariance. Much more recently, Berwick-Evans and Tripathy [1] developed a coherent theory for all compact Lie groups, but still over the complex numbers. In [4], Lurie gave a sketch how to obtain a theory without restricting to complex coefficients. Our work follows the same outline Lurie gives and is also heavily based on Lurie's foundational work on spectral algebraic geometry.

We will present the general form of our theory in a way that is inspired by the axiomatics from Ginzburg–Kapranov–Vasserot [2]. As they already mention, there are many advantages to work in the derived context, which means here in the context of spectral algebraic geometry. A *spectral scheme* is a topological space X together with a sheaf $\mathcal{O}_X$ of E_∞-ring spectra such that $(X, \pi_0\mathcal{O}_X)$ is a usual scheme, plus two more technical conditions. Like in usual algebraic geometry, we can talk about quasi-coherent sheaves on a spectral scheme. We refer to [5] for further details.

Using this language, let us explain the outline of our theory. Let S be a spectral base scheme such that $\mathcal{O}_S$ is even-periodic and let E be a spectral elliptic curve over S, plus a futher piece of datum, called an orientation. Our construction gives us:

(1) for every G, a spectral S-scheme X_G;
(2) for every G, a functor $\mathcal{E}ll_G\colon$ finite G-CW complexes$^{\mathrm{op}} \to \mathrm{QCoh}(X_G)$, sending G-homotopy equivalences to equivalences;
(3) for each group homomorphism $\varphi\colon G \to H$ an affine morphism
$$X_\phi\colon X_G \to X_h$$
such that $(X_\varphi)_*\mathcal{E}ll_G(Z) \simeq \mathcal{E}ll_H(G \times_H Z)$.

In the case of K-theory, X_G corresponds to $\operatorname{Spec} K_G(\mathrm{pt})$ and the functor $\mathcal{E}ll_G$ corresponds to viewing $K_G(X)$ as a $K_G(\mathrm{pt})$-module and hence a quasi-coherent sheaf on $\operatorname{Spec} K_G(\mathrm{pt})$.

In our case, the spectral schemes X_G are in general hard to describe explicitely. For G abelian, however, we have $X_G \simeq \mathrm{Hom}(\widehat{G}, E)$, where $\widehat{G}$ is the Pontryagin dual. As the Pontryagin dual of S^1 is $\mathbb{Z}$, this gives us in particular $X_{S^1} \simeq E$. In the case $G = U_n$ one can identify X_G with the Hilbert scheme of length n divisors on E. In general, we have the structural result that X_G is always proper over S, which gives strong finiteness results.

Theorem A (Gepner–M.). *Assume that $S = \mathrm{Spec} R$ is affine. Then for every finite G-CW-complex X, the global sections $\Gamma(\mathcal{E}ll_G(X))$ are a finite R-module.*

The relevance of these global sections is that the composite functor
$$\text{finite } G\text{-CW complexes}^{\mathrm{op}} \xrightarrow{\mathcal{E}ll_G} \mathrm{QCoh}(X) \xrightarrow{\Gamma} \mathrm{Mod}_R \xrightarrow{\pi_{-n}} \mathrm{AbelianGroups}$$
is an equivariant cohomology theory in the classical sense, having the finiteness properties we asked for.

By results of Lurie, the most canonical spectral elliptic curve with an orientation is a spectral refinement of the universal elliptic curve. While the base is not an (affine) spectral scheme, the results above apply, mutatis mutandis. In particular, this yields a genuine equivariant refinement of the spectrum TMF of topological modular forms such that all of its G-fixed points are finite TMF-modules.

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New results and conjectures on slice knots

Marco Marengon

(joint work with Thomas Hockenhull and Michael Willis, and partly also with Nathan M. Dunfield and Sherry Gong)

A *knot* is a closed, connected, 1-submanifold of S^3 up to isotopy. If we view S^3 as the boundary of the 4-ball B^4, we say that a knot $K \subset S^3$ is called *smoothly* (resp. *topologically*) *slice* if it bounds a smooth (resp. locally flat) properly embedded disc $D \subset B^4$. Note that smoothly slice implies topologically slice.

By a classical result of Fox-Milnor [1], if K is a topologically slice knot, then its Alexander polynomial $\Delta_K(t) \in \mathbb{Z}[t, t^{-1}]$ admits a factorisation as

$$\Delta_K(t) = f(t) \cdot f(t^{-1})$$

for some $f(t) \in \mathbb{Z}[t, t^{-1}]$. This in particular implies that the unsigned determinant $|\det K| := |\Delta_K(-1)|$ is an (odd) square. ($|\det K|$ is an odd number for any knot K.)

Knot Floer homology ($\widehat{\mathrm{HFK}}$) is a bigraded vector space over $\mathbb{F}_2$, and its graded Euler chacteristic recovers the Alexander polynomial [4, 6]. The naive categorification of Fox-Milnor that if K is (smoothly) slice then $\widehat{\mathrm{HFK}}(K) = V \otimes V^*$ for some (bigraded) vector space V is known to be false. In fact, even the weaker statement that $\operatorname{rk} \widehat{\mathrm{HFK}}(K)$ should be an odd square if K is slice is false, a counterexample being the Kinoshita-Terasaka knot K_{KT}, for which $\operatorname{rk} \widehat{\mathrm{HFK}}(K_{KT}) = 33$.

Nonetheless, in collaboration with Hockenhull and Willis, we prove that for a certain family of slice knots $\mathcal{F}$ the rank is an odd square.

Theorem A (Hockenhull-M.-Willis)**.** *If $K \in \mathcal{F}$, then $\operatorname{rk} \widehat{\mathrm{HFK}}(K)$ and $\operatorname{rk} \widetilde{\mathrm{Kh}}_{\mathbb{F}_2}(K)$ are square integers.*

Here $\widetilde{\mathrm{Kh}}_{\mathbb{F}_2}(K)$ denotes *reduced Khovanov homology* with $\mathbb{F}_2$ coefficients, another homology theory for knots that shares some formal properties with $\widehat{\mathrm{HFK}}$ [3].

The proof of Theorem A is based upon the existence of certain long exact sequence in $\widehat{\mathrm{HFK}}$ and $\widetilde{\mathrm{Kh}}_{\mathbb{F}_2}$, and on the fact that the above homologies can be given a structure of modules over $\mathbb{F}_2[X]/(X^2)$.

$\widehat{\mathrm{HFK}}$ has an analogue for 3-dimensional oriented closed manifolds called *Heegaard Floer homology*, and denoted $\widehat{\mathrm{HF}}$. The same technique used to prove Theorem A can be adapted to prove the following result.

Theorem B (Hockenhull-M.-Willis)**.** *Let K be an essential knot in $S^1 \times S^2$, and let $Y(K, f)$ denote the result of surgery on K with a longitudinal framing f.*

Then $\operatorname{rk} \widehat{\mathrm{HF}}(Y(K, f))$ is independent of f. In particular, if $Y(K, f)$ is an L-space for some f, then it is an L-space for all choices of longitudinal framing f.

Recall that an L-space is a 3-manifold with Heegaard Floer homology as simple as possible, and that conjecturally Y not being an L-space should be equivalent to

$\pi_1(Y)$ being left-orderable and also to the existence of a co-oriented taut foliation on Y.

Going back to slice knots, and keeping in mind that there is a slice knot (namely K_{KT}) with $\operatorname{rk}\widehat{\mathrm{HFK}} = 33$, we asked whether there is a statement weaker than $\operatorname{rk}\widehat{\mathrm{HFK}}$ being an odd square that holds for all slice knots. In joint work in progress with Dunfield, Gong, Hockenhull, and Willis, we formulate the following conjecture.

Conjecture 1 (Dunfield-Gong-Hockenhull-M.-Willis). Let $\mathcal{R}$ be $\mathbb{F}_p$ or $\mathbb{Z}$. Given a knot $K \subset S^3$, the following numbers are concordance invariants of K:

(1) $\operatorname{rk}\widehat{\mathrm{HFK}}(K) \pmod 8$;
(2) $\operatorname{rk}\widetilde{\mathrm{Kh}}_{\mathcal{R}}(K) \pmod 8$.

Thus, there are homomorphisms from the concordance group

$$\begin{aligned} \rho_{\widehat{\mathrm{HFK}}} \colon \mathcal{C} &\to (\mathbb{Z}/8\mathbb{Z})^* \\ K &\mapsto [\operatorname{rk}\widehat{\mathrm{HFK}}(K)]_8 \end{aligned} \qquad \begin{aligned} \rho_{\widetilde{\mathrm{Kh}}_{\mathcal{R}}} \colon \mathcal{C} &\to (\mathbb{Z}/8\mathbb{Z})^* \\ K &\mapsto [\operatorname{rk}\widetilde{\mathrm{Kh}}_{\mathcal{R}}(K)]_8 \end{aligned}$$

Conjecture 1 is equivalent to the following, seemingly weaker, conjecture.

Conjecture 2 (Dunfield-Gong-Hockenhull-M.-Willis). Let $\mathcal{R}$ be $\mathbb{F}_p$ or $\mathbb{Z}$. Given a ribbon knot $R \subset S^3$, we have:

(1) $\operatorname{rk}\widehat{\mathrm{HFK}}(R) \equiv 1 \pmod 8$;
(2) $\operatorname{rk}\widetilde{\mathrm{Kh}}_{\mathcal{R}}(R) \equiv 1 \pmod 8$.

Recall that a knot R is *ribbon* if it is obtained from the $(n+1)$-component unlink by performing n band surgeries. The minimum such n is called the *fusion number* of R.

Potential applications of these conjectures are detection of topologically slice but not smoothly slice knots, for example certain Whitehead doubles. Very interestingly, if the Khovanov side of the conjecture were true, this would give a new proof that the Piccirillo knot (hence the Conway knot) is not slice. The only known proof so far is via Rasmussen's s-invariant [5].

Building on an argument of Hom-Kang-Park [2], we prove Conjecture 2.(1) for ribbon knots with fusion number 1. Moreover, we tested that Conjecture 2.(1) holds on a list of 400000 ribbon knots compiled by Dunfield-Gong.

Regarding the Khovanov side of the conjecture, in the list of Dunfield-Gong there are two counterexamples (with $\mathbb{Z}$ coefficients) to Conjecture 2.(2). They both satisfy the weaker condition that $\operatorname{rk}\widetilde{\mathrm{Kh}}_{\mathbb{Z}} \equiv 1 \pmod 4$. Thus, we propose to weaken Conjectures 2.(2) and 1.(2) from $(\mathrm{mod}\, 8)$ to $(\mathrm{mod}\, 4)$. The weaker conjectures would still suffice for the potential applications we mentioned above.

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Equivariant dual Steenrod algebras

Michael A. Hill
(joint work with Michael J. Hopkins)

We describe an approach to computing the Steenrod algebra that is applicable to computing equivariant versions for cyclic 2-groups. We begin with a non-equivariant story, then indicate how it changes with the additional data of a group action.

The basic approach is simple: we have a pushout square of commutative ring spectra

$$\begin{array}{ccc} H\mathbb{F}_2 \wedge MU & \longrightarrow & H\mathbb{F}_2 \wedge H\mathbb{F}_2 \\ \downarrow & & \downarrow \\ H\mathbb{F}_2 & \longrightarrow & H\mathbb{F}_2 \underset{MU}{\wedge} H\mathbb{F}_2. \end{array}$$

We have great computational control over three of these pieces, and we can use this to describe the fourth. The bottom left corner gives just the homology of a point, which non-equivariantly is easy to understand and equivariantly we may just black-box as a given. The upper left corner can be computed via the Thom isomorphism, since $H\mathbb{F}_2$ is complex oriented:

$$H\mathbb{F}_2 \wedge MU \simeq H\mathbb{F}_2 \wedge BU_+,$$

and the homotopy is polynomial on classes in even degrees.

The story becomes less classical at the bottom right. Here, we can use the presentation of $H\mathbb{Z}$ as an MU-module given by the (non-equivariant form of the) Reduction Theorem of [5]. Choosing algebra generators of $\pi_* MU$, the Lazard ring, we write down a map of associative algebras

$$A := \bigwedge_{i=1}^{\infty} \mathbb{S}^0[S^{2i}] \to MU,$$

where here the associative ring spectrum $\mathbb{S}^0[S^{2i}]$ is the free associative ring spectrum on the space S^{2i}. The Reduction theorem says that we have an equivalence (of MU-module spectra)

$$H\mathbb{Z} \simeq MU \underset{A}{\wedge} \mathbb{S}^0,$$

where we give $\mathbb{S}^0$ an A-module structure via the augmentation map $A \to \mathbb{S}^0$. This gives an identification as spectra

$$H\mathbb{F}_2 \underset{MU}{\wedge} H\mathbb{Z} \simeq H\mathbb{F}_2 \wedge \left(\prod_{i=1}^{\infty} S^{2i+1}\right)_+ .$$

Alternatively, the relative Thom isomorphism as in work of Beardsley [2] using work of Basu–Sagave–Schlichtkrull [1] gives an equivalence

$$H\mathbb{F}_2 \underset{MU}{\wedge} H\mathbb{Z} \simeq H\mathbb{F}_2 \wedge BBU_+ \simeq H\mathbb{F}_2 \wedge SU_+.$$

The final piece is a Greenlees–Serre style spectral sequence for augmented algebras [3]. Since everything is flat over the homology of a point here, the spectral sequence has the form

$$E_2 = \pi_*\big(H\mathbb{F}_2 \underset{MU}{\wedge} H\mathbb{F}_2\big) \underset{\mathbb{F}_2}{\otimes} \pi_*\big(H\mathbb{F}_2 \wedge MU\big) \Rightarrow \pi_*\big(H\mathbb{F}_2 \wedge H\mathbb{F}_2\big).$$

The differentials here are simply given by analyzing the Hurewicz image of elements in the homotopy of MU, and extensions can be resolved by direct computations in basic spaces.

Framed this way, the computation goes through quite similarly equivariantly!

Replacing MU with the Fujii–Landweber spectrum $MU_{\mathbb{R}}$ of Real bordism gives the C_2-equivariant case, reproducing the computation of the C_2-equivariant dual Steenrod algebra originally due to Hu–Kriz [6]. As has become standard with these computations, the even spheres S^{2i} are replaced by regular representation sphere $S^{i\rho_2}$, where ρ_2 is the regular representation of C_2, but otherwise, the additive story goes through essentially without change. There is additionally an equivariant lift of the Greenlees–Serre spectral sequence which works the same way as the non-equivariant one.

Replacing $MU_{\mathbb{R}}$ with the norm to a finite cyclic 2-group C_{2^n} then can be used to study the C_{2^n}-equivariant case. Here, very little was known. If we assume known the homology of a point, then the top left and bottom right corners can be computed using essentially the same techniques. The interesting piece of data is the more initial computation

$$H\underline{\mathbb{Z}} \underset{N_{C_2}^{C_{2^n}} MU_{\mathbb{R}}}{\wedge} H\underline{\mathbb{Z}} \simeq H\underline{\mathbb{Z}} \wedge \left(\prod_{i=1}^{\infty} Map^{C_2}\big(G, S^{i\rho_2+1}\big)\right)_+ ,$$

where here $Map^{C_2}(G,-)$ is the functor of coinduction. These computations are still fairly straightforward, using the RO-graded algebra developed in [4]. Unfortunately, the Greenlees–Serre spectral sequence is much more complicated, and the computations to date suggest that the resulting algebra is not flat as a module over the homology of point!

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K-theory of Z/p^k and prismatic cohomology

Thomas Nikolaus

We will explain recent results about the computation of algebraic K-theory groups of the ring $\mathbb{Z}/p^k$ for $k > 1$ (and more generally $\mathcal{O}_K/\pi^k$ for K a p-adic number field).

The basis for these results is the newly developed prismatic cohomology of Bhatt–Scholze. The main part of the talk focuses on defining prisms and prismatic cohomology. A key insight is that prismatic cohomology can be defined relative to arbitrary δ-rings and not just prisms. This in particular gives additional functoriality needed for our computations.

Finally we will also explain a new universal property of prismatic cohomology which exhibits $\Delta_{R/H}$ as an initial object and thereby gives a better way of thinking about prismatic cohomology.

Stability of concordance embeddings

Manuel Krannich
(joint work with Thomas Goodwillie and Alexander Kupers)

Concordance embeddings and diffeomorphisms. Fix a smooth d-manifold M and a compact submanifold $P \subset M$ meeting ∂M transversely. Writing $I := [0,1]$, a *concordance embedding* of P into M is a smooth embedding

$$e\colon P \times I \longhookrightarrow M \times I$$

that satisfies $e^{-1}(M \times \{i\}) = P \times \{i\}$ for $i = 0, 1$ and agrees with the inclusion $P \times I \subset M \times I$ in a neighbourhood of $P \times \{0\} \cup (P \cap \partial M) \times I$. The space of such embeddings, equipped with the smooth topology, is denoted $\mathrm{CE}(P, M)$. In the case $P = M$, every concordance embedding is in fact a *concordance diffeomorphism*, that is a diffeomorphism of $M \times I$ that is the identity on a neighbourhood of $M \times \{0\} \cup \partial M \times I$. One writes $\mathrm{C}(M)$ for the space of concordance diffeomorphisms. $\mathrm{C}(M)$ and $\mathrm{CE}(P, M)$ are closely related: a concordance diffeomorphism of M yields by restriction to $P \times I \subset M \times I$ a concordance embedding of P into M, and on the level of spaces, this observation leads to a fibre sequence of the form

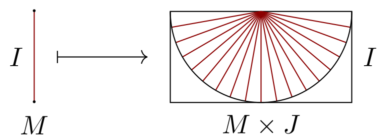
$$\mathrm{C}(M\backslash\nu(P)) \longrightarrow \mathrm{C}(M) \longrightarrow \mathrm{CE}(P,M) \tag{1}$$

where $\nu(P) \subset M$ is an open tubular neighbourhood of P.

Stabilisation. Concordance embeddings can be stabilised: there is a map

$$\sigma\colon \mathrm{CE}(P,M) \longrightarrow \mathrm{CE}(P\times J, M\times J)$$

by taking products with $J := [-1,1]$ and bending the result appropriately to make it satisfy the conditions for concordance embeddings, schematically like this:



It was Igusa [6] who, building Hatcher's work [5], showed that this stabilisation map is at least about $d/3$-connected—one of the key ingredients in studying manifolds and their diffeomorphism groups via surgery theory and pseudoisotopy theory. He phrased his result for concordance diffeomorphisms (i.e. the case $P = M$), but the version for general concordance embeddings follows from this, using (1).

The stability theorem. One consequence of the work with T. Goodwillie and A. Kupers that the talk was about is that, under a certain assumption on $P \subset M$, the stabilisation map is significantly more connected than the known $d/3$-bound. This "certain assumption" is a requirement on the *handle dimension* of the inclusion $P \subset M$, which is the minimal number p so that P can be built from a closed collar on $P \cap \partial M$ by attaching handles of index $\leq p$.

Theorem A (Goodwillie–Krannich–Kupers)**.** *If the handle dimension p of $P \cap \partial M \subset P$ satisfies $p \leq d-3$, then the stabilisation map*

$$\sigma\colon \mathrm{CE}(P,M) \longrightarrow \mathrm{CE}(P\times J, M\times J)$$

is $(2d-p-5)$-connected.

Remark. The case $P = *$ was previously known from work of G. Meng [8].

An application. One use case of Theorem A is the following: via the fibre sequence (1) for various choices of submanifolds $P \subset M$, this result puts one in the position to transfer information on the stabilisation map for concordance diffeomorphisms of a specific manifold M (for example lower or upper connectivity bounds) to other manifolds. For instance, together with O. Randal-Williams, I computed as part of [7] the rationalised relative homotopy groups $\pi_*(\mathrm{C}(M\times J), \mathrm{C}(M)) \otimes \mathbb{Q}$ for high-dimensional closed discs $M = D^d$ in a range of degrees beyond that in which these groups vanish, and Theorem A allows for an extension of this computation from discs to any high-dimensional simply-connected spin manifold M [3, Corollary C].

The multirelative stability theorem. Theorem A is a special case of our main theorem, which is a more general "multirelative" version. Interestingly, the proof of the more general version involves an induction that would fail if one tried to only prove the special case stated as Theorem A. Said differently, the more general version is not only more general, but also necessary (at least for our proof).

In addition to the submanifold $P \subset M$, the statement of the multirelative version involves compact submanifolds $Q_1, \ldots, Q_r \subset M$ that are pairwise disjoint as well as disjoint from P. Writing $M_S := M \backslash \cup_{i \notin S} Q_i$ for subsets $S \subset \underline{r}$ of $\underline{r} := \{1, \ldots, r\}$, there are inclusions $\mathrm{CE}(P, M_S) \subset \mathrm{CE}(P, M_{S'})$ whenever $S \subset S'$. This enhances the space $\mathrm{CE}(P, M)$ to an *r-cube*—a space-valued functor on the poset of subsets of $\underline{r}$. This functor $\underline{r} \supset S \mapsto \mathrm{CE}(P, M_S)$ is denoted by $\mathrm{CE}(P, M_\bullet)$. Note that the value at the empty set recovers $\mathrm{CE}(P, M)$. Defined suitably, the stabilisation map extends to a *map of r-cubes* (meaning, a natural transformation)

$$\sigma\colon \mathrm{CE}(P, M_\bullet) \longrightarrow \mathrm{CE}(P \times J, (M \times J)_\bullet)$$

whose target is the r-cube involving the submanifolds $Q_i \times J \subset M \times J$.

Our multirelative stability theorem (which specialises to Theorem A by setting $r = 0$) is an estimate on the connectivity of this map of r-cubes in terms of the handle dimensions p and q_i of the inclusions $\partial M \cap P \subset P$ and $\partial M \cap Q_i \subset Q_i$.

Theorem B (Goodwillie–Krannich–Kupers). *If the handle dimensions satisfy $p \leq d-3$ and $q_i \leq d-3$ for all i, then the stabilisation map of r-cubes*

$$\sigma\colon \mathrm{CE}(P, M_\bullet) \longrightarrow \mathrm{CE}(P \times J, (M \times J)_\bullet)$$

is $(2d - p - 5 + \sum_{i=1}^{r}(d - q_i - 2))$-connected.

Here are the relevant definitions: an r-cube $X_\bullet$ is k-cartesian if the natural map $X_\emptyset \to \mathrm{holim}_{\emptyset \neq S \subset \underline{r}} X_S$ is k-connected in the usual sense, and a map of r-cubes $X_\bullet \to Y_\bullet$ is k-connected if a certain $(r+1)$-cube is k-cartesian, namely the $(r+1)$-cube that maps $S \subset \underline{r+1}$ to X_S if $S \subset \underline{r}$ and to $Y_{S \backslash \{r+1\}}$ otherwise.

Analyticity and calculus. Theorem B is in line with previous multirelative connectivity results in geometric topology. Let me mention the two that are the closest to Theorem B (incidentally both used in our proof). The statement involves the quantity $\Sigma := \sum_{i=1}^{r}(d - q_i - 2)$ and the space $\mathrm{E}(P, M)$ of ordinary embeddings $P \hookrightarrow M$ that agree with the inclusion in a neighbourhood of $P \cap \partial M$.

(a) If $p, q_i \leq d-3$, then the r-cube $\mathrm{CE}(P, M_\bullet)$ is $(d - p - 2 + \Sigma)$-cartesian, by [1].
(b) If $p, q_i \leq d-3$, then the r-cube $\mathrm{E}(P, M_\bullet)$ is $(1 - p + \Sigma)$-cartesian, by [2].

These two results as well as Theorem B may be viewed as analyticity results in the sense of Goodwillie–Weiss' *manifold calculus* [4, 9] for functors on a suitable poset category of compact submanifolds of M, namely the functors sending $P \subset M$ to $\mathrm{E}(P, M)$, $\mathrm{CE}(P, M)$, or $\mathrm{hofib}(\mathrm{CE}(P, M) \to \mathrm{CE}(P \times I, M \times I))$ respectively. There is also an intriguing connection to the approach to studying diffeomorphism groups by means of Weiss' *orthogonal calculus* [10], waiting to be explored.

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