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What is Pattern?

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Pattern is ubiquitous and seems totally familiar. Yet if we ask what it is, we find a bewildering collection of answers. Here we suggest that there is a common thread, and it revolves around dynamics.

1 Introduction

Mathematics has been called the science of patterns, and information has been described as ultimately being pattern. There are weather patterns, patterns of behavior, swarming patterns like the one shown in Figure 1(a), and thousands more. The brain is described as the most fantastic pattern recognizer and there is the vast field of pattern recognition, so important to modern artificial intelligence and deep learning.

All of this seems perfectly familiar, but it does bring up the rather natural question of what exactly it is that is being recognized. Pattern is obviously important, but as a "thing" it is elusive. In fact – and this may be the first clue – it is perhaps not a "thing" at all. Pattern as a concept is hardly separable from the processes of being recognized or being recognizable, and recognition is process. It is inherently dynamical in nature.

Figure 1(b) shows an example of a pattern that emerges mathematically. Representing each of the *odd* numbers in Pascal's triangle of binomial coefficients by a black square leads to the famous Sierpiński triangle fractal.

If mathematics is described as the science of patterns, perhaps it is possible to reverse the roles and let mathematics offer some deeper insight into what pattern is really all about. We think that it can, and indeed there is a venerable area of mathematics that is well equipped to do so.

2 Difference and change

Underlying any process of recognition there is the concept of *difference.* This goes beyond the simple fact that we are forever conceiving of differences and dwelling upon them. Difference is the foundation upon which our senses operate, and how we come to experience the world. For example, the sense of sound is based on differences in air pressure experienced in the changing shape of the tympanic membrane, its transference to the ossicles in the middle ear, and from there to the hair cells that generate electric signals for further processing by the brain. And so it is with all of our other senses. They operate on the basis of difference. The fact that our entire empirical knowledge of the world is based on difference surely points to something fundamental about pattern.

Difference is inherently *dynamical* in nature, because it is ultimately connected with *change.* It is not really possible to talk about difference if there is no aspect of change. If there is a difference, then something has to be different and so implicitly there is change. Reciprocally, how can we speak of change if there is no difference? So at the heart of any pattern recognition, this fundamental pair – difference and change – appears. Beyond this, there is an implicit assumption of "context". Difference and change can only carry some "meaning" if there is a context in which they are taking place. Telephone answering systems, which are limited to a tiny contextual framework and which completely fail outside of it, provide a familiar example.

3 From the simplest system to dynamics

The simplest system that offers a context and within it the possibility for change is the two-element set

$$\mathcal{A} = \{0, 1\}$$

along with the simple operation T of interchange $0 \overset{T}{\longleftrightarrow} 1$. This is the familiar on/off, yes/no, yin/yang pattern. It may be simplistic in the extreme, but still, it is the basis of all our modern computers, our digital cameras, and our communications systems. It is the simplest pattern system.

There is a shift in view here. Pattern is understood less as a thing and more as a feature that arises within the context of a system. In this sense, it connects with classical Chinese ideas that emphasize change over pure existence. In terms of modern mathematical ideas, it brings us to the subject of *dynamical systems.*

The basic components of a dynamical system are simple: a non-empty set X, usually referred to as the *state space*, and a set S of transformations that permute the elements of X. The elements of X are called *states* and its transformations, that map states to states, are referred to as *operators.* In *continuous dynamics*, the operators are often based on time, parameterized by the real numbers. Each

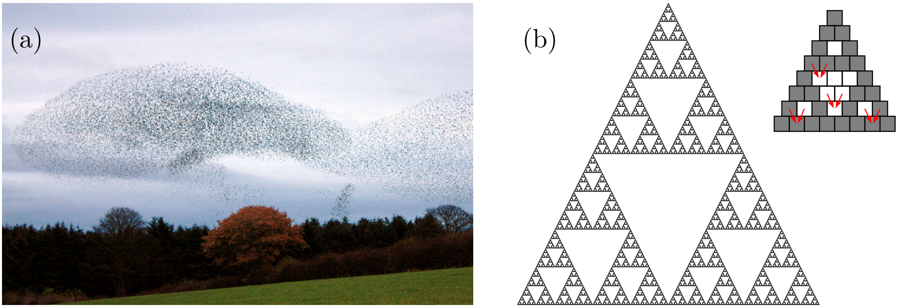


Figure 1: (a) Flocking (murmuration) of starlings. Photo: © Adam. (b) The Sierpiński triangle emerges from the construction indicated on the right, which corresponds to Pascal's triangle with black squares for odd numbers and white for even.

value t represents the effect of the continuous dynamical change due to the advance of t units of time. In our digital age, discrete [1] dynamics has also come to prominence.

Discrete dynamics is often based on a single operator of change, s, which can be iterated indefinitely to produce the full dynamics. Think of the internal clock of a computer, whose stepwise increments drive the changes in its registers. We denote the action of s on a state x by $s \triangleright x$, and its repeated action by $s^2 \triangleright x$, $s^3 \triangleright x$, and so on. Further divisions can be made along topological/measure-theoretical lines. All are relevant. In this short preview of [7], which is joint work with Deng Ming-Dao, we illustrate how effectively dynamical systems can formalize our intuitive ideas of pattern. In what follows, the approach is measure-theoretic in nature.

4 Shift spaces

Perhaps the most fundamental discrete pattern system is the *finite-shift system*. This is the natural extension of the tiny two-element 0/1 system, where we now go to long strings of 0's and 1's. We are familiar with memory sticks, CDs and DVDs, and the streaming of data in a communications channel. These use long strings of binary bits and rely on the patterning that is embedded or encoded into finite successions of bits. The mathematical idealization of this allows the

[1] Discrete can be seen as a natural counterpart to continuous, similar to stepweise versus flowlike.

strings to continue indefinitely: the state space is taken to be

$$X = \{0,1\}^{\mathbb{N}},$$

consisting of *infinite* strings $x = x_1x_2x_3x_4\ldots$, where each x_i is either 0 or 1, and the dynamics is supplied by the *left shift* s,

$$s \triangleright x = x_2x_3x_4\ldots.$$

The left shift simply transforms one infinite string into another by dropping the first symbol. Think of it as reading the first symbol off a long tape on which the symbols are written. This models a system without memory – once read, the symbol cannot be recovered. We call it the 0/1-*shift system*. This provides a suitable state space in which many discrete forms of patterning can be represented.[2]

5 Information

In the late 1940s, Claude Shannon, an engineer at Bell Laboratories in the USA, took up the problem of optimal transmission of English text down a noisy binary communications channel – in other words, transmitting binary bits representing ordinary words through a communication system that makes random mistakes [8]. This was not just about sending binary bits as fast as possible, but actually about the optimal transmission of *information*, given that this information was in the form of words in English, written in the usual Roman alphabet. There are two sources of randomness for this problem. The obvious one is that it explicitly includes the possibilities of random errors in transmission. The other is that this is not about transmitting just one particular text, but, at least from the engineer's point of view, any randomly chosen piece of English text.

This raises an important question, and one that remains relevant today – how can we define and quantify the slippery term "information"? What interests the engineer is not the semantic content of language. What meanings people want to communicate is not the problem. It has more to do with *distinguishing*, and the intrinsic potential of the patterning of a particular language to do this.

The basic idea is already familiar: The simple pattern system $\mathcal{A} = \{0,1\}$ from above is capable of holding one (binary) *bit* of information. The enlarged system $\mathcal{A}\times\mathcal{A}$, consisting of the four pairs $00, 01, 10, 11$, amounts to two bits (of

[2] There is a standard variation of this where one takes bi-infinite strings (extending endlessly both to the left and to the right), which allows the process of change (shifting) to be undone. In either model, the set $\{0,1\}$ can be replaced by any finite set of symbols.

information). And so it goes, with three bits, four bits, and so on. The number of bits is the logarithm (base 2) of the number of states of the system.

This is a start, but it is not yet enough. If we flip a coin, there is an intrinsic uncertainty about the outcome: heads or tails? If we know in advance that it is weighted so that it comes up heads nine times out of ten, the uncertainty is reduced. We know where to put our bets! Shannon's idea is that information is the opposite to uncertainty, specifically minus the entropy. The relevance of information is directly related to the uncertainty of the situation in which it arises. The unit of measurement is the binary bit.

The entropy of a system depends on the possible states and their probabilities of appearance. For a state space with states $A_1, A_2, \ldots, A_n$ and positive probabilities $p_1, p_2, \ldots, p_n$ (with $p_1 + p_2 + \cdots + p_n = 1$), the *entropy* is defined to be

$$H = -\sum_{j=1}^{n} p_j \log_2(p_j)\,.$$

Shannon defined his measure of *information* by precisely this formula. Information and entropy were to be seen as the two sides of a single conceptual idea. There are good arguments to show that this definition [3] is really the only one possible [6].

Up to this point, we have not yet included dynamics. As a system evolves, it passes from state to state and it is the transition probabilities that become the relevant feature. Compare the letter transition "th" with "ht". The first occurs far more frequently in standard English text. The complexity increases, and following the evolving increase in information, we can ask for the *average information gain per step*. This is the *dynamical information* (or dynamical entropy). For precise details, see [11].

So, at this point, our view of a pattern system can be described as consisting of a state space, a set of dynamical operators that act on this state space, and a probability distribution (or measure) $\boldsymbol{p}$ on the state space. This probability measure can be thought of either as an emerging outcome of the dynamics or, more importantly, as a vital constituent of the system that shapes the way the dynamics proceeds, making some events common and others rare. The latter is the reason this measure is often called the *law* of the process.

6 Events and measures

The 0/1-shift space has a natural partition into two parts, $X = [0] \cup [1]$, which distinguishes the states according to whether the first letter is 0 or 1. From the

[3] This is subject to using logarithms to different bases, which simply results in a uniform scaling of the values of information.

point of view of “reading” the contents of a state, this is the first step. The result of two steps is the division into four parts $X = [00] \cup [01] \cup [10] \cup [11]$. This can be continued to three steps, four steps, and so on. These types of subsets are called *cylinder sets.* The idea of a cylinder, say in three-dimensional space, is that of constraining space in two directions, say into a circle, and allowing the other direction to be completely unconstrained. Thus the cylinder set [10] constrains the first two symbols of an infinite string to 1 and 0, and leaves all the rest free to take any 0/1-values.

These cylinder sets are referred to as *elementary events.* Further events, described as subsets of X, are constructed out of unions, intersections, and complements of these sets. This is intended to include both finite and countable unions and intersections. The entire resulting collection of sets is called the *event space* $\mathcal{E}$, which is a special case of a σ-algebra. Events play two separate roles in our approach.

In the first place, we can think of the event space as consisting of those subsets of X that we can actually describe. This is more important than it first seems. We are dealing with states that individually can only be described in terms of *infinite* resolution. An important aspect of pattern has to do with the difference between idealizations like the states of X and the actual world we live in, for which such distinctions may not make sense. But one thing we can do is to talk about the *finite* number of symbols that lie at the start of a state.

The other important role is that the event space is the natural vehicle for the concept of *probability.* Specifying any measure on a set requires a σ-algebra of subsets of that set on which we define it. The event space can serve that role. A *probability measure* on X is a function

$$\boldsymbol{p}\colon\ \mathcal{E} \longrightarrow \mathbb{R}_{\geq 0} := \{x \in \mathbb{R} : x \geq 0\}$$

with the properties

- $\boldsymbol{p}(\varnothing) = 0$;
- $\boldsymbol{p}$ is additive, in the sense that its value on a finite or countable disjoint union of events is the sum of its values on those events;
- $\boldsymbol{p}(X) = 1$.

The first two say that $\boldsymbol{p}$ is a *positive measure*, and the last restricts it to be a *probability measure.*

The simplest example is the coin-tossing scenario, or *Bernoulli trial.* We assume that X models the process of conducting infinite coin-tossing trials, each state being the outcome of one such trial. Without assuming that we have a fair coin, but still assuming that the successive tosses are completely independent of each other, we can choose some value p, with $0 \leq p \leq 1$, and

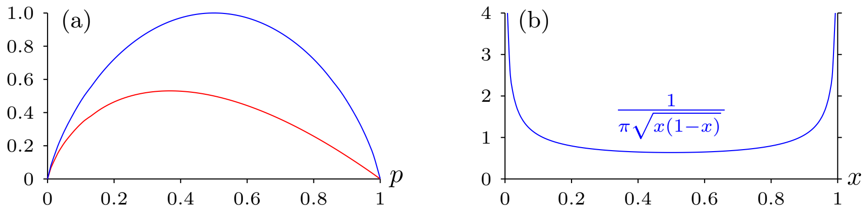


Figure 2: (a) Plot of $-p\log_2(p)$ (in red) and of $-\big(p\log_2(p) + (1-p)\log_2(1-p)\big)$ (in blue, with maximum at $p = \frac{1}{2}$). Both are zero at $p = 0$ and $p = 1$. (b) The invariant probability distribution of the chaotic logistic pattern system.

take the probability measure $\boldsymbol{p}$ defined on cylinder sets by

$$\boldsymbol{p}([0]) = p\,; \quad \boldsymbol{p}([1]) = 1 - p\,;$$
$$\boldsymbol{p}([a_1\,a_2\,\ldots\,a_n]) = \boldsymbol{p}([a_1])\,\boldsymbol{p}([a_2])\cdots\boldsymbol{p}([a_n])\,.$$

When $p = \frac{1}{2}$, the amount of information gained in the evolution $[x_1]$, $[x_1\,x_2]$, $[x_1\,x_2\,x_3]\ldots$ of states is exactly one bit per step, so the dynamical entropy is 1, as one might expect. For general p, it is $-\big(p\log_2(p) + (1-p)\log_2(1-p)\big)$, see Figure 2(a). Notice that if $p = 0$ or $p = 1$, this reduces to 0. No information can be gained from a system that produces the same value with probabilistic certainty!

7 Invariance

Although we have emphasized change as a fundamental constituent of pattern, lack of change, or *invariance*, is equally important. It is true that change is a universal feature of reality, but within the processes of change, there are aspects that do not change. The most obvious are what we call the laws of physics, which we view as unchanging both in time and space. Invariance can appear in pattern systems in many ways, and finding invariants is a crucial part of understanding any pattern system. The primary form of invariance is often found in the probability measure $\boldsymbol{p}$. For a shift system with shift operator s and event space $\mathcal{E}$, the probability measure $\boldsymbol{p}$ is said to be *invariant* if

$$\boldsymbol{p}(s^{-1} \triangleright E) = \boldsymbol{p}(E) \quad \text{holds for all } E \in \mathcal{E}.$$

Though s itself is not invertible, s^{-1} is well defined at the level of subsets of X,

$$s^{-1} \triangleright E := \{x \in X \;:\; s \triangleright x \in E\}.$$

The Bernoulli measures that we have just seen are invariant. For example, if $\boldsymbol{p}([0]) = p$ in a Bernoulli shift, one has $s^{-1} \triangleright [0] = [00] \cup [10]$ (all states that have first component 0 after one shift) and

$$\boldsymbol{p}(s^{-1} \triangleright [0]) = \boldsymbol{p}([00] \cup [10]) = \boldsymbol{p}([00]) + \boldsymbol{p}([10]) = p^2 + (1-p)\,p = p = \boldsymbol{p}([0]).$$

In essence, invariance simply means that the probability of events does not change as the present moves into the future. Similarly, our engineer may well assume that the frequency of the letter pairs in English is something that does not depend on where in the text one is, or what particular piece of text is being processed.[4]

8 Pattern systems and repetition

Now, we can write down the entire definition of a *pattern system* $\mathcal{X}$, which is a system consisting of a state space X, a set of operators S on it, an event space $\mathcal{E}$ of subsets of X, and a measure (often an invariant probability measure) $\boldsymbol{p}$:

$$\mathcal{X} = (X, S, \mathcal{E}, \boldsymbol{p})\,.$$

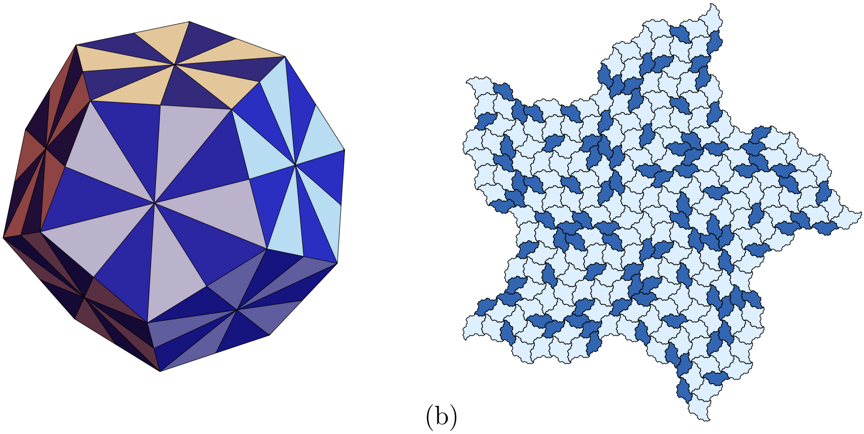
If S consists of a single operator s, we speak of this as a *simple iterative pattern system.* The existence of an invariant measure is often not obvious, and part of the problem is to find one. For example, there is a simple iterative system on the unit interval $[0, 1]$ on the real line defined by $s \triangleright x = 4x(1-x)$. This so-called *logistic system* is chaotic, compare [9], and there is no sign of any invariant measure. However, it does have one, as illustrated in Figure 2(b).

When people talk of pattern, the idea of some form of *repetition* is often in their minds. Repetition somehow seems essential if there is to be recognition (re-cognition). There is not a hint of it in sight anywhere in our definition, but amazingly, it is implicit in it. This is the famous Poincaré Recurrence Theorem.

Theorem 1. *Suppose that* $\mathcal{X} = (X, S, \mathcal{E}, \boldsymbol{p})$ *is a simple iterative pattern system with an invariant probability measure* $\boldsymbol{p}$*, and suppose that* E *is any event with* $\boldsymbol{p}(E) > 0$*. Then, almost all states of* E *return to* E *infinitely often.*

This means that in choosing any $x \in E$, we see, with probabilistic certainty (as measured by $\boldsymbol{p}$), the trajectory $x, s \triangleright x, s^2 \triangleright x, s^3 \triangleright x, \ldots$ to pass through E infinitely many times. The theorem tells us neither how long we have to wait for the first recurrence, nor whether there is any uniform bound on consecutive waiting times. But we can be probabilistically sure that it will return.

[4] This is no doubt an assumption. In the context of communications systems, it is appropriate to lump all English texts together with common statistics. If we were interested in the differences between William Shakespeare and Christopher Marlowe, it may not be.



(a) (b)

Figure 3: (a) The symmetry group of the dodecahedron is simply transitive on the 120 triangles that are indicated here. (b) A five-fold symmetric patch of the chiral GLB tiling.

9 Pattern of symmetry

Surely symmetry is one of the most familiar forms of pattern, and indeed symmetry is very much included in what we have been discussing so far. As an example, let us consider the dodecahedron, the last and quintessential fifth shape in the ancient Greek ideas about the elements (and sometimes viewed as the ultimate achievement of Euclid's *Elements*). Its symmetry group of 120 elements is made rather evident in Figure 3(a), see also [3]. We can take the state space as the set D of points that constitute the surface of the dodecahedron, the operators as the group of rotations and reflections that act on it, and the ordinary surface area as the invariant measure. As for a suitable event space, a natural one is to consider the subsets consisting of the vertices, the edges, and the faces. Then we have a nice pattern system. The same sort of scheme can be arranged for any group acting as symmetries.

Frank Wilczek[5] likes to think of symmetry as "change without change". For instance, a rotation certainly moves the points around, but it preserves the overall form. It is a deep idea of modern physics that many laws can be viewed from this perspective. If a law does not change under temporal or spatial transformations, it is an invariant of the group generated by those transformations. The physics of Galileo and Newton diverges from that of Einstein on just such a point [12]. The famous Emmy Noether, who is known

[5] Nobel Prize for Physics, 2004.

among algebraists for her work on what are now called Noetherian rings, is better known in the physics community for Noether's theorem, which hinges precisely on the point that symmetry and invariant laws of physics are, in effect, the two sides of a single coin.[6]

10 Periodic symmetry

We know that the function $t \mapsto e^{2\pi it}$ traces out a circle $\mathbb{U}$ of radius 1 in the complex plane, and that it circles around counterclockwise through one cycle every time t increases by 1. This suggests a little dynamical system with $\mathbb{U}$ as the state space and $\mathbb{R}$ acting as operators that rotate points around it:

$$t \triangleright e^{2\pi \mathrm{i}x} = e^{2\pi \mathrm{i}(t+x)} .$$

It even has a natural invariant measure based on the arcs on the circle: arc length is not changed by rotation. This is a universally appearing pattern system: the pattern of cyclic or periodic repetition. Any periodic function with period 1 can be viewed as a function on $\mathbb{U}$, by writing it in the form $f(e^{2\pi \mathrm{i}t})$. The simplest periodic functions are those of the form $e^{2\pi \mathrm{i}kt} = \cos(2\pi kt) + \mathrm{i}\sin(2\pi kt)$, where k is an integer. These are the pure harmonics, and Fourier analysis serves to express arbitrary periodic functions in terms of these harmonics. This reveals a remarkable connection between the continuous and the discrete: the continuous periodic function f is expressed in terms of its frequencies (which are integers).

The natural extension of this to two dimensions is the study of functions that are periodic in two different directions: wallpaper patterns and floor tilings are familiar examples [5]. The formalism is now based on $\mathbb{U}^2 = \mathbb{U}\times\mathbb{U}$, the set product of two circles, which is the torus. Points are expressed in the form $(e^{2\pi \mathrm{i}x}, e^{2\pi \mathrm{i}y})$, and the dynamics is expressed by the two-component action of $\mathbb{R}^2$. A wallpaper pattern can be represented as a pattern on the torus, with the action of $\mathbb{R}^2$ opening it up in the plane. The same ideas extend to the (harder to visualize) 3-torus $\mathbb{U}^3$ and to the important area of crystallography with its periodic repetition in three dimensions. All this suggests that repetitivity can be deeply encoded into the topological structure of the dynamical system.

11 Long-range order

Repetition of local features that are far apart can bring overall coherence to an extensive planar or spatial pattern even if there is no strict periodicity. This is

[6] The WIKIPEDIA entry on Noether's theorem gives a good account of this result.

called *long-range order*, and is another form of patterning. Figure 3(b) shows part of a tiling[7] $\mathcal{T}$ that is aperiodic but still *strongly repetitive*, meaning that finite patches of its tiles repeat with *bounded* gaps under translations. The tiling can be translated so that large finite patches of tiles perfectly align – in fact, the finite patches can be chosen of any size whatsoever and such translations are possible. Nonetheless, there is no overall translational symmetry; see [4, 2] for further examples, also including some randomness.

The *hull* of $\mathcal{T}$ is defined as the set X of all tilings $\mathcal{T}'$ that are indistinguishable from $\mathcal{T}$ by any local considerations: any finite patch of tiles of $\mathcal{T}'$ agrees, subject to translation, with one appearing in $\mathcal{T}$, and vice-versa. By the very construction of X, there is a natural translation action of $\mathbb{R}^2$ on X. This translational action provides the dynamics. Given any bounded region R of the plane, there are only finitely many distinct tile patches that can be put into R, and this partitions X into a finite number of parts, according to these arrangements. The frequency of these parts, via cylinder sets, then produces an invariant measure. This way, we have recovered a full pattern system. In fact, this very much formalizes what we intuitively understand as the patterns that these tiles produce. Aperiodic tilings have been motivational in this exploration of pattern [1, 2, 7].

12 Emergence of pattern

Our conception of pattern is not so much about things as it is about process. What we take as pattern is frequently a feature that emerges out of process, often from a few rather simple rules. The probability distribution of the logistic system is an example, as is the emergence of the Sierpiński triangle out of the famous Pascal triangle of binomial coefficients, see Figure 1(b).

An amazing sight is to witness the complex patterns that emerge in the flocking or swarming of thousands of birds as they swirl together in flight, Figure 1(a). In spite of its apparent complexity, mathematical modeling has shown that this sort of patterning can emerge if each bird follows three simple rules: (i) separation: avoid crowding neighbors (short-range repulsion); (ii) alignment: steer towards average heading of neighbors; (iii) cohesion: steer towards average position of neighbors (long-range attraction).[8] To some extent, this type of situation is the dream of science – that in spite of the seeming complexity of our world, beneath it is the outcome of interaction of a few basic principles.

[7] It is a fractal version of the Godrèche–Lançon–Billard (GLB) tiling; see [2, Sec. 6.5.1].

[8] See, for instance, `https://en.wikipedia.org/wiki/Flocking_(behavior)`.

13 Going beyond

If we think about the simple dynamics of Section 10, we can see that it is the starting point for the whole subject of periodic functions and the search for the harmonics that underlie them (spectral theory). This same idea extends to pattern systems in general. Having established a pattern system in the form $\mathcal{X} = (X, S, \mathcal{E}, \boldsymbol{p})$, we can go on to study the functions on X and how they behave under the dynamics. This goes beyond what we can discuss in this snapshot, but a spectral theory of pattern systems exists and is, for instance, a crucial part of the study of aperiodic and almost-periodic order [2, 11].

Beyond this, there is the story of pattern in general. Nothing in this world is unchanging, except perhaps the fundamental laws of physics. Wherever there is change, there will be pattern in the sense that we have described, for ultimately pattern is nothing more than the recognition of difference and change within some context. Almost anything that we look at, from neural networks to quantum mechanics, must somehow imply these basic concepts. In this way, we suggest, there is a common thread to our perception of pattern: it is dynamical in origin, it is ubiquitous, and to some significant extent it can be formalized.

Dedication

To the memory of Uwe Grimm, dear friend, colleague, and co-author of this paper, who tragically passed away during its completion.

Image credits

Figure 1(a) © Adam. Licensed under cc-by/2.0. `https://www.flickr.com/photos/aaddaamn/5196833268/in/photostream/`, cropped from original.

All other images created by the authors.

References

[1] M. Baake, D. Damanik and U. Grimm, *Aperiodic order and spectral properties*, Snapshots of modern mathematics from Oberwolfach, 3/2017. https://publications.mfo.de/handle/mfo/1310.

[2] M. Baake and U. Grimm, *Aperiodic Order. Vol. 1: A Mathematical Invitation*, Cambridge University Press, 2013.

[3] R. O. Buchweitz and E. Faber, *Swallowtail on the shore*, Snapshots of modern mathematics from Oberwolfach, 7/2014. https://publications.mfo.de/handle/mfo/433.

[4] É. Fusy, *Random sampling of domino and lozenge tilings*, Snapshots of modern mathematics from Oberwolfach, 2/2016. https://publications.mfo.de/handle/mfo/457.

[5] T. Holm, *Friezes and tilings*, Snapshots of modern mathematics from Oberwolfach, 4/2015. https://publications.mfo.de/handle/mfo/447.

[6] A. I. Khinchin, *Mathematical Foundations of Information Theory*, Dover, 1957.

[7] R. V. Moody and M.-D. Deng, *The Pattern of Change*, in preparation.

[8] C. E. Shannon, *A mathematical theory of communication*, The Bell System Technical Journal **27** (1948), 379–423 and 623–656.

[9] T. Solomon, *Chaos and chaotic fluid mixing*, Snapshots of modern mathematics from Oberwolfach, 5/2015. https://publications.mfo.de/handle/mfo/448.

[10] H. Tong, *Statistics and dynamical phenomena*, Snapshots of modern mathematics from Oberwolfach, 6/2014. https://publications.mfo.de/handle/mfo/432.

[11] P. Walters, *An Introduction to Ergodic Theory*, Springer, 2000.

[12] F. Wilczek, *A Beautiful Question: Finding Nature's Deep Design*, Penguin, 2015.

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