Mathematisches Forschungsinstitut Oberwolfach

Report No. 47/2022

DOI: 10.4171/OWR/2022/47

Arbeitsgemeinschaft: Higher Rank Teichmüller Theory

Organized by
Fanny Kassel, Bures-sur-Yvette
Beatrice Pozzetti, Heidelberg
Andrés Sambarino, Paris
Anna Wienhard, Heidelberg

9 October – 14 October 2022

Abstract. Higher rank Teichmüller theory is the study of certain connected components of character varieties of surface groups in higher rank semisimple Lie groups, with the property that all elements in these components correspond to faithful representations with discrete image. Like classical Teichmüller theory, this relatively recent theory is very rich and builds on a combination of methods from various areas of mathematics. Its many facets were explored in detail during the Arbeitsgemeinschaft.

Mathematics Subject Classification (2020): 30F60, 32G15.

Introduction by the Organizers

Riemann surfaces are of fundamental importance in many areas of mathematics and theoretical physics. The study of the moduli space of Riemann surfaces of a fixed topological type is intimately related to the study of the Teichmüller space of that surface, together with the action of the mapping class group. Classical Teichmüller theory has many facets and involves the interplay of various methods from geometry, analysis, dynamics, and algebraic geometry.

Through the Uniformization theorem, the Teichmüller space of a surface can be realized as the space of marked hyperbolic structures. In this way, the Teichmüller space can be identified with a subset of the character variety of the fundamental group of the surface to the Lie group $\mathsf{PSL}(2,\mathbb{R})$, corresponding to conjugacy classes of faithful representations with discrete image. It turns out that this subset of the character variety is actually a full connected component.

This is part of a more general phenomenon: given a closed surface of genus at least two, there are also higher rank semisimple real Lie groups admitting connected components of the character variety of the surface corresponding only to faithful representations with discrete image. These connected components are often called *higher rank Teichmüller spaces*, and their study *higher rank Teichmüller theory*.

Like classical Teichmüller theory, higher rank Teichmüller theory builds on a combination of methods from various areas of mathematics: bounded cohomology, Higgs bundles, positivity, cluster algebras, harmonic maps, incidence structures, geodesic currents, real algebraic geometry, dynamics... The variety of techniques involved adds to the richness of the topic.

In the Arbeitsgemeinschaft, organized by Fanny Kassel (CNRS & IHES), Beatrice Pozzetti (Heidelberg), Andrés Sambarino (CNRS & Jussieu) and Anna Wienhard (Heidelberg), almost 50 participants, ranging from beginning graduate students to established professors, came together to learn and explore some aspects of higher rank Teichmüller theory. Starting from a short review of key properties of the classical Teichmüller space, the program focused on geometric aspects, but explored also relations to algebraic structures (e.g. cluster algebras), Higgs bundles, and dynamical properties. Several talks were given by graduate students, who had been assigned a postdoc or faculty member with more experience as a mentor. The quality of the talks was excellent, and the atmosphere was very conducive to learning. Many questions were asked during and after the talks, discussion continued during the afternoon break, and in the evenings we had some lively question and answer sessions.

Arbeitsgemeinschaft: Higher Rank Teichmüller Theory

Table of Contents

Abstracts

Teichmüllerspace I: Definitions and parametrizations

Raphael Appenzeller
(mentored by V. Disarlo)

Let S be a smooth oriented closed surface of genus $g \geq 2$. The surface S admits a Riemannian metric of constant curvature -1. The goal of Teichmüller-theory is to describe the space of all such hyperbolic structures. We give two definitions of Teichhmüller space $\mathrm{Teich}(S)$ and indicate why they are equivalent. General references include [3, 7, 2, 6].

1. Definitions

Definition 1. The *Teichmüller space of* S is

$$\mathrm{Teich}(S) = \{(X, \varphi) \text{ marked hyperbolic structures of } S\} / \text{homotopy}.$$

A *marked hyperbolic structure* (X, φ) of S consists of a Riemannian surface X with constant curvature -1 and a diffeomorphism $\varphi\colon S \to X$, called the *marking*. Two marked hyperbolic structures $(X, \varphi), (X', \varphi')$ are *homotopic* if there is an isometry $g\colon X \to Y$ such that $g \circ \varphi$ is homotopic to φ'.

Definition 2. The *Teichmüller space* $\mathrm{Teich}(S)$ *of* S is one of the two connected components consisting entirely of discrete and faithful representations of the *character variety*

$$\mathrm{Hom}(\pi_1(S), \mathrm{PSL}_2(\mathbb{R}))/\,\mathrm{PSL}_2(\mathbb{R}),$$

where the quotient is by the action of $\mathrm{PSL}_2(\mathbb{R})$ by conjugation.

Since $\pi_1(S)$ is generated by $2g$ elements, representations $\pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$ are determined by $2g$ images in $\mathrm{PSL}_2(\mathbb{R})$. Viewing $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}_2(\mathbb{R}))$ as a subspace of the topological space $\mathrm{PSL}_2(\mathbb{R})^{2g}$ induces the quotient topology on $\mathrm{Teich}(S)$. After showing that the two definitions agree, we can pull back the topology to the set in the first definiton. We first note that $\mathrm{PSL}_2(\mathbb{R})$ can be identified with the group of orientation preserving isometries $\mathrm{Isom}^+(\mathbb{H}^2)$ of the hyperbolic plane $\mathbb{H}^2$.

Proposition 3 ([1, 4])**.** *Let* $\Gamma < \mathrm{PSL}_2(\mathbb{R}) \cong \mathrm{Isom}^+(\mathbb{H}^2)$ *be a subgroup. The following are equivalent:*

(1) Γ *is a torsion-free Fuchsian (i.e. discrete) group.*
(2) $\Gamma \curvearrowright \mathbb{H}^2$ *freely and properly discontinuously.*
(3) $\mathbb{H}^2/\Gamma$ *is Hausdorff and* $\mathbb{H}^2 \to \mathbb{H}^2/\Gamma$ *is a covering map.*

Theorem 4 ([1])**.** *For every hyperbolic surface* X*, there is a homomorphism* $\mathrm{Hol}\colon \pi_1(X) \to \mathrm{Isom}^+(\mathbb{H}^2)$*, called the* Holonomy representation*, such that*

(1) Hol *is injective.*
(2) $\mathrm{Hol}(\pi_1(X))$ *is discrete.*

(3) $\mathbb{H}^2/\operatorname{Hol}(\pi_1(X))$ *is isometric to* X.
(4) Hol *is unique up to conjugation in* $\operatorname{Isom}(\mathbb{H}^2)$.

To go from the first definition to the second, we start with a marked hyperbolic surface (X, φ) and use the holonomy representation Hol to obtain a discrete and faithful representation $\rho\colon \pi_1(S) \cong \pi_1(X) \to \operatorname{Isom}^+(\mathbb{H}^2) \cong \mathrm{PSL}_2(\mathbb{R})$. Taking a different marked hyperbolic structure related to (X, φ) by homotopy, results in the same identification $\pi_1(S) \cong \pi_1(X)$. Since the holonomy representation Hol is only unique up to conjugation in $\operatorname{Isom}(\mathbb{H}^2)$, representations that are conjugated have to be identified. Finally since in the character variety, we quotient by the target group $\mathrm{PSL}_2(\mathbb{R}) = \operatorname{Isom}^+(\mathbb{H}^2)$ instead of $\operatorname{Isom}(\mathbb{H}^2)$, we obtain two copies of Teichmüller space, corresponding to two orientations on $\mathbb{H}^2$.

If we start with a discrete and faithful representation ρ on the other hand, then we can define a hyperbolic surface $X = \mathbb{H}^2/\rho(\pi_1(S))$. This way we get a group isomorphism $\pi_1(S) \cong \pi_1(X)$, which we can improve to a homotopy equivalence $S \to X$, which induces the isomorphism $\pi_1(S) \cong \pi_1(X)$, since S and X are Eilenberg-McLane classifying spaces. The homotopy equivalence can then be upgraded to first a homeomorphism and then a diffeomorphism $\varphi\colon S \to X$, which we can use as a the marking for the marked hyperbolic surface (X, φ). We note that starting with equivalent representations produces isometric hyperbolic surfaces X and the same marking up to homotopy. Thus the result is only defined up to homotopy, as stated in Definition 1. Details can be found in [1, 2].

2. Parametrizations

In this section we follow [2]. We will discuss the Fenchel-Nielsen-coordinates for Teichmüller space, from which it follows that $\operatorname{Teich}(S)$ is homeomorphic to $\mathbb{R}^{6g-6}$. We first have to choose a *framing*, consisting of

(1) A collection of $3g-3$ oriented curves $\Gamma = \{\gamma_1, \ldots, \gamma_{3g-3}\}$, decomposing S into pairs of pants,
(2) A transverse multicurve μ that cuts each pair of pants into hexagons.

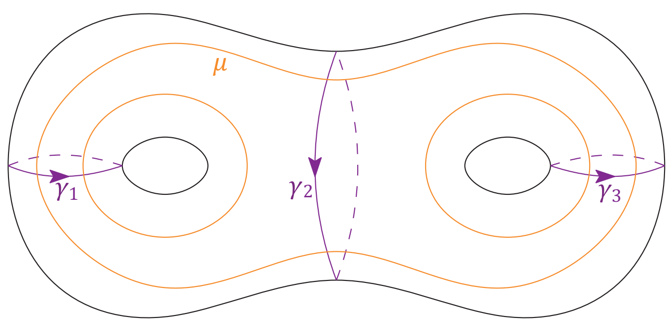


Figure 1. A framing (Γ, μ) of a genus 2 surface.

Given a framing (Γ, μ), we define a map

$$\begin{aligned} \mathrm{Teich}(S) &\to \mathbb{R}_{>0}^{3g-3} \times \mathbb{R}^{3g-3} \\ [(X,\varphi)] &\mapsto (\ell_i, \theta_i)_{i\in\{1,\dots,3g-3\}} \end{aligned}$$

by considering the unique geodesic representatives c_i of $\varphi(\gamma_i) \subseteq X$. The parameters ℓ_i are defined to be the lengths of c_i as measured in the hyperbolic surface X. For the parameters θ_i, we consider the two pairs of pants on the left and the right of c_i and two connected components μ_L, μ_R of $\varphi(\mu) \subseteq X \setminus \{c_i \colon i = 1, \dots, 3g-3\}$ that intersect in a common point on c_i. Comparing μ_L and μ_R with their geodesic representatives (making a right angle with the boundary components of the pairs of pants), results in twists t_L and t_R, whose difference we normalize as

$$\theta_i = \frac{t_L - t_R}{\ell_i} \cdot 2\pi.$$

to obtain the twist parameters θ_i in the Fenchel-Nielsen map.

Theorem 5 (Fricke)**.** *The Fenchel-Nielsen map*

$$\mathrm{Teich}(S) \to \mathbb{R}_{>0}^{3g-3} \times \mathbb{R}^{3g-3} \cong \mathbb{R}^{6g-6}$$

is a homeomorphism.

Further topics include Shear-coordinates [6], the mapping class group [2], the $(9g-9)$-theorem [2] and Thurston's compactification of Teichmüller space [5].

References

[1] J. Aramayona, *Hyperbolic structures on surfaces*, Geometry, topology and dynamics of character varieties (2012), 65–94.

[2] B. Farb, D. Margalit, *A Primer on Mapping Class Groups*, Princeton university press, 2011.

[3] J. Hubbard, *Teichmüller theory and applications to geometry, topology, and dynamics. Volume 1: Teichmüller Theory.*, Matrix Editions, Ithaca, NY, 2006. with contributions by A. Douady, W. Dunbar, R. Roeder, S. Bonnot, D. Brown, A. Hatcher, C. Hruska, S. Mitra.

[4] S. Katok, *Fuchsian Groups*, Chicago Lectures in Mathematics, 1992.

[5] C. Leininger, *Degenerations of hyperbolic structures on surfaces*, Geometry, topology and dynamics of character varieties (2012), 95–138.

[6] B. Martelli, *An Introduction to Geometric Topology*, Independently published, 2nd Edition, 2022.

[7] A. Papadopoulos, *Introduction to Teichmüller theory, old and new.*, Handbook of Teichmüller theory I, volume 11 of IRMA Lectures in Mathematics and Theoretical Physics. European Mathematical Society, Zürich, 2007.

Teichmüller space II

Balthazar Fléchelles
(mentored by I. Calderon)

Given the definition of Teichmüller space as the space of marked hyperbolic structures on S considered up to homotopy, one may wonder how to add more structure. It turns out the analytic point of view provides a lot of structure on $\mathcal{T}_S$, in particular, an analytic structure giving, and a Kähler structure.

We follow John Hubbard [1] in our exposition, if one needs further details.

1. $\mathcal{T}_S$ as a complete metric space

The first step is to define the Teichmüller metric, which relies on the notion of quasi-conformal maps between two Riemann surfaces X and Y.

Definition 1. An application $f : X \to Y$ is K-quasi-conformal (for $K \geq 1$) if it admits distributional derivatives $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$ satisfying

$$\left|\frac{\partial f}{\partial \bar{z}}\right| \leq \mu \left|\frac{\partial f}{\partial z}\right|, \tag{1}$$

for $\mu = \frac{K-1}{K+1}$.

Observe that the closer K is to 1, the closer f is to being conformal, which explains the terminology.

Given a Riemann surface of finite volume X, we can consider X "up to quasi-conformal mappings", and call the result a quasi-conformal surface S of finite type. It is known that quasi-conformal maps distinguish between cusps and funnels, so that any analytic structure on S compatible with its quasi-conformal structure will be of finite volume.

This motivates a new definition of Teichmüller space, without the need to choose a base point X.

Definition 2. The Teichmüller space $\mathcal{T}_S$ of S is

$$\mathcal{T}_S = \{\varphi : S \to X \text{ quasi-conformal isomorphism}\}/\sim, \tag{2}$$

where two *markings* $\varphi_1 : S \to X_1$ and $\varphi_2 : S \to X_2$ are equivalent if there exists an analytic map $f : X_1 \to X_2$ such that $f \circ \varphi_1 = \varphi_2$.

In this framework, we can define the Teichmüller metric: the distance between $\varphi_1 : S \to X_1$ and $\varphi_2 : S \to X_2$ is then given by

$$d_{\mathcal{T}_S}([\varphi_1], [\varphi_2]) = \inf\{\log K(f \circ \varphi_1 \circ \varphi_2^{-1}), f : X_1 \to X_2 \text{ quasi-conformal}, f \circ \varphi_1 = \varphi_2\}$$

With this metric, $\mathcal{T}_S$ is a complete metric space. Using Beltrami forms, and holomorphic quadratic forms, we can add an additional analytic structure.

2. Analytic structure on $\mathcal{T}_S$

A first step is to describe the geodesics of the Teichmüller metric. Using a compactness argument on families of quasi-conformal applications, one can show that there always exists a quasi-conformal map realizing the distance between two given points of $\mathcal{T}_S$, called a *Teichmüller mapping.*

2.1. Teichmüller's theorem.

The Teichmüller theorem describes precisely these Teichmüller mappings, using holomorphic quadratic forms. These are given in coordinates in the neighborhood of non-cusped points as

$$q = \mu(z)\mathrm{d}z^2, \tag{3}$$

where μ is a holomorphic function, and has a pole of order 1 at cusped points.

Up to a holomorphic change of coordinates, one can find *natural coordinates* in which one can write

$$q = \mathrm{d}z^2. \tag{4}$$

In these coordinates, there is a natural notion of *horizontal* and *vertical* directions. This defines a structure of *half-translation surface* on the surface. The space of holomorphic quadratic forms on a Riemann surface X is denoted by $\mathcal{Q}(X)$.

Theorem 3 (Teichmüller's theorem)**.** *An application $f : X_1 \to X_2$ is a K-Teichmüller mapping if and only if there exist holomorphic quadratic forms $q_k \in \mathcal{Q}(X_k)$ $(k = 1, 2)$ such that f sends zeroes (resp. poles) of q_1 to zeroes (resp. poles) of q_2, and, in natural coordinates $x_k + iy_k$ $(k = 1, 2)$ for q_1 and q_2, f can be expressed as*

$$f(x_1 + iy_1) = x_2 + \frac{i}{K}y_2 \tag{5}$$

This, in some sense, relates holomorphic quadratic forms on X to tangent vectors at the associated point in $\mathcal{T}_S$.

One can make this intuition more precise by using Beltrami forms to define an analytic structure on $\mathcal{T}_S$.

2.2. Beltrami forms.

Definition 4. A Beltrami form $\mu \in \mathrm{Bel}(X)$ is a L^∞ $(-1, 1)$-form on a Riemann surface X. It is written in coordinates as

$$\mu = \xi(z)\frac{\mathrm{d}\bar{z}}{\mathrm{d}z}, \tag{6}$$

where ξ is a L^∞ function on X of norm smaller than 1.

The full vector space of L^∞ $(-1, 1)$-forms on X is denoted by $\mathrm{bel}(X)$, in which $\mathrm{Bel}(X)$ lies as the unit open ball.

Observe that if $f : X \to X$ is a quasi-conformal application, then $\mu = (\partial f)^{-1}\bar{\partial} f \in \mathrm{Bel}(X)$ by (1), and f is a solution of the Beltrami equation

$$\bar{\partial} f = \partial f \circ \mu. \tag{7}$$

Conversely, given a Beltrami form $\mu \in \mathrm{Bel}(X)$, the mapping theorem says:

Theorem 5 (Mapping theorem)**.** *Given $\mu \in \mathrm{Bel}(X)$, the Beltrami equation* (7) *given by μ locally has solutions on charts. Moreover, two solutions differ by an analytic change of coordinates.*

Therefore, a Beltrami form $\mu \in \mathrm{Bel}(X)$ determines a new analytic structure X_μ on X, such that the identity map $\mathrm{id} : X \to X_\mu$ is quasi-conformal of constant $\frac{1+\|\mu\|}{1-\|\mu\|}$.

Observe that we have a bilinear form

$$\begin{cases} \mathrm{bel}(X) \times \mathcal{Q}^1(X) \to \mathbb{C} \\ \qquad\qquad (\mu, q) \mapsto \int_X \mu q \end{cases} \tag{8}$$

(where $\mathcal{Q}^1(X)$ denotes $\mathcal{Q}(X)$ endowed with the L^1 norm).

2.3. **Analytic structure on $\mathcal{T}_S$.** Theorem 5 yields a new definition for $\mathcal{T}_S$, in terms of Beltrami forms on S.

Definition 6. A Beltrami form $\mu \in \mathrm{Bel}(S)$ is an equivalent class of $\mu = [\varphi : S \to X, \nu]$ where $\nu \in \mathrm{Bel}(X)$ and φ is quasi-conformal. (φ_1, ν_1) and (φ_2, ν_2) are equivalent if $\nu_1 = (\varphi_2 \circ \varphi_1^{-1})^* \nu_2$.

Observe that the group $\mathbf{QC}(S)$ of quasi-conformal isomorphisms of S acts naturally on $\mathrm{Bel}(S)$ by composition. Using the last paragraph, we obtain the identification

$$\mathcal{T}_S = \mathrm{Bel}(S)/\mathbf{QC}^0(S), \tag{9}$$

where $\mathbf{QC}^0(S)$ is the group of quasi-conformal isomorphisms of S that are homotopic to the identity.

Hence, the *mapping class group* $\mathrm{MCG}(S) := \mathbf{QC}(S)/\mathbf{QC}^0(S)$ has a natural action on $\mathcal{T}_S$.

The bilinear form (8) induces a pairing $T_\tau \mathcal{T}_S \times \mathcal{Q}^1(X) \to \mathbb{C}$ (where $\tau = [\varphi : S \to X]$) using the identification (9).

After dividing by the proper power of the volume element on X, one also gets a pairing $\mathcal{Q}^1(X) \times \mathcal{Q}^\infty(X^*) \to \mathbb{C}$, where X^* is X with the opposite orientation, obtained through conjugating the coordinates.

Therefore, we can identify $T_\tau \mathcal{T}_S$ with $\mathcal{Q}^\infty(X^*)$. One can show that this is compatible with an analytic structure on $\mathcal{T}_S$.

3. The Weil-Petersson metric

In the same manner, one gets a pairing $\mathcal{Q}^2(X) \times \mathcal{Q}^2(X^*) \to \mathbb{C}$ which is a Hermitian metric on $\mathcal{T}_S$. In fact, it makes $\mathcal{T}_S$ a Kähler manifold. This metric is different from the Teichmüller metric.

Several results show that it is a very natural metric on $\mathcal{T}_S$. Wolpert's formula [2] links the imaginary part of this pairing to the Fenchel-Nielsen coordinates. Goldman's symplectic form on the character variety restricts to its imaginary part [4]. Moreover, Bonahon's inner product on the space of geodesic currents coincides with its real part on $\mathcal{T}_S$ [3].

References

[1] J. Hubbard, *Teichmüller Theory and Applications to Geometry, Topology and Dynamics. Volume 1: Teichmüller theory*, Matrix editions.

[2] S. Wolpert, *Thurston's Riemannian metric for Teichmüller space*, Journal of Differential Geometry **23** (1986), 143–174.

[3] F. Bonahon, *The geometry of Teichmüller space via geodesic currents*, Inventiones mathematicae **92** (1988), 139–162.

[4] W. M. Goldman, *The Symplectic Nature of Fundamentals Groups of Surfaces*, Advencements in Mathematics **54** (1984), 200-225.

The Hitchin component

Jacques Audibert

(mentored by K. Tsouvalas)

Let S be a closed orientable surface of genus $g \geq 2$ and let G be a connected semisimple Lie group. We are interested in connected components of the character variety

$$\mathfrak{X}(S, G) = \mathrm{Hom}(\pi_1(S), G)/G$$

consisting only of discrete and faithful representations. For $G = \mathrm{PSL}(2, \mathbb{R})$ there is exactly two such components. They are copies of the Teichmüller space of S. When G has rank at least 2, connected components consisting only of discrete and faithful representations are called *higher Teichmüller spaces*. If G is compact or complex, no such components exist [2] [3]. When G is real however, higher Teichmüller spaces can exist. The prototypical example is the *Hitchin component*, named after Hitchin's seminal work [1].

Let $n \geq 3$. When n is odd, the character variety $\mathfrak{X}(S, \mathrm{PSL}(n, \mathbb{R}))$ has three connected components. The Hitchin component is one of them. When n is even, $\mathfrak{X}(S, \mathrm{PSL}(n, \mathbb{R}))$ has 6 connected components and there are two Hitchin components. In both cases Hitchin showed that those components are balls of dimension $(2g-2)(n^2-1)$.

Let us define the Hitchin component for $G = \mathrm{PSL}(n, \mathbb{R})$. There exists a unique (up to conjugation) irreducible representation of $\mathrm{SL}(2, \mathbb{R})$ of dimension n. We can describe it by identifying $\mathbb{R}^n$ with the space of homogeneous polynomials in two variables X and Y of degree $n-1$. The irreducible representation

$$\tau_n : \mathrm{SL}(2, \mathbb{R}) \to \mathrm{SL}(n, \mathbb{R})$$

is defined by the following action:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} X^{n-i-1} Y^i = (aX + cY)^{n-i-1} (bX + dY)^i$$

for all $0 \leq i \leq n-1$. Let $j : \pi_1(S) \to \mathrm{SL}(2, \mathbb{R})$ be a discrete and faithful representation. The Hitchin component is the connected component of $\mathfrak{X}(S, \mathrm{PSL}(n, \mathbb{R}))$ that contains $\tau_n \circ j$.

Similar to the Teichmüller space, the Hitchin component of $\mathfrak{X}(S, \mathrm{PSL}(3, \mathbb{R}))$ has an interpretation in terms of geometric structures on the surface S. As showed by Koszul [4], Goldman [5] and Choi-Goldman [6] it parametrizes convex real projective structures on S: for every Hitchin representation $\rho : \pi_1(S) \to \mathrm{PSL}(3, \mathbb{R})$ there exists a properly convex domain $\Omega \subset \mathbb{RP}^2$ which is invariant by ρ and such that $\Omega/\rho(\pi_1(S))$ is homeomorphic to S. It follows that Hitchin representations in $\mathrm{PSL}(3, \mathbb{R})$ are discrete and faithful. Note that hyperbolic structures on S are convex real projective structures as can be seen using the hyperboloid model of the hyperbolic plane. It corresponds to embedding the Teichmüller space in the Hitchin component of $\mathrm{PSL}(3, \mathbb{R})$ using τ_3.

In 2006, Labourie [7] and Fock-Goncharov [8] showed that Hitchin representations $\rho : \pi_1(S) \to \mathrm{PSL}(n, \mathbb{R})$ admit a boundary map

$$\xi : \partial_\infty \pi_1(S) \to \mathrm{Flag}(\mathbb{R}^n)$$

which is continuous, ρ-equivariant and transverse. This implies that Hitchin representations are discrete and faithful. Labourie investigated the dynamical properties of Hitchin representations. He defined the notion of *Anosov representations* and showed that Hitchin representations are Anosov. On the other hand, Fock and Goncharov proved that additionally to the above mentioned properties, the boundary map is *positive*; a notion that has been generalized for other higher Teichmüller spaces.

Hitchin representations can be defined whenever G is a centerless split real Lie group. Such Lie groups admit a special copy of $\mathrm{PSL}(2, \mathbb{R})$ which is called *principal*. It allows to embed the Teichmüller space in $\mathfrak{X}(S, G)$ by postcomposition. The Hitchin component is the connected component of $\mathfrak{X}(S, G)$ that contains it. Each of the previous properties have a suitable generalization in this setting. In particular, it is always a higher Teichmüller space.

References

[1] N.J. Hitchin, *Lie groups and Teichmüller space*, Topology **31** (1992), 449–473.

[2] M.F. Atiyah, R. Bott, *The Yang-Mills Equations over Riemann Surfaces*, Philosophical Transactions of the Royal Society of London **308** (1983), 523–615

[3] J. Li, *The space of surface group representations*, manuscripta mathematica **78** (1993), 223–243

[4] J.-L. Koszul, *Déformations de connexions localement plates*, Ann. Inst. Fourier **18** (1968), 103–114

[5] W. Goldman, *Convex real projective structures on compact surfaces*, J. Differential Geom. **31** (1990), 791–845

[6] S. Choi, W. Goldman, *Convex real projective structures on closed surfaces are closed*, Proc. Amer. Math. Soc. **118** (1993) 657–661

[7] F. Labourie, *Anosov flows, surface groups and curves in projective space*, Invent. Math. **165** (2006) 51–114

[8] V. Fock, A. Goncharov, *Moduli spaces of local systems and higher Teichmüller theory*, Publ. Math. Inst. Hautes Etudes Sci. **103** (2006) 1–211

Maximal Representations

Lisa Ricci
(mentored by J.-P. Burelle)

Let S be a closed connected orientable surface of genus $g \geq 2$ and let G be a Lie group of Hermitian type. We define the *Toledo number* $T(\rho)$ of a representation $\rho : \pi_1(S) \to G$, which coincides with the Euler number when $G =$PSL$(2, \mathbb{R})$ and we show that it satisfies an analogous Milnor-Wood type inequality. In particular it is bounded, and representations for which the Toledo number attains the biggest possible value $|\chi(S)|\text{rank}(G)$ are called *maximal*. In [3] Goldman shows that maximal representations into PSL$(2, \mathbb{R})$ are precisely the holonomies of complete hyperbolic metrics on S, so that they coincide with the classical Teichmüller space considered as a subset of Hom$(\pi_1(S)$, PSL$(2, \mathbb{R}))$. In [2] Burger, Iozzi and Wienhard prove that the map $T : \text{Hom}(\pi_1(S), G) \to \mathbb{R}$ is continuous and takes finitely many values in the interval $[\chi(S)\text{rank}(G), -\chi(S)\text{rank}(G)]$. This implies that the set of maximal representations forms a union of connected components. Moreover, in [1] Burger, Iozzi and Wienhard show that a representation $\rho : \pi_1(S) \to G$ into a tube-type Hermitian Lie group G is maximal if and only if there exists a continuous ρ-equivariant map $\mathbb{RP}^1 \to \check{S}$ that sends positive triples in $\mathbb{RP}^1$ to maximal triples in the Shilov boundary $\check{S}$. A consequence of this result is that maximal representations are injective and have discrete image. In particular, they are an example of a higher Teichmüller space. Another example are *Hitchin components*, and we prove that Hitchin representations are maximal when $G = \text{Sp}(2n, \mathbb{R})$, which is the only Lie group both split real simple and of Hermitian type. We conclude by showing that the diagonal embedding of $\text{SL}(2, \mathbb{R}) \to \text{Sp}(2n, \mathbb{R})$ precomposed with a faithful, discrete and orientation-preserving homomorphism $\pi_1(S) \to \text{SL}(2, \mathbb{R})$ is a further example of a maximal representation.

References

[1] M. Burger, A. Iozzi, A. Wienhard, *Surface group representations with maximal Toledo invariant*, Ann. of Math. (2) **172** (2010), no. 1, 517-566.

[2] M. Burger, A. Iozzi, A. Wienhard, *Higher Teichmüller spaces: from SL$(2, \mathbb{R})$ to other Lie groups*, in Handbook of Teichmüller theory, Vol. IV, volume 19 of IRMA Lect. Math. Theor. Phys., 539-618, Eur. Math. Soc., Zürich, 2014.

[3] W. M. Goldman, *Discontinuous groups and the Euler class*, Thesis, University of California at Berkeley, 1980.

Introduction to Anosov representations

Timothée Bénard
(mentored by F. Zhu)

Anosov representations were first introduced by Labourie in [7] to show that Hitchin representations are faithful and discrete. It is now a fundamental tool in Higher Teichmüller Theory. It is also a relevant generalization of the notion of convex cocompact representation to the higher rank setting.

1. Definitions of Anosov representation

Notations. We denote by $\mathbb{D}$ the Poincaré disc, and identify its unit tangent bundle $T^1\mathbb{D}$ with $PSL_2(\mathbb{R})$. The right-action of $a_t = \mathrm{diag}(e^{t/2}, e^{-t/2})$ corresponds to the geodesic flow on $T^1\mathbb{D}$. Given $x \in T^1\mathbb{D}$, we write $(\varepsilon_+(x), \varepsilon_-(x)) \in \partial\mathbb{D} \times \partial\mathbb{D}$ its limit points obtained by following forward/backward the geodesic defined by x. We let Γ be a surface group, i.e. a discrete torsion-free cocompact subgroup of $PSL_2(\mathbb{R})$, and fix a representation $\rho : \Gamma \to SL_d(\mathbb{R})$.

1.1. Definition in terms boundary maps ξ_k, ξ_{d-k}.

Fix a Γ-invariant continuous family of norms on the trivial bundle $T^1\mathbb{D} \times \mathbb{R}^d$ and write $||.||_x$ the norm on the fibre $\{x\} \times \mathbb{R}^d$ above $x \in T^1\mathbb{D}$.

Definition 1.1. *We say that ρ is P_k-Anosov ($k \in \{1, \dots, d-1\}$) if there exist boundary maps*

$$\xi_k : \partial\mathbb{D} \to Gr_k(\mathbb{R}^d) \qquad \xi_{d-k} : \partial\mathbb{D} \to Gr_{d-k}(\mathbb{R}^d)$$

satisfying

(i) *ξ_k, ξ_{d-k} are continuous ρ-equivariant, and transverse: $\forall p \neq q \in \partial\mathbb{D}$,*

$$\mathbb{R}^d = \xi_k(p) \oplus \xi_{d-k}(q)$$

(ii) *There exists $C, c > 0$ such that for every $x \in T^1\mathbb{D}$, $v \in \xi_k(\varepsilon_+(x))$, $w \in \xi_{d-k}(\varepsilon_-(x))$, $v, w \neq 0$,*

$$\frac{||v||_{xa_t}}{||v||_x} \leq Ce^{-ct} \frac{||w||_{xa_t}}{||w||_x}$$

Remark that by cocompactness of Γ, this definition does not depend on the choice of norms $||.||_x$. It also makes sense if ρ takes values in $PSL_d(\mathbb{R})$ instead of $SL_d(\mathbb{R})$. Finally, one can check that the inclusion map $\Gamma \hookrightarrow PSL_2(\mathbb{R})$ is indeed P_1-Anosov.

1.2. Definition in terms of singular values.

Every element g of $SL_d(\mathbb{R})$ can be written

$$g = k \begin{pmatrix} e^{t_1} & & \\ & \ddots & \\ & & e^{t_d} \end{pmatrix} l$$

where $k, l \in SO_d(\mathbb{R})$ and the t_i's are real number of sum 0, and such that $t_1 \geq \cdots \geq t_d$. The tuple $(t_1, \dots, t_d)$ is unique and called the *Cartan projection* of g, we write it $\mu(g) = (\mu_1(g), \dots, \mu_d(g))$.

Theorem 1.2. [5, 4, 6, 1] *Fix a word distance d on Γ. Then ρ is P_k-Anosov if and only if there exists $C, c > 0$ such that for every $\gamma \in \Gamma$,*

$$\mu_k(\rho(\gamma)) - \mu_{k+1}(\rho(\gamma)) \geq cd(1, \gamma) - C$$

We stress a mental image to be associated to the above inequality. By definition, the Cartan projection belongs to the Weyl chamber

$$\mathfrak{a}^+ = \{t \in \mathbb{R}^d, \sum_i t_i = 0, t_1 \geq \cdots \geq t_d\}$$

This is a convex cone, delimited by "walls", namely $\mathfrak{a}^+ \cap \{t_1 = t_2\}, \ldots, \mathfrak{a}^+ \cap \{t_{d-1} = t_d\}$. The above inequality yields that

$$\mu_k(\rho(\gamma)) - \mu_{k+1}(\rho(\gamma)) \geq c'||\mu(\gamma)|| - C'$$

for some $C', c' > 0$ (here we use that $||\mu(g_1g_2) - \mu(g_2)|| \ll ||\mu(g_1)||$ so $||\mu(\rho(\gamma))|| \ll d(1, \gamma)$). In other words, the Cartan projection of $\rho(\Gamma)$ drifts away linearly from the wall $\mathfrak{a}^+ \cap \{t_k = t_{k+1}\}$ of $\mathfrak{a}^+$.

1.3. **Level of generality.** In greater generality, we can give meaning to the sentence

"$\rho : \Gamma \to G$ is P-Anosov"

when Γ is hyperbolic, G a non compact reductive real linear algebraic group, and $P \subseteq G$ a parabolic subgroup (see for instance [5] or [4]).

2. Properties of Anosov representations

We cite a few properties of Anosov representations. Proofs can be found in [3].

2.1. Quasi-isometric embedding.

Proposition 2.1. *Let $\rho : \Gamma \to SL_d(\mathbb{R})$ be a P_k-Anosov representation. Then ρ is a quasi-isometric embedding for any left invariant metric on $SL_d(\mathbb{R})$. In particular, it is discrete and faithful (here Γ is torsion-free).*

2.2. **Proximality.** The image of a P_k-Anosov representation has a particular dynamics on the Grasmannian variety $Gr_k(\mathbb{R}^d)$. We introduce the Jordan projection of $g \in SL_d(\mathbb{R})$

$$\lambda(g) = \lim_{n \to +\infty} \frac{1}{n}\mu(g^n)$$

As $\mu_1(g) = \log ||g||$, we get that $\lambda_1(g)$ stands for the logarithm of the spectral radius of g. Considering exterior products representations $\Lambda^i\mathbb{R}^d$, we see as well that $\lambda(g)$ is (well defined and is) the logarithm of the modules of the eigenvalues of g, ordered by decreasing order.

Say that $g \in SL_d(\mathbb{R})$ is *P_k-proximal* if $\lambda_k(g) > \lambda_{k+1}(g)$. This translates dynamically by asking that $\mathbb{R}^d$ decomposes as $\mathbb{R}^d = \xi_k^+(g) \oplus \xi_{d-k}^-(g)$ where $\xi_k^+(g) \in Gr_k(\mathbb{R}^d)$, $\xi_{d-k}^-(g) \in Gr_{d-k}(\mathbb{R}^d)$ and $\xi_k^+(g)$ is attracting in the sense that for every k-plane V in direct sum with $\xi_{d-k}^-(g)$, we have $g^nV \to \xi_k^+(g)$ as $n \to +\infty$. We call $\xi_k^+(g)$ the *attracting space* of g, and $\xi_k^-(g)$ the *repelling space* of g. Of course they are unique.

For example, every non-trivial $\gamma \in \Gamma$ is P_1-proximal, and this reads on the boundary $\partial\mathbb{D} \equiv \mathbb{P}(\mathbb{R}^2)$: the action of γ on $\partial\mathbb{D}$ has 2 distinct fixed points γ^-, γ^+ such that $\partial\mathbb{D} \setminus \{\gamma^-\}$ is contracted toward γ^+ under the action of γ.

The following proposition states that Anosov representations respect the dynamics on the boundary.

Proposition 2.2. *Assume $\rho : \Gamma \to SL_d(\mathbb{R})$ is P_k-Anosov, with limit maps ξ_k, ξ_{d-k}. Then every element of $\rho(\gamma)$ is P_k-proximal. Moreover ξ_k sends the attracting point γ^+ of γ to the attracting k-plane of $\rho(\gamma)$. Analogously, and $\xi_{d-k}(\gamma^-)$ is the repelling $d-k$-plane of $\rho(\gamma)$.*

Using the continuity of ξ_k and that the set $\{\gamma^+, \gamma \in \Gamma\}$ is dense in $\partial\mathbb{D}$, (and the analog for ξ_{d-k}, γ^-) we get

Corollary 2.3. *The limit maps ξ_k, ξ_{d-k} of a P_k-Anosov representation are unique.*

2.3. **Stability.**

Proposition 2.4. *Assume $\rho : \Gamma \to SL_d(\mathbb{R})$ is a P_k-Anosov representation. Then any representation $\rho' : \Gamma \to SL_d(\mathbb{R})$ close enough from ρ on a fixed generating set of Γ is also P_k-Anosov.*

3. Examples

Theorem 3.1 ([5])**.** *A representation $\rho : \Gamma \to G$ to a rank-one simple connected real Lie group G is Anosov if and only if it is convex cocompact (or equivalently a quasi-isometric embedding).*

Theorem 3.2 ([7])**.** *If G is a split simple Lie real group, then every representation in the Hitchin component of the character variety $Hom(\Gamma, G)/G$ is P-Anosov for every parabolic subgroup P.*

Theorem 3.3 ([2])**.** *Let G be a real Lie group of Hermitian type, and $\rho : \Gamma \to G$ a maximal representation. Then ρ is P-Anosov with respect to the stabilizer of a point in the Shilov boundary associated to G.*

References

[1] J. Bochi, R. Potrie; A. Sambarino, *Anosov representations and dominated splittings*, J. Eur. Math. Soc. (JEMS) 21, No. 11, 3343-3414 (2019).
[2] M. Burger, A. Iozzi, F. Labourie, A. Wienhard, *Maximal representations of surface groups: symplectic Anosov structures.* Pure Appl. Math. Q. 1, No. 3, 543-590 (2005).
[3] D. Canary *Anosov representations: informal lecture notes.*
[4] F. Guéritaud, O. Guichard, F. Kassel, A. Wienhard *Anosov representations and proper actions*, Geom. Topol. 21, No. 1, 485–584 (2017)
[5] O. Guichard, and A. Wienhard, *Anosov representations: domains of discontinuity and applications*, Invent. Math. 190, No. 2, 357–438 (2012)
[6] M. Kapovich, B. Leeb, J. Porti, *A Morse lemma for quasigeodesics in symmetric spaces and Euclidean buildings.* Geom. Topol. 22, No. 7, 3827-3923 (2018).
[7] F. Labourie *Anosov flows, surface groups and curves in projective space*, Invent. Math. 165, No. 1, 51-114 (2006).

Domains of discontinuity and geometric structures

Sami Douba

(mentored by F. Stecker)

Let Z be a compact metric space and Γ a discrete group acting continuously on Z. Suppose the action of Γ on Z is a *convergence action*, that is, suppose that for every sequence $\gamma_n \to \infty$ in Γ, there are (not necessarily distinct) points $z^\pm \in Z$ and a subsequence (γ_{n_k}) of (γ_n) such that $\gamma_{n_k}\big|_{Z-\{z^-\}} \to z^+$ uniformly on compacta. In this setting, the *limit set* $\Lambda_\Gamma \subset Z$ of the Γ-action is defined as the set of all points in Z of the form z^+ as above; this is a closed Γ-invariant subset of Z on whose (possibly empty) complement Γ acts properly.

Convergence actions arose as abstractions of Kleinian group actions on the Riemann sphere. More generally, let X be a rank-one symmetric space of noncompact type and $\partial_\infty X = G/P$ its visual boundary, where $G = \mathrm{Isom}(X)$ and P is a proper parabolic subgroup of G. Then a proper isometric action of a discrete group Γ on X induces a convergence action of Γ on $\partial_\infty X$. The quotient $\Gamma\backslash(\partial_\infty X - \Lambda_\Gamma)$ is then an orbifold endowed with a $(G, \partial_\infty X)$-structure. This provides a useful link between discrete representations $\Gamma \to G$ and $(G, \partial_\infty X)$-structures on $\Gamma\backslash(\partial_\infty X - \Lambda_\Gamma)$.

If we now remove the restriction on the rank of X, it is no longer true that proper isometric actions on X give rise to convergence actions on $\partial_\infty X$. To restrict to a setting where the action at infinity nevertheless retains some desirable features of convergence dynamics, one considers *P-divergent* actions on X; if $\overline{\mathfrak{a}}^+$ is a Weyl chamber of X with origin $o \in X$, and P is the stabilizer $G_{[\xi]}$ in G of the asymptoticity class $[\xi] \in \partial_\infty X$ of a geodesic ray $\xi \subset \overline{\mathfrak{a}}^+$ emanating from o, then an isometric action of a discrete group Γ on X is *P-divergent*, or *P-regular*, if for every sequence $\gamma_n \to \infty$ in Γ, the $\overline{\mathfrak{a}}^+$-valued distances $d_{\overline{\mathfrak{a}}^+}(o, \gamma_n o)$ diverge from each wall of $\overline{\mathfrak{a}}^+$ not containing the ray ξ. Since a P-divergent group action on X is also P^--divergent for any parabolic subgroup $P^- < G$ opposite to P, we will make the simplifying assumption that P is conjugate within G to its opposite. We say an action on X is *divergent* if the action is P-divergent for some choice of $P < G$.

If X has rank one, then a divergent action of Γ on X is nothing but a proper isometric action. If X is a product $X_1 \times \ldots \times X_n$ of symmetric spaces X_i of noncompact type, then a diagonal isometric action of Γ on X is divergent if and only if the action of Γ on at least one of the factors X_i is divergent. A proper isometric action of Γ on X preserving a totally geodesic rank-one subspace $Y \subset X$ is $G_{[\xi]}$-divergent, where ξ is any geodesic ray in Y. On the other hand, if $\mathrm{rank}(X) \geq 2$ and some subgroup of Γ preserves and acts cocompactly on a maximal flat in X (for instance, if Γ is a lattice in G), then the action of Γ on X is not divergent (with respect to any proper parabolic subgroup of G). Important examples of P-divergent actions on X are those coming from P-Anosov representations $\Gamma \to G$.

For a P-divergent action of Γ on X, the associated action of Γ on G/P satisfies the following weaker notion of convergence: for any sequence $\gamma_n \to \infty$ in Γ, there are points $z^\pm$ and a subsequence (γ_{n_k}) of (γ_n) such that $\gamma_{n_k}\big|_{C(z^-)} \to z^+$ uniformly on compacta, where $C(z^-) \subset G/P$ denotes the set of all points in G/P opposite to z^-. One analogously defines the *limit set* $\Lambda_\Gamma \subset G/P$ in this setting as the set of all points in G/P of the form z^+ as above. An elementary consequence of the above weak form of convergence is that Γ preserves and acts properly on the (possibly empty) open subset of G/P consisting of all points in G/P that are opposite to each point in Λ_Γ.

We rephrase the preceding statement so that we can later refine it. For any proper parabolic subgroup $Q < G$, the set $P\backslash G/Q$ of positions that a point in G/Q can occupy relative to a point in G/P is a finite poset on which the longest element w_0 of the restricted Weyl group of G acts as an order-reversing involution $PgQ \mapsto Pw_0gQ$. Given an ideal $\mathrm{Th} \subset P\backslash G/Q$ and a subset $S \subset G/P$, we denote by $\mathrm{Th}(S)$ the set of all points in G/Q whose position relative to some point in S lies in Th. Taking $Q = P$ and $\mathrm{Th} \subset P\backslash G/P$ to be the ideal consisting of all elements in $P\backslash G/P$ except the greatest element Pw_0P, the conclusion of the previous paragraph reads as follows: the subset $\mathrm{Th}(\Lambda_\Gamma) \subset G/P$ is closed and Γ-invariant, and Γ acts properly on the complement $G/P - \mathrm{Th}(\Lambda_\Gamma)$.

With the aim of constructing larger domains of proper discontinuity for Γ, Kapovich, Leeb, and Porti [3] observed that, in fact, the weak form of convergence satisfied by the Γ-action on G/P implies that for any proper parabolic subgroup $Q < G$ and any ideal $\mathrm{Th} \subset P\backslash G/Q$ with the property that $P\backslash G/Q = \mathrm{Th} \cup w_0\mathrm{Th}$, the subset $\mathrm{Th}(\Lambda_\Gamma) \subset G/Q$ is closed and Γ-invariant, and Γ acts properly on the complement $G/Q - \mathrm{Th}(\Lambda_\Gamma)$. The quotient $\Gamma\backslash(G/Q - \mathrm{Th}(\Lambda_\Gamma))$ is then an orbifold endowed with a $(G, G/Q)$-structure. This establishes a link between P-divergent representations $\Gamma \to G$ and $(G, G/Q)$-structures on $\Gamma\backslash(G/Q - \mathrm{Th}(\Lambda_\Gamma))$ that had already been fruitfully explored in the higher-rank setting by Guichard and Wienhard [1, 2].

References

[1] O. Guichard and A. Wienhard, *Convex foliated projective structures and the Hitchin component for* $\mathrm{PSL}_4(\mathbb{R})$, Duke Math. J., 144:381-445, 2008.

[2] O. Guichard and A. Wienhard, *Anosov representations: Domains of discontinuity and applications*, Invent. Math., 190:357-438, 2012.

[3] M. Kapovich, B. Leeb, and J. Porti. *Dynamics on flag manifolds: Domains of proper discontinuity and cocompactness*, Geom. Topol., 22:157-234, 2018.

Pressure metrics

Cagri Sert

The talk consisted of three parts. In the first part, we stated the main results and indicated the proof roadmap; this consists of roughly three parts: first, associating a Hölder function (via a flow) to a projective Anosov representation; second, bringing in the convexity of pressure functional and relating the pressure semi-definite form to the intersection numbers; and third, showing that the form is positive definite. In the last two parts of the talk, we spoke about the first two stages of the proof (in the reverse order). For the exposition, we mainly followed the original reference [4] as well as the more recent paper and survey [3, 5].

1. Statements of results and proof roadmap

We start by introducing some terminology and notation needed to state the main results.

Let Γ be a finitely generated word-hyperbolic group and G be a closed reductive subgroup of $\mathrm{SL}_d(\mathbb{R})$. The main example for such a group Γ is the fundamental group $\pi_1(S)$ of an orientable, closed surface of genus ≥ 2, but the class of word-hyperbolic groups contains many other examples (virtually free groups, free products etc.). An element g of G is called *generic* if its centralizer is a maximal torus in G (for example, if $G = \mathrm{SL}_n(\mathbb{R})$ any complex-diagonalizable matrix with distinct eigenvalues is generic). A representation $\Gamma \to G$ is called generic if its image contains a generic element. Finally, a representation $\Gamma \to G$ is called *regular* if its a smooth point of the algebraic variety $\mathrm{Hom}(\Gamma, G)$. Let $C(\Gamma, G)$ denote the space of (conjugacy classes) of regular, irreducible, projective Anosov representations of Γ in $G < \mathrm{SL}_d(\mathbb{R})$. Let $C_g(\Gamma, G)$ be the space of (conjugacy classes of) those representations in $C(\Gamma, G)$ that are additionally generic.

Remark. 1. It is shown in [4, §6] that the sets $C(\Gamma, G)$ and $C_g(\Gamma, G)$ have natural analytic structures compatible with the algebraic structure of $\mathrm{Hom}(\Gamma, G)$ (as quotients by a free, proper and algebraic action).
2. When $\Gamma = \pi_1(S)$, a Hitchin component for representations of Γ in $\mathrm{PSL}_d(\mathbb{R})$ lifts to component of $C_g(\Gamma, G)$.

The following is the main result of [4] obtained in combination with the previous work of Wolpert [7] and Bonahon [1].

Theorem 1 (Bridgeman–Canary–Labourie–Sambarino [4])**.** *Keep the above setting.*
1. There exists a Riemmanian metric, called the pressure metric, on $C_g(\Gamma, G)$ that is Out(Γ)-invariant.
2. When $\Gamma = \pi_1(S)$ and $G = SL_n(\mathbb{R})$, the restriction of this metric to the Fuchsian locus coincides with Weil–Petersson metric.

The proof of this result is very dynamical in nature and consists of several parts. The outcome of the dynamical argument consist of two things. One of them Theorem 2 below which yields a semi-metric on $C(\Gamma, G)$. The other part of the outcome is a characterization of when the semi-metric degenerates. Theorem 1 is then deduced by showing that the degeneration does not happen when the semi-metric given by Theorem 2 is restricted to $C_g(\Gamma, G)$. In this note, we will focus on the elements of the proof of the following.

Theorem 2. *Keep the above setting. There exists a positive, Out(Γ)-invariant analytic map $J : C(\Gamma, G) \times C(\Gamma, G) \to \mathbb{R}$ such that for every analytic path γ :*

$(-1,1) \to C(\Gamma, G)$, *letting* $J_\gamma(t) := J(\gamma(0), \gamma(t))$, *we have*

$$\frac{d}{dt}_{|t=0} J_\gamma = 0 \qquad \textit{and} \qquad \frac{d^2}{dt^2}_{|t=0} J_\gamma \geq 0.$$

In other words, the Hessian of J on yields a non-negative analytic 2-tensor field on $C(\Gamma, G)$ (pressure form). Theorem 1 is deduced from Theorem 2 by showing additionally that the restriction of the pressure form to $C_g(\Gamma, G)$ is positive definite.

The rest of the note will focus on exposing elements of proof of the previous theorem. The proof goes by associating a flow (on a compact metric space) to a representation in $C(\Gamma, G)$ which is shown to be a Hölder reparametrization of the standard (or Gromov) flow of Γ on $U_0\Gamma$ which therefore yields a (Livsic class of) Hölder function(s). The authors show that this association is analytic and after constructing the pressure form on the pressure zero Hölder functions and relating it with an intersection number functional J_0, one obtains J by precomposiong with the above-described association.

2. Metric Anosov flows, pressure and intersection numbers

2.1. **Metric Anosov flows.** Let X be a compact Riemannian manifold and Φ_t a smooth flow. Recall that the flow is called *uniformly hyperbolic* (in particular *Anosov*) if there exists a flow-invariant splitting of the tangent bundle $TX = E^s \oplus E^o \oplus E^u$ such that E^u is the flow direction, E^s is uniformly contracted in the future (as $t \to \infty$) and E^u is uniformly contracted in the past (as $t \to -\infty$).

A *metric Anosov flow* is a generalization of an Anosov flow in a setting where the underlying space X has only a metric space structure. We refer to [4, 6] for its definition noting that many of the ergodic properties of Anosov flows were shown to hold for metric Anosov flows by Pollicott [6].

2.2. **Hölder reparametrizations of flows.** Let Φ_t be a topologically transitive metric Anosov flow on a compact metric space X. Given a Hölder function $f : X \to R_{>0}$, we define the f-reparametrization Φ^f of Φ by $\Phi^f_t(x) = \Phi_{\alpha_f(t,x)}(x)$, where $\alpha_f(t,x)$ is defined as the unique real satisfying $t = \int_0^{\alpha_f(t,x)} f(\Phi_s(x))ds$. This is another continuous flow Hölder orbit equivalent to Φ.

2.3. **Topological entropy and pressure.** For $T > 0$, let $R_T(\Phi)$ be the set of periodic orbits of Φ of period at most T. Following Bowen [2], we define the *topological entropy* $h(\Phi)$ as $\limsup_{T\to\infty} \frac{\log \# R_T(\Phi)}{T}$.

Given a Hölder function (a potential) $f : X \to R$, we define its *topological pressure* $P_\Phi(t)$ with respect to the flow Φ to be

$$\limsup_{T\to\infty} \frac{1}{T} \log \sum_{a \in R_T(\Phi)} e^{\ell_f(a)},$$

where $\ell_f(a)$ is the period of $a \in R_T(\Phi)$ for the flow Φ^f. By the variational principle, $P_\Phi(t)$ is equal to the supremum of $h_\mu(\Phi) + \int f d\mu$, where μ varies over Φ-invariant probability measures and $h_\mu(\Phi)$ is the Kolmogorov entropy of (Φ, μ).

A measure attaining the supremum in the variational principle is called an equilibrium measure for f. In our setting, there is a unique equilibrium measure [4, 6], we denote it by m_f.

Theorem 3 (Ruelle, Parry–Pollicott). *Let $f, g : X \to \mathbb{R}$ be Hölder functions.*
1. $\frac{\partial P(f+tg)}{\partial t}_{|t=0} = \int g dm_f$.
2. If $\int g dm_f = 0$, *then* $\frac{\partial^2 P(f+tg)}{\partial t^2}_{|t=0} =: var(g, m_f) \geq 0$.
3. $var(g, m_f) = 0$ *if and only if g is Livsic cohomologous to zero.*

To define the pressure semi-norm; on considers the space $P(X)$ of pressure zero Hölder functions (for some fixed exponent). For $f \in P(X)$, one identifies the tangent space $T_f P(X)$ with $\{g : \int g dm_f = 0\}$. On $T_f P(X)$, one defines the pressure semi-norm by setting $\|g\|_P^2 = \text{var}(g, m_f)$.

2.4. **Intersection numbers.** Let f, g be positive Hölder functions on X. We define their *intersection number* $I(f, g)$ as

$$\lim_{T\to\infty} \frac{1}{\#R_T(\Phi^f)} \sum_{a\in R_T(\Phi^f)} \frac{\ell_g(a)}{\ell_f(a)}.$$

The limit exists in view of an equidistribution result of Bowen [2]. Finally, we define the *normalized intersection number* $J_0(f, g)$ as $\frac{h_g(\Phi)}{h_f(\Phi)} I(f, g)$.

It turns out that for two positive Hölder functions f, g, we have $J_0(f, g) \geq 1$ and $J_0(f, g) = 1$ if and only if the Hölder reparametrized flows associated to $h_f(\Phi)f$ and $h_g(\Phi)g$ are Hölder conjugate (i.e. there is Hölder homeomorphism of X equivariant with respect to two flows). The following result relates the pressure sermi-norm to the normalized intersection (and therefore relates the pressure semi-norm to the Weil–Petersson metric via the results of Wolpert [7] and Bonahon [1]).

Proposition 4. *Let $(f_t)_{t\in(-1,1)}$ be an analytic family of positive Hölder functions of a fixed exponent on X. Let $\psi_t = -h_{f_t}(\Phi) f_t$ the pressure zero form of f_t. Then,* $\frac{\partial^2}{\partial t^2}_{|t=0} J_0(f_0, f_t) = \|\dot{\psi}_0\|_p^2$.

In other words, the pressure semi-norm on $T_{\Psi_0} P(X)$ is given by the symmetric 2-tensor corresponding to the Hessian of J_0.

3. From representations to flows

Given a representation $\Gamma \to G$ in $C(\Gamma, G)$, the goal here is to define a flow on a compact metric space which is Hölder conjugate to a Hölder reparametrization of the standard flow of Γ. This allows to pull-back the pressure form to $C(\Gamma, G) \times C(\Gamma, G)$ appearing in Theorem 2. Let $\rho \in C(\Gamma, G)$. Recall that there exist boundary maps $\xi : \partial\Gamma \to P(\mathbb{R}^n)$ and $\theta : \partial\Gamma \to P(\mathbb{R}^n)^*$ where $\partial\Gamma$ is the Gromov- boundary of Γ and $P(\mathbb{R}^n)$ and $P(R^n)^*$ are respectively the projective space of $\mathbb{R}^n$ and its dual. For $x \in \partial\Gamma$, choose $v_x \in \mathbb{R}^n$ such that $\mathbb{R}v_x = \xi(x)$. Let $F_\rho = \partial\Gamma^{(2)} \times R_{>0}$ where $\partial\Gamma^{(2)}$ is the Cartesian product of $\partial\Gamma$ with itself minus the diagonal. The space F_ρ is endowed with commuting Γ and $\mathbb{R}$ actions given respectively by $\gamma(x,y,s) = (\gamma x, \gamma y, \frac{\|\rho(\gamma)v_x\|}{\|v_x\|}s)$ and $t(x,y,s) = (x,y,e^t s)$. Before stating the following main result of this part, recall that the spaces $\widetilde{U_0\Gamma}$ and F_ρ has metrics adapted to their topology with respect to which Γ acts by isometries.

Theorem 5. [4, §4 and §5] *Let $\rho \in C(\Gamma, G)$.*
1.The Γ-action on F_ρ is proper and cocompactly — denote the quotient space F_ρ/Γ by $U_\rho\Gamma$.
2. The $\mathbb{R}$-action on $U_\rho\Gamma$ is a metric Anosov flow — denote it by Φ_t.
3. The flow Φ_t is Hölder conjugate to a Hölder reparametrization of the standard flow on $U_0\Gamma$.
4. The Φ-orbit associated to an infinite order primitive element γ has period equal to the logarithm of the spectral radius of $\rho(\gamma)$.

Therefore, this result allows us to associate a Hölder function (or a Livsic class of such functions) on $U_0\Gamma$ to a representation $\rho \in C(\Gamma, G)$. The authors also show that this association varies analytically when ρ varies in an analytic family [4, §6 and §7]. This constitute the second piece in the proof of Theorem 2 combined with Proposition 4.

References

[1] Bonahon, F. *The geometry of Teichmüller space via geodesic currents.* Invent. Math. 92(1988), 139–162.
[2] Bowen, R. *Periodic orbits of hyperbolic flows.* Amer. J. Math. 94(1972), 1–30.
[3] Bridgeman, M., Canary, R. and Sambarino, A. *An introduction to pressure metrics for higher Teichmüller spaces.* Ergodic Theory and Dynamical Systems 38, no. 6 (2018): 2001–2035.
[4] Bridgeman, M., Canary, R., Labourie, F. and Sambarino. A. *The pressure metric for Anosov representations.* Geometric and Functional Analysis 25, no. 4 (2015): 1089–1179.
[5] Bridgeman, M., Canary, R., Labourie, F. and Sambarino. A. *Simple root flows for Hitchin representations.* Geometriae Dedicata 192, no. 1 (2018): 57–86.
[6] Pollicott, M. *Symbolic dynamics for Smale flow.* Am. J. of Math. 109(1987), 183–200.
[7] Wolpert, S. *Thurston's Riemannian metric for Teichmüller space.* J. Diff. Geom. 23(1986), 143–174.

Fock-Goncharov coordinates and cluster varieties

Jared Miller
(mentored by D. Kaufmann)

In [1], Fock and Goncharov show that if S is a closed surface then a positive representation $\rho : \pi_1(S) \to \mathrm{PSL}_m(\mathbb{R})$ is discrete and faithful. Further, they show that for a surface S without boundary and G a split semisimple Lie group with trivial center, the moduli space of positive representations from $\pi_1(S)$ into G coincides with the Hitchin component in the representation space of $\pi_1(S)$ into $\mathrm{PSL}_m(\mathbb{R})$. Given this, we aim to understand the moduli space of framed representations in the style of Fock-Goncharov. In this talk, we describe the moduli space of framed representations for the fundamental group of a surface with boundary into $\mathrm{PGL}_m(\mathbb{C})$, which generalizes to closed surfaces without boundary.

In the first part of the talk, we introduce spaces of representations. Given some compact surface S with boundary and the group $G = \mathrm{PGL}_m(\mathbb{C})$, a representation of $\pi_1(S)$ into G is a homomorphism $\rho \in \hom(\pi_1(S), G)$ and the moduli space of representations is $\mathcal{R}_G(S) = \hom(\pi_1(S), G)/G$ where the action of G is by conjugation. We can decorate ∂S with marked points $\{p_1, \cdots, p_j\}$ and consider the so called ciliated surface $\hat{S} = S \setminus \{p_1, \cdots p_j\}$. Such a surface has boundary $\partial\hat{S}$ which consists of arcs and loops, with loops coming from boundary components of S without any marked points.

We define a flag as a nested sequence of m many vector subspaces in $\mathbb{C}^m$ such that the index of a subspace in the sequence is equal to its dimension. Given a ciliated surface $\hat{S}$, a framed representation consists of

(1) a representation $\rho \in \hom(\pi_1(S), \mathrm{PGL}_m(\mathbb{C}))$
(2) flags $(F^{(1)}, \cdots, F^{(t)})$ in $\mathbb{C}^m$ associated to each connected component of $\partial\hat{S}$ such that if $C_j \in \pi_1(S)$ corresponds to a boundary curve with no marked points, the associated flag is invariant under left multiplication by $\rho(C_j)$

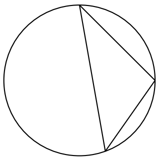
The moduli space of framed representations is defined as the quotient of the set of framed representations by the action of $\mathrm{PGL}_m(\mathbb{C})$, where $\mathrm{PGL}_m(\mathbb{C})$ acts on representations by conjugation and on flags by left multiplication.

The second part of the talk focuses on the so called Fock-Goncharov coordinates. In particular, we describe how to obtain Fock-Goncharov coordinates from a framed representation and triangulation of the associated ciliated surface. We define triangulations of a ciliated surface $\hat{S}$ as a maximal collection of disjoint, nonhomotopic, simple curves joining connected components of $\partial\hat{S}$ and discuss configuration spaces of triples and quadruples of flags in $\mathbb{C}^m$. Given a flag F, let F_i denote the i dimensional subspace in the flag. We say that a collection of k flags $\{A^{(1)}, A^{(2)}, \cdots, A^{(k)}\}$ in $\mathbb{C}^m$ is in generic position if for every $i_1 + i_2 + \cdots + i_k = m$ we have $A^{(1)}_{i_1} \oplus A^{(2)}_{i_2} \oplus \cdots \oplus A^{(k)}_{i_k} = \mathbb{C}^m$. Given a quadruple of flags (A, B, C, D) in $\mathbb{C}^2$ in generic position, taken as points in $\mathbb{C}P^1$, by transitivity of $\mathrm{PGL}_2(\mathbb{C})$ we can map (A, B, C) to $(0, -1, \infty)$ with D sent to $x \in \mathbb{C}P^1$. x is given by the cross ratio of the four points (A, B, C, D) and can be computed as

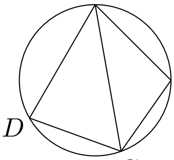
$$\frac{(A-D)(B-C)}{(A-B)(C-D)}$$

It is well known that the cross ratio is invariant under action by $\mathrm{PGL}_2(\mathbb{C})$ and that the cross ratio of four points in $\mathbb{C}P^1$ is real if and only if the four points lie on a circle or line. We note that with this normalization we have that a real cross ratio is positive if and only if the triangles (A, B, C) and (A, C, D) are disjoint.

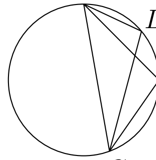
For a framed representation into $\mathrm{PGL}_2(\mathbb{C})$, to each interior edge (A, C) between triangles (A, B, C) and (A, C, D) in a triangulation we assign the cross ratio of (A, B, C, D) to the edge (A, C). These cross ratios are the Fock-Goncharov coordinates in dimension 2. We describe how for a framed representation into



$\mathbb{R}P^1$



Positive Real



Negative Real

$\mathrm{PGL}_m(\mathbb{C})$, for each interior edge of the triangulation we project the four flags at the vertices of the adjacent triangles to $m-1$ different subspaces of dimension 2 to obtain $m-1$ cross ratios assigned to the edge (A, C). These $m-1$ edge invariants are assigned to each interior edge of the triangulation and are one of two types of Fock-Goncharov coordinates in dimension $m \geq 3$.

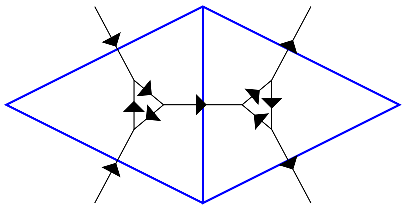
We then discuss triples of flags (A, B, C) in $\mathbb{C}^3$, given by a point and line in $\mathbb{C}P^2$. If we choose direction vectors v_a, v_b, v_c for the one dimensional subspaces in flags A, B, C and linear form representatives $f_a, f_b, f_c \in (\mathbb{C}^3)^*$ defining the two dimensional subspaces in A, B, C, then the triple ratio of (A, B, C) is given by

$$\frac{f_a(v_b)\ f_b(v_c)\ f_c(v_a)}{f_a(v_c)\ f_b(v_a)\ f_c(v_b)}$$

As with the cross ratio, the triple ratio is invariant under action by $\mathrm{PGL}_3(\mathbb{C})$. For a framed representation into $\mathrm{PGL}_3(\mathbb{C})$, to each triangle (A, B, C) in a triangulation, we assign the triple ratio $r_3(A, B, C)$ to that triangle. These triple ratios are the second type of Fock-Goncharov coordinates in dimension 3. We describe how to project a triple of flags in $\mathbb{C}^m$ to $(m-1)(m-2)/2$ subspaces of dimension 3, to obtain triangle invariants assigned to the triangle. Given a framed representation and triangulation, the collection of $m-1$ cross ratios on each interior edge of the triangulation and $(m-1)(m-2)/2$ triple ratios on each triangle are the Fock-Goncharov coordinates in dimension m.

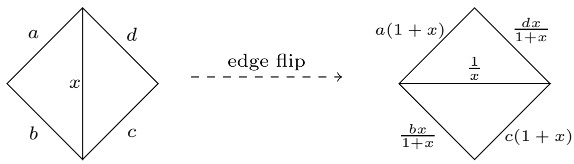
In part three of the talk, we describe how to construct a framed representation given a triangulation of a ciliated surface $\hat{S}$ and collection of Fock-Goncharov coordinates therein. The construction essentially consists of finding two things: the representation and the flags associated to each connected component of $\partial\hat{S}$. Determining the flags up to action by $\mathrm{PGL}_m(\mathbb{C})$ is done by choosing a triangle in the triangulation and normalizing so that the flags at two vertices and the one dimensional part of the flag at the third vertex are arbitrary and in generic position. The Fock-Goncharov coordinates then describe the remaining flags, which can be computed explicitly. For the representation, we construct an oriented graph dual to the triangulation, as illustrated in the figure below representing part of a triangulation and embedded graph. To each edge of the embedded graph, we assign a matrix in $\mathrm{PGL}_m(\mathbb{C})$. These matrices have explicit formulas, but matrices associated to edges of the graph inside a triangle in the triangulation can be seen as the transformations mapping the flags (A, B, C) at the vertices to (B, C, A)

and matrices associated to edges of the graph crossing an edge (A, C) in the triangulation can be seen as transformations mapping the flags $A \leftrightarrow C$ and $B_1 \mapsto D_1$.



Each curve in $\pi_1(S)$ is homotopic to a curve on the embedded graph and the representation can be found by taking the product of matrices associated to the corresponding edges in the embedded graph.

In the final part of the talk, we discuss positive representations and the role that the triangulation plays in the Fock-Goncharov coordinates. The triangulation does affect the coordinates, but it does so in a well defined way. For a framed representation into $\mathrm{PGL}_2(\mathbb{C})$, we show that an edge flip in the triangulation changes the Fock-Goncharov coordinates in a subtraction free way.



We quickly describe quivers and how to use a sequence of cluster mutations to determine the Fock-Goncharov coordinates of a framed representation into $\mathrm{PGL}_m(\mathbb{C})$ corresponding to a different triangulation and again note that these change of coordinate formulas are subtraction free. We conclude the talk by defining positive representations as framed representations which have all positive Fock-Goncharov coordinates with respect to any triangulation and stating relevant results of [1].

References

[1] V. V. Fock and A. B. Goncharov, *Moduli spaces of local systems and higher Teichmüller theory*, Publ. Math. Inst. Hautes Etudes Sci., 103:1–211, 2006

[2] D. Kaufman, *Invariant Functions On Cluster Ensembles*, Preprint: https://arxiv.org/abs/2010.15179, 2020

[3] I. Le, *An approach to higher Teichmüller spaces for general groups*, Int. Math. Res. Not. IMRN, (16):4899–4949, 2019

[4] I. Le, *Cluster structures on higher Teichmüller spaces for classical groups*, Forum Math. Sigma, 7:Paper No. e13, 165, 2019

[5] F. Palesi, *Introduction to Positive Representations and Fock-Goncharov Coordinates*, working paper or preprint, 2013

Higgs Bundles

Tianqi Wang

Let X be a closed Riemann surface of genus at least 2. We denote the collection of smooth (p,q)-forms with coefficients in a smooth $\mathbb{C}$-vector bundle E over X by $A^{p,q}(E)$.

Let (E,h) be a hermitian vector bundle over X. For a connection ∇ on E, there is a unique decomposition $\nabla = \nabla_h + \Phi_h$, such that ∇_h is a hermitian connection and Φ_h is symmetric with respect to h, that is, the adjoint $\Phi_h^{*_h}$ equals Φ_h. Therefore, we have a one-to-one correspondence defined as

$$\begin{aligned}\{\bar\partial\text{-operator on } E\} \times A^{1,0}(\mathrm{End}(E)) &\longleftrightarrow \{\text{connections on } E\}\\ (\bar\partial_E, \varphi) &\longmapsto \bar\partial_E + \partial_h + \varphi + \varphi^{*_h}\\ (\nabla_h^{0,1}, \Phi_h^{1,0}) &\longleftarrow\!\shortmid \nabla = \nabla_h + \Phi_h,\end{aligned} \tag{1}$$

where $\nabla_h = \bar\partial_E + \partial_h$ is the Chern connection of the holomorphic hermitian vector bundle $(E, \bar\partial_E, h)$, $\nabla_h^{0,1}$ is the $(0,1)$-part of the connection ∇_h, and $\Phi_h^{1,0}$ is the $(1,0)$-part of Φ_h.

Definition 1 (Hitchin[3])**.** Let $(E, \bar\partial_E)$ be a holomorphic vector bundle and $\varphi \in A^{1,0}(\mathrm{End}(E))$. The triple $(E, \bar\partial_E, \varphi)$ is called a *Higgs bundle* if $\bar\partial_E \varphi = 0$. Such φ is called a *Higgs field*.

The nonabelian Hodge correspondence states that there is a one-to-one correspondence between the conjugacy classes of semisimple representations of the fundamental group $\pi_1(X)$ into $\mathrm{GL}(r, \mathrm{C})$, and the isomorphism classes of polystable Higgs bundles of rank r and degree 0 over X.

We can replace the conjugacy classes of semisimple representations by the isomorphism classes of flat $\mathbb{C}$-vector bundles of rank r over X, followed from the Riemann-Hilbert correspondence.

The relation of flat vector bundles and polystable Higgs bundles can be built by introducing an intermediate notion, called the harmonic bundles.

Definition 2. We call a flat hermitian vector bundle (E, ∇, h) a *harmonic bundle* if $\nabla_h^{0,1}\Phi_h^{1,0} = 0$, up to the correspondence (1). Equivalently, we say a Higgs bundle with a hermitian metric $(E, \bar\partial_E, \varphi, h)$ is a harmonic bundle if the connection $\nabla = \bar\partial_E + \partial_h + \varphi + \varphi^{*_h}$ is flat, up to the correspondence (1).

Actually, $\nabla = \bar\partial_E + \partial_h + \varphi + \varphi^{*_h}$ being flat is equivalent to the conditions that

$$\begin{cases} \bar\partial_E \varphi = 0 \\ F_{\bar\partial_E, h} + [\varphi, \varphi^{*_h}] = 0, \end{cases} \tag{2}$$

where the second equation is called the Hitchin equation (the self-dual equation in [3]) and $F_{\bar\partial_E,h}$ denotes the curvature form of the Chern connection with respect to the pair $(\bar{\partial}_E, h)$. The first equation is equivalent to that h is a harmonic metric, that is, it minimizes the energy functional

$$\mathbb{E}(\nabla, h) = \int_X \mathrm{tr}(\varphi \wedge \varphi^{*_h}).$$

The nonabelian Hodge correspondence can be shown by the following series of theorems (see [1],[2],[3],[5],[7],[8] etc.).

Theorem 3 (Donaldson[2],Corlette[1])**.** *Let (E, ∇) be a flat irreducible vector bundle over X, then there exists a hermitian metric h, unique up to scaling, such that (E, ∇, h) is harmonic, i.e., $\nabla_h^{0,1}\Phi_h^{1,0} = 0$.*

The idea of the proof is finding the hermitian metric h that minimizes the energy function. The problem can be simplified by fix a hermitian metric and apply “the gauge transformations”. The proper “gauge transformation” can be found by applying Uhlenbeck’s compactness theorem [9].

Proposition 4 (Corlette [1])**.** *Let (E, ∇, h) be a harmonic vector bundle, then the flat vector bundle (E, ∇) is semisimple.*

Theorem 5 (Kobayashi[5])**.** *Let $(E, \bar{\partial}_E, h), \varphi, h)$ be a harmonic vector bundle, then the Higgs bundle $(E, \bar{\partial}_E, \varphi)$ is polystable.*

Theorem 6 (Hitchin[3],Simpson[7],[8])**.** *Let $(E, \bar{\partial}_E, \varphi)$ be a stable Higgs bundle, then there exists a hermitian metric h, unique up to scaling, such that $(E, \bar{\partial}_E, \varphi, h)$ is harmonic, i.e., $F_{\bar{\partial}_E,h} + [\varphi, \varphi^{*_h}] = 0$.*

The proof of the above theorem is similar to the proof of Corlette’s theorem by minimizing the Yang-Mills-Higgs functional

$$\mathrm{YMH}(\bar{\partial}'_E, \varphi') = \| F_{\bar{\partial}'_E,h} + [\varphi', \varphi'^{*_h}] \|^2$$

in the space $\mathrm{HP} = \{(\bar{\partial}'_E, \varphi') \ : \ \bar{\partial}'_E \varphi' = 0\}$.

One important example is the Hitchin section introduced in [4] (see also [6]). Let $\mathcal{M}_{Higgs}(\mathrm{SL}(n, \mathbb{C}))$ denote the isomorphism classes of polystable Higgs bundles corresponding to the conjugacy classes of semisimple representations of $\pi_1(X)$ into $\mathrm{SL}(r, \mathrm{C})$ by the nonabelian Hodge correspondence. Let K denote the canonical bundle over X. The *Hitchin fibration* is the map

$$\begin{aligned} h : \mathcal{M}_{Higgs}(\mathrm{SL}(n, \mathbb{C})) &\longrightarrow \bigoplus_{j=2}^{n} \mathbb{H}^0(K^j) \\ (E, \varphi) &\longmapsto (\mathrm{tr}(\varphi^2), \mathrm{tr}(\varphi^3), ..., \mathrm{tr}(\varphi^n)), \end{aligned}$$

and the *Hitchin section* is given by

$$s(q_2, q_3, ..., q_n) = (K^{\frac{n-1}{2}} \oplus K^{\frac{n-3}{2}} \oplus ... \oplus K^{\frac{3-n}{2}} \oplus K^{\frac{1-n}{2}}, \begin{pmatrix} 0 & q_2 & q_3 & ... & q_{n-1} & q_n \\ r_1 & 0 & q_2 & ... & ... & q_{n-1} \\ 0 & r_2 & 0 & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & 0 & q_2 \\ 0 & ... & ... & 0 & r_{n-1} & 0 \end{pmatrix}).$$

The conjugacy classes of representations, that correspond to the points in the image of the Hitchin section by the nonabelian Hodge correspondence, have holonomy in $\mathrm{SL}(n,\mathbb{R})$, which actually give the Hitchin component $\mathrm{Hit}_n(X)$ in the character variety.

References

[1] K. Corlette. Flat G-bundles with canonical metrics. *J. Differential Geom.,* 28(3):361–382, 1988.

[2] S. K. Donaldson. Twisted harmonic maps and the self-duality equations. *Proc. London Math. Soc. (3)*, 55(1):127–131, 1987.

[3] N. J. Hitchin. The self-duality equations on a Riemann surface. *Proc. London Math. Soc. (3),* 55(1):59–126, 1987.

[4] N. J. Hitchin. Lie groups and Teichm¨uller space. *Topology,* 31(3):449–473, 1992.

[5] S. Kobayashi. Curvature and stability of vector bundles, *Proc. Japan Acad. Ser. A. Math. Sci.,* 58, 158–162, 1982.

[6] Q. Li. An introduction to Higgs bundles via harmonic maps.*SIGMA Symmetry Integrability Geom. Methods Appl.,* 15:Paper No. 035, 30, 2019.

[7] C.T. Simpson, Constructing variations of Hodge structure using Yang–Mills theory and applications to uniformization, *J. Amer. Math. Soc.* 1, 867–918, 1988.

[8] C. T. Simpson. Higgs bundles and local systems. *Inst. Hautes ´Etudes Sci. Publ. Math.,* (75):5–95, 1992.

[9] K. K. Uhlenbeck. Connections with L^p, bounds on curvature. *Communications in Mathematical Physics,* 83(1):31–42, 1982.

Projections to Teichmüller space

Nicholas Rungi

Let S be a closed oriented surface of genus $g \geq 2$. For any complex structure J on S we denote with $\Sigma = (S, J)$ the associated Riemann surface structure, seen as a point in Teichmüller space $\mathcal{T}(S)$. For any fixed $\Sigma \in \mathcal{T}(S)$, given a connected simple complex Lie group G with adjoint split real form G_0, Hitchin ([12]) found a connected component in the associated G_0-character variety parametrized by a complex vector space and diffeomorphic to an open ball of real dimension $-\chi(S)\mathrm{dim}G_0$. Such a component $\mathrm{Hit}(S, G_0)$ is now-known as the *Hitchin component* of the G_0-character variety and it contains a copy of $\mathcal{T}(S)$ induced by the principal embedding $\mathrm{PSL}(2,\mathbb{R}) \hookrightarrow G_0$ ([13]). These were the first examples of higher rank Teichmüller components. The aforementioned parametrization gives rise to a complex structure on $\mathrm{Hit}(S, G_0)$ which is not invariant under the action of the mapping class group $\mathrm{Mod}(S)$ since it depends on the initial choice of a Riemann surface structure. In [15] Labourie proposed a strategy to solve this issue when $G_0 = \mathrm{PSL}(n,\mathbb{R})$, which can be easily generalized to any split real form. Let $\rho : \pi_1(S) \to G_0$ be a reductive representation (i.e. the Zariski clousure of $\rho(\pi_1(S))$ is dense in G_0), then for any $\Sigma \in \mathcal{T}(S)$ let us denote with $f_\rho : \widetilde{\Sigma} \to G_0/K_0$ the (unique) ρ-equivariant harmonic map from the universal cover of Σ to the symmetric space of G_0 ([10],[9]). For any reductive ρ, let e_ρ be the *energy functional* which associates to any $\Sigma = (S, J) \in \mathcal{T}(S)$ the *energy* of f_ρ. The critical points of this functional are called *conformal maps* and they can be characterized as those

f_ρ's for which the *Hopf differential* $\mathrm{H}(f_\rho)$ (the (2,0) part of $f_\rho^* g_{G_0/K_0}$) vanishes ([17]). Labourie proved that for any ρ in the Hitchin component (actually for any Anosov representation) the energy functional e_ρ is smooth and proper ([15]). In particular, there exists a Riemann surface structure $\Sigma \in \mathcal{T}(S)$ such that the associated ρ-equivariant harmonic map f_ρ is conformal.

Labourie's conjecture (split case): For any $\rho \in \mathrm{Hit}(S, G_0)$ the associated ρ-equivariant conformal map $f_\rho : \widetilde{\Sigma} \to G_0/K_0$ is unique.

It is clear from the preceding argument that such conjecture can be stated also for simple real Lie groups of Hermitian type replacing Hitchin representations with the maximal ones. In order to understand how this conjecture is related to the problem of the $\mathrm{Mod}(S)$-invariant parametrization of $\mathrm{Hit}(S)$, let us denote with $\mathcal{E}$ the holomorphic vector bundle on Teichmüller space whose fiber over a point J is $\mathcal{E}_J = \bigoplus_{i=2}^{l} H^0(\Sigma, K^{m_i+1})$, where l is the rank of G_0 and $(1, m_2, \dots, m_l)$ are its exponents. For any element $\big(J, (q_{m_2+1}, \dots, q_{m_l+1})\big) \in \mathcal{E}$, let $\rho_{0, q_{m_2+1}, \dots, q_{m_l+1}}$ be the monodromy of the solution to the self-duality equations of the Higgs bundles constructued by Hitchin ([12]) using the t-uple
$(0, q_{m_2+1}, \dots, q_{m_l+1})$. This gives a well-defined map

$$\Phi : \mathcal{E} \longrightarrow \mathrm{Hit}(S, G_0)$$

Moreover, since $\mathrm{Mod}(S)$ acts on the representations space by pre-composition and on $\mathcal{E}$ by pull-back, from the previous construction follows that the map Φ is $\mathrm{Mod}(S)$-equivariant. By the work of Labourie Φ is surjective since for any Hitchin representation ρ there exists a ρ-equivariant conformal map $f_\rho : \widetilde{\Sigma} \to G_0/K_0$ and the t-uple $(q_2, q_{m_2+1}, \dots, q_{m_l+1})$ corresponding to ρ via the *Hitchin section* has the holomorphic quadratic differential q_2 equal to zero ($q_2 = c \cdot \mathrm{H}(f_\rho)$, for some $c \in \mathbb{C}$). This implies that the element $\big(J, (0, q_{m_2+1}, \dots, q_{m_l+1})\big)$ belongs to $\mathcal{E}$ and it is mapped to ρ by Φ. In the end, if the conjecture were true, the above map would be injective (unique conformal map f_ρ) and it would induce a $\mathrm{Mod}(S)$-invariant parametrization of $\mathrm{Hit}(S, G_0)$ as a holomorphic vector bundle over Teichmüller space.

Theorem 1 ([18],[14],[16])**.** *If G_0 is a split simple real Lie group of rank 2 with finite center, then Labourie's conjecture holds.*

Theorem 2 ([7],[1],[8])**.** *If G_0 is a simple real Lie group of rank 2 and of Hermitian type, then Labourie's conjecture holds.*

It should also be added that the uniqueness of the conformal map holds in the semisimple case with $G = \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ ([21]), and it was the first group, after $\mathrm{PSL}(2,\mathbb{R})$ ([23]), for which existence and uniqueness could be proved. Historically, the first proof for a simple group of higher rank was found independently by Loftin ([18]) and Labourie ([14]) when $G_0 = \mathrm{PSL}(3,\mathbb{R})$. In this case the argument is geometric and there is no need to use the tools of harmonic and conformal maps in symmetric spaces. Indeed, the connected component $\mathrm{Hit}(S, \mathrm{PSL}(3,\mathbb{R}))$ can be parametrized as the space of (properly) convex $\mathbb{RP}^2$-structure on S ([11],[6]). In particular, thanks to a highly non-trivial theorem ([4],[5]) the datum of a (properly) convex $\mathbb{RP}^2$-structure on the surface is the same as the datum of an immersion $\widetilde{S} \to \mathbb{R}^3$ as a hyperbolic (equivariant) affine sphere. Such immersion is completely determined by the Blaschke metric h (second fundamental form) and the Pick form, i.e. the difference between the restriction of the flat connection on $\mathbb{R}^3$ and the h-Levi-Civita connection. Such tensors descend on the surface by equivariance and they give rise to a pair (J, q) with q a cubic holomorphic differential with $\Re(q)$ equal to the Pick form (up to a contraction with the metric). This gives a $\mathrm{Mod}(S)$-invariant parametrization of $\mathrm{Hit}(S, \mathrm{PSL}(3, \mathbb{R}))$ as the holomorphic vector bundle of cubic differentials over $\mathcal{T}(S)$. The general case of a simple split rank 2 real Lie group G_0 has been established by Labourie some years ago ([16]). Up to isomorphism these are: $\mathrm{PSL}(3,\mathbb{R}), \mathrm{Psp}(4,\mathbb{R})$ and $\mathrm{G}_2^{\mathbb{R}}$, where the last one is the real split form of the exceptional G_2. The proof is general and consists of defining a new class of maps (called *cyclic maps*) from the universal cover $\widetilde{\Sigma}$ to the homogeneous space G_0/T, where $T < K_0$ is a maximal torus, that has the structure of a fibre bundle over the G_0-symmetric space. Due to the fact that in the case of rank 2, every Higgs bundle in the image of the Hitchin section coming from a pair (q_2, q_{m_2+1}) with $q_2 = 0$ is cyclic, Labourie showed that the datum of a cyclic map in G_0/T is equivalent to the datum of a conformal map in the symmetric space. Finally, studying the infinitesimal deformations of cyclic maps, seen as solutions to a Pfaffian system, he proved that the holomorphic vector bundle $\mathcal{E}$ of cyclic Higgs bundles is $\mathrm{Mod}(S)$-equivariant isomorphic to $\mathrm{Hit}(S, G_0)$. Notice that this is the same vector bundle which appears in the definition of the map Φ, since in rank 2 we only have two non-zero exponents: $m_1 = 1$ and m_2. When $G_0 = \mathrm{PSL}(3,\mathbb{R})$ we recover the previous result, i.e. $\mathcal{E}$ is the holomorphic bundle of cubic differentials. When $G_0 = \mathrm{Psp}(4,\mathbb{R})$ the second exponent is $m_2 = 3$ and the fibre of $\mathcal{E}$ over $J \in \mathcal{T}(S)$ is $H^0(\Sigma, K^4)$. Finally, when $G_0 = \mathrm{G}_2^{\mathbb{R}}$ the second exponent is $m_2 = 5$, hence $\mathcal{E}_J \cong H^0(\Sigma, K^6)$.

The case of maximal representations for Lie groups of Hermitian type of rank 2 is somewhat more challenging because in general the related connected components in the character variety are neither smooth nor contractible. Imitating the cyclic surfaces approach, Collier proved that Labourie's conjecture is true for every representation in the $2g-3$ Gothen components of the $\mathrm{Sp}(4,\mathbb{R})$-character variety ([7]). Three years later, Alessandrini and Collier generalized the previous result to any maximal representation in the $\mathrm{Sp}(4,\mathbb{R})$-character variety ([1]). In both cases, they provided a $\mathrm{Mod}(S)$-invariant parametrization of the maximal connected components as a holomorphic fibre bundle over Teichmüller space. In 2019, Collier, Tholozan and Toulisse proved that the Labourie's conjecture is true for any simple Lie groups of Hermitian type of rank 2 ([8]). Thanks to the classification in rank 2 and to the work in [22],[2],[3], they reduced to study only maximal representations in $\mathrm{SO}_0(2, n+1)$, for $n \geq 2$. Then, with the use of Higgs bundles and the study of the pseudo-Riemannian geometry of $\mathbb{H}^{2,n}$ (a homogeneous space for $\mathrm{SO}_0(2, n+1), n \geq 2$) they proved the existence and uniqueness of the conformal map, respectively.

Labourie's conjecture was also thought to be true for rank 3 and above, but it has very recently been shown to be false in both the split and Hermitian type case.

Theorem 3 ([19]). *Let $n \geq 3$. For every closed oriented surface S of genus $g \geq 2$ there exists a maximal representation $\rho : \pi_1(S) \to \Pi_{i=1}^n \mathrm{PSL}(2, \mathbb{R})$ such that the energy functional $e_\rho : \mathcal{T}(S) \to \mathbb{R}_{\geq 0}$ admits an unstable critical point. In particular, there are at least two conformal surfaces in the product of hyperbolic surfaces determined by ρ.*

Theorem 4 ([20]). *Let G_0 be a simple real split Lie group and let S be a closed oriented surface of genus $g \geq 3$. Then, there exists a $\rho \in \mathrm{Hit}(S, G_0)$ with at least two area minimizing ρ-equivariant conformal maps in G_0/K_0.*

Although Labourie's conjecture in general is false, it should be noted that it may still be possible to find a $\mathrm{Mod}(S)$-invariant parameterization of the Hitchin/maximal connected components using different methods from those proposed here.

References

[1] D. Alessandrini, B. Collier, *The geometry of maximal components of the* $\mathrm{Psp}(4, \mathbb{R})$ *character variety*, Geom. Top. **23** (2019), pp. 1251–1337

[2] S. B. Bradlow, O. García-Prada, P. B. Gothen, *Surface group representations and* $\mathrm{U}(p, q)$-*Higgs bundles*, Journal of Differential Geometry **64** (2003), pp. 111–170

[3] M. Burger, A. Iozzi, A. Wienhard, *Surface group representations with maximal Toledo invariant*, Ann. of Math. (2) **172** (2010), pp. 517–566

[4] S.Y. Cheng, S.T. Yau, *On the regularity of the Monge-Ampére equation det* $(\frac{\partial^2 u}{\partial x_i \partial x_j}) = f(x, u)$, Comm. on Pure and Appl. Math. **30** (1) (1977), pp. 41–68

[5] S.Y. Cheng, S.T. Yau, *Complete affine hypersurfaces. Part I. The completeness of affine metrics*, Comm. on Pure and Appl. Math. **39** (6) (1986), pp. 839–866

[6] S. Choi, W. M. Goldman, *Convex real projective structures on closed surfaces are closed*, Proc. of the Amer. Math. Soc. **118** (2) (1993), pp. 657–661

[7] B. Collier, *Maximal* $\mathrm{Sp}(4, \mathbb{R})$ *surface group representations, minimal immersions and cyclic surfaces*, Geom. Dedicata **180** (2016), pp. 241–285

[8] B. Collier, N. Tholozan, J. Toulisse, *The geometry of maximal representations of surface groups into* $\mathrm{SO}_0(2, n)$, Duke Math. J. **168** (2019), pp. 2873–2949

[9] K. Corlette, *Flat G-bundles with canonical metrics*, Journal of Differential Geometry, **28** (1988), pp. 361–382

[10] S. K. Donaldson, *Twisted harmonic maps and the self-duality equations*, Proc. London Math. Soc. **55** (1987), pp. 127–131

[11] W. M. Goldman, *Convex real projective structures on compact surfaces*, Journal of Differential Geometry **31** (1990), pp. 791–845

[12] N. J. Hitchin, *Lie groups and Teichmüller space*, Topology **31** (1992), 449–473

[13] B. Konstant, *The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group*, American Journal of Mathematics **81** (1959), 973–1032

[14] F. Labourie, *Flat projective structures on surfaces and cubic holomorphic differentials*, Pure Appl. Math. Q. **3** (4, Special Issue: in honor of Grigory Margulis. Part 1, 2007), pp. 1057–1099

[15] F. Labourie, *Cross ratios, Anosov representations and the energy functional on Teichmüller space*, Ann. Sci. Éc. Norm. Supér. (4), **41** (2008), pp. 437–469

[16] F. Labourie, *Cyclic surfaces and Hitchin components in rank 2*, Ann. of Math. (2) **185** (2017), pp. 1–57

[17] Q. Li, *An introduction to Higgs bundles via harmonic maps*, SIGMA Symmetry, Integrability Geom. Methods and Appl. **15** (2019), n. 035, 30 pp

[18] J. C. Loftin, *Affine sphere and convex $\mathbb{RP}^n$-manifolds*, Amer. J. Math. **123** (2001), pp. 255–274

[19] V. Marković, N. Sagman, P. Smillie, *Unstable minimal surfaces in $\mathbb{R}^n$ and in products of hyperbolic surfaces*, arxiv preprint (2022)

[20] N. Sagman, P. Smillie, *Unstable minimal surfaces in symmetric space of non-compact type*, arxiv preprint (2022)

[21] R. M. Schoen, *The role of harmonic mappings in rigidity and deformation problems*, Complex Geometry, Osaka (1990), Dekker, New York (1993), pp. 179–200

[22] D. Toledo, *Representations of surface groups in complex hyperbolic space*, Journal of Differential Geometry **29** (1989), pp. 125–133

[23] M. Wolf, *The Teichmüller theory of harmonic maps*, Journal of Differential Geometry **29** (1989), pp. 449–479

Θ-Positivity I

Merik Niemeyer

(mentored by T. Weisman)

1. Motivation

As always, assume that S is a closed oriented surface with genus $g \geq 2$. If G is a split real semisimple Lie group, Fock and Goncharov showed that Hitchin representations in $\mathrm{Hom}(\pi_1(S), G)$ can be characterised by the existence of a continuous positive equivariant boundary map from the boundary of the fundamental group to the generalised flag variety G/B, for B a Borel subgroup, where a map is called *positive* if it maps any finite cyclic subset of the cyclic set $\partial\pi_1(S)$ to a positive configuration of flags [2]. The underlying notion of positivity in G/B is that of *total positivity* as introduced by Lusztig [5].

On the other hand, if G is Hermitian of tube type, maximal representations can be characterised in a similar way due to Burger, Iozzi and Wienhard [1]: A representation is maximal if and only if there exists a continuous positive equivariant boundary map to the Shilov boundary of the Hermitian symmetric space. This Shilov boundary can be viewed as a partial flag variety G/Q for some parabolic subgroup Q of G and the notion of positivity is given by maximality of the Maslov index.

The similarity of these characterisations makes it plausible to consider positivity as a unifying framework for higher Teichmüller theory, which leads to the notion of Θ-positivity as introduced by Guichard and Wienhard [3, 4].

2. Lusztig's total positivity

A matrix in $\mathrm{GL}(n, \mathbb{R})$ is called *totally positive* if all of its minors are positive numbers. The subset consisting of such elements is denoted $\mathrm{GL}(n, \mathbb{R})^{>0}$. A. Whitney proved a decomposition theorem, namely

$$\mathrm{GL}(n, \mathbb{R})^{>0} = U_-^{\geq 0} H^\circ U_+^{\geq 0}, \tag{1}$$

where $U_+^{\geq 0}$ and $U_-^{\geq 0}$ denote the subsets of upper and lower triangular unipotent matrices whose minors are all positive unless they are zero by triangularity, and H° denotes the identity component of the subgroup of diagonal matrices.

This result restricts to $\mathrm{SL}(n,\mathbb{R})$ and Lusztig introduced a notion of positivity for arbitrary split real semisimple Lie groups [5] by reversing the above: Given such a group G, after fixing a pinning, and thus in particular a pair of opposite Borel subgroups $B_\pm$, for G, he defines subsets $U_\pm^{\geq 0}$ of the unipotent radicals $U_\pm$ of $B_\pm$. These are constructed as the image of a map, which depends on choosing a reduced expression for the longest element in the Weyl group W of G in terms of the standard generators corresponding to the simple roots of the Lie algebra $\mathfrak{g}$ of G. Lusztig proved that this construction is independent of the chosen reduced expression and that the resulting subsets are actually semigroups, called the *totally positive subsemigroup* of $U_\pm$, respectively.

These are then used to define the *totally positive semigroup* of G,

$$G^{>0} = U_-^{\geq 0} H^\circ U_+^{\geq 0}, \tag{2}$$

where H° denotes the identity component of the maximal torus $H = B_+ \cap B_-$, just like before.

Finally, this gives a notion of positive triples (or more generally tuples) in the flag variety $\mathcal{F} = G/B_+$, which shows up in the work of Fock and Goncharov as mentioned in the introduction. Recall that a pair (F_1, F_2) of flags is called *transverse* if it lies in the G-orbit of $([B_+],[B_-])$ in $\mathcal{F}^2$, and that a triple of flags is called *generic* if the flags are pairwise transverse. Let us denote by Ω_F the set of flags transverse to F.

Lusztig proved that the subset $\mathcal{F}^{>0} := G^{>0}[B_+]$ of $\mathcal{F}$ is a connected component of $\Omega_{[B_+]} \cap \Omega_{[B_-]}$, which carries the structure of a semigroup [5]. Moreover, this subset is used to define *positive triples* in $\mathcal{F}$, which are triples (F_1, F_2, F_3) for which there exists a $g \in G$ such that $gF_1 = [B_+], gF_3 = [B_-]$ and $gF_2 \in \mathcal{F}^{>0}$.

3. Θ-positivity

Θ-positivity, recently introduced by Guichard and Wienhard [3, 4], provides a further generalization of Lusztig's total positivity. However, as mentioned in the beginning it also encompasses the notion of positivity provided by the Maslov index in the case where G is Hermitian of tube type.

The setup is the following: Let G be a semisimple real Lie group with finite center. Let $K < G$ be a maximal compact subgroup with Lie algebra $\mathfrak{k}$. We have a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}^\perp$ of the Lie algebra of G. Choose a maximal abelian Cartan subspace $\mathfrak{a} \in \mathfrak{k}^\perp$, and consider the corresponding restricted root system $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$. Let Δ be a set of simple roots and choose a subset $\Theta \subset \Delta$.

This choice gives us a decomposition

$$\mathfrak{g} = \mathfrak{l}_\Theta \oplus \mathfrak{u}_\Theta \oplus \mathfrak{u}_\Theta^{\mathrm{opp}}$$

of the Lie algebra, where $\mathfrak{u}_\Theta$ is the sum of all root spaces for positive roots that lie in $\Sigma \setminus \mathrm{Span}(\Delta - \Theta)$ and $\mathfrak{u}_\Theta^{\mathrm{opp}}$ is the sum of the corresponding negative root spaces. Finally, $\mathfrak{l}_\Theta$ is the sum of all remaining root spaces, i.e. $\mathfrak{g}_0$ and those for which the root lies in $\mathrm{Span}(\Delta - \Theta)$. Associated to these is a pair of opposite parabolic subgroups $P_\Theta = N_G(\mathfrak{u}_\Theta)$ and $P_\Theta^{\mathrm{opp}} = N_G(\mathfrak{u}_\Theta^{\mathrm{opp}})$ of G and the *Levi subgroup* $L_\Theta = P_\Theta \cap P_\Theta^{\mathrm{opp}}$ with Lie algebra $\mathfrak{l}_\Theta$.

Let us denote the center of $\mathfrak{l}_\Theta$ by $\mathfrak{z}_\Theta$ and its intersection with the Cartan subspace by $\mathfrak{t}_\Theta = \mathfrak{z}_\Theta \cap \mathfrak{a}$. This acts on $\mathfrak{u}_\Theta$ via the adjoint action and $\mathfrak{u}_\Theta$ decomposes into weight spaces $\mathfrak{u}_\beta$ for this with $\beta \in \mathfrak{t}_\Theta^*$. Since the roots of $\mathfrak{g}$ restrict to $\mathfrak{t}_\Theta^*$, we can talk about weight spaces $\mathfrak{u}_\beta$ for $\beta \in \Theta$ (a slight abuse of notation).

In this setup Guichard and Wienhard make the following definition: G is said to *admit a Θ-positive structure* if for all $\beta \in \Theta$ the weight space $\mathfrak{u}_\beta$ contains an L_Θ°-invariant acute non-trivial open convex cone [3].

The easiest example to think about is the case where G is split real and $\Theta = \Delta$. Then the weight spaces are simply the root spaces, which are one-dimensional, and the cones correspond to $\mathbb{R}_+$.

Remarkably, Guichard and Wienhard have given a classification of all pairs (G, Θ) consisting of a *simple* real Lie group and a subset Θ of the simple roots such that G admits a Θ-positive structure [3, 4]. Of course, this includes the cases where G is split real or Hermitian of tube type but also the new case of G being locally isomorphic to $\mathrm{SO}(p+1, p+k)$ with $p, k > 1$ and some exceptional cases.

Assume that (G, Θ) falls into one of these cases. With the Θ-positive structure in place, it is possible to define the Θ-Weyl group, which is isomorphic to the Weyl group of some simple root system. One can use a reduced expression of the longest word in this group to define a map which is analogous to the map Lusztig uses in his construction and obtain *Θ-positive subsemigroups* $U_\Theta^{>0} \subset U_\Theta = \exp(\mathfrak{u}_\Theta)$ and $U_\Theta^{\mathrm{opp},>0} \subset U_\Theta^{\mathrm{opp}} = \exp(\mathfrak{u}_\Theta^{\mathrm{opp}})$, which finally allows one to define the *Θ-positive subsemigroup* $G_\Theta^{>0}$ of G as the semigroup generated by $U_\Theta^{>0}$, $U_\Theta^{\mathrm{opp},>0}$ and L_Θ°.

This gives rise to a notion of positivity on the flag variety G/P_Θ, where we have results which are very similar to the ones discussed in the case of total positivity.

References

[1] M. Burger, A. Iozzi & A. Wienhard, *Surface group representations with maximal Toledo invariant*, Annals of mathematics (2010), 517–566.

[2] V. Fock & A. Goncharov, *Moduli spaces of local systems and higher Teichmüller theory*, Publications Mathématiques de l'IHÉS **103** (2006), 1–211.

[3] O. Guichard & A. Wienhard, *Positivity and higher Teichmüller theory*, European Congress of Mathematics (2018), 289–310.

[4] O. Guichard & A. Wienhard, *Generalizing Lusztig's total positivity*, arXiv preprint arXiv:2208.10114 (2022).

[5] G. Lusztig, *Total positivity in reductive groups*, Lie theory and geometry (1994), 531–568.

Θ-positivity II

Guillermo Barajas
(mentored by M. Riestenberg)

1. Θ-positive structures

Let $G^{\mathbb{R}}$ be a connected simple real Lie group with Lie algebra $\mathfrak{g}^{\mathbb{R}}$. Choose a set of (restricted) simple roots Δ and fix a subset $\Theta \subset \Delta$. Let $P := P_\Theta$ be the corresponding parabolic subgroup, $U \leq P$ its unipotent radical. Assume that $G^{\mathbb{R}}$ has a Θ-positive structure $N \subset U$ (defined as in 'Θ-positivity I´). A Θ-positive structure on $G^{\mathbb{R}}$ provides a notion of positive tuples on $G^{\mathbb{R}}/P$.

Example 1.1. If we take $G^{\mathbb{R}} = \mathrm{SL}(2,\mathbb{R})$ then P is the parabolic subgroup consisting of upper triangular matrices and we get a notion of positive triples and quadruples in $\mathrm{SL}(2,\mathbb{R})/P \cong S^1$. A triple $(a,b,c) \in S^1$ is positive if its elements are pairwise disjoint, and a quadruple (a,b,c,d) is positive if b and d lie between a and c.

Now let S be a compact real surface of genus $g \geq 2$. Recall that its fundamental group $\pi_1(S)$ acts on its Shilov boundary, which is isomorphic to S^1.

Definition 1.2. *A representation $\pi_1(S) \to G^{\mathbb{R}}$ is Θ-**positive** if there exists a $\pi_1(S)$-invariant subset $A \subset S^1$ and a $\pi_1(S)$-equivariant map $A \to G^{\mathbb{R}}/P$ such that positive quadruples are sent to positive quadruples.*

Definition 1.2 was introduced in [4] to find **higher Teichmüller components** in the representation variety $\mathfrak{R}(S,G^{\mathbb{R}}) := \mathrm{Hom}(\pi_1(S),G^{\mathbb{R}}) /\!\!/ G^{\mathbb{R}}$. These are connected components consisting entirely of faithful representations with discrete image. The set of classes of Θ-positive representations is denoted $\mathfrak{R}_{\Theta-\mathrm{pos}}(S,G^{\mathbb{R}})$.

Theorem 1.3 (Guichard–Labourie–Wienhard)**.** *Every Θ-positive representation $\pi_1(S) \to G^{\mathbb{R}}$ is Anosov, in particular discrete and faithful. Moreover, the set $\mathfrak{R}_{\Theta-pos}(S,G^{\mathbb{R}})$ is open in $\mathfrak{R}(S,G^{\mathbb{R}})$ and it is closed in the subspace $\mathfrak{R}^*(S,G^{\mathbb{R}})$ corresponding to representations $\pi_1(S) \to G^{\mathbb{R}}$ which do not virtually factor through a parabolic subgroup.*

In other words, $\mathfrak{R}^*(S,G^{\mathbb{R}})$ consists of conjugacy classes of representations $\pi_1(S) \to G^{\mathbb{R}}$ which, when restricted to any finite index subgroup of $\pi_1(S)$, do not factor through a parabolic subgroup of $G^{\mathbb{R}}$.

Theorem 1.4 (Beyrer–Pozzetti)**.** *The set $\mathfrak{R}_{\Theta\text{-}positive}\,(S,\mathrm{PO}(p,q))$ is closed in $\mathfrak{R}(S,\mathrm{PO}(p,q))$.*

In general the question of whether $\mathfrak{R}_{\Theta\text{-positive}}\,(S,G^{\mathbb{R}})$ is closed in $\mathfrak{R}(S,G^{\mathbb{R}})$ is not known. To find higher Teichmüller components we introduce Higgs bundle techniques following [1].

2. Magical $\mathfrak{sl}(2)$-triples

Let G be a connected simple complex Lie group. By the Jacobson–Morozov Theorem there exists a one-to-one correspondence between conjugacy classes of nilpotent elements $e \in \mathfrak{g}$ (such that some power of $\mathrm{ad}(e)$ is 0) and conjugacy classes of $\mathfrak{sl}(2)$-triples (f,h,e). Given an $\mathfrak{sl}(2)$-triple (f,h,e), the corresponding $\mathfrak{sl}(2)$-subalgebra induces two decompositions of $\mathfrak{g}$, namely a decomposition into irreducible representations:

$$\mathfrak{g} = \bigoplus_j W_j, \tag{1}$$

where W_i is a direct sum of n_i copies of the (unique) irreducible representation of dimension $i+1$, and a decomposition into eigenspaces of $\mathrm{ad}(h)$:

$$\mathfrak{g} = \bigoplus_{j=-l}^{l} \mathfrak{g}_j.$$

There is a vector space involution θ_e of $\mathfrak{g}$ whose restriction to $\mathrm{ad}(f)^k(W_i \cap \mathfrak{g}_i)$ is equal to $(-1)^{k+1}$ if $i > 0$ and is trivial on W_0.

Definition 2.1. *(f,h,e) is a **magical triple** if θ_e is a Lie algebra involution.*

A **real form** of $\mathfrak{g}$ is the subalgebra of fixed points under an antiholomorphic involution. We sometimes call the involution itself a real form and we denote the set of real forms $\mathrm{conj}(\mathfrak{g})$. Recall that there is a bijection

$$\mathrm{conj}(\mathfrak{g})/\operatorname{Int}(\mathfrak{g}) \leftrightarrow \mathrm{Aut}_2(\mathfrak{g})/\operatorname{Int}(\mathfrak{g}),$$

where Aut_2 is the group of Lie algebra (holomorphic) involutions. More precisely, for any conjugation ι there exists a compact form τ of $\mathfrak{g}$ that commutes with ι, which defines a **Cartan involution** $\iota\tau$ of ι.

Remark 1. An (antiholomorphic/holomorphic) involution of $\mathfrak{g}$ does not, in general, integrate to an involution form of G. This is true, for example, if G is simply connected or the adjoint form of $\mathfrak{g}$. However, we always assume integrability.

Definition 2.2. *The **canonical real form** $\mathfrak{g}^{\mathbb{R}}$ associated to the magical triple (f,h,e) is the fixed point subalgebra of $\mathfrak{g}$ under a real form τ_e having σ_e as a Cartan involution. There is an associated real subgroup $G^{\mathbb{R}} \subset G$.*

Theorem 2.3 (Bradlow–Collier–Garcia-Prada–Gothen–Oliveira)**.** *The simple real Lie groups which are the canonical form of a magical triple of some complex simple Lie group are exactly the ones with a Θ-positive structure for some Θ. More precisely, a magical triple in $\mathfrak{g}$ determines a parabolic subgroup of $G^{\mathbb{R}}$ which is the same as P_Θ for some set of simple restricted roots Θ featuring a Θ-positive structure, and vice-versa.*

See [1] for a precise list of the weighted Dynkin weighted diagrams determined by the respective magical triples.

3. Higgs bundles and Cayley components

Equip S with a complex structure, making it a compact Riemann surface X. Let $\mathfrak{g}^{\mathbb{R}} = \mathfrak{h}^{\mathbb{R}} \oplus \mathfrak{m}^{\mathbb{R}}$ be a Cartan decomposition of $G^{\mathbb{R}}$ (i.e. $\mathfrak{h}^{\mathbb{R}}$ is maximal compact and $\mathfrak{m}^{\mathbb{R}}$ is its orthogonal subspace according to the Killing form). Let $\mathfrak{h}$ and $\mathfrak{m}$ be the respective complexifications, and let $H < G$ be subgroup with Lie algebra $\mathfrak{h}$.

Definition 3.1. *Let $L \to X$ be a holomorphic line bundle. An **L-twisted $G^{\mathbb{R}}$-Higgs bundle** is a pair (E, φ), where E is a holomorphic H-bundle and $\varphi \in H^0(X, E(\mathfrak{m}) \otimes K)$ is a holomorphic section (the Higgs field). Here $E(\mathfrak{m}) := E \times \mathfrak{m}/H$, where H acts on the first factor via the bundle action and on $\mathfrak{m}$ via the restriction of the adjoint representation of G on $\mathfrak{g}$; this is a holomorphic vector bundle with fiber $\mathfrak{m}$. If $L = K$, the canonical bundle of X, we omit L.*

There is a moduli space $\mathcal{M}_L(X, G^{\mathbb{R}})$ of L-twisted $G^{\mathbb{R}}$-Higgs bundles parameterising isomorphism classes of polystable L-twisted $G^{\mathbb{R}}$-Higgs bundles. If $L = K$ we omit the subscript.

Theorem 3.2 (Non-abelian Hodge correspondence)**.** *There exists a real analytic isomorphism*

$$\mathcal{M}(X, G^{\mathbb{R}}) \xrightarrow{\sim} \mathfrak{R}(X, G^{\mathbb{R}}).$$

In particular, connected components can be studied from both points of view.

Theorem 3.3. *There exists an open and closed complex algebraic embedding*

$$\widehat{\Psi}_e : \mathcal{M}_{K^{m_c+1}}(G^{\mathbb{R}}_{0,ss}) \times \prod_{j=1}^{\mathrm{rk}(\mathfrak{g}(e))} \mathcal{M}_{K^{l_j+1}}(\mathbb{R}^+) \to \mathcal{M}(G^{\mathbb{R}}), \tag{2}$$

*where we have dropped X from the notation for simplicity and: $\{m_c\} \cup \{l_j\}_j$ is a subset of the exponents j appearing in (1) and the group $G^{\mathbb{R}}_{0,ss}$ is a real form of the subgroup $G_{0,ss} < G$ whose Lie algebra is the semisimple part of $\mathfrak{g}_0$. Moreover, each connected component in $\mathcal{M}_e(G^{\mathbb{R}})$, which we call a **Cayley component**, contains elements which are mapped, via the non-abelian Hodge correspondence, to Θ-positive -representations, and it consists entirely of representations which do not virtually factor through a parabolic subgroup of $G^{\mathbb{R}}$.*

Using Theorems 1.3 and 3.3 together, we find the following important result:

Theorem 3.4. *Cayley components consist entirely of Θ-positive representations, in particular they are higher Teichmüller components.*

References

[1] S. Bradlow, B. Collier, O. Garcia-Prada, P. B. Gothen and A. Oliveira, *A general Cayley correspondence and higher Teichmüller spaces*, arXiv:2101.09377, 2021.

[2] J. Beyrer and M. B. Pozzetti, *Positive surface group representations in* $\mathrm{SO}(p,q)$, arXiv:2106.14725, 2021.

[3] O. Guichard, F. Labourie and A. Wienhard, *Positivity and representations of surface groups*, arXiv:2106.14584, 2021.

[4] O. Guichard and A. Wienhard, *Positivity and higher Teichmüller theory*, in European congress of mathematics, pages 289–310. Eur. Math. Soc, Zürich, 2018.

Compactifications via currents

Xenia Flamm
(mentored by G. Martone)

In this talk, we will use Bonahon's space of projectivized geodesic currents to construct compactifications of higher rank Teichmüller spaces. A recipe is the following. Let $\mathcal{T}$ be a higher rank Teichmüller space. Assume that a continuous map

$$f : \mathcal{T} \to \{\text{geodesic currents}\} \to \mathbb{P}(\{\text{geodesic currents}\})$$

exists. Then $\overline{f(\mathcal{T})}$ provides a compactification of $\mathcal{T}$, since the space of projectivized geodesic currents is compact. A *boundary point* will be a point in $\overline{f(\mathcal{T})} \setminus f(\mathcal{T})$. The first part of this talk will be concerned with the properties of the space of (projectivized) geodesic currents following [3]. In the second part we will discuss the question on how to assign to a representation in a higher rank Teichmüller space a geodesic current following [5] and [9].

Let S be a closed, orientable, connected surface of genus $g \geq 2$, endowed with an auxiliary hyperbolic metric. We will denote its fundamental group by $\pi_1(S)$. A *geodesic current* on S is a locally finite, $\pi_1(S)$-invariant, regular Borel measure on $\mathcal{G}$, the space of (unoriented and unparametrized) geodesics in the universal cover $\tilde{S}$ of S. Denote by $\mathcal{C}$ the space of geodesic currents on S endowed with the weak*-topology. It is not difficult to see that $\mathcal{C}$ is independent of the choice of hyperbolic metric on S [3, Fact 1]. Many seemingly different objects are in fact geodesic currents, e.g. homotopy classes of closed curves on S or isotopy classes of hyperbolic metrics, i.e. points in the Teichmüller space $T(S)$ of S [3, Lemma 9]. The induced map $f : T(S) \to \mathcal{C}$ is continuous, injective and a homeomorphism onto its image, see [3, Corollary 11]. Consider $D\mathcal{G} \subseteq \mathcal{G}\times\mathcal{G}$ the set of pairs of transversely intersecting geodesics. We define the *intersection number* $i : \mathcal{C} \times \mathcal{C} \to \mathbb{R}_{\geq 0}$ as $i(\mu, \eta) := (\mu \times \eta)(D\mathcal{G}/\pi_1(S))$. Then i is finite, continuous, symmetric and bilinear, and generalizes the geometric intersection number of homotopy classes of closed curves on S [2, Proposition 4.5]. In order to prove that the space $\mathbb{P}\mathcal{C} := \mathcal{C}/\mathbb{R}_{>0}$ of projectivized geodesic currents is compact, we make us of the *compactness criterion*: Let $\alpha \in \mathcal{C}$ be such that every geodesic in $\mathcal{G}$ intersects a geodesic in the support of α, then $\{\beta \in \mathcal{C} \mid i(\alpha, \beta) \leq 1\}$ is compact [3, Proposition 4]. The continuous map $\mathcal{C} \setminus \{0\} \to \mathcal{C}$, $\beta \mapsto \frac{\beta}{i(\alpha,\beta)}$ factors through $\mathbb{P}\mathcal{C}$, hence the latter is compact [3, Corollary 5].

The above considerations cummulate in the following results [3, Corollary 16, Theorem 17, Propostion 18].

Theorem 1. *The map $f : T(S) \to \mathbb{P}\mathcal{C}$ is a homeomorphism onto its image, and $\overline{f(T(S))}$ is homeomorphic to Thurston's compactification.*

To associate to a representation in a higher rank Teichmüller space a geodesic current, we will make use of positive cross-ratios. A *positive cross-ratio* is a $\pi_1(S)$-invariant continuous function $B : \partial_\infty \pi_1(S)^{[4]} \to \mathbb{R}_{>0}$, defined on distinct ordered four-tuples of points, that is symmetric and additive. The *B-period* of a non-trivial $\gamma \in \pi_1(S)$ is $\ell_B(\gamma) := B(\gamma^+, \gamma^-, x, \gamma x)$ for some (any) $x \in \partial_\infty \pi_1(S) \setminus \{\gamma^\pm\}$, where γ^+ respectively γ^- is the attracting respectively repelling fixed point of γ. A geodesic current $\mu \in \mathcal{C}$ is an *intersection current* for B if for every non-trivial $\gamma \in \pi_1(S)$ we have $\ell_B(\gamma) = i(\mu, \gamma)$. If B is positive there exists a unique intersection current for B [4]. Let $\rho : \pi_1(S) \to \mathrm{PSL}(n, \mathbb{R})$ be P_k-Anosov. Martone-Zhang prove in [5, Proposition 2.24] that there exists a unique cross-ratio B_k^ρ such that for all non-trivial $\gamma \in \pi_1(S)$

$$\ell_{B_k^\rho}(\gamma) = \log \frac{\lambda_1(\rho(\gamma)) \cdot \ldots \cdot \lambda_k(\rho(\gamma))}{\lambda_{n-k+1}(\rho(\gamma)) \cdot \ldots \cdot \lambda_n(\rho(\gamma))},$$

where $\lambda_1(\rho(\gamma)) \geq \ldots \geq \lambda_n(\rho(\gamma))$ are the absolute values of the generalized eigenvalues of $\rho(\gamma)$. They define a P_k-Anosov representation ρ to be *P_k-positively ratioed* if B_k^ρ is positive [5, Definition 2.25]. Putting everything together we conclude that to a positively ratioed representation ρ we can assign its unique intersection current $\mu_\rho \in \mathcal{C}$. Examples of such representations include Hitchin representations in $\mathrm{PSL}(n, \mathbb{R})$ [5, Theorem 3.4], maximal representations in $\mathrm{Sp}(2n, \mathbb{R})$ [6, Section 4.2], and Θ-positive representations in $\mathrm{PO}(p, q)$ [1, Theorem 4.9]. To summarize we obtain the following. If $\mathcal{T}$ is a higher rank Teichmüller space that consists entirely of P_k-positively ratioed representations, then the map

$$f : \mathcal{T} \to \mathbb{P}\mathcal{C},\ [\rho] \mapsto [\mu_\rho]$$

is continuous with relatively compact image, and $\overline{f(\mathcal{T})}$ provides a compactification of $\mathcal{T}$. Contrary to the case of Teichmüller space, f is in general not injective. For example, a Hitchin representation in $\mathrm{PSL}(n, \mathbb{R})$ and its contragradient representation have the same intersection current.

In rank two, there is yet another way of associating to a representation in a higher rank Teichmüller space a geodesic current as described in [8], [9] and [10]. We focus here on the case of $\mathrm{PSL}(3, \mathbb{R})$ as in [9]. Denote by $\mathcal{T}_3$ the Hitchin component in $\mathrm{PSL}(3, \mathbb{R})$. For a Hitchin representation $\rho : \pi_1(S) \to \mathrm{PSL}(3, \mathbb{R})$ the associated equivariant hyperbolic affine sphere carries a strictly negatively curved Riemannian metric, the *Blaschke metric*, which descends to a strictly negatively curved metric m_ρ on S. Since strictly negatively curved metrics can be viewed as geodesic currents [7, Theorem 1], we obtain the following continuous map

$$f : \mathcal{T}_3 \to \mathbb{P}\mathcal{C},\ [\rho] \mapsto [\mu_{m_\rho}].$$

Ouyang-Tamburelli identify the boundary points, i.e. points in $\overline{f(\mathcal{T}_3)} \setminus f(\mathcal{T}_3)$, with projectivized mixed structures [9, Theorem A]. A *mixed structure* is a geodesic current, that comes from a singular flat metric on a subsurface of S and a measured lamination on the complementary subsurface.

References

[1] J. Beyrer, B. Pozzetti, *Positive surface group representations in PO(p,q)*, ArXiv.2106.14725 (2021).

[2] F. Bonahon, *Bouts des variétés hyperboliques de dimension 3*, Ann. of Math. (2) **124** (1986), 71–158.

[3] F. Bonahon, *The geometry of Teichmüller space via geodesic currents*, Inven. Math. **92** (1988), 139–162.
[4] U. Hamenstädt, *Cocycles, symplectic structures and intersection*, Geom. Funct. Anal. **9** (1999), 90–140.
[5] G. Martone, T. Zhang, *Positively ratioed representations*, Comment. Math. Helv. **94** (2019), 273–345.
[6] F. Labourie, *Cross ratios, Anosov representations and the energy functional on Teichmüller space*, Ann. Sci. Éc. Norm. Supér. (4) **41** (2008), 437–469.
[7] J. P. Otal, *Le spectre marqué des longueurs des surfaces à courbure négative*, Ann. of Math. (2) **131** (1990), 151–162.
[8] C. Ouyang, A. Tamburelli, *Length spectrum compactification of the $SO_0(2,3)$-Hitchin component*, arXiv:2010.03499, 2020.
[9] C. Ouyang, A. Tamburelli, *Limits of Blaschke metrics*, Duke Math. J. **170** (2021), 1683 – 1722.
[10] C. Ouyang, *High energy harmonic maps and degeneration of minimal surfaces*, arXiv:1910.06999, 2019.

Compactifications, buildings and geometric structures

Martin Ulirsch

Let A be a commutative ring with one. A *prime cone* P in A is a subset $P \subseteq A$ that fulfils the following axioms:

- P is closed under addition and multiplication, or in other words $P+P \subseteq P$ and $P \cdot P \subseteq P$;
- $P \cup (-P) = A$;
- the *support* $P \cap (-P)$ of P is a prime ideal in A.

We write $f >_P 0$ if $-f \notin P$. The *real spectrum* $\mathrm{Sper}(A)$ of A may be defined as the set of prime cones in A together with the coarsest topology that makes the sets $\{P \in \mathrm{Spec} A \mid f >_P 0\}$ for all $f \in A$ open.

What makes this object surprisingly interesting to Higher Rank Teichmüller Theory is the following: Let $A = A(V)$ be the coordinate ring of an affine real algebraic set V. Then the set of points in V naturally embed as a dense and open set into the subset of closed points $\mathrm{Sper}^{cl}(A)$ of $\mathrm{Sper}(A)$. But there are additional closed points "at infinity" that make $\mathrm{Sper}^{cl}(A)$ into a compact Hausdorff space. Therefore one may think of $\mathrm{Sper}^{cl}(A)$ as a canonical compactification of an affine real algebraic set V. We refer the reader to [3] as the standard reference on the many beautiful properties of the real spectrum.

Classical Teichmüller space is well-known to carry the structure of a real semi-algebraic set. In [2] uses the real spectrum to construct a canonical compactification of Teichmüller space and gives an interpretation of its boundary points in terms of actions of the fundamental group Γ of a compact closed surface of genus $g \geq 2$ on certain $\mathbb{R}$-trees.

In [1] the authors use the same principle to construct a canonical *real spectrum compactification* for character varieties associated to a finitely generated group Γ with values in a connected semisimple group $G \leq \mathrm{SL}_n$ defined over $\mathbb{Q}$. Among their many announced results there is an interpretation of the boundary points using operations of Γ on the building associated to $G(K)$ (for a Robinson field K).

This is compatible with another construction of a compactification of the character variety given in [4]. In [4] Parreau, in particular, shows that the operation of $G(K)$ on the building of $G(K)$ is naturally an ultralimit of the operation of G on the symmetric space of $G(\mathbb{R})$. In this sense, this provides us with a geometric interpretation for points in the boundary of both compactifications.

References

[1] Marc Burger, Alessandra Iozzi, Anne Parreau, and Maria Beatrice Pozzetti, *The real spectrum compactification of character varieties: characterizations and applications*, Comptes Rendus Mathématique. Académie des Sciences. Paris **359** (2021), 439–463.

[2] G. W. Brumfiel, *The real spectrum compactification of Teichmüller space*, Contemp. Math. **74** (1988), 51–75.

[3] Jacek Bochnak, Michel Coste, and Marie-Françoise Roy, *Real algebraic geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **36** (1998), x+430.

[4] Anne Parreau, *Compactification d'espaces de représentations de groupes de type fini*, Mathematische Zeitschrift **272** (2012), 51–86.

The Symplectic Geometry of Higher Teichmüller Spaces

Fernando Camacho Cadena

(mentored by F. Mazzoli)

Let π denote the fundamental group of a compact oriented closed surface S of genus $g \geq 2$, and let G be a connected Lie group. The goal is to present Goldman's construction in [1] of a symplectic form on the space of conjugacy classes $\chi(\pi, G) := \mathsf{Hom}(\pi, G)/\mathsf{Inn}(G)$. The form is invariant under the group of outer automorphisms of π, which in particular includes the mapping class group of S.

We begin by briefly recalling some symplectic geometry. A differential 2-form ω on a manifold M is symplectic if it is non-degenerate and closed. Flows on symplectic manifolds can be constructed in the following way. Let $f\colon M \to \mathbb{R}$ be a smooth function. Then the Hamiltonian vector field of f is the unique vector field H_f on M such that $df(\cdot) = \omega(H_f, \cdot)$, and the flow of H_f is the Hamiltonian flow of f. For $\chi(\pi, G)$ we will discuss the tangent space to define the Goldman symplectic form. With that, we describe some explicit Hamiltonian flows on $\chi(\pi, G)$.

1. The Zariski tangent space

Note that $\mathsf{Hom}(\pi, G)$ is an algebraic variety when G is an algebraic group, and has singular points. Nevertheless, there is a notion of Zariski tangent space at a representation ρ, which we find in two steps. Let $u\colon \pi \to \mathfrak{g}$ be a function, and define a small path ρ_t of representations given by $\rho_t(\gamma) = \mathsf{exp}(tu(\gamma) + \mathcal{O}(t^2))\rho(\gamma)$. Since ρ_t is also a representation, one deduces that

$$(\diamondsuit) \qquad u(\alpha\beta) = u(\alpha) + \mathsf{Ad}_{\rho(\alpha)} u(\beta).$$

The Zariski tangent space to ρ is thus identified with $\{u\colon \pi \to \mathfrak{g} \text{ satisfying } (\diamondsuit)\}$.

Now we turn to the Zariski tangent space to an $\mathsf{Inn}(G)$ orbit. Let $g_t = \mathsf{exp}(tu_0 + \mathcal{O}(t^2))$ be a path in G, for some $u_0 \in \mathfrak{g}$. The path $\rho_t(\gamma) = g_t^{-1}\rho(\gamma)g_t$ is in the $\mathsf{Inn}(G)$ orbit. Rewriting ρ_t as before with a function $u\colon \pi \to \mathfrak{g}$, one deduces that

$$(\heartsuit) \qquad u(\gamma) = \mathsf{Ad}_{\rho(\gamma)}u_0 - u_0.$$

Hence, the Zariski tangent space at ρ to $\mathsf{Inn}(G).\rho$ is identified with $\{u\colon \pi \to \mathfrak{g} \text{ satisfying } (\heartsuit)\}$.

The Lie algebra $\mathfrak{g}$ is turned into a π module through the adjoint action, denoted by $\mathfrak{g}_{\mathsf{Ad}_\rho}$. The equations $(\diamondsuit)$ and $(\heartsuit)$ define respectively the cocycles and coboundaries in the group cohomology of π with coefficients in $\mathfrak{g}_{\mathsf{Ad}_\rho}$. The Zariski tangent space to $[\rho]$ is therefore identified with $H^1(\pi, \mathfrak{g}_{\mathrm{Ad}_\rho})$.

What is meant by a symplectic form in our case, is a bilinear, alternating, non-degenerate pairing at each Zariski tangent space, which is a closed form when restricted to the smooth points. It is well known that Hitchin components are smooth. More generally, Goldman found in [1] that $\rho \in \mathsf{Hom}(\pi, G)$ is a smooth point if and only if $\mathsf{dim}(Z(\rho)/Z(G)) = 0$, and that $\mathsf{Inn}(G)$ acts freely on the set of smooth points.

2. The Goldman Symplectic Form

To define the Goldman symplectic form, we require that G admit an orthogonal structure. That is, a bilinear, symmetric, non-degenerate, and Ad_G invariant pairing $\mathfrak{B}\colon \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$. Semisimple Lie groups admit an orthognal structure through the Killing form. More generally, reductive groups also admit such structures.

The Goldman pairing $\omega_{[\rho]}$ on Zariski tangent spaces at an equivalence class $[\rho]$ is defined through $H^1(\pi; \mathfrak{g}_{\mathrm{Ad}_\rho}) \times H^1(\pi; \mathfrak{g}_{\mathrm{Ad}_\rho}) \xrightarrow{\mathfrak{B}(\cdot \smile \cdot)} H^2(\pi, \mathbb{R}) \xrightarrow{\frown [\pi]} \mathbb{R}$ where $\smile$ is the cup product on cohomology, and $\frown [\pi]$ is integration with a fundamental class (coming from the orientation of S) inducing Poincaré duality. The following theorem is due to Goldman in [1].

Theorem. *The pairing $\omega_{[\rho]}$ is bilinear, alternating and non-degenerate for every $[\rho] \in \chi(\pi, G)$. Moreover, the form ω is closed on the smooth part of $\chi(\pi, G)$, and is $\mathsf{Out}(\pi)$ invariant.*

In the case when $G = \mathsf{PSL}(2, \mathbb{R})$ and $\mathfrak{B}(X, Y) = \mathsf{trace}(XY)$ the symplectic form ω restricts to a symplectic form on Teichmüller space $\mathcal{T}(\mathcal{S})$. Goldman then obtains that $\omega = -2\,\omega_{WP}$, where ω_{WP} is the Weil-Petersson symplectic form.

The Weil-Petersson metric is also a Kähler form on $\mathcal{T}(S)$. Labourie and Wentworth showed in [3] that the pressure metric on Hitchin components obtained by using the highest eigenvalue is not compatible with the Goldman symplectic form. Nevertheless, Kim and Zhang in [4] and Labourie in [5] show that when G has real rank 2, the Hitchin component carries a family of Kähler structures, but their relationship to the Goldman form remains mysterious.

3. Hamiltonian Flows

We can now describe some Hamiltonian flows in $\chi(\pi, G)$, generalizing the well known twist flows in Teichmüller space. Fix a simple separating curve α in S. A twist along α in $\mathcal{T}(S)$ is a deformation of a hyperbolic structure given by cutting S along α into S_1 and S_2, and then regluing after twisting S_2. Let ρ be the holonomy of a hyperbolic structure and $X_\alpha \in \mathfrak{sl}(2,\mathbb{R})$ such that $\rho(\alpha) = \mathsf{exp}(\ell_\alpha(\rho)X_\alpha/2)$, where $\ell_\alpha(\rho)$ is the length of α in the hyperbolic metric induced by ρ. Then the twist is given by $\rho_t(\gamma) = \mathsf{exp}(tX_\alpha)\rho(\gamma)\mathsf{exp}(-tX_\alpha)$ if $\gamma \in \pi_1(S_1)$ and by $\rho_t(\gamma) = \rho(\gamma)$ if $\gamma \in \pi_1(S_2)$. The key observation here is that $X_\alpha \in Z(\rho(\alpha))$, which allows us to reglue the representations along α. In general, given $\rho \in \mathsf{Hom}(\pi, G)$, and a path $\zeta_t \in Z(\rho(\alpha))$, a generalized twist flow is defined by $\rho_t(\gamma) = \zeta_t\rho(\gamma)\zeta_t^{-1}$ when $\gamma \in \pi_1(S_2)$ and by $\rho_t(\gamma) = \rho(\gamma)$ if $\gamma \in \pi_1(S_1)$. Goldman obtains in [2] that such flows are Hamiltonian flows for functions of the form $f_\alpha\colon [\rho] \mapsto f(\rho(\alpha))$, where $f\colon G \to \mathbb{R}$ is a conjugation invariant function. In particular, when $G = \mathsf{PSL}(2,\mathbb{R})$ and ℓ_α is the length of α on a hyperbolic structure, the Hamiltonian flow is the twist flow in $\mathcal{T}(S)$. One can use this fact to justify Wolpert's magic formula $\omega_{WP} = \sum_{i=1}^{3g-3} d\ell_{c_i} \wedge d\tau_{c_i}$ in Fenchel-Nielsen coordinates.

An interesting observation is that in $\mathcal{T}(S)$, the hyperbolic structure on a pair of pants is completely determined by the lengths of the boundary curves, i.e. by the conjugacy classes for the corresponding representation. It is no longer true that in higher rank, a representation on a pair of pants is fully determined by the conjugacy classes of the boundary curves.

New Hamiltonian flows were defined by Wienhard and Zhang in [6] for $\mathsf{Hit}(\pi, \mathsf{PGL}(3,\mathbb{R}))$, called eruption flows. These flows deform the interior of a pair of pants, while preserving the conjugacy classes of the boundary. They are roughly described as follows, with the figure inspired from [6]. Take an ideal triangulation T, T' (in teal and green) of a pair of pants. A Hitchin representation determines a boundary map $\xi\colon \partial_\infty \pi \to \mathcal{F}(\mathbb{R}^3)$. The endpoints of lifts of T are sent to flags whose 2 dimensional parts determine a triangle Δ (in black) in $\mathbb{RP}^2$. Connecting the one dimensional part of the flag to the vertex of Δ opposite of it, one obtains a smaller equilateral (with respect to the Hilbert metric induced by Δ) triangle Λ (in purple). The eruption flow increases the side-length of Λ by shifting the one dimensional parts of ξ along the two dimensional parts. The process is repeated for T' but decreasing the length, and then repeated for the other lifts.

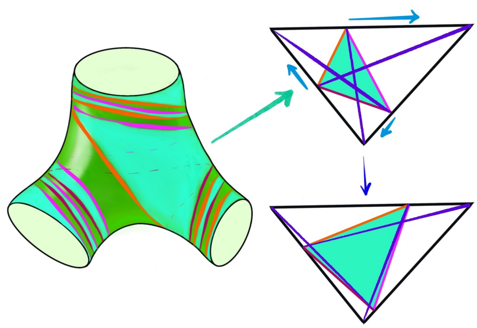


Figure 1. Eruption Flow

References

[1] W. M. Goldman, *The symplectic nature of fundamental groups of surfaces*, Advances in Mathematics **54** (1984), 200–225.

[2] W. M. Goldman, *Invariant functions on Lie groups and Hamiltonian flows of surface group representations*, Inventiones mathematicae **85** (1986), 263–302.

[3] F. Labourie, R. A. Wentworth, *Variations along the Fuchsian locus*, Annales Scientifiques de l'ENS **51** (2015), 487–547.

[4] I. Kim, G. Zhang *Kähler metric on the space of convex real projective structures on surface*, J. Differential Geometry **106** (2017), 127–137.

[5] F. Labourie, *Cyclic surfaces and Hitchin components in rank 2*, Annals of Mathematics **185** (2017), 1–58.

[6] A. Wienhard, T. Zhang, *Deforming convex real projective structures*, Geometriae Dedicata **192** (2018), 327–360.

Higher Degree Complex Structures

Alexander Nolte

(mentored by A. Thomas)

The aim of this talk is to present the current state of the theory of higher complex structures, as introduced by Fock and Thomas in [1]. Our starting point is the following perspective on complex structures on a closed, oriented surface S.

A complex structure on S is equivalent to an endomorphism $J \in \mathrm{End}(T^*S)$ so that $J^2 = -\mathrm{Id}$. At a point $x \in S$, the action of J on the complexified cotangent space $T_x^{*\mathbb{C}}$ diagonalizes along two eigenspaces $V_x^{\pm i}$ with eigenvalues $\pm i$. As J is an endomorphism of the *real* cotangent bundle, V_x^i and V_x^{-i} must be conjugate under the natural conjugation on $T^{*\mathbb{C}}(S)$. As a consequence, J is entirely determined by its $-i$ eigenspaces, which are linear directions within the complexified cotangent bundle.

The idea of higher complex structures is to replace these linear directions with "polynomial directions." These are encoded as sections of a bundle whose fibers are special ideals in algebras of polynomials on fibers of T^*S. The "higher" in "higher complex structures" refers to raising the degree of polynomials considered: degree $n-1$ polynomials correspond to n-complex structures.

In our exposition, the relevant algebras of polynomials on T^*S consist of appropriate restrictions along the zero section Z^*S of the cotangent bundle of jets of functions $f : T^*S \to \mathbb{C}$ that vanish on Z^*S, as in [2]. Diffeomorphisms of T^*S that map Z^*S to itself act on these functions on the right by pre-composition, and this gives rise to an action on higher complex structures.

The symplectic structure of T^*S allows us to pick out the right group of these diffeomorphisms to use as an equivalence relation in our setting. This turns out to be the group $\mathrm{Ham}_c^0(T^*S)$ consisting of Hamiltonian diffeomorphisms of T^*S generated by compactly supported Hamiltonian flows that setwise fix the zero section

This material is based upon work supported by the National Science Foundation under Grant No. 1842494.

at all times. The group $\mathrm{Ham}_c^0(T^*S)$ is called the *higher degree diffeomorphism group.*

The action of the higher degree diffeomorphism group on n-complex structures has a rather different flavor than the action of the group $\mathrm{Diff}_0(S)$ of diffeomorphisms isotopic to the identity on complex structures. A new difficulty in working with the action of $\mathrm{Ham}_c^0(T^*S)$ is that for $n > 2$, the quotient of $\mathrm{Ham}_c^0(T^*S)$ by the kernel of its action on degree-n complex structures acts neither properly nor freely. On the other hand, the action of $\mathrm{Ham}_c^0(T^*S)$ on general n-complex structures has some useful algebraic structure which is not present in the $n = 2$ case.

In any matter, the key definition of this talk is:

Definition 1. The *degree-*n *Fock-Thomas space* $\mathcal{T}^n(S)$ is the quotient of the collection of all orientation-compatible[1] degree-n complex structures on S by the action of the higher degree diffeomorphism group.

This all may seem rather abstract, but there are coordinate systems in which higher complex structures can be worked with quite concretely. Following [1] (c.f. [2], §6), one way to do this begins by fixing a reference complex structure Σ_c. Then take a local holomorphic coordinate z, and write $p := \partial_z, \overline{p} := \partial_{\overline{z}}$. Any orientation-compatible n-complex structure I has a unique expression of the form $I = \langle -\overline{p} + \mu_2 p + \cdots + \mu_n p^{n-1}\rangle$ with $|\mu_2| < 1$. The coefficients μ_k $(k = 2, ..., n)$ transform as $(-k+1, 1)$-tensors on the reference Riemann surface Σ_c and give coordinates for the space of all orientation-compatible n-complex structures on S. In particular, μ_2 is a Beltrami differential on Σ_c.

It is worth mentioning that in this perspective, the orbit of the μ_2-coordinate of a n-complex structure under $\mathrm{Ham}_c^0(T^*S)$ coincides exactly with the orbit of the Beltrami differential μ_2 on Σ_c under the standard action of $\mathrm{Diff}_0(S)$ on complex structures on S. This gives rise to a natural identification of the degree-2 Fock-Thomas space $\mathcal{T}^2(S)$ and the Teichmüller space $\mathcal{T}(S)$.

In the following, suppose that S has genus at least 2. The basic structural features of Fock-Thomas spaces are:

Theorem 2 (Fock-Thomas [1]). *The degree-n Fock-Thomas space $\mathcal{T}^n(S)$ has:*

(1) *Natural projections $p_k : \mathcal{T}^n(S) \to \mathcal{T}^k(S)$, $(k < n)$ and $p : \mathcal{T}^n(S) \to \mathcal{T}(S)$,*
(2) *A natural injection $i_n : \mathcal{T}(S) \to \mathcal{T}^n(S)$,*
(3) *A mapping class group action.*

The projections in Theorem 2 come from truncating polynomials, the inclusion is in the above coordinates $[\mu] \mapsto [(\mu_2, 0, ..., 0)]$, and the mapping class group action is analogous to the mapping class group action on Teichmüller space.

Further structural results follow from recent analysis by the speaker of the structure of the action of higher degree diffeomorphisms on n-complex structures:

Theorem 3 (Nolte [2]). *Let $\mathcal{T}^n(S)$ be the degree-n Fock-Thomas space.*

(1) *$\mathcal{T}^n(S)$ has a natural smooth structure, diffeomorphic to $\mathbb{R}^{(2g-2)(n^2-1)}$,*

[1] Degree-n complex structures have underlying complex structures. This means that the orientation induced by the underlying complex structure agrees with the orientation of S.

(2) *$\mathcal{T}^n(S)$ has a natural complex structure and a real-dimension $n-2$ family of compatible Kähler metrics,*
(3) *The projections $p_k : \mathcal{T}^n(S) \to \mathcal{T}^k(S)$ and $p : \mathcal{T}^n(S) \to \mathcal{T}(S)$ are holomorphic vector-bundle fibrations. The zero section of the vector bundle structure over $\mathcal{T}(S)$ is the image of the injection $i_n : \mathcal{T}(S) \to \mathcal{T}^n(S)$.*
(4) *The mapping class group action on $\mathcal{T}^n(S)$ is proper, and preserves all above structures.*

The guiding conjecture in the study of higher complex structures, and the point of connection to the theme of this arbeitgemeinschaft, is:

Conjecture 4 (Fock-Thomas)**.** *There is a canonical, mapping class group equivariant diffeomorphism between $\mathcal{T}^n(S)$ and the $\mathrm{PSL}(n,\mathbb{R})$ Hitchin component.*

We mention that analogues of n-complex structures have been defined for general semi-simple Lie groups by Thomas [4], and are conjectured to parametrize other higher Teichmüller spaces.

Fock and Thomas' conjecture is known to be true for $n = 2$ and $n = 3$ ([1], [2] respectively), though the proof for $n = 3$ relies strongly on tools specific to rank 2 Hitchin components. Some other evidence in favor of the conjecture is that the two spaces are non-canonically diffeomorphic, both possess distinguished copies of Teichmüller space, and both have natural proper mapping class group actions. Additionally, Thomas has introduced a program to address the conjecture in general [3].

Confirmation of Fock and Thomas' conjecture would answer a number of open problems in higher Teichmüller theory. In particular, a positive resolution would give $\mathrm{PSL}(n,\mathbb{R})$ Hitchin components natural vector bundle structures over $\mathcal{T}(S)$, mapping class group invariant complex structures, and natural mapping class group invariant Kähler structures, as Fock-Thomas spaces have all of these. Additionally, this would substantially further our understanding of the mapping class group action on $\mathrm{PSL}(n,\mathbb{R})$ Hitchin components, since the mapping class group action on $\mathcal{T}^n(S)$ is well-understood.

Most compellingly, though, a proof of Fock and Thomas' conjecture would open up an avenue of investigation of Hitchin components analogous to the Ahlfors-Bers development of Teichmüller theory. The interplay between the complex-geometric, hyperbolic-geometric, and algebraic perspectives on Teichmüller space is at the center of Teichmüller theory. Fock and Thomas' conjecture gives a compelling candidate for an analogue of the complex-geometric perspective on Teichmüller space in the higher rank setting, where it is currently largely absent.

References

[1] V.V. Fock, A. Thomas, Higher Complex Structures, *Int. Math. Res. Not. IMRN.* p. 15873-15893, 2021.
[2] A. Nolte, Canonical Maps from Spaces of Higher Complex Structures to Hitchin Components, preprint, 2022.

[3] A. Thomas, Higher Complex Structures and Flat Connections, preprint, 2020.
[4] A. Thomas, Generalized Punctual Hilbert Schemes and $\mathfrak{g}$-complex Structures, *Int J. of Mathematics* Vol. 32, No. 1, 2022.

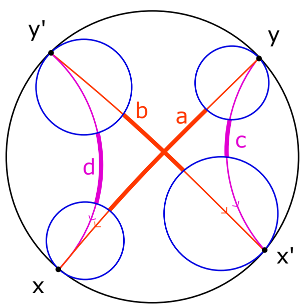
Universal Higher Teichmuller Spaces

Charlie Reid

In this talk we discussed two spaces which are claimed to be $n = \infty$ versions of the $SL(n, \mathbb{R})$ Hitchin components for a Riemann surface S: Labourie's space [4] of homomorphisms $\pi_1(S) \to \mathrm{Diff}^h(S^1) \ltimes C^{1,h}$, and Hitchin's space [2] of Hyperkahler disk bundles in T^*S. Labourie's space is analogous to finite rank Hitchin components in that it parametrizes a collection of homomorphisms out of $\pi_1(S)$, and is universal in the sense that it contains all the $SL_n\mathbb{R}$ Hitchin components. Hitchin's space on the other hand depends on a complex structure on S, and parametrizes geometric objects which are are analogous to solutions to Hitchin's equation. The relationship between these two spaces is still mysterious, but there are intriguing connections.

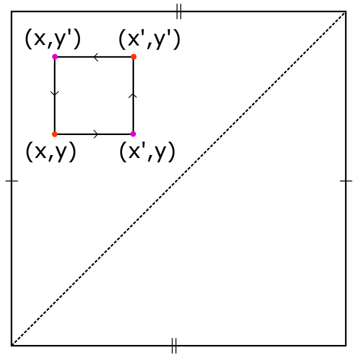
Labourie first proves that all $SL_n\mathbb{R}$ Hitchin components of S, as well as the space of negatively curved metrics on S, naturally embed into the space of cross ratios on $\partial\pi_1(S) \simeq S^1$. He defines a cross ratio to be a hölder function of tuples of 4 points in $\partial\pi_1(S)$ satisfying a list of 5 axioms.

The negatively curved metric case [5] elucidates these axioms, and the way Labourie goes beyond cross ratios later. Fix a negatively curved metric g on S, and choose 4 distinct points x, x', y, y' in the Gromov boundary $\partial\pi_1(S)$. The metric lets us define a visual boundary of the universal cover, which gets identified with $\partial\pi_1(S)$. After choosing horospheres around the 4 points we define the cross ratio to be the following combination of lengths of geodesic segments.



$$b_g(x, y, x', y') = a + b - c - d$$

Note that this quantity does not depend on the choice of horocycles. Amazingly, $b(x, y, x', y')$ can be interpreted as a holonomy. The unit tangent bundle $T^1\tilde{S}$ maps to $\mathcal{G} = \partial\pi_1(S) \times \partial\pi_1(S)\backslash\Delta$, via extending a unit vector to a complete geodesic and recording the endpoints. Geodesic flow makes $T^1\tilde{S}$ into a principal $\mathbb{R}$ bundle over $\mathcal{G}$. The horocyclic and antihorocyclic distributions on $T^1\tilde{S}$ span a horizontal distribution on this principal $\mathbb{R}$ bundle, and $b(x, y, x', y')$ is its holonomy around the rectangle $[x, x'] \times [y, y'] \subset \mathcal{G}$.



The axioms Labourie gives for a cross ratio are simply the equations satisfied by the holonomy of a connection of a principal $\mathbb{R}$ bundle on $\mathcal{G}$, with some non-degeneracy and completeness conditions on its curvature.

We can also associate cross ratios to Hitchin representations. Let V be a real vector space with volume form, and let ρ be a Hitchin representation $\pi_1(S) \to SL(V)$. Let $\xi : \partial\Gamma \to \mathbb{P}(V)$ and $\xi^* : \partial\Gamma \to \mathbb{P}(V^*)$ be the limit curves.

$$B_\rho(x, y, x', y') := \frac{\langle \xi(x), \xi^*(y) \rangle \langle \xi(x'), \xi^*(y') \rangle}{\langle \xi(x), \xi^*(y') \rangle \langle \xi(x'), \xi^*(y) \rangle}$$

Labourie shows that Hitchin representations are determined by their cross ratios, and gives a complete characterization of the cross ratios arising from $SL_n\mathbb{R}$ Hitchin representations.

We define $b_\rho = log|B_\rho|$ to get an additive cross ratio which we can compare with b_g from above. Labourie raises the interesting question of wether cross ratios associated to negatively curved metrics can be obtained as limits of cross ratios associated with Hitchin representations.

B_ρ turns out to be the holonomy of a natural $\mathbb{R}^*$ bundle $X \to \mathcal{G}$ with connection. The total space X is the set of pairs $(v, \alpha) \in V \times V^*$ such that $[v]$ is in the image of ξ, $[\alpha]$ is in the image of ξ^*, and $\alpha(v) = 1$. The connection is again spanned by two natural foliations: the foliation by curves of constant v, and the foliation by curves of constant α. The quotient $\Gamma\backslash X$ has all the salient features of a unit tangent bundle of S for a negatively curved metric, except for a projection to S.

Labourie could have declared the universal Hitchin component of S to be the space of all Γ invariant cross ratios on $\partial\Gamma$, but he decided to single out cross ratios which are holonomies of equivariant principal $\mathbb{R}$ bundles $X \to \mathcal{G}$ such that $\Gamma\backslash X$ is a compact holder manifold, and the $\mathbb{R}$ action descends to an Anosov flow. The horocyclic and anti-horocyclic foliations are determined by this Anosov flow, and give rise to the connection. In this setting, there is a unique, (up to $\mathrm{Diff}^h(S^1) \ltimes C^{1,h}(S^1)$), way to identify X with the space of 1-jets of functions on S^1, such that $\pi_1(S)$ gets taken to a subgroup of $\mathrm{Diff}^h(S^1) \ltimes C^{1,h}(S^1)$. This is roughly how Labourie is lead to studying conjugacy classes of homomorphisms $\pi_1(S) \to \mathrm{Diff}^h(S^1) \ltimes C^{1,h}(S^1)$.

While Labourie studies dynamical structures on the unit tangent bundle, Hitchin studies differential-geometric structures on the unit disk bundle in T^*S. The starting point is the standard $SU(2)$ harmonic bundle coming from the hyperbolic metric g on S.

$$E = K^{1/2} \oplus K^{-1/2} \qquad \varphi = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad h = g^{-1/2} \oplus g^{1/2}$$

$SU(2)$ acts on $\mathbb{C}P^1$ so we can take the associated $\mathbb{C}P^1$ bundle. The total space is simply $P(K^{1/2} \oplus K^{-1/2})$. Hitchin explains how the harmonic bundle (E, φ, h) translates to a hyperkahler structure on $P(K^{1/2} \oplus K^{-1/2})$ which has a "fold singularity" along the equatorial circle bundle. For instance, the Chern connection D_h corresponds to the horizontal distribution given by Riemannian complements to the fibers, and the circle of flat connections

$$\{D_h + \zeta^{-1}\varphi + \zeta\varphi^\dagger \ \ : \ \ \zeta \in U(1)\}$$

corresponds to the circle of complex structures perpendicular to the standard one. The real structure on (E, φ, h) corresponds to an anti-holomorphic involution of $P(K^{1/2} \oplus K^{-1/2})$ which fixes this circle bundle, and swaps northern and southern hemispheres. Quotienting by this involution leaves us with a hyperkahler manifold M, identified holomorphic-symplectically with the unit disk bundle in T^*S, with hyperkahler metric satisfying a particular boundary condition.

Hitchin proposed that the $SL(\infty, \mathbb{R})$ Hitchin component is the space of all deformations of this structure, and conjectured that it is parametrized by an infinite dimensional vector space.

$$\bigoplus_{n \geq 2} H^0(S, K^n)$$

Biquard [1] has shown that indeed an open neighborhood in the moduli space of these folded hyperkahler disk bundles in T^*S is parametrized by an open neighborhood in this vector space.

A disk bundle in T^*S, which is a small deformation of the unit disk bundle of a hyperbolic metric, defines a negatively curved Finsler metric [3], which has a geodesic flow, and thus defines a point in Labourie's universal Hitchin component. The extent to which this map from Hitchin's space to Labourie's space is or isn't globally defined, injective, and surjective, are all interesting questions.

References

[1] O. Biquard. *Metriques hyperkähleriennes pliées*, Bull. Soc. Math. France 147 (2019) 303–340.

[2] N. Hitchin. *Higgs bundles and diffeomorphism groups*, In Surveys in differential geometry 2016. Advances in geometry and mathe-matical physics, volume 21 of Surv. Differ. Geom., pages 139–163.
Int. Press, Somerville, MA, 2016.

[3] N. Hitchin. *A Teichmüller space for negatively curved surfaces.* arXiv:2010.07186, 2020.

[4] F. Labourie, *Cross ratios, surface groups, $PSL(n, \mathbb{R})$ and diffeomorphisms of the circle*, Publ. Math. IHES (2007), no. 106, 139–213.
[5] J. P. Otal. *Le spectre marqué des longueurs des surfaces à courbure négative*, Ann. Math. (2), 131 (1990), 151–162.

Reporter: Enya Hsiao

Participants

Raphael Appenzeller
Forschungsinstitut für Mathematik
ETH-Zürich
ETH Zentrum
Rämistr. 101
8092 Zürich
SWITZERLAND

Jacques Audibert
CNRS
Sorbonne Université
Box 169
4 Place Jussieu
75252 Paris Cedex
FRANCE

Dr. Mathieu Ballandras
Instituto de Matematicas y Fisica
Fundamental
CSIC
c/Serrano, 113 bis 123
28006 Madrid
SPAIN

Guillermo Barajas Ayuso
ICMAT
Instituto de Ciencias Matematicas
Campus de Cantoblanco, UAM
Nicolás Cabrera, nº 13-15
28049 Madrid
SPAIN

Dr. Timothée Bénard
Centre for Mathematical Sciences
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM

Dr. Jean-Philippe Burelle
Département de Mathématiques
Faculté des Sciences
Université de Sherbrooke
2500 boulevard de l'université
Sherbrooke QC J1K 2R1
CANADA

Irving Calderón
Dept. of Mathematical Sciences
Durham University
Science Laboratories
South Road
Durham DH1 3LE
UNITED KINGDOM

Fernando Camacho Cadena
Mathematisches Institut,
Universität Heidelberg,
Im Neuenheimer Feld 205
Heidelberg 69117
GERMANY

Dr. Valentina Disarlo
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 205
69120 Heidelberg
GERMANY

Sami Douba
Institut des Hautes Études Scientifiques
35 Rte de Chartres
91440 Bures-sur-Yvette
FRANCE

Dr. Spencer Dowdall
Department of Mathematics
Vanderbilt University
1326 Stevenson Center
Nashville TN 37240-0001
UNITED STATES

Parker Evans
Department of Mathematics
Rice University
P.O. Box 1892
Houston, TX 77005-1892
UNITED STATES

Prof. Dr. Livio Flaminio
Département de Mathématiques
Université de Lille
Cité Scientifique - Bâtiment M2
59655 Villeneuve d'Ascq Cédex
FRANCE

Xenia Flamm
Departement Mathematik
ETH Zürich
Rämistr. 101
8092 Zürich
SWITZERLAND

Balthazar Flechelles
I.H.E.S.
Le Bois Marie
35, route de Chartres
91440 Bures-sur-Yvette
FRANCE

Blandine Galiay
95880 Enghien les Bains
FRANCE

Guillermo Gallego Sanchez
Facultad de Ciencias Matematicas
Universidad Complutense de Madrid
Plaza de las Ciencias, 3
28040 Madrid
SPAIN

Enya Hsiao
Mathematisches Institut
Ruprecht-Karls-Universität Heidelberg
Im Neuenheimer Feld 205
69120 Heidelberg
GERMANY

Victor Théo Jaeck
Departement Mathematik
ETH-Zürich
ETH Zentrum
Rämistrasse 101
8092 Zürich
SWITZERLAND

Prof. Fanny Kassel
I.H.E.S.
Le Bois Marie
35, route de Chartres
91440 Bures-sur-Yvette
FRANCE

Dr. Dani Kaufman
Department of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
2100 København
DENMARK

Max Lahn
Department of Mathematics
University of Michigan - Ann Arbor
East Hall
530 Church Street
Ann Arbor MI 48109-1043
UNITED STATES

Dr. Jialun Li
Institut für Mathematik
Universität Zürich
Winterthurerstrasse 190
8057 Zürich
SWITZERLAND

Dr. Keivan Mallahi-Karai
School of Science
Jacobs University Bremen
Campus Ring I
28725 Bremen 28759
GERMANY

Dr. Giuseppe Martone
Department of Mathematics
Yale University
10 Hillhouse Ave
New Haven CT 06511
UNITED STATES

Dr. Filippo Mazzoli
Department of Mathematics
University of Virginia
Kerchof Hall
141 Cabell Drive
P.O. Box 400137
Charlottesville VA 22904-4137
UNITED STATES

Abderrahim Mesbah
Université du Luxembourg
Unite de Recherche en Mathématiques
Av. de l'Universite
P.O. Box 2
4365 Esch-sur-Alzette
LUXEMBOURG

Jared Miller
208 Love Building
Department of Mathematics
Florida State University
1017 Academic Way
Tallahassee, FL 32306-4510
UNITED STATES

Alex Moriani
Laboratoire de Mathématiques
Université de Nice
Parc Valrose
06108 Nice Cedex
FRANCE

Merik Niemeyer
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 205
69120 Heidelberg
GERMANY

Alex Nolte
Department of Mathematics
Rice University
P.O. Box 1892
Houston TX 77005-1892
UNITED STATES

Jun-Prof. Dr. Maria Beatrice Pozzetti
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 205
69120 Heidelberg
GERMANY

Charles A. Reid
Department of Mathematics
The University of Texas at Austin
2515 Speedway
PMA 8.100 Austin 78705
UNITED STATES

Ulysse Remfort-Aurat
Institut de Mathématiques de Marseille
Aix-Marseille Université (I2M)
39, rue F. Joliot-Curie
13453 Marseille Cedex 13
FRANCE

Lisa Ricci
Forschungsinstitut für Mathematik
ETH-Zürich
ETH Zentrum
Rämistr. 101
8092 Zürich
SWITZERLAND

Dr. Max Riestenberg
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 205
69120 Heidelberg
GERMANY

Nicholas Rungi
SISSA
International School for Advanced Studies
Via Bonomea 265
34136 Trieste
ITALY

Dr. Andrés Sambarino
CNRS
Sorbonne Université
Box 169
4 Place Jussieu
75252 Paris Cedex
FRANCE

Dr. Cagri Sert
Institut für Mathematik
Universität Zürich
Winterthurerstrasse 190
8057 Zürich
SWITZERLAND

Florian Stecker
Department of Mathematics
The University of Texas at Austin
1 University Station C1200
Austin, TX 78712-1082
UNITED STATES

Prof. Dr. Andreas B. Thom
Institut für Geometrie
Fakultät für Mathematik
Technische Universität Dresden
01062 Dresden
GERMANY

Dr. Alexander Thomas
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 205
69120 Heidelberg
GERMANY

Dr. Konstantinos Tsouvalas
IHÉS
Institut des Hautes Ètudes Scientifiques
35, route de Chartres
91440 Bures-sur-Yvette
FRANCE

Prof. Dr. Martin Ulirsch
Institut für Mathematik
Goethe-Universität Frankfurt
Robert-Mayer-Str. 6-8
Postfach 111932
60325 Frankfurt am Main
GERMANY

Tianqi Wang
Department of Mathematics
National University of Singapore
10 Lower Kent Ridge Road
Singapore 119076
SINGAPORE

Theodore Weisman
Department of Mathematics
The University of Texas at Austin
1 University Station C1200
Austin TX 78712-1082
UNITED STATES

Prof. Dr. Anna Katharina Wienhard
Mathematisches Institut
Ruprecht-Karls-Universität Heidelberg
Im Neuenheimer Feld 205
69120 Heidelberg
GERMANY

Dr. Feng Zhu
Department of Mathematics
University of Wisconsin-Madison
480 Lincoln Drive
Madison WI 53706-1388
UNITED STATES