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Arbeitsgemeinschaft: Quantitative Stochastic Homogenization

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Abstract. Homogenization means approximating the effective, i. e. macroscopic, behavior of a heterogeneous medium by a homogeneous one, which amounts to a substantial conceptual and practical reduction of complexity. Stochastic homogenization means that one is considering an ensemble of, i. e. a probability measure on, such heterogeneities (typically expressing a lack of knowledge of the details); and that the effective behavior is also deterministic next to being homogeneous. The aim of this Arbeitsgemeinschaft is to present the recent progress in this field.

Mathematics Subject Classification (2020): 35B27, 60H07, 60H15, 76M50, 78M40.

Introduction by the Organizers

We mostly focus on media that are described by a linear elliptic operator in divergence form, which may describe the conductive, elastic or viscous properties of the medium. Quantitative stochastic homogenization means that we focus on quantitative aspects of this theory, where there has been much recent progress. The subject, which is motivated by materials science, combines the theory of linear partial differential equations (PDE) with hands-on probability theory, and also has a life in computational mathematics and statistical physics.

We propose a program that starts with the general theory of oscillatory coefficients (with its rational mechanics flavor), and then addresses the qualitative theory of stochastic homogenization (with its functional analytic flavor). We then connect to the topic of random walks in random environments. Quantitative stochastic homogenization is intimately connected to elliptic regularity theory in Hölder and L^p-spaces: Large-scale regularity emerges and helps in the quantification.

Stochastic estimates are typically derived from spectral gap or from finite range assumptions. The rate of the convergence depends on whether it is expressed in terms of strong or weak topologies. All these topics are covered. We then branch out to computational aspects, to the design of electro-magnetic (meta-)materials, to applications in fluids, to boundary layer effects, and to wave propagation.

Arbeitsgemeinschaft: Quantitative Stochastic Homogenization

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Abstracts

Two-scale Expansion and Convergence

Harprit Singh, David Wiedemann

1. Two-scale expansion (by H. Singh)

We study the ϵ-dependent family of equations

$$-\nabla \cdot (A(\cdot/\epsilon)\nabla u_\epsilon) = f, \qquad \int_{\mathbb{T}^d} u_\epsilon = 0 \tag{1}$$

on the d-dimensional torus $\mathbb{T}^d$, where $A = \{a_{i,j}\}_{i,j=1}^d \in L^\infty(\mathbb{T}^d)$ is a symmetric[1] matrix which is uniformly elliptic (i.e. there exists a constant $\lambda_0 > 0$ such that for every $\xi \in \mathbb{R}^d$ and almost every $x \in \mathbb{T}^d$ one has $\sum_{i,j=1}^d a_{i,j}(x)\xi_i\xi_j \geq \lambda|\xi|^2$) and $f \in L^2(\Pi^d)$ satisfies $\int_{\mathbb{T}^d} f = 0$.

It follows by standard PDE arguments that this problem is well posed for every $\epsilon > 0$. The aim of the talk is to study the ϵ dependence of the solutions $\{u_\epsilon\}_{\epsilon\in(0,1)} \in H^1(\Pi^d)$ as $\epsilon \to 0$. The centerpiece of the first half of the talk is the *two scale expansion ansatz*:

$$u_\epsilon(x) \sim u_0(x, x/\epsilon) + \epsilon u_1(x, x/\epsilon) + \epsilon^2 u_2(x, x/\epsilon) + \dots \,. \tag{2}$$

Formally writing $y = x/\epsilon$, inserting the ansatz into (1) and equating powers of epsilon (and ignoring the summands supressed in ...) one finds the following.

- The function $u_0(x, y)$ depends only on the fist variable, i.e. $u(x, y) = u(x)$.
- If one defines the *first order correctors* $\{\phi_i\}_{i=1}^d$ as the solutions of

$$-\nabla \cdot (A(\cdot)\nabla\phi_i) = \nabla \cdot (A(\cdot)e_i) \,, \qquad \int_{\mathbb{T}^d} \phi_i = 0 \,, \tag{3}$$

  then $u_1(x, y)$ has the form

$$u_1(x, y) = \sum_{i=1}^d \phi_i(y)\frac{\partial u_0}{\partial x_i}(x) \,. \tag{4}$$

- If one defines the *homogenized matrix* $\bar{A} = \{\bar{a}_{i,j}\}_{i,j=1}^d$ as

$$\bar{a}_{i,j} := \int_{\mathbb{T}^d} \langle A(y)(e_j + \phi_j(y)), e_i\rangle \, dy \,, \tag{5}$$

  then u_0 solves the *homogenized* equation

$$-\nabla \cdot (\bar{A}\nabla u_0) = f \tag{6}$$

  with $\int_{\mathbb{T}^d} u_0 = 0$. In particular $\bar{A}$ is elliptic, so that this equation is well posed.

[1] The matrix being symmetric simplifies certain arguments, but is not crucial.

Thus (at least heuristically)

$$u_\epsilon(x) \sim u_0(x) + \epsilon \sum_{i=1}^{d} \phi_i(x/\epsilon) \frac{\partial u_0}{\partial x_i}(x) \ .$$

Indeed, in the second part of the talk we shall see that this is a good approximation.

Lastly, we mention that it is possible to follow the ansatz (2) to higher order (encountering *higher order correctors* and *higher order homogenised equations*) and refer to the main theorem of [1].

2. Two-scale convergence (by D. Wiedemann)

In this second part, we consider two-scale convergence introduced in [2] and [3], which provides from the start a rigorous approach to periodic homogenisation. We consider the weak formulation of (1) for a general domain $\Omega \subset \mathbb{R}^d$, zero Dirichlet boundary conditions, a coefficient field a as above and a source term $f \in L^2(\Omega)$, i.e. we study $u_\epsilon \in H_0^1(\Omega)$ such that

$$\int_\Omega a\left(\frac{x}{\epsilon}\right) \nabla u_\epsilon(x) \cdot \nabla \varphi(x)\, dx = \int_\Omega f(x)\varphi(x)\, dx \tag{7}$$

for all $\varphi \in H_0^1(\Omega)$.

Classical L^p-theory only provides weak convergence of $a\left(\frac{\cdot}{\epsilon}\right) \rightharpoonup \int_Y a(y)dy$ and weak-compactness results for u_ϵ, i.e. $\nabla u_\epsilon \rightharpoonup \nabla u_0$ for $\epsilon \to 0$. Therefore, we can not pass to the limit $\epsilon \to 0$ on the left-hand side of (7) by taking the product of the limits. *Two-scale convergence*, nevertheless, allows one to pass to a limit.

A sequence u_ϵ in $L^p(\Omega)$, for $p, q \in (1, \infty)$, with $\frac{1}{p} + \frac{1}{q} = 1$, is said to *(weakly) two-scale converge* to $u_0 \in L^p(\Omega \times Y)$ (where $Y = (0,1)^d$) if

$$\lim_{\epsilon \to 0} \int_\Omega u_\epsilon(x) \varphi\left(x, \frac{x}{\epsilon}\right) dx = \int_\Omega \int_Y u_0(x,y)\varphi(x,y)\, dy\, dx \tag{8}$$

for all test functions $\varphi \in L^q(\Omega; C_\#(Y))$. Motivated by strictly convex Banach spaces, u_ϵ is said to *strongly two-scale converge* to u_0 if, furthermore,

$$\lim_{\epsilon \to 0} ||u_\epsilon||_{L^p(\Omega)} = ||u_0||_{L^p(\Omega \times Y)} \ .$$

This notion is justified by the following compactness result. Let u_ϵ be a bounded sequence in $L^p(\Omega)$ for $p \in (1, \infty)$. Then, there exists a subsequence u_ϵ and $u_0 \in L^p(\Omega \times Y)$ such that u_ϵ two-scale converges to u_0.

It can be shown, that the coefficients as well as the test functions of (8) strongly two-scale converge. Thus, we can interchange the limit and the product, i.e. let u_ϵ be a sequence in $L^p(\Omega)$ two-scale converging to u_0 and v_ϵ a sequence in $L^q(\Omega)$ strongly two-scale converge to v_0 for $p, q, r \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then, $u_\epsilon v_\epsilon$ two-scale converges to $u_0 v_0$.

Having this property and the compactness result, we can pass to the limit $\epsilon \to 0$ in (7). However, this is not enough in order to derive a well posed limit problem, since we lose the gradient structure for the two-scale limit of ∇u_ϵ. Therefore, we consider the following two-scale compactness result for gradients (cf. [3]). Let u_ϵ

be sequence which weakly converges to u_0 in $H^1(\Omega)$. Then, there exists $u_1 \in L^2(\Omega; H^1_\#(Y)/\mathbb{R})$ and a subsequence of ∇u_ϵ which two-scale converges to $\nabla_x u_0 + \nabla_y u_1$.

Homogenisation of (7) **by two-scale convergence:** Standard elliptic PDE theory provides existence of a unique solution $u_\epsilon \in H^1_0(\Omega)$ to (7) and uniform boundedness in $H^1_0(\Omega)$. Therefore, we can pass to a subsequence such that u_ϵ converges weakly to $u_0 \in H^1(\Omega)$. By the previous result, there exists $u_1 \in L^2(\Omega; H^1_\#(Y)/\mathbb{R})$ and a subsequence such that ∇u_ϵ two-scale converges to $\nabla_x u_0 + \nabla_y u_1$. Moreover, we note that $a\left(\frac{\cdot}{\epsilon}\right)$ strongly two-scale converges to a.

Finally, we test (7) with $\varphi(x)+\epsilon\varphi_1\left(x, \frac{x}{\epsilon}\right)$ for $\varphi \in D(\Omega)$ and $\varphi_1 \in D(\Omega; C^\infty_\#(Y))$ and note that $\nabla\epsilon\varphi_1\left(x, \frac{x}{\epsilon}\right) = \epsilon\nabla_x\varphi_1\left(x, \frac{x}{\epsilon}\right) + \nabla_y\varphi_1\left(x, \frac{x}{\epsilon}\right)$. Thus, we can pass to the limit in (7) and obtain the *two-scale limit problem*:

$$\int_\Omega \int_Y a(y)(\nabla_x u_0(x) + \nabla_y u_1(x,y)) \cdot (\nabla_x\varphi(x) + \nabla_y\varphi_1(x,y))\, dy\, dx = \int_Y f(x)\varphi(x)\, dx$$

for all $\varphi \in D(\Omega)$ and $\varphi_1 \in D(\Omega; C^\infty_\#(Y))$ (and by density for all $\varphi \in H^1_0(\Omega)$ and $\varphi_1 \in L^2(\Omega; H^1_\#(Y)/\mathbb{R})$). It can be shown that the two scale limit problem has a unique solution $(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega; H^1_\#(Y)/\mathbb{R})$, see [3], which implies that the whole sequence converges. Moreover, by choosing $\varphi = 0$, we obtain (4). Then, choosing $\varphi_1 = 0$ and inserting (4) yields (6).

References

[1] Kamotski, Vladimir and Matthies, Karsten and Smyshlyaev, Valery P., *Exponential Homogenization of Linear Second Order Elliptic PDEs with Periodic Coefficients*, SIAM Journal on Mathematical Analysis **38** (2007), 1565-1587.

[2] G. Nguetseng, *A general convergence result for a functional related to the theory of homogenization*, SIAM J. Math. Anal. **20** (1989), 608–623.

[3] G. Allaire, *Homogenization and Two-scale Convergence*, SIAM J. Math. Anal. **23** (1992), 1482–1518.

H-Convergence

Adina Ciomaga, Markus Schmidtchen

H-convergence was introduced by Spagnolo [1] under the name of G-convergence, and later generalized by Tartar [4], Murat and Tartar [2] (see also [3]). It is a type of convergence that allows to establish a general compactness result for a family of uniformly elliptic and uniformly bounded operators, without making any structural assumptions such as periodicity, randomness, ergodicity, etc.

Setting. Given two positive constants $\alpha, \beta > 0$, let $\mathcal{M}_{\alpha,\beta}$ denote the space of all real-valued, uniformly coercive square matrices with uniformly coercive inverses, i.e., matrices which satisfy

$$\forall \xi \in \mathbb{R}^N \quad A\xi \cdot \xi \geq \alpha|\xi|^2 \quad \text{and} \quad A^{-1}\xi \cdot \xi \geq \beta|\xi|^2.$$

Let $D \subset \mathbb{R}^N$ be a bounded and open subset of $\mathbb{R}^N$. Consider a sequence of matrices $A^\varepsilon \in L^\infty(D; \mathcal{M}_{\alpha,\beta})$ with $\varepsilon > 0$, and introduce the following second order elliptic PDE with homogeneous Dirichlet boundary condition:

$$\left\{ \begin{array}{rl} -\mathrm{div}\left(A^\varepsilon(x)\nabla u^\varepsilon(x)\right) &= f(x), \quad \text{in } D \\ u^\varepsilon(x) &= 0, \quad \text{on } \partial D. \end{array} \right. \tag{1}$$

Facts. For any $f \in H^{-1}(D)$, the Lax-Milgram theorem provides a unique solution, $u^\varepsilon \in H_0^1(D)$, for problem (1), which satisfies the following weak formulation

$$\forall \varphi \in H_0^1(D) \quad \int_D A^\varepsilon \nabla u^\varepsilon \cdot \nabla \varphi dx = \langle f, \varphi \rangle_{H^{-1}, H_0^1(D)}.$$

In view of the uniform coercivity of A^ε, one obtains uniform bounds on the sequence of gradients, i.e.,

$$\|\nabla u^\varepsilon\|_{L^2(D)^N} \leq \frac{C}{\alpha} \|f\|_{H^{-1}(D)},$$

which implies the existence of a *subsequence*, still denoted by ε, and of a limiting function $u \in H_0^1(D)$, such that,

$$u^\varepsilon \rightharpoonup u, \text{ weakly in } H_0^1(D). \tag{2}$$

On the other hand, the uniform coercivity of the inverse matrices implies uniform boundness of A^ε, and this, in turn, gives uniform bounds on the fluxes, i.e.

$$\|A^\varepsilon \nabla u^\varepsilon\|_{L^2(D)^N} \leq \frac{C}{\alpha\beta} \|f\|_{H^{-1}(D)},$$

hence there exists a subsequence and a limiting flux $\sigma \in L^2(D)^N$, such that,

$$A^\varepsilon \nabla u^\varepsilon \rightharpoonup \sigma, \text{ weakly in } L^2(D)^N. \tag{3}$$

Goal. The main challenge is to establish a relationship between the limit $u \in H_0^1(D)$, provided by (2) and the limiting flux $\sigma \in L^2(D)^N$, provided by (3). More precisely, one would like to show there exists an 'average' matrix A^* such that $\sigma = A^* \nabla u$. If this were true, one could pass to the limit in the equation and identify u as a solution of

$$\left\{ \begin{array}{rl} -\mathrm{div}\left(A^*(x)\nabla u(x)\right) &= f(x), \quad \text{in } D \\ u(x) &= 0, \quad \text{on } \partial D. \end{array} \right. \tag{4}$$

This leads to the following definition of H convergence, for the coefficients of an elliptic PDE, defined in terms of properties of the solution.

Definition 1 (H-convergence)**.** *Let $(A^\varepsilon)_\varepsilon \subset L^\infty(D; \mathcal{M}_{\alpha,\beta})$. We say that the sequence A^ε H-converges to a matrix $A^* \in L^\infty(D; \mathcal{M}_{\alpha,\beta})$ if, for any $f \in H^{-1}(D)$, the sequence u^ε of solutions of* (1) *satisfies*

$$\left\{ \begin{array}{ll} u^\varepsilon \rightharpoonup u, & \text{weakly in } H_0^1(D), \\ A^\varepsilon \nabla u^\varepsilon \rightharpoonup A^* \nabla u, & \text{weakly in } L^2(D)^N, \end{array} \right. \tag{5}$$

where u is the solution of the limiting problem (4)*.*

This definition makes it now possible to establish the following sequential compactness theorem.

Theorem 1 (Main result). *For any sequence $(A^\varepsilon)_\varepsilon \subset L^\infty(D;\mathcal{M}_{\alpha,\beta})$ there exists a subsequence $(A^{\varepsilon'})_{\varepsilon'}$ and a homogenized matrix $A^* \in L^\infty(D;\mathcal{M}_{\alpha,\beta})$, such that $A^{\varepsilon'}$ H-converges to A^*.*

In contrast to the periodic setting, this compactness result does not provide any explicit formula for the limit A^*. Indeed, in the periodic case, the entire sequence convergences and the limit is given by the formula

$$A^*_{ij} = \int_Y A(y)(e_i + \nabla\phi_i(y)) \cdot (e_j + \nabla\phi_j(y))dy,$$

for all $i, j = 1 \dots N$. Here, Y is the periodic cell, $(e_i)_{i=1,\dots,N}$ a canonical basis of $\mathbb{R}^N$, and $(\phi_i)_{i=1,\dots,N} \subset H^1_\#(Y)$ is the family of periodic correctors, corresponding to each direction e_i, i.e., solutions of the periodic cell problems

$$-\mathrm{div} A(y)(e_i + \nabla\phi_i) = 0, \quad \text{in } Y. \tag{6}$$

Introducing the perturbed test functions $\phi_i^\varepsilon(x) = x_i + \varepsilon\phi_i(\varepsilon^{-1}x)$ and re-scaling the cell-problem, it is possible to see that A^* is in fact a weak limit of the corresponding fluxes

$$A^\varepsilon \nabla\phi_i^\varepsilon \rightharpoonup \int_Y A(y)(e_i + \nabla\phi_i)dy = A^* e_i, \quad \text{weakly in } L^2(D)^N.$$

While weak H^1-compactness of solutions $(u_\varepsilon)_\varepsilon$ and weak L^2-compactness of the fluxes $(A^\varepsilon \nabla u_\varepsilon)_\varepsilon$ are readily obtained, the difficulty lies in the identification of the limiting flux, which is a limit of a product of weakly convergent sequences. It is well-known that the weak limit of a product need not agree with the product of the weak limits. Using a compensated compactness method (the celebrated div-curl lemma), it is possible to identify the product of the weakly convergent sequences.

Lemma 1 (Div-Curl). *Let $(f_\varepsilon)_\varepsilon, (g_\varepsilon)_\varepsilon \subset L^2(D)^N$ be uniformly bounded in $L^2(D)^N$. Let $f, g \in L^2(D)^N$ such that $f_\varepsilon \rightharpoonup f$, $g_\varepsilon \rightharpoonup g$, as $\varepsilon \to 0$. Furthermore assume*

(i) *$(\mathrm{div}\ f_\varepsilon)_\varepsilon$ is precompact in $H^{-1}(D)$,*
(ii) *$(\mathrm{curl}\ g_\varepsilon)_\varepsilon$ is precompact in $H^{-1}(D)^{N\times N}$.*

Then, there holds

$$f_\varepsilon \cdot g_\varepsilon \longrightarrow f \cdot g, \quad \text{in} \quad \mathcal{D}'(D).$$

Remark 1 (Properties of H-convergence).
We list some properties of H-convergence:

(1) *A^* does not depend on the right-hand side, f.*
(2) *the coercivity constants $\alpha, \beta > 0$ are the same for A^**
(3) *in the compactness proof it is necessary to extract a subsequence — this is due to the dependence of the oscillation speed on ε, in some sense. In the periodic setting, the whole sequence converges.*
(4) *If $A^\varepsilon \overset{H}{\to} A^*$, then any subsequence H-converges to the same limit, i.e., $A^{\varepsilon'} \overset{H}{\to} A^*$.*

(5) *Let* $A^\varepsilon \overset{H}{\to} A^\star$, $B^\varepsilon \overset{H}{\to} B^\star$ *and* $S \subset D$, *such that* $\bar{S}$ *is compact in* D. *If* $A^\varepsilon|_S = B^\varepsilon|_S$, *then* $A^*|_S = B^*|S$, *i.e., H-convergence is local and H-limits are unique.*

(6) *H-convergence can be applied no matter the boundary conditions*

(7) *H-convergence is metrisable*

(8) *the energy density and the energy converge, i.e.,*

$$(A^\varepsilon \nabla u^\varepsilon) \cdot \nabla u^\varepsilon \to (A^* \nabla u) \cdot \nabla u,$$

in the sense of distributions, and

$$\int_D (A^\varepsilon \nabla u^\varepsilon) \cdot \nabla u^\varepsilon dx \to \int_D (A^* \nabla u) \cdot \nabla u dx.$$

References

[1] S. Spagnolo, *Sulla convergenza di soluzione di equazione paraboliche ed ellitiche*, Ann. Sc. Norm. Sup. Pisa **22** (1968), 577–597.

[2] F. Murat and L. Tartar, *H-convergence*. In *Topics in the mathematical modelling of composite materials*, Progr. Nonlinear Differential Equations Appl., **3** (1997), 21–43.

[3] L. Tartar, *The general theory of homogenization*, Lecture Notes of the Unione Matematica Italiana **7** (2009).

[4] L. Tartar, *Quelques remarques sur l'homogeneisation*, Proc. of the Japan-France Seminar 1976 "Functional Analysis and Numerical Analysis", Japan Society for the Promotion of Sciences pp.469-482 (1978).

Probability measures on the space of coefficients, Γ-convergence, and subadditive ergodic theorem.

Stefano Decio, Lorenzo Portinale

In order to study stochastic homogenization it is indispensable to have a σ-algebra and a probability measure on the space of coefficients. Here we introduce a σ-algebra that respects the topology of H-convergence, which in turn is convenient for homogenization purposes; see [7] for the original source on the theory of H-convergence, and [6] for its use in the context of stochastic homogenization. We then define the fundamental properties of stationarity and ergodicity, which are the minimal requirements for a qualitative theory of homogenization.

The probability space of coefficients. We want to treat equations of the form $\nabla \cdot A\nabla u = 0$ in a domain $D \subset \mathbb{R}^d$, where A is a matrix valued function (the 'coefficient field'). We consider coefficients fields that are uniformly elliptic and bounded; namely, for $\lambda, \Lambda \geq 0$, let

$$\Omega_{\lambda,\Lambda} = \left\{A : D \to \mathbb{R}^{d\times d} \colon \xi A(x)\xi \geq \lambda|\xi|^2, |A(x)\xi|^2 \leq \Lambda \xi \cdot A(x)\xi,\ \forall x \in D, \xi \in \mathbb{R}^d\right\}.$$

The topology $\mathcal{T}$ associated to H-convergence is generated by the functionals

$$F_{h,\tilde{h}}(A) = \int_D \tilde{h} \cdot (\nabla u, A\nabla u),$$

where $A \in \Omega_{\lambda,\Lambda}$, $h, \tilde{h} \in L^2(D;\mathbb{R}^d)$, and u is the solution to $-\nabla \cdot A\nabla u = \nabla \cdot h$ which vanishes outside D. That is to say, $\mathcal{T}$ is the coarsest topology on Ω for which all the

functionals $F_{h,\tilde{h}}$ are continuous. We now let $\mathcal{F}$ be the Borel σ-algebra generated by $\mathcal{T}$. Then $(\Omega_{\lambda,\Lambda},\mathcal{F})$ is a measurable space, and we can consider probability measures $\mathbb{P}$ on it. The triplet $(\Omega_{\lambda,\Lambda},\mathcal{F},\mathbb{P})$ is a probability space of coefficents.

Remark 1. *With the σ-algebra $\mathcal{F}$, the evaluation map $x \to A(x)$ is measurable.*

Stationarity and ergodicity. For $y \in \mathbb{R}^d$, let $T_y : \Omega_{\lambda,\Lambda} \to \Omega_{\lambda,\Lambda}$ be defined by $T_y A = A(\cdot + y)$. T_y can be extended to $\mathcal{F}$ simply by setting, for $E \in \mathcal{F}$, $T_y E = \{T_y A\colon A \in E\}$. Finally, for an integrable random variable F, we define $F^y(A) = F(T_y A)$.

Definition 1. *We say that a probability measure $\mathbb{P}$ on $(\Omega_{\lambda,\Lambda},\mathcal{F})$ is **stationary** if for all $z \in \mathbb{Z}^d$, F^z and F have the same law for any integrable random variable F.*

Definition 2. *We say that a probability measure $\mathbb{P}$ on $(\Omega_{\lambda,\Lambda},\mathcal{F})$ is **ergodic** if $\mathbb{P}[E] \in \{0,1\}$ for any $E \in \mathcal{F}$ such that $E = \cap_{z\in\mathbb{Z}^d} T_z E$.*

As mentioned above, stationarity and ergodicity are somewhat minimal requirements for a theory of homogenization: stationarity guarantees that the homogenized matrix $\overline{A}$ is a constant coefficient matrix, while ergodicity guarantees that it is a deterministic matrix.

The variational approach to homogenization. Periodic and stochastic homogenization result can be obtained by means of study of correctors and via compensated compactness. For symmetric diffusion coefficients, an alternative approach to is to use a variational interpretation of the solutions of PDEs as minimisers of integral functionals, and use the notion of Γ-convergence.

An example. Let $D \subset \mathbb{R}^d$ be an open and bounded subset of $\mathbb{R}^d$ and fix $u_0 \in \mathrm{W}^{1,2}(D)$ which represents a boundary datum. Let $u^* \in \mathrm{W}^{1,2}(D)$ be solving

$$u^* \in \arg\min_u \left\{ \frac{1}{2}\int_D \langle A(x)\nabla u(x), \nabla u(x)\rangle \,:\, u - u_0 \in \mathrm{W}^{1,2}_0(D) \right\}, \tag{1}$$

where $A : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ is a measurable, symmetric, and uniformly elliptic diffusion operator (in particular $A \in \Omega_{\lambda,\Lambda}$). The associated Euler–Lagrange equation reads

$$\begin{cases} \Delta u^* = 0 & \text{on } D\,, \\ u^* = u_0 & \text{on } \partial D\,. \end{cases} \tag{2}$$

This example shows the link between minimisation of integral functionals as in (1) and solutions of the Laplace equation with Dirichlet boundary conditions in (2).

Γ-convergence topology. Fix $p \in (1,+\infty)$ and $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ which is measurable in the first variable and convex in the second one. Assume there exist $c_1, c_2 \in (0,+\infty)$ such that

$$c_1|\xi|^p \le f(x,\xi) \le c_2|\xi|^p\,, \quad \forall x \in \mathbb{R}^d,\, \xi \in \mathbb{R}^d\,. \tag{3}$$

Denote by $\mathcal{O}$ the collection of all the open and bounded subset of $\mathbb{R}^d$ and define

$$F : \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^d) \times \mathcal{O} \to [0, +\infty]$$
$$(u, D) \mapsto \begin{cases} \displaystyle\int_D f(x, \nabla u(x))\,\mathrm{d}x & \text{if } \ u \in \mathrm{W}^{1,p}(D)\,, \\ +\infty & \text{otherwise}\,. \end{cases}$$

We denote by $\mathcal{F} = \mathcal{F}(p, c_1, c_2)$ the class of all such functionals F.

Definition 3. *We say that a sequence $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$ is Γ-convergent to some $F \in \mathcal{F}$ as $n \to \infty$ if, for every $D \in \mathcal{O}$, the following two conditions hold:*

- *For every $u_n \to u$ in $L^p(D)$, one has that $\liminf_{n\to\infty} F_n(u_n) \geq F(u)$.*
- *For every $u \in L^p(D)$, there exists $u_n^* \to u$ in $L^p(D)$ with $\limsup_{n\to\infty} F_n(u_n^*) \leq F(u)$.*

Remark 2. *As shown in* [4]*, the Γ-convergence is in fact induced by distance on the space $\mathcal{F}$, which defines a topology τ_Γ. The obtained topological space $(\mathcal{F}, \tau_\Gamma)$ is compact, and for a given $D \in \mathcal{O}$ and $u_0 \in W^{1,p}(D)$, the maps*

$$F \in \mathcal{F} \mapsto m(F, D, u_0) := \min\Big\{F(u, D) \,:\, u - u_0 \in W_0^{1,p}(D)\Big\} \tag{4}$$

are continuous with respect to τ_Γ. Furthermore, if $F_N \xrightarrow{\tau_\Gamma} F$ and $\{u_n^*\}_n$ is such that $u_n^* \in \arg\min\Big\{F_N(u, D) \,:\, u - u_0 \in W_0^{1,p}(D)\Big\}$, then $\{u_n^*\}_n$ is compact in $L^p(D)$ and any limit point u^* satisfies $u^* \in \arg\min\Big\{F(u, D) \,:\, u - u_0 \in W_0^{1,p}(D)\Big\}$. In particular, in the setting of quadratic integral functionals induced by elliptic diffusion operators (as in the example above), the Γ-convergence turns out to be equivalent to the H convergence for the operator, see e.g. [5, Proposition 1.7].

Homogenization of random integral functionals. Denote by Σ_Γ the Borel σ-algebra generated by τ_Γ. A *random integral functional* on $\mathcal{F}$ is given by

$$F : (\Omega, \Sigma, \mathbb{P}) \to (\mathcal{F}, \Sigma_\Gamma) \quad \text{measurable}\,,$$

where $(\Omega, \Sigma, \mathbb{P})$ is a given probability space. Consider the action of $\mathbb{Z}^d$ and $\mathbb{R}_+$ on $\mathcal{F}$ given by, for every $z \in \mathbb{Z}^d$, $\varepsilon \in \mathbb{R}_+$, $D \in \mathcal{O}$, and $u \in L^p(\mathbb{R}^d)$,

$$\big(\tau_z F\big)(u, D) := F(u(\cdot - z), D + z)\,, \quad \big(\rho_\varepsilon F\big)(u, D) := F\Big(\frac{1}{\varepsilon} u(\varepsilon \cdot), \frac{1}{\varepsilon} D\Big)\,.$$

A random functional F is called *stationary* if the law of $\tau_z F$ is independent of z. It is called *ergodic* if for every invariant set $S = \tau_z S \subset \mathcal{F}$ one has $\mathbb{P}(F \in S) \in \{0, 1\}$.

Theorem 1 ([3])**.** *Let F be a random, stationary functional and set $F_\varepsilon := \rho_\varepsilon F$. Then $F_\varepsilon \xrightarrow{\tau_\Gamma} F_{hom}$ as $\varepsilon \to 0$ $\mathbb{P}$-almost surely, where the limit functional is*

$$F_{hom}(\omega; u, D) = \int_D f_{hom}(\omega; \nabla u(x))\, dx\,, \quad \omega \in \Omega\,.$$

The limit density admits the following representation: for $\mathbb{P}$-a.e. $\omega \in \Omega$, we have

$$f_{hom}(\omega; \xi) = \lim_{R\to+\infty} \frac{m(F(\omega), l_\xi, Q_R)}{R^d}\,, \quad \xi \in \mathbb{R}^d\,, \tag{5}$$

where Q_R denotes the cube of size length $R > 0$, $l_\xi(x) := \xi \cdot x$ the linear map with slope ξ, and m is the map defined in (4). If F is ergodic, then $f_{hom}(\cdot, \xi)$ is $\mathbb{P}$-a.e. constant with value $f_{hom}(\xi) := \mathbb{E}_{\mathbb{P}}[f_{hom}(\cdot, \xi)]$.

The proof of the previous theorem is based on the compactness of the Γ-convergence topology [2, Theorem IV], and an application of the so-called *subadditive ergodic theorem*, in the spirit of [1]. In particular, for $\xi \in \mathbb{R}^d$ one considers

$$\mu_\xi(\omega)(D) := m(F(\omega), l_\xi, D)\,, \quad \omega \in \Omega\,, D \in \mathcal{O}\,.$$

The subadditivity of μ_ξ follows from the properties of the min and the fact that $F(\omega) \in \mathcal{F}$. The stationarity (resp. ergodicity) of μ follows from the one of F.

References

[1] M. A. Akcoglu and U. Krengel. Ergodic theorems for superadditive processes. *J. Reine Angew. Math.*, 323, 53–67, 1981.

[2] G. Dal Maso and L. Modica. Integral functionals determined by their minima. *Rend. Sem. Mat. Univ. Padova*, 76, 255–267, 1986.

[3] G. Dal Maso and L. Modica. Nonlinear stochastic homogenization and ergodic theory. *J. Reine Angew. Math.*, 368, 28–42, 1986.

[4] G. Dal Maso and L. Modica. Nonlinear stochastic homogenization. *Ann. Mat. Pura Appl.*, 144, 347–389, 1986.

[5] E. De Giorgi and G. Dal Maso. Γ-convergence and calculus of variations. In *Mathematical theories of optimization (Genova, 1981)*, volume 979 of *Lecture Notes in Math.*, pages 121–143. Springer, Berlin, 1983.

[6] A. Gloria and F. Otto. The corrector in stochastic homogenization: optimal rates, stochastic integrability, and fluctuations. Preprint, arXiv:1510.08290.

[7] F. Murat and L. Tartar. H-convergence. In *Topics in the mathematical modeling of composite materials*, volume 31 of *Progr. Nonlinear Differential Equations Appl.*, pages 21–43. Birkhäuser Boston, Boston, MA, 1997.

Construction of correctors

Mikhail Cherdantsev, Roberta Marziani

Correctors are the key objects in homogenisation. Given $a\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$ uniformly elliptic and $\xi \in \mathbb{R}^d$ the corresponding corrector ϕ is the unique solution to

$$\nabla \cdot a(\nabla\phi + \xi) = 0 \quad \text{in} \quad \mathbb{R}^d, \tag{1}$$

having zero average on the unit ball. In the case of periodic homogenisation, i.e., if a is periodic, the construction of correctors turns out to be rather simple. Indeed it is sufficient to look for solutions to (1) in the periodic cell with periodic boundary conditions for which existence and uniqueness is ensured by Poincaré's inequality together with Lax-Milgram theorem. Unfortunately the same argument doesn't apply to the case of stochastic homogenisation, i.e., when a belongs to the sample space Ω of random matrix-valued variables and is stationary with respect to some probability measure $\mathbb{P}$. There are two approaches to constructing correctors with stationary gradients in stochastic homogenisation: one based on infra-red regularization [3], another on decomposition of a vector field in solenoidal and potential parts, see [2, Chapter 7] and [1, Lemma 1].

With the infra-red approach, in order to overcome the lack of coercivity in the corrector problem (1) one modifies the problem by adding a "massive" term:

$$\nabla \cdot a(\nabla \phi_T + \xi) + \frac{1}{T}\phi_T = 0.$$

By Lax-Milgram lemma this problem has a unique solution such that both ϕ_T and $\nabla \phi_T$ are stationary. Moreover, it satisfies the bound

$$\mathbb{E}[|\nabla \phi_T|^2 + \frac{1}{T}|\phi_T|^2] \leq C.$$

Passing to the limit as T goes to infinity, we observe that $\nabla \phi_T$ converges weakly to the stationary solution $\nabla \phi$ of (1), whose potential, however, is no longer stationary in general (notice that the bound on ϕ_T blows up). Moreover $\nabla \phi$ has zero expectation and bounded second moment.

The second approach has a more probabilistic flavour and can be divided into two main steps. The first one consists in lifting the corrector equation (1) to the probability space. Precisely we look for a solution $\Phi \in L^2_{\mathrm{pot}}(\Omega)$ to

$$\mathbb{E}[a(0)(\Phi + \xi)\Psi] = 0 \quad \forall\ \Psi \in L^2_{\mathrm{pot}}(\Omega). \tag{2}$$

Here $L^2_{\mathrm{pot}}(\Omega)$ denotes the Hilbert space of curl-free vector fields $\Psi = (\Psi_i)_{i=1}^d \in L^2(\Omega; \mathbb{R}^d)$ having zero expectation (see [2, Chapter 7] for its precise definition). Again by Lax-Milgram there exists a unique solution $\Psi \in L^2_{\mathrm{pot}}(\Omega)$ to (2). The second step consists in showing that the stationary random field $\bar{\Phi}(x, a) := \Phi(a(x + \cdot))$ is the gradient of a random variable $\phi(x, a)$ (with zero average on the unit ball) which in turn satisfies (1).

The fact that ϕ has zero average on the unit ball together with the properties of $\nabla \phi$ imply sublinear growth at unfinity, that is, almost surely

$$\lim_{R \to \infty} \frac{1}{R^2} \fint_{B_R} |\phi|^2 = 0. \tag{3}$$

Having the sublinear corrector at hand, we can define the homogenised coefficients

$$a_{\mathrm{hom}}\xi := \mathbb{E}[a(\nabla \phi + \xi)].$$

There is a number of ways to prove the convergence of solutions to the problem

$$-\nabla \cdot a(x/\varepsilon)\nabla u_\varepsilon = f, \quad u_\varepsilon \in H^1_0(S),$$

to the solution to the homogenised one,

$$-\nabla \cdot a_{\mathrm{hom}} \nabla u_0 = f, \quad u_0 \in H^1_0(S).$$

One has

$$u_\varepsilon \rightharpoonup u_0 \text{ weakly in } H^1(S) \text{ as } \varepsilon \to 0.$$

Moreover, the convergence of fluxes holds:

$$a(x/\varepsilon)\nabla u_\varepsilon \rightharpoonup a_{\mathrm{hom}} \nabla u_0 \text{ weakly in } L^2(S) \text{ as } \varepsilon \to 0.$$

Note that the above is exactly the H-convergence of $a(x/\varepsilon)$ to a_{hom}.

References

[1] A. Gloria, S. Neukamm, and F. Otto, *A regularity theory for random elliptic operators*, Milan J. Math. **88(1)** (2020), 99–170.

[2] V. V. Jikov, S. M. Kozlov, and O. A. Oleĭnik, *Homogenization of differential operators and integral functionals*, Springer-Verlag, Berlin (1994).

[3] G. C. Papanicolaou and S. R. S. Varadhan, *Boundary value problems with rapidly oscillating random coefficients*, In Random fields, Vol. I, II (Esztergom, 1979), Colloq. Math. Soc. János Bolyai **27**, North-Holland, Amsterdam (1981), 835–873.

Invariance principle for the random walk in random environment

Armand Bernou

Donsker invariance principle. Let $(Y_k)_{k\geq 0}$ be i.i.d. random variables of mean 0 and variance 1. Once properly rescaled, the random walk $X_k = \sum_{\ell=1}^{k} Y_\ell$ behaves like a Brownian motion. Here, $\mathcal{D}(0,1)$ is the space of càdlàg functions from $[0,1]$ to $\mathbb{R}$ endowed with the Skorohod metric that turns it into a Polish metric space.

Theorem 1 (Donsker's, 1950's)**.** *Let, for all $t \in [0,1]$, $n \geq 1$, $X_t^n := \frac{X_{[n^2 t]}}{n}$. In $\mathcal{D}(0,1)$, we have convergence in the sense of distributions towards a standard Brownian motion with diffusion matrix I_d (we write sBM(I_d)) $(W_t)_{t\geq 0}$:*

$$(X_t^n)_{t\in[0,1]} \xrightarrow{d} (W_t)_{t\in[0,1]}.$$

Let $(e_1, \dots, e_d)$ be the canonical basis of $\mathbb{R}^d$ and $E = \{e_1, \dots, e_d, -e_1, \cdots - e_d\}$. On the grid $\mathbb{Z}^d$, we put random conductances: at each site $x \in \mathbb{Z}^d$, the conductance in direction $e \in E$ is given by $a_e(x)$ with $a_e(x)$ in $[b_1, b_2]$, $0 < b_1 < b_2 < \infty$ although those uniform ellipticity hypotheses can be relaxed. We impose a **symmetry condition**:

$$a_e(x) = a_{-e}(x+e), \qquad \forall x \in \mathbb{Z}^d, \quad \forall e \in E.$$

The random walk in random environment (RWRE) is a stochastic process (that is, a collection of random vectors indexed by $t \in \mathbb{R}_+$) denoted $(X_t)_{t\geq 0}$ that evolves as follows: assume at time $t \geq 0$, $X_t = x$ with $x \in \mathbb{Z}^d$. We consider $2d$ random clocks, τ_e with $e \in E$ s.t. τ_e has exponential distribution with mean $a_e(x)$. Then

$$X_{t+s} = X_t, \quad s \in [0, \min_{e\in E} \tau_e], \qquad X_{t+\min_{e\in E}\tau_e} = X_t + \left(\operatorname{argmin}_{e\in E} \tau_e\right)$$

i.e. the random walk moves in the direction indicated by the first ringing clock.

The increments of the RWRE are not i.i.d. because they depend on the conductances. Can we still derive an invariance principle for the RWRE ? There are two sources of randomness:

(1) the environment itself, since the $a_e(x)$, $x \in \mathbb{Z}^d$, $e \in E$ are random.
(2) The random walks performed on a given environment, that is the way the clocks described above ring (and how the RWRE thus behaves).

Quenched results describe the behavior of the random walk in a given environment $a := \{a_e(x), x \in \mathbb{Z}^d, e \in E\}$ in the space of environments $\mathbf{N}$ (see the next report), while *annealed results*, on which we focus here, consider a distribution μ on $\mathbf{N}$ and provide results for the annealed measure given, for any event A, by

$$\mathbb{P}(A) = \int_{\mathbf{N}} P^a(A)\mu(da).$$

Some notions about Markov processes. The RWRE is a Markov process, and the argument of [1] and [2] rely on an abstract result concerning those.

Let us introduce a bit of probabilistic machinery. Let $(Z_t)_{t\geq 0}$ be a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{F}_s = \sigma(Z_u, 0 \leq u \leq s)$ for all $s \geq 0$ so that $(\mathcal{F}_s)_{s\geq 0}$ is the canonical filtration of $(Z_t)_{t\geq 0}$. We say that $(Z_t)_{t\geq 0}$ is a Markov process if we have,

$$\forall A \in \mathcal{F}, \quad \mathbb{P}(Z_{t+s} \in A|\mathcal{F}_t) = \mathbb{P}(Z_{t+s} \in A|Z_t), \qquad t, s \geq 0.$$

In words, the information contained in $\mathcal{F}_t$ (i.e. everything that happened to the process up to time t) is exactly as relevant to predict the future value Z_{t+s} as the value Z_t itself, which is a priori a much smaller information. Markov processes appear in numerous contexts and share many key properties. We consider processes with values in a state space $(G, \mathcal{G})$ that are time-homogeneous. In particular one can identify a *transition kernel* or *transition semigroup* $(S^t)_{t\geq 0}$ such that

$$\forall B \in \mathcal{G}, \qquad S^t(z, B) = \mathbb{P}(Z_t \in B|Z_0 = z).$$

From there we can also introduce the *generator* of the process, which is formally $\mathcal{L} = \partial_t S^t_{|t=0}$ and the notion of stationary measure: if μ is a stationary measure for $(Z_t)_{t\geq 0}$, and if $Z_0 \sim \mu$ (i.e. Z_0 has distribution μ), then $Z_t \sim \mu$ for all $t \geq 0$. The precise formulation writes, for all $B \in \mathcal{G}$,

$$\int_G S^t(z, B)\mu(dx) = \mu(B).$$

The stationary measure μ is *ergodic* if for all $B \in \mathcal{G}$ such that $S^t(z, B) = 1$ for all $z \in B$, $\mu(B) \in \{0, 1\}$. This means that any absorbing set (i.e. a set that, if reached, captures the process forever) is either somewhere where the process spends all the time or no time when starting from the invariant measure. We will need the notion of reversibility: the process is as likely to go from z to y in a time $t > 0$ than it is to go from y to z. Mathematically, for all time $t > 0$

$$S^t(z, dy)\mu(dz) = S^t(y, dz)\mu(dy).$$

At last, roughly, a Markov process $(Z_t)_{t\geq 0}$ is a *martingale* if $Z_t \in L^1$ and if $\mathbb{E}[Z_t|\mathcal{F}_s] = Z_s$ almost surely, for all $t \geq s \geq 0$.

Theorem 2 ([1])**.** *Let $(Z_t)_{t\geq 0}$ be a reversible Markov process with generator $\mathcal{L}$ and stationary measure μ, write $(\mathcal{F}_t)_{t\geq 0}$ for the corresponding filtration. Assume that μ is translation invariant and ergodic. Let X be a family indexed by closed bounded intervals of $\mathbb{R}$ with values in $\mathbb{R}^d$, anti-symmetric, i.e. if $I = [a, b]$,*

$$X_I((Z_s)_{s\in I}) = -X_I((Z_{b+a-s})_{s\in I}).$$

Assume that the following strong L^1 limit exists,

$$\lim_{\delta\to 0} \tfrac{1}{\delta}\mathbb{E}_\mu\big[X_{[0,\delta]}\big|\mathcal{F}_0\big] =: b(Z_0)$$

and that the martingale $M_t = X_t - \int_0^t b(Z_s)ds$ is square integrable. Defining $D_{ij} = C_{ij} + 2(b_i, \mathcal{L}^{-1}b_j)$, where C satisfies $e^T Ce = \mathbb{E}_\mu[(e\cdot M_1)^2]$, we have

$$\frac{1}{n}X_{[0,n^2t]} \to W_D,$$

in the sense of finite-dimensional distributions, where W_D is a sBM(D).

The idea here is that one can obtain an invariance principle when considering increments related to a reversible process having a stationary distribution that is mixing and translation invariant. The proof is based on [2] and is subtle: in particular showing that all quantities admitting a drift b so that M is a square-integrable martingale are such that $(b_i, \mathcal{L}^{-1}b_j)$ is well-defined is difficult. One can however relate to the usual central limit theorem: to be able to describe the behavior at the limit one needs a second moment. The condition on the drift here are analogous to this requirement.

An invariance principle for the RWRE. To conclude on the RWRE, it only remains to find such a reversible process $(Z_t)_{t\geq 0}$ and the appropriate additive functionals X. We introduce the process of the *environment seen by the particle* (see [2]). Keeping the particle centered at 0 along the walk, and translating the environment to compensate its jumps, how does the latter evolve ? We thus consider the process $(A_t)_{t\geq 0}$ with generator

$$\mathcal{L}f(a) = \sum_{e\in E} a_e(0)(f(S_{-e}a) - f(a))$$

i.e. at rate $a_e(0)$, the environment a is replaced by the one translated by $-e$: $(S_{-e}a)_{e'}(x) = a_{e'}(x+e)$ for all $x\in\mathbb{Z}^d$, $a\in\mathbf{N}$, $e, e'\in E$. If now μ is a distribution on $\mathbf{N}$ ergodic and translation invariant, satisfying the symmetry assumption, then the process $(A_t)_{t\geq 0}$ with $A_0\sim\mu$ is stationary, ergodic and reversible. How to reconstruct the random walk from A? Coming back to the discrete setting, if $(Y_n)_{n\geq 0}$ denote the successive positions of the random walk, we can introduce a random time $n^*(t)$ such that $n^*(t) = n$ if $X_t = Y_n$, as well as a discrete environment process $(B_n)_{n\geq 0}$. One can easily reconstruct $(Y_n)_{n\geq 0}$ from $(B_n)_{n\geq 0}$: we have

$$Y_0 = 0, \qquad Y_{n+1} = Y_n + x \quad \text{if } B_{n+1} = S_{-x}B_n.$$

Once the chain $(Y_n)_{n\geq 0}$ is identified, one can rebuild $(X_t)_{t\geq 0}$ from it. The previous theorem then applies, and we find that for all bounded continuous functions $F\in\mathcal{D}(0,\infty)$ the Skorohod space, setting

$$(X_t^n)_{t\geq 0} := (\tfrac{1}{n}X_{n^2t})_{t\geq 0}, \qquad \text{one has} \quad \mathbb{E}_\mu[F(X^n)] \to \mathbb{E}[F(W_D)],$$

with W_D is a sBM(D), where, writing $\langle c\rangle_\mu = \int_{\mathbf{N}} c(a)\mu(da)$ for $c:\mathbf{N}\to\mathbb{R}$,

$$(1)\qquad D_{ij} = 2\Big\langle a_{e_i}(0)\delta_{ij}\Big\rangle_\mu + 2\Big\langle \big(a_{e_i}(0) - a_{e_i}(-e_i)\big)\mathcal{L}^{-1}\big(a_{e_j}(0) - a_{e_j}(-e_j)\big)\Big\rangle_\mu.$$

A comparison with the PDE viewpoint. In the PDE setting, focusing on the diagonal terms for simplicity, the homogenized coefficient $\bar{a}$ is given by

$$\bar{a}e_i = \mathbb{E}[a(0)e_i] + \mathbb{E}\big[a(0)\nabla\phi_i(0)\big], \tag{2}$$

where ϕ is the corrector. With $\mathbf{L}$ the operator, the equation for the corrector writes $\mathbf{L}\phi_i = \mathrm{div}(ae_i)$. We can see here (see the discussion about the discrete gradient), that considering ∇ to be the discrete gradient instead, formally, we obtain $\phi_i(0) = \mathbf{L}^{-1}\mathrm{div}(a_{e_i}(0)) = -\mathbf{L}^{-1}(a_{e_i}(0) - a_{e_i}(-e_i))$. Coming back to (2), we get for the second term, using the translation invariance and as expected

$$\begin{aligned}\mathbb{E}[a(0)\nabla\phi_i(0)] &= \mathbb{E}\Big[\big(a_{e_i}(-e_i) - a_{e_i}(0)\big)\phi_i(0)\Big] \\ &= \mathbb{E}\Big[\big(a_{e_i}(0) - a_{e_i}(-e_i)\big)\mathbf{L}^{-1}\big(a_{e_i}(0) - a_{e_i}(-e_i)\big)\Big].\end{aligned}$$

References

[1] A. De Masi and P.A. Ferrari and S. Goldstein and W. D. Wick, *An Invariance Principle for Reversible Markov Processes. Applications to Random Motions in Random Environments*, Journal of Statistical Physics **55** (1989), 787–855.

[2] C. Kipnis and S. R. S. Varadhan *Central Limit Theorem for Additive Functionals of Reversible Markov Processes and Applications to Simple Exclusions*, Communication in Mathematical Physics **104** (1986), 1–19.

Quenched invariance principle

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We show the quenched invariance principle for random walks in stationary and ergodic environment, with speed that is symmetric (i.e. random conductance model (RCM)) and having finite p-q moment. Precisely, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space for the random environment $\omega \equiv (\omega(x,y))_{(x,y)\in\vec{E}_d}$ on $(\mathbb{Z}^d, \vec{E}_d)$, the d-dimensional Euclidean lattice equipped with oriented edges. We consider $d \geq 2$. The random environment encodes the speed of the random walk. We assume

(A1) ω is positive, symmetric and finite

$$\forall (x,y) \in \vec{E}_d : 0 < \omega(x,y) = \omega(y,x) < \infty,$$

(A2) $\mathbb{P}$ is stationary and ergodic with respect to the space shifts $(\tau_x)_{x\in\mathbb{Z}^d}$,

(A3) ω satisfies the following $p-q$ moment condition

$$\forall (x,y) \in \vec{E}_d : \mathbb{E}[(\omega(x,y))^p] < \infty, \mathbb{E}[(\omega(x,y))^{-q}] < \infty,$$

where $p, q \in [1,\infty]$, and $\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$.

Further let $(X_t)_{t\geq 0}$ be the continuous time random walk of a particle that starts from the origin of a $\mathbb{P}-$distributed random environment ω and jumps at rate ω to the nearest-neighbor sites. Let $(X_t^{(n)})_{t\geq 0}$ denote the *diffusively scaled* random process, with $X_t^{(n)} := \frac{1}{n}X_{n^2t}, \forall t \geq 0$.

Theorem 1 (Quenched invariance principle (QIP) [2]). *Suppose $d \geq 2$, and (A1), (A2), (A3) are satisfied. Then for $\mathbb{P}-a.e.\,\omega$, $(X_t^{(n)})_{t\geq 0}$ converges weakly under the quenched law P_0^ω to a Brownian motion on $\mathbb{R}^d$ with a deterministic non-degenerate covariance matrix Σ.*

In i.i.d. conductance setting (relaxation of (A2)), QIP has been studied, for instance, for $p = q = \infty$ ([9]); $q = \infty$ and without p-th moment condition ([3], [1]); percolation clusters, in particular, without q-th inverse moment condition ([5], [7]). For more details on RCM see [6],[8]. Recently, under (A1), (A2), [4] shows QIP while relaxing (A3) with $\frac{1}{p} + \frac{1}{q} < \frac{2}{d-1}$, $d \geq 3$. For $d = 2$, QIP has been addressed by [6] under the minimal moment condition $p = q = 1$.

One of the motivations to study QIP is that, it can be used to approximate solutions to problems in continuous domain, such as Dirichlet problem for Laplace's equation, Poisson equation with zero boundary condition, and Cauchy problem for heat equation with coefficients, where the Laplace operator in above mentioned problems is replaced by the quenched generator (will be defined later).

We discuss essential elements in [2]'s proof for QIP.

First, the proof relies on the random process $(\omega_t)_{t\geq 0}$, the process of the *environment viewed from the particle*, i.e. $\omega_t := \tau_{X_t}\omega, \forall t \geq 0$. $(\omega_t)_{t\geq 0}$ is stationary and ergodic in time with respect to $\mathbb{P}$ provided (A1), (A2), and the finiteness of the first moment; and it is non-explosive $P_0^\omega - a.s.$, provided the finiteness of the first inverse moment. The environment process has the *generator* $\mathcal{L}$, with

$$\mathcal{L}F(\omega) = \sum\nolimits_{z\sim 0} \omega(0,z)(F(\tau_z\omega) - F(\omega)),$$

for all $F : \Omega \to \mathbb{R}$ measurable and bounded.

Second, denote by L^ω the *quenched generator*, with $L^\omega f(x) := \sum_{y\sim x} \omega(x,y)$ $(f(y) - f(x))$, for $f : \mathbb{Z}^d \to \mathbb{R}$ suitable. By Helmholtz type decomposition, the trajectory $\Phi(\omega, X_t) = X_t$ can be written as the sum of $\Phi(\omega, X_t)$ and $\chi(\omega, X_t)$, such that for $\mathbb{P} - a.e.\,\omega$, $\forall x \in \mathbb{Z}^d$,

$$\begin{aligned} L^\omega\chi(\omega,x) &= L^\omega\Pi(\omega,x), \\ L^\omega\Phi(\omega,x) &= 0. \end{aligned} \tag{1}$$

In contrary to the convention in the homogenization community, we denote by $\Phi(\omega,\cdot) : \mathbb{Z}^d \to \mathbb{R}^d$ the harmonic coordinate, and by $\chi(\omega,\cdot) : \mathbb{Z}^d \to \mathbb{R}^d$ the corrector. An advantage of ω being symmetric is, χ, Φ are the orthogonal projection of Π via $L^2_{\text{cov}} = L^2_{\text{pot}} \bigoplus L^2_{\text{sol}}$ (also see [6, p. 328]).

By the property (1), $\Phi(\omega, X_t)$ is a martingale under P_0^ω. With Helland's FCLT for martingales, one can show the diffusively scaled process $M_t^{(n)} := \frac{1}{n}\Phi(\omega, X_{n^2t})$, $t \geq 0$, converges weakly under P_0^ω to a Brownian motion with non-degenerate covariance matrix Σ. On the other hand, QIP is attainable if $\chi(\omega, X_t)$ vanishes under the diffusive scaling, in particular, when it holds that

- (vanishing corrector) for $\mathbb{P} - a.e.\,\omega$,

$$\forall T > 0 : \lim_{n\to\infty} \sup_{0\leq t\leq T} \frac{1}{n} |\chi(\omega, X_{n^2t})| = 0, \text{ in } P_0^\omega. \tag{2}$$

To show (2), it is sufficient to have

- (sublinearity of the corrector) for $\mathbb{P}-a.e.\,\omega$,

$$\lim_{n\to\infty}\frac{1}{n}\sup_{x\in B(n)}|\chi(\omega,x)|=0, \tag{3}$$

where $B(n):=\{x\in\mathbb{Z}^d:|x|\le n\}$. For a finite subset $D\subset\mathbb{Z}^d$, $r\in[1,\infty]$, let $\|\cdot\|_{r,D}$ be the space-averaged ℓ^r norm: for $f:\mathbb{Z}^d\to\mathbb{R}$,

$$\|f\|_{r,D}:=\left(\frac{1}{|D|}\sum_{x\in D}|f(x)|^r\right)^{\frac{1}{r}}.$$

The property (3) can be derived via a two-scale argument and spatial ergodic theorem from two ingredients

- (quenched maximum inequality (QMI)) for $f:\mathbb{Z}^d\to\mathbb{R}$ that is non-negative and L^ω-subharmonic on $B(n)$, and $\eta:\mathbb{Z}^d\to[0,1]$, with $\mathrm{supp}(\eta)\subset B(n-1)$. Then there exist constants $\kappa=\kappa(d,p,q)\in(\frac{1}{2},\infty)$ and $C=C(d,p,q)$ such that for all $\frac{1}{2}\le\sigma'<\sigma\le1$,

$$\max_{x\in B(\sigma' n)}|f(x)|\le C\left(\frac{1\vee\|\mu^\omega\|_{p,B(n)}\|\nu^\omega\|_{q,B(n)}}{(\sigma-\sigma')^2}\right)^{\kappa}\|f\|_{2p_*,B(\sigma n)}, \tag{4}$$

  where p_* is the Hölder conjugate of p, and $\mu^\omega(x)=\sum_{y\sim x}\omega_z(x)$, $\nu^\omega(x)=\sum_{y\sim x}\frac{1}{\omega_z(x)}$.
- (sublinearity of the corrector in space-averaged $\ell^{2\rho}$ norm) for $\mathbb{P}-a.e.\,\omega$

$$\lim_{n\to\infty}\frac{1}{n}\|\chi(\omega,\cdot)\|_{2\rho,B(n)}=0, \tag{5}$$

  where $\rho:=\frac{d}{d-2+d/q}$.

To obtain (QMI), one apply a PDE technique called *Moser iteration*, together with the following two estimates related to the *quenched Dirichlet energy* $\mathcal{E}^\omega$, with $\mathcal{E}^\omega(f,g):=\langle f,-L^\omega g\rangle_{\ell^2(\mathbb{Z}^d)}=\sum_{x\in\mathbb{Z}^d}f(x)(-L^\omega g)(x)$, for all $f,g:\mathbb{Z}^d\to\mathbb{R}$ suitable.

- (weighted Soblolev inequality (WSI)) for $f:\mathbb{Z}^d\to\mathbb{R}$, $\eta:\mathbb{Z}^d\to[0,1]$, with $\mathrm{supp}(\eta)\subset B(n-1)$, there exists a constant $C=C(d)$, such that

$$\|(\eta f)\|^2_{2\rho,B(n)}\le C\,|B(n)|^{\frac{2}{d}}\,\|\nu^\omega\|_{q,B(n)}\frac{\mathcal{E}^\omega(\eta f)}{|B(n)|},$$

- (quenched energy estimate (QEE)) let f,η be the same as in (QMI), then we have for $\forall p\in[1,\infty]$, there exists a constant C

$$\frac{\mathcal{E}^\omega(\eta f)}{|B(n)|}\le C\|\nabla\eta\|^2_{\ell^\infty(\vec{E}d)}\|\mu^\omega\|_{p,B(n)}\|f^2\|_{p_*,B(n)}. \tag{6}$$

The choice of ρ in (5) is intricate. On the one hand, ρ is chosen during the fabrication of the (WSI) from the classical unweighted version, and an application of Hölder's inequality. On the other hand, it allows to perform Moser iteration

for the purpose of (QMI) by utilizing a sequence of increasing exponents $\alpha_k := (\frac{\rho}{p_*})^k > 1$, while $1/p + 1/q < 2/d$ is equivalent to $\rho > p_*$.

Finally, we remark that for random walks in non-symmetric random environments, the QIP proof scheme in [2] is applicable to a class of doubly stochastic environments.

References

[1] S. Andres, M.T. Barlow, J.-D. Deuschel, B.M. Hambly, *Invariance principle for the random conductance model*, Probab. Theory Relat. Fields **156** (2013), 535–580.

[2] S. Andres, J.-D. Deuschel, M. Slowik, *Invariance Principle for the random conductance model in a degenerate ergodic environment*, Ann. Probab. 43(4) (2015), 1866–1891.

[3] M.T. Barlow, J.-D. Deuschel, *Invariance principle for the random conductance model with unbounded conductances*, Ann. Probab. **38**(1) (2010), 234–276.

[4] P. Bella, M. Schäffner, *Quenched invariance principle for random walks among random degenerate conductances*, Ann. Probab. **48**(1) (2020), 296–316.

[5] N. Berger, M. Biskup, *Quenched invariance principle for simple random walk on percolation clusters*, Probab. Theory Relat. Fields **137** (2007), 83–120.

[6] M. Biskup, *Recent progress on the Random Conductance Model*, Probab. Surveys **8** (2011), 294–373.

[7] P. Mathieu, A. Piatniski, *Quenched invariance principles for random walks on percolation clusters*, Proc. R. Soc. A. **463** (2007) 2287–2307.

[8] T. Kumagai, *Random walks on disordered media and their scaling limits*, Cham: Springer (2014).

[9] V. Sidoravicius, AS. Sznitman, *Quenched invariance principles for walks on clusters of percolation or among random conductances*, Probab. Theory Relat. Fields **129** (2004), 219–244.

Statistical Mechanics: The $\nabla\phi$-model

Paul Dario, Florian Schweiger

1. Overview

The $\nabla\phi$ model is a stochastic model of a d-dimensional interface in $\mathbb{R}^{d+1}$. We consider the lattice $\mathbb{Z}^d$ with nearest-neighbor edges. Given a finite subset Λ of $\mathbb{Z}^d$, the model is given by the probability measure

$$\mu_\Lambda(d\phi) = \frac{1}{Z_\Lambda} \exp\left(- \sum_{\substack{x\sim y \\ \{x,y\}\cap\Lambda\neq\emptyset}} V(\phi(x)-\phi(y)) \right) \prod_{x\in\Lambda} d\phi(x) \prod_{x\in\mathbb{Z}^d\setminus\Lambda} \delta_0(d\phi(x))$$

(where we take 0-boundary values for simplicity). Here V is a function $V\colon \mathbb{R}\to\mathbb{R}$ that is symmetric around 0 and grows sufficiently fast at ∞. The case $V(a) = \frac{a^2}{2}$ corresponds to the discrete Gaussian free field (DGFF), the best-studied such model. In the talk, we discussed a more general model, where we just assume that $V \in C^2(\mathbb{R})$ and $0 < c_- \leq V''(a) \leq c_+ < \infty$ for some constants $c_\pm$. If $d \geq 3$ (or if $d \geq 2$ and one passes to gradient fields) one can take the thermodynamic limit $\Lambda \nearrow \mathbb{Z}^d$ that we denote by μ.

This model is known in the literature as the $\nabla\phi$-*model* or the *Ginzburg-Landau model*. It is expected that this model behaves similarly to the DGFF, and in the last 25 years there has been an intense research effort to make this intuition rigorous, starting with the pioneering works [1, 3] (see also [5] for an overview on the topic).

The key tool for the study of the $\nabla\phi$ model is the so-called *Helffer-Sjöstrand representation*. One can understand this representation from a PDE perspective or from a probabilistic perspective. Both viewpoints have in common that they show a homogenization effect. Thus on large scales the $\nabla\phi$-model should behave like a scalar multiple of the DGFF. This heuristic stands behind most of the rigorous works on the $\nabla\phi$-model.

The PDE viewpoint was introduced by Naddaf-Spencer [1]. To state it, introduce the vertical derivatives $\partial_x f(\phi) := \frac{\partial}{\partial\phi(x)} f(\phi)$ and define the operator $\mathcal{L}$ (the so-called Witten Laplacian) on functions $u\colon \mathbb{R}^\Lambda \times \Lambda \to \mathbb{R}$ defined by

$$\mathcal{L}u(x,\phi) := -\sum_y \partial_y u(x,\phi) + \sum_{y\sim z} V'(\phi(z)-\phi(y))\partial_y u(x,\phi) + \nabla\cdot\mathbf{a}\nabla u(x,\phi)$$

where $\mathbf{a}((x,y)) := V''(\phi(y)-\phi(x))$. Then one has (under mild regularity assumptions) the representation

$$\mathbb{E}_\mu(F(\phi)G(\phi)) = (DF(\phi), \mathcal{L}^{-1}DG(\phi))$$

where $DF(\phi) := (\partial_y F(\phi))_{y\in\mathbb{Z}^d}$.

The probabilistic viewpoint was introduced by Deuschel-Giacomin-Ioffe [4] and Giacomin-Olla-Spohn [2]. Using Duhamel's formula, one can rewrite the preceding PDE representation in terms of random walk in time-dependent random environment, and we will state a simplified version below.

2. Scaling limit of the model

To showcase how these methods are used, we present a result on the scaling limit of the model. In the following statement and for $f \in C_c^\infty(\mathbb{R}^d)$, we denote by

$$\left(f, (-\Delta)^{-1} f\right) := \int_{\mathbb{R}^d\times\mathbb{R}^d} f(x)G(x-y)f(y)dx\,dy,$$

where $G : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ is the standard Green's function.

Theorem 1 (Scaling limit for the $\nabla\phi$ model, $d \geq 3$)**.** *Fix $d \geq 3$ and let $\phi : \mathbb{Z}^d \to \mathbb{R}$ be a random surface distributed according to the measure μ. Then, there exists a constant $\overline{\mathbf{a}} := \overline{\mathbf{a}}(d, V) > 0$ such that, for any $f \in C_c^\infty(\mathbb{R}^d)$,*

$$\frac{1}{L^{\frac{d}{2}+1}} \sum_{x\in\mathbb{Z}^d} f\left(\frac{x}{L}\right)\phi(x) \overset{\text{(law)}}{\underset{L\to\infty}{\longrightarrow}} N\left(0, \overline{\mathbf{a}}^{-1}\left(f, (-\Delta)^{-1}f\right)\right).$$

This result was originally proved, in greater generality, by Naddaf-Spencer [1] and Giacomin-Olla-Spohn [2]. The proof sketched below is a simplified version of [2].

Sketch of proof. For $L \in \mathbb{N}$ and $f \in C_c^\infty(\mathbb{R}^d)$, we introduce the notation

$$S_L := \frac{1}{L^{\frac{d}{2}+1}} \sum_{x \in \mathbb{Z}^d} f\left(\frac{x}{L}\right) \phi(x).$$

The proof is decomposed in two steps:

(i) Proving convergence of the variance, i.e., showing that

$$\operatorname{var}_\mu [S_L] \underset{L\to\infty}{\longrightarrow} \overline{\mathbf{a}}^{-1} \left(f, (-\Delta)^{-1} f\right). \tag{1}$$

(ii) Proving that S_L converges in distribution to a Gaussian random variable.

We only sketch the proof of (1). The argument relies on the Helffer-Sjöstrand identity in its probabilistic version. Formally, the identity applied to the random variable S_L reads as follows

$$\operatorname{var}_\mu [S_L] = \int_0^\infty \frac{1}{L^d} \sum_{x \in \frac{1}{L}\mathbb{Z}^d} f(x)\, \mathbb{E}\left[H_L(L^2 t, Lx)\right]\, dt. \tag{2}$$

where the function $H_L : (0,\infty) \times \mathbb{Z}^d \to \mathbb{R}$ is defined to be the solution of the discrete parabolic equation

$$\begin{cases} \partial_t H_L - \nabla \cdot \mathbf{a} \nabla H_L = 0 & \text{in } (0,\infty] \times \mathbb{Z}^d, \\ H_L(0,\cdot) = f\left(\frac{\cdot}{L}\right) & \text{in } \mathbb{Z}^d, \end{cases}$$

where the environment $\mathbf{a}$ is random, depends on the space and time variables and is formally defined by $\mathbf{a}(t,(x,y)) := V''(\phi_t(y) - \phi_t(x))$, where $(\phi_t)_{t\geq 0}$ is the solution of the *Langevin dynamics*

$$\begin{cases} d\phi_t(x) := \sum_{y\sim x} V'(\phi_t(y) - \phi_t(x))dt + \sqrt{2} dB_t(x) & \text{for } (t,x) \in (0,\infty) \times \mathbb{Z}^d, \\ \phi_0(x) = \phi(x) & \text{for } x \in \mathbb{Z}^d. \end{cases} \tag{3}$$

where $\left\{B_t(x) : t \geq 0, x \in \mathbb{Z}^d\right\}$ is a collection of independent Brownian motions and the initial condition ϕ is distributed according to μ independently of the Brownian motions. The symbol $\mathbb{E}$ in (2) then refers to the expectation with respect to the dynamics $(\phi_t)_{t\geq 0}$.

It can be proved (see [2, Lemma 4.2]) that the process $(\phi_t)_{t\geq 0}$ is ergodic with respect to the space and time variables. This implies that the same ergodicity property holds for the environment $\mathbf{a}$ and allows to prove the following homogenization theorem: there exists a deterministic coefficient $\overline{\mathbf{a}} := \overline{\mathbf{a}}(d,V) > 0$ such that, if we let $\bar{H}$ be the solution of the (continuous) parabolic equation

$$\begin{cases} \partial_t \bar{H} - \overline{\mathbf{a}} \Delta \bar{H} = 0 & \text{in } (0,\infty] \times \mathbb{R}^d, \\ \bar{H}(0,\cdot) = f & \text{in } \mathbb{Z}^d, \end{cases}$$

then the following convergence holds

$$\int_0^\infty \frac{1}{L^d} \sum_{x \in \frac{1}{L}\mathbb{Z}^d} \left|\mathbb{E}\left[H_L(L^2 t, Lx)\right] - \bar{H}(t,x)\right|^2 dt \underset{L\to\infty}{\longrightarrow} 0.$$

Combining the previous result with (2), we obtain

$$\mathrm{var}_\mu\left[S_L\right] \underset{L\to\infty}{\longrightarrow} \int_0^\infty \int_{\mathbb{R}^d} f(x)\bar{H}(t,x)\,dx = \overline{\mathbf{a}}^{-1}\left(f,(-\Delta)^{-1}f\right),$$

completing the proof of (1).

□

References

[1] A. Naddaf, T. Spencer, *On homogenization and scaling limit of some gradient perturbations of a massless free field*, Comm. Math. Phys. **183** (1997), no. 1, 55–84.

[2] G. Giacomin, S. Olla, H. Spohn, *Equilibrium fluctuations for $\nabla\phi$ interface model*, Ann. Probab. **29** (2001), no. 3, 1138–1172.

[3] T. Funaki, H. Spohn, *Motion by mean curvature from the Ginzburg-Landau $\nabla\phi$ interface model*, Comm. Math. Phys. **185** (1997), no. 1, 1–36.

[4] J.-D. Deuschel, G. Giacomin, D. Ioffe, *Large deviations and concentration properties for $\nabla\phi$ interface models*, Probab. Theory Related Fields **117** (2000), no. 1, 49–111.

[5] T. Funaki, *Stochastic interface models*, Lectures on Probability Theory and Statistics, volume 1869 of *Lecture Notes in Math.*, pages 103–274. Springer, Berlin, 2005.

Variational approach to small convergence rates

Annika Bach

In this talk I presented a variational approach to quantitative stochastic homogenisation of elliptic PDE introduced in [3] (see also [1, Chapter 2] and [2, Chapter 4]). In the case considered in this talk the coefficient fields $a : \mathbb{R}^d \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ are assumed to be uniformly elliptic and symmetric, while the probability measure on the sample space Ω of coefficient fields a (equipped with a suitable σ-algebra) is assumed to be $\mathbb{Z}^d$-stationary, ergodic and with uniform range of dependence[1]. The approach is then based on the variational formulation of Dirichlet problems associated to the elliptic equation $-\nabla\cdot a\left(\frac{x}{\varepsilon}\right)\nabla u(x) = 0$. In particular, it uses the characterisation of the homogenised coefficient field $\overline{a}$ as the limit of suitable subadditive quantities. The latter have been introduced in [4] in the context of Γ-convergence of random integral functionals. More precisely, for any open bounded set with Lipschitz boundary and any direction $p \in \mathbb{R}^d$ one considers the quantity

$$\mu(U,p) := \inf\left\{\frac{1}{|U|}\int_U \tfrac{1}{2}a\nabla v\cdot\nabla v\,dx \colon v \in \ell_p + H^1_0(U)\right\} \tag{1}$$

with $\ell_p(x) := p\cdot x$ denoting the linear function with gradient p. Thanks to the prescription of boundary conditions, $\mu(\,\cdot\,,p)$ is subadditive as a set function for every $p\in\mathbb{R}^d$. Using subadditive ergodic theorems one can thus characterise the homogenised coefficients via (see, e.g., [4, Theorem I] or [2, Section 1.4])

$$\frac{1}{2}\overline{a}p\cdot p = \lim_{n\to+\infty}\mathbb{E}\big[\mu(Q_n,p)\big] = \inf_{n\in\mathbb{N}}\mathbb{E}\big[\mu(Q_n,p)\big], \tag{2}$$

[1] See [1, Chapter 10] for the case of non-symmetric coefficients, [3] and [1, Chapter 11] for the nonlinear uniformly elliptic setting, and [2, Chapter 3 and 4] for weaker concentration properties of the coeffiecient fields.

where $Q_n := \big(-\frac{3^n}{2}, \frac{3^n}{2}\big)^d$. The last equality in (2) holds thanks to subadditivity, which together with stationarity implies monotonicity of the expectation.

One can further rewrite (2) by the following observation: For every $n \in \mathbb{N}$ and every $p \in \mathbb{R}$ the infimum defining $\mu(Q_n, p)$ is realised for a unique a-harmonic function $v_{n,p} \in \ell_p + H^1_0(Q_n)$. Moreover, $p \mapsto v_{n,p}$ is linear, so that $\mu(Q_n, \cdot)$ is actually a quadratic form and may be written as $\mu(Q_n, p) = \frac{1}{2} a(Q_n) p \cdot p$ for some *coarse-grained* coefficients $a(Q_n) \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$. In this way, (2) states that $\overline{a}$ is approximated from above by $\mathbb{E}\big[a(Q_n)\big]$. There is however no quantitative information in this monotonicity argument

To obtain such information, the key idea in [3] is to couple μ with a *superadditive* quantity, which should approximate $\overline{a}$ from below, and to control the difference between the two quantities. More in detail, for any $q \in \mathbb{R}^d$ the authors consider the optimisation problem

$$\mu^*(U, q) := \sup\left\{ \frac{1}{|U|} \int_U \Big(q \cdot \nabla u - \tfrac{1}{2} a \nabla u \cdot \nabla u \Big)\, dx \colon u \in H^1(U) \right\}, \tag{3}$$

so that $q \cdot p - \mu^*(\,\cdot\,, q)$ is superadditive for every p, q and we have

$$J(Q_n, p, q) := \mu(Q_n, p) + \mu^*(Q_n, q) - q \cdot p \geq 0 \ \text{ for every } n, p, q\,. \tag{4}$$

(4) almost looks like a duality relation between μ and μ^*. Indeed, if a was constant, then μ and μ^* would be dual to each other and we would have $\mu^*(Q_n, q) = \frac{1}{2} a^{-1} q \cdot q$ and $J(Q_n, p, ap) = 0$. Although the argument fails in the inhomogeneous case, due to the homogenisation procedure one still expects that $J(Q_n, p, \overline{a}p) \to 0$ as $n \to +\infty$. This turns out to be true and the convergence can be made quantitative in $L^1(\Omega)$. In fact, it can be shown (see, e.g., [1, Proposition 2.8]) that there exist a constant $C > 0$ and an exponent $\alpha \in (0, \frac{1}{2}]$ (both depending only on the dimension and the ellipticity contrast) such that

$$\mathbb{E}\big[J(Q_n, p, \overline{a}p)\big] \leq C 3^{-n\alpha} \ \text{ for every } n \in \mathbb{N} \text{ and } p \in B_1. \tag{5}$$

One can read (5) as an algebraic decay of the expected duality defect between the quantities μ and μ^*. As a consequence, it provides an algebraic rate of convergence for the homogenisation error $|\overline{a} - a(Q_n)|$ in $L^1(\Omega)$, since an optimisation procedure shows that $|\overline{a} - a(Q_n)| \leq C \sup_{p\in B_1} J(Q_n, p, \overline{a}p)$. Thus, (5) also implies

$$\mathbb{E}\big[|\overline{a} - a(Q_n)|\big] \leq C 3^{-n\alpha}\,. \tag{6}$$

Moreover, using the unit range of dependence, from (5) one can deduce path-wise estimates on the homogenisation error, where at the cost of worsening the algebraic control on the deterministic part one gets optimal estimates on the decay of the tail distribution. In fact, for every $s \in (0, d)$ one can find a constant C depending now on the dimension, the ellipticity contrast, and on s such that

$$\mathbb{P}\Big(|\overline{a} - a(Q_n)| \geq C 3^{-n\alpha(d-s)} + t \Big) \leq C \exp(-3^{ns} t) \tag{7}$$

for every $n \in \mathbb{N}$ and every $t > 0$ (cf. [1, Theorem 2.4 and Remark 2.5]).

The key steps to obtain (5) are a control of the variance of $\|\nabla(u_{n,q} - v_{n,p})\|_{L^2(Q_n)}$ and based on that a control of the expected flatness of $(u_{n,q} - v_{n,p})$, where $u_{n,q}$ is a

maximizer for $\mu^*(Q_n, q)$. The control of the first quantity relies on the unit range of dependence, while the second quantity is estimated using a multiscale Poincaré inequality [1, Proposition 1.7 and Corollary 1.9] together with a decomposition of scales argument. The latter techniques also allow to give a rate on the sublinearity of the *finite-volume correctors* $\phi_{n,p} := v_{n,p} - \ell_p$, as they provide suitably weighted estimates of $\|\phi_{n,p}\|_{L^2(Q_n)}$ in terms of $|\overline{a} - a(Q_m)|$ on scales $m = 0, \dots, n$, which in turn can be controlled thanks to (6) and (7). Eventually, the rate on the sublinearity yields a rate of convergence for solutions u_ε of Dirichlet problems associated to $-\nabla \cdot a\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon(x) = 0$.

References

[1] S. N. Armstrong, T. Kuusi, and J.-C. Mourrat *Quantitative stochastic homogenization and large-scale regularity*, volume 352 of *Grundlehren der Mathematischen Wissenschaften*, Springer, Cham, 2019.

[2] S. N. Armstrong and T. Kuusi *Elliptic homogenization from qualitative to quantitative*, preprint, arXiv:2210.06488 (2022).

[3] S. N. Armstrong and C. K. Smart *Quantitative stochastic homogenization of convex integral functionals*, Ann. Sci. Éc. Norm. Supér. (4) **49** (2016), 423–481.

[4] G. Dal Maso and L. Modica *Nonlinear stochastic homogenization and ergodic theory*, J. Reine Angew. Math. **368** (1986), 28–42.

Functional Calculus in Probability

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We provide a basic introduction to Malliavin calculus and how it naturally arises as a useful quantitative tool in stochastic homogenization (see, e.g., [9, 7, 8, 4, 5, 6], as well as some most recent extensions to more general settings [1, 2, 3]).

We start with a warm-up example, the so-called Efron-Stein inequality, which can be stated as follows: for an i.i.d. sequence $a = (a_z)_{z\in\mathbb{Z}^d}$ and a smooth function $X = X(a) \in \mathbb{R}$, it holds

$$\mathrm{Var}\,[X] := \mathbb{E}\left[X^2\right] - \mathbb{E}\,[X]^2 \le \sum_{z\in\mathbb{Z}^d} \mathbb{E}\left[\,(X - \mathbb{E}_z\,[X])^2\,\right]\,, \tag{1}$$

where $\mathbb{E}_z\,[X] := \mathbb{E}\,[X \mid (a_w)_{w\neq z}]$ denotes conditional expectation . The variance inequality (1) is a first instance of a concentration inequality for the random variable X, controlling a *global* measure of deviation, i.e., $\mathrm{Var}[X]$, in terms of sums of mean *local* displacements $\mathbb{E}[\big(X - \mathbb{E}_z\,[X]\,\big)^2]$. In this sense, the quantity $D_z X := X - \mathbb{E}_z\,[X]$ plays the role of "partial derivative" of X with respect to a_z.

Building on this rather intuitive discrete example, we focus our presentation on the most standard context of Malliavin calculus in the "continuum" Gaussian setting. We start with a Hilbert space $\mathfrak{H}$ and a centered, translation-invariant Gaussian field $G = G(x)$ on $\mathbb{R}^d$ isonormal over $\mathfrak{H}$; then, we consider a sufficiently rich class of random variables $X = X(G)$ (namely, the cylindrical and smooth ones); finally, for such functionals, we introduce the notion of Malliavin derivative DX, a random $\mathfrak{H}$-valued element. The operator D is closable in the space of

square-integrable random variables; in what follows, we let $\mathbb{D}^{1,2}$ — the infinite-dimensional analogue of the most common Sobolev space $W^{1,2}$ — denote the domain of its closure.

Along with the $\mathfrak{H}$-valued DX, we introduce the divergence operator D^* as the adjoint of D, and the corresponding infinite-dimensional Laplacian (or Ornstein–Uhlenbeck operator) $L := D^*D$. Finally, on the Malliavin-Sobolev space $\mathbb{D}^{1,2}$, we present the Gaussian analogue of (1), i.e., the first-order Poincaré inequality: for all square-integrable $X = X(G)$ in $\mathbb{D}^{1,2}$, it holds

$$\operatorname{Var}[X] \leq \mathbb{E}\left[\|DX\|_{\mathfrak{H}}^2\right] . \tag{2}$$

This inequality proves to be useful in deriving quantitative Laws of Large Numbers: having a quantitative control of the smallness of DX would guarantee that the random variable X is approximately constant with respect to the underlying randomness. Log-Sobolev and second-order Poincaré inequalities hold in this setting as well, and turn out to be effective when establishing higher-order moments estimates and quantitative Central Limit Theorems, respectively; we refer the reader to [5, Sec. 4] for further details.

With inequality (2) in hand, we show one of its basic applications (more involved applications in the context of quantitative stochastic homogenization will be presented in subsequent sessions): the optimal decay for the spatial–average random variable $X_R := \frac{1}{|B_R|}\int_{B_R} a$, with $a(x) := a_0(G(x))$, $a_0 \in \mathcal{C}_c^\infty(\mathbb{R})$.

References

[1] Mitia Duerinckx and Antoine Gloria. Multiscale second-order Poincaré inequalities in probability. *arXiv:1711.03158*, 2017.

[2] Mitia Duerinckx and Antoine Gloria. Multiscale functional inequalities in probability: concentration properties. *ALEA Lat. Am. J. Probab. Math. Stat.*, 17(1):133–157, 2020.

[3] Mitia Duerinckx and Antoine Gloria. Multiscale functional inequalities in probability: constructive approach. *Ann. H. Lebesgue*, 3:825–872, 2020.

[4] Mitia Duerinckx, Antoine Gloria and Felix Otto. The structure of fluctuations in stochastic homogenization. *Comm. Math. Phys.*, 377(1):259–306, 2020.

[5] Mitia Duerinckx and Felix Otto. Higher-order pathwise theory of fluctuations in stochastic homogenization. *Stoch. Partial Differ. Equ. Anal. Comput.*, 8(3):625–692, 2020.

[6] Mitia Duerinckx. On the size of chaos via Glauber calculus in the classical mean-field dynamics. *Comm. Math. Phys.*, 382(1):613–653, 2021.

[7] Antoine Gloria and Felix Otto. An optimal variance estimate in stochastic homogenization of discrete elliptic equations. *Ann. Probab.*, 39(3):779–856, 2011.

[8] Antoine Gloria, Stefan Neukamm and Felix Otto. Quantification of ergodicity in stochastic homogenization: optimal bounds via spectral gap on Glauber dynamics. *Invent. Math.*, 199(2):455–515, 2015.

[9] Ali Naddaf and Thomas Spencer. On homogenization and scaling limit of some gradient perturbations of a massless free field. *Comm. Math. Phys.*, 183(1):55–84, 1997.

Bounds on correctors

Laure Giovangigli, Lorenzo Marino

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $a : \Omega \times \mathbb{R}^d \to \mathcal{M}_d(\mathbb{R})$ be an ergodic stationary uniformly elliptic field. For $f \in C_c^\infty(\mathbb{R}^d)$ and $\varepsilon > 0$, we consider u_ε solution in $W = \{v \in H^1(\mathbb{R}^d), \nabla v \in L^2(\mathbb{R}^d)\}/\mathbb{R}$ of

$$-\nabla \cdot a\left(\frac{\cdot}{\varepsilon}\right)\nabla u_\varepsilon = \nabla \cdot f \quad \text{in } \mathbb{R}^d. \tag{1}$$

Let $\bar{u}$ denote the solution in W to the homogenized problem and (ϕ, σ) the extended correctors associated to (1) (Definition 2 in [2]).

Our goal is to prove the convergence of u_ε to its two-scale expansion $\bar{u} + \varepsilon\phi_i\left(\frac{\cdot}{\varepsilon}\right)\partial_i\bar{u}$ in W and estimate the rate of convergence.

The error $z_\varepsilon := u_\varepsilon - \bar{u} - \varepsilon\phi_i\left(\frac{\cdot}{\varepsilon}\right)\partial_i\bar{u}$ verifies

$$-\nabla \cdot a\left(\frac{\cdot}{\varepsilon}\right)\nabla z_\varepsilon = \nabla \cdot \varepsilon(a\phi_i - \sigma_i)\left(\frac{\cdot}{\varepsilon}\right)\nabla\partial_i\bar{u} \quad \text{in } \mathbb{R}^d. \tag{2}$$

The convergence of z_ε is then a direct consequence of the sublinearity of the extended corrector. In order to quantify this convergence, we need to establish bounds on the correctors.

In the case of Gaussian coefficient fields a with integrable correlations, the following estimates can be proven [1].

Theorem 1. *Under the previous assumptions, for any $r \in [1, +\infty)$, there holds*

$$\mathbb{E}\left[|\nabla(\phi, \sigma)|^{2r}\right]^{\frac{1}{r}} \lesssim 1, \tag{3}$$

and the increments of the extended correctors are controlled as follows

$$\forall x \in \mathbb{R}^d, \quad \mathbb{E}\left[|(\phi, \sigma)(x) - (\phi, \sigma)(0)|^{2r}\right]^{\frac{1}{r}} \lesssim \mu_d^2(|x|), \tag{4}$$

where the function μ_d is defined as

$$\mu_d(r) := \begin{cases} \sqrt{1+r} & \text{if } d = 1, \\ \ln^{\frac{1}{2}}(r+2) & \text{if } d = 2, \\ 1 & \text{if } d = 3. \end{cases}.$$

We outline below the proof for the results on ϕ presented in [1]. The estimates on σ can be proven following the same method.

Proof. Step 1 Let $g \in C_c^\infty(\mathbb{R}^d))$. We first derive a representation formula for the Malliavin derivative

$$\frac{\partial}{\partial a}\int g \cdot \nabla\phi = \nabla v \otimes \nabla w,$$

where $w := \phi_i + e_i$, $i \in [|1, d|]$ and v is the solution in W of

$$-\nabla \cdot a^*\nabla v = \nabla \cdot g \quad \text{in } \mathbb{R}^d.$$

Step 2 We plug this representation formula into the L^r version of the spectral gap and obtain thanks to an annealed Calderon-Zygmund estimate for $r \gg 1$

$$\mathbb{E}\left[\left|\int g\cdot\nabla\phi\right|^{2r}\right]^{\frac{1}{r}} \lesssim \mathbb{E}\left[\left(\fint_{B_1}|\nabla w|^2\right)^r\right]^{\frac{1}{r}}\int|g|^2. \tag{5}$$

Step 3 We apply a PDE argument, the Caccioppoli estimate, to control the r-th moment of $\fint_{B_1}|\nabla w|^2$ by spatial averages of ∇w on large balls B_R with $R\gg 1$

$$\mathbb{E}\left[\left(\fint_{B_1}|\nabla w|^2\right)^r\right]^{\frac{1}{r}} \lesssim R^{d(1-\frac{1}{r})}\mathbb{E}\left[\left|\fint_{B_R}\nabla w\right|^{2r}\right]^{\frac{1}{r}}. \tag{6}$$

Step 4 We choose $g:=\nabla\phi_{B_R}$ in (5) and use it to bound the rhs of (6).

$\mathbb{E}\left[\left(\fint_{B_1}|\nabla w|^2\right)^r\right]^{\frac{1}{r}}$ appears then both in the left and rhs and we can buckle to obtain

$$\mathbb{E}\left[\left(\fint_{B_1}|\nabla w|^2\right)^r\right]^{\frac{1}{r}} \lesssim 1. \tag{7}$$

Thanks to a local regularity estimate, we can deduce (3) for ϕ. Combining (7) and (5), we moreover get that the spatial averages of $\nabla\phi$ present cancellations

$$\mathbb{E}\left[\left|\int g\cdot\nabla\phi\right|^{2r}\right]^{\frac{1}{r}} \lesssim \int|g|^2. \tag{8}$$

Step 5 Lastly, we use (3) to get

$$\mathbb{E}\left[\left|\phi(0)-\fint_{B_1(0)}\phi\right|^{2r}\right] \lesssim \mathbb{E}\left[\fint_{B_1(0)}|\nabla\phi|^{2r}\right] \lesssim 1. \tag{9}$$

Moreover, we have the following representation formula

$$\fint_{B_1(x)}\phi-\fint_{B_1(0)}\phi=\int\nabla h\cdot\nabla\phi,$$

where h is the decaying solution of

$$-\Delta h=\frac{1}{|B_1|}(\nabla\phi_{B_1(x)}-\nabla\phi_{B_1(0)}).$$

From (8) and classical potential theory, we then deduce

$$\mathbb{E}\left[\left|\fint_{B_1(x)}\phi-\fint_{B_1(0)}\phi\right|^{2r}\right]^{\frac{1}{r}} \lesssim \int|\nabla h|^2\lesssim\mu_d^2(|x|). \tag{10}$$

Combining (9) and (10), we obtain (4). □

Equipped with the bounds on correctors established in Theorem 1, we can now deduce from (2) the rate of convergence of z_ε in W [1].

Corollary 1. *Under the previous assumptions, the following estimate holds for* $\varepsilon > 0$

$$\mathbb{E}\left[\|\nabla z_\varepsilon\|^2_{L^2(\mathbb{R}^d)}\right]^{\frac{1}{2}} \lesssim \varepsilon \mu_d\left(\frac{1}{\varepsilon}\right).$$

References

[1] M. Josien and F. Otto, *The annealed Calderon-Zygmund estimate as convenient tool in quantitative stochastic homogenization*, Preprint, arXiv:2005.08811 (2020).

[2] A. Gloria, S. Neukamm, and F. Otto, *Quantitative estimates in stochastic homogenization for correlated coefficient fields*, Analysis & PDE **14.8** (2021) 2497–2537.

Large-scale $C^{1,\alpha}$ regularity

Cole Jeznach

Following the work in [1], we discuss large-scale Schauder theory for solutions to random, divergence form elliptic operators in $\mathbb{R}^d$. In what follows, we assume for simplicity that we have a random field of uniformly elliptic, bounded coefficients $a(\cdot)$ (with some constant $\lambda > 1$) on a probability space $(\Omega, \mathbb{P})$. Moreover, we assume stationarity of the field of coefficients as well as ergodicity. First, let us recall some results in the classical, deterministic setting.

Regularity theory for uniformly elliptic equations says roughly that if $u \in H^1(B_1)$ is a solution to the divergence-form equation

$$-\nabla \cdot (A\nabla u) = 0, \quad u \in H^1(B_1), \tag{1}$$

then in the appropriate Hölder spaces, u gains one extra degree of differentiability over the coefficients A. That is, if one knows that $A \in C^{k,\alpha}_{\text{loc}}(B_1)$ for $\alpha \in (0,1)$, then $u \in C^{k+1,\alpha}_{\text{loc}}(B_1)$ with appropriate estimates coming from A (there are also results for solutions to the inhomogeneous problem, but again, for brevity, we omit these). Let us also focus on the case $k = 0$, and assume for the future that $A \in C^{0,\alpha}_{\text{loc}}(\mathbb{R}^n)$ and u is a solution to (1).

There are several methods to proving such Schauder estimates: one involves potential theoretic estimates, the other maximum principles and Harnack inequalities, and the last energy estimates for divergence-form equations. Morally speaking, each of the proofs boils down to the fact that since the coefficients A are Hölder continuous, at small scales A behaves quantitatively constant, and thus solutions to (1) should behave like solutions to constant-coefficient divergence form elliptic equations.

One way to approach the energy-method of Schauder estimates is to show decay of the so-called "tilt-excess": for $0 < r < R$ and $B_R(x_0) \subset B_1$,

$$\text{Exc}(\nabla u; B_r(x_0)) \leq C\left(\frac{r}{R}\right)^{2\alpha} \text{Exc}(\nabla u; B_R(x_0)) \tag{2}$$

where, for any vector field $g \in L^2_{\text{loc}}$,

$$\text{Exc}(g; B) = \inf_{\xi \in \mathbb{R}^n} |B|^{-1} \int_B |g - \xi|^2 \, dx. \tag{3}$$

That this excess-decay implies that ∇u is locally $C^{0,\alpha}$ follows readily from Campanato's Theorem (see, for example, Theorem 5.5 in [2]).

For the classical case of harmonic functions v (which are of course, analytic), one has the the excess decay

$$\mathrm{Exc}(\nabla v; B_r(x_0)) \leq C\left(\frac{r}{R}\right)^2 \mathrm{Exc}(\nabla v; B_R(x_0))$$

and in fact, one can show this directly using Caccioppoli's inequality, the mean-value property of harmonic functions, and the fact that linear functions are harmonic. For non-constant coefficient equations, the fact that linear functions are not a-harmonic (i.e., do not solve $-\nabla \cdot (a\nabla v) = 0$) causes some difficulty in extending this proof. In the setting of homogenization, this is where the extended first-order correctors come into play.

Under the assumptions of stationarity and ergodicity mentioned above, one can prove ($\mathbb{P}$-a.e.) the existence of the extended correctors. That is, almost surely there are random tensor fields $\{\phi_i\}, \{\sigma_{ijk}\}$ for which $\nabla(\phi, \sigma)$ have bounded second moment, zero expectation, are stationary, and which solve the corrector equations

$$\begin{aligned} -\nabla \cdot (a(e_i + \nabla\phi_i)) &= 0 \\ \nabla \cdot \sigma_i &= q_i \\ \Delta\sigma_{ijk} &= \partial_j q_{ik} - \partial_k q_{ij} \end{aligned}$$

in all of $\mathbb{R}^d$. Here and in the future e_i is the standard i^{th} basis vector of $\mathbb{R}^d$, $q_i = a(e_i + \nabla\phi_i) - a_{\text{hom}} e_i$, and $a_{\text{hom}} = \mathbb{E}[a(e_i + \nabla\phi_i)]$ is the homogenized, constant coefficient matrix.

In many ways, the functions $x_i + \phi_i$ play the role that linear functions do for harmonic functions. Indeed, the correct analogue for "tilt-excess" in this setting is instead the distance of solutions (in energy norm) from a-linear functions. That is, we set

$$\mathrm{Exc}_a(\nabla u; B) = \inf_{\xi \in \mathbb{R}^d} |B|^{-1} \int_B |\nabla u - (\xi + \nabla\phi_\xi)|^2 \, dx \tag{4}$$

where $\phi_\xi = \sum_{i=1}^d \xi_i \phi_i$. In particular, for this deviation from a-linear functions, one can show the following quenched result, which holds $\mathbb{P}$-a.e., depending only on the realization of (ϕ, σ).

Theorem 1 ([1]). *For each $\alpha \in (0,1)$, there is a $C > 0$ depending only on d, α and the uniform ellipticity bounds on the field $a(\cdot)$ so that for $r^* = r^*(C)$ defined by*

$$r^* = \inf\{t > 0 \ : \ \text{for all } s > t, \ s^{-d-2} \int_{B_s} \left| (\phi, \sigma) - \left(\fint_{B_s} (\phi, \sigma) \right) \right|^2 \, dx \leq 1/C\},$$

the following holds. Whenever $r^* \leq r \leq R$ and u solves

$$-\nabla \cdot (a\nabla u) = 0 \quad \text{in } B_R,$$

then one has the excess decay

$$\mathrm{Exc}_a(\nabla u; B_r) \leq C(r/R)^{2\alpha} \mathrm{Exc}_a(\nabla u; B_R).$$

On the one hand, it can be shown that the extended correctors satisfy

$$\limsup_{R\to\infty} R^{-d-2}\int_{B_R}\left|(\phi,\sigma)-\left(|B_R|^{-1}\int_{B_R}(\phi,\sigma)\right)\right|^2\,dx=0$$

almost surely, so then $r^*<\infty$ almost surely and Theorem 1 gives a qualitative statement on the $C^{1,\alpha}$ regularity of u at large scales. On the other hand, if one can obtain estimates on the size of r^*, then the conclusion of the Theorem can be made quantitative.

As a final remark, the excess decay from Theorem 1 can be used to prove the following Liouville-type theorem for global solutions with sub-linear growth at infinity, as in the following result.

Corollary 1 ([1])**.** *Suppose that u solves $-\nabla\cdot(a\nabla u)=0$ in $\mathbb{R}^d$, and that for some $\alpha\in(0,1)$, one has*

$$\limsup_{R\to\infty} R^{-d-2\alpha}\int_{B_R}u^2\,dx=0.$$

Then almost surely, there is some $\xi\in\mathbb{R}^d$ and $c\in\mathbb{R}$ so that $u(x)=c+x\cdot\xi+\phi_\xi(x)$, almost everywhere in $\mathbb{R}^d$.

References

[1] A. Gloria, S. Neukamm, and F. Otto, *A regularity theory for random elliptic operators.* Milan J. Math. **88** (2020), no. 1, 99–170.

[2] M. Giaquinta and L. Martinazzi, *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs.* Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) (2012).

Systematic Argument for Bounds on the Minimal Radius

Xiaoqin Guo, Peter Morfe

Recall that for a given stationary, ergodic coefficient field $a:\mathbb{R}^d\to\mathbb{R}^{d\times d}$, the *minimal radius* $r_*>0$ is the random scale above which large-scale regularity begins to kick in. More precisely, in [2], Gloria, Neukamm, and Otto define, for an arbitrary $A>0$, $r_*=r_*(A)$ as follows:

$$r_*(A)=\inf\left\{r>0\;\middle|\;\forall R\geq r,\ R^{-(d+2)}\int_{B_R}\left\|(\phi,\sigma)-\fint_{B_R}(\phi,\sigma)\right\|^2\,dx\leq A^{-1}\right\}.$$

Ergodicity is enough to imply that $r_*<\infty$ holds almost surely. In case a is a "random checkerboard," formal arguments readily suggest that $\langle\exp(C^{-1}r_*^\beta)\rangle<\infty$ only if $\beta\leq d$. This integrability property was proven to hold in the regime $\beta<d$ by Armstrong and Smart [1]. In [2], the optimal exponent $\beta=d$ was achieved.

Theorem 1. *If a is a pointwise bounded and uniformly elliptic random field that satisfies the logarithmic Sobolev inequality, then, for any $A>0$, the minimal radius*

$r_* = r_*(A)$ defined as above satisfies the following exponential moment bound for some $C > 0$ depending only on A, the ellipticity constants, and the dimension:

$$\left\langle \exp\left(C^{-1} r_*^d\right)\right\rangle < \infty.$$

There are three main steps in the proof. In order to exploit the statistical properties of the field a (in particular, the log-Sobolev inequality), one would like to consider random variables that depend more-or-less only locally on a, whereas the correctors ϕ and σ have a non-local dependence. Therefore, in the first step, it is shown that the sub-linear growth of the correctors (ϕ, σ) can be controlled by those of the massive correctors (ϕ_T, σ_T) obtained by adding a massive cut-off at scale $\sqrt{T}$. More precisely, in [2, Proposition 4.1], it is proved that if, for some radius $r_{**} > 0$ and exponent $\nu > 0$, one has

$$\int_{B_{\sqrt{T}}} \frac{1}{T} \|(\phi_T, \sigma_T)\|^2 \, dx \le \left(\frac{r_{**}}{\sqrt{T}}\right)^{2\nu} \quad \text{for } \sqrt{T} \ge r_{**}, \tag{1}$$

then this behavior transfers over to (ϕ, σ) in the sense that

$$R^{-(d+2)} \int_{B_R} \left\| (\phi, \sigma) - |B_R|^{-1} \int_{B_R} (\phi, \sigma) \right\|^2 dx \le C \left(\frac{r_{**}}{R}\right)^{2\nu} \quad \text{for } R \ge r_{**}.$$

Note, in particular, that this means r_{**} controls r_*, i.e., $r_* \le C r_{**}$. The reason such a statement is at least plausible is $\mathbb{E}(\|\nabla\phi_T - \nabla\phi\|^2) \to 0$ as $T \to \infty$; the proof exploits this observation using a combination of Campanato iteration and large-scale regularity estimates.

The next step of the proof provides deterministic ingredients used in the final probabilistic argument. To prove decay as in (1), it is shown that, for any $0 < t < T$, one has a bound

$$\int_{\mathbb{R}^d} \frac{1}{T} \|(\phi_T, \sigma_T)\|^2 \omega_T(x) \, dx \le C \int_{\mathbb{R}^d} \left(\frac{1}{t}\phi_t^2 + \frac{1}{t}\|g_t\|^2 + \|\nabla g_t\|^2\right) \omega_T(x) \, dx, \tag{2}$$

where here $\omega_T(x) = (C_d A^d T^{\frac{d}{2}})^{-1} \exp(-\|x\|/(AT))$ is normalized to be a probability density and the auxiliary corrector g_t is defined so that the triple $(\phi_t, g_t, \nabla g_t)$ serves as an approximate (homogeneous) H^{-1} norm of the gradient field $\nabla\phi_T$ and corresponding flux. The advantage of this last estimate is it holds for any $t < T$, the idea being that (1) can be obtained with high probability by treating the behavior at scale $\sqrt{T}$ as an approximately-independent sum of $(T/t)^{d/2}$ random variables.

Finally, concentration inequalities are applied to obtain the stochastic integrability of the minimal radius. One defines a stationary field F_t by

$$F_t(x) = \int_{\mathbb{R}^d} \left(\frac{1}{t}\phi_t(y)^2 + \frac{1}{t}\|g_t(y)\|^2 + \|\nabla g_t(y)\|^2\right) \omega_t(y - x) \, dy.$$

It turns out that F_t is uniformly bounded (in t) and well-localized (i.e, with a rapid decay of the dependence on environments beyond distance $\sqrt{t}$). This can be

exploited using the log-Sobolev inequality to show that F_t satisfies the following concentration-type estimate

$$\left\langle I\left\{\left|\int_{\mathbb{R}^d}[F_t(x)-\langle F_t\rangle]\omega_T(x)\,dx\right|>\delta\right\}\right\rangle \leq C\exp\left(-C^{-1}\delta^2\left(\frac{\sqrt{T}}{\sqrt{t}}\right)^d\right).$$

Furthermore, using sensitivity estimates (obtained by Widman's hole-filling technique for uniformly elliptic equations in divergence form), the log-Sobolev inequality yields an algebraic decay of the average:

$$\langle F_t\rangle \leq Ct^{-\epsilon}.$$

The theorem is proved upon combining these last two estimates with (1) and (2).

References

[1] S. N. Amrstrong and C. K. Smart, "Quantitative stochastic homogenization of convex integral functionals," *Ann. Sci. Éc. Norm. Supér.* 49-2 (2016): 423-481.

[2] A. Gloria, S. Neukamm, and F. Otto, "A regularity theory for random elliptic operators," *Milan Journal of Mathematics* 88-1 (2020): 99-170.

Quenched and Annealed Calderón-Zygmund estimates

Christopher Irving, David Lee

A consequence of the large-scale $C^{1,1-}$ regularity theory is that we obtain Calderón-Zygmund type estimates for the inhomogenous problem

$$-\operatorname{div} a\nabla u = \operatorname{div} g. \tag{1}$$

In the deterministic setting where a is continuous and uniformly elliptic, this asserts that we have estimates in the L^p scales taking the form

$$\int_{\mathbb{R}^d}|\nabla u|^p\,\mathrm{d}x \lesssim \int_{\mathbb{R}^d}|g|^p\,\mathrm{d}x \tag{2}$$

for $1<p<\infty$. In the setting of homogenization, we will see similar estimates hold in the large scales, and also in a suitably averaged sense. We will present two analogues of these L^p estimates following [2, 3]; the quenched and annealed estimates.

We first fix some notation; we consider a probability space $(\Omega,\mathcal{F},\mathbb{P})$ of λ-uniformly elliptic coefficient fields. Let a be a instance of our random coefficient field, and suppose the associated extended correctors (ϕ,σ) exist. Then for each $x\in\mathbb{R}^n$ and $C>0$ to be determined, we define the *minimal radius* as

$$r_*(x,C)=\inf\left\{r>0\colon \forall R\geq r, \frac{1}{R^2}\fint_{B_R(x)}|(\phi,\sigma)-\fint_{B_R(x_0)}(\phi,\sigma)|^2\leq\frac{1}{C}\right\}. \tag{3}$$

We remark that if we assume our coefficient field is stationary and ergodic, with respect to translations, then (ϕ,σ) exists and $r_*<\infty$ for $\mathbb{P}$-a.e. a. If this occurs, we obtain the followed quenched, or "pathwise" regularity estimates.

Theorem 1 (Quenched Calderón-Zygmund estimates)**.** *Let a be λ-uniformly elliptic, and $C_0 > 0$ so the large-scale regularity results in* [3, Theorem 1] *holds. Then there exists a $\frac{1}{8}$-Lipschitz stationary field $\underline{r_*}$ satisfying $r_*(C_0) \le \underline{r_*} \le r_*(3^{d+2}C_0)$ for which the following holds. Suppose u, g satisfies* (1)*, then for any $1 < p < \infty$ we have*

$$\left(\int_{\mathbb{R}^d} \left(\fint_{B_*(x)} |\nabla u|^2 \right)^{\frac{p}{2}} \mathrm{d}x \right)^{\frac{1}{p}} \lesssim \left(\int_{\mathbb{R}^d} \left(\fint_{B_*(x)} |g|^2 \right)^{\frac{p}{2}} \mathrm{d}x \right)^{\frac{1}{p}}, \tag{4}$$

where $B_*(x) = B_{\underline{r_*}(x)}(x)$ and the implicit constant depends on d, λ, p only.

Here we cannot expect (2) to hold without assuming a is more regular. However we know that regularity holds in sufficiently large scales, namely when $r > \underline{r_*}$, so we replace any $f \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ by

$$f_*(x) := \left(\fint_{B_*(x)} |f|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}. \tag{5}$$

We remark that it is useful to replace the minimal radius by a regularised version $\underline{r_*}$, but this is more of a technical step.

We will outline the proof, following the strategy in [2, Section 6]. This will combine standard energy estimates (namely (1) with $p = 2$) together with uniform estimates in the large scale that takes the form

$$\fint_{B_r} |\nabla u|^2 \lesssim \fint_{B_R} |\nabla u|^2 \quad \text{for all} \quad r_*(x) \le r \le R. \tag{6}$$

The result then follows by an interpolation argument; a key technical tool will be the following Calderón-Zygmund type estimate due to Shen [5], which is based on ideas of Caffarelli & Peral [1]. One can also pass through a BMO estimate as is done in [3, Section 3.7], however the approach we detail will also apply in the annealed case.

We note that one can also obtain weighted estimates. In [3, Corollary 5] one can allow for radial weights of type $|x|^\gamma$ which belong to the Muckenhoupt A_p class. This follows from the corresponding unweighted estimates applied on annular domains, along with decay properties of a-harmonic functions that follow from the mean value property (6).

Lemma 1. *Let $1 \le p_0 < p_1 < \infty$, and $C_0 > 0$, $\theta \in (0, 1)$. Suppose we have $F, G \in L^{p_0} \cap L^{p_1}(\mathbb{R}^d)$ with the property that for all balls $B \subset \mathbb{R}^d$, there exists a measurable decomposition $F = F_{B,0} + F_{B,1}$ in B such that*

$$\left(\fint_B |F_{B,0}|^{p_0} \right)^{\frac{1}{p_0}} \le C_0 \left(\fint_{\theta^{-1}B} |G|^{p_0} \right)^{\frac{1}{p_0}} \tag{7}$$

$$\left(\fint_{\theta B} |F_{B,1}|^{p_1} \right)^{\frac{1}{p_1}} \le C_0 \left(\fint_B |F_{B,1}|^{p_0} \right)^{\frac{1}{p_0}}. \tag{8}$$

Then for all $q \in (p_0, p_1)$ we have

$$\left(\int_{\mathbb{R}^d} |F|^q\right)^{\frac{1}{q}} \leq C(d, p_0, p_1, q, \theta, C_0) \left(\int_{\mathbb{R}^d} |G|^q\right)^{\frac{1}{q}}. \tag{9}$$

We will apply this with $F = (\nabla u)_*$ and $G = g_*$ using (5) to establish the case $p \geq 2$. For a ball $B \subset \mathbb{R}^n$ we decompose u to solve

$$-\operatorname{div} a\nabla u_{B,0} = \operatorname{div}(g\mathbb{1}_B), \quad -\operatorname{div} a\nabla u_{B,1} = \operatorname{div}(g\mathbb{1}_{\mathbb{R}^d\setminus B}). \tag{10}$$

Then $F_{B,0} = (\nabla u_{B,0})_*$ satisfies (7) with $p_0 = 2$ by straightforward energy estimates, and for $F_{B,1} = (\nabla u_{B,1})_*$ we use the mean value property (6) to show (8) for any $p_1 > p$, noting $u_{B,1}$ is harmonic in B. The subquadratic case then follows from a duality argument.

We can upgrade the quenched estimates by incorporating the stochastic integrability, which gives rise to the *annealed* estimates.

Theorem 2 (Annealed Calderón-Zygmund estimates)**.** *Under the assumptions of Theorem 1, for all $1 < q \leq p < \infty$ we have*

$$\left(\int_{\mathbb{R}^d} \mathbb{E}\left[\left(\fint_{B_*(x)} |\nabla u|^2\right)^{\frac{q}{2}}\right]^{\frac{p}{q}} \mathrm{d}x\right)^{\frac{1}{p}} \lesssim \left(\int_{\mathbb{R}^d} \mathbb{E}\left[\left(\fint_{B_*(x)} |g|^2\right)^{\frac{q}{2}}\right]^{\frac{p}{q}} \mathrm{d}x\right)^{\frac{1}{p}}. \tag{11}$$

The proof uses a similar strategy as in the quenched case, appealing to Lemma 2. In particular one takes the same decomposition $u = u_{B,0} + u_{B,1}$ specified by (10), but now we estimate $u_{B,0}$ using the quenched estimates in L^q.

The benefit of the annealed version is that, subject to moment bounds on r_*, we can pass from the averages on the scale $\underline{r_*}$ to the unit ball. In particular in [3, Theorem 2], it is shown that if the coefficient field additionally satisfies a suitable logarithmic Sobolev inequality (which is satisfied for instance in the Gaussian setting), we have the moment bound $\mathbb{E}\left[\exp\left(\frac{1}{C} r_*^d\right)\right] < 2$ for some $C > 0$. Using this, for all $\delta \in (0, \frac{1}{2})$ we infer from the (11) that

$$\left(\int_{\mathbb{R}^d} \mathbb{E}\left[\left(\fint_{B_1(x)} |\nabla u|^2\right)^{\frac{q}{2}}\right]^{\frac{p}{q}} \mathrm{d}x\right)^{\frac{1}{p}} \lesssim_\delta \left(\int_{\mathbb{R}^d} \mathbb{E}\left[\left(\fint_{B_1(x)} |g|^2\right)^{\frac{q+\delta}{2}}\right]^{\frac{p}{q+\delta}} \mathrm{d}x\right)^{\frac{1}{p}}, \tag{12}$$

where the implicit constant blows up at $\delta \to 0$. Here we have a loss in stochastic integrability, which arises as we write

$$\left(\fint_{B_*(x)} |g|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \simeq \sum_{n=0}^{\infty} 1_{\{2^n - 1 < r_*(x) \leq 2^{n+1} - 1\}} \left(\fint_{B_{2^{n+1}}(x)} |g|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}. \tag{13}$$

This can be estimated using Hölder's inequality and the moment bounds of r_*, however we must pass to the $L^{q+\delta}$-norm to do so.

The annealed estimates in particular serve as a powerful tool in the study of quantitative stochastic homogenisation, such as in the study of fluctuations. This will be discussed in future talks, and we will also refer the reader to the discussion in [4].

References

[1] L. Caffarelli and I. Peral, *On $W^{1,p}$ estimates for ellliptic equations in divergence form*, Commun. Pure Appl. Math., **51.1** (1998), 1–21

[2] M. Duerinckx and F. Otto, *Higher-order pathwise theory of fluctuations in stochastic homogenization*, Stoch. Partial Differ. Equ. Anal. Comput., **8.3** (2020), 625–692

[3] A. Gloria, S. Neukamm, and F. Otto, *A regularity theory for random elliptic operators*, Milan J. Math., **88** (2020), 99–170.

[4] M. Joisen and F. Otto, *The annealed Calderón-Zygmund estimate as convenient tool in quantitative stochastic homogenization*, J. Func. Anal., **283.7** (2022).

[5] Z. Shen, *Bounds for the Riesz transforms on L^p spaces for second order elliptic operators.* Ann. Inst. Fourier, **55.1** (2005) 173–197.

Pathwise structure of fluctuations

Quentin Goepfert

We consider, for $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, u_ϵ, the solution of:

$$-\nabla \cdot a_\epsilon \nabla u_\epsilon = \nabla \cdot f$$

In stochastic homogenization, not only the oscillations of the solution u_ϵ are an important topic to study, but the fluctuations of macroscopic observable also matter. One can think of

$$F_\epsilon := \int g \cdot \nabla u_\epsilon$$

for a deterministic mask g. To study this object, a first intuition would be to use the two-scale expansion. Though, from work from Mourrat-Otto [1], Gu-Mourrat [2] and Mourrat-Nolen [3], one knows that, in the Gaussian settings with integrable covariance, both $\epsilon^{-d/2} F_\epsilon$ and its two-scaling expansion

$$\epsilon^{-d/2} \bar{F} := \int g \cdot (e_i + \nabla \phi_i) \partial_i \bar{u}$$

have Gaussian limits. But those two Gaussian limits differ. From that observation, one can see that a new quantity must be studied in order to get information on F_ϵ. Note that the scaling $\epsilon^{-d/2}$ is quite natural and corresponds to the scaling of the coefficient field a_ϵ itself.

It turns out that the quantity one needs to consider is the homogenization commutator

$$\Xi_\epsilon[\nabla u_\epsilon] := (a_\epsilon - \bar{a}) \nabla u_\epsilon$$

This quantity is remarkable in several ways. We can highlight three principles:

- *Principle 1*, the fluctuations of F_ϵ (and of $\int g \cdot a_\epsilon \nabla u_\epsilon$) can be deduced from the fluctuations of $\int g \cdot \Xi_\epsilon[\nabla u_\epsilon]$. We have a direct link between F_ϵ and the commutator:

  $$\int g \cdot (\nabla u_\epsilon - \nabla \bar{u}) = \int \nabla \bar{v} \cdot \Xi_\epsilon[\nabla u_\epsilon]$$

  where $\bar{v}$ is the solution of:

  $$-\nabla a^* \nabla \bar{v} = \nabla \cdot g$$

- *Principle 2*, the two-scale expansion of the commutator is precise in the fluctuations scaling *i.e* the fluctuations of $\int g \cdot \Xi_\epsilon[\nabla u_\epsilon]$ are *a.s.* close to the fluctuations of $\int g \cdot \Xi_i(\frac{\cdot}{\epsilon})\partial_i \bar{u}$ where $\Xi_i := (a - \bar{a})(e_i + \nabla \phi_i)$ is the so-called standard homogenization commutator.
- *Principle 3*, Ξ_i is almost local w.r.t. a.

As a consequence of the first two principles, one can develop a *a.s*-theory of the fluctuations governed by this intrinsic quantity Ξ_i. From the first two principles and this remark, one can deduce that this initial problem of understanding the complex object F_ϵ can be reduced to the understanding of the standard homogenization commutator. Though, if this quantity were not easier to study, theses principles and observations would be of no use in our main quest.

We remind that the difficulty of the studying of F_ϵ comes from the non-local dependency of a_ϵ. Indeed, a local variation of a_ϵ implies variations of F_ϵ in the whole domain (this can be computed thanks to Malliavin's calculus). On the other hand, $\int g \cdot \Xi_\epsilon[\nabla u_\epsilon]$ is much more local. It is local up to an error of magnitude ϵ. Recalling that ϕ^* is the adjoint of ϕ, the Malliavin derivative can be computed as:

$$\begin{aligned} D_z \Xi_\epsilon[\nabla u_\epsilon] = &\left(Id + \nabla \phi(\frac{\cdot}{\epsilon})^* \right) \cdot D_z a_\epsilon \nabla u_\epsilon \\ &- \epsilon \nabla \cdot \left(\left(\phi(\frac{\cdot}{\epsilon})^* a_\epsilon + \sigma(\frac{\cdot}{\epsilon})^* \right) \nabla D_z u_\epsilon + \phi(\frac{\cdot}{\epsilon})^* D_z a_\epsilon \nabla u_\epsilon \right) \end{aligned}$$

From that computation, one can see that the standard homogenization commutator will be a good approximation of the commutator. Indeed, taking a look at the main, local, part of $D_z(\Xi_\epsilon[\nabla u_\epsilon] - \Xi_i \partial_i \bar{u})$, one finds:

$$D_z(\Xi_\epsilon[\nabla u_\epsilon] - \Xi_i \partial_i \bar{u}) \approx \left(Id + \nabla \phi(\frac{\cdot}{\epsilon})^* \right) \cdot D_z a_\epsilon \left(\nabla u_\epsilon - (e_i + \nabla \phi_i) \partial_i u_0 \right) \approx O(\epsilon)$$

The local, and non-local, error terms, are shown to be of the same magnitude $O(\epsilon)$. This is summarized in the following theorem:

Theorem 1. *In the Gaussian setting with integrable covariance, for $p < \infty$,*

$$\begin{aligned} &\mathbb{E}\left[\left| \epsilon^{d/2} \int g \cdot \Xi_\epsilon[\nabla u_\epsilon] - \mathbb{E}[\Xi_\epsilon[\nabla u_\epsilon]] - \epsilon^{-d/2} \int g \cdot \Xi_i \partial_i \bar{u} \right|^p \right]^{\frac{1}{p}} \\ &\qquad \leq C(p, g, f)\ \epsilon \times \begin{cases} 1 & \text{if } d > 2 \\ |\log \epsilon|^{1/2} & \text{if } d = 2 \\ \epsilon^{-1/2} & \text{if } d = 1. \end{cases} \end{aligned}$$

These results are rigorously shown in [4] in the simplified random conductance model and shown in the continuous case in [5]. Furthermore, this results also apply for non Gaussian fields (even when Malliavin calculus is not available), as long as one has a multiscale functional inequality, with a weight that has a sufficient decay, see [5, Remark 2.1].

References

[1] J. Mourrat and F. Otto, *Correlation structure of the corrector in stochastic homogenization*, Annals of Probability **44** (2016), 3207–3233.

[2] Y. Gu,and J. Mourrat, *Scaling limit of fluctuations in stochastic homogenization*, Multiscale Modeling & Simulation **14** (2016), 452–481.
[3] J. Mourrat and J. Nolen, *Scaling limit of the corrector in stochastic homogenization*, The Annals of Applied Probability **27** (2017) 944–959.
[4] M. Duerinckx, A. Gloria and F. Otto, *The structure of fluctuations in stochastic homogenization*, Communications in Mathematical Physics **377** (2020), 259–306.
[5] A. Gloria, M. Duerinckx, and F. Otto, *Robustness of the pathwise structure of fluctuations in stochastic homogenization*, Probability theory and related fields (2020) 531–566.

Scaling law of commutator

Georgiana Chatzigeorgiou

Our purpose is to describe the (large-scale) fluctuations of the solution to a linear and uniformly elliptic equation in divergence form

$$\nabla \cdot \left(a\left(\frac{\cdot}{\epsilon}\right) \nabla u_\epsilon + f \right) = 0, \quad \text{in } \mathbb{R}^d$$

with random (stationary and ergodic) coefficient field. The pathwise theory of fluctuations (see [3] for the i.i.d discrete case and [2] for the continuum underlying Gaussian case) reduces this analysis to the characterization of the fluctuations of the so-called standard homogenization commutator

$$\Xi^o\left(\frac{\cdot}{\epsilon}\right) := \left(a\left(\frac{\cdot}{\epsilon}\right) - \bar{a}\right)\left(Id + \nabla\phi\left(\frac{\cdot}{\epsilon}\right)\right)$$

where ϕ is the usual first order corrector and $\bar{a}$ is the homogenized coefficient. The main focus of this talk is to describe the scaling limit of this key quantity (as in [1]).

In order to have access to Malliavin calculus, we consider a weakly correlated (underlying) Gaussian coefficient field as follows.

Assumptions. We assume that the coefficient field a has the form

$$a(x) := a_0(G(x))$$

where $a_0 \in C_b^2(\mathbb{R})^{d\times d}$ is such that the boundedness and ellipticity conditions are satisfied and G is some real-valued centered stationary Gaussian random field characterized by its covariance function $c(x) := \mathbb{E}(G(x)G(0))$, which we assume to have integrable decay at infinite, i.e. $|c(x)| \sim (1+|x|)^{-\beta}$ for some $\beta > d$.

Our main result is divided into the following two parts,

Theorem 1. *For $F \in C_c^\infty(\mathbb{R}^d)^{d\times d}$ we denote by*

$$I_\epsilon(F) := \epsilon^{-d/2} \int_{\mathbb{R}^d} F(x) : \Xi^o\left(\frac{x}{\epsilon}\right) \mathrm{d}x.$$

(i) Limiting variance structure: *There exists a constant four-tensor Q such that*

$$\left| \mathrm{Var}(I_\epsilon(F)) - \int_{\mathbb{R}^d} F(x) : QF(x)\mathrm{d}x \right| \le C \begin{cases} \epsilon, & \text{if } d > 2, \beta \ge d+1 \\ \epsilon|\ln\epsilon|^{1/2}, & \text{if } d = 2, \beta \ge d+1 \\ \epsilon^{\beta-d}, & \text{if } d < \beta < d+1. \end{cases}$$

Moreover we have a representation formula for the effective fluctuation tensor Q in terms of the correctors. Let

$$K_{ijkl}(x) := \mathbb{E}[(a_0(G)(\nabla\phi_j^* + e_j)\cdot(\nabla\phi_i + e_i))(x) \times (1+\mathcal{L})^{-1}(a_0(G)(\nabla\phi_l^* + e_l)\cdot(\nabla\phi_k + e_k))(0)]$$

where $\mathcal{L}$ the infinite-dimensional Laplacian (or Ornstein-Uhlenbeck operator). Then Q is given by

$$Q_{ijkl} := \int_{\mathbb{R}^d} K_{ijkl}(x)c(x)dx.$$

(ii) Approximate normality:

$$(\mathrm{d_W} + \mathrm{d_{TV}})\left(\frac{I_\epsilon(F)}{\mathrm{Var}^{1/2}(I_\epsilon(F))}; \mathcal{N}\right) \leq C\frac{\epsilon^{d/2}|\ln\epsilon|}{\mathrm{Var}(I_\epsilon(F))}.$$

where $\mathrm{d_W}$ and $\mathrm{d_{TV}}$ are the 2-Wasserstein and the total variation distance respectively and $\mathcal{N}$ is a standard Gaussian law .

Remark 1. *Observe that, if Q is non-degenerate then **(i)** and **(ii)** give us a CLT-type result for Ξ_ϵ^o, namely it turns out that Ξ_ϵ^o converges in law to a (2-tensorial) Gaussian white noise with covariance given by Q. However, as it is explained in* [1]*, Q could happen to be degenerate, but in the same work the authors give sufficient additional conditions on the coefficient field under which the non-degeneracy of Q is guaranteed.*

The property that makes Ξ^o a particularly useful quantity when it comes to fluctuations is the fact that it is an approximately local function of the coefficient field which suggest that we could relate the large-scale behaviour of Ξ^o with that of a itself. In our framework this property is seen on the level of the Malliavin derivative of $I_\epsilon(g) := \epsilon^{-d/2}\int_{\mathbb{R}^d} g(x)\Xi_{ij}^o\left(\frac{x}{\epsilon}\right)\mathrm{d}x = \int_{\mathbb{R}^d} g_\epsilon(x)\Xi_{ij}^o(x)\,\mathrm{d}x$ (here we take $F := ge_i\otimes e_j$ for convenience and set $g_\epsilon(x) := \epsilon^{d/2}(\epsilon x)$),

$$DI_\epsilon(g) = a_0(G)(\nabla\phi_i + e_i)\otimes\big((\nabla\phi_j^* + e_j)g_\epsilon + \phi_j^*\nabla g_\epsilon + \nabla h_{\epsilon j}\big)$$

with $-\nabla\cdot a^*\nabla h_{\epsilon j} = \nabla\cdot\big((a^*\phi_j^* - \sigma_j^*)\nabla g_\epsilon\big)$. We observe that the only non-local term is the last one but since the right-hand side of the equation that $h_{\epsilon j}$ satisfies is given in terms of the derivative of g_ϵ, this term is expected to vanish in the limit as $\epsilon\to 0$ (with order 1). To study the behaviour of $h_{\epsilon j}$ one needs to invoke annealed Calderón-Zygmund estimates (see [4] and [5]) together with moment bounds for the corrector (see for instance [5]), tools that play central role in our analysis.

Once we have the aforementioned formula for $DI_\epsilon(g)$ we employ ingredients from Malliavin calculus. For part **(i)**, we use a tool which is very convenient when one tries to characterize the limit of a variance, namely the Helffer-Sjöstrand representation formula,

$$\mathrm{Cov}[I,J] = \mathbb{E}\left[\int\int c(x-y)DI(x)((1+\mathcal{L})^{-1}DJ)(y)\mathrm{d}x\mathrm{d}y\right]$$

where $\mathcal{L}$ the infinite dimensional Laplacian. For part **(ii)**, we appeal to second-order Poincaré inequality which measures the distance to a normal distribution in terms of the second derivative,

$$(\mathrm{d_W} + \mathrm{d_{TV}})(I;\mathcal{N}) \leq C\, \mathbb{E}^{1/4}(||DI||_{L^2}^4) \times \mathbb{E}^{1/4}\left[\left(\sup_{||\zeta||_{L^2}=1}\left|\int\int \zeta(x)\zeta(y)D^2I(x,y)\mathrm{d}x\mathrm{d}y\right|\right)^4\right].$$

We refer to section 4 in [4], for more details on Malliavin calculus.

References

[1] M. Duerinckx, J. Fischer, A. Gloria, *Scaling limit of the homogenization commutator for gaussian coefficient fields*, Ann. Appl. Probab. (2021), to appear.

[2] M. Duerinckx, A. Gloria, F. Otto, *Robustness of the pathwise structure of fluctuations in stochastic homogenization*, Probab. Theory Related Fields **178(1-2)** (2020), 531–566.

[3] M. Duerinckx, A. Gloria, F. Otto, *The structure of fluctuations in stochastic homogenization*, Comm. Math. Phys. **377(1)** (2020), 259–306.

[4] M. Duerinckx, F. Otto, *Higher-order pathwise theory of fluctuations in stochastic homogenization*, Stoch. Partial Differ. Equ. Anal. Comput. **8(3)** (2020), 625–692.

[5] M. Josien, F. Otto, *The annealed Calderon-Zygmund estimate as convenient tool in quantitative stochastic homogenization*, J. Funct. Anal. **283(7)** (2022), No. 109594.

Bourgain's surprising result in stochastic homogenization

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Let A be a uniformly elliptic, stationary and ergodic random coefficient field, constructed on a probability space $(\Omega,\mathbb{P})$. Homogenization theory has been focussing on the fine description of the solution ∇u_f^ε of the rescaled elliptic problem

$$-\nabla\cdot A(\tfrac{\cdot}{\varepsilon})\nabla u_f^\varepsilon = \nabla\cdot f, \qquad \text{in } \mathbb{R}^d,$$

in the limit of fast oscillating coefficients $\varepsilon \downarrow 0$, for a given deterministic force field $f \in L^2(\mathbb{R}^d)^d$. A different perspective on the topic has been recently initiated by Sigal [6], based on the following observation (see [4]).

Lemma 1. *There exist a bounded convolution operator $\bar{\mathcal{A}}(\nabla)$ on $L^2(\mathbb{R}^d)$ and a bounded pseudo-differential operator $\mathcal{F}(\cdot,\nabla): L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d\times\Omega)$ with centered stationary random symbol, related by $\bar{\mathcal{A}}(\nabla) = \mathbb{E}[A(\mathrm{Id}+\mathcal{F}(\cdot,\nabla))]$, such that:*

- *the averaged solution $\mathbb{E}[\nabla u_f^\varepsilon]$ satisfies*

$$-\nabla\cdot\bar{\mathcal{A}}(\varepsilon\nabla)\mathbb{E}[\nabla u_f^\varepsilon] = \nabla\cdot f, \qquad \text{in } \mathbb{R}^d; \tag{1}$$

- *the deviation is described by*

$$\nabla u_f^\varepsilon - \mathbb{E}[\nabla u_f^\varepsilon] = \mathcal{F}(\tfrac{\cdot}{\varepsilon},\varepsilon\nabla)\mathbb{E}[\nabla u_f^\varepsilon]. \tag{2}$$

This result is obtained as a simple consequence of stationarity and of the Schur complement formula, starting from the block decomposition of the operator $L = -\nabla \cdot A\nabla$ on $L^2(\mathbb{R}^d \times \Omega)$ with respect to projections $P := \mathbb{E}$ and $P^\perp := \mathrm{Id} - P$,

$$L = \begin{pmatrix} PLP & PLP^\perp \\ P^\perp LP & P^\perp LP^\perp \end{pmatrix}. \tag{3}$$

Homogenization can be reformulated in these terms as the regularity of the symbols $\mathbb{R}^d \to \mathbb{R}^{d\times d} : i\xi \mapsto \bar{\mathcal{A}}(i\xi)$ and $\mathbb{R}^d \to L^2(\Omega)^{d\times d} : i\xi \mapsto \mathcal{F}(\cdot, i\xi)$ at $i\xi = 0$. Indeed, this regularity allows to transform the equation (1) for the averaged solution perturbatively into the usual form of (higher-order) effective PDEs, and to transform the relation (2) into (higher-order) two-scale expansions; see [3] for a precise equivalence. Unlike the case of periodic homogenization, we recall that two-scale expansions in the random setting cannot be pursued to arbitrary order, corresponding to the problem of existence of higher-order correctors: under the strongest mixing conditions, the two-scale expansion of the solution ∇u_f^ε is only possible to accuracy $O(\varepsilon^{d/2})$ in $L^2(\mathbb{R}^d \times \Omega)$, which implies that the symbols $\bar{\mathcal{A}}$ and $\mathcal{F}$ are a priori only (almost) of class $\mathcal{C}^{d/2}$ at 0. This regularity is optimal for $\mathcal{F}$, but an improvement can be expected for $\bar{\mathcal{A}}$ as it is an averaged quantity. By refined homogenization techniques, the regularity of $\bar{\mathcal{A}}$ has been shown in [2] to be indeed at least twice better, that is, (almost) of class $\mathcal{C}^d$. Very surprisingly, Bourgain [1] and Kim and Lemm [5] proved in a perturbative regime that it is actually *four times better*, that is, (almost) of class $\mathcal{C}^{2d}$, thus yielding an effective description of the averaged solution by an effective PDE to accuracy (almost) $O(\varepsilon^{2d})$. Still no understanding of this result is available by non-perturbative homogenization techniques, and it has led to formulate the following conjecture.

Conjecture (Bourgain–Spencer)**.** If A satisfies strong enough mixing conditions, then the symbol $\bar{\mathcal{A}}$ is of class (almost) $\mathcal{C}^{2d}$ in a neighborhood of 0.

The rest of this note is devoted to a brief description of Bourgain's perturbative argument. As in [1, 5], we focus on the discrete iid setting for simplicity (see [4] for more general results): we consider the operator $L = \nabla^* A \nabla$ on $L^2(\mathbb{Z}^d)$, where ∇ stands now for the discrete gradient and where $A = \{A_x\}_{x\in\mathbb{Z}^d}$ is a sequence of iid uniformly elliptic random conductivities. The description of the averaged solution still holds as above in that case, and we use the same notation $\bar{\mathcal{A}}(\nabla)$ for the corresponding convolution operator. The perturbative regularity result by Bourgain [1] and Kim and Lemm [5] takes on the following guise.

Theorem 1 (Bourgain [1], Kim–Lemm [5])**.** *Let $d \geq 2$ and assume $A_x = 1 + \delta B_x$ with ellipticity ratio $\delta \ll 1$ and with $|B_x| \leq 1$ and $\mathbb{E}[B_x] = 0$. Then we have $\bar{\mathcal{A}}(\nabla) = \mathrm{Id} + \mathbb{L}_\delta$ where $\mathbb{L}_\delta$ is a convolution operator on $\mathbb{Z}^d$ with kernel satisfying*

$$|\mathbb{L}_\delta(x,y)| \leq C\delta^2 \langle x-y\rangle^{C\delta - 3d}, \qquad (\langle x\rangle := 1 + |x|) \tag{4}$$

for some universal constant $C > 0$, meaning that its symbol is of class $\mathcal{C}^{2d - C\delta}$.

This result is essentially optimal in the sense that the decay of the kernel cannot be improved beyond $\langle\cdot\rangle^{-3d}$, as will be clear from the proof, but we emphasize that

it does not solve the above conjecture even in the perturbative regime due to the loss $C\delta$ in the exponent. This indicates that the conjecture might, in fact, be false and that the non-perturbative $\mathcal{C}^d$ regularity in [2] might be optimal in general.

The proof starts from the Schur complement formula for the block decomposition (3), combined with a Neumann expansion, which allows to represent $(PL^{-1}P)^{-1} = \nabla^*(\mathrm{Id} + \mathbb{L}_\delta)\nabla$ with

$$\mathbb{L}_\delta = \delta \sum_{n=1}^\infty \delta^n \mathbb{L}^{(n)}, \qquad \mathbb{L}^{(n)} := PB(\mathbb{K}P^\perp B)^n P, \tag{5}$$

with the short-hand notation $\mathbb{K} := \nabla \triangle^{-1} \nabla^*$, and we then proceed by analyzing this perturbation series. Expanding the composition of operators, the kernel for the nth term in the series takes the form

$$\mathbb{L}^{(n)}(x_0, x_n) = \sum_{\underline{x} \in (\mathbb{Z}^d)^{n-1}} PB_{x_0} \mathbb{K}(x_0 - x_1) P^\perp B_{x_1} \dots \mathbb{K}(x_{n-1} - x_n) P^\perp B_{x_n}, \tag{6}$$

where the sum runs over all 'paths' $\underline{x} = (x_1, \dots, x_{n-1})$ in $\mathbb{Z}^d$ connecting x_0 to x_n. A direct estimate of the series, using the pointwise decay $|\mathbb{K}(x,y)| \lesssim \langle x - y \rangle^{-d}$, would yield

$$|\mathbb{L}^{(n)}(x,y)| \le C^n \langle x - y \rangle^{-d} \log(2 + |x - y|)^n,$$

where the logarithms come from estimating integrals with borderline decay. For all $\eta > 0$, using $\log t \le \eta^{-1} t^\eta$ for $t \ge 1$, this bound translates into

$$|\mathbb{L}^{(n)}(x,y)| \le n^n (\tfrac{C}{\eta})^n \langle x - y \rangle^{\eta - d}. \tag{7}$$

The combinatorial factor n^n destroys any possible use of this direct estimate in the perturbation series. In [1], Bourgain made a more clever use of the global structure of the paths, together with Calderón–Zygmund theory in form of the L^p-boundedness of $\mathbb{K}$, to show that this factor can, in fact, be removed.

Lemma 2 (Bourgain's deterministic lemma)**.** *For all $\eta \in (0,1)$ and $x \neq y$,*

$$|\mathbb{L}^{(n)}(x,y)| \lesssim \eta (\tfrac{C}{\eta})^n \langle x - y \rangle^{\eta - d}.$$

Choosing $\eta = 2C\delta$, this bound can now be used to estimate the perturbation series (5), to the effect of

$$|\mathbb{L}_\delta(x,y)| \le \delta\eta \sum_{n=1}^\infty (\tfrac{C\delta}{\eta})^n \langle x - y \rangle^{\eta - d} \le 2C\delta^2 \langle x - y \rangle^{2C\delta - d}.$$

To prove the stated decay (4), this naive bound needs to be improved by taking advantage of stochastic cancellations. We indeed easily realize that many paths do not contribute in the sum (6): for instance,

$$PB(x_0)P^\perp B(x_1) \dots P^\perp B(x_n) = 0$$
$$\text{whenever } \{x_0, \dots, x_j\} \cap \{x_{j+1}, \dots, x_n\} = \varnothing, \text{ for some } 0 \le j \le n.$$

The sum in (6) can thus be restricted to the so-called 'irreducible' paths that do not satisfy this condition. Further cancellations exist but are not needed in the analysis. For $x \neq y$, we note for instance that there is no irreducible path

with $n \leq 2$ edges from x to y, and that for $n = 3$ the only irreducible path is (x, y, x, y), that is,

A simple combinatorial argument shows that an irreducible path from x to y can always be decomposed into three disjoint paths from x to y. Evaluating the sum (6) by summing separately over these three paths, a direct estimate as in (7) would then yield the following, for all $n \geq 1$ and $\eta > 0$,

$$|\mathbb{L}^{(n)}(x_0, x_n)| \leq n^n (\tfrac{C}{\eta})^n \langle x - y \rangle^{\eta - 3d}.$$

This captures the optimal decay $\langle \cdot \rangle^{-3d}$ as stated in (4), but the factor n^n again makes this direct estimate useless in the perturbation series. Since the restriction to irreducible paths breaks the special oscillatory structure of the composition of Calderón–Zygmund kernels in (6), it is a priori unclear how to improve on such direct estimates. In a nutshell, the main contribution of Bourgain's work in [1] is to show how simple enough restrictions on the summations still allow to appeal to the Calderón–Zygmund theory.

References

[1] J. Bourgain. On a homogenization problem. *J. Stat. Phys.*, 172(2):314–320, 2018.
[2] M. Duerinckx. Non-perturbative approach to the Bourgain–Spencer conjecture in stochastic homogenization. Preprint, arXiv:2102.06319, 2021.
[3] M. Duerinckx, A. Gloria, and M. Lemm. A remark on a surprising result by Bourgain in homogenization. *Comm. Part. Diff. Eq.*, 44(2):1345–1357, 2019.
[4] M. Duerinckx, M. Lemm, and F. Pagano. On Bourgain's result in stochastic homogenization. In preparation.
[5] J. Kim and M. Lemm. On the Averaged Green's Function of an Elliptic Equation with Random Coefficients. *Arch. Ration. Mech. Anal.*, 234:1121–1166, 2019.
[6] I. M. Sigal. Homogenization problem. Unpublished preprint.

Numerical approaches to homogenization

Nicolas Clozeau, Marc Josien, Matti Schneider, Lihan Wang

The goal of this talk was to give a brief introduction to techniques which serve as the basis for numerical computation of the effective conductivity matrix in stochastic homogenization. The presentation had an overview character and was intended to convey the underlying ideas, pointing the audience to the relevant literature.

The contribution was split into four parts, and exclusively focused on numerical homogenization of the linear operator $-\nabla \cdot a\nabla$, and mostly on the computation of the constant homogenized coefficient a_{hom}. The latter can be expressed thanks to the correctors ϕ_i as $(a_{\mathrm{hom}})_{ij} = \langle e_i \cdot a(\nabla\phi_j + e_j)\rangle$.

For a start, even though the corrector equation is initially set on the whole space, we discussed the necessity to work on domains of finite size due to practical constraints – in general, on cells $[0, L]^d$. This gives rise to the so-called *Representative Volume Element* (RVE) approaches. Appropriate boundary conditions

need to be applied, replacing the sublinearity condition on the whole-space correctors. It can be shown [1, 2] in the general stationary and ergodic setting that for the most commonly used boundary conditions – Dirichlet, Neumann and periodic boundary conditions – the computed properties converge to the effective properties when the size L of the cell goes to infinity.

When working on cells of finite size, the computed properties are still random variables. Therefore, the error of the RVE method naturally decomposes into two parts [3, 4]: a *random error*, which quantifies the fluctuations of the computed properties around its mean, and a *systematic error*, which is deterministic and quantifies the error introduced by working on cells of finite size. The scalings of the systematic and the random error of the RVE method, which are generically different, can be investigated numerically [3].

We wanted to estimate such scalings rigorously. We placed ourselves in the simplest possible setting and restricted our analysis to *periodized* ensembles which moreover admit a spectral gap inequality [4, 5], uniformly in the size of the cell. Given the original (stationary ergodic) ensemble generating the coefficient field a, a periodized ensemble can be seen as conditioning the original ensemble on the coefficients a being periodic of period $[0, L]^d$. Unlike the more naive strategy consisting of taking a snapshot of the coefficient field a and then restricting it to the cell, periodized ensembles are stationary. (However, periodizing might prove impossible in practice.)

Under these assumptions, the random error scales with the central-limit theorem scaling $L^{-d/2}$, which is a direct consequence of inserting the spatially averaged energy into the spectral gap inequality and invoking a uniform bound on the quartic moment of the gradient of the corrector [4]. The systematic error scales as L^{-d} (possibly, up to logarithmic factors) [4], but the analysis is more subtle. We discussed it in the second presentation for low spatial dimensions $d \leq 4$. It requires a coupling between the periodized ensembles and the whole-space ensemble, e.g., by requiring that both ensembles generate coefficient fields that coincide on half of the cell. Then, the idea of Gloria et al. [4] is to benefit from the exponential localization of the inverse *massive* operator $\nabla(1/T - \nabla \cdot a\nabla)^{-1}\nabla\cdot$ by inserting a massive term with prefactor $1/T$ into the equation of the corrector. On the level of the massive equation, the difference between homogenized coefficients of both ensembles is exponentially small in $L/\sqrt{T}$. Using the spectral gap assumption, the difference between the massive and the non-massive homogenized coefficients scales like $T^{-d/2}$. Selecting the massive parameter T appropriately yields the result.

The third part of our talk was concerned with *reducing the variance* of our RVE approximation. This is of important interest for practical computations of the effective tensor by Monte-Carlo methods, since the achieved accuracy depends strongly on the magnitude of the fluctuations. We considered the Special Quasi-random Structures (SQS) introduced by Le Bris et al. [6] which are based on a suitable selection approach of the random structure. Due to stationarity and ergodicity, every realization may be used to compute the effective conductivity

(with probability one). However, in practice some realizations may be more suitable than others. The SQS conditions are a family of criteria which are cheap to check numerically and select samples that realize certain statistical properties of the entire material already on the small cell $[0, L]^d$ in an exceptionally accurate way. For instance, in the case of random inclusions, the first SQS condition selects a sample for which the volume fraction matches its statistical average (up to a tolerance error). For general microstructures, SQS conditions can be derived in a systematic way by a formal expansion of the effective tensor in terms of the ellipticity contrast. The rigorous justification of variance reduction under the SQS selection criteria was established by Fischer [7] for general random microstructures with a finite range of dependence.

The final part of our presentation was concerned with computing multiscale problems with low scale separation. More precisely, to solve for Dirichlet problem $-\nabla \cdot a\nabla u = f$ on a large domain Q_L with $L \gg 1$, Armstrong et al. [8] provide an iterative scheme for computing the sought solution, assuming that the effective conductivity a_{hom} was obtained beforehand. In particular, in each iteration step, suppose we have an initial guess v of the solution, then we obtain $\hat{v}$ by solving the following three equations:

$$\begin{cases} \left(\dfrac{1}{T} - \nabla \cdot a\nabla\right) u_0 = f + \nabla \cdot a\nabla v, \\ -\nabla \cdot a_{\mathrm{hom}}\nabla \bar{u} = \dfrac{1}{T} u_0, \\ \left(\dfrac{1}{T} - \nabla \cdot a\nabla\right) u_1 = \left(\dfrac{1}{T} - \nabla \cdot a_{\mathrm{hom}}\nabla\right) \bar{u}, \end{cases}$$

and finally, the field $\hat{v} = v + u_0 + u_1$ should be a better approximation of the true solution u compared to the field v we started out with. This method, similar to the V-cycle in the multigrid method, is based on a decomposition of the contributing frequencies into low-frequency and high-frequency parts. The first equation already contains the high-frequency parts of the error $u - v$; and to further approximate the remaining error $u - v - u_0$, one solves the homogenized equation that accounts for its low-frequency component and finally add back high-frequency details in the third equation. By inverting only the homogenized operator $-\nabla \cdot a_{\mathrm{hom}}\nabla$ and the massively corrected operator $1/T - \nabla \cdot a_{\mathrm{hom}}\nabla$ (for large parameter T but much smaller than L^2), the algorithm reduces the condition number of the operators from L^2 to T, thus substantially decreasing the computation cost. The result of Armstrong et al. [8] shows that with high probability, the error $\|\nabla(\hat{v} - u)\|_{L^2}$ of the new iterate is smaller than the error $\|\nabla(v - u)\|_{L^2}$ of the old iterate for sufficiently large parameter T, with a contraction rate independent of the parameter L but dependent on the field v. Furthermore, the recent result of Gu [9], which was also a part of our presentation, shows that the contraction rate may be bounded independently of the field v but with a logarithmic correction error in the parameter L. Hence, as long as massive parameter T is larger than some logarithmic power of the length-scale L, one can iterate the procedure and the iterates converge to the correct solution with overwhelming probability.

References

[1] A. Bourgeat and A. Piatnitski, *Approximations of effective coefficients in stochastic homogenization*, Annales de l'Intitut H. Poincaré **40** (2004), 153–165.

[2] H. Owhadi, *Approximation of the effective conductivity of ergodic media by periodization*, Probability Theory and Related Fields **125** (2003), 225–258.

[3] T. Kanit, S. Forest, I. Galliet, V. Mounoury, and D. Jeulin, *Determination of the size of the representative volume element for random composites: statistical and numerical approach*, International Journal of Solids and Structures **40:13–14** (2003), 3647–3679.

[4] A. Gloria, S. Neukamm, and F. Otto, *Quantification of ergodicity in stochastic homogenization: optimal bounds via spectral gap on Glauber dynamics*, Inventiones mathematicae **199:2** (2015), 455–515.

[5] A. Naddaf and T. Spencer, *Estimates On the Variance of Some Homogenization Problems*, Unpublished Preprint (1998), 1–14.

[6] C. Le Bris, F. Legoll, and M. Minvielle, *Special quasirandom structures: A selection approach for stochastic homogenization*, Monte Carlo Methods and Applications **22:1** (2016), 25–54.

[7] J. Fischer, *The choice of representative volumes in the approximation of effective properties of random materials*, Archive of Rational Mechanics and Analysis **234:2** (2019), 635–726.

[8] S. Armstrong, A. Hannukainen, T. Kuusi, and J.-C. Mourrat , *An iterative method for elliptic problems with rapidly oscillating coefficients*, ESAIM:M^{2}AN **55** (2021), 37–55.

[9] C. Gu , *Uniform estimate of an iterative method for elliptic problems with rapidly oscillating coefficients*, Stochastics and Partial Differential Equations: Analysis and Computations **8** (2020), 787–818.

Bounds on homogenized coefficients, metamaterials

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It is a classical question to ask which homogenized properties can be achieved by combining a number of individual materials in a composite. To be more concrete, and following the general outline of [1], assume that a and b are given tensors specifying for example the conductivity of two materials, and a fixed local volume fraction θ of material a, find the set

$$\mathcal{G}_\theta = \{A^{\mathrm{hom}} : A^\epsilon = \chi^\epsilon a + (1-\chi^\epsilon)b \overset{H}{\to} A^{\mathrm{hom}}, \chi^\epsilon \overset{*}{\rightharpoonup} \theta\},$$

for any sequence of characteristic functions χ^ϵ such that H-convergence holds. In general, the characterization of the set $\mathcal{G}_\theta$, commonly referred to the G-closure problem is an open, but a number of different bounds can be found.

Clearly, the harmonic and arithmetic mean are two such bounds (commonly referred to as the Reuss and Voigt bounds, respectively, in the case of elasticity). Those bounds are sharp in the sense that both can be achieved by a simple laminate microstructure, i.e., a layered structure of the two materials. However, if in one direction the Reuss bound is saturated, then in all other directions the Voigt bound must be saturated. Thus, much shaper bounds can be found, for example, when assuming that the homogenized material must be isotropic.

A great simplification comes from the fact that, instead of general H-convergent composite materials, it is enough to consider the case of periodic homogenization. This was first shown by Tartar [2] in a special case. The general result was first

proved by Dal Maso and Kohn in an unpublished manuscript, but a proof in full generality – also extended to the nonlinear setting – can be found in [3].

Sharp bounds can be established using the so-called Hashin-Strikman variational principle [4], which was further developed by a number of authors. In some special cases, in particular in the scalar case involving two isotropic materials, these bounds do in fact characterize the G-closure [2, 5]. Generally, the Hashin-Strikman bounds can be realized by a procedure of sequential lamination. Very different homogenized material properties can, however, be realized when one removes the assumption that the contrast between the two materials remains bounded.

In physical and engineering applications, designing media with properties not commonly found in nature has been the main rationale for the development of *metamaterials*. The related story that began in the 1960s with a question on the nature of material properties with changing sign [6] and has since had a major impact no materials science, nanotechnologies, and applied mathematics. In optics, light propagating through metamaterials is subject to negative refraction and cloaking [7]. Other important applications are found in acoustics and mechanics, in particular for sonar or seismic waves absorption. The key idea for metamaterial design is based on considering large periodic arrangements of sub-wavelength inclusions, or cells. Each cell contains a combination of materials with varying moduli and geometries.

The question of designing periodic heterogeneous structures is related to homogenisation, i.e. linking the mesoscopic structure of a metamaterial to a hypothetical macroscopic homogeneous material. The latter is described by some "effective" material parameters, which are spatially homogeneous. In the literature, a number of homogenisation techniques for metamaterials can be found. In the mathematics community, most methods are based on asymptotic homogenisation, utilising multiscale expansions or two-scale convergence [8, 9, 10]. Within this approach, local constitutive relations have been considered, in which the electric displacement $\mathbf{D}$ and the magnetic field $\mathbf{H}$ depend on the macroscopic electric field $\mathbf{E}$ and the magnetic induction $\mathbf{B}$ as follows ($\mathbf{r} \in \mathbb{R}^3$, $t \geq 0$):

$$\begin{aligned}\mathbf{D}(\mathbf{r},t) &= \mathbf{E}(\mathbf{r},t) + \mathbf{P}[\mathbf{E},\mathbf{B}](\mathbf{r},t),\\ \mathbf{H}(\mathbf{r},t) &= \mathbf{B}(\mathbf{r},t) - \mathbf{M}[\mathbf{B},\mathbf{E}](\mathbf{r},t),\end{aligned}$$

where $\mathbf{P}$ and $\mathbf{M}$ are electric polarisation and magnetisation, respectively. When the wavelength and the spatial period are of the same order of magnitude, physicists recommend to use nonlocal constitutive relations [11, 12], which for the electric displacement have the form of a convolution of the electric field and a response function $\mathbf{R}$:

$$\mathbf{D}(\mathbf{r},\omega) = \int_{\mathbb{R}^3} \mathbf{R}(\mathbf{r}-\mathbf{r}',\omega)\mathbf{E}(\mathbf{r}',\omega)\ \mathrm{d}\mathbf{r}',$$

where ω is the frequency. This relation describes the effective response of optical metamaterials to an incident electric field.

In the context of wave propagation, metamaterials (acoustic, electromagnetic) can be viewed as composite media with a dependence of the effective wavespeed on

frequency. This is a general feature of (multi-component) composites in which the wavelength becomes comparable to the spatial size for some of the components.

For example, for each $\varepsilon > 0$, $z \in \mathbb{C} \setminus \mathbb{R}_+$, $f \in L^2(\mathbb{R}^3)$ consider the following resolvent problem for $u = u_\varepsilon$:

$$-\nabla \cdot A^\varepsilon(\cdot/\varepsilon)\nabla u - zu = f,$$

where

$$A^\varepsilon(y) = \begin{cases} \varepsilon^2 I, \ y \in Q_1, \\ I, \ y \in Q_0. \end{cases}$$

Here $Q_0 \subset Q := [0,1)^3$ is such that $\operatorname{dist}(Q_0, \partial Q) > 0$ and $\overline{Q}_0 \cup \overline{Q}_1 = \overline{Q}$.

For each $\varepsilon > 0$, define the set

$$U_\varepsilon^1 := \varepsilon \bigcup_{n \in \mathbb{Z}^3} (Q_1 + n).$$

Denote by λ_j, φ_j the eigenvalues (arranged in the increasing order) and the normalised eigenfunctions of the Dirichlet Laplacian on Q_0.

Theorem 1. *For $R > \sigma > 0$, there exists a constant $C = C(R, \sigma) > 0$ such that one has*

$$\|u_\varepsilon - u_{\mathrm{hom}}\|_{L^2(U_\varepsilon^1)} \le C\varepsilon \|f\|_{L^2(U_\varepsilon^1)},$$

where

$$-\nabla \cdot A^{\mathrm{hom}} \nabla u_{\mathrm{hom}} - \beta(z) u_{\mathrm{hom}} = f. \tag{1}$$

Here the matrix A^{hom} is obtained by classical homogenisation from the "perforated" set U_ε^1, and

$$\beta(z) := z + z^2 \sum_{j=1}^{\infty} \frac{\langle \varphi_j \rangle^2}{\lambda_j - z}.$$

The function β ("Zhikov function" [14]) describes the effective time dispersion: if one formally sets $z = \omega^2$, then the relation between the wavenumber k and frequency ω for such an effective medium can be written as $k^2 = \beta(\omega^2)$. The corresponding version of the wave equation for is obtained by taking the inverse Fourier transform of (1) with respect to the frequency ($W = W(x,t)$, $x \in \mathbb{R}^3$, $t \ge 0$):

$$W_{tt} - \int_0^\infty a(\tau) W(t-\tau) d\tau - \nabla \cdot A^{\mathrm{hom}} \nabla W = f,$$

where a is the inverse Fourier transform (in ω) of the function $\beta(\omega^2) - \omega^2$.

References

[1] G. Allaire, *Shape Optimization by the Homogenization Method*, Springer (2012)

[2] L. Tartar, Estimations Fines des Coefficients Homogeneises, Ennio de Giorgi colloquium, P. Kree ed., *Pitman Research Notes in Math.*, **125** (1985) 168–187.

[3] U. Raitums, On the local representation of G-closure, *Arch. Ration Mech. Anal.* **158** (2001) 213–234.

[4] Z. Hashin and S. Shtrikman, A variational approach to the theory of the elastic behavior of multiphase materials, *J. Mech. Phys. Solids*, **11** (1963), 127–140.

[5] K. Lurie and A. Cherkaev, Exact estimates of conduetivity of composites formed by two isotropically conducting media, taken in pre-scribed proportion, *Proc. Royal Soc. Edinburgh* **99A** (1984) 71–87.
[6] V.G. Veselago, Computing certain invariants of topological spaces of dimension three, *Usp. Fiz. Nauk* **92**(7) (1967), 509–514.
[7] J. Pendry, Negative refraction makes a perfect lens, *Phys, Rev. Lett.* **85**(18) (2000), 3966.
[8] G. Bouchitté, C. Bourel, and D. Felbacq, Homogenization near resonances and artificial magnetism in three dimensional dielectric metamaterials, *Arch. Ration. Mech. Anal.* **225**(3) (2017), 1233–1277.
[9] G. Bouchitté, B. Schweizer. Homogenization of Maxwell's equations in a split ring geometry, *Multiscale Model. Simul.* **8**(3) (2010), 717–750.
[10] K. Cherednichenko, J. A. Evans. Full two-scale asymptotic expansion and higher-order constitutive laws in the homogenization of the system of quasi-static maxwell equations, *Multiscale Model. Simul.* **14**(4) (2016), 1513–1539.
[11] F. Z. Goffi, K. Mnasri, M. Plum, C. Rockstuhl, and A. Khrabustovskyi. Towards more general constitutive relations for metamaterials: A checklist for consistent formulations, *Phys. Rev. B* **101**(19) (2020), 195411.
[12] F. Z. Goffi, A. Khrabustovskyi, R. Venkitakrishnan, C. Rockstuhl, M. Plum. Higher order constitutive relations and interface conditions for metamaterials with strong spatial dispersion, *Phys. Lett. A* **412** (2021), 127570.
[13] K. D. Cherednichenko, Y. Ershova, A. Kiselev. Effective behaviour of critical-contrast PDEs: micro-resonances, frequency conversion, and time dispersive properties. I. *Comm. Math. Phys.* **375** (2016), 1833–1884.
[14] V. V. Zhikov. On an extension of the method of two-scale convergence and its application. *Sb. Math.* **191**(7) (2000), 973–1014.

Homogenization in Fluids – From Navier-Stokes to Brinkman

Peter Bella

We discuss homogenization of Poisson and Stokes equation in a domain finely perforated by many small holes. Assuming homogeneous Dirichlet boundary condition on the holes, the limiting problem depends on the size and distributions of the holes. More precisely, denoting $\varepsilon > 0$ the (average) distance between the holes and ε^α the scaling parameter for the radii, in the regime of tiny holes $\alpha > \frac{d}{d-2}$ the limiting problem as $\varepsilon \to 0$ is the same – here $d \geq 3$ denotes the dimension. In the case of large holes $\alpha < \frac{d}{d-2}$ solutions converge to zero, and their proper rescaling (by a power of ε) converge to a limit described by the Darcy law.

Focusing on the critical case $\alpha = \frac{d}{d-2}$, we obtain the same limiting equation (Poisson or (Navier)-Stokes equation) with an additional "friction" zeroth-order term. For periodically arranged holes and scalar problem (Poisson equation) this was shown by Cioranescu and Murat [2, 3], who called this additional term "a strange term brought from somewhere else". Instead of modifying (possibly non-regular) weak solution, one modifies (truncate) the test function – a strategy called an oscillating test function method [6]. The material being viscous, this truncation produces a friction related to the capacity of the holes.

The vectorial case (Stokes system) deals with solenoidal (divergence-free) functions, introducing additional difficulty when dealing with divergence-free truncations. In contrast to the scalar situation, where one can define truncation near

each hole separately, take the minimum of those truncations and use that capacity is subadditive, in the solenoidal case the modification near close-by holes is highly non-trivial. The simpler case of periodically arranged holes was analysed long time ago by Allaire [1], while the case of randomly arranged holes is quite recent (see [4] for the Poisson and [5] for the Stokes problem) – the main difference being that in the random case the holes might be very close to each other or even overlap with positive probability.

In the random case, the idea is to split the holes into 2 groups – the good ones which have enough space around them (i.e. one can "truncate" one hole without modifying other trucations) and the bad ones, which are close to some other holes (and possibly form clusters). The trick is then to show that the bad set is small in the sense that their capacity (cost of truncation) goes to 0 and therefore does not contribute to the limiting problem. As already mentioned, in the Stokes case one needs to be more careful to preserve the divergence-free condition. In [5] this is done by additionally splitting the bad holes into finitely-many groups of similarly large holes and treating them group by group.

References

[1] G. Allaire, *Homogenization of the Navier-Stokes equations in open sets perforated with tiny holes. I. Abstract framework, a volume distribution of holes*, Arch. Rational Mech. Anal. **113** (1990), no. 3, 209–259. MR 1079189

[2] D. Cioranescu and F. Murat, *Un terme étrange venu d'ailleurs. II*, Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. III (Paris, 1980/1981), Res. Notes in Math., vol. 70, Pitman, Boston, Mass.-London, 1982, pp. 154–178, 425–426. MR 670272

[3] ———, *A strange term coming from nowhere [MR0652509 (84e:35039a); MR0670272 (84e:35039b)]*, Topics in the mathematical modelling of composite materials, Progr. Nonlinear Differential Equations Appl., vol. 31, Birkhäuser Boston, Boston, MA, 1997, pp. 45–93. MR 1493040

[4] Arianna Giunti, Richard Höfer, and Juan J. L. Velázquez, *Homogenization for the Poisson equation in randomly perforated domains under minimal assumptions on the size of the holes*, Comm. Partial Differential Equations **43** (2018), no. 9, 1377–1412. MR 3915491

[5] Arianna Giunti and Richard M. Höfer, *Homogenisation for the Stokes equations in randomly perforated domains under almost minimal assumptions on the size of the holes*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **36** (2019), no. 7, 1829–1868. MR 4020526

[6] L. Tartar, *Quelques remarques sur les systèmes semi-linéaires*, Étude numérique des grands systèmes (Proc. Sympos., Novosibirsk, 1976), Méthodes Math. Inform., vol. 7, Dunod, Paris, 1978, pp. 198–212. MR 517856

Sedimentation of random suspensions

Shenglan Yuan

If the particles in the random suspension are heavier than the fluid, they fall by gravity. The quantities of interest are the speed of sedimentation and its variance, which are investigated in [3], and require linear analysis and stochastic cancellations.

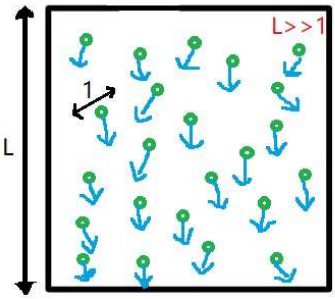

FIGURE 1. Two-dimensional tank

The tank $Q_L := (-\frac{L}{2}, \frac{L}{2}]^d$ of side length $L \geq 1$ with periodic boundary conditions is filled with a (steady) Stokes fluid, together with a monodisperse collection of disjoint spherical suspended particles,

$$\mathcal{I}_L := \bigcup_n I_{n,L},$$

where the particle $I_{n,L} := B(x_{n,L})$ is the unit ball centered at $x_{n,L}$ and $\mathcal{P}_L := \{x_{n,L}\}_n$ is a collection of positions in the tank Q_L; see Figure 1.

The fluid with the suspension is characterized by the following steady Stokes equation

$$(1) \qquad \begin{cases} -\triangle \phi_L + \nabla \Pi_L = -\alpha_L e, & \text{in } Q_L \setminus \mathcal{I}_L; \\ \operatorname{div} \phi_L = 0, & \text{in } Q_L \setminus \mathcal{I}_L; \\ D(\phi_L) = 0, & \text{in } \mathcal{I}_L; \\ e|I_{n,L}| + \int_{\partial I_{n,L}} \sigma(\phi_L, \Pi_L)\nu = 0, & \forall\, n; \\ \int_{\partial I_{n,L}} \Theta\nu \cdot \sigma(\phi_L, \Pi_L)\nu = 0, & \forall\, n,\ \forall\, \Theta \in \mathbb{M}^{\text{skew}}. \end{cases}$$

Here, ϕ_L stands for the velocity of the fluid that is incompressible with $\operatorname{div}\phi_L = 0$ in the fluid domain $Q_L \setminus \mathcal{I}_L$. It also describes the velocity of the particles with the rigidity constraint

$$D(\phi_L) = 0, \quad \text{in } \mathcal{I}_L,$$

where $D(\phi_L) := \frac{1}{2}(\nabla\phi_L + (\nabla\phi_L)')$ denotes the symmetrized gradient of ϕ_L. This means that $\phi_L(x) = V_{n,L} + \Theta_{n,L}(x - x_n, L)$ inside each particle $I_{n,L}$, for some translational velocity $V_{n,L} \in \mathbb{R}^d$ and some rotational velocity (skew-symmetric) matrix $\Theta_{n,L}$. The term Π_L represents the pressure in experiments. In the fluid, $-\alpha_L e$ accounts for the multiparticle backflow in the opposite direction to gravity $e \in \mathbb{R}^d$ with the relevant factor $\alpha_L := \frac{\lambda_L}{1-\lambda_L}$ and the total volume fraction $\lambda_L := L^{-d}|\mathcal{I}_L|$. The Cauchy stress tensor $\sigma(\phi_L, \Pi_L) = 2D(\phi_L) - \Pi_L \mathrm{Id}$ appears through the momentum equilibrium on particles, where ν denotes the outward unit normal vector at particle boundaries. Moreover, ϕ_L and Π_L satisfy the vanishing average

conditions

$$\int_{Q_L} \phi_L = 0, \qquad \int_{Q_L \setminus \mathcal{I}_L} \Pi_L = 0.$$

Well-posedness for the steady Stokes system (1) is standard with $\phi_L \in H^1_{\rm per}(Q_L)^d$ and $\Pi_L \in L^2_{\rm per}(Q_L \setminus \mathcal{I}_L)$ based on [4].

We are interested in statistical properties of the effective sedimentation speed and its variance. We compute the velocities of particles via the averaged boundary values

$$V_{n,L} := \fint_{I_{n,L}} \phi_L.$$

The aim is to analyze the mean settling speed and the fluctuations of individual velocities,

$$\bar{V}_L := \frac{e}{|e|} \cdot \mathbb{E}[V_{n,L}], \qquad \sigma_L := \big|\mathrm{Var}[V_{n,L}]\big|^{\frac{1}{2}}, \tag{2}$$

in the large-volume limit $L \uparrow \infty$.

A linear analysis allows to compute the scalings for the mean settling speed and the velocity fluctuations by neglecting the multibody interactions in the dilute regime $\lambda_L \ll 1$. The Stokes model (1) can be formally reduced to $\phi_L \approx \phi_L^\circ$,

$$-\triangle \phi_L^\circ + \nabla \Pi_L^\circ = \Big(\sum_n \mathbb{1}_{I_{n,L}} - \lambda_L\Big) e, \quad \mathrm{div} \phi_L^\circ = 0, \quad \text{in } Q_L, \tag{3}$$

and particle velocities are approximated by $V_{n,L} \approx V_{n,L}^\circ := \int_{I_{n,L}} \phi_L^\circ$. For this simplified linear model (3), the mean settling speed is estimated by

$$\lambda_L |e| \bar{V}_L^\circ \overset{L\uparrow\infty}{\sim} \mathbb{E}\Big[\big|\nabla \phi_L^\circ\big|^2\Big] = \mathbb{E}\Big[\Big|\sum_n \nabla U_L(x_{n,L})\Big|^2\Big] = \Big|\mathrm{Var}\Big[\sum_n \nabla U_L(x_{n,L})\Big]\Big|,$$

in terms of the periodic (locally averaged) Stokeslet,

$$-\triangle U_L + \nabla P_L = (\mathbb{1}_{B_1(0)} - L^{-d}|B_1(0)|)e, \quad \text{in } Q_L.$$

The velocity fluctuations are calculated as

$$(\sigma_L^\circ)^2 \approx |\mathrm{Var}[\phi_L^\circ]| = \Big|\mathrm{Var}\Big[\sum_n U_L(x_{n,L})\Big]\Big|.$$

The family $(\mathcal{P}_L)_{L\geq 1}$ of point processes is constructed on some probability space $(\Omega, \mathbb{P})$. It may have the following properties:

$(\mathbf{H}_\delta)$ – General conditions:

- ***Periodicity in law*:** For all $L \geq 1$, the point process $\mathcal{P}_L$ is stationary with respect to shifts, i.e., there exists a measure-preserving group action $\{\tau_{L,x}\}_{x\in Q_L}$ of $(\mathbb{R}^d/L\mathbb{Z}^d, +)$ such that $\mathcal{P}_L^\omega + x = \mathcal{P}_L^{\tau_{L,x}\omega}$ for all x and ω.
- ***Stabilization*:** The restricted point set $\mathcal{P}_L \cap K$ converges almost surely as $L \uparrow \infty$ for any compact set $K \subset \mathbb{R}^d$. The limiting point process is denoted by $\mathcal{P}$, which is assumed stationary (on $\mathbb{R}^d$) and ergodic, and we denote by $\mathcal{I} := \cup_n I_n$ the corresponding particle suspension.

- ***Hardcore condition*:** The point process $\mathcal{P}_L$ has a minimal interparticle distance

$$\inf_{m\neq n} |x_m - x_n|_L \geq 2(1+\delta) \qquad \text{almost surely.}$$

(Mix) – Mixing condition: The pair correlation function $g_{2,L}$ of $\mathcal{P}_L$ is integrable with

$$\sup_{L\geq 1} \int_{Q_L} |x|_L^2 |g_{2,L}(x)| dx < \infty.$$

(Hap) – Mixing and hyperuniformity conditions: The sedimenting suspension still displays fast decaying correlation

$$\sup_{L\geq 1} \int_{Q_L} |g_{2,L}| < \infty,$$

and hyperuniformity holds in the sense that the total pair correlation $h_{2,L}$ satisfies

$$\sup_{L\geq 1} L^2 \Big| \int_{Q_L} h_{2,L} \Big| < \infty.$$

(Mix)$^+$ – Improved mixing condition: There is a non-increasing weight function $\pi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\pi(\ell) \leq C_p \langle \ell \rangle^{-p}$ for all $p \geq 1$ such that the point process $(\mathcal{P}_L)_{L\geq 1}$ satisfies, for all $\sigma(\mathcal{P}_L)$-measurable random variables $Y(\mathcal{P}_L)$,

$$\mathrm{Var}[Y(\mathcal{P}_L)] \leq \mathbb{E}\Big[\int_0^L \int_{Q_L} \Big(\partial^{\mathrm{osc}}_{\mathcal{P}_L} Y(\mathcal{P}_L) \Big)^2 dx \langle \ell \rangle^{-d} \pi(\ell) d\ell \Big],$$

where the "oscillation" derivative ∂^{osc} is defined by

$$\begin{aligned} \partial^{\mathrm{osc}}_{\mathcal{P}, B_\ell(x)} Y(\mathcal{P}) := &\sup \mathrm{ess} \Big\{ Y(\mathcal{P}') : \mathcal{P}'|_{Q_L \setminus B_\ell(x)} = \mathcal{P}|_{Q_L \setminus B_\ell(x)} \Big\} \\ &- \inf \mathrm{ess} \Big\{ Y(\mathcal{P}') : \mathcal{P}'|_{Q_L \setminus B_\ell(x)} = \mathcal{P}|_{Q_L \setminus B_\ell(x)} \Big\}. \end{aligned}$$

(Hap$^+$) – Improved mixing and hyperuniformity conditions: For all $L \geq 1$ the point process $\mathcal{P}_L$ satisfies, for all $\sigma(\mathcal{P}_L)$-measurable random variables $Y(\mathcal{P}_L)$,

$$\mathrm{Var}[Y(\mathcal{P}_L)] \leq \mathbb{E}\Big[\int_0^L \int_{\mathbb{R}^d} \Big(\partial^{\mathrm{hyp}}_{\mathcal{P}_L, B_\ell(x)} Y(\mathcal{P}_L) \Big)^2 dx \langle \ell \rangle^{-d} \pi(\ell) d\ell \Big],$$

where the "hyperuniform" derivative is determined by

$$\partial^{\mathrm{hyp}}_{\mathcal{P}_L, B_\ell(x)} Y(\mathcal{P}_L) = \partial^{\mathrm{mov}}_{\mathcal{P}_L, B_\ell(x)} Y(\mathcal{P}_L) + L^{-1} \partial^{\mathrm{osc}}_{\mathcal{P}_L, B_\ell(x)} Y(\mathcal{P}_L)$$

and the "move-point" derivative is given by

$$\begin{aligned} \partial^{\mathrm{mov}}_{\mathcal{P}, B_\ell(x)} Y(\mathcal{P}) := &\sup \mathrm{ess} \Big\{ Y(\mathcal{P}') : \mathcal{P}'|_{Q_L \setminus B_\ell(x)} = \mathcal{P}|_{Q_L \setminus B_\ell(x)}, \sharp\mathcal{P}'|_{B_\ell(x)} = \sharp\mathcal{P}|_{B_\ell(x)} \Big\} \\ &- \inf \mathrm{ess} \Big\{ Y(\mathcal{P}') : \mathcal{P}'|_{Q_L \setminus B_\ell(x)} = \mathcal{P}|_{Q_L \setminus B_\ell(x)}, \sharp\mathcal{P}'|_{B_\ell(x)} = \sharp\mathcal{P}|_{B_\ell(x)} \Big\}. \end{aligned}$$

Let the random point processes $(\mathcal{P}_L)_{L\geq 1}$ satisfy $\mathbf{(H_\delta)}$ for some $\delta > 0$. The intensity of $\mathcal{P}_L$ is defined by $\rho_L := \mathbb{E}[L^{-d} \sharp\mathcal{P}_L]$ with $\mathbb{E}[\lambda_L] = |B_1(0)| \rho_L$.

Theorem 1. *Under **(Mix)** and **(Mix)**$^+$, the following bounds are expected to be sharp,*

$$\frac{\bar{V}_L}{\rho_L|e|} \lesssim \begin{cases} 1, & d>2; \\ (\log L)^{\frac{1}{2}}, & d=2; \\ L^{\frac{1}{2}}, & d=1; \end{cases} \qquad \text{and} \qquad \frac{\sigma_L}{\rho_L|e|} \lesssim \begin{cases} 1, & d>4; \\ (\log L)^{\frac{1}{2}}, & d=4; \\ L^{\frac{1}{2}}, & d=3; \end{cases}$$

respectively.

For a mixing ensemble of particles without long-range order, the mean settling speed and velocity fluctuations are well-defined in the large-volume limit only in dimensions $d > 2$ and $d > 4$, respectively. More precisely, the boundedness of $\bar{V}_L$ for $d > 2$ explicitly justifies Batchelor's analysis [1]. The linear divergence of σ_L^2 for $d = 3$ provides a rigorous version of the celebrated calculation by Caflisch and Luke [2].

Theorem 2. *Under **(Hap)** and **(Hap**$^+$**)**, the critical dimensions are shifted by 2,*

$$\frac{\bar{V}_L}{\rho_L|e|} \lesssim 1 \qquad \text{and} \qquad \frac{\sigma_L}{\rho_L|e|} \lesssim \begin{cases} 1, & d>2; \\ (\log L)^{\frac{1}{2}}, & d=2; \\ L^{\frac{1}{2}}, & d=1; \end{cases}$$

respectively.

The linear analysis yields a good description of the nonlinear long-range interactions, even by hyperuniform statistics. In particular, this rigorously illustrates the screening of hydrodynamic interactions in dimension $d = 3$ from Koch and Shaqfeh [5].

References

[1] G.K. Batchelor, *Sedimentation in a dilute dispersion of spheres*, J. Fluid Mech. **52** (1972), 245–268.

[2] R.E. Caflisch and J.H.C. Luke, *Variance in the sedimentation speed of a suspension*, Phys. Fluids **28** (1985), 759–760.

[3] M. Duerinckx and A. Gloria, *Sedimentation of random suspensions and the effect of hyperuniformity*, Annals of PDE **8** (2022), 1–66.

[4] G. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations: Steady-state problems*, Springer Science & Business Media, 2011.

[5] D.L. Koch and E.S.G. Shaqfeh, *Screening in sedimenting suspensions*, J. Fluid Mech. **224** (1991), 275–303.

Boundary Layers in Periodic Homogenization

Claudia Raithel

Consider the problem of obtaining homogenization rates for the Dirichlet problem:

$$\begin{aligned} -\nabla \cdot (a(\cdot/\epsilon)\nabla u_\epsilon) &= f \quad \text{in } \mathcal{D}, \\ u &\equiv 0 \quad \text{on } \partial\mathcal{D}, \end{aligned} \tag{1}$$

where $\mathcal{D} \subset \mathbb{R}^d$ is a bounded domain and $d \geq 2$; f and a are smooth. We, furthermore, assume that the coefficient field, $a : \mathbb{R} \to \mathbb{R}^{d\times d}$ is uniformly elliptic and bounded and $\mathbb{Z}^d$-periodic, in the sense that $A(y+\xi) = A(y)$ for all $y \in \mathbb{R}^d$ and $\xi \in \mathbb{Z}^d$. Recalling the standard 2-scale expansion in the absence of boundaries,

$$u_\epsilon(x) = u_0(x, x/\epsilon) + \epsilon u_1(x, x/\epsilon) + \epsilon^2 u_2(x, x/\epsilon) + ...,$$

we would expect that

$$\|u_\epsilon - (u_0(x) + \epsilon u_1(x, x/\epsilon))\|_{H^1(\mathcal{D})} \leq C\epsilon. \tag{2}$$

Instead of (2), it is a classical result that

$$\|u_\epsilon - (u_0(x) + \epsilon u_1(x, x/\epsilon))\|_{H^1(\mathcal{D})} \leq C\epsilon^{1/2} \tag{3}$$

holds and that this estimate is in general optimal. This loss of $\epsilon^{1/2}$ is the result of a boundary layer in which the full-space 2-scale expansion fails to give a good approximation of u_ϵ – in particular, notice that $u_\epsilon - (u_0(x) + \epsilon u_1(x, x/\epsilon)) \neq 0$ on $\partial\mathcal{D}$.

To remedy this issue it is standard to introduce a boundary layer correction

$$\begin{aligned} -\nabla \cdot (a(\cdot/\epsilon)\nabla u^{bl}_{1,\epsilon}) &= 0 && \text{in } \mathcal{D}, \\ u^{bl}_{1,\epsilon} &\equiv -u_1(x, x/\epsilon) && \text{on } \partial\mathcal{D}. \end{aligned} \tag{4}$$

Using the energy estimate for the divergence form equation satisfied by $u_\epsilon - (u_0(x)+\epsilon(u_1(x, x/\epsilon)+u^{bl}_{1,\epsilon})$ – where u_0 is the homogenized solution and $u_1(x, x/\epsilon) = \phi_i(x/\epsilon)\partial_i u_0$ – and that $u_\epsilon - (u_0(x) + \epsilon(u_1(x, x/\epsilon) + u^{bl}_{1,\epsilon}) = 0$ on $\partial\mathcal{D}$, one can then easily show that

$$\|u_\epsilon - (u_0(x) + \epsilon u_1(x, x/\epsilon) + u^{bl}_{1,\epsilon})\|_{H^1(\mathcal{D})} \leq C\epsilon. \tag{5}$$

Clearly, the "corrected" two scale expansion including the boundary layer correction does a better job of approximating u_ϵ – however, a new problem is introduced in the fact that $u^{bl}_{1,\epsilon}$ itself solves (4), a Dirichlet problem with not only oscillating coefficients, but also oscillating boundary data. Therefore, in order to give a good approximation for u_ϵ, we must consider the homogenization of this "oscillating Dirichlet problem".

The homogenization of the oscillating Dirichlet problem (4) is much more subtle than that of the standard Dirichlet problem (1). In particular, while the u_ϵ solving (1) satisfy a uniform H^1 estimate, due to the oscillating boundary data in (4) this is not the case for the $u^{bl}_{1,\epsilon}$. The homogenization of (4) also turns out to be very sensitive to the geometry of $\mathcal{D}$ – we now restrict ourselves to polygonal domains, which may be expressed as the intersection of half spaces:

$$\mathcal{D} = \cap_{k=1}^N \{x : n_k \cdot x > c_k\}, \tag{6}$$

where n_k are the inner-normal vectors and $c_k \in \mathbb{R}^d$. There are two main cases that are studied in the literature: 1) When the n_k are rational, studied by Allaire and Amar [2] and, in a slightly different context, also by Vogelius and Santosa [8, 9] and Vogelius and Moskow [7]. In [2] the authors consider the case that $\mathcal{D} = [0, 1]^d$, constructing a second-order (interior) approximation of u_ϵ, but only

along the sequence $\epsilon_n = 1/n$. The results of Vogelius and coauthors imply that the approximation of u_ϵ constructed in [2] depends on the sequence of ϵ. And, 2) When the n_k are diophantine. This case was first studied by Gérard-Varet and Masmoudi [4] and led to considering uniformly convex domains [5, 1, 3, 6]. In [3], Armstrong, Kuusi, Mourrat, and Prange are able to show convergence rates in L^q, $q \in [2, \infty)$, for the homogenization of the oscillating Dirichlet problem on uniformly convex domains that are essentially optimal for $d \geq 4$. Following the work of Shen and Zhuge [6] in which they were able to obtain better regularity for the homogenized boundary data, the methods in [3] are also shown to yield the optimal rates for $d = 2, 3$.

Returning to the setting of a polygonal domain (6), to address the homogenization of (4) the solution is decomposed into contributions corresponding to the various half-spaces used to define the domain. Each one of these contributions $v^{bl}_{\epsilon,k}(x, x/\epsilon)$ is found to solve a half-space problem

$$\begin{aligned} -\nabla_y \cdot (a(y)\nabla_y v^{bl}_{\epsilon,k}) &= 0 && \text{in } n_k \cdot y > c_k/\epsilon, \\ v^{bl}_{\epsilon,k} &\equiv -u_1(x, y) && \text{on } n_k \cdot y = c_k/\epsilon, \end{aligned} \tag{7}$$

where x now only appears as a parameter. The aim is to show that these problems are well-posed, and – more to the point – that the solutions $v^{bl}_{\epsilon,k}$ converge to constants $a_{\epsilon,k}$ as $n_k \cdot y \to \infty$ with a suitable rate of convergence. The $a_{\epsilon,k}$ are then used to construct the homogenized boundary data – this, of course, only makes sense if the $a_{\epsilon,k}$ do not actually depend on ϵ.

In the case that the n_k are rational, one can quite easily see that, while the boundary layer tails $v^{bl}_{\epsilon,k}$ converge to constants, these $a_{\epsilon,k}$ do depend on ϵ. This is due to how the half-space cuts the periodicity cell of the boundary data. When the n_k are instead diophantine, as treated by Gérard-Varet and Masmoudi [4], it can be shown that the convergence of the boundary layer tails is faster than any polynomial and, furthermore, that the $a_{\epsilon,k}$ are independent of ϵ. This yields the existence of homogenized boundary data.

References

[1] H. Aleksanyan, H. Shahgholian, P. Sjölin. *Applications of Fourier analysis in homogenization of Dirichlet problem I. Pointwise estimates*, J. Differential Equations **256** (2013), 2626–2637.

[2] G. Allaire, M. Amar *Boundary layer tails in periodic homogenization*, ESAIM: Control, Optimisation, and Calculus of Variations **4** (1999), 209–243.

[3] S. Armstrong, T. Kuusi, J.-C. Mourrat, C. Prange *Quantitative Analysis of Boundary Layers in Periodic Homogenization*, Arch. Ration. Mech. Anal. **226** (2017), 695-741.

[4] D. Gérard-Varet, N. Masmoudi *Homogenization in polygonal domains*, J. Eur. Math. Soc. **13** (2011), 1477-1503.

[5] D. Gérard-Varet, N. Masmoudi *Homogenization and boundary layers*, Acta. Math. **209** (2012), 133-178.

[6] Z. Shen, J. Zhuge *Boundary layers in periodic homogenization of Neumann problems*, Comm. Pure Appl. Math. **71** (2018) 2163–2219

[7] S. Moskow, M. Vogelius *First-order corrections to the homogenised eigenvalues of a periodic composite medium. A convergence proof*, Proc. Roy. Soc. Edinburgh Sect. A **127** (1997), 1263–1299.

[8] F. Santosa, M. Vogelius *First-order corrections to the homogenized eigenvalues of a periodic composite medium*, SIAM J. Appl. Math. **53** (1993), 1636–1668.
[9] F. Santosa, M. Vogelius *Erratum to the paper: First-order corrections to the homogenized eigenvalues of a periodic composite medium*, SIAM J. Appl. Math. **55** (1995), 864.

Effective boundary and interface conditions

Sonia Fliss

In this talk, we are interested in explaining how we can derive

- effective boundary conditions in order to replace a rough boundary by a flat one;
- effective interface conditions in order to replace heterogeneous thin layers.

These conditions are derived in general by studying the behaviour at infinity of some boundary layers terms. The accuracy of such effective conditions depends also on this behaviour. A lots of works in fluid mechanics (see among other [9, 2, 10, 19, 20, 5, 22]), for reaction-diffusion processes (see [21, 15]) or for wave propagation phenomenon (see for instance [1, 14, 11, 13]) address this question. In general the boundary and/or the layer are periodic, only few works deals with the random setting [7, 16, 12, 4].

Understanding these works is important in order to propose similar effective conditions for the homogenization of heterogeneous media in presence of boundaries or interfaces. For the homogenization of periodic media, effective boundary conditions were derived in [3, 17, 18, 6] for Dirichlet problems, in [23] for Neumann problems and in [8] for interface problems. To our knowledge, nothing similar exists for the homogenization of random media.

References

[1] Abboud, Toufic, and Habib Ammari. "Diffraction at a curved grating: TM and TE cases, homogenization." Journal of mathematical analysis and applications 202.3 (1996): 995-1026.
[2] Achdou, Yves, Olivier Pironneau, and Frederic Valentin. "Effective boundary conditions for laminar flows over periodic rough boundaries." Journal of Computational Physics 147.1 (1998): 187-218.
[3] Allaire, Grégoire, and Micol Amar. "Boundary layer tails in periodic homogenization." ESAIM: Control, Optimisation and Calculus of Variations 4 (1999): 209-243.
[4] Amirat, Youcef, et al. "Effective boundary condition for Stokes flow over a very rough surface." Journal of Differential Equations 254.8 (2013): 3395-3430.
[5] Amirat, Youcef, et al. "Asymptotic approximation of the solution of the Laplace equation in a domain with highly oscillating boundary." SIAM journal on mathematical analysis 35.6 (2004): 1598-1616.
[6] Armstrong, Scott, et al. "Quantitative analysis of boundary layers in periodic homogenization." Archive for Rational Mechanics and Analysis 226.2 (2017): 695-741.
[7] Basson, Arnaud, and David Gérard-Varet. "Wall laws for fluid flows at a boundary with random roughness." Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences 61.7 (2008): 941-987.
[8] Cakoni, Fioralba, et al. "Scattering by a bounded highly oscillating periodic medium and the effect of boundary correctors." SIAM Journal on Applied Mathematics 79.4 (2019): 1448-1474.

[9] Chechkin, Gregory A., and Andrey L. Piatnitski. "Homogenization of boundary-value problem in a locally periodic perforated domain." Applicable Analysis 71.1-4 (1998): 215-235.
[10] Chechkin, Gregory A., Avner Friedman, and Andrey L. Piatnitski. "The boundary-value problem in domains with very rapidly oscillating boundary." Journal of Mathematical Analysis and Applications 231.1 (1999): 213-234.
[11] Claeys, Xavier, and Bérangère Delourme. "High order asymptotics for wave propagation across thin periodic interfaces." Asymptotic Analysis 83.1-2 (2013): 35-82.
[12] Dalibard, Anne-Laure, and David Gérard-Varet. "Effective boundary condition at a rough surface starting from a slip condition." Journal of Differential Equations 251.12 (2011): 3450–3487.
[13] Delourme, Bérangère. "High-order asymptotics for the electromagnetic scattering by thin periodic layers." Mathematical Methods in the Applied Sciences 38.5 (2015): 811-833.
[14] Delourme, Bérangère, Houssem Haddar, and Patrick Joly. "Approximate models for wave propagation across thin periodic interfaces." Journal de mathématiques pures et appliquées 98.1 (2012): 28-71.
[15] Gahn, Markus, Maria Neuss-Radu, and Peter Knabner. "Homogenization of reaction–diffusion processes in a two-component porous medium with nonlinear flux conditions at the interface." SIAM Journal on Applied Mathematics 76.5 (2016): 1819-1843.
[16] Gérard-Varet, David. "The Navier wall law at a boundary with random roughness." Communications in mathematical physics 286.1 (2009): 81-110.
[17] Gérard-Varet, David, and Nader Masmoudi. "Homogenization in polygonal domains." Journal of the European Mathematical society 13.5 (2011): 1477-1503.
[18] Gérard-Varet, David, and Nader Masmoudi. "Homogenization and boundary layers." Acta mathematica 209.1 (2012): 133-178.
[19] Mikelic, Andro, and Willi Jäger. "On the interface boundary condition of Beavers, Joseph, and Saffman." SIAM Journal on Applied Mathematics 60.4 (2000): 1111-1127.
[20] Mikelic, Andro, and Willi Jäger. "On the roughness-induced effective boundary conditions for an incompressible viscous flow." Journal of Differential Equations 170.1 (2001): 96-122.
[21] Neuss-Radu, Maria, and Willi Jäger. "Effective transmission conditions for reaction-diffusion processes in domains separated by an interface." SIAM Journal on Mathematical Analysis 39.3 (2007): 687-720.
[22] Neuss, Nicolas, Maria Neuss-Radu, and Andro Mikelić. "Effective laws for the Poisson equation on domains with curved oscillating boundaries." Applicable Analysis 85.05 (2006): 479-502.
[23] Shen, Zhongwei, and Jinping Zhuge. "Boundary layers in periodic homogenization of Neumann problems." Communications on Pure and Applied Mathematics 71.11 (2018): 2163-2219.

Homogenization of boundary conditions in fluids

Umberto Pappalettera, Jules Pitcho

We consider a two-dimensional rough channel

$$\Omega^\epsilon \ = \ \Omega \cup \Sigma \cup R^\epsilon$$

where $\Omega = \mathbb{R} \times (0,1)$ is the smooth part, R^ϵ is the rough part, and $\Sigma = \mathbb{R} \times \{0\}$ is their interface. The rough part has typical size ϵ and is given by

$$R^\epsilon \ = \ \left\{ x, \ x_2 > \epsilon\omega\left(\frac{x_1}{\epsilon}\right) \right\},$$

where $\omega : \mathbb{R} \to (-1,0)$ is a random process on some probability space $(P, \mathcal{C}, \pi)$ with smooth trajectories.

In this channel, the fluid is modeled by the solution of the steady-state Navier-Stokes equations:

$$\begin{cases} -\Delta u + u \cdot \nabla u + \nabla p = 0, \ x \in \Omega^\epsilon, \\ \operatorname{div} u = 0, \ x \in \Omega^\epsilon, \\ \displaystyle\int_{\sigma^\epsilon} u_1 = \phi, \\ u|_{\partial\Omega^\epsilon} = 0, \end{cases} \tag{1}$$

where σ^ϵ denotes any vertical cross-section of Ω^ϵ and $\phi > 0$ is the flux across the channel.

The problem we are interested in is the justification the **Navier-slip boundary condition**, acting at the artificial boundary Σ, as the homogenized effect of the rough boundary. More precisely, we want to justify that for some constant α the solution of

$$\begin{cases} -\Delta u + u \cdot \nabla u + \nabla p = 0, \ x \in \Omega, \\ \operatorname{div} u = 0, \ x \in \Omega, \\ \displaystyle\int_{\sigma} u_1 = \phi, \quad u|_{x_2=1} = 0, \\ (u_1 - \epsilon\alpha\partial_2 u_1, \, u_2) = 0 \text{ at } \Sigma, \end{cases} \tag{2}$$

provides an $O(\epsilon^{3/2}|\log(\epsilon)|^{1/2})$ approximation in $L^2(P \times \Omega)$ of (1).

In the case when the irregularity ω is periodic, this was already proved in [4]. The hypothesis of periodic irregularity of the boundary is unphysical and technically easier, whereas [3] deals with the case when the irregularity $\omega \in P$ is randomly distributed and stationary, with uniform bound given by

$$P := \{\omega : \mathbb{R} \to (-1, 0) \text{ such that } \|\omega\|_{C^{2,\nu}} \leq K\}$$

for some $\nu > 0$ and $K \in (0, \infty)$. Here, we additionally require that the σ-fields

$$\sigma(s \mapsto \omega(s), \, s \leq a) \qquad \text{and} \qquad \sigma(s \mapsto \omega(s), \, s \geq b)$$

are independent under π for $b - a \geq \kappa$, for some $\kappa > 0$.

The starting point for the justification of the Navier-slip boundary is a formal expansion of u^ϵ:

$$u^\epsilon(x) \sim u^0(x) + 6\phi\epsilon v(x/\epsilon) + \dots$$

The leading term u^0 satisfies (1) with the no-slip condition $u^0 = 0$ at Σ, and is given by the Poiseuille flow:

$$u^0(x) = (U(x_2), 0), \qquad U(x_2) = 6\phi x_2(1 - x_2).$$

In the random setting, it was justified in [2] that no-slip boundary condition provides a $O(\epsilon)$ approximation of u^ϵ in L^2. This however does not take into account the behaviour of u^ϵ near the boundary, and is refined by a boundary layer corrector $\epsilon\phi v(x/\epsilon)$ in the expansion. This corrector is defined on the rescaled domain

$$\Omega^{bl} = \{x \in \mathbb{R}^2 : \, x_2 > \omega(x_1)\},$$

and satisfies the Stokes problem:

$$\begin{cases} -\Delta v + \nabla q = 0,\ x \in \Omega^{bl}, \\ \operatorname{div} v = 0,\ x \in \Omega^{bl}, \\ v(x_1, \omega(x_1)) \ = \ -(\omega(x_1), 0). \end{cases} \tag{3}$$

The boundary condition on v is imposed to compensate the value of $u^0(x)$ at boundary points. It is critical to observe that solutions of (3) have the following representation formula

$$\begin{aligned} v(\omega, y) &= \int_{\mathbb{R}} G(t, y_2) v(\omega, y_1 - t, 0) dt \\ &= -\int_{\mathbb{R}} t \partial_t G(t, y_2) \frac{1}{t} \int_0^t v(\omega, y_1 - s, 0) ds dt. \end{aligned}$$

G is the matrix-valued Green function associated to the Stokes problem:

$$G(y_1, y_2) = \frac{2y_2}{\pi(y_1^2 + y_2^2)^2} \begin{pmatrix} y_1^2 & y_1 y_2 \\ y_1 y_2 & y_2^2 \end{pmatrix}.$$

In conjuction with

$$\frac{1}{t} \int_0^t v(\omega, y_1 - s, 0) ds \to (\alpha, 0),$$

which follows from from the ergodic theorem, this gives a strategy to prove the following central limit theorem type of convergence

$$y_2 \mathbb{E}\left[(v(\cdot, 0, y_2) - (\alpha, 0)) \right] \to \sigma \geq 0. \tag{4}$$

Now, for $n \in \mathbb{N}$, we denote by $\tau_n : \omega(\cdot) \mapsto \omega(\cdot + h)$ the translation operator on P, and we observe that the difficulty in obtaining (4) is that the random variables

$$X^n(\omega) = F \circ \tau_n(\omega), \quad \text{where} \quad F(\omega) = \int_0^1 (v(\omega, t, 0) - \alpha) dt,$$

are not independant. This is because of infinite speed of propagation of information in the Stokes system. However, by obtaining a good decay of correlations on the random variables X^n as n goes to infinity, we can still prove a central limit theorem for non-independant random variables, from which (4) follows. The decay of correlations on the random variables X^n is obtained using an argument of homogenization of the Green's function for the Stokes operator in a domain with an oscillating boundary, in analogy with the work of Avellanada and Lin [1] for homogenization of elliptic systems.

We then conclude by introducing the approximation

$$u^\epsilon_{app}(\omega, x) = u^0(x) + 6\epsilon\phi v(\omega, x/\epsilon) + 6\epsilon\phi u^1(\omega, x) + 6\epsilon\phi r^\epsilon(\omega, x),$$

where u^1 and r^ϵ respectively solve

$$\begin{cases} u^0 \cdot \nabla u^1 + u^1 \cdot \nabla u^0 - \Delta u^1 + \nabla p = 0, \ x \in \Omega, \\ \operatorname{div} u^1 = 0, \ x \in \Omega, \\ \displaystyle\int_0^1 u^1 \cdot e_1 dx_2 = -\alpha, \\ u^1|_{y_2=0} = 0, \ u^1|_{y_2=1} = -(\alpha, 0), \end{cases}$$

and

$$\begin{cases} r^\epsilon(\omega, x_1, 0) = 0, \\ r^\epsilon(\omega, x_1, 1) = \alpha - v(\omega, x_1/\epsilon, 1/\epsilon), \\ \operatorname{div} r^\epsilon = 0, \ x \in \Omega, \end{cases}$$

one can make use of (4) to prove both $u^\epsilon - u^\epsilon_{app}$ and $u^N - u^\epsilon_{app}$ to be of order $O(\epsilon^{3/2}|\log(\epsilon)|^{1/2})$ in $L^2(P \times \Omega)$, where u^ϵ is the unique solution of (1) and u^N is the unique solution of (2), thus implying the desired result.

References

[1] Marco Avellaneda and Fang-Hua Lin. Compactness methods in the theory of homogenization. *Comm. Pure Appl. Math.*, 40(6):803–847, 1987.

[2] Arnaud Basson and David Gérard-Varet. Wall laws for fluid flows at a boundary with random roughness. *Comm. Pure Appl. Math.*, 61(7):941–987, 2008.

[3] David Gérard-Varet. The Navier wall law at a boundary with random roughness. *Comm. Math. Phys.*, 286(1):81–110, 2009.

[4] Willi Jäger and Andro Mikelić. On the roughness-induced effective boundary conditions for an incompressible viscous flow. *J. Differential Equations*, 170(1):96–122, 2001.

Homogenisation of the wave equation for long times

Matteo Capoferri, Mikhail Cherdantsev, Igor Velcic

The goal of this talk is to give an overview of the existing result on the homogenisation of the wave equation for long times, both in the periodic and in the stochastic setting.

Working in Euclidean space $\mathbb{R}^d$, for $\varepsilon > 0$ consider the initial value problem

$$\begin{cases} (\partial_{tt}^2 - \nabla \cdot a(\cdot/\varepsilon)\nabla)u_\varepsilon = 0, \\ u_\varepsilon|_{t=0} = u_0, \\ \partial_t u_\varepsilon|_{t=0} = 0, \end{cases} \tag{1}$$

for some sufficiently regular initial datum u_0.

Suppose a is a $[0,1]^d$-periodic, bounded, uniformly elliptic symmetric matrix-function on $\mathbb{R}^d$. It is a classical result in homogenisation theory that, given $T > 0$,

u_ε can be approximated on the time interval $[0,T]$ by the solution u_{hom} of a non-dispersive wave equation with constant coefficients:

$$\begin{cases} (\partial_{tt}^2 - \nabla \cdot a_{\mathrm{hom}} \nabla) u_{\mathrm{hom}} = 0, \\ u_{\mathrm{hom}}|_{t=0} = u_0, \\ \partial_t u_{\mathrm{hom}}|_{t=0} = 0. \end{cases} \tag{2}$$

Efforts have been made over the years to improve the above results in two main directions: (i) quantifying the rate of convergence of u_ε to u_{hom} as $\varepsilon \to 0$ and (ii) increasing the time window in which the approximation holds.

For periodic problems, Dohnal–Lamacz–Schweizer [3, 4], using techniques reliant on the Bloch-wave decomposition in the homogenisation regime [5, 2], were able to show the following.

Theorem 1 (Thms. 2.2 and 2.4 in [4])**.** *Let u_ε be the solution of* (1) *for some $u_0 \in H^2(\mathbb{R}^d)$, and let w_ϵ be the solution of the well-posed weakly dispersive equation*

$$\begin{cases} (\partial_{tt}^2 - \nabla \cdot a_{\mathrm{hom}} \nabla) w_\varepsilon = \varepsilon^2 \nabla \cdot D_{\mathrm{hom}} \nabla \partial_t^2 w_\varepsilon - \varepsilon^2 F_{\mathrm{hom}} \nabla^{\otimes 4} w_\varepsilon, \\ w_\varepsilon|_{t=0} = u_0, \\ \partial_t w_\varepsilon|_{t=0} = 0, \end{cases} \tag{3}$$

where D_{hom} and F_{hom} are constant positive semi-definite symmetric tensors of order two and four, respectively, which can be determined explicitly. Then, for every $T > 0$ there exists a constant $C = C(a, u_0, T)$ such that

$$\sup_{t \in [0, \varepsilon^{-2} T]} \| u_\varepsilon - w_\varepsilon \|_{L^2(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)} \leq C\varepsilon.$$

Remark 1. *Note that, in fact, up to an additional error of order $O(\varepsilon)$, w_ε can be written down explicitly in terms of the coefficients in the analytic expansion near the origin of the first Bloch eigenvalue and eigenfunction, cf.* [4, Theorem 2.3 and Eqn. (1.9)].

One calls (3) a higher order homogenised wave equation. It is worth emphasising that for large times the interplay between the time derivative and the oscillatory nature of the spatial part bring about dispersive effects, see [4].

Recently, Benoit–Gloria [1] developed new interesting techniques inspired by the Bloch method (which is not applicable if the spatial operator is not periodic) to obtain homogenisation results for long times in the stochastic setting. These techniques consist in devising an approximate Bloch theory in the whole space.

The key observation here — one that comes from the periodic setting and, as is shown, can be successfully adapted to the stochastic case — is that in the homogenisation limit all information important for a large time approximation up to an $O(\varepsilon)$-error is contained in the expansion of the first Bloch eigenpair (approximate Taylor-Bloch eigenpair in non-periodic case) for small values of the quasimomentum. Remarkably, the first order term in the expansion of the first eigenfunction is precisely the first order homogenisation corrector. One can make

a step further, and construct an expansion of any given order, provided that sufficiently many higher order correctors (how many one needs depends on how far one wants to go in time) are well defined.

Based on this expansion, one can write down a higher order homogenised wave equation:

$$\begin{cases} (\partial_{tt}^2 + \mathcal{L}_{\mathrm{hom},\varepsilon,\ell})w_{\varepsilon,\ell} = 0, \\ w_{\varepsilon,\ell}|_{t=0} = u_0, \\ \partial_t w_{\varepsilon,\ell}|_{t=0} = 0. \end{cases}$$

Here $\mathcal{L}_{\mathrm{hom},\varepsilon,\ell}$ is a positive elliptic operator whose order depends on ℓ with higher order derivatives multiplied by appropriate powers of ε. Benoit–Gloria showed that the solution $w_{\varepsilon,\ell}$ approximates u_ε up to times of order $\varepsilon^{-\alpha(\ell)}T$, where $\alpha(\ell)$ is a positive number depending on ℓ (provided that the ℓ-th order corrector is regular enough).

In case of periodic coefficients $a \in L^\infty$ there exist well defined correctors of any order $\ell \in \mathbb{N}$, thus, once can construct an approximate solution up to any prescribed scaling of time $\varepsilon^{-\alpha}T$, $\alpha > 0$. The analogous statement is true for quasi-periodic coefficients under the additional assumption that a is sufficiently smooth. In purely stochastic setting, in general one can construct an approximate solution up to times of order $\varepsilon^{-d/2}T$ only. The latter restriction stems from the fact that one needs to verify the existence of higher order correctors with prescribed growth.

Benoit–Gloria's approach also works for more general initial conditions, in the presence of forcing terms, and for systems of PDEs.

References

[1] A. Benoit and A. Gloria, *Long-time homogenization and asymptotic ballistic transport of classical waves*, Annales scientifiques de l'Ecole Normale Superieure **52** (2019), 703–759.

[2] C. Conca and M. Vanninathan, *Homogenization of periodic structures via Bloch decomposition*, SIAM Journal on Applied Mathematics **57** no. 6 (1997) 1639–1659.

[3] T. Dohnal, A. Lamacz and B. Schweizer, *Bloch-Wave homogenization on large time scales and dispersive effective wave equations*, Multiscale Modeling & Simulation **12** no. 2 (2014), 488–513.

[4] T. Dohnal, A. Lamacz and B. Schweizer, *Dispersive homogenized models and coefficient formulas for waves in general periodic media*, Asymptotic Analysis **93** no. 1-2 (2015), 21–49.

[5] F. Santosa and W.W. Symes, *A dispersive effective medium for wave propagation in periodic composites*, SIAM Journal on Applied Mathematics **51** no. 4 (1991), 984–1005.

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