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HUTCHINSON'S INTERVALS AND ENTIRE FUNCTIONS FROM THE LAGUERRE–PÓLYA CLASS

THU HIEN NGUYEN AND ANNA VISHNYAKOVA

Abstract. We find the intervals $[\alpha, \beta(\alpha)]$ such that if a univariate real polynomial or entire function $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ with positive coefficients satisfy the conditions $\frac{a_{k-1}^2}{a_{k-2}a_k} \in [\alpha, \beta(\alpha)]$ for all $k \geq 2$, then f belongs to the Laguerre–Pólya class. For instance, from J.I. Hutchinson's theorem, one can observe that f belongs to the Laguerre–Pólya class (has only real zeros) when $q_k(f) \in [4, +\infty)$. We are interested in finding those intervals which are not subsets of $[4, +\infty)$.

1. Introduction

We study zero localization of real univariate polynomials and entire functions $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ with positive coefficients. In 1923, J.I. Hutchinson proved that, if the inequalities $a_{k-1}^2 \geq 4a_{k-2}a_k$, for all $k \geq 2$, are valid, then the function f belongs to the Laguerre–Pólya class. In this short note, the chief object is to extend the sufficient conditions for a polynomial or an entire function to belong to the Laguerre–Pólya class obtained by J.I. Hutchinson, or, more precisely, to find the intervals $[\alpha, \beta(\alpha)]$ which are not subsets of $[4, +\infty)$.

Let us recall some facts from the theory of entire functions.

1.1. The Laguerre–Pólya class.

We begin with the definitions of hyperbolic polynomials, the Laguerre–Pólya class and the Laguerre–Pólya class of type I.

Definition 1. *A real polynomial P is said to be* ***hyperbolic****, written $P \in \mathcal{HP}$, if all its zeros are real.*

Definition 2. *A real entire function f is said to be in the* ***Laguerre–Pólya class****, written $f \in \mathcal{L} - \mathcal{P}$, if it can be expressed in the form*

$$f(z) = cz^n e^{-\alpha z^2 + \beta z} \prod_{k=1}^{\infty} \left(1 - \frac{z}{x_k}\right) e^{z x_k^{-1}}, \tag{1}$$

where $c, \alpha, \beta, x_k \in \mathbb{R}$, $x_k \neq 0$, $\alpha \geq 0$, n is a nonnegative integer and $\sum_{k=1}^{\infty} x_k^{-2} < \infty$.

Definition 3. *A real entire function f is said to be in the* ***Laguerre–Pólya class of type I****, written $f \in \mathcal{L} - \mathcal{P}I$, if it can be expressed in the following form*

$$f(z) = cz^n e^{\beta z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{x_k}\right), \tag{2}$$

where $c \in \mathbb{R}, \beta \geq 0, x_k > 0$, n is a nonnegative integer, and $\sum_{k=1}^{\infty} x_k^{-1} < \infty$.

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Note that the product on the right-hand sides in both definitions can be finite or empty (in the latter case, the product equals 1).

Various important properties and characterizations of the Laguerre–Pólya class and the Laguerre–Pólya class of type I can be found in works by I.I. Hirshman and D.V. Widder [6], B.Ja. Levin [16], G. Pólya and G. Szegö [23], G. Pólya and J. Schur [22], monograph by N. Obreshkov [20, Chapter II] and many other works. These classes are essential in the theory of entire functions since it appears that the polynomials with only real zeros (or only real and nonpositive zeros) converge locally uniformly to these and only these functions. The following prominent theorem provides an even stronger result.

Theorem A (E. Laguerre and G. Pólya, see, for example, [6, p. 42–46] and [16, chapter VIII, §3])**.**

(i) *Let* $(P_n)_{n=1}^{\infty}$, $P_n(0) = 1$, *be a sequence of hyperbolic polynomials which converges uniformly on the disc* $|z| \leq A, A > 0$. *Then this sequence converges locally uniformly in* $\mathbb{C}$ *to an entire function from the* $\mathcal{L} - \mathcal{P}$ *class.*

(ii) *For any* $f \in \mathcal{L} - \mathcal{P}$ *there exists a sequence of hyperbolic polynomials, which converges locally uniformly to* f.

(iii) *Let* $(P_n)_{n=1}^{\infty}$, $P_n(0) = 1$, *be a sequence of hyperbolic polynomials having only negative zeros which converges uniformly on the disc* $|z| \leq A, A > 0$. *Then this sequence converges locally uniformly in* $\mathbb{C}$ *to an entire function from the class* $\mathcal{L} - \mathcal{P}I$.

(iv) *For any* $f \in \mathcal{L} - \mathcal{P}I$ *there is a sequence of hyperbolic polynomials with only negative zeros which converges locally uniformly to* f.

For a real entire function (not identically zero) of the order less than 2 the property of having only real zeros is equivalent to belonging to the Laguerre–Pólya class. Similarly, for a real entire function with positive coefficients of the order less than 1 having only real nonpositive zeros is equivalent to belonging to the Laguerre–Pólya class of type I. Strikingly, the situation changes for the functions of order 2 in the case of the Laguerre–Pólya class and for the functions of order 1 in the case of the Laguerre–Pólya class of type I. For instance, the entire function $f(x) = e^{-x^2}$ belongs to the $\mathcal{L} - \mathcal{P}$ class while the entire function $g(x) = e^{x^2}$ does not.

1.2. **Hutchinson's constant.** The problem of understanding whether a given polynomial or entire function has only real zeros is considered subtle and complicated. A simply verified description of this class, in terms of the coefficients of a series, is impossible since it is determined by an infinite number of discriminant inequalities. In 1923, J. I. Hutchinson found a simple sufficient condition in terms of coefficients for an entire function with positive coefficients to have only real zeros, which was a generalization of the results by M. Petrovitch [21] and G. Hardy [3], or [4, pp. 95 - 100].

To formulate the theorem, let us define the second quotients of Taylor coefficients of f. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function with real nonzero coefficients, then

$$q_n = q_n(f) := \frac{a_{n-1}^2}{a_{n-2}a_n}, \quad \forall n \geq 2. \tag{3}$$

In addition, it follows straightforwardly from this definition that

$$a_n = a_1 \left(\frac{a_1}{a_0}\right)^{n-1} \frac{1}{q_2^{n-1} q_3^{n-2} \cdots q_{n-1}^2 q_n}. \tag{4}$$

Theorem B (J.I. Hutchinson, [7])**.** *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k > 0$ *for all* k, *be an entire function. Then* $q_k(f) \geq 4$, *for all* $k \geq 2$, *if and only if the following two conditions are fulfilled:*

(i) *The zeros of* f *are all real, simple and negative.*

(ii) *The zeros of any polynomial* $\sum_{k=m}^{n} a_k z^k$, $m < n$, *formed by taking any number of consecutive terms of* f, *are all real and non-positive.*

For some extensions of Hutchinson's result see, for example, the paper by T. Craven and G. Csordas, [2, §4]. From Hutchinson's theorem (Theorem B) we see that f has only real zeros when $q_k(f) \in [4, +\infty)$.

1.3. **Some results related to Hutchinson's constant.** Strikingly, there are many results which are stated in the following style: there exists a constant $c > 1$ such that if a polynomial or an entire function f with nonzero coefficients satisfies the conditions $|q_k(f)| \geq c$ for all k, then we can formulate a statement about the localization of the zeros of f. For example, in [2] the authors obtained an analogue of the Hutchinson's theorem for polynomials decomposed in the Pochhammer basis. In [5], it was proved that, if for some constant $c > 0$ a polynomial P with positive coefficients satisfies the conditions $q_k(P) > c$ for all k, then all the zeros of P lie in a special sector depending on c. In [10], the smallest possible constant $c > 0$ was found such that if a polynomial P with positive coefficients satisfies the conditions $q_k(P) > c$ for all k, then P is stable (all the zeros of P lie in the left half-plane). In [8], the smallest possible constant $c > 0$ was found such that if a polynomial P with positive coefficients satisfies the conditions $q_k(P) > c$ for all k, then P is a sign-independently hyperbolic polynomial. In [1], the smallest possible constant $c > 0$ was found such that if a polynomial P with complex coefficients satisfies the conditions $|q_k(P)| > c$ for all k, then P has only simple zeros.

The following special function

$$g_a(z) = \sum_{k=0}^{\infty} z^k a^{-k^2}, \ a > 1,$$

which is called the **partial theta function**, plays a significant role in the mentioned circle of problems. Strikingly, $q_k(g_a) = a^2$ for all $k \geq 2$. One of the interesting questions is, for which values of a this function belongs to the Laguerre–Pólya class. The paper [9] by O.M. Katkova, T. Lobova-Eisner, and A.M. Vishnyakova gives an exhaustive answer to this question. In particular, it is proved that there exists a constant $q_\infty \approx 3.23363666$ such that $g_a \in \mathcal{L} - \mathcal{P}$ if and only if $a^2 \geq q_\infty$. Moreover, the authors studied analogous questions for the Taylor sections of the function g_a. For more details on the partial theta function, see a series of works by V.P. Kostov dedicated to its various properties [11, 12, 13, 14], his joint work with B. Shapiro [15], and a fascinating historical review by S.O. Warnaar [24].

It is easy to show that, if the estimation of $q_k(f)$ only from below is given then the constant 4 in $q_k(f) \geq 4$ is the smallest possible to conclude that $f \in \mathcal{L} - \mathcal{P}$ (that is, Theorem B remains valid when omitting (ii)). However, if we only have the estimation of q_k from below and require monotonicity, then the constant 4 in the condition $q_k \geq 4$ can be reduced to conclude that $f \in \mathcal{L} - \mathcal{P}$. As an example, in [17], it was proved that if the entire functions have the decreasing q_k such that $\lim_{n\to\infty} q_k = c \geq q_\infty$, then the function belongs to the Laguerre–Pólya class.

In this work, we show that if the estimations on $q_k(f)$ from below and from above are given, then the constant 4 can be decreased. We would like to investigate such problems where assumption $q_k(f) \geq c$ for all k is changed by $q_k(f) \in [\alpha, \beta]$ for all k for some given segment $[\alpha, \beta]$. As far as we know, the first result of such kind was obtained in [9] where the following theorem was proved.

Theorem C (O.M. Katkova, T. Lobova, and A.M. Vishnyakova, [9])**.** *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k > 0$, *be an entire function and* $\alpha \in [3.43; 4]$. *Then* $q_k(f) \in \left[\alpha, \frac{0.95}{2\sqrt{\alpha} - \alpha}\right]$ *for all* $k \geq 2$ *implies* $f \in \mathcal{L} - \mathcal{P}$.

2. Hutchinson's intervals

We present our main result.

Theorem 2.1. *Let* $P(x) = \sum_{k=0}^{n} a_k x^k$, $a_k > 0$, *be a polynomial, and* $n \geq 4$. *Suppose that there exists* $\alpha, 1 + \sqrt{5} \leq \alpha < 4$, *such that* $q_k(P) \in \left[\alpha, \frac{8}{\alpha(4-\alpha)}\right]$ *for all* $k = 2, 3, \ldots, n$. *Then* $P \in \mathcal{HP}$.

The following statement is a simple corollary of the above result.

Corollary 2.2. *Let* $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $a_k > 0$, *be an entire function. Suppose that there exists* $\alpha, 1 + \sqrt{5} \leq \alpha < 4$, *such that* $q_k(f) \in \left[\alpha, \frac{8}{\alpha(4-\alpha)}\right]$ *for all* $k = 2, 3, \ldots$ *Then* $f \in \mathcal{L} - \mathcal{P}$.

Remark 2.3. As it follows from the result about the partial theta-function by [9], the constant α in the statement of Theorem 2.1 can not be less than $q_\infty \approx 3.23363666$. We observe that $1 + \sqrt{5} \approx 3.23606797$.

2.1. **Proof of Theorem 2.1.** For a polynomial $P(x) = \sum_{k=0}^{n} a_k x^k$ with positive coefficients, without loss of generality, we can assume that $a_0 = a_1 = 1$, since we can consider a polynomial $T(x) = a_0^{-1} P(a_0 a_1^{-1} x)$ instead of $P(x)$, due to the fact that such rescaling of P preserves its property of having real zeros and preserves the second quotients: $q_k(T) = q_k(P)$ for all k. For the sake of brevity, we further use notation q_k instead of $q_k(P)$. Thereafter, we consider a polynomial

$$Q(x) = T(-x) = 1 - x + \sum_{k=2}^{n} \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \cdots q_{k-1}^2 q_k} \tag{5}$$

instead of P (see (4) for the formulas for coefficients).

Our proof is based on the following lemma.

Lemma 2.4. *Let* $[\alpha, \beta]$, $0 < \alpha < \beta$, *be a given segment. Then the following two statements are equivalent:*

(a) *For every polynomial*

$$S_{q_2,q_3,q_4}(x) = 1 - x + \frac{x^2}{q_2} - \frac{x^3}{q_2^2 q_3} + \frac{x^4}{q_2^3 q_3^2 q_4}$$

such that $q_j \in [\alpha, \beta]$ *for all* $j = 2, 3, 4$, *there exists a point* $x_0 \in (1, \alpha)$ *such that* $S_{q_2,q_3,q_4}(x_0) \leq 0$.

(b) *The following inequalities are valid:* $\alpha \geq 1 + \sqrt{5}$, *and, if* $\alpha < 4$ *then* $\beta \leq \frac{8}{\alpha(4-\alpha)}$.

Proof. Suppose that the Statement (a) of Lemma 2.4 is valid. Then for the polynomial $S_{\alpha,\alpha,\alpha}(x) = 1 - x + \frac{x^2}{\alpha} - \frac{x^3}{\alpha^3} + \frac{x^4}{\alpha^6}$, there exists a point $x_0 \in (1, \alpha)$ such that $S_{\alpha,\alpha,\alpha}(x_0) \leq 0$. It is easy to check the following identity

$$S_{\alpha,\alpha,\alpha}(x) = \left(\frac{x^2}{\alpha^3} - \frac{x}{2} + 1\right)^2 - \left(\frac{1}{4} + \frac{2}{\alpha^3} - \frac{1}{\alpha}\right) x^2. \tag{6}$$

If $\left(\frac{1}{4} + \frac{2}{\alpha^3} - \frac{1}{\alpha}\right) < 0$, then for every $x \in \mathbb{R}$ we have $S_{\alpha,\alpha,\alpha}(x) > 0$. Thus,

$$\left(\frac{1}{4} + \frac{2}{\alpha^3} - \frac{1}{\alpha}\right) = \frac{1}{4\alpha^3}(\alpha - 1 + \sqrt{5})(\alpha - 2)(\alpha - 1 - \sqrt{5}) \geq 0,$$

whence $\alpha \in (0, 2] \cup [1 + \sqrt{5}, \infty)$.

First we consider the case $\alpha \in (0, 2]$. By (6), we have

$$S_{\alpha,\alpha,\alpha}(x) = \left(\frac{x^2}{\alpha^3} - \left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2}{\alpha^3} - \frac{1}{\alpha}}\right) x + 1\right) \times \left(\frac{x^2}{\alpha^3} - \left(\frac{1}{2} - \sqrt{\frac{1}{4} + \frac{2}{\alpha^3} - \frac{1}{\alpha}}\right) x + 1\right).$$

We have two quadratic polynomials in brackets with the following discriminants:

$$D_{\pm} = \left(\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{2}{\alpha^3} - \frac{1}{\alpha}}\right)^2 - \frac{4}{\alpha^3}.$$

If $D_+ < 0$ and $D_- < 0$, then for every $x \in \mathbb{R}$ we have $S_{\alpha,\alpha,\alpha}(x) > 0$. Thus, at least one of these two discriminants is nonnegative, whence $D_+ \geq 0$, and we obtain

$$\sqrt{\frac{1}{4} + \frac{2}{\alpha^3} - \frac{1}{\alpha}} \geq \frac{4 - \alpha^{3/2}}{2\alpha^{3/2}}. \tag{7}$$

We observe that $4 - \alpha^{3/2} \geq 0$ for $\alpha \in (0, 2]$. Thus, the inequality (7) implies

$$\psi(\alpha) := -\alpha^2 + 2\alpha^{3/2} - 2 \geq 0.$$

The derivative of $\psi(\alpha)$ has a unique positive root $\alpha_0 = \frac{9}{4}$, and the maximal value of ψ for $\alpha > 0$ is $\psi\left(\frac{9}{4}\right) = -\frac{5}{16} < 0$. Thus, if the statement (a) of Lemma 2.4 is valid then $\alpha \geq 1 + \sqrt{5}$.

Further, we assume that $\alpha \geq 1 + \sqrt{5}$, and suppose that the statement (a) of Lemma 2.4 is valid. Let $S_{q_2,q_3,q_4}(x)$ be an arbitrary polynomial such that $q_j \in [\alpha, \beta]$ for all $j = 2, 3, 4$. We want to investigate whether there exists $x_0 \in (1, \alpha)$ such that $S_{q_2,q_3,q_4}(x_0) \leq 0$. We observe that for all $x > 0$

$$S_{q_2,q_3,q_4}(x) \leq S_{q_2,q_3,\alpha}(x) := 1 - x + \frac{x^2}{q_2} - \frac{x^3}{q_2^2 q_3} + \frac{x^4}{q_2^3 q_3^2 \alpha}.$$

Thus, for every polynomial $S_{q_2,q_3,q_4}(x)$ with $q_j \in [\alpha, \beta]$ for all $j = 2, 3, 4$, there exists a point $x_0 \in (1, \alpha)$ such that $S_{q_2,q_3,q_4}(x_0) \leq 0$ if and only if for every polynomial $S_{q_2,q_3,\alpha}(x)$ with $q_j \in [\alpha, \beta]$ for all $j = 2, 3$, there exists a point $x_0 \in (1, \alpha)$ such that $S_{q_2,q_3,\alpha}(x_0) \leq 0$.

Next, we compute the derivative of $S_{q_2,q_3,\alpha}$ with respect to q_3. We get

$$\frac{\partial}{\partial q_3} S_{q_2,q_3,\alpha}(x) = \frac{x^3}{q_2^2 q_3^2} - \frac{2x^4}{q_2^3 q_3^3 \alpha}.$$

We observe that $\frac{x^3}{q_2^2 q_3^2} - \frac{2x^4}{q_2^3 q_3^3 \alpha} > 0 \Leftrightarrow x < \frac{q_2 q_3 \alpha}{2}$, so for all $x \in (1, \alpha)$ we get that $S_{q_2,q_3,\alpha}$ is increasing in q_3. Whence we have

$$S_{q_2,q_3,\alpha}(x) \leq S_{q_2,\beta,\alpha}(x) = 1 - x + \frac{x^2}{q_2} - \frac{x^3}{q_2^2 \beta} + \frac{x^4}{q_2^3 \beta^2 \alpha}.$$

Analogously, we consider the derivative of $S_{q_2,\beta,\alpha}(x)$ with respect to q_2 to understand the monotonicity and we get

$$\frac{\partial}{\partial q_2} S_{q_2,\beta,\alpha}(x) = -\frac{x^2}{q_2^2} + \frac{2x^3}{q_2^3 \beta} - \frac{3x^4}{q_2^4 \beta^2 \alpha}.$$

We show that $\frac{\partial}{\partial q_2} S_{q_2,\beta,\alpha}(x) < 0$ for $x \in (1, \alpha)$, or, equivalently,

$$3x^2 - 2q_2 \alpha \beta x + q_2^2 \alpha \beta^2 > 0. \tag{8}$$

Under our assumption that $\alpha \geq 1 + \sqrt{5}$, we compute the discriminant of the lefthand side of (8)and observe that $\frac{D}{4} = q_2^2 \beta^2 \alpha(\alpha - 3) > 0$, so the quadratic expression has two positive roots

$$x_{\pm} = \frac{q_2 \alpha \beta \pm q_2 \beta \sqrt{\alpha(\alpha - 3)}}{3}.$$

To prove (8), it is sufficient to check that $\alpha < x_-$, or $q_2 \beta \sqrt{\alpha(\alpha - 3)} < q_2 \alpha \beta - 3\alpha$. The last inequality is equivalent to $q_2^2 \beta^2 + 3\alpha - 2q_2 \alpha \beta = q_2 \beta (q_2 \beta - 2\alpha) + 3\alpha > 0$, which holds under our assumptions since $q_2 \geq \alpha$, and $\beta > \alpha > 2$. Thus, we have proved that for all $x \in (1, \alpha)$:

$$S_{q_2,\beta,\alpha}(x) \leq S_{\alpha,\beta,\alpha}(x) = 1 - x + \frac{x^2}{\alpha} - \frac{x^3}{\alpha^2 \beta} + \frac{x^4}{\alpha^4 \beta^2}.$$

Consequently, for every polynomial $S_{q_2,q_3,q_4}(x)$ such that $q_j \in [\alpha, \beta]$ for all $j = 2, 3, 4$, there exists a point $x_0 \in (1, \alpha)$ such that $S_{q_2,q_3,q_4}(x_0) \leq 0$ if and only if for the polynomial $S_{\alpha,\beta,\alpha}(x)$ there exists a point $x_0 \in (1, \alpha)$ such that $S_{\alpha,\beta,\alpha}(x_0) \leq 0$.

Now we consider the polynomial $S_{\alpha,\beta,\alpha}(x)$ for $x \in (1, \alpha)$. Set $x =: \alpha\sqrt{\beta} y$. Since $x \in (1, \alpha)$, we have $y \in (\frac{1}{\alpha\sqrt{\beta}}, \frac{1}{\sqrt{\beta}})$. Hence, after change of variables, we get a self-reciprocal polynomial

$$\begin{aligned}\widetilde{P}(y) := S_{\alpha,\beta,\alpha}(\alpha\sqrt{\beta} y) &= 1 - \alpha\sqrt{\beta} y + \alpha\beta y^2 - \alpha\sqrt{\beta} y^3 + y^4 \\ &= y^2 \Big((y^{-2} + y^2) - \alpha\sqrt{\beta}(y^{-1} + y) + \alpha\beta \Big).\end{aligned}$$

Set $w := y^{-1} + y$. We want to investigate whether there exists a point $w_0 \in (\sqrt{\beta} + \frac{1}{\sqrt{\beta}}, \alpha\sqrt{\beta} + \frac{1}{\alpha\sqrt{\beta}})$ such that

$$\widetilde{\widetilde{P}}(w_0) = w_0^2 - \alpha\sqrt{\beta} w_0 + \alpha\beta - 2 \leq 0.$$

We consider the vertex of the parabola $w_v = \frac{\alpha\sqrt{\beta}}{2}$, and check if it lies in $\left(\sqrt{\beta} + \frac{1}{\sqrt{\beta}}, \alpha\sqrt{\beta} + \frac{1}{\alpha\sqrt{\beta}}\right)$. Obviously, $w_v = \frac{\alpha\sqrt{\beta}}{2} < \alpha\sqrt{\beta} + \frac{1}{\alpha\sqrt{\beta}}$. We show that the following inequality is fulfilled $\frac{\alpha\sqrt{\beta}}{2} > \sqrt{\beta} + \frac{1}{\sqrt{\beta}}$, or, equivalently, $\beta > \frac{2}{\alpha - 2}$. It is sufficient to prove that $\alpha > \frac{2}{\alpha - 2}$, and it is equivalent to $\alpha^2 - 2\alpha - 2 > 0$, which is fulfilled under our assumption $\alpha \geq 1 + \sqrt{5}$. Since $w_v \in \left(\sqrt{\beta} + \frac{1}{\sqrt{\beta}}, \alpha\sqrt{\beta} + \frac{1}{\alpha\sqrt{\beta}}\right)$, there exists $w_0 \in \left(\sqrt{\beta} + \frac{1}{\sqrt{\beta}}, \alpha\sqrt{\beta} + \frac{1}{\alpha\sqrt{\beta}}\right)$ such that $\widetilde{\widetilde{P}}(w_0) \leq 0$ if and only if the discriminant of this quadratic function is non-negative:

$$D = \alpha^2 \beta - 4\alpha\beta + 8 = \beta\alpha(\alpha - 4) + 8 \geq 0. \tag{9}$$

The inequality above is equivalent to the following statement: if $\alpha < 4$ then $\beta \leq \frac{8}{\alpha(4-\alpha)}$. $\square$

Remark 2.5. Lemma 2.4 is an analog of Theorem 1.5 from [18]. This theorem states that if $f(x) = 1 + x + \sum_{k=2}^{\infty} a_k x^k$ is an entire function with positive coefficients, and $3 \leq q_2(f) < 4$, $q_4(f) \geq 3$ and $2 \leq q_3(f) \leq \frac{8}{d(4-d)}$, where $d = \min(q_2(f), q_4(f))$, then there exists $x_0 \in [-q_2(f), 0]$ such that $f(z_0) \leq 0$.

Now we can prove Theorem 2.1. Let $Q(x) = 1 - x + \sum_{k=2}^{n} \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \cdots q_{k-1}^2 q_k}$ be a polynomial, $n \geq 4$, and there exists $\alpha \in [1 + \sqrt{5}, 4)$ such that $q_k \in \left[\alpha, \frac{8}{\alpha(4-\alpha)}\right]$ for all $k = 2, 3, \ldots, n$. Let us fix an arbitrary j such that $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$, and suppose that $x \in (q_2 q_3 \cdots q_j, q_2 q_3 \cdots q_j q_{j+1})$ (for $j = 1$ we assume that $1 < x < q_2$). Then we observe that

$$1 < x < \frac{x^2}{q_2} < \frac{x^3}{q_2^2 q_3} < \cdots < \frac{x^j}{q_2^{j-1} q_3^{j-2} \cdots q_j}, \tag{10}$$

and

$$\frac{x^j}{q_2^{j-1}q_3^{j-2}\cdots q_j} > \frac{x^{j+1}}{q_2^{j}q_3^{j-1}\cdots q_j^2 q_{j+1}} > \frac{x^{j+2}}{q_2^{j+1}q_3^{j}\cdots q_j^3 q_{j+1}^2 q_{j+2}} > \cdots > \frac{x^n}{q_2^{n-1}q_3^{n-2}\cdots q_{n-1}^2 q_n}. \tag{11}$$

We have the following representation:

$$\begin{aligned}(-1)^{j-1}Q(x) &= \left((-1)^{j-1} - (-1)^{j-1}x + \sum_{k=2}^{j-2}\frac{(-1)^{k+j-1}x^k}{q_2^{k-1}q_3^{k-2}\cdots q_{k-1}^2 q_k}\right)\\ &\quad + \sum_{k=j-1}^{j+3}\frac{(-1)^{k+j-1}x^k}{q_2^{k-1}q_3^{k-2}\cdots q_{k-1}^2 q_k} + \left(\sum_{k=j+4}^{n}\frac{(-1)^{k+j-1}x^k}{q_2^{k-1}q_3^{k-2}\cdots q_{k-1}^2 q_k}\right)\\ &:= \Sigma_{1,j}(x) + g_j(x) + \Sigma_{2,j}(x).\end{aligned} \tag{12}$$

We note that, for some j the sum $\Sigma_{2,j}(x)$ can be empty (and equal to zero), but for $n \geq 5$ we have $j+3 \leq \lfloor \frac{n}{2} \rfloor + 3 \leq n$, so all 5 summands in $g_j(x)$ are nonzero. We later consider the case $n = 4$.

For $x \in (q_2q_3\cdots q_j, q_2q_3\cdots q_jq_{j+1})$, we observe that the terms in $\Sigma_{1,j}(x)$ are alternating in sign and and their moduli are increasing, while the summands in $\Sigma_{2,j}(x)$ are alternating in sign and their moduli are decreasing. Hence, $\Sigma_{1,j}(x) < 0$ and $\Sigma_{2,j}(x) < 0$ for all $x \in (q_2q_3\cdots q_j, q_2q_3\cdots q_jq_{j+1})$, whence we get

$$(-1)^{j-1}Q(x) < g_j(x) \quad \forall x \in (q_2q_3\cdots q_j, q_2q_3\cdots q_jq_{j+1}). \tag{13}$$

We have

$$\begin{aligned}g_j(x) = \frac{x^{j-1}}{q_2^{j-2}q_3^{j-3}\cdots q_{j-2}^2 q_{j-1}}\Bigg(1 &- \frac{x}{q_2q_3\cdots q_{j-1}q_j} + \frac{x^2}{q_2^2q_3^2\cdots q_{j-1}^2 q_j^2 q_{j+1}}\\ &- \frac{x^3}{q_2^3q_3^3\cdots q_{j-1}^3 q_j^3 q_{j+1}^2 q_{j+2}} + \frac{x^4}{q_2^4q_3^4\cdots q_{j-1}^4 q_j^4 q_{j+1}^3 q_{j+2}^2 q_{j+3}}\Bigg).\end{aligned}$$

Set $y = \frac{x}{q_2q_3\cdots q_{j-1}q_j}$, and for $x \in (q_2q_3\cdots q_j, q_2q_3\cdots q_jq_{j+1})$ we have $y \in (1, q_{j+1})$. Therefore, we obtain

$$\begin{aligned}h_j(y) := g_j(q_2q_3\cdots q_{j-1}q_jy) &= q_2q_3^2\cdots q_{j-1}^{j-2}q_j^{j-1}y^{j-1}\\ &\times\left(1 - y + \frac{y^2}{q_{j+1}} - \frac{y^3}{q_{j+1}^2 q_{j+2}} + \frac{y^4}{q_{j+1}^3 q_{j+2}^2 q_{j+3}}\right).\end{aligned}$$

Since $q_{j+1}, q_{j+2}, q_{j+3} \in \left[\alpha, \frac{8}{\alpha(4-\alpha)}\right]$ for some α, where $1+\sqrt{5} \leq \alpha < 4$, the polynomial in brackets satisfies the assumptions of Lemma 2.4. Thus, there exists $y_j \in (1, q_{j+1})$ such that $h_j(y_j) \leq 0$. Hence, there exists $x_j = q_2q_3\cdots q_{j-1}q_jy_j \in (q_2q_3\cdots q_j, q_2q_3\cdots q_jq_{j+1})$, such that $g_j(x_j) \leq 0$. Taking into account (13), for every $n \geq 5$ we obtain

$$\forall j,\ 1 \leq j \leq \lfloor \frac{n}{2} \rfloor, \quad \exists x_j \in (q_2q_3\cdots q_j, q_2q_3\cdots q_jq_{j+1}): \quad (-1)^{j-1}Q(x_j) < 0. \tag{14}$$

The only problem for $n = 4$ is when $j = 2$. In the latter case, we have

$$-Q(x) = -1 + \left(x - \frac{x^2}{q_2} + \frac{x^3}{q_2^2q_3} - \frac{x^4}{q_2^3q_3^2q_4}\right) = -1 + x\left(1 - \frac{x}{q_2} + \frac{x^2}{q_2^2q_3} - \frac{x^3}{q_2^3q_3^2q_4}\right).$$

We highlight that the polynomial in brackets is of degree 3, however, we can estimate it from above with a polynomial of degree 4 as follows:

$$-Q(x) < x\left(1-\frac{x}{q_2}+\frac{x^2}{q_2^2q_3}-\frac{x^3}{q_2^3q_3^2q_4}\right) < x\left(1-\frac{x}{q_2}+\frac{x^2}{q_2^2q_3}-\frac{x^3}{q_2^3q_3^2q_4}+\frac{x^4}{q_2^4q_3^3q_4^2q_4}\right).$$

Therefore, we can further reason in the same way as before. Thus, we conclude that (14) is valid for all $n \geq 4$.

Now let us fix an arbitrary j, $\lfloor \frac{n}{2} \rfloor+1 \leq j \leq n-1$, and suppose that $x \in (q_2q_3\cdots q_j, q_2q_3\cdots q_jq_{j+1})$. Then both (10) and (11) are valid, and we have the following representation:

$$\begin{aligned}(15) \qquad (-1)^{j-1}Q(x) &= \left((-1)^{j-1}-(-1)^{j-1}x+\sum_{k=2}^{j-4}\frac{(-1)^{k+j-1}x^k}{q_2^{k-1}q_3^{k-2}\cdots q_{k-1}^2q_k}\right)\\ &+\sum_{k=j-3}^{j+1}\frac{(-1)^{k+j-1}x^k}{q_2^{k-1}q_3^{k-2}\cdots q_{k-1}^2q_k}+\left(\sum_{k=j+2}^{n}\frac{(-1)^{k+j-1}x^k}{q_2^{k-1}q_3^{k-2}\cdots q_{k-1}^2q_k}\right)\\ &:= \Sigma_{1,j}(x)+g_j(x)+\Sigma_{2,j}(x).\end{aligned}$$

We note that for some j, the sum $\Sigma_{1,j}(x)$ can be empty (and equal to zero), although for $n \geq 4$, we have $j-3 \geq \lfloor \frac{n}{2} \rfloor - 2 \geq 0$, so all 5 summands in $g_j(x)$ are nonzero.

For $x \in (q_2q_3\cdots q_j, q_2q_3\cdots q_jq_{j+1})$, we obeserve that the terms in $\Sigma_{1,j}(x)$ are alternating in sign and and their moduli are increasing, while the summands in $\Sigma_{2,j}(x)$ are alternating in sign and their moduli are decreasing. Hence, $\Sigma_{1,j}(x) < 0$ and $\Sigma_{2,j}(x) < 0$ for all $x \in (q_2q_3\cdots q_j, q_2q_3\cdots q_jq_{j+1})$, whence we get

$$(16) \qquad (-1)^{j-1}Q(x) < g_j(x) \quad \forall x \in (q_2q_3\cdots q_j, q_2q_3\cdots q_jq_{j+1}).$$

We have

$$\begin{aligned}g_j(x) = \frac{x^{j+1}}{q_2^jq_3^{j-1}\cdots q_j^2q_{j+1}} \times \Bigg(1 &- \frac{q_2q_3\cdots q_{j-1}q_jq_{j+1}}{x}+\frac{q_2^2q_3^2\cdots q_{j-1}^2q_j^2q_{j+1}}{x^2}\\ &-\frac{q_2^3q_3^3\cdots q_{j-2}^3q_{j-1}^3q_j^2q_{j+1}}{x^3}+\frac{q_2^4q_3^4\cdots q_{j-3}^4q_{j-2}^4q_{j-1}^3q_j^2q_{j+1}}{x^4}\Bigg).\end{aligned}$$

Set $y = \frac{q_2q_3\cdots q_{j-1}q_jq_{j+1}}{x}$, and we observe that for $x \in (q_2q_3\cdots q_j, q_2q_3\cdots q_jq_{j+1})$ we have $y \in (1, q_{j+1})$. Thus, we obtain

$$\begin{aligned}h_j(y) := g_j\left(\frac{q_2q_3\cdots q_{j-1}q_jq_{j+1}}{y}\right) &= \frac{q_2q_3^2\cdots q_{j-1}^{j-2}q_j^{j-1}q_{j+1}^j}{y^{j+1}}\\ &\times\left(1-y+\frac{y^2}{q_{j+1}}-\frac{y^3}{q_{j+1}^2q_j}+\frac{y^4}{q_{j+1}^3q_j^2q_{j-1}}\right).\end{aligned}$$

Since $q_{j+1}, q_j, q_{j-1} \in \left[\alpha, \frac{8}{\alpha(4-\alpha)}\right]$ for some $\alpha \in [1+\sqrt{5}, 4)$, the polynomial in brackets satisfies the assumptions of Lemma 2.4. Thus, there exists $y_j \in (1, q_{j+1})$ such that $h_j(y_j) \leq 0$. Whence, there exists $x_j = \frac{q_2q_3\cdots q_{j-1}q_jq_{j+1}}{y_j} \in (q_2q_3\cdots q_j, q_2q_3\cdots q_jq_{j+1})$, such that $g_j(x_j) \leq 0$. Taking into account (13), for every $n \geq 4$ we obtain

$$(17) \qquad \forall j, \lfloor\frac{n}{2}\rfloor+1 \leq j \leq n-1,\ \exists x_j \in (q_2q_3\cdots q_j, q_2q_3\cdots q_{j+1}):\ (-1)^{j-1}Q(x_j) < 0.$$

Since $q_j > 1$ for all $j = 2, 3, \ldots, n$, we get $1 < q_2 < q_2q_3 < q_2q_3q_4 < \ldots < q_2q_3q_4\cdots q_n$, whence $x_1 < x_2 < \ldots < x_{n-1}$. By (14) and (17) we have

$$Q(0) > 0, -Q(x_1) > 0, Q(x_2) > 0, -Q(x_3) > 0, \ldots,$$

$$(-1)^{n-1}Q(x_{n-1}) > 0, (-1)^n Q(+\infty) > 0.$$

Thus, we have proved that all the zeros of Q are real.

Theorem 2.1 is proved.

Remark 2.6. Assumptions on q_j in Theorem 2.1 could be slightly weakened for entire functions and polynomials of higher degrees if we obtain an analogue of Lemma 2.4 for polynomials of even degrees that are greater than 4. As it is shown in the proof of Lemma 2.4, an important role in such considerations is played by special polynomials which have the following property: $\alpha = q_2 = q_4 = q_6 = \dots$, and $\beta = q_3 = q_5 = q_7 = \dots$, when $\alpha < \beta$.

The paper [19] by T.H. Nguyen and A. Vishnyakova studies the entire functions with alternating second quotients of Taylor coefficients. Let $f_{a,b}(x) = 1 - x + \sum_{k=2}^{\infty} \frac{(-1)^k x^k}{q_2^{k-1} q_3^{k-2} \cdots q_k}$, be an entire function such that $q_2 = q_4 = q_6 = \dots = \alpha$, $q_3 = q_5 = q_7 = \dots = \beta$, and $1 < \alpha < \beta$. In [19], it is proved that the function $f_{a,b}$ belongs to the Laguerre–Pólya class if and only if there exists $x_0 \in [0, q_2]$ such that $f_{a,b}(x_0) \leq 0$. In addition, it is proved that if the function $f_{a,b}$ belongs to the Laguerre–Pólya class, then $\alpha \geq q_\infty$.

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