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CONVOLUTION IN DUAL CESÀRO SEQUENCE SPACES

GUILLERMO P. CURBERA AND WERNER J. RICKER

Abstract. We investigate convolution operators in the sequence spaces d_p, for $1 \le p < \infty$. These spaces, for $p > 1$, arise as dual spaces of the Cesàro sequence spaces ces_p thoroughly investigated by G. Bennett. A detailed study is also made of the algebra of those sequences which convolve d_p into d_p. It turns out that such multiplier spaces exhibit features which are very different to the classical multiplier spaces of ℓ^p.

1. Introduction

In 1966, in a celebrated paper, [16], N. K. Nikolskii initiated the study of multipliers acting on the classical sequence spaces $\ell^p = \ell^p(\mathbb{N}_0)$, with $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, where

$$\ell^p := \Big\{ a = (a_n)_{n=0}^\infty \in \mathbb{C}^{\mathbb{N}_0} : \sum_{n=0}^\infty |a_k|^p < \infty \Big\}, \quad 1 \le p < \infty.$$

A sequence $b = (b_n)_{n=0}^\infty \in \mathbb{C}^{\mathbb{N}_0}$ defines a *multiplier* on ℓ^p if the *convolution* $a * b \in \mathbb{C}^{\mathbb{N}_0}$, defined by

$$(a * b)_n := \sum_{j=0}^n a_j b_{n-j}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

belongs to ℓ^p, for every $a \in \ell^p$. The *multiplier algebra* $\mathscr{M}(\ell^p)$ of ℓ^p is the collection of all such $b \in \mathbb{C}^{\mathbb{N}_0}$. Nikolskii established the following fundamental properties of these multiplier algebras:

a) $\ell^1 \subsetneq \mathscr{M}(\ell^p) \subsetneq \ell^p$, for $1 < p < \infty$;
b) $\mathscr{M}(\ell^p) = \mathscr{M}(\ell^{p'})$, for $1/p + 1/p' = 1$;
c) $\mathscr{M}(\ell^{p_1}) \subsetneq \mathscr{M}(\ell^{p_2})$, for $1 \le p_1 < p_2 \le 2$.

These multiplier algebras, except when $p \in \{1, 2\}$, are not well understood and their investigation is far from finalized. Important contributions were made by Vinogradov,

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Verbitskii and others; see, for example, [4, §6.41–6.43], and [8] for a recent account of the state of the art.

The Cesàro sequence spaces ces_p, for $1 < p < \infty$, are intimately connected to the spaces ℓ^p via the Cesàro averaging operator which maps each element of ℓ^p to the sequence of its averages (again an element of ℓ^p). The spaces ces_p were throughly investigated by G. Bennett, [2]; see also [12] and the references therein. They have the property that $\ell^p \subsetneq ces_p$, for all $1 < p < \infty$. However, in contrast to ℓ^p, the situation regarding the multipliers of ces_p is completely different: the multiplier algebra $\mathscr{M}(ces_p) = \ell^1$, for *every* $1 < p < \infty$, [10, Theorem 4.1].

The purpose of this note is to investigate the multiplier algebras $\mathscr{M}(d_p)$ of the sequence spaces d_p, also spaces closely related to ℓ^p, which are defined by

$$d_p := \Big\{a = (a_n)_{n=0}^{\infty} \in \mathbb{C}^{\mathbb{N}_0} : \sum_{n=0}^{\infty} \sup_{k\geq n} |a_k|^p < \infty\Big\}, \quad 1 \leq p < \infty. \tag{1.2}$$

They were defined and studied by G. Bennett, [2], when he obtained a tractable identification of the dual Banach space of ces_p. More precisely, the dual Banach space $(ces_p)^*$ is isomorphic to d_q, for $p \in (1,\infty)$, where $\frac{1}{p} + \frac{1}{q} = 1$; [2, Corollary 12.17]. Despite having similarities in their definition, the spaces ℓ^p and d_p are rather different. A significant difference is that the canonical vectors $e_n := (\delta_{n,k})_{k=0}^{\infty}$, for $n \in \mathbb{N}_0$, are all *unit vectors* in every space ℓ^p, for $p \in [1,\infty]$, but they have norm $\|e_n\|_{d_p} = (n+1)^{1/p}$ whenever $1 \leq p < \infty$ and $n \in \mathbb{N}_0$. For further properties of the spaces d_p, see [5], for example. Note that $d_p \subsetneq \ell^p \subsetneq ces_p$, for $1 < p < \infty$.

The multiplier algebras $\mathscr{M}(d_p)$ of d_p consist of all $b \in \mathbb{C}^{\mathbb{N}_0}$ which convolve d_p into itself. Differences between the spaces ℓ^p and d_p induce drastically different features between their respective multiplier spaces $\mathscr{M}(\ell^p)$ and $\mathscr{M}(d_p)$. In contrast to property a) above, we have that

$$\mathscr{M}(d_p) \subsetneq \ell^1 \quad \text{and} \quad \mathscr{M}(d_1) = d_1 \subsetneq \mathscr{M}(d_p) \subsetneq d_p, \quad 1 < p < \infty;$$

see Theorem 4.2 and Corollary 4.3. That is, *all* the spaces $\mathscr{M}(d_p)$ are inside ℓ^1. In contrast to properties b) and c) above, it turns out that

$$\mathscr{M}(d_{p_1}) \subsetneq \mathscr{M}(d_{p_2}), \quad 1 \leq p_1 < p_2 < \infty;$$

see Theorem 4.5. That is, there is no largest space with the role that $\mathscr{M}(\ell^2)$ has in the ℓ^p setting.

As for $\mathscr{M}(\ell^p)$, with $p \notin \{1,2\}$, no characterization of the entire algebra $\mathscr{M}(d_p)$ is known (except for $p = 1$). Nevertheless, we devote some effort to identify natural classes of elements which do belong to $\mathscr{M}(d_p)$. For example, the weighted Banach algebra $\ell^1(w_p)$ with $w_p(n) = (n+1)^{1/p}$ for $n \in \mathbb{N}_0$ is contained in $\mathscr{M}(d_p)$ for every $1 \leq p < \infty$; see Proposition 4.4. A characterization of those elements from ℓ^1 which belong to $\mathscr{M}(d_p)$ is presented in Theorem 5.1. A more tractable sufficient condition for a sequence $b \in \ell^1$ to be a multiplier for d_p, in terms of its coefficients, namely that

$$\sum_{n=0}^{\infty} 2^{np} \sup_{2^n \le k < 2^{n+1}} |b_k|^p < \infty,$$

is established in Theorem 5.2.

Together with $\mathscr{M}(d_p)$ we also consider the associated algebra $\mathscr{M}_{\mathrm{op}}(d_p)$ of all (necessarily) bounded, linear convolution operators T_b on d_p induced by the elements b of $\mathscr{M}(d_p)$; see Section 2 for the definitions. As for the spaces ℓ^p, the right-shift operator S (which maps an element $(a_0, a_1, \dots)$ to $(0, a_0, a_1, \dots)$) also plays an important role for the spaces d_p. For instance, it turns out that the commutant algebra $\mathscr{M}_{\mathrm{op}}(d_p)^c$ of $\mathscr{M}(d_p)$ equals

$$\mathscr{M}_{\mathrm{op}}(d_p)^c = \Big\{ T \in \mathscr{L}(d_p) : TS = ST \Big\}, \quad 1 \le p < \infty, \tag{1.3}$$

where $\mathscr{L}(d_p)$ is the space of all bounded linear operators of d_p into itself. A crucial difference between the ℓ^p and the d_p setting is that the operator norm of $S^n \in \mathscr{L}(d_p)$ equals $(n+1)^{1/p}$ for each $n \in \mathbb{N}_0$ and $1 \le p < \infty$, whereas $S^n \in \mathscr{M}(\ell^p)$ is an isometry for all such n and p. Consequences of (1.3) are that $\mathscr{M}(d_p)$ is complete for the weak operator topology (cf. Section 3) and that the spectrum of an operator in the unital, commutative Banach algebra $\mathscr{M}_{\mathrm{op}}(d_p)$, for $1 \le p < \infty$, coincides with its spectrum as an element of $\mathscr{L}(d_p)$. The topic of the spectrum of operators belonging to $\mathscr{M}_{\mathrm{op}}(d_p)$ is pursued in the final section. Of particular relevance are the distinct subspaces d_1, $\ell^1(w_p)$ and $d_{pp} \cap \ell^1$ of $\mathscr{M}(d_p)$ because, if $b = (b_n)_{n=0}^\infty$ belongs to any one of these subspaces, then the corresponding multiplier operator $T_b \in \mathscr{M}_{\mathrm{op}}(d_p)$ can be approximated in the operator norm by the polynomial operators $\{\sum_{k=0}^n b_k S^k\}_{n=0}^\infty$; see Remark 6.6(ii) and Proposition 6.7.

The paper is organized as follows. Section 2 presents the necessary preliminaries required in the sequel. Section 3 treats various relevant properties of the operator algebras $\mathscr{M}_{\mathrm{op}}(d_p)$, whereas Section 4 concentrates on the multiplier algebras $\mathscr{M}(d_p)$. In Section 5 we identify various subspaces of $\mathscr{M}(d_p)$. The final Section 6 is devoted to spectral and Banach algebra properties of $\mathscr{M}_{\mathrm{op}}(d_p)$.

2. Preliminaries

For each $p \in [1, \infty)$ the sequence space d_p defined in (1.2) is a Banach space for the norm

$$\|a\|_{d_p} := \Big(\sum_{n=0}^{\infty} \sup_{k \ge n} |a_k|^p \Big)^{1/p}, \quad a \in d_p. \tag{2.1}$$

A direct consequence of (2.1) is that $d_p \subseteq \ell^p$ with a continuous inclusion. Given $a = (a_n)_{n=0}^\infty \in \ell^\infty$, the *least decreasing majorant of* a is the sequence $\hat{a} := (\sup_{k \ge n} |a_k|)_{n=0}^\infty$, [2, (3.7)]. Then, $a \in d_p$ precisely when $\hat{a} \in \ell^p$ and $\|a\|_{d_p} = \|\hat{a}\|_p$, where $\|\cdot\|_p$ is the usual norm in ℓ^p. The canonical vectors $\{e_n : n \in \mathbb{N}_0\}$ satisfy

$$\|e_n\|_{d_p} = \|\widehat{e_n}\|_{\ell^p} = \Big\|(1,\dots,1,\overbrace{1}^{\text{position } n},0,0,\dots)\Big\|_p = (n+1)^{1/p}.$$

For every $p \in [1,\infty)$, the vectors $\{e_n : n \in \mathbb{N}_0\}$ form an unconditional basis in d_p, [5, Proposition 2.1]; see Section 4 for the case $p = 1$.

A combination of Cauchy's condensation test for series and Abel's summation formula implies the following two useful equivalent expressions for the norm (2.1) in d_p:

$$\|a\|_{d_p} \asymp \left(\sup_{k\ge 0}|a_k|^p + \sum_{n=0}^{\infty} 2^n \sup_{2^n\le k<2^{n+1}} |a_k|^p\right)^{1/p}, \tag{2.2}$$

$$\|a\|_{d_p} \asymp \left(\sup_{k\ge 0}|a_k|^p + \sup_{k\ge 1}|a_k|^p + \sum_{n=0}^{\infty} 2^n \sup_{2^n< k\le 2^{n+1}} |a_k|^p\right)^{1/p}, \tag{2.3}$$

where $A \asymp B$ means that there exist absolute constants $c, C > 0$ such that $cA \le B \le CA$; see also [12, Example 13.2] and [1, (3)].

As noted in Section 1, the space d_q is isomorphic to $(ces_p)^*$, where ces_p, [2], is defined, for each $1 < p \le \infty$, by

$$ces_p := \left\{a = (a_n)_{n=0}^{\infty} \in \mathbb{C}^{\mathbb{N}_0} : \|a\|_{ces_p} := \left(\sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n}|a_k|\right)^p\right)^{1/p}\right\}, \tag{2.4}$$

that is, $a \in ces_p$ if and only if $\left(\frac{1}{n+1}\sum_{k=0}^{n}|a_k|\right)_{n=0}^{\infty} \in \ell^p$.

The *convolution* of $a, b \in \mathbb{C}^{\mathbb{N}_0}$ is the sequence $a * b \in \mathbb{C}^{\mathbb{N}_0}$ defined by (1.1). According to Section 1 the *multiplier algebra*

$$\mathscr{M}(d_p) := \left\{b \in \mathbb{C}^{\mathbb{N}_0} : a * b \in d_p, \forall a \in d_p\right\}.$$

Each $b \in \mathscr{M}(d_p)$ defines a convolution operator $a \mapsto a * b \in d_p$, for $a \in d_p$, which is continuous (due to the closed graph theorem). The multiplier algebra $\mathscr{M}(d_p)$ endowed with the norm

$$\|b\|_{\mathscr{M}(d_p)} := \sup_{0\ne a\in d_p} \frac{\|a*b\|_{d_p}}{\|a\|_{d_p}}, \tag{2.5}$$

is a Banach algebra; see Section 3. Since $e_0 \in d_p$ satisfies $e_0 * b = b$ for every $b \in \mathbb{C}^{\mathbb{N}_0}$, it is clear that $\mathscr{M}(d_p) \subseteq d_p$. This implies (as mentioned above) that $\mathscr{M}(d_p)$ is a unital, commutative algebra under convolution. Moreover, for each $b \in \mathscr{M}(d_p)$, we have that $\|b\|_{d_p} = \|e_0 * b\|_{d_p}/\|e_0\|_{d_p} \le \|b\|_{\mathscr{M}(d_p)}$. Since $\|e_0\|_{\mathscr{M}(d_p)} = 1 = \|e_0\|_{d_p}$, it follows that the operator norm of the natural inclusion $\mathscr{M}(d_p) \subseteq d_p$ is precisely 1.

3. The operator algebra $\mathscr{M}_{\mathrm{op}}(d_p)$

Convolution operators on d_p will be considered within the unital (non-commutative) Banach algebra $\mathscr{L}(d_p)$ of all bounded linear operators on d_p equipped with the operator norm. Given $b \in \mathscr{M}(d_p)$, denote by T_b the convolution operator defined by $T_b(a) := a*b \in d_p$, for each $a \in d_p$, and set

$$\mathscr{M}_{\mathrm{op}}(d_p) := \Big\{ T_b \in \mathscr{L}(d_p) : b \in \mathscr{M}(d_p) \Big\}.$$

Observe that $\|T_b\|_{\mathscr{M}_{\mathrm{op}}(d_p)} = \|b\|_{\mathscr{M}(d_p)}$ for all $b \in \mathscr{M}(d_p)$. Clearly, $\mathscr{M}_{\mathrm{op}}(d_p)$ is a commutative, unital subalgebra of $\mathscr{L}(d_p)$, with the identity operator $I = T_{e_0}$ as its unit. Equipped with the operator norm from $\mathscr{L}(d_p)$, which we denote by $\|\cdot\|_{\mathscr{M}_{\mathrm{op}}(d_p)}$, it becomes a normed algebra.

The *commutant algebra* of $\mathscr{M}(d_p)$ is defined by

$$\mathscr{M}_{\mathrm{op}}(d_p)^c := \Big\{ R \in \mathscr{L}(d_p) : T_b R = R T_b, \ \forall b \in \mathscr{M}(d_p) \Big\}.$$

The right-shift $S \colon d_p \to d_p$ is the linear map given by

$$Sa = (0, a_0, a_1, \dots) = e_1 * a = T_{e_1} a, \quad a \in d_p.$$

It follows, for $n \in \mathbb{N}_0$, that

$$S^n a = (0, \dots, 0, \overbrace{a_0}^{\text{position } n}, a_1, \dots) = e_n * a = T_{e_n} a, \quad a \in d_p.$$

Direct calculation yields $\|e_n\|_{d_p} = \|S^n\|_{\mathscr{M}_{\mathrm{op}}(d_p)} = (n+1)^{1/p}$, for $n \in \mathbb{N}_0$ and $p \in [1,\infty)$; see [11, Lemma 4.12]. This is distinctly different to the situation for the spaces ℓ^p, where $\|e_n\|_p = \|S^n\|_{\mathscr{L}(\ell^p)} = 1$, for all $n \in \mathbb{N}_0$ and $p \in [1,\infty]$.

Proposition 3.1. *Let $p \in [1,\infty)$. Then*

$$\mathscr{M}_{\mathrm{op}}(d_p) = \Big\{ R \in \mathscr{L}(d_p) : RS = SR \Big\}. \tag{3.1}$$

Moreover,

$$\mathscr{M}_{\mathrm{op}}(d_p) = \mathscr{M}_{\mathrm{op}}(d_p)^c = \mathscr{M}_{\mathrm{op}}(d_p)^{cc}. \tag{3.2}$$

Proof. Let $T \in \mathscr{L}(d_p)$ satisfy $TS = ST$ and set $b := Te_0 \in d_p$. Since $e_1 = Se_0$, we have $Te_1 = TSe_0 = STe_0 = Sb = b * e_1$. In a similar way, using $e_{n+1} = Se_n$, it follows that $Te_n = S^n b = b * e_n$ for all $n \in \mathbb{N}_0$. Hence, $Ta = b * a$ for all a belonging to the linear span of $\{e_n : n \in \mathbb{N}_0\}$. Since the canonical vectors $\{e_n : n \in \mathbb{N}_0\}$ form a basis for d_p, for every $a = (a_n)_{n=0}^\infty \in d_p$ we have $a^N \to a$ in d_p, where $a^N = \sum_{j=0}^N a_j e_j$. Then $Ta^N \to Ta$ in d_p and so $b * a^N \to Ta$ in d_p. Since convergence in d_p implies coordinatewise convergence, for each fixed $n \in \mathbb{N}_0$, we have

$$(b * a^N)_n = \Big(b * \sum_{j=0}^N a_j e_j \Big)_n = \Big(\sum_{j=0}^N a_j (b * e_j) \Big)_n \to (Ta)_n.$$

Note, for $N \geq n$, that

$$\Big(\sum_{j=0}^{N} a_j (b * e_j)\Big)_n = \Big(\sum_{j=0}^{n} a_j (b * e_j)\Big)_n = \Big(\sum_{j=0}^{n} a_j S^j b\Big)_n = (b * a)_n.$$

Hence, $(b * a)_n = (Ta)_n$ for $n \in \mathbb{N}_0$, that is, $b * a = Ta$ and so, $b * a \in d_p$. Since $a \in d_p$ is arbitrary, we have $b \in \mathscr{M}(d_p)$ and $T = T_b$.

The reverse inclusion in (3.1) follows easily as $S = T_{e_1} \in \mathscr{M}_{\rm op}(d_p)$.

Since $\mathscr{M}_{\rm op}(d_p)$ is commutative, it is contained in $\mathscr{M}_{\rm op}(d_p)^c$. On the other hand, if $R \in \mathscr{M}_{\rm op}(d_p)^c$, then $S = T_{e_1}$ implies that $RS = SR$ and so, by (3.1), the operator $R \in \mathscr{M}_{\rm op}(d_p)$. Hence, $\mathscr{M}_{\rm op}(d_p) = \mathscr{M}_{\rm op}(d_p)^c$. It then follows that

$$\mathscr{M}_{\rm op}(d_p)^{cc} = (\mathscr{M}_{\rm op}(d_p)^c)^c = \mathscr{M}_{\rm op}(d_p)^c = \mathscr{M}_{\rm op}(d_p),$$

which is precisely (3.2). □

Remark 3.2. (i) For the spaces ℓ^p in place of d_p, with $p \in [1,\infty)$, the identity (3.1) is known, [16, Theorem 2(2)]. Also, for ces_p in place of d_p, with $p \in (1,\infty)$, the same proof as in Proposition 3.1 applies to show that identities (3.1) and (3.2) hold. However, unlike for ℓ^p and d_p, we have the remarkable fact that

$$\mathscr{M}_{\rm op}(ces_p) = \{T_b : b \in \ell^1\}, \quad p \in (1,\infty),$$

and that $\|T_b\|_{ces_p \to ces_p} = \|b\|_1$ for $a \in \ell^1$; see [10, Theorem 4.1].

(ii) In view of (3.2) it is well known that $\mathscr{M}_{\rm op}(d_p)$ is *inverse closed* in $\mathscr{L}(d_p)$, [6, I Proposition 2.3], that is, if $T \in \mathscr{M}_{\rm op}(d_p)$ is invertible in $\mathscr{L}(d_p)$, then its inverse operator $T^{-1} \in \mathscr{L}(d_p)$ actually belongs to $\mathscr{M}_{\rm op}(d_p)$. In particular, the spectrum $\sigma(R; \mathscr{M}_{\rm op}(d_p))$ of an operator $R \in \mathscr{M}_{\rm op}(d_p)$ coincides with its spectrum $\sigma(R; \mathscr{L}(d_p))$ as an element of $\mathscr{L}(d_p)$. For the definition of the spectrum of an element in a unital Banach algebra we refer to [6], [15], for example.

Corollary 3.3. *For each $p \in [1,\infty)$ the algebra $\mathscr{M}_{\rm op}(d_p)$ is closed in $\mathscr{L}(d_p)$ for the weak operator topology and hence, also for the strong operator topology and the operator norm topology. In particular, $\mathscr{M}_{\rm op}(d_p)$ is a commutative Banach algebra (i.e., it is complete).*

Proof. Let $\{T^{(\alpha)}\} \subseteq \mathscr{M}_{\rm op}(d_p)$ be a net and $T \in \mathscr{L}(d_p)$ such that $T^{(\alpha)} \xrightarrow{\alpha} T$ for the weak operator topology. Proposition 3.1 yields $T^{(\alpha)} S = S T^{(\alpha)}$ for all α. Fix $a \in d_p$ and $y^* \in d_p^*$. Then, with $S^* \in \mathscr{L}(d_p^*)$ denoting the adjoint operator of S, we have

$$\begin{aligned}\langle STa, y^*\rangle &= \langle Ta, S^* y^*\rangle = \lim_\alpha \langle T^{(\alpha)} a, S^* y^*\rangle \\ &= \lim_\alpha \langle S T^{(\alpha)} a, y^*\rangle = \lim_\alpha \langle T^{(\alpha)} S a, y^*\rangle = \langle TSa, y^*\rangle.\end{aligned}$$

It follows that $TS = ST$ and hence, that $T \in \mathscr{M}_{\rm op}(d_p)$. □

4. The multiplier algebra $\mathscr{M}(d_p)$

In this section we study various properties of the multiplier algebras $\mathscr{M}(d_p)$. We begin with $p = 1$ which is simpler and is already known. Recall that

$$d_1 := \Big\{ a = (a_n)_{n=0}^\infty \in \mathbb{C}^{\mathbb{N}_0} : \|a\|_{d_1} := \sum_{n=0}^\infty \sup_{k\geq n} |a_k| < \infty \Big\},$$

which can be traced back to the work of Beurling, [3]; see Remark 4.1 below. The canonical vectors $\{e_n : n \in \mathbb{N}_0\}$ form an unconditional basis in d_1. This follows from a necessary condition for a sequence $a = (a_n)_{n=0}^\infty$ to belong to d_p, namely that

$$\lim_n n \sup_{k\geq n} |a_k|^p = 0,$$

which is a consequence of Pringsheim's theorem for convergent series of positive decreasing terms. Indeed, for $a \in d_1$, we have for $N \to \infty$ that

$$\Big\| a - \sum_{n=0}^N a_n e_n \Big\|_{d_1} = \Big\| \Big(\sup_{k\geq N+1} |a_k|, \ldots, \overbrace{\sup_{k\geq N+1} |a_k|}^{\text{position } N+1}, \sup_{k\geq N+2} |a_k|, \ldots \Big\|_{\ell^1}$$
$$= N \sup_{k\geq N+1} |a_k| + \sum_{n=N+1}^\infty \sup_{k\geq n} |a_k| \to 0.$$

The bounded multiplier test ensures the unconditionality of the basis. The space d_1 is known to be an algebra for convolution with unit e_0 (see the proof of [1, Proposition 1]). So, $\mathscr{M}(d_1)$ and d_1 coincide as sets and have equivalent norms, that is, for some $C > 0$ we have

$$\|b\|_{d_1} \leq \|b\|_{\mathscr{M}(d_1)} \leq C\, \|b\|_{d_1}, \quad b \in d_1,$$

where we have used $\|b\|_{d_1} = \|T_b e_0\|_{d_1}$ and (2.5). In particular, $\mathscr{M}(d_1) \subsetneq \ell^1$ (since $|a| \leq \hat{a}$ and [11, Remark 4.20(i)] imply that $d_1 \subsetneq \ell^1$).

Remark 4.1. A result of Beurling concerning the absolute convergence of contracted Fourier series is based on imposing on the Fourier coefficients $(a_n)_{-\infty}^\infty$ of an integrable function on $[0, 2\pi]$ the condition

$$\sum_{n=0}^\infty \sup_{|k|\geq n} |a_k| < \infty,$$

[3, Theorem V]. Note that d_1 corresponds to this condition when $a_n = 0$ for $n < 0$.

The following result already indicates how different the multiplier algebras $\mathscr{M}(d_p)$ and $\mathscr{M}(\ell^p)$ are.

Theorem 4.2. *For each $p \in [1, \infty)$, the following continuous inclusion holds:*

$$\mathscr{M}(d_p) \subseteq \ell^1.$$

Proof. For $p = 1$ this is $d_1 \simeq \mathscr{M}(d_1) \subseteq \ell^1$. For $p \in (1, \infty)$, let $0 \neq b \in \mathscr{M}(d_p)$. Denote by n_0 the smallest $n \in \mathbb{N}_0$ such that $b_n \neq 0$. Fix $n \geq n_0$. For any $a \in d_p$, it follows from (2.2) that

$$\|a * b\|_{d_p}^p \geq 2^n \sup_{2^n \leq k < 2^{n+1}} |(a * b)_k|^p \geq 2^n |(a * b)_{2^n}|^p = 2^n \Big| \sum_{j=0}^{2^n} b_j a_{2^n - j} \Big|^p.$$

Define $a = (a_n)_{n=0}^\infty \in d_p$ via $a_{2^n-j} = |b_j|/b_j$ for $0 \leq j \leq 2^n$ (with $a_{2^n-j} = 0$ if $b_j = 0$) and $a_j = 0$ for $j > 2^n$. Then

$$\sum_{j=0}^{2^n} b_j a_{2^n-j} = \sum_{j=0}^{2^n} |b_j|.$$

Note that $\|a\|_{d_p}^p \leq (2^n + 1)$. Consequently,

$$\|b\|_{\mathscr{M}(d_p)}^p = \sup_{0 \neq a \in d_p} \frac{\|a * b\|_{d_p}^p}{\|a\|_{d_p}^p} \geq \frac{2^n \Big(\sum_{j=0}^{2^n} |b_j| \Big)^p}{2^n + 1} \geq \frac{1}{2} \Big(\sum_{j=0}^{2^n} |b_j| \Big)^p.$$

It follows that $b \in \ell^1$ and $\sum_{j=0}^\infty |b_j| \leq 2^{1/p} \|b\|_{\mathscr{M}(d_p)}$. □

Corollary 4.3. *Let $p \in (1, \infty)$. The following assertions hold.*

(i) $\mathscr{M}(d_p) \subsetneq d_p$.
(ii) $\mathscr{M}(d_p) \neq \ell^1$.

Proof. (i) We have already seen in Section 2 that $\mathscr{M}(d_p) \subseteq d_p$. Let $a = (1/(n+1))_{n=0}^\infty$. Since it is a decreasing sequence and $a \in \ell^p$, we see that $a \in d_p$. However, since $a \notin \ell^1$, we have $a \notin \mathscr{M}(d_p)$. Note that a is the sequence of Taylor coefficients of the analytic function $\log(1 - z) \notin H^\infty(\mathbb{D})$.

(ii) Suppose that $\mathscr{M}(d_p) = \ell^1$. Since $\mathscr{M}(d_p) \subseteq d_p$ this would imply that $\ell^1 \subseteq d_p$, which is not the case; see [5, Remark 2.8(i)]. □

Consider the weight $w_p := ((n+1)^{1/p})_{n=0}^\infty$ and the corresponding weighted ℓ^1-space

$$\ell^1(w_p) := \Big\{ (a_n)_{n=0}^\infty : \sum_{n=0}^\infty (n+1)^{1/p} |a_n| < \infty \Big\},$$

equipped with the norm $\|a\|_{1,w_p} := \sum_{n=0}^\infty (n+1)^{1/p} |a_n|$. Observe that $w_p(m+n) \leq w_p(m) w_p(n)$ for all $m, n \in \mathbb{N}_0$.

Proposition 4.4. *For each $p \in [1, \infty)$ the following continuous embedding holds:*

$$\ell^1(w_p) \subseteq \mathscr{M}(d_p).$$

Proof. Let $m \in \mathbb{N}_0$. The canonical vector $e_m \in d_p$ defines a multiplier in d_p. Indeed, fix $a \in d_p$. Since

$$e_m * a = (\overbrace{0, \dots, 0}^{m}, a_0, a_1, \dots),$$

the least decreasing majorant of $e_m * a$ is

$$(e_m * a)\hat{} = \Big(\overbrace{\sup_{k\geq 0} |a_k|, \dots, \sup_{k\geq 0} |a_k|}^{m+1}, \sup_{k\geq 1} |a_k|, \dots \Big).$$

But, $a \in d_p$ and so $\hat{a} \in \ell^p$. By the previous identity it is clear that $(e_m * a)\hat{} \in \ell^p$ and

$$\big\|e_m * a\big\|_{d_p} = \Big\|(e_m * a)\hat{}\,\Big\|_p = \Big(m\big(\sup_{k\geq 0} |a_k|\big)^p + \|a\|_{d_p}^p\Big)^{1/p}.$$

In particular, $\big\|e_m * a\big\|_{d_p} \leq (m+1)^{1/p}\|a\|_{d_p}$. Consequently, $e_m \in \mathscr{M}(d_p)$ and $\|e_m\|_{\mathscr{M}(d_p)} \leq (m+1)^{1/p}$. This bound is sharp as can be seen by selecting $a = e_0$, in which case $e_m * e_0 = e_m$ with $\hat{e_m} = \sum_{n=0}^m e_n$. So, $\|e_m\|_{\mathscr{M}(d_p)} \geq (m+1)^{1/p}$. Hence, $\|e_m\|_{\mathscr{M}(d_p)} = (m+1)^{1/p}$.

Let $a = (a_n)_{n=0}^\infty \in \ell^1(w_p)$. Consider in $\mathscr{M}(d_p)$ the series $\sum_{n=0}^\infty a_n e_n$. It is absolutely convergent in $\mathscr{M}(d_p)$ because

$$\sum_{n=0}^\infty \|a_n e_n\|_{\mathscr{M}(d_p)} = \sum_{n=0}^\infty |a_n| \|e_n\|_{\mathscr{M}(d_p)} = \sum_{n=0}^\infty |a_n|(n+1)^{1/p} = \|a\|_{1,w_p}.$$

Since the space $\mathscr{M}(d_p) \simeq \mathscr{M}_{\rm op}(d_p)$ is complete (cf. Corollary 3.3), it follows that the series is convergent in $\mathscr{M}(d_p)$. □

Theorem 4.5. *Let $1 \leq p_1 < p_2 < \infty$. Then $\mathscr{M}(d_{p_1}) \subsetneq \mathscr{M}(d_{p_2})$. In particular, $d_1 \subseteq \mathscr{M}(d_p)$ for all $1 \leq p < \infty$.*

Proof. We first show, for $1 \leq p_1 < p_2 < \infty$, that d_{p_2} is an interpolation space between d_{p_1} and ℓ^∞. More precisely, we will show that

$$(d_{p_1})^\theta (\ell^\infty)^{1-\theta} = d_{p_2}, \quad \text{for} \quad \theta := \frac{p_1}{p_2} \in (0,1), \tag{4.1}$$

where $(d_{p_1})^\theta (\ell^\infty)^{1-\theta}$ is a Calderón space, [7, 13.5]. Observe that each space d_p is the Tandori space corresponding to ℓ^p since, in the notation of [13], for $a = (a_n)_{n=0}^\infty \in \ell^\infty$, we have $\widetilde{a} = \hat{a}$, [13, §1]. Recall that $\hat{a}$ is the decreasing majorant of a (cf. §2). Consequently, $\widetilde{\ell^p} = d_p$, for $1 \leq p < \infty$; see [13, (1.6)]. It is clear that $\widetilde{\ell^\infty} = \ell^\infty$.

Theorem 4 in [13] states, for suitable spaces X_0, X_1 and an adequate function φ (cf. [13, §3]), that

$$\varphi(\widetilde{X_0}, \widetilde{X_1}) = [\varphi(X_0, X_1)]\tilde{}.$$

We apply this result to the spaces $X_0 = \ell^{p_1}$, $X_1 = \ell^\infty$ and the function $\varphi(s,t) := s^\theta t^{1-\theta}$ with $\theta := p_1/p_2 \in (0,1)$. Then, $\widetilde{X_0} = d_{p_1}$, $\widetilde{X_1} = \ell^\infty$ and $\varphi(X_0, X_1) = (\ell^{p_1})^\theta (\ell^\infty)^{1-\theta} = \ell^{p_2}$, so that $[\varphi(X_0, X_1)]\tilde{} = d_{p_2}$. Thus, the equality (4.1) follows.

Let $b \in \mathscr{M}(d_{p_1})$. Then $T_b\colon d_{p_1} \to d_{p_1}$. Theorem 4.2 yields that $b \in \ell^1$. This implies, for $a \in \ell^\infty$ and every $n \in \mathbb{N}_0$, that $|(a * b)_n| \le \sum_{j=0}^n |a_j b_{n-j}| \le \|a\|_\infty \|b\|_1$, that is, $T_b a \in \ell^\infty$. Hence, $T_b\colon \ell^\infty \to \ell^\infty$. The equality (4.1) implies that d_{p_2} is a Calderón θ-space for d_{p_1} and ℓ^∞. So, d_{p_2} is an interpolation space between d_{p_1} and ℓ^∞, [7, 33.5]. This yields that $T_b\colon d_{p_2} \to d_{p_2}$, that is, $b \in \mathscr{M}(d_{p_2})$.

To show that $\mathscr{M}(d_{p_1}) \neq \mathscr{M}(d_{p_2})$, let $b = (b_n)_{n=0}^\infty$ be defined by $b_n = 2^{-k/p_1}$ when $n = 2^k$ (for $k \in \mathbb{N}_0$) and $b_n = 0$ otherwise. Since $\frac{1}{p_1} > \frac{1}{p_2}$, it follows that

$$\sum_{n=0}^\infty |b_n|(n+1)^{1/p_2} = \sum_{k=0}^\infty \frac{(2^k+1)^{1/p_2}}{2^{k/p_1}} < \infty,$$

and so $b \in \ell^1(w_{p_2})$. From Proposition 4.4 we have $\ell^1(w_{p_2}) \subseteq \mathscr{M}(d_{p_2})$, that is, $b \in \mathscr{M}(d_{p_2})$. However, $b \notin d_{p_1}$ because

$$\sum_{n=0}^\infty 2^n \sup_{2^n \le k < 2^{n+1}} |b_k|^{p_1} = \sum_{n=0}^\infty 2^n |b_{2^n}|^{p_1} = \sum_{n=0}^\infty \frac{2^n}{(2^{n/p_1})^{p_1}} = \infty.$$

Since $\mathscr{M}(d_{p_1}) \subseteq d_{p_1}$, it follows that $b \notin \mathscr{M}(d_{p_1})$. Hence, $\mathscr{M}(d_{p_1}) \subsetneq \mathscr{M}(d_{p_2})$.

By the discussion prior to Remark 4.1 we have that $d_1 = \mathscr{M}(d_1)$, which implies that $d_1 \subseteq \mathscr{M}(d_p)$ for all $1 \le p < \infty$. □

Remark 4.6. (i) We also refer to [14, §15 p.176] for spaces of the form $X_0^\theta X_1^{1-\theta}$ and [20, Theorem 3] for an interpolation theorem for these spaces.

(ii) In the proof of Theorem 4.5, an alternative way of showing that d_{p_2} is an interpolation space between d_{p_1} and ℓ^∞, for $1 \le p_1 < p_2 < \infty$, is via an interpolation result for Wiener-Beurling spaces. More precisely, Theorem 5.1(i) in [17] applied to $WB^{1/p_1}_{\infty,p_1}(\mathbb{N}_0) = d_{p_1}$, $WB^0_{\infty,\infty}(\mathbb{N}_0) = \ell^\infty$ and $WB^{1/p_2}_{\infty,p_2}(\mathbb{N}_0) = d_{p_2}$ yields $(d_{p_1}, \ell^\infty)_{1-\frac{p_1}{p_2},p_2} = d_{p_2}$.

Let $H(\mathbb{D})$ denote the space of all analytic functions on $\mathbb{D}$. Consider the space of those functions in $H(\mathbb{D})$ whose Taylor coefficients belong to d_p, namely,

$$H(d_p) := \Big\{ f_a(z) := \sum_{n=0}^\infty a_n z^n : (a_n)_{n=0}^\infty \in d_p \Big\} \subseteq H(\mathbb{D}),$$

where the notation f_a indicates that $a = (a_n)_{n=0}^\infty$ is the sequence of Taylor coefficients of f_a. Since $d_p \subseteq \ell^\infty$, it is clear that f_a is indeed analytic in $\mathbb{D}$ for each $a \in d_p$. The norm in $H(d_p)$ is defined by

$$\|f_a\|_{H(d_p)} = \Big\| \sum_{n=0}^\infty a_n z^n \Big\|_{H(d_p)} := \|(a_n)_{n=0}^\infty\|_{d_p}, \quad f_a \in H(d_p).$$

Accordingly, as Banach spaces d_p and $H(d_p)$ are linearly isomorphic and isometric via the map $a \leftrightarrow f_a$. Consequently, the dual space $H(d_p)^*$ of $H(d_p)$ is isomorphic to the space $H(ces_q)$ of analytic functions with Taylor coefficients in ces_q.

Given $z \in \mathbb{D}$ the point evaluation functional δ_z on $H(d_p)$, for $p \in [1, \infty)$, is defined by

$$f_a \in H(d_p) \longmapsto \delta_z(f_a) := f_a(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{C}.$$

Proposition 4.7. *Let $p \in [1, \infty)$. For each $z \in \mathbb{D}$ the functional δ_z on $H(d_p)$ is linear and bounded, that is, $\delta_z \in H(d_p)^*$. For $p \in (1, \infty)$ its norm satisfies*

$$\frac{1/p}{1-|z|}\Big(\sum_{n=0}^{\infty}\Big(\frac{1-|z|^{n+1}}{n+1}\Big)^q\Big)^{1/q} \leq \|\delta_z\|_{H(d_p)^*} \leq \frac{(q-1)^{1/q}}{1-|z|}\Big(\sum_{n=0}^{\infty}\Big(\frac{1-|z|^{n+1}}{n+1}\Big)^q\Big)^{1/q},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. In particular,

$$\frac{1}{p}\zeta(q)^{1/q} \leq \|\delta_z\|_{H(d_p)^*} \leq \frac{(q-1)^{1/q}}{1-|z|}\zeta(q)^{1/q}.$$

For $p = 1$, the functional δ_z acting on $H(d_1)$ has norm one.

Proof. Fix $z \in \mathbb{D}$. Consider $f_a(z) = \sum_{n=0}^{\infty} a_n z^n \in H(d_p)$. Then

$$\delta_z(f_a) = f_a(z) = \sum_{n=0}^{\infty} a_n z^n = \big\langle (z^n)_{n=0}^{\infty}, (a_n)_{n=0}^{\infty}\big\rangle. \tag{4.2}$$

For $p \in (1, \infty)$, we have $a \in d_p$ and $(z^n)_{n=1}^{\infty} \in \ell^q \subseteq ces_q$, which is isomorphic to d_p^*. Thus, δ_z acting on $H(d_p)$ can be identified with the sequence $(z^n)_{n=0}^{\infty} \in (d_p)^*$ acting on d_p. Since $H(d_p)$ and d_p are isometric, the norms of δ_z as an element of $H(d_p)^*$ and of $(z^n)_{n=0}^{\infty}$ as an element of d_p^* coincide. The equivalence of the norms between d_q and $(ces_p)^*$ is given by

$$\frac{1}{q}\|a\|_{d_q} \leq \|a\|_{(ces_p)^*} \leq (p-1)^{1/p}\|a\|_{d_q}, \quad a \in (ces_p)^*, \tag{4.3}$$

where p and q are conjugate indices, i.e., $\frac{1}{p} + \frac{1}{q} = 1$, [2, p. 61 and Corollary 12.17]. From (4.3) it follows that the equivalence of the norms between $(d_p)^*$ and ces_q is given by

$$\frac{1}{p}\|a\|_{ces_q} \leq \|a\|_{(d_p)^*} \leq (q-1)^{1/q}\|a\|_{ces_q}, \quad a \in (d_p)^*.$$

In our case this yields

$$\frac{1}{p}\|(z^n)_{n=0}^{\infty}\|_{ces_q} \leq \|\delta_z\|_{H(d_p)^*} \leq (q-1)^{1/q}\|(z^n)_{n=0}^{\infty}\|_{ces_q}. \tag{4.4}$$

The norm of $(z^n)_{n=0}^{\infty}$ in ces_q is given by

$$\|(z^n)_{n=0}^{\infty}\|_{ces_q}^q = \sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n}|z^k|\right)^q = \frac{1}{(1-|z|)^q}\sum_{n=0}^{\infty}\left(\frac{1-|z|^{n+1}}{n+1}\right)^q.$$

Since

$$(1-|z|)^q\sum_{n=0}^{\infty}\frac{1}{(n+1)^q} \leq \sum_{n=0}^{\infty}\left(\frac{1-|z|^{n+1}}{n+1}\right)^q \leq \sum_{n=0}^{\infty}\frac{1}{(n+1)^q},$$

we can conclude that

$$\zeta(q) \leq \|(z^n)_{n=0}^\infty\|_{ces_q}^q \leq \frac{\zeta(q)}{(1-|z|)^q}.$$

The claim now follows from (4.4).

For $p = 1$, from (4.2) we have $a \in d_1$ and $(z^n)_{n=1}^\infty \in ces_\infty$, which is isometric to d_1^*, [10, Remark 6.3]. Thus, δ_z acting on $H(d_1)$ can be identified with the sequence $(z^n)_{n=0}^\infty \in (d_1)^*$ acting on d_1. Hence, the norm of δ_z equals the norm of $(z^n)_{n=0}^\infty$ in ces_∞, that is,

$$\|(z^n)_{n=0}^\infty\|_{ces_\infty} = \sup_{n\geq 0} \frac{1}{n+1} \sum_{k=0}^n |z|^k = 1.$$

□

In view of the proof of the above result and the isomorphism $d_p \simeq H(d_p)$, it is clear, for each $z \in \mathbb{D}$, that $\delta_z \in H(d_p)^*$ corresponds to the element of d_p^* given by $a \mapsto \sum_{n=0}^\infty a_n z^n$, for $a \in d_p$.

The Taylor coefficients of the pointwise product of two analytic functions f_a and f_b in $\mathbb{D}$ are obtained via the convolution of a and b, that is, $f_a f_b = f_{a*b}$. Consequently, the space

$$\mathscr{M}(H(d_p)) := \Big\{\varphi \in H(\mathbb{D}) : \varphi f \in H(d_p), \forall f \in H(d_p)\Big\}$$

of analytic multipliers for $H(d_p)$ is linearly isomorphic and isometric to the space $H(\mathscr{M}(d_p))$ of analytic functions on $\mathbb{D}$ with Taylor coefficients in the algebra $\mathscr{M}(d_p)$, that is, to the algebra

$$H(\mathscr{M}(d_p)) := \Big\{\varphi_a(z) = \sum_{n=0}^\infty a_n z^n : (a_n)_{n=0}^\infty \in \mathscr{M}(d_p)\Big\} \subseteq H(\mathbb{D})$$

equipped with the norm $\|\varphi_a\|_{H(\mathscr{M}(d_p))} := \|a\|_{\mathscr{M}(d_p)}$. Note the identification between $\mathscr{M}(H(d_p))$ and $H(\mathscr{M}(d_p))$. Observe that $H(\mathscr{M}(d_p)) \subseteq H(d_p)$ because $\mathscr{M}(d_p) \subseteq d_p$.

With obvious notation (that is, interchanging $d_p \leftrightarrow \ell^p$) it is known that

$$\ell^1 \subseteq \mathscr{M}(\ell^p) \simeq \mathscr{M}(H(\ell^p)) \subseteq H^\infty(\mathbb{D}), \quad 1 < p < \infty, \tag{4.5}$$

where $H^\infty(\mathbb{D})$ is the space of all bounded analytic functions on $\mathbb{D}$, [16, Theorem 4]. The containment in the right-side of (4.5) can be sharpened when we consider d_p in place of ℓ^p. This is because $f_a(z) = \sum_{n=0}^\infty a_n z^n \in H(\mathscr{M}(d_p))$ implies, via Theorem 4.2, that $a = (a_n)_{n=0}^\infty \in \ell^1$, and so in (4.5) we can replace the space $H^\infty(\mathbb{D})$ by the classical (one-sided) analytic Wiener algebra, [15, §11.6], denoted by ℓ^1_A in [16], consisting of all analytic functions on $\mathbb{D}$ with absolutely convergent Taylor coefficients. That is,

$$d_1 \subseteq \mathscr{M}(d_p) \simeq \mathscr{M}(H(d_p)) \subseteq \ell^1_A, \quad 1 < p < \infty.$$

5. Subspaces of $\mathscr{M}(d_p)$

Theorem 4.2 shows for $b \in \mathbb{C}^{\mathbb{N}_0}$ that a necessary condition for being a multiplier for d_p is that $b \in \ell^1$. This fact allows the formulation of a necessary and sufficient condition for $b \in \ell^1$ to belong to $\mathscr{M}(d_p)$, which has the advantage that, for each $n \in \mathbb{N}_0$, in the n-th term of the series in (5.1) below only the terms b_j for $2^{n-1} < j < 2^{n+1}$ occur.

Theorem 5.1. *Let $p \in (1,\infty)$ and $b \in \ell^1$. Then $b \in \mathscr{M}(d_p)$ if and only if*

$$\sum_{n=0}^{\infty} 2^n \sup_{2^n \le k < 2^{n+1}} \Big| \sum_{\frac{k}{2} < j \le k} b_j a_{k-j} \Big|^p < \infty, \quad a \in d_p. \tag{5.1}$$

Proof. Recall that $b \in \mathscr{M}(d_p)$ if and only if $a * b \in d_p$, for every $a \in d_p$. This is equivalent, via (2.2), to

$$\sup_{n\ge 0} |(a*b)_n|^p + \sum_{n=0}^{\infty} 2^n \sup_{2^n \le k < 2^{n+1}} \big|(a*b)_k\big|^p < \infty, \quad a \in d_p.$$

Since $b \in \ell^1$, given any $a \in d_p \subseteq \ell^p$ it follows that $a * b \in \ell^p$ and so, $a * b$ is bounded. Hence, $b \in \mathscr{M}(d_p)$ if and only if

$$\sum_{n=0}^{\infty} 2^n \sup_{2^n \le k < 2^{n+1}} \big|(a*b)_k\big|^p < \infty, \quad a \in d_p. \tag{5.2}$$

First assume that the condition (5.1) is satisfied. To prove that $b \in \mathscr{M}(d_p)$ it suffices to establish (5.2). Let $a \in d_p$. Then, for each $k \in \mathbb{N}_0$, we have

$$\begin{aligned} \big|(a*b)_k\big| &= \Big|\sum_{j=0}^{k} b_j a_{k-j}\Big| = \Big| \sum_{0 \le j \le \frac{k}{2}} b_j a_{k-j} + \sum_{\frac{k}{2} < j \le k} b_j a_{k-j} \Big| \\ &\le \Big(\sum_{0 \le j \le \frac{k}{2}} |b_j| \Big) \sup_{0 \le j \le \frac{k}{2}} |a_{k-j}| + \Big| \sum_{\frac{k}{2} < j \le k} b_j a_{k-j} \Big| \\ &\le \|b\|_1 \sup_{\frac{k}{2} \le j \le k} |a_j| + \Big| \sum_{\frac{k}{2} < j \le k} b_j a_{k-j} \Big|. \end{aligned} \tag{5.3}$$

Fix $n \in \mathbb{N}_0$. It follows from (5.3) that

$$\begin{aligned} \sup_{2^n \le k < 2^{n+1}} \big|(a*b)_k\big|^p &= \Big(\sup_{2^n \le k < 2^{n+1}} \big|(a*b)_k\big| \Big)^p \\ &\le \Big(\sup_{2^n \le k < 2^{n+1}} \|b\|_1 \sup_{\frac{k}{2} \le j \le k} |a_j| + \sup_{2^n \le k < 2^{n+1}} \Big| \sum_{\frac{k}{2} < j \le k} b_j a_{k-j} \Big| \Big)^p \\ &= \Big(\|b\|_1 \sup_{2^{n-1} \le k < 2^{n+1}} |a_j| + \sup_{2^n \le k < 2^{n+1}} \Big| \sum_{\frac{k}{2} < j \le k} b_j a_{k-j} \Big| \Big)^p. \end{aligned} \tag{5.4}$$

The inequality (5.4) implies that

$$\sum_{n=0}^{\infty} 2^n \sup_{2^n\le k<2^{n+1}} \big|(a*b)_k\big|^p \le \sum_{n=0}^{\infty} 2^n \Big(\|b\|_1 \sup_{2^{n-1}\le k<2^{n+1}} |a_j| + \sup_{2^n\le k<2^{n+1}} \Big| \sum_{\frac{k}{2}<j\le k} b_j a_{k-j} \Big| \Big)^p$$

Applying Minkowski's inequality yields

$$\Big(\sum_{n=0}^{\infty} 2^n \sup_{2^n\le k<2^{n+1}} \big|(a*b)_k\big|^p \Big)^{1/p} \le \Big(\sum_{n=0}^{\infty} 2^n \big(\|b\|_1 \sup_{2^{n-1}\le k<2^{n+1}} |a_j| \big)^p \Big)^{1/p} + \Big(\sum_{n=0}^{\infty} 2^n \sup_{2^n\le k<2^{n+1}} \Big| \sum_{\frac{k}{2}<j\le k} b_j a_{k-j} \Big|^p \Big)^{1/p}. \tag{5.5}$$

The second term in the right-side of (5.5) is finite because of (5.1). Regarding the first term in the right-side of (5.5), note that

$$\begin{aligned} \sum_{n=0}^{\infty} 2^n \sup_{2^{n-1}\le k<2^{n+1}} |a_j|^p &\le \sum_{n=0}^{\infty} 2^n \sup_{2^{n-1}\le k<2^{n}} |a_j|^p + \sum_{n=0}^{\infty} 2^n \sup_{2^n\le k<2^{n+1}} |a_j|^p \\ &= 2\sum_{n=0}^{\infty} 2^{n-1} \sup_{2^{n-1}\le k<2^{n}} |a_j|^p + \sum_{n=0}^{\infty} 2^n \sup_{2^n\le k<2^{n+1}} |a_j|^p \\ &\le 3\sum_{n=0}^{\infty} 2^n \sup_{2^n\le k<2^{n+1}} |a_j|^p. \end{aligned} \tag{5.6}$$

Then

$$\Big(\sum_{n=0}^{\infty} 2^n \big(\|b\|_1 \sup_{2^{n-1}\le k<2^{n+1}} |a_j| \big)^p \Big)^{1/p} \le \|b\|_1 3^{1/p} \Big(\sum_{n=0}^{\infty} 2^n \sup_{2^n\le k<2^{n+1}} |a_j|^p \Big)^{1/p}, \tag{5.7}$$

which is also finite since $b \in \ell^1$ and $a \in d_p$. Hence, (5.2) is finite for every $a \in d_p$ and so, $b \in \mathscr{M}(d_p)$.

Conversely, we need to show that condition (5.1) is necessary. So, assume that $b \in \mathscr{M}(d_p)$. Fix $a \in d_p$. Then

$$\begin{aligned} \sum_{n=0}^{\infty} 2^n \sup_{2^n\le k<2^{n+1}} \Big| \sum_{\frac{k}{2}<j\le k} b_j a_{k-j} \Big|^p &= \sum_{n=0}^{\infty} 2^n \sup_{2^n\le k<2^{n+1}} \Big| \sum_{0\le j\le k} b_j a_{k-j} - \sum_{0\le j\le \frac{k}{2}} b_j a_{k-j} \Big|^p \\ &\le \sum_{n=0}^{\infty} 2^n \sup_{2^n\le k<2^{n+1}} \Big(\big|(a*b)_k\big| + \Big| \sum_{0\le j\le \frac{k}{2}} b_j a_{k-j} \Big| \Big)^p \end{aligned}$$

$$\leq \sum_{n=0}^{\infty} 2^n \sup_{2^n \leq k < 2^{n+1}} \Big(\big|(a*b)_k\big| + \Big(\sum_{0\leq j\leq \frac{k}{2}} |b_j| \Big) \sup_{0 \leq j \leq \frac{k}{2}} |a_{k-j}| \Big)^p$$

$$\leq \sum_{n=0}^{\infty} 2^n \sup_{2^n \leq k < 2^{n+1}} \Big(\big|(a*b)_k\big| + \|b\|_1 \sup_{\frac{k}{2} \leq j \leq k} |a_j| \Big)^p$$

$$\leq \sum_{n=0}^{\infty} 2^n \Big(\sup_{2^n \leq k < 2^{n+1}} \big|(a*b)_k\big| + \|b\|_1 \sup_{2^n \leq k < 2^{n+1}} \sup_{\frac{k}{2} \leq j \leq k} |a_j| \Big)^p$$

$$= \sum_{n=0}^{\infty} 2^n \Big(\sup_{2^n \leq k < 2^{n+1}} \big|(a*b)_k\big| + \|b\|_1 \sup_{2^{n-1} \leq k < 2^{n+1}} |a_j| \Big)^p.$$

Minkowski's inequality and (5.6) yield

$$\Big(\sum_{n=0}^{\infty} 2^n \sup_{2^n \leq k < 2^{n+1}} \Big| \sum_{\frac{k}{2} < j \leq k} b_j a_{k-j} \Big|^p \Big)^{1/p} \leq \Big(\sum_{n=0}^{\infty} 2^n \sup_{2^n \leq k < 2^{n+1}} \big|(a*b)_k\big|^p \Big)^{1/p}$$
$$+ \|b\|_1 \Big(\sum_{n=0}^{\infty} 2^n \sup_{2^{n-1} \leq k < 2^{n+1}} |a_j|^p \Big)^{1/p}$$
$$\leq \|a*b\|_{d_p} + 3\|b\|_1 \|a\|_{d_p}.$$

So, (5.1) holds. □

The equivalent norms for d_p given in (2.2) and (2.3) suggest, for each $1 \leq p < \infty$, to introduce the sequence space

$$d_{pp} := \Big\{ a = (a_n)_{n=0}^{\infty} \in \mathbb{C}^{\mathbb{N}_0} : \sum_{n=0}^{\infty} 2^{np} \sup_{2^n \leq k < 2^{n+1}} |a_k|^p < \infty \Big\}, \tag{5.8}$$

equipped with the norm

$$\|a\|_{d_{pp}} := \Big(\sup_{k \geq 0} |a_k|^p + \sum_{n=0}^{\infty} 2^{np} \sup_{2^n \leq k < 2^{n+1}} |a_k|^p \Big)^{1/p}, \quad a \in d_{pp}. \tag{5.9}$$

The canonical vectors $\{e_n : n \in \mathbb{N}_0\}$ form an unconditional basis in d_{pp}. To see this fix $a = (a_n)_{n=0}^{\infty} \in d_{pp}$. For each $N \in \mathbb{N}_0$ let $n_0 \in \mathbb{N}_0$ satisfy $2^{n_0} \leq N < 2^{n_0+1}$. Then, for $N \to \infty$, we have

$$\Big\| a - \sum_{n=0}^{N} a_n e_n \Big\|_{d_{pp}}^p \leq \sup_{k > N} |a_k|^p + \sum_{n > n_0} 2^{np} \sup_{2^n \leq k < 2^{n+1}} |a_k|^p \to 0.$$

The bounded multiplier test ensures the unconditionality of the basis.

Theorem 5.2. *Let $p \in [1, \infty)$. Then $d_{pp} \cap \ell^1 \subsetneq \mathscr{M}(d_p)$ with a continuous inclusion.*

Proof. Since $\mathscr{M}(d_1) = d_1 = d_{11}$, we only need to consider the case when $p \in (1, \infty)$. Fix $b \in d_{pp} \cap \ell^1$. We apply Theorem 5.1 by verifying that (5.1) holds. Given $a \in d_p$ we have

$$\Big| \sum_{\frac{k}{2} < j \leq k} b_j a_{k-j} \Big| \leq \Big(\sum_{j=0}^{k/2} |a_j| \Big) \sup_{\frac{k}{2} < j \leq k} |b_j|, \quad k \geq 1,$$

and so Hölder's inequality together with $d_p \subseteq \ell^p$ yields

$$\begin{aligned} \sup_{2^n \leq k < 2^{n+1}} \Big| \sum_{\frac{k}{2} < j \leq k} b_j a_{k-j} \Big|^p &\leq \sup_{2^n \leq k < 2^{n+1}} \Big(\sum_{j=0}^{k/2} |a_j| \Big)^p \sup_{\frac{k}{2} < j \leq k} |b_j|^p \\ &\leq \Big(\sum_{j=0}^{2^n - 1} |a_j| \Big)^p \sup_{2^{n-1} < j < 2^{n+1}} |b_j|^p \\ &\leq 2^{n(p/q)} \|a\|_{d_p}^p \sup_{2^{n-1} \leq j < 2^{n+1}} |b_j|^p. \end{aligned}$$

Hence, arguing as in (5.6), it follows that

$$\begin{aligned} \sum_{n=0}^{\infty} 2^n \sup_{2^n \leq k < 2^{n+1}} \Big| \sum_{\frac{k}{2} < j \leq k} b_j a_{k-j} \Big|^p &\leq \sum_{n=0}^{\infty} 2^n 2^{n(p/q)} \|a\|_{d_p}^p \sup_{2^{n-1} \leq j < 2^{n+1}} |b_j|^p \\ &= \|a\|_{d_p}^p \sum_{n=0}^{\infty} 2^{np} \sup_{2^{n-1} \leq j < 2^{n+1}} |b_j|^p \\ &\leq 3 \|a\|_{d_p}^p \sum_{n=0}^{\infty} 2^{np} \sup_{2^n \leq j < 2^{n+1}} |b_j|^p < \infty, \end{aligned} \tag{5.10}$$

which is finite since $b \in d_{pp}$. So, $d_{pp} \cap \ell^1 \subseteq \mathscr{M}(d_p)$.

In view of (5.5) and (5.9), it follows from (5.7) and (5.10) that there exists a constant $K > 0$ such that

$$\|b * a\|_{d_p} \leq K \|a\|_{d_p} \max\big\{ \|b\|_1, \|b\|_{d_{pp}} \big\}, \quad a \in d_p.$$

Since the space $d_{pp} \cap \ell^1$ is normed by $\|b\|_{d_{pp} \cap \ell^1} := \max\{\|b\|_1, \|b\|_{d_{pp}}\}$, it follows that the natural inclusion $d_{pp} \cap \ell^1 \subseteq \mathscr{M}(d_p)$ is continuous.

It remains to show that there exists $b \in \mathscr{M}(d_p) \setminus d_{pp}$. Consider $b = (b_n)_{n=0}^{\infty}$ defined by $b_n = 1/n$ for $n = 2^k$ with $k \in \mathbb{N}_0$, and $b_n = 0$ elsewhere. Then $b \notin d_{pp}$ since

$$\sum_{n=0}^{\infty} 2^{np} \sup_{2^n \leq k < 2^{n+1}} |b_k|^p = \sum_{n=0}^{\infty} 2^{np} \Big(\frac{1}{2^n} \Big)^p = \infty.$$

However, $b \in \mathscr{M}(d_p)$. Indeed, via Theorem 5.1 and the fact that $b \in \ell^1$ we have

$$\sum_{n=0}^{\infty} 2^n \sup_{2^n \le k < 2^{n+1}} \Big| \sum_{\frac{k}{2} < j \le k} b_j a_{k-j} \Big|^p = \sum_{n=0}^{\infty} 2^n \Big| \frac{a_0}{2^n} \Big|^p < \infty, \quad a \in d_p.$$

□

The containment $d_1 \subseteq d_{pp}$ follows directly from (5.8) because of (2.2), (5.9) and

$$\sum_{n=0}^{\infty} 2^{np} \sup_{2^n \le j < 2^{n+1}} |a_j|^p \le \Big(\sum_{n=0}^{\infty} 2^n \sup_{2^n \le j < 2^{n+1}} |a_j| \Big)^p.$$

Thus, Theorem 5.1 and the fact that $d_1 = \mathscr{M}(d_1)$ imply the following result (a strengthening of part of Theorem 4.5).

Corollary 5.3. *Let $p \in [1, \infty)$. The following continuous inclusion holds:*

$$d_1 \subseteq \mathscr{M}(d_p).$$

Let $H(\overline{\mathbb{D}})$ denote the algebra, under pointwise multiplication, of all $\mathbb{C}$-valued functions which are holomorphic in some open set containing $\overline{\mathbb{D}}$.

Corollary 5.4. *Let $p \in [1, \infty)$. The following inclusions hold:*

$$\Big\{ b = (b_n)_{n=0}^{\infty} : f_b \in H(\overline{\mathbb{D}}) \Big\} \subseteq d_1 \subseteq \mathscr{M}(d_p).$$

Proof. Given $f_b \in H(\overline{\mathbb{D}})$, the power series of f_b has radius of convergence $r > 1$ and so its Taylor coefficients satisfy $|b_n| \le C/r^n$, for some $C > 0$ and all $n \in \mathbb{N}_0$. Hence, $b \in d_1 \subseteq \mathscr{M}(d_p)$ for all $p \in [1, \infty)$. □

Corollary 5.5. *Let $p \in [1, \infty)$. For $b = (b_n)_{n=0}^{\infty}$ belonging to any one of the spaces $\ell^1(w_p)$ or $d_{pp} \cap \ell^1$ or d_1, it is the case, for $N \to \infty$, that*

$$\Big\| b - \sum_{n=0}^{N} b_n e_n \Big\|_{\mathscr{M}(d_p)} \to 0.$$

Equivalently,

$$\Big\| T_b - \sum_{n=0}^{N} b_n S^n \Big\|_{\mathscr{M}_{\mathrm{op}}(d_p)} \to 0.$$

Proof. The sequence $\{e_n : n \in \mathbb{N}_0\}$ is a basis for each of these spaces. This, together with Proposition 4.4, Theorem 5.2 and Corollary 5.3, proves the result. □

Remark 5.6. We compare the various subspaces of $\mathscr{M}(d_p)$ which have already appeared.

(i) For every $p \in [1, \infty)$ the spaces d_1 and $\ell^1(w_p)$ are different. Indeed, $b = (b_n)_{n=0}^{\infty}$ given by $b_n := 1/(n+1)^{1+\frac{1}{p}}$, for $n \in \mathbb{N}_0$, satisfies $b \in d_1$ but $b \notin \ell^1(w_p)$. So, $b \in \mathscr{M}(d_p) \setminus \ell^1(w_p)$. On the other hand, the example b in the proof of Theorem 5.2 satisfies $b \in \ell^1(w_p)$ but $b \notin d_1$ as $b \notin d_{pp}$. So, $b \in \mathscr{M}(d_p) \setminus d_1$.

(ii) For every $p \in (1, \infty)$ we have $d_{pp} \subsetneq d_p$. The containment is direct from (2.2) and (5.8). To see that it is strict, consider again the example b in the proof of Theorem 5.2. Then $b \in d_p$ but $b \notin d_{pp}$.

(iii) For every $p \in (1, \infty)$ we have $\ell^1 \not\subseteq d_{pp}$. The proof of Corollary 4.3(ii) yields $\ell^1 \not\subseteq d_p$. To see that $d_{pp} \not\subseteq \ell^1$, consider $b = (b_n)_{n=0}^{\infty}$ with $b_0 = 0$ and $b_n = 1/(k2^k)$ when $2^k \leq n < 2^{k+1}$ and $k \in \mathbb{N}_0$. Then $b \in d_{pp}$ but $b \notin \ell^1$. This sequence b shows that $d_1 \subsetneq d_{pp} \cap \ell^1$, since it satisfies $b \in d_{pp} \cap \ell^1$ and $b \notin d_1$.

(iv) For every $p \in [1, \infty)$ the spaces d_{pp} and $\ell^1(w_p)$ are different. Indeed, $b = (b_n)_{n=0}^{\infty}$ given by $b_n := 1/(n+1)^{1+\frac{1}{p}}$, for $n \in \mathbb{N}_0$, satisfies $b \in d_{pp}$ but $b \notin \ell^1(w_p)$. On the other hand, the example b in the proof of Theorem 5.2 satisfies $b \in \ell^1(w_p)$ and $b \notin d_{pp}$.

6. Spectral properties of $\mathscr{M}(d_p)$

It was noted in Section 1 that the multiplier algebra $\mathscr{M}(ces_p) = \ell^1$ for every $1 < p < \infty$. For elements $b \in \ell^1$, the spectrum of the corresponding operator $T_b \in \mathscr{L}(ces_p)$ is precisely known, [18, Theorem 2]. The proof requires a knowledge of the spectrum of the right-shift $S \in \mathscr{L}(ces_p)$, which is identified in [18, Proposition 6]. The aim of this section is to investigate the spectrum of multiplier operators $T_b \in \mathscr{M}(d_p)$ for $1 \leq p < \infty$. Due to the more involved nature of the Banach algebras $\mathscr{M}(d_p)$ this is significantly more complicated than the situation for ces_p. We begin with the right-shift $S \in \mathscr{L}(d_p)$. The spectrum of $S \in \mathscr{L}(d_p)$ is well known, [9, VII Proposition 6.5].

Proposition 6.1. *Let $p \in [1, \infty)$. The right-shift operator $S \colon d_p \to d_p$ satisfies*

$$\sigma(S; \mathscr{L}(d_p)) = \overline{\mathbb{D}}. \tag{6.1}$$

Moreover, the point spectrum

$$\sigma_{pt}(S; \mathscr{L}(d_p)) = \emptyset$$

and the residual spectrum satisfies

$$\mathbb{D} \subseteq \sigma_r(S; \mathscr{L}(d_p)).$$

Whenever $p \in (1, \infty)$, the continuous spectrum satisfies

$$\sigma_c(S; \mathscr{L}(d_p)) = \overline{\mathbb{D}} \setminus \mathbb{D}. \tag{6.2}$$

Proof. The proof proceeds via a series of steps. All steps, but for for the last one, concern $p \in [1, \infty)$.

Step 1. We have that

$$\sigma_{\mathrm{pt}}(S; \mathscr{L}(d_p)) = \emptyset.$$

To prove this, suppose that $\lambda \in \sigma_{\mathrm{pt}}(S; \mathscr{L}(d_p))$. Then there exist $0 \neq a \in d_p$ such that $Sa = \lambda a$. Since $a \in \ell^p$ this implies that a is an eigenvalue of $S \colon \ell^p \to \ell^p$. This cannot be since $\sigma_{\mathrm{pt}}(S; \mathscr{L}(\ell^p)) = \emptyset$; see [9, Proposition VII.6.5].

Step 2. For the range $R(S - \lambda I)$ of $S - \lambda I$ it is the case that

$$e_0 \notin R(S - \lambda I) \subseteq d_p, \quad \lambda \in \overline{\mathbb{D}}.$$

To prove this, fix $\lambda \in \overline{\mathbb{D}}$. Suppose there exists $a \in d_p$ such that $(S-\lambda I)a = e_0$. Necessarily $a \neq 0$. If $\lambda = 0$, then $Sa = e_0$, which is impossible. For $0 < |\lambda| \leq 1$ we have

$$-\lambda a_0 = 1, \quad -\lambda a_{n+1} = a_n, \quad n \in \mathbb{N}_0.$$

Proceeding recursively yields $a_n = 1/\lambda^{n+1}$ for $n \in \mathbb{N}_0$. But, then $a \notin d_p$ as $1/|\lambda| \geq 1$.

Step 3. The same calculations as in Step 2, for ℓ^p in place of d_p and the right-shift operator $S \in \mathscr{L}(\ell^p)$ show that

$$e_0 \notin R(S - \lambda I) \subseteq \ell^p, \ \ \lambda \in \overline{\mathbb{D}}.$$

Step 4. For each $\lambda \in \mathbb{D}$, it is the case that

$$e_0 \notin \overline{R(S - \lambda I)} \subseteq d_p,$$

where the bar denotes closure. To prove this, fix $\lambda \in \mathbb{D}$. Suppose, on the contrary, that there exists a sequence $\{a^m\}_{m=0}^{\infty} \subseteq d_p$ such that $(S - \lambda I)a^m \to e_0$ in d_p. Then also $e_0 \in \ell^p$ and the sequence $\{a^m\}_{m=0}^{\infty} \subseteq \ell^p$ satisfies $(S - \lambda I)a^m \to e_0$ in ℓ^p. But, the range $R(S - \lambda I)$ is *closed* in ℓ^p; see Proposition VII.6.5 in [9]. Hence, $e_0 \in R(S - \lambda I) \subseteq \ell^p$ which contradicts Step 3.

Step 5. For the residual spectrum we have the inclusion

$$\mathbb{D} \subseteq \sigma_{\mathrm{r}}(S; \mathscr{L}(d_p)).$$

To prove this note, by Step 1, that $S - \lambda I$ is injective for every $\lambda \in \mathbb{D}$. Accordingly, for each $\lambda \in \mathbb{D}$, Step 4 shows that $\overline{R(S - \lambda I)} \neq d_p$ and hence, that $\lambda \in \sigma_{\mathrm{r}}(S; \mathscr{L}(d_p))$.

Step 6. The claim is that

$$\sigma(S; \mathscr{L}(d_p)) \subseteq \overline{\mathbb{D}}.$$

To prove this, recall that $\|S^n\|_{\mathscr{L}(d_p)} = (n+1)^{1/p}$ for $n \in \mathbb{N}_0$. Accordingly, the spectral radius $r(S) = \lim_n \|S^n\|_{\mathscr{L}(d_p)}^{1/n} = 1$ from which the result follows, [6, I Theorem 5.8].

Step 7. The identity (6.1) is valid, that is,

$$\sigma(S; \mathscr{L}(d_p)) = \overline{\mathbb{D}}.$$

To prove this, note that Steps 5 and 6 yield

$$\mathbb{D} \subseteq \sigma_{\mathrm{r}}(S; \mathscr{L}(d_p)) \subseteq \sigma(S; \mathscr{L}(d_p)) \subseteq \overline{\mathbb{D}}.$$

Since the spectrum of S is a closed set in $\mathbb{C}$ the desired conclusion follows.

Step 8. For every $\lambda \in \mathbb{C} \setminus \{0\}$ it is the case that

$$\Big\{ -\lambda e_0 + \frac{1}{\lambda^n} e_{n+1} : n \in \mathbb{N}_0 \Big\} \subseteq R(S - \lambda I) \subseteq d_p.$$

To verify this define, for each $n \in \mathbb{N}_0$, the element

$$a^{[n]} := \sum_{j=0}^{n} \frac{1}{\lambda^j} e_j = \Big(1, \frac{1}{\lambda}, \dots, \overbrace{\frac{1}{\lambda^n}}^{\text{position } n+1}, 0, \dots\Big) \in d_p.$$

Direct calculation yields

$$(S-\lambda I)a^{[n]}=\Big(-\lambda,0,\dots,0,\overbrace{\frac{1}{\lambda^n}}^{\text{position } n+2},0,\dots\Big)=-\lambda e_0+\frac{1}{\lambda^n}e_{n+1}.$$

Step 9. Consider now $p\in(1,\infty)$. Then

$$\sigma_{\mathrm{c}}(S;\mathscr{L}(d_p))=\overline{\mathbb{D}}\setminus\mathbb{D}.$$

To prove this, recall that $d_p^*=ces_q$, with $\frac{1}{p}+\frac{1}{q}=1$. Fix $\lambda\in\overline{\mathbb{D}}\setminus\mathbb{D}$. Let $y^*=(y_n)_{n=0}^\infty\in d_p^*$ satisfy

$$\Big\langle -\lambda e_0+\frac{1}{\lambda^n}e_{n+1},y^*\Big\rangle=0,\quad n\in\mathbb{N}_0. \tag{6.3}$$

Substituting $n=0,1,\dots$ successively into (6.3) yields $y_n=\lambda^n y_0$, for all $n\in\mathbb{N}_0$, and so $y^*=(y_0\lambda^n)_{n=0}^\infty$. Then $|y^*|=(|y_0|)_{n=0}^\infty\in d_p^*=ces_q$. The definition of ces_q in (2.4) implies that $|y^*|=\mathcal{C}|y^*|\in\ell^q$ which implies that $y_0=0$, that is, $y^*=0$.

Now let $y^*\in d_p^*$ satisfy $\langle a,y^*\rangle=0$ for all $a\in R(S-\lambda I)$. According to Step 8, y^* also satisfies (6.3) and hence, $y^*=0$. It follows that $\overline{R(S-\lambda I)}=d_p$. Since $\lambda\in\sigma(S;\mathscr{L}(d_p))$, due to Step 7, and $S-\lambda I$ is injective (see Step 1), we can conclude that $\lambda\in\sigma_{\mathrm{c}}(S;\mathscr{L}(d_p))$. That is, $\overline{\mathbb{D}}\setminus\mathbb{D}\subseteq\sigma_{\mathrm{c}}(S;\mathscr{L}(d_p))$. Now Steps 5 and 7 yield $\sigma_{\mathrm{c}}(S;\mathscr{L}(d_p))=\overline{\mathbb{D}}\setminus\mathbb{D}$.

The proof is thereby complete. □

The omission of $p=1$ in (6.2) is necessary, as seen by the following result.

Proposition 6.2. *For $p=1$ we have that*

$$\sigma(S;\mathscr{L}(d_1))=\sigma_r(S;\mathscr{L}(d_1))=\overline{\mathbb{D}}.$$

In particular,

$$\sigma_{pt}(S;\mathscr{L}(d_1))=\sigma_c(S;\mathscr{L}(d_1))=\emptyset.$$

Proof. According to Proposition 6.1 we only need to show that if $|\lambda|=1$, then $\lambda\in\sigma_{\mathrm{r}}(S;\mathscr{L}(d_1))$. Recall that $d_1^*=(ces_0)^{**}=ces_\infty$, [10, Remark 6.3]. Set $y^*:=(\lambda^n)_{n=0}^\infty$. Observe that $|y^*|=(1)_{n=0}^\infty$ and, for $\mathcal{C}$ the Cesàro averaging operator, that $\mathcal{C}|y^*|=(1)_{n=0}^\infty\in\ell^\infty$. Hence, by definition $y^*\in ces_\infty=d_1^*$.

Let $a\in d_1$ be arbitrary. Then

$$\begin{aligned}\langle (S-\lambda I)a,y^*\rangle&=\big\langle(-\lambda a_0,a_0-\lambda a_1,a_1-\lambda a_2,\dots),(1,\lambda,\lambda^2,\dots)\big\rangle\\&=-\lambda a_0+\lambda(a_0-\lambda a_1)+\lambda^2(a_1-\lambda a_2)+\cdots\\&=0.\end{aligned}$$

That is, $y^*\neq 0$ in d_1^* satisfies $\langle u,y^*\rangle=0$ for all $u\in R(S-\lambda I)\subseteq d_1$. Accordingly, $\overline{R(S-\lambda I)}\neq d_1$. Since $S-\lambda I$ is injective, we can conclude that $\lambda\in\sigma_{\mathrm{r}}(S;\mathscr{L}(d_1))$. □

The above knowledge of the spectrum for the right-shift operator has implications for other multipliers. Given $f\in H(\overline{\mathbb{D}})$, let $b_f=(b_n)_{n=0}^\infty$ denote the sequence of its Taylor coefficients.

Proposition 6.3. *Let $p \in [1, \infty)$. For every $f \in H(\overline{\mathbb{D}})$ we have that $b_f \in \mathscr{M}(d_p)$ and*

$$\sigma(T_{b_f}; \mathscr{M}_{\rm op}(d_p)) = \sigma(T_{b_f}; \mathscr{L}(d_p)) = f(\overline{\mathbb{D}}).$$

Proof. Fix $f \in H(\overline{\mathbb{D}})$. We know (cf. Corollary 5.4) that $b_f \in \mathscr{M}(d_p)$ and so $T_{b_f} \in \mathscr{M}_{\rm op}(d_p)$. Via the functional calculus for unital Banach algebras, [6, Ch.I, §7], [19, Ch.10 & 11], the operator $f(S) \in \mathscr{M}_{\rm op}(d_p)$ is defined by the Cauchy integral formula

$$f(S) := \frac{1}{2\pi i} \int_\gamma f(z)(zI - S)^{-1} dz$$

for a suitable contour γ surrounding $\overline{\mathbb{D}} = \sigma(S; \mathscr{M}_{\rm op}(d_p))$, where we use Remark 3.2(ii) and (6.1).

Fix $n \in \mathbb{N}_0$. Given $z \in \gamma$ a direct calculation yields (as $|z| > 1$) that

$$(zI - S)^{-1} e_n = \Big(0, \dots, 0, \overbrace{\frac{1}{z}}^{\text{position } n}, \frac{1}{z^2}, \frac{1}{z^3}, 0, \dots\Big) \in d_1 \subseteq d_p.$$

Accordingly,

$$f(S) e_n = \frac{1}{2\pi i} \int_\gamma f(z)(zI - S)^{-1} e_n \, dz = \sum_{k=0}^{\infty} \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z^{k+1}} \, dz \cdot e_{k+n}.$$

Since $b_f = (\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z^{k+1}} \, dz)_{k=0}^\infty$, it follows that

$$f(S) e_n = \Big(0, \dots, 0, \overbrace{b_0}^{\text{position } n}, b_1, b_2, \dots\Big) = b_f * e_n.$$

But, $b_f \in \mathscr{M}(d_p)$, that is, $T_{b_f} \in \mathscr{M}_{\rm op}(d_p)$ and so $f(S)e_n = T_{b_f} e_n$ for all $n \in \mathbb{N}_0$. Since $\{e_n : n \in \mathbb{N}_0\}$ is basis for d_p, we can conclude that $f(S) = T_{b_f}$. By the spectral mapping theorem for $f(S)$ and (6.1) we have

$$\sigma(f(S); \mathscr{L}(d_p)) = f(\sigma(S; \mathscr{L}(d_p))) = f(\overline{\mathbb{D}}).$$

Since $\sigma(f(S); \mathscr{L}(d_p)) = \sigma(f(S); \mathscr{M}_{\rm op}(d_p)) = \sigma(T_{b_f}; \mathscr{M}_{\rm op}(d_p))$, the proof is complete. □

Proposition 6.4. *The maximal ideal space of $\mathscr{M}(d_1)$ is homeomorphic to $\overline{\mathbb{D}}$. Moreover, for each $b \in \mathscr{M}(d_1) = d_1$, its spectrum is given by*

$$\sigma(b; \mathscr{M}(d_1)) = \sigma(T_b; \mathscr{M}_{op}(d_1)) = f_b(\overline{\mathbb{D}}).$$

Proof. Recall that d_1 is an algebra, that is, $\mathscr{M}(d_1) = d_1$ with equivalence of norms. Moreover, the unital Banach algebra $\mathscr{M}(d_1)$ is generated by e_1. To see this, let $b = (b_n)_{n=0}^\infty \in \mathscr{M}(d_1) = d_1$. Recall that $e_m = e_1^m$ for all $m \geq 1$ and so each element $b^n := b_0 e_0 + \sum_{j=1}^n b_j e_j$, for $n \in \mathbb{N}_0$, belongs to the algebra $\langle e_0, e_1 \rangle$ generated by e_0 and e_1. Since $\{e_n : n \in \mathbb{N}_0\}$ is a basis for d_1 and $\mathscr{M}(d_1) = d_1$, it follows that $b^n \to b$ in the norm of d_1 and hence, in the norm of $\mathscr{M}(d_1)$. So, the closure of $\langle e_0, e_1 \rangle$ in $\mathscr{M}(d_1)$ is $\mathscr{M}(d_1)$.

Theorem 2 on p. 98 of [6] implies that the maximal ideal space Φ of $\mathscr{M}(d_1)$ is homeomorphic with the spectrum $\sigma(e_1;\mathscr{M}(d_1))$ of the generator e_1. Since $\mathscr{M}(d_1)$ is isometric to $\mathscr{M}_{\rm op}(d_1)$ we know from Proposition 6.1 that

$$\sigma(e_1;\mathscr{M}(d_1)) = \sigma(T_{e_1};\mathscr{M}_{\rm op}(d_1)) = \sigma(S;\mathscr{M}_{\rm op}(d_1)) = \sigma(S;\mathscr{L}(d_1)) = \overline{\mathbb{D}}.$$

More explicitly, each $z \in \overline{\mathbb{D}} \simeq \Phi$ defines the multiplicative, linear functional on $\mathscr{M}(d_1)$ via point evaluation, namely

$$b \mapsto f_b(z), \quad b \in \mathscr{M}(d_1) = d_1.$$

Since $b \in d_1 \subseteq \ell^1$, the continuity is immediate from $|f_b(z)| = |\sum_{n=0}^{\infty} b_n z^n| \le \sum_{n=0}^{\infty} |b_n| \le \|b\|_{d_1}$, for $b \in \mathscr{M}(d_1)$. The Gelfand transform $\hat{b}\colon \Phi \to \mathbb{C}$, of each $b \in \mathscr{M}(d_1)$ is given by $\hat{b}(z) = f_b(z)$, for $z \in \overline{\mathbb{D}}$. It follows from Theorem 11.9.(c) in [19] that $\sigma(b;\mathscr{M}(d_1)) = \hat{b}(\Phi) = f_b(\overline{\mathbb{D}})$ for each $b \in \mathscr{M}(d_1)$. □

Fix $p \in [1,\infty)$ and let $\mathscr{A}(S,d_p)$ denote the closure in $\mathscr{M}_{\rm op}(d_p)$ of the algebra $\langle I, S\rangle$ consisting of all operators which are polynomials in S.

Proposition 6.5. *Let $p \in [1,\infty)$. The maximal ideal space of $\mathscr{A}(S,d_p)$ is homeomorphic to $\overline{\mathbb{D}}$. Moreover, for each $T_b \in \mathscr{A}(S,d_p)$, that is, for each $b \in \mathscr{M}(d_p)$ such that $T_b \in \mathscr{A}(S,d_p)$, its spectrum is given by*

$$\sigma(T_b;\mathscr{A}(S,d_p)) = f_b(\overline{\mathbb{D}}).$$

Proof. The discussion at the beginning of the proof of Proposition 6.4 shows that $\mathscr{A}(S,d_1) = \mathscr{M}_{\rm op}(d_1) = d_1$ and so Proposition 6.4 establishes the desired identity.

Next consider $p \in (1,\infty)$. Since the multiplication in any Banach algebra is jointly continuous, it follows that $\mathscr{A}(S,d_p)$ is a *closed* subalgebra of $\mathscr{M}_{\rm op}(d_p)$. Moreover, $\sigma(S;\mathscr{M}_{\rm op}(d_p)) = \overline{\mathbb{D}}$; see Remark 3.2(ii) and Proposition 6.1. Since $\mathbb{C}\setminus\overline{\mathbb{D}}$ is a connected set, it follows from [6, I Proposition 5.14] that also $\sigma(S;\mathscr{A}(S,d_p)) = \overline{\mathbb{D}}$. In particular, the maximal ideal space of $\mathscr{A}(S,d_p)$ is homeomorphic to $\overline{\mathbb{D}}$ (cf. [6, II Theorem 19.2]) and so, for any polynomial f, we have that

$$\sigma(f(S);\mathscr{A}(S,d_p)) = \sigma(f(S);\mathscr{M}_{\rm op}(d_p)) = f(\overline{\mathbb{D}}).$$

Every $T \in \mathscr{A}(S,d_p) \subseteq \mathscr{M}_{\rm op}(d_p)$ is of the form $T = T_b$ for some unique element $b \in \mathscr{M}(d_p)$. Each $z \in \overline{\mathbb{D}}$ defines the linear, multiplicative functional on $\mathscr{A}(S,d_p)$ via

$$T_b \mapsto f_b(z), \quad T_b \in \mathscr{A}(S,d_p),$$

which is automatically continuous, [6, II Proposition 16.3]. The Gelfand transform $\widehat{T_b}\colon \overline{\mathbb{D}} \to \mathbb{C}$, of each $T_b \in \mathscr{A}(S,d_p)$, is given by $\widehat{T_b}(z) = f_b(z)$, for $z \in \overline{\mathbb{D}}$. Again by Theorem 11.9(c) in [19] we can conclude that $\sigma(T_b;\mathscr{A}(S,d_p)) = \widehat{T_b}(\overline{\mathbb{D}})$. □

Remark 6.6. (i) Let $b \in \mathscr{M}(d_1)$ belong to the radical. Proposition 6.4 together with Theorem 11.9.(c) in [19] imply, for the Gelfand transform $\hat{b}$, that $\|\hat{b}\|_\infty = 0$, that is, $f_b(\overline{\mathbb{D}}) = 0$ and so $b = 0$. Hence, $\mathrm{rad}(\mathscr{M}(d_1)) = \{0\}$, that is, $\mathscr{M}(d_1)$ is *semisimple*. An analogous argument (now using Proposition 6.5) shows that also $\mathscr{A}(S,d_p)$ is a semisimple algebra for all $p\in[1,\infty)$.

(ii) Given $p\in[1,\infty)$, which elements $b\in\mathscr{M}(d_p)$ satisfy $T_b\in\mathscr{A}(S,d_p)$? According to Corollary 5.5, this includes the space d_1 (hence, also the Taylor coefficients b_f of any function $f\in H(\overline{\mathbb{D}})$ via Corollary 5.4), the weighted space $\ell^1(w_p)$ and $d_{pp}\cap\ell^1$. Actually, for every $b=(b_n)_{n=0}^\infty$ belonging to any one of these spaces, the approximation of T_b can be achieved by using the *Taylor polynomials* of b. That is, for $n\to\infty$, we have

$$\Big\|T_b-\sum_{j=0}^n b_jS^j\Big\|_{\mathscr{A}(S,d_p)}=\Big\|T_b-\sum_{j=0}^n b_jS^j\Big\|_{\mathscr{M}_{\mathrm{op}}(d_p)}\to 0.$$

The following identities occur in Proposition 6.3, namely

$$\sigma(T_{b_f};\mathscr{A}(S,d_p))=\sigma(T_{b_f};\mathscr{M}_{\mathrm{op}}(d_p))=f(\overline{\mathbb{D}}),\quad f\in H(\overline{\mathbb{D}}).$$

For certain other multipliers an inclusion is possible.

Proposition 6.7. *Let $p\in[1,\infty)$ and $b\in\mathscr{M}(d_p)$ satisfy*

$$\Big\|T_b-\sum_{j=0}^n b_jS^j\Big\|_{\mathscr{M}_{\mathrm{op}}(d_p)}\to 0\quad \textit{for } n\to\infty. \tag{6.4}$$

Then

$$\sigma(T_b;\mathscr{A}(S,d_p))=\Big\{\sum_{n=0}^\infty b_n\lambda^n:\lambda\in\overline{\mathbb{D}}\Big\}\subseteq\sigma(T_b;\mathscr{M}_{\mathrm{op}}(d_p))=\sigma(T_b;\mathscr{L}(d_p)).$$

Proof. Fix $\lambda\in\overline{\mathbb{D}}$. Since $b\in\ell^1$ (cf. Theorem 4.2) the series $\sum_{j=0}^\infty b_j\lambda^j$ converges absolutely in $\mathbb{C}$. Define $\alpha_n:=\sum_{j=0}^n b_j\lambda^j$, for $n\in\mathbb{N}_0$, in which case $\alpha_n\to\alpha:=\sum_{j=0}^\infty b_j\lambda^j$ for $n\to\infty$. Moreover, setting $R_n:=\sum_{j=0}^n b_jS^j$ we have that $R_n\in\mathscr{M}_{\mathrm{op}}(d_p)$ and so

$$\sigma(R_n;\mathscr{M}_{\mathrm{op}}(d_p))=\Big\{\sum_{j=0}^n b_jz^j:z\in\overline{\mathbb{D}}=\sigma(S;\mathscr{M}_{\mathrm{op}}(d_p))\Big\},\quad n\in\mathbb{N}_0.$$

That is, $\alpha_n\in\sigma(R_n;\mathscr{M}_{\mathrm{op}}(d_p))$ for $n\in\mathbb{N}_0$. For $\mathscr{A}:=\mathscr{M}_{\mathrm{op}}(d_p)$ it follows from (6.4) that $R_n\to T_b$ in $\mathscr{A}$ and so [9, Ex. 5, p.199] implies that $\sum_{j=0}^\infty b_j\lambda^j\in\sigma(T_b;\mathscr{M}_{\mathrm{op}}(d_p))$. □

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