Snapshots of modern mathematics
from Oberwolfach

№ 7/2023

The geometry of fair division

Florian Frick [1]

How can we fairly divide a necklace with various types of beads? We use this problem as a motivating example to explain how geometry naturally appears in solutions of non-geometric problems. The strategy we develop to solve this problem has been used in several other contexts.

1 Introduction

Famously, Wigner[2] called the effectiveness of mathematics in the natural sciences "unreasonable." This snapshot explores the entirely reasonable effectiveness of geometry in understanding non-local information: Geometry, as the study of shape, measures phenomena that are global instead of local. If the solution to a mathematical problem depends on the aggregate of the data that is given to us, instead of just a myopic view, one may expect that geometry will be useful in the problem's resolution—even if the problem itself is not a geometric one.

There is a, by now well-established, approach to finding solutions for such problems, which has the following outline:

1. Parametrize the space of all potential solutions. (This space is a geometric object!)
2. Define a function on this space that measures to what extent a potential solution differs from being actually a solution.

[1] Florian Frick is supported by NSF grant DMS 1855591 and a Sloan Research Fellowship.

[2] Eugene Paul Wigner (1902–1995) was a Hungarian physicist and mathematician. He received the Nobel Prize in 1963.

3. Use symmetry to establish that this function equals zero at some point, and thus an actual solution exists.

This proof scheme is of central importance in the field of *Geometric and Topological Combinatorics* and has been used for a multitude of problems ranging from applications in economics (such as the existence of Nash equilibria [7]) to cutting a sandwich with three ingredients with one straight cut such that both halves have the same amount of each ingredient; see Matoušek's book [5] for an excellent introduction. That several problems follow the theme of fair division is not a coincidence; after all, a solution is fair if it is symmetric in some sense. Here we focus on a simple "toy problem" to showcase how modern mathematics applies geometry to numerous (often non-geometric) problems.

In Section 2, we introduce the problem of fairly dividing a necklace with white and black beads between two people. In Section 3, we add a third type of bead, which complicates the problem. We thus start following the proof scheme above, and visualize the space of potential solutions as a parallelogram. Section 4 defines the function that measures how close we are to a fair division; Section 5 exploits symmetries to show that a solution must always exist. In Section 6, we collect a few examples of other problems, where this proof scheme has been successfully applied.

2 Fairly splitting a necklace

Alice and Bob inherited a necklace that consists of two different kinds of beads: Black beads and white beads. The types of beads are not arranged in any particular pattern, and Alice and Bob are unsure about which type of bead is more valuable. Nevertheless, they want to fairly divide the necklace. The only way to ensure that the division is indeed fair is if both Alice and Bob receive the same number of white beads and the same number of black beads. Of course, they would like to disturb the integrity of the necklace as little as possible, that is, they want to use as few cuts as needed to achieve this fair division.

Perhaps the necklace to be divided looks like the one depicted in Figure 1 with twelve black beads and twelve white beads. One cut will not suffice to achieve a fair division: This cut would have to be precisely in the middle of the necklace, but then there are seven black beads on the left and only five black beads on the right—unfair! Are two cuts sufficient?

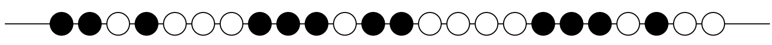

Figure 1: A necklace with two kinds of beads.

Let us first observe that with two cuts, we separate a piece from the middle of the necklace (and hand it to Alice, say), while Bob gets the outer two pieces.

Since Alice is supposed to receive twelve beads in total, the two cuts must be twelve beads apart. This means that the first cut uniquely determines the second: If the first cut is after the bead in position j, the second cut must occur after position $j + 12$. In general, if the necklace has $2n$ beads, the two cuts must be at distance n.

A second simple observation is that if Alice receives the correct amount of beads in total (twelve in our example) and she receives the fair amount of black beads (that is, six), she automatically has the right number of white beads too.

These two observations give us a way of seeing that indeed two cuts suffice in general: Call n consecutive beads along the necklace a *window*. Now slide this window, starting with the first n beads, bead-by-bead, all the way to the last n beads. If there was an excess of black beads among the first n beads, then there must be a deficit among the others, the last n beads. Since excess flips to deficit or vice versa, somewhere along the way the window contained neither an excess nor a deficit of black beads; it has the right number of black beads and so the right number of white beads too. Thus two cuts (three pieces) are always sufficient.

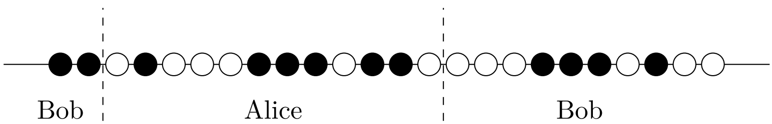

Figure 2: A fair division with two cuts.

Sliding the two cuts bounding Alice's piece along the necklace was sufficient to establish the existence of a fair division for two types of beads. Next we will investigate this problem for three types of beads. Here exhibiting the hidden geometry will be crucial. Existence of fair divisions for necklaces with several types of beads was shown by Goldberg and West [3] with simplified proofs by Alon and West [1]. Here we present an elementary proof that is similar in spirit to [3] and [1]. The concepts presented are elementary, but the proof will require some bookkeeping (bead-counting, one might say), so follow carefully the different values of the functions we will present.

3 A parallelogram of possibilities

The previous fair division problem becomes more intriguing once we allow a third kind of bead. You can check that our sliding window trick already fails for a boring necklace such as the one depicted in Figure 3. At least three cuts are necessary if we want that Alice and Bob receive equal numbers of black, gray, and white beads, respectively.

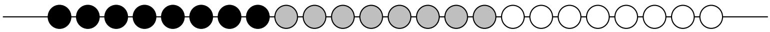

Figure 3: A necklace with three kinds of beads.

In order to show that, in general, three cuts do indeed suffice to achieve a fair division, we first describe the general situation: We are given a necklace with $2b$ black beads, $2g$ gray beads, and $2w$ white beads – even numbers – so we could give Alice and Bob the same number of each. Our goal is to cut the necklace into four connected pieces P_1, P_2, P_3, P_4, such that P_1 and P_3 together contain b black beads, g gray beads, and w white beads. (And thus the same is true for parts P_2 and P_4.) We then hand pieces P_1 and P_3 to Alice and pieces P_2 and P_4 to Bob.

As in the case of only black and white beads, we first need to parametrize all possible cuts of the necklace. For two types of beads, these possible cuts appeared in a linear order, and we could slide a window through the necklace from the first possible division to the last. Now the possible divisions make up a two-dimensional object.

Let $n = b + g + w$. We can think of the necklace as having a bead at every integer between 1 and $2n$. Consider the parallelogram Q with vertices at $(0,0), (n,n), (n,2n)$, and $(0,n)$.

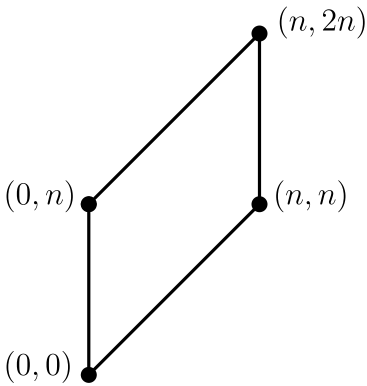

Figure 4: The parallelogram Q parametrizing certain divisions of the necklace.

To any point (x, y) with integer coordinates within Q, we associate a division of the necklace into four connected pieces P_1, P_2, P_3, P_4, such that the total number of beads in pieces P_1 and P_3 is equal to the total number of beads in pieces P_2 and P_4. Given such a point (x, y), place the first cut after the bead in position x and the second cut after the bead in position y. (Here cutting after the zeroth bead simply means to cut before the first bead.) By definition of Q, we have $x \leq y$, so the cuts actually appear in this order along the necklace. We allow equality $x = y$, which wastes a cut by performing it twice.

The third cut is now uniquely determined by the first two and the requirement that the first and third pieces have the same total length as the second and fourth pieces. The third cut needs to occur after the bead in position $z = y - x + n$. These three cuts split the necklace into four pieces. Their lengths are x, $y - x$, $z - y = n - x$, and $2n - z = x - y + n$. Indeed, the first and third lengths sum to n, as do the second and fourth lengths. Conversely, any such division of the necklace into four pieces corresponds to an integer point within the parallelogram Q. Non-integer points in Q also correspond to divisions of the necklace. These divisions, however, may cut through beads and not only between them. Note that a division corresponding to a point on the boundary of Q cuts the necklace into two or three pieces instead of four.

To summarize, the point (x, y) in Q corresponds to the division of the necklace, where we cut in three (not necessarily distinct) points: At position x, at position y, and at position $y - x + n$. Think of the necklace as the interval $(0, 2n]$ with a bead at every positive integer. This divides the interval (necklace) into pieces $P_1 = (0, x]$, $P_2 = (x, y]$, $P_3 = (y, y - x + n]$, and $P_4 = (y - x + n, 2n]$. Here $(a, a]$ will mean an empty piece.

4 Testing fairness

Now that we have parametrized all valid divisions, let us associate two numbers to every point in Q: The excess of black beads and the excess of gray beads in the union $P_1 \cup P_3$ of the first and third pieces. That is, to a point (x, y) in Q, we associate the pair $(\beta - b, \gamma - g)$, where β is the number of black beads up to position x in the necklace plus the number of black beads between positions y and $z = y - x + n$. The number γ is defined in the same way for gray beads. Denote by $f(x, y)$ the pair associated to the point (x, y). In other words, $f(x, y)$ encodes the difference between the number of black and gray beads in the division determined by (x, y) and a fair division.

Suppose that for some point (x, y) in Q, we have $f(x, y) = (0, 0)$. Then this means that the first piece P_1 and the third piece P_3 of the corresponding division contain together b black beads and g gray beads—the correct number. Since their combined length is n beads and $b + g + w = n$, this also implies that we have the right number w of white beads in $P_1 \cup P_3$. Thus such a point corresponds to a fair division. We might be worried that if x and y are not integer points, then the division might cut through beads instead of between them. However, if a bead of some color is cut, then since beads appear an even number of times, another bead of that color must also be cut. We can adjust such a pair of cuts simultaneously to not go through beads. (We sometimes may have to adjust more than two cuts simultaneously.) Thus $f(x, y)$ measures the fairness of the division determined by the point (x, y). To find a fair division, we need to find a a point where the function f equals zero. To find this zero we will investigate the values of f on the boundary of Q.

Let us walk along the boundary of Q. The vertex $(0, 0)$ of Q is associated with the division where we "cut" twice before the first bead, and thus the third cut is after bead $z = n$. So the first n beads belong to piece P_3, while the remaining n beads belong to P_4. As we diagonally walk up the edge toward the vertex (n, n), we increase the length of the piece P_1, which contains all beads up to position x. Since $x = y$ along this entire edge, the piece P_2 always has length zero. Thus along the edge from $(0, 0)$ to (n, n), the union $P_1 \cup P_3$ always occupies the first half of the necklace. In other words, all divisions along this edge let us give the same beads to Alice – those from the first half of the necklace. So f is constant here.

Trace up from (n, n) to $(n, 2n)$ and keep track of the divisions parametrized by the points along this edge. Since $x = n$ along this edge, piece P_1 remains unchanged; it always consists of the first n beads. The third cut is at position $z = y - x + n = y$, and so it coincides with the second cut. This means that P_3 has length zero along this edge, and again $P_1 \cup P_3$ is constantly equal to the first half of the necklace. We have established that f is constant all the way from $(0, 0)$ via (n, n) to $(n, 2n)$.

The other two edges of Q are more interesting. What is the value of f at the vertex $(0, n)$? Recall that $(0, n)$ corresponds to the division where we cut at positions 0, n, and $n - 0 + n = 2n$. Thus pieces P_1 and P_4 have length zero, P_2 extends from position 0 to n, and P_3 covers the rest, positions n to $2n$. Compared to the situation in vertex $(0, 0)$ and $(n, 2n)$, the roles of $P_1 \cup P_3$ and $P_2 \cup P_4$ have swapped; $P_1 \cup P_3$ covers the second half of the necklace in vertex $(0, n)$ and $(n, 2n)$, but the first half in vertex $(0, 0)$. Thus, an excess of black (or gray) beads for one division flips into a deficit for the other, or more concisely $f(0, n) = -f(0, 0) \ = -f(n, 2n)$.

This symmetry extends from the vertices to the two edges incident to $(0, n)$. Points along the top edge of Q are of the form $(a, n + a)$ with $0 \leq a \leq n$. Such a point corresponds to the division where P_1 is the interval $(0, a]$, piece P_2 is $(a, n + a]$, and P_3 is $(n + a, 2n]$ while P_4 is empty. Points along the left edge of Q are of the form $(0, a)$ for $0 \leq a \leq n$, and such a point gives the division where P_1 is empty, P_2 is $(0, a]$, piece P_3 is $(a, n + a]$, and P_4 is $(n + a, 2n]$. That is, from point $(a, n + a)$ to $(0, a)$ the roles of $P_1 \cup P_3$ and $P_2 \cup P_4$ flip. This means $f(a, n + a) = -f(0, a)$ for all a between 0 and n.

5 Following paths to find zeros

The topologically inclined reader might have noticed that we are done: The symmetry of f on the boundary of Q implies that f has odd mapping degree, and so must pass through zero. Thus there is a fair division. Here we prove this central topological fact in an elementary way.

We will exploit the symmetry of the function f on the boundary of Q to find a zero, or equivalently a fair division of the necklace. If $f(0,0) = (0,0)$, meaning that one cut precisely in the middle of the necklace already constitutes a fair division, we are done. If not, we may assume that the first coordinate of $f(0,0)$ is non-zero. Because if only the second coordinate is non-zero, we simply switch the roles of black and gray beads. Denote the first coordinate of f, which counts the excess or deficit of black beads in parts P_1 and P_3, by f_1. Similarly, the second coordinate of f, measuring the excess or deficit of gray beads in P_1 and P_3, will be denoted by f_2.

Keep track of f_1 as we go up the edge of Q that joins $(0,0)$ to $(0,n)$. Since, as we showed in the previous section, $f(0,n) = -f(0,0)$, the sign of f_1 flips. This means that along this edge the sign of f_1 has to change an odd number of times. Thus, along the edge from $(0,0)$ to $(0,n)$, f_1 passes through zero an odd number of times. There is only one minor problem with this reasoning: f_1 could be zero infinitely many times by being zero along an entire interval; for example, by being negative, reaching zero and then going back again without changing sign. We remedy this by instead considering $f_1 + \frac{1}{2}$. Since $\frac{1}{2} < 1$ and $f_1(0,0)$ is a non-zero integer, the sign of $f_1 + \frac{1}{2}$ still flips along the left edge of Q. Now, since f_1 cannot be equal to a non-integer along an interval, we avoid the problem of having infinitely many zeros of $f_1 + \frac{1}{2}$.[3]

By the same reasoning, now using that $f(0,n) = -f(n,2n)$, the function $f_1 + \frac{1}{2}$ has an odd number of zeros along the edge joining $(0,n)$ to $(n,2n)$. Moreover, remembering that for each a between 0 and n, we have $f(a,n+a) = -f(0,a)$, we can deduce that $f_1 + \frac{1}{2}$ is positive (negative) along the upper edge of Q for every integer a such that it is negative (positive) along the left edge. As a result, $f_1 + \frac{1}{2}$ changes sign the same number of times along the upper edge as it does along the left one.

Since f is constant on the other two edges of Q, there are no more zeros of $f_1 + \frac{1}{2}$ on the boundary of Q. We have established that $f_1 + \frac{1}{2}$ has $2k$ zeros on the boundary of Q, where k is odd. Furthermore, because of the symmetry of f, these zeros come in pairs: For every zero of $f_1 + \frac{1}{2}$ with positive second coordinate f_2, there is such a zero with negative f_2. So there are exactly k zeros of $f_1 + \frac{1}{2}$ on the boundary of Q, where f_2 is positive.

[3] Try to draw this on paper and see why it works.

We are now in a position to understand the zeros of $f_1 + \frac{1}{2}$ on all of Q. Split Q into triangles such that the vertices of the triangles are the points with integer coordinates in Q and edges that only connect vertices whose coordinates differ by at most one, as in Figure 5. Since f_1 achieves integer values at points with integer coordinates, $f_1 + \frac{1}{2}$ is never zero at a vertex of one of the small triangles. A zero of $f_1 + \frac{1}{2}$ on an edge of a small triangle occurs if and only if f_1 changes sign from one endpoint of the edge to the other. Going along a small triangle, the sign of f_1 can only change two times or none at all: If f_1 changes sign from, say, the first endpoint to the second, it has to change back between the second and the third or the third and the first, but cannot at both – as with a third change we will end up with a different sign from what we started with. Thus if $f_1 + \frac{1}{2}$ has a zero somewhere on a small triangle, this triangle contains a line segment of zeros, joining the two zeros of $f_1 + \frac{1}{2}$ on the boundary of the triangle. This implies that zeros of $f_1 + \frac{1}{2}$ come in non-branching paths that either start and end in the boundary of Q or close up to loops.

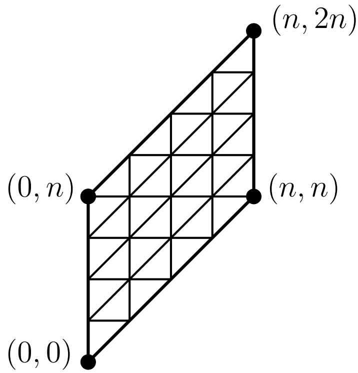

Figure 5: Splitting Q into triangles. Here $n = 4$.

We have established that the $2k$ zeros of $f_1 + \frac{1}{2}$ on the boundary of Q are joined in pairs by paths of zeros. Since k is odd, the k zeros of $f_1 + \frac{1}{2}$ with positive f_2 cannot be joined in pairs. So at least one of these paths must connect a point on the boundary of Q with positive f_2 to a point with negative f_2. Since the sign of f_2 flips along this path, there must be a point where $f_2 = 0$. Since the path consists of points with $f_1 + \frac{1}{2} = 0$, we have found a point with $f_1 + \frac{1}{2} = 0$ and $f_2 = 0$. The corresponding division of the necklace fairly divides the gray beads (since $f_2 = 0$) and up to half a bead fairly divides the black beads (since $f_1 + \frac{1}{2} = 0$). Thus the division must cut through a black bead and we can adjust the corresponding cut in such a way that the division of black beads is precisely fair. Then, as we argued before, the white beads are also fairly divided. Finally, we have found a fair division of the necklace using only three cuts.

6 Further problems

The reader may be worried that since it took some effort to establish fair division for necklaces with three kinds of beads, the reasoning will be much more involved for four or even five kinds of beads. However, the good news is that we have developed all ideas required to fairly divide necklaces with any number of types of beads. For a necklace that consists of beads with k different colors, the space of divisions with k cuts will be a $(k-1)$-dimensional geometric object. Define the function f as before, now measuring the fairness of the division with respect to the first $k-1$ kinds of beads. To establish that f has a zero, either appeal to a topological fact, or give an elementary proof following paths of zeros of the first $k-2$ coordinates of f, where the first coordinate was shifted by $\frac{1}{2}$.

The ideas we have presented here can be used in many other contexts and for a variety of problems. Some examples where this proof scheme has been applied include:

- The square peg problem: Does any simple closed curve in the plane have four points that are the vertices of a square? This is still unknown in general, but has been settled with various restricting conditions on the curve; see [6].
- Lovász [4] used this proof scheme to show the following result about intersections of finite sets: If one wants to partition the collection of k-element subsets of $\{1, 2, \ldots, n\}$ into c parts, such that in each part all sets intersect pairwise, then $c \geq n - 2k + 2$ parts are needed.
- The rent of a 3-bedroom apartment can be split among the three rooms in such a way that the rooms may be assigned to three roommates with their own subjective preferences so that no roommate is envious of another. The same holds true for n roommates in an n-bedroom apartment [8], even if the preferences of one roommate are unknown [2].
- The sandwich we mentioned in the introduction: where the objects we want to divide are arranged in three-dimensional space rather than along a line as the beads.

The reader is invited to identify why these problems benefit from detecting global phenomena and how they depend on inherent symmetries. Finding the relevant space of potential solutions can be easy (for the square peg problem, one may argue with the space of four points on a curve) or the main difficulty: Lovász used the space of probability measures supported on k-element sets disjoint from a common k-element set to prove his result. New applications of this powerful proof scheme are found regularly.

References

[1] N. Alon and D. West, *The Borsuk–Ulam theorem and bisection of necklaces*, Proceedings of the American Mathematical Society **98** (1986), 623–628.

[2] F. Frick, K. Houston-Edwards, and F. Meunier, *Achieving rental harmony with a secretive roommate*, American Mathematical Monthly **126** (2019), 18–32.

[3] C. Goldberg and D. West, *Bisection of circle colorings*, SIAM Journal on Algebraic and Discrete Methods **6** (1985), 93–106.

[4] L. Lovász, *Kneser's conjecture, chromatic number, and homotopy*, Journal of Combinatorial Theory. Series A **25** (1978), 319–324.

[5] J. Matoušek, *Using the Borsuk–Ulam Theorem: Lectures on Topological Methods in Combinatorics and Geometry*, Universitext, Springer, 2003.

[6] B. Matschke, *A survey on the square peg problem*, Notices of the American Mathematical Society **61** (2014), 346–351.

[7] J. Nash, *Non-cooperative games*, Annals of Mathematics **54** (1951), 286–295.

[8] F. Su, *Rental harmony: Sperner's lemma in fair division*, American Mathematical Monthly **106** (1999), 930–942.

Florian Frick *received his Ph.D. at TU Berlin and is now an associate professor at Carnegie Mellon University.*

Mathematical subjects
Discrete Mathematics and Foundations, Geometry and Topology

Connections to other fields
Finance, Humanities and Social Sciences

DOI
10.14760/SNAP-2023-007-EN

Snapshots of modern mathematics from Oberwolfach provide exciting insights into current mathematical research. They are written by participants in the scientific program of the Mathematisches Forschungsinstitut Oberwolfach (MFO). The snapshot project is designed to promote the understanding and appreciation of modern mathematics and mathematical research in the interested public worldwide. All snapshots are published in cooperation with the IMAGINARY platform and can be found on www.imaginary.org/snapshots and on www.mfo.de/snapshots.

ISSN 2626-1995

Junior Editor
Daniel Kronberg
junior-editors@mfo.de

Senior Editor
Anja Randecker
senior-editor@mfo.de

Mathematisches Forschungsinstitut
Oberwolfach gGmbH
Schwarzwaldstr. 9 – 11
77709 Oberwolfach
Germany

Director
Gerhard Huisken

Mathematisches Forschungsinstitut Oberwolfach

Member of Leibniz Association

IMAGINARY
open mathematics