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Darcy's law and groundwater flow modelling

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Formulations of natural phenomena are derived, sometimes, from experimentation and observation. Mathematical methods can be applied to expand on these formulations, and develop them into better models. In the year 1856, the French hydraulic engineer Henry Darcy performed experiments, measuring water flow through a column of sand. He discovered and described a fundamental law: the linear relation between pressure difference and flow rate – known today as Darcy's law. We describe the law and the evolution of its modern formulation. We furthermore sketch some current mathematical research related to Darcy's law.

1 Introduction

Henry Darcy (1803–1858) was interested in the description of flow through a porous medium such as sand. Everyday experience tells us that water can travel through sand. Most people are happy with this qualitative fact, few people ask how the flow can be described in a quantitative way (it seems that nobody asked the question before Darcy). Darcy designed an experiment to *measure* the flow rate in dependence of other physical parameters. His experimental finding (reported in [4]) is an important law – the linear relation between pressure differences and flow rate. Until today, his law is used as a basis in several scientific disciplines concerned with porous media. In the mathematical community, the name "Darcy" is mentioned in almost 1000 publications of the last 10 years in title or abstract (as gathered from MathSciNet and Zentralblatt Math[1]).

Linear relations (such as the one of Darcy) are fundamental to the description of many phenomena of our world. One example is Newton's law $F = ma$, which relates force F and acceleration a with a factor m of proportionality (the factor m turns out to be another important quantity, the mass of the body). Newton's law is from 1687; in this sense, Darcy's discovery from 1856 came late. Certainly, people have been less interested in porous media than in the description of stars or moving objects, since water in sand cannot be observed directly. On the other hand, Fourier's law of heat conduction (1822) and Ohm's law of electric currents (1827) have also been found earlier than Darcy's law, even though neither heat-flux nor electric current can be observed directly. The technical restriction imposed by the fact that important progress in the development of manometers (devices for the measurement of pressures) has been made only in the 1840s, might also be of relevance.

2 Darcy's experiment and his law

Darcy used a (vertical) cylindrical column of 2.5m height and 0.35m diameter. The column was filled with sand. At the top (which was otherwise closed) was a water supply tube and a manometer to measure the water pressure. At the bottom end, a filter kept the sand in place and water could exit through a tube. The bottom end was also equipped with a manometer. Darcy measured (with the same sand) the amount of water leaving the column at the bottom per time, denoted by J (in liter per minute), and the difference between the pressure of water entering at the top (p_{top}) and the pressure of the water leaving the column (p_{bottom}). We must take into account the fact that if the pressure at the bottom is higher than at the top (namely by the pressure of a water column of height 2.5m), there will be no water flow through the bottom. Therefore, it is reasonable (and Darcy obviously did so) to measure the *water head*, that is, to measure p_{top} relative to the atmospheric pressure, and to measure p_{bottom} relative to the gravity-induced expected pressure of a 2.5m water column. After performing several runs of the experiment with different

[1] Two comprehensive databases of mathematical publications. The first maintained by the American Mathematical Society; the second by the Leibniz Institute for Information Infrastructure (FIZ Karlsruhe) and has a partially free access.

pressures, Darcy calculated the ratio and observed it to be essentially constant:

$$\frac{J}{p_{\text{top}} - p_{\text{bottom}}} = \text{constant}. \tag{1}$$

We should additionally take into account the dimensions of the experimental setup. In a column of doubled cross-section, the flow rate is doubled. Furthermore, a column of doubled height requires a doubled pressure difference to produce the same flow rate (each half-column is then driven by the original pressure difference). Denoting by A the area of a cross section of the column and by d its height, and then dividing by these and rearranging (1), we get a number $k > 0$, dependent on the type of sand that is used but not on the dimensions of the experiment, for which

$$\frac{J}{A} = k\, \frac{p_{\text{top}} - p_{\text{bottom}}}{d}. \tag{2}$$

2.1 A differential form of the law

In equation (2), we have essentially written so-called "differential quantities" on both sides. The term implies a semblance to the method we use to get derivatives: measuring the difference of quantities over small ranges and averaging. On the left we have the flow of water (volume divided by time) that crosses an arbitrary surface element divided by surface area – the *water flux* denoted by j. This quantity can be assigned to every point in the column: $j(x)$ describes the water flux in the point x (imagine a thin line of water flowing through x). On the right, we have a pressure difference divided by the distance of the measurement points. In a point x (writing $p(x)$ for the pressure at x), and for small distances around it, this corresponds to a derivative of the pressure with respect to the spatial variable. Assuming, for the moment, that these quantities work one-dimensionally (water-flow and pressure move only up/down) and denoting the vertical variable by x_3, the right hand side of (2) corresponds to $k\,\frac{\partial p}{\partial x_3}$.

We make one final step and move to a three-dimensional framework. The flux $j(x)$ should be regarded as a vector with length being the amount of water flowing, and the direction is that of the motion of the water. Also, the right hand side of (2) is related to a vector called the *gradient* of p and denoted by ∇p. Regarding the pressure as a spatially distributed quantity, $\nabla p(x)$ is the vector that points in the direction of the largest pressure variation at x, its length is the pressure variation per distance in this direction. We can expect that the direction of the two vectors coincides (up to sign): the water flows in the direction of the largest pressure variations (towards lower pressures). This is obviously the case in Darcy's experiment. The pressure gradient points upwards, in x_3-direction, the flux is exactly downward. Performing these two steps (writing the law in differential form and with vectors), we obtain the modern form of Darcy's law:

$$j(x) = -k\,\nabla p(x) \quad \text{for every point } x. \tag{3}$$

We emphasize that we have used some assumptions to obtain (3) from (2). We have assumed that the law (2) holds in each point x (as noted before, the law does not depend on the size of the column, so we may think of arbitrarily small columns containing x). Furthermore, we have assumed that the direction of the flow coincides with the direction of the pressure gradient (this is justified if the medium has a rotational symmetry). Based on (1), (3) is a convincing relation, but we should distinguish between the two equations: While Darcy obtained (1) as the result of experiments, our modern and stronger law (3) was obtained from additional assumptions. Nonetheless, today, the law (3) is known as "Darcy's law".

When we compare Darcy's law with the above-mentioned laws of Fourier and Ohm, we observe a striking similarity – In all three laws, a flux is related to a gradient through a conductivity factor (k in the case of Darcy's law). Fourier's law relates the heat-flux q in a medium with the temperature gradient, $q(x) = -\lambda\nabla T(x)$, where the proportionality factor λ is known as the heat conductivity. Ohm's law relates the current I in a conductor with a voltage difference U at the end-points, $I = R^{-1}U$, where R is the (Ohmic) resistance and R^{-1} is the conductivity, similar to (1).

3 Partial Differential Equations based on Darcy's law

From now on, we truly regard our quantities as functions in three-dimensional space. Points in the porous medium are denoted by $x \in \mathbb{R}^3$. Associated with each point x is a pressure $p(x) \in \mathbb{R}$ and a flux $j(x) \in \mathbb{R}^3$. Inside the medium, water can neither vanish nor appear. We can imagine an arbitrary cube $Q \subset \mathbb{R}^3$ in the medium and expect that the total flux into and out of Q vanishes. There is a mathematical object to describe the behavior of the of the flux j on the cube, it is called the *divergence* of j, denoted by $\nabla \cdot j$ (not to be confused with the gradient), and defined as $\nabla \cdot j := \frac{\partial}{\partial x_1} j_1 + \frac{\partial}{\partial x_2} j_2 + \frac{\partial}{\partial x_3} j_3$. Since the volume Q can be chosen arbitrarily, the famous Theorem of Gauß (also known as the Divergence Theorem) implies that $\nabla \cdot j = 0$. This law can also be understood in an elementary way: If x_1 is the coordinate to the right, the number $\frac{\partial}{\partial x_1} j_1$ measures the difference between outflow on the right side of Q and inflow on the left side of Q. Adding over the three coordinates, we obtain the net fluid flow into and out of the cube Q. We demand that this quantity should vanish.

Combining Darcy's law with the law of *conservation of mass*, we find the equation

$$-\nabla \cdot (k\,\nabla p) = 0. \tag{4}$$

The equation describes a relation between functions (p and j) and their derivatives (encoded in ∇). At first look, this seems impossible to solve, that is, to find the exact form of the functions. After all, we need to know the function in order to get the derivative; while, in order to solve the equation (find the function) we need to know the derivative. Such equations are called *differential equations*[2]. As it turns out, under certain circumstances, some differential equations can be solved (or, at least, we can define some properties of possible solutions). Such circumstances include, for example, adding information by setting *initial* and *boundary conditions* (knowing the values of the function and derivative at specific points), or by complementing the equation to a *system* of equations[3]. Luckily, equation (4) is one of those equations that can be solved using these methods. Once it is assigned with boundary conditions (one must prescribe, e.g., the pressure at the boundaries of the porous medium), given k, equation (4) has a unique solution $p = p(x)$.

3.1 Richards' equation

More interesting than (4) is the following variant, which describes *unsaturated flow.* If we do not assume that the porous medium (the sand) is everywhere fully saturated with water, we have an additional unknown, the *saturation* $s = s(x)$, measuring the volume-fraction of the pores that is occupied by water (and not air). In this situation, the divergence of the water-flux does not necessarily vanish. Instead, if water enters a test volume, the saturation in this test volume must increase, leading to a positive time derivative $\partial_t s = \frac{\partial}{\partial t} s$. All variables now depend additionally on the time variable $t \in (0, \infty)$ and we write $s = s(x,t)$, $p = p(x,t)$, $j = j(x,t)$. The combination of Darcy's law with an adapted law of conservation of mass, $\partial_t s + \nabla \cdot j = 0$, now yields

$$\partial_t s - \nabla \cdot (k\,\nabla p) = 0. \tag{5}$$

This equation is known as Richards' equation (formulated in 1931, see [9]; we suppressed here some physical parameters such as porosity of the medium or

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[2] If we have a function of several variables, we can take a derivative with respect to only one of its variables. This derivative is called a *partial derivative.* A differential equation which relates a multivariable function and its partial derivatives (as in the case of (4)) is then called a *partial* differential equation.

[3] This is similar to when we are faced with linear equations concerning more than one variable. Adding more equations describing relations between the variables makes it possible to fined their values.

density of the fluid). Since we now have two unknowns (p and s), equation (5) is not sufficient to determine the solution.

3.2 Capillary pressure

Darcy's law (3) is commonly accepted as a good model for the relation between pressure and flux (it is experimentally confirmed and can also be justified with mathematical methods, see below). In order to complete Richards' equation to a solvable system, we additionally have to have a relation between pressure and saturation. Let us imagine that we want to push water into the medium (imbibition[4]). We assume here for simplicity that the medium is water repellent, meaning that water does not like to be in contact with the medium. In order to push the water inside, we need some pressure. In the beginning (for small saturation s), this pressure is not very high, since water can easily fill the large pores of the medium. The more water is inside the medium (larger saturation s), the more pressure we must exert (larger p), since now we must push the water also into the small pores[5]. This reasoning suggests that the pressure is a function of the saturation. For example, we may assume that for some given function $p_c : [0, 1] \to \mathbb{R}$ there holds, at every space-time point (x, t):

$$p(x,t) = p_c(s(x,t)). \tag{6}$$

This law is known as the *capillary pressure relation.*

Experiments confirm (to a certain extent; see next section) the law (6). Given an increasing (with s) function p_c (obtained from experiments), the two equations (5) and (6) (together with boundary conditions and an initial condition) can be solved. This fact makes the approach sketched here (initiated by Darcy) so extremely useful. One can measure k and p_c for a given medium, some rock type, say. Solving (5)–(6) we find s, p and j and can *predict* the flow of water through the medium. This method is used extensively in oil recovery: In order to plan where to drill the production wells, one studies the medium, determines the parameters k and p_c, and solves (5)–(6). This allows to optimize the recovery.

[4] The process of absorbing water but not forming a solution.

[5] In reality, many times the medium is actually hydrophilic (water likes to stay in contact with it), and we want to push the water out by, say, pushing air in. In these cases, the principle is the same, but the process is opposite: the higher the saturation – the less pressure we need, since it is easier to push the water out of the larger pores; while the lower the saturation – the more pressure we need to push the remaining water out of the small pores.

4 Current research

We now finally indicate in some aspects how modern mathematics is concerned with the above equations.

1. Numerical solution of (4). Computing solutions to equations using methods of approximation is called *numerical analysis* (one famous example would be Newton's method to find the zeros of polynomials). There are standard methods that can be used to solve (4) numerically, and many students of mathematics usually implement these at some point of their studies. Difficulties appear if the conductivity coefficient k is allowed to vary, $k = k(x)$, and indeed has large variations (varying by many orders of magnitude, possibly even across small distances). For such situations, new numerical methods are currently developed by mathematicians; see, for example, [2, 7] and the references therein.
2. The conductivity coefficient k was determined by Darcy not only for some given type of sand, but also for some fixed saturation, namely $s = 1$, full saturation. In the situation of the Richards equation, the *permeability* k will depend on s, $k(x) = k_0(x, s(x,t))$. This additional nonlinearity poses questions regarding existence and uniqueness of solutions of (5)–(6). This is particularly interesting because the system becomes *degenerate:* For a low saturation s, the medium has a low permeability, since there are few water-conducting channels in the medium. This results in $k_0(x, 0) = 0$, a degeneracy that makes the system of equations hard to analyze; see, for example, [6].
3. Derivation of Darcy's law (or variants thereof) from more elementary equations. The method of *homogenization* allows to consider models for porous media. Let us assume that a porous medium is nothing but free space with many tiny obstacles (the sand grains). Between the obstacles, water flows according to a free-flow equation (Stokes or Navier–Stokes). Homogenization theory allows us to study the limit of a vanishing obstacle size. The resulting limit equation is, in a specific scaling, indeed Darcy's law (3); see, for example, [1].

4.1 Hysteresis

Another part of research addresses the capillary pressure law (6). So far, this law cannot be obtained from other, more fundamental, physical laws. Indeed, the law is even known to be incorrect. Performing an experiment with imbibition provides one function p_c, a subsequent drainage experiment (extracting the water from the medium) provides another function p_c. The medium shows *hysteresis,* it "remembers" its history (the previous wetting or de-wetting processes). A simple and crude way to take this well-known fact into account is to work with two curves, p_c^{imb} and p_c^{drain}. But in transitions from drainage to imbibition and vice versa, even more information should be included.

Different models have been suggested for the hysteresis effect in porous media. We advocate here a model that was suggested in [3], which reads

$$p \in p_c(s) + \gamma \text{sign}(\partial_t s) + \tau \partial_t s. \tag{7}$$

This equation relates the evolution of s (via the imbibition and drainage) to that of p. The multi-valued function sign is defined as $\text{sign}(a) = \{1\}$ for $a > 0$, $\text{sign}(a) = \{-1\}$ for $a < 0$, and $\text{sign}(0) = [-1, 1]$. The coefficient γ encodes the influence of the saturation change. If the saturation s is currently increasing ($\partial_t s > 0$, imbibition), then a high value of the pressure is needed (adding γ to $p_c(s)$). Instead, if the saturation s is decreasing ($\partial_t s < 0$, drainage), then a low value of the pressure is needed (subtracting γ from $p_c(s)$). Finally, $\tau > 0$ introduces a dependence on the rate of imbibition. If the imbibition is faster, then $\partial_t s$ is larger and so is $\tau \partial_t s$, so more pressure is needed. Simply put, the hysteresis law (7) states that the pressure at any point during the imbibition-drainage process is around $p_c(s) \pm \gamma$ (plus the influence of $\tau \partial_t s$).

At spacetime points where $\partial_t s = 0$ (the saturation does not change), it is convenient to allow the value of p to range over the interval $[p_c(s) - \gamma, p_c(s) + \gamma]$. This is why we use in (7) the symbol "$\in$" ("an element of") rather than the symbol "$=$" ("equals") and also the function $\text{sign}(a)$ (which makes the second summand into $[-\gamma, \gamma]$ when $\partial_t s = 0$). The pressure function p thus becomes multi-valued at such points.

4.2 Gravity fingering

Some research of the author is concerned with the analysis of the hysteresis model described by the system (5) and (7). We have shown the existence of solutions in various settings; see for example, [5, 11]. Furthermore, numerical schemes to solve the system have been developed and implemented; see [8] for results and further references.

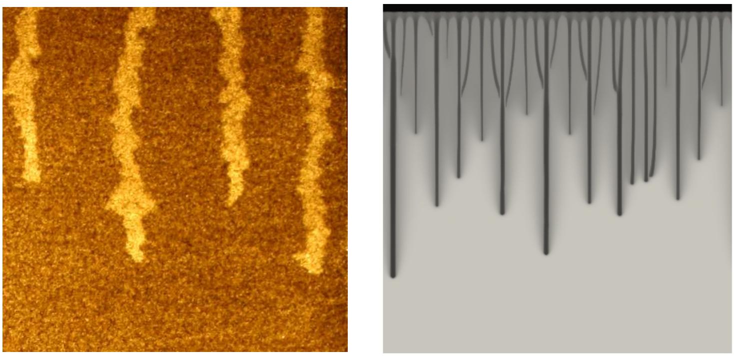
But let us take another perspective: If we replace (6) by (7), we obtain a new system of equations to describe water flow in porous media. We ask: Does this new system provide a *better model* for water flow in porous media? And: Can mathematics help to answer this question?

To answer the first question, we refer to modern experiments. Under certain conditions, one can observe *gravity fingering*. Pouring water on initially dry sand in a thin, almost two-dimensional, glass container, one observes that the water does not travel downward with a single front (showing the same saturation at each point along a horizontal line). Instead, fingers occur, well-defined long and thin regions that have a much higher saturation than the surrounding sand.

A mathematical analysis reveals (see [10]) that the original system (5) and (6) cannot explain the fingering effect; a mathematical stability mechanism of Richards' equation prevents the system from forming fingers. In contrast, system (5) and (7) does not have the stability mechanism for $k = k(s)$ and $\gamma > 0$. It is, therefore, potentially a better model. The numerical analysis of the system provides results as in Figure 1; they confirm that system (5) and (7) shows, at least qualitatively, the physically adequate, behavior of solutions.

We see that mathematics provides a bunch of information: (A) The existence of solutions for both systems. (B) A numerical scheme to solve the systems, providing results as in Figure 1(b). (C) A stability analysis that helps to understand the qualitative features of the two systems.

In closing, we remark that the capillary pressure relation only accompanied equation (5). Darcy's law, found more than 150 years ago, is beyond question!



(a) (b)

Figure 1: (a) Fingered flow in a homogenous layer as observed with transmitted light. (b) A numerical solution of the Richards equation with hysteresis law (7). In dark regions, the saturation is higher than in bright regions. We clearly see the formation of gravity fingers.

Image credits

Figure 1(a) Appears in: F. Rezanezhad, H.-J. Vogel and K. Roth, *Experimental study of fingered flow through initially dry sand*, Hydrology and Earth System Sciences Discussions **3** (2006), 2595–2620. http://www.hydrol-earth-syst-sci-discuss.net/3/2595/2006/ [visited on June 23, 2015].

Figure 1(b) Author: Andreas Rätz, TU Dortmund.

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