Oberwolfach Preprints

JORDY TIMO VAN VELTHOVEN AND FELIX VOIGTLAENDER

Coorbit Spaces and Dual Molecules: the Quasi-Banach Case

Oberwolfach Preprints (OWP)

The MFO publishes a preprint series **Oberwolfach Preprints (OWP)**, ISSN 1864-7596, which mainly contains research results related to a longer stay in Oberwolfach, as a documentation of the research work done at the MFO. In particular, this concerns the Oberwolfach Research Fellows program (and the former Research in Pairs program) and the Oberwolfach Leibniz Fellows (OWLF), but this can also include an Oberwolfach Lecture, for example.

A preprint can have a size from 1 - 200 pages, and the MFO will publish it in electronic and printed form. Every OWRF group or Oberwolfach Leibniz Fellow may receive on request 20 free hard copies (DIN A4, black and white copy) by surface mail.

The full copyright is left to the authors. With the submission of a manuscript, the authors warrant that they are the creators of the work, including all graphics. The authors grant the MFO a perpetual, irrevocable, non-exclusive right to publish it on the MFO's institutional repository. Since the right is non-exclusive, the MFO enables parallel or later publications, e.g. on the researcher's personal website, in arXiv or in a journal. Whether the other journals also accept preprints or postprints can be checked, for example, via the Sherpa Romeo service.

In case of interest, please send a **pdf file** of your preprint by email to *owrf@mfo.de*. The file should be sent to the MFO within 12 months after your stay at the MFO.

The preprint (and a published paper) should contain an acknowledgement like: *This research was supported through the program "Oberwolfach Research Fellows" (resp. "Oberwolfach Leibniz Fellows") by the Mathematisches Forschungsinstitut Oberwolfach in [year].*

There are no requirements for the format of the preprint, except that the paper size (or format) should be DIN A4, "letter" or "article".

On the front page of the hard copies, which contains the logo of the MFO, title and authors, we shall add a running number (20XX - XX). Additionally, each preprint will get a Digital Object Identifier (DOI).

We cordially invite the researchers within the OWRF and OWLF program to make use of this offer and would like to thank you in advance for your cooperation.

COORBIT SPACES AND DUAL MOLECULES:
THE QUASI-BANACH CASE

JORDY TIMO VAN VELTHOVEN AND FELIX VOIGTLAENDER

ABSTRACT. This paper provides a self-contained exposition of coorbit spaces associated with integrable group representations and quasi-Banach function spaces. It extends the theory in [Studia Math., 180(3):237–253, 2007] to locally compact groups that do not necessarily possess a compact, conjugation-invariant unit neighborhood. Furthermore, the present paper establishes the existence of dual molecules of frames and Riesz sequences as in [J. Funct. Anal., 280(10):56, 2021] for the setting of quasi-Banach spaces. To ensure the direct applicability to various well-studied examples, the theory is developed for possibly projective and reducible unitary representations.

CONTENTS

1. INTRODUCTION

For a continuous representation $(\pi, \mathcal{A})$ of a locally compact group G on a Banach or Fréchet space $\mathcal{A}$ and a vector space Y of measurable functions on G, the coorbit method provides a procedure for constructing an associated distribution space $\mathrm{Co}(Y)$, namely

$$\mathrm{Co}(Y) := \big\{ f \in \mathcal{A}^* \; : \; V_g f \in Y \big\} \quad \text{with} \quad V_g f(x) = \langle f, \pi(x) g \rangle, \tag{1.1}$$

where $g \in \mathcal{A} \setminus \{0\}$ is a fixed vector and where $\mathcal{A}^*$ denotes the associated (anti)-linear dual space. Common choices for $\mathcal{A}$ are the space of integrable vectors (resp. smooth vectors) whenever π is an integrable representation (resp. G is a Lie group).

The influential series of papers [18–20,34] introduced the spaces $\mathrm{Co}(Y)$ for an irreducible, integrable unitary representation π and a Banach function space Y and established general properties of these spaces. Among others, it was shown that the spaces (1.1) are independent of the defining vector g, that each space $\mathrm{Co}_\pi(Y)$ admits an atomic decomposition in terms of a suitably sampled family $(\pi(x_i)g)_{i \in I}$, and that properties such as inclusions, embeddings and minimality/maximality can be completely characterized by the corresponding properties of associated sequence spaces. In addition, the papers [18–20,34] revealed that many classical

function spaces in complex and harmonic analysis—such as Bergman spaces, Fock spaces, Hardy spaces, and (homogeneous) Besov and Sobolev spaces—can be realized as a coorbit space (1.1).

The purpose of the present paper is to provide a self-contained exposition of coorbit spaces $\mathrm{Co}(Y)$ associated with integrable representations π and *quasi*-Banach function spaces Y, i.e., spaces where the triangle inequality of a norm is replaced by $\|F_1 + F_2\|_Y \leq C \cdot (\|F_1\|_Y + \|F_2\|_Y)$. To some extent this has already been done in [46], but only for the restrictive setting of groups with a conjugation-invariant compact unit neighborhood (IN groups). The present paper removes this assumption. In addition, it establishes the existence of dual coorbit molecules as in [50] for the full range of *quasi*-Banach spaces.

1.1. **Motivation.** The motivation for quasi-Banach coorbit spaces is two-fold.

One motivation is that all of the above mentioned classical function spaces have aside their Banach range also a natural range of parameters yielding quasi-Banach spaces. For example, the Hardy spaces $H^p(\mathbb{R}^d)$, where $p \in (0, \infty]$, are Banach spaces for $p \in [1, \infty]$, but merely quasi-Banach spaces for $p \in (0, 1)$. Therefore, in order to treat these examples for the full range of parameters in the setting of coorbit theory, it is essential for the latter theory to also apply to quasi-Banach spaces. It should be mentioned that the Hardy spaces can be identified with coorbit spaces associated with the (non-unimodular) affine group, and hence they cannot be treated in the setting of [46] as it only applies to IN groups.

Another motivation stems from applications of coorbit theory to *non-linear approximation*. Here, given a family $(f_i)_{i \in I} \subset \mathcal{H}_\pi$ of elements of a Hilbert (or Banach) space $\mathcal{H}_\pi$, the objective is to seek for given $f \in \mathcal{H}_\pi$ a good approximation $\widetilde{f} = \sum_{i \in I_f} c_i \, f_i$ to f under the restriction $|I_f| \leq K$, i.e., using only a fixed number of elements. The associated map $f \mapsto \widetilde{f}$ is not necessarily linear. In the context of the finite-dimensional space $\mathcal{H}_\pi = \mathbb{C}^N$ and with $(f_i)_i = (e_i)_{i=1}^N$ being the standard basis, the best approximation (with respect to any ℓ^p-norm on $\mathbb{C}^N$) to $v \in \mathbb{C}^N$ is obtained by $\widetilde{v} = v \cdot \mathbb{1}_{I_v}$, with $\mathbb{1}_{I_v}$ being the indicator of the set I_v containing the indices of the K largest entries of v (in absolute value). The associated minimal approximation error is denoted by

$$\sigma_K(v)_p = \min \left\{ \|v - u\|_{\ell^p} \ : \ u \in \mathbb{C}^N \ \text{is } K\text{-sparse} \right\},$$

where $u \in \mathbb{C}^N$ is called K-*sparse* if at most K entries of u are non-zero. This error obeys the following bound, sometimes referred to as *Stechkin's inequality*:

$$\sigma_K(v)_q \leq K^{\frac{1}{q} - \frac{1}{p}} \cdot \|v\|_{\ell^p} \quad \text{for} \quad 0 < p \leq q \leq \infty, \tag{1.2}$$

see, e.g., [25, Proposition 2.3]. Hence, $\sigma_K(v)_q$ decays fast for those $v \in \mathbb{C}^N$ that have $\|v\|_{\ell^p}$ of reasonable size with a *small* value of $p > 0$. In the setting of coorbit spaces, one can similarly show (based on the discretization theory of coorbit spaces) that elements of $\mathrm{Co}(L^p)$ with small $p > 0$ can be well approximated by "sparse vectors" (i.e., by linear combinations of the family $\big(\pi(x_i)g\big)_{i \in I}$ with at most K terms) as elements of $\mathrm{Co}(L^q)$ for $q \gg p$, where it is important to note $\mathrm{Co}(L^2) = \mathcal{H}_\pi$. Hence, the *quasi*-Banach spaces $\mathrm{Co}(L^p)$ with $p \in (0, 1)$ play an important role in non-linear approximation. See also the motivating discussions [29, Section 1.1] and [46, Section 7] for considering coorbit spaces $\mathrm{Co}(L^p)$ with $p < 1$ for the purpose of non-linear approximation.

1.2. **Related work.** The theory of coorbit spaces $\mathrm{Co}(Y)$ with a Banach function space Y as developed in [18–20,34] crucially relies on convolution relations of the form

$$Y * L_w^1(G) \hookrightarrow Y \quad \text{and} \quad L_w^1(G) * Y \hookrightarrow Y \tag{1.3}$$

for a suitable weight function $w : G \to [1, \infty)$, often called a *control weight* for Y. If Y is merely a *quasi*-Banach space, then relations such as (1.3) cannot be expected to hold, e.g., for $Y = L^p(G)$ with $p \in (0, 1)$.

The paper [46] considered coorbit spaces $\mathrm{Co}(Y)$ associated with quasi-Banach spaces Y and used instead of the embeddings (1.3) certain convolution relations for so-called (left) *Wiener amalgam spaces* $W^L(Y)$ (see, e.g., [5,17,26,37] for such Banach spaces), as proven in [47]. In [46], the coorbit theory is developed only for invariant neighborhood (IN) groups, i.e., groups G admitting a compact unit neighborhood $U \subset G$ satisfying $xUx^{-1} = U$ for all $x \in G$. In the setting of IN groups, several of the convolution relations in [47] for general locally compact groups possess simpler versions, but this setting is very restrictive. For example, it excludes simply connected nilpotent Lie groups (e.g., the non-reduced Heisenberg group) and connected locally compact groups with exponential volume growth (e.g., affine group), see [38,44]

It should be pointed out that the preprint version [45] of [46] also proposed a coorbit theory valid on general (possibly, non-IN) locally compact groups. However, as observed in the PhD thesis [54] of the second named author, an essential convolution relation asserted in [47] (which is used in [45,46]) is incorrect and fails for general groups. More precisely, two examples presented in Section 3.3 below (both taken from [54]) show that

$$W^L(L_v^p) * \left[W^L(Y^\vee)\right]^\vee \nsubseteq W^L(Y)$$

with weights $v(x) = \|L_{x^{-1}}\|_{W^L(Y) \to W^L(Y)}$ and $v(x) = \|L_x\|_{W^L(Y) \to W^L(Y)}$, for the left-translation operator $L_x F(y) = F(x^{-1}y)$ and involution $F^\vee(x) = F(x^{-1})$. These examples show that the asserted [47, Theorem 5.2] fails in general.

A corrected and modified version of the theory proposed in [45,46], valid for general locally compact groups, is contained in [54]. The theory in [54] shows that essentially all the basic properties of coorbit spaces known in the Banach space setting [18–20,34] remain valid for quasi-Banach spaces, if essential modifications are made at appropriate places in both the definition of coorbit spaces and the proofs of their basic properties.

Lastly, a very general theory of *quasi*-Banach coorbit spaces has been developed in [40], where also coorbit spaces associated with general reproducing formulae not necessarily arising from group representations can be treated, see also [24,48]. Some of the basic properties of coorbit spaces for the group case can be extracted from the general theory [40], but this can be a rather daunting task (especially for interested non-experts) due to the many technical assumptions required in [40]. In addition, as the present article shows, many of these technicalities can simply be avoided or considerably simplified in the group case considered here. There are also significant parts of the present article that are currently not available outside the group setting, most notably the theory of coorbit molecules.

1.3. **Contributions.** The present paper provides a self-contained, greatly simplified and generalized, exposition of coorbit spaces associated with integrable representations and quasi-Banach function spaces. The exposition is similar to [54], but contains important further simplifications and improvements, which will be commented on at the relevant places throughout the text. As one of the biggest simplifications, it is not assumed here (in contrast to [45,47,54]) that the so-called control weight w satisfies $w(x) \geq \|R_x\|_{W^L(Y) \to W^L(Y)}$; instead, it is only required that $w(x) \geq \|R_x\|_{Y \to Y}$. Although it is true that the Wiener amalgam space $W^L(Y)$ is right-invariant whenever Y is (cf. [47, Corollary 4.2]), it does not seem possible to *readily* estimate $\|R_x\|_{W^L(Y) \to W^L(Y)}$ in terms of $\|R_x\|_{Y \to Y}$ whenever G is a non-IN group, see, e.g., [54, Lemma 2.3.18]. This simplification of using $\|R_x\|_{Y \to Y}$ instead of $\|R_x\|_{W^L(Y) \to W^L(Y)}$ is important, since having a control weight of only moderate growth is essential for obtaining sharp explicit conditions on atoms yielding atomic decompositions. To see that just estimating $\|R_x\|_{Y \to Y}$ instead of $\|R_x\|_{W^L(Y) \to W^L(Y)}$ is advantageous, we mention that large parts of the recent PhD thesis [4] are concerned with deriving usable bounds for the operator norm $\|R_x\|_{W^L(Y) \to W^L(Y)}$ for settings in which $\|R_x\|_{Y \to Y}$ can be readily estimated.

The biggest improvements in the present article compared to [46,54] are the results on atomic decompositions. Instead of adapting the classical sampling techniques from [19,34] as done in [46,54], the present article provides an extension of the recent results on dual molecules

as in [50] to the setting of quasi-Banach spaces. This approach is more easily accessible and at the same time yields much stronger conclusions on the localization of the dual system by showing that it also forms a family of molecules satisfying similar envelope conditions as the basic atoms. In particular, the obtained results on molecules close a gap between what was known for general (quasi-Banach) coorbit spaces [54] and the concrete setting of Besov-Triebel-Lizorkin spaces [27,32].

Lastly, several arguments presented in this article are different and often simpler than the classical arguments in [18–20,34], even for the setting of Banach spaces. For example, the present article avoids the use of an atomic decomposition to prove that the space of integrable vectors (so-called analyzing vectors) is dense, and instead provides a direct proof using Bochner integration, cf. Section 4.2; in particular, Lemma 4.3.

1.4. **Extensions.** To ensure a wide applicability, all the results in the present article will be proved for possibly *reducible* and/or *projective* group representations. This flexibility is essential for treating several key examples. For example, reducible representations occur naturally in high-dimensional wavelet theory [28,30,41], whereas the use of (possibly non-continuous) projective representations is convenient for treating weighted Bergman spaces [7,8,10] or representations of nilpotent Lie groups that are only square-integrable *modulo a central subgroup* [21,35].

As possible extensions of the theory [18–20,34], it was already mentioned in [18] that the case of quasi-Banach spaces, reducible and/or projective representations would be desirable for treating several key examples. For coorbit spaces associated with Banach spaces, extensions to reducible and projective representations can be found in [9,14] and [7,11], respectively. For *quasi*-Banach spaces, the present article seems to be the first to develop these extensions.

1.5. **Overview.** Section 2 introduces general notation used throughout the paper and provides background on quasi-Banach function spaces and local maximal functions. Convolution relations for Wiener amalgam spaces are proved in Section 3. In addition, Section 3.3 contains two counterexamples to convolution relations asserted in [47]. Section 4 is devoted to the definition of coorbit spaces and to studying their basic properties as quasi-Banach spaces. Convolution-dominated integral operators and matrices form the subject of Section 5. Among others, it will be shown that these classes of operators and matrices form algebras and possess a local holomorphic functional calculus. The results on convolution-dominated operators will be exploited in Section 6 to prove the existence of dual coorbit molecules of frames and Riesz sequences and to derive associated decompositions of coorbit spaces. Section 7 presents several applications of the obtained results to the boundedness of operators. Simplified statements of the main results for irreducible, square-integrable representations are given in Section 8. The proofs of several miscellaneous and technical results are postponed to the appendix.

2. QUASI-BANACH FUNCTION SPACES AND LOCAL MAXIMAL FUNCTIONS

This section provides background on solid function spaces, twisted convolution and quasi-Banach spaces.

2.1. **General notation.** Throughout this article, G denotes a second countable locally compact group and $Q \subset G$ denotes a fixed open, relatively compact symmetric neighborhood of the identity element $e_G \in G$. In addition, a left Haar measure μ_G on G is fixed and the associated modular function is denoted by $\Delta : G \to (0, \infty)$.

We write $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. For $x \in \mathbb{R}^d$ or $x \in \mathbb{C}$, we denote by $B_r(x)$ the open ball (with respect to the usual Euclidean norm) of radius $r > 0$ around x. For a subset V of a "base set" X that is usually implied by the context, the notation $\mathbb{1}_V$ denotes the indicator function of V, i.e., $\mathbb{1}_V(x) = 1$ if $x \in V$ and $\mathbb{1}_V(x) = 0$ otherwise. A pairwise disjoint union of sets will be denoted by the symbol $\biguplus$.

For functions $F_1, F_2 : X \to [0, \infty)$ on a set X (which in most cases will be implied by the context), the notation $F_1 \lesssim F_2$ means that there exists a constant $C > 0$ such that $F_1(x) \leq C \cdot F_2(x)$ for all $x \in X$. The notation $F_1 \asymp F_2$ is used if $F_1 \lesssim F_2$ and $F_2 \lesssim F_1$. For a function $F : G \to \mathbb{C}$, its involution $F^\vee : G \to \mathbb{C}$ is defined by $F^\vee(x) = F(x^{-1})$ for $x \in G$. A function $w : G \to (0, \infty)$ will be called *submultiplicative* if $w(xy) \leq w(x)w(y)$ for all $x, y \in G$.

The Lebesgue space of r-integrable functions is denoted by $L^r(G)$ for $0 < r \leq \infty$ and defined relative to μ_G. For a (measurable) function $w : G \to (0, \infty)$, the associated weighted spaces $L_w^r(G)$ are equipped with the norm $\|F\|_{L_w^r} := \|F \cdot w\|_{L^r}$. For $r \in [1, \infty)$, the bi-linear dual pairing between $L_w^r(G)$ and its dual space $L_{1/w}^{r'}(G)$ is denoted $(\cdot, \cdot)_{L_w^r, L_{1/w}^{r'}}$, while the sesquilinear pairing (which is anti-linear in the second component) is denoted by $\langle \cdot, \cdot \rangle_{L_w^r, L_{1/w}^{r'}}$. If the involved spaces are clear from the context, these pairings will also sometimes simply be denoted by $(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$.

2.2. Quasi-Banach function spaces.

A vector space Y is called a *quasi-normed space* if it is equipped with a map $\| \cdot \| : Y \to [0, \infty)$ that is positive definite (i.e., $\|f\| > 0$ for $f \neq 0$), absolute homogeneous (i.e., $\|\alpha f\| = |\alpha| \, \|f\|$) and such that there exists a constant $C > 0$ satisfying $\|f + g\| \leq C \cdot (\|f\| + \|g\|)$ for all $f, g \in Y$; such a map $\| \cdot \|$ is called a *quasi-norm* on Y with *triangle constant* $C > 0$. For $p \in (0, 1]$, a map $\| \cdot \| : Y \to [0, \infty)$ is a *p-norm* if $\|f + g\|^p \leq \|f\|^p + \|g\|^p$. If $\| \cdot \| : Y \to [0, \infty)$ satisfies the p-norm property, then it is a quasi-norm with triangle constant $C = 2^{\frac{1}{p} - 1}$; see, e.g. [33, Exercise 1.1.4].

By the Aoki-Rolewicz theorem (see, e.g., the proof of [15, Chapter 2, Theorem 1.1]), given a quasi-norm $\| \cdot \| : Y \to [0, \infty)$, there exists some $p \in (0, 1]$ such that

$$\||f\|| := \inf \left\{ \left(\sum_{i=1}^n \|f_i\|^p \right)^{\frac{1}{p}} \; : \; n \in \mathbb{N}, f = \sum_{i=1}^n f_i, \; f_1, ..., f_n \in Y \right\} \tag{2.1}$$

defines a p-norm on Y which is equivalent to $\| \cdot \|$, i.e., $\| \cdot \| \asymp \||\cdot\||$. A vector space Y with quasi-norm $\| \cdot \|$ is a *quasi-Banach space* if it is complete with respect to the metric $d(f, g) = \||f - g\||^p$, where $\||\cdot\||$ is any p-norm (for some $p \in (0, 1]$) equivalent to $\| \cdot \|$.

Let $L^0(G)$ be the space of (μ_G-equivalence classes of) measurable functions $f : G \to \mathbb{C}$. A *quasi-Banach function space* $(Y, \| \cdot \|_Y)$ is a quasi-Banach space satisfying $Y \subset L^0(G)$. It is called *solid* if for each measurable $f : G \to \mathbb{C}$ satisfying $|f| \leq |g|$ μ_G-a.e. for some $g \in Y$, it follows that $f \in Y$ with $\|f\|_Y \leq \|g\|_Y$.

It is not difficult to see (cf. [54, Corollary 2.2.12] for a proof) that if $(Y, \| \cdot \|_Y)$ is a solid quasi-Banach function space, then the p-norm $\||\cdot\||$ defined in Equation (2.1) is solid as well. Hence, we can (and will) always assume that solid quasi-Banach function spaces are equipped with a solid p-norm, for some $p \in (0, 1]$ depending on Y.

The space $Y^\vee = \{F^\vee : F \in Y\}$ associated with a quasi-normed space $Y \subset L^0(G)$ is equipped with the quasi-norm $\|F\|_{Y^\vee} = \|F^\vee\|_Y$.

2.3. Discrete sets.

Let $\Lambda = (\lambda_i)_{i \in I}$ be a countable family in G. Then Λ is called *relatively separated* if

$$\mathrm{Rel}(\Lambda) := \sup_{x \in G} \# \{i \in I : \lambda_i \in xQ\} = \sup_{x \in G} \sum_{i \in I} \mathbb{1}_{\lambda_i Q}(x) < \infty.$$

Let $U \subset G$ be a relatively compact unit neighborhood. The family Λ is said to be *U-dense* if $G = \bigcup_{i \in I} \lambda_i U$, and is called *$U$-separated* if $\lambda_i U \cap \lambda_{i'} U = \varnothing$ for all $i, i' \in I$ with $i \neq i'$. If Λ is separated, then it is also relatively separated. A family Λ is called *relatively dense* if it is V-dense for some unit neighborhood V. A relatively separated and U-dense family exists for any chosen U, cf. [3].

For a measurable unit neighborhood U, and any countable, U-dense family $\Lambda = (\lambda_i)_{i \in I}$ in G, there exists a family $(U_i)_{i \in I}$ of measurable sets $U_i \subset \lambda_i U$ satisfying $G = \biguplus_{i \in I} U_i$; see, e.g., [50, Lemma 2.1]. Any such family $(U_i)_{i \in I}$ is called a *disjoint cover* associated with Λ and U.

2.4. **Sequence spaces.** Let Y be a solid quasi-Banach function space on G with p-norm $\| \cdot \|_Y$. For a relatively separated family Λ in G, the *discrete sequence space* associated with Y and Λ is defined as

$$Y_d(\Lambda) := Y_d(\Lambda, Q) := \left\{ c = (c_i)_{i \in I} \in \mathbb{C}^I : \sum_{i \in I} |c_i| \mathbb{1}_{\lambda_i Q} \in Y \right\}$$

and equipped with the p-norm

$$\|c\|_{Y_d(\Lambda)} := \left\| \sum_{i \in I} |c_i| \mathbb{1}_{\lambda_i Q} \right\|_Y, \quad c = (c_i)_{i \in I} \in Y_d(\Lambda). \tag{2.2}$$

The space $Y_d(\Lambda)$ is a quasi-Banach space. If Y is right-invariant, then $Y_d(\Lambda, Q)$ is independent of the choice of the neighborhood Q, with equivalent quasi-norms for different choices. In fact, the implied constant of the norm equivalence for different choice of Q is independent of Λ. To see this, note that if Q' is a relatively compact unit neighborhood, then $Q \subset \bigcup_{j=1}^n Q' x_j^{-1}$ for a certain $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in G$. Hence, for any family $\Lambda = (\lambda_i)_{i \in I} \subset G$,

$$\left\| \sum_{i \in I} |c_i| \mathbb{1}_{\lambda_i Q} \right\|_Y \leq \left\| \sum_{i \in I} |c_i| \sum_{j=1}^n \mathbb{1}_{\lambda_i Q' x_j^{-1}} \right\|_Y \lesssim_{n,Y} \sum_{j=1}^n \left\| R_{x_j} \sum_{i \in I} |c_i| \mathbb{1}_{\lambda_i Q'} \right\|_Y$$

$$\lesssim_{n,Y} \left(\sum_{j=1}^n \|R_{x_j}\|_{Y \to Y} \right) \left\| \sum_{i \in I} |c_i| \, \mathbb{1}_{\lambda_i Q'} \right\|_Y, \tag{2.3}$$

where the implied constants only depend on Q, Q', Y and are independent of Λ.

If $Y = L_v^p(G)$ for $p \in (0, \infty]$ and a (measurable) weight $v : G \to (0, \infty)$ satisfying $v(xy) \leq w(x)v(y)$ and $v(xy) \leq v(x)w(y)$ for a submultiplicative, measurable $w : G \to (0, \infty)$, then $Y_d(\Lambda) = \ell_u^p(I)$, where $u(i) := v(\lambda_i)$ for $i \in I$. The space of finite sequences on Λ will be denoted by $c_{00}(\Lambda)$.

For further properties and proofs, cf. [47, Section 2] and [54, Section 2.3.2].

2.5. **Cocycles and twisted convolution.** A *cocycle* or *multiplier* on G is a Borel measurable function $\sigma : G \times G \to \mathbb{T}$ satisfying the properties

  (1) For all $x, y, z \in G$, it holds $\sigma(x, yz)\sigma(y, z) = \sigma(xy, z)\sigma(x, y)$;
  (2) For any $x \in G$, it holds $\sigma(x, e_G) = \sigma(e_G, x) = 1$.

Given a cocycle σ, the associated *twisted translation operators* L_x^σ and R_x^σ, where $x \in G$, act on a measurable function $F : G \to \mathbb{C}$ as

$$(L_x^\sigma F)(y) = \sigma(x, x^{-1}y)F(x^{-1}y) \quad \text{and} \quad (R_x^\sigma F)(y) = \overline{\sigma(y, x)}F(yx), \quad y \in G. \tag{2.4}$$

If $\sigma \equiv 1$, then L_x^σ (resp. R_x^σ) will simply be denoted by L_x (resp. R_x) and called a translation operator. The *twisted convolution* of two measurable functions $F_1, F_2 : G \to \mathbb{C}$ is defined by

$$F_1 *_\sigma F_2(x) = \int_G F_1(y)[L_y^\sigma F_2](x) \, d\mu_G(y) = \int_G F_1(y)\sigma(y, y^{-1}x)F_2(y^{-1}x) \, d\mu_G(y) \tag{2.5}$$

whenever the integral converges. As for twisted translation, if $\sigma \equiv 1$, the ordinary convolution product will be denoted by $*$. Note that

$$|F_1 *_\sigma F_2(x)| \leq (|F_1| * |F_2|)(x)$$

for all $x \in G$.

2.6. **Local maximal functions.** For a measurable (resp. continuous) function $F : G \to \mathbb{C}$, the *left* and *right (local) maximal functions* defined by

$$M^L F(x) = \operatorname*{ess\,sup}_{y \in xQ} |F(y)| \quad \text{resp.} \quad M^R F(x) = \operatorname*{ess\,sup}_{y \in Qx} |F(y)|, \tag{2.6}$$

are measurable (resp. continuous) on G. The maximal functions satisfy the commutation relations

$$M^L[L_x F] = L_x[M^L F] \qquad \text{and} \qquad M^R[R_x F] = R_x[M^R F] \qquad (2.7)$$

for arbitrary $x \in G$.

The notation $M^L_{Q'}$ (resp. $M^R_{Q'}$) will be used if maximal functions are defined as in (2.6), but relative to a (relatively compact) unit neighborhood Q' that is possibly different from the canonical neighborhood Q (cf. Section 2.1).

By symmetry of Q, the left and right maximal functions are related by $(M^L F)^\vee = M^R F^\vee$. For any measurable functions $F_1, F_2 : G \to \mathbb{C}$ for which $F_1 *_\sigma F_2$ is (almost everywhere) well-defined, the estimates

$$M^L(F_1 *_\sigma F_2)(x) \leq (|F_1| * M^L F_2)(x) \quad \text{and} \quad M^R(F_1 *_\sigma F_2)(x) \leq (M^R F_1 * |F_2|)(x) \qquad (2.8)$$

hold for all $x \in G$.

For a solid quasi-Banach function space Y that is invariant under left- and right translation and with p-norm $\|\cdot\|_Y$, the associated left and right *Wiener amalgam spaces* $W^L(Y)$ and $W^R(Y)$ are defined by

$$W^L(Y) = \{F \in L^\infty_{\text{loc}}(G) : M^L F \in Y\} \quad \text{and} \quad W^R(Y) = \{F \in L^\infty_{\text{loc}}(G) : M^R F \in Y\},$$

respectively. These spaces are equipped with the canonical p-norms $\|F\|_{W^L(Y)} = \|M^L F\|_Y$ and $\|F\|_{W^R(Y)} = \|M^R F\|_Y$. Note that $W^R(Y) = [W^L(Y^\vee)]^\vee$, as can be deduced from the identity $(M^L F)^\vee = M^R(F^\vee)$. The spaces $W^L(Y)$ and $W^R(Y)$ are complete and continuously embedded into Y; in notation, $W^L(Y), W^R(Y) \hookrightarrow Y$. In addition, the spaces $W^L(Y)$ and $W^R(Y)$ are independent of the choice of defining neighborhood Q, with equivalent norms for different choices. Furthermore, $W^L(Y), W^R(Y)$ are invariant under left- and right-translations, and satisfy the following estimates

$$\|L_x\|_{W^L(Y) \to W^L(Y)} \leq \|L_x\|_{Y \to Y} \qquad \text{and} \qquad \|R_x\|_{W^R(Y) \to W^R(Y)} \leq \|R_x\|_{Y \to Y}. \qquad (2.9)$$

These estimates are easy consequences of Equation (2.7).

The following simple pointwise estimates for continuous functions will be used repeatedly. For a proof, see, e.g., [50, Lemma 2.4].

Lemma 2.1. *Let $\Lambda = (\lambda_i)_{i \in I} \subset G$ be relatively separated. If $F_1, F_2 : G \to [0, \infty)$ are continuous functions, then*

$$\sum_{i \in I} F_1(\lambda_i^{-1} x) F_2(y^{-1} \lambda_i) \leq \frac{\text{Rel}(\Lambda)}{\mu_G(Q)} \cdot (M^L F_2 * M^R F_1)(y^{-1} x) \qquad (2.10)$$

$$\sum_{i \in I} F_1(y^{-1} \lambda_i) \leq \frac{\text{Rel}(\Lambda)}{\mu_G(Q)} \|F_1\|_{W^L(L^1)} \qquad (2.11)$$

for all $x, y \in G$.

In addition, if $F_1 \in L^1(G)$ is continuous and satisfies $F_1^\vee \in W^L(L^1)$, then the mapping

$$D_{F_1, \Lambda} : \quad \ell^2(\Lambda) \to L^2(G), \quad (c_i)_{i \in I} \mapsto \sum_{i \in I} c_i L_{\lambda_i} F_1$$

is well-defined and bounded, with absolute convergence of the defining series a.e. on G. Its operator norm satisfies $\|D_{F_1, \Lambda}\|^2_{\ell^2 \to L^2} \leq \frac{\text{Rel}(\Lambda)}{\mu_G(Q)} \|F_1\|_{L^1} \|F_1^\vee\|_{W^L(L^1)}$.

The *(two-sided) maximal function* is defined by $M := M^L M^R = M^R M^L$. The associated two-sided Wiener amalgam space is the subspace of $W^L(Y) \cap W^R(Y)$ defined by

$$W(Y) := \{F \in L^\infty_{\text{loc}}(G) : MF \in Y\}.$$

The (closed) subspace of $W(Y)$ consisting of continuous functions is denoted by $W_C(Y)$. Both $W(Y)$ and $W_C(Y)$ are quasi-Banach function spaces when equipped with the p-norm

$\|F\|_{W(Y)} = \|MF\|_Y$. These properties follow from $W(Y) = W^R(W^L(Y))$. In addition, note that

$$[W(Y)]^\vee = W^L([W^L(Y)]^\vee) = W^L(W^R(Y^\vee)) = W(Y^\vee).$$

For further properties and proofs, see [47, Sections 2 and 3] and [54, Section 2.3].

3. Convolution relations for Wiener amalgam spaces

The purpose of this section is to provide several convolution relations and simple embeddings of Wiener amalgam spaces into Lebesgue spaces. These results will play an essential role in the development of coorbit theory in the subsequent sections.

3.1. **Embeddings into Lebesgue spaces.** The following conditions on a weight function will often be assumed in the sequel.

Definition 3.1. Let $p \in (0, 1]$. A function $w : G \to (0, \infty)$ will be called a *p-weight* if it satisfies the following conditions:

(w1) w is measurable and satisfies $w \geq 1$,
(w2) w is *submultiplicative*, i.e., $w(xy) \leq w(x)w(y)$ for all $x, y \in G$,
(w3) $w(x) = w(x^{-1}) \cdot \Delta^{1/p}(x^{-1})$ for all $x \in G$.

A p-weight is called a *control weight* for a quasi-Banach function space $(Y, \|\cdot\|_Y)$ with p-norm $\|\cdot\|_Y$ if Y is *translation-invariant* (i.e., invariant under left- and right translations) and w satisfies

(w4) $\|R_y\|_{Y \to Y} \leq w(y)$ for all $y \in G$.

A control weight w for Y satisfying additionally

(w5) $\|L_{y^{-1}}\|_{Y \to Y} \leq w(y)$ for all $y \in G$

is called a *strong control weight* for Y.

Remark 3.2. A measurable, submultiplicative weight is automatically locally bounded; see, e.g., [39, Lemma 1.3.3].

Lemma 3.3. *Let $r \in (0, \infty]$, let $w : G \to (0, \infty)$ be measurable and submultiplicative, and let Y be a solid, translation-invariant quasi-Banach function space on G. Then the following hold:*

(i) If $v : G \to (0, \infty)$ is measurable and satisfies $\|L_{y^{-1}}\|_{Y \to Y} \leq v(y)$ for all $y \in G$, then $W^L(Y) \hookrightarrow L^\infty_{1/v}$.
(ii) $W^L(L^r_w) \hookrightarrow L^s_w$ for all $s \in [r, \infty]$,
(iii) If $p \in (0, 1]$ and w is a p-weight, then $W^R(L^p_w) \hookrightarrow [L^s_w]^\vee$ for all $s \in [p, \infty]$.

Remark 3.4. In fact, the lemma implies something slightly stronger: If $F \in W^L(L^r_w)$, then $M^L F \in W^L(L^r_w) \hookrightarrow L^s_w$ and hence $F \in W^L(L^s_w)$. This argument easily implies that $W^L(L^r_w) \hookrightarrow W^L(L^s_w)$ for $s \in [r, \infty]$.

Proof. For proving assertions (i)-(iii), it will be used that for every measurable $F : G \to \mathbb{C}$, there exists a null-set $N \subset G$ such that for every $x \in G \setminus N$, it holds that

$$\mathbb{1}_{xQ^{-1}} \cdot |F(x)| \leq M^L F \quad \text{a.e.} \tag{3.1}$$

To show this, let $\Omega := \{(x, z) \in G \times G : \mathbb{1}_{xQ^{-1}}(z) \cdot |F(x)| > M^L F(z)\}$. Since $M^L F$ is measurable (cf. Section 2.6), it is easy to see that Ω belongs to the Borel σ-algebra on $G \times G$. Since G is second countable, the Borel σ-algebra on $G \times G$ coincides with the product σ-algebra on $G \times G$; see e.g. [23, Theorem 7.20]. Let $\Omega_x = \{z \in G : (x, z) \in \Omega\}$ and $\Omega^z = \{x \in G : (x, z) \in \Omega\}$. Then Tonelli's theorem shows that $\int_G \mu_G(\Omega_x) \, d\mu_G(x) = \int_G \mu_G(\Omega^z) \, d\mu_G(z)$. Therefore, it suffices to show that Ω^z is a null-set for arbitrary $z \in G$. For this, let $z \in G$. By definition of M^L, there exists a null-set $N_z \subset Q$ such that $M^L F(z) \geq |F(zq)|$ for all $q \in Q \setminus N_z$.

For $x \in z(G \setminus N_z)$ there are two cases. First, if $x \notin zQ$, then $\mathbb{1}_{xQ^{-1}}(z) = 0$ and hence trivially $\mathbb{1}_{xQ^{-1}}(z) \cdot |F(x)| \leq M^L F(z)$. Second, if $x \in zQ$, then $q := z^{-1}x \in Q \setminus N_z$ and hence $M^L F(z) \geq |F(zq)| = |F(x)| \geq |F(x)|\mathbb{1}_{xQ^{-1}}(z)$. Overall, this shows that $\Omega^z \subset zN_z$ is a null-set, and establishes inequality (3.1).

(i) Define $C := 1/\|\mathbb{1}_{Q^{-1}}\|_Y$, with the understanding that $C = 0$ if $\mathbb{1}_{Q^{-1}} \notin Y$. Let $F \in W^L(Y)$. It will be shown that $\|F\|_{L^\infty_{1/v}} \leq C \, \|F\|_{W^L(Y)}$. For this, let $N = N(F) \subset G$ be a null set such that (3.1) holds for all $x \in G \setminus N$. For $x \in G \setminus N$, it follows then by the solidity of Y that

$$|F(x)| \cdot \|\mathbb{1}_{xQ^{-1}}\|_Y \leq \|M^L F\|_Y = \|F\|_{W^L(Y)} < \infty.$$

If $|F(x)| \neq 0$, then this implies $\mathbb{1}_{xQ^{-1}} \in Y$, and

$$\|\mathbb{1}_{Q^{-1}}\|_Y = \|L_{x^{-1}}\mathbb{1}_{xQ^{-1}}\|_Y \leq v(x) \cdot \|\mathbb{1}_{xQ^{-1}}\|_Y,$$

so that $|F(x)|/v(x) \leq C \cdot \|F\|_{W^L(Y)}$. If $|F(x)| = 0$, then this trivially holds. In conclusion, it follows that $|F(x)|/v(x) \leq C \cdot \|F\|_{W^L(Y)}$ for all $x \in G \setminus N$.

(ii) It will be shown that $W^L(L^r_w) \hookrightarrow L^\infty_w$. Since $W^L(L^r_w) \hookrightarrow L^r_w$ (cf. Section 2.6), this will then imply that $W^L(L^r_w) \hookrightarrow L^s_w$ for all $s \in [r, \infty]$. Let $F \in W^L(L^r_w)$. Choose a null-set $N = N(F) \subset G$ such that Equation (3.1) holds for all $x \in G \setminus N$. Since L^r_w is solid, it follows that, for all $x \in G \setminus N$,

$$|F(x)| \cdot \|\mathbb{1}_{xQ^{-1}}\|_{L^r_w} \leq \|M^L F\|_{L^r_w} = \|F\|_{W^L(L^r_w)} < \infty.$$

There exists $C = C(Q, w) > 0$ such that $w(q) \leq C$ for all $q \in Q$ (cf. Remark 3.2). Hence, if $z \in Q^{-1}$, then $w(x) \leq w(xz)w(z^{-1}) \leq C \cdot w(xz)$ for all $x \in G$. Therefore,

$$\|\mathbb{1}_{xQ^{-1}}\|^r_{L^r_w} = \int_{xQ^{-1}} [w(y)]^r d\mu_G(y) = \int_{Q^{-1}} [w(xz)]^r d\mu_G(z) \geq C^{-r} \cdot [w(x)]^r \cdot \mu_G(Q^{-1})$$

for all $x \in G$. Overall, this shows $|F| \cdot w \lesssim \|F\|_{W^L(L^r_w)}$ a.e. on G, as required.

(iii) Condition (w3) combined with the identity $\int_G F(x) \, d\mu_G(x) = \int_G F(x^{-1})\Delta(x^{-1}) \, d\mu_G(x)$ easily implies that $[L^p_w]^\vee = L^p_w$, with identical quasi-norms. Hence,

$$W^R(L^p_w) = [W^L((L^p_w)^\vee)]^\vee = [W^L(L^p_w)]^\vee,$$

with identical quasi-norms (cf. Section 2.6). Therefore, Part (ii) implies for all $s \in [p, \infty]$ that $W^R(L^p_w) = [W^L(L^p_w)]^\vee \hookrightarrow [L^s_w]^\vee$. $\qquad\square$

3.2. Convolution relations.
This subsection is devoted to convolution relations for Wiener amalgam spaces. The following compatibility condition between a quasi-Banach function space and a weight function will play an important role in the sequel.

Definition 3.5. Let $w : G \to [1, \infty)$ be a p-weight for some $p \in (0, 1]$ and let Y be a solid quasi-Banach function space Y on G with p-norm $\| \cdot \|_Y$. The space Y is said to be L^p_w-*compatible* if

(c1) Y is translation-invariant.
(c2) The convolution relation $W^L(Y) * W(L^p_w) \hookrightarrow W^L(Y)$ holds.
(c3) The embedding $W^L(Y) \hookrightarrow L^\infty_{1/w}(G)$ holds.

Remark 3.6. In Corollary 3.9, it will be shown that Y is L^p_w-compatible if w is a strong control weight for Y. The reason for introducing the concept of L^p_w-compatibility is that there are some (important) cases in which L^p_w-compatibility holds even if w is not a strong control weight for Y. A case in point are the spaces $Y = W^L(L^p_w)$ and $Y = W^R(L^p_w)$, as well as $Y = W(L^p_w)$, which are always L^p_w-compatible if w is a p-weight; see Corollary 3.10.

The following result provides a general convolution relation for Amalgam spaces. The most important consequences of this result are stated in Corollary 3.9.

Theorem 3.7. *Let $p \in (0,1]$ and let $w : G \to (0,\infty)$ be a p-weight. Let Y be a solid, translation-invariant quasi-Banach function space and assume that w is a control weight for Y. Suppose there exists a submultiplicative measurable weight $v : G \to [1,\infty)$ satisfying $W^L(Y) \hookrightarrow L^\infty_{1/v}$ and $W^R(L^p_w) \hookrightarrow [L^1_v]^\vee$. Then the convolution relation*

$$W^L(Y) * W^R(L^p_w) \hookrightarrow Y$$

holds. More precisely, there exists a constant $C = C(p,w,Q) > 0$ such that

$$\|F_1 * F_2\|_Y \le \||F_1| * |F_2|\|_Y \le C\|F_1\|_{W^L(Y)}\|F_2\|_{W^R(L^p_w)}$$

for all $F_1 \in W^L(Y)$ and $F_2 \in W^R(L^p_w)$. In addition, $|F_1| * |F_2|(x) < \infty$ for all $x \in G$ and $F_1 * F_2 : G \to \mathbb{C}$ is continuous.

Proof. The proof will be split into three steps.

Step 1. Since $F_1 * F_2(x) = (F_1, L_x[F_2^\vee])_{L^\infty_{1/v}, L^1_v}$, it follows that $|F_1| * |F_2|(x) < \infty$ for all $x \in G$. Furthermore, the map $x \mapsto L_x[F_2^\vee]$ is continuous from G into L^1_v, see, e.g., [39, Lemma 1.3.6] or [49, Proposition 3.7.6]. Hence, $F_1 * F_2 : G \to \mathbb{C}$ is continuous as well.

Step 2. In this step, it will be shown that there exists a countable family $(x_i)_{i \in I} \subset G$ and a measurable partition of unity $(\varphi_i)_{i \in I}$ on G with $\varphi_i^{-1}(\mathbb{C} \setminus \{0\}) \subset x_i Q$ and a constant $C = C(p,Q,w) > 0$ such that

$$\left\| \left(w(x_i) \cdot \|F \cdot \varphi_i\|_{L^\infty} \right)_{i \in I} \right\|_{\ell^p} \le C\|F\|_{W^L(L^p_w)} \tag{3.2}$$

for all $F \in W^L(L^p_w)$.

For constructing $(\varphi_i)_{i \in I}$, an application of [3, Lemma 1] yields a subset $X \subset G$ and some $N \in \mathbb{N}$ such that $G = \bigcup_{x \in X} xQ$ and such that each $x \in G$ belongs to at most N of the sets xQ for $x \in X$. Since G is second-countable and hence σ-compact, it is possible to extract a countable subset of X that still satisfies these conditions, so that it may be assumed that X is countable. Thus, for a suitable $M \in \mathbb{N} \cup \{\infty\}$ and for $I = \{i \in \mathbb{N}: i < M\}$, we can write $X = \{x_i: i \in I\}$ with $x_i \ne x_j$ for $i \ne j$. Define $\Omega_i := x_i Q \setminus \bigcup_{\ell=1}^{i-1} x_\ell Q$ inductively and set $\varphi_i := \mathbb{1}_{\Omega_i}$. Then $\varphi_i^{-1}(\mathbb{C} \setminus \{0\}) = \Omega_i \subset x_i Q$ and $\sum_{i \in I} \varphi_i \equiv 1$ on G.

For showing (3.2), define $Q' := Q^{-1}Q$ and let $F \in W^L(L^p_w)$. Note that if $x \in x_i Q$, then $\varphi_i^{-1}(\mathbb{C} \setminus \{0\}) \subset x_i Q \subset xQ^{-1}Q = xQ'$. This implies that $\|F \cdot \varphi_i\|_{L^\infty} \le M^L_{Q'}F(x)$. On the other hand, we have $w \lesssim 1$ everywhere on Q (cf. Remark 3.2), and thus there exists $C_1 > 0$ such that $w(x_i) \le w(x)w(x^{-1}x_i) \le C_1 w(x)$. Overall, these considerations show that

$$[w(x_i) \cdot \|F \cdot \varphi_i\|_{L^\infty}]^p \cdot \mathbb{1}_{x_i Q} \lesssim [w \cdot M^L_{Q'}F]^p \quad \text{for all} \quad i \in I.$$

By the choice of $\{x_i: i \in I\} = X$, it holds that $\sum_{i \in I} \mathbb{1}_{x_i Q} \le N$, and hence

$$\sum_{i \in I} [w(x_i) \cdot \|F \cdot \varphi_i\|_{L^\infty}]^p \cdot \mathbb{1}_{x_i Q} \lesssim [w \cdot M^L_{Q'}F]^p \cdot \sum_{i \in I} \mathbb{1}_{x_i Q} \lesssim [w \cdot M^L_{Q'}F]^p.$$

Integrating this estimate over G gives

$$\sum_{i \in I} [w(x_i) \cdot \|F \cdot \varphi_i\|_{L^\infty}]^p = \frac{1}{\mu_G(Q)} \sum_{i \in I} \int_G [w(x_i) \cdot \|F \cdot \varphi_i\|_{L^\infty}]^p \cdot \mathbb{1}_{x_i Q}(x) d\mu_G(x)$$

$$\lesssim \|M^L_{Q'}F\|^p_{L^p_w} \lesssim \|F\|^p_{W^L(L^p_w)},$$

where the last estimate follows from the independence of $W^L(L^p_w) = W^L_Q(L^p_w)$ from the choice of the set Q (cf. Section 2.6.) The preceding estimate easily shows that Equation (3.2) holds.

Step 3. With notation as in Step 2, set $\psi_i := \varphi_i^\vee$ and $F_2^{(i)} := F_2 \cdot \psi_i$ for $i \in I$. Note that $(F_2^{(i)})^\vee = F_2^\vee \cdot \varphi_i$ and $\psi_i^{-1}(\mathbb{C} \setminus \{0\}) \subset Q^{-1}x_i^{-1}$. Furthermore, note that $L^p_w = [L^p_w]^\vee$,

which follows from condition (w3) of Definition 3.1. Hence, applying the estimate (3.2) to the function $F_2^\vee \in W^L([L_w^p]^\vee) = W^L(L_w^p)$ shows that

$$\left\| \left(w(x_i) \cdot \|F_2^{(i)}\|_{L^\infty} \right)_{i \in I} \right\|_{\ell^p} \lesssim \|F_2^\vee\|_{W^L(L_w^p)} = \|F_2\|_{[W^L(L_w^p)]^\vee} = \|F_2\|_{W^R(L_w^p)}. \qquad (3.3)$$

If $i \in I$ and $F_2^{(i)}(y^{-1}x) \neq 0$, then $y^{-1}x \in (F_2^{(i)})^{-1}(\mathbb{C} \setminus \{0\}) \subset Q^{-1}x_i^{-1}$ and hence $\mathbb{1}_{xx_iQ}(y) = 1$. Thus, $F_2^{(i)}(y^{-1}x) = F_2^{(i)}(y^{-1}x)\mathbb{1}_{xx_iQ}(y)$, and

$$
\begin{aligned}
(|F_1| * |F_2^{(i)}|)(x) &= \int_G |F_1(y)| \cdot |F_2^{(i)}(y^{-1}x)| d\mu_G(y) \\
&\leq \|F_2^{(i)}\|_{L^\infty} \|F_1 \cdot \mathbb{1}_{xx_iQ}\|_{L^\infty} \cdot \mu_G(xx_iQ) \\
&= \mu_G(Q) \cdot \|F_2^{(i)}\|_{L^\infty} \cdot (M^L F_1)(xx_i) \\
&= \mu_G(Q) \cdot \|F_2^{(i)}\|_{L^\infty} \cdot R_{x_i}[M^L F_1](x)
\end{aligned}
$$

for all $x \in G$. By the solidity of Y and since $\|R_{x_i}\|_{Y \to Y} \leq w(x_i)$ by assumption (cf. condition (w4) of Definition 3.1), this implies

$$\big\| |F_1| * |F_2^{(i)}| \big\|_Y \lesssim \|F_2^{(i)}\|_{L^\infty} \|R_{x_i}[M^L F_1]\|_Y \leq w(x_i) \cdot \|F_2^{(i)}\|_{L^\infty} \|F_1\|_{W^L(Y)}, \quad i \in I.$$

Using the estimate (3.3) and Lemma A.2, it follows that $F := \sum_{i \in I} |F_1| * |F_2^{(i)}| \in Y$ and

$$\|F\|_Y \lesssim \|F_1\|_{W^L(Y)} \cdot \left\| \left(w(x_i) \cdot \|F_2^{(i)}\|_{L^\infty} \right)_{i \in I} \right\|_{\ell^p} \lesssim \|F_1\|_{W^L(Y)} \|F_2\|_{W^R(L_w^p)}.$$

Since $F_2 = \sum_{i \in I} F_2^{(i)}$, an application of the monotone convergence theorem yields

$$|F_1 * F_2| \leq |F_1| * |F_2| \leq \sum_{i \in I} |F_1| * |F_2^{(i)}| = F,$$

which easily implies the claim. $\qquad\square$

Remark 3.8. The assumptions $W^L(Y) \hookrightarrow L_{1/v}^\infty$ and $W^R(L_w^p) \hookrightarrow [L_v^1]^\vee$ in Theorem 3.7 are not actually needed for deriving the convolution relation $W^L(Y) * W^R(L_w^p) \hookrightarrow Y$; these assumptions are only used to ensure that $F_1 * F_2$ is well-defined everywhere and continuous.

As a consequence of Theorem 3.7, it follows, in particular, that Y is L_w^p-compatible whenever w is a strong control weight for Y. This sufficient condition for compatibility is particularly convenient for applications. The precise statement is as follows.

Corollary 3.9. *Let $w : G \to [1, \infty)$ be a p-weight for some $p \in (0, 1]$.*

(i) *The following convolution relations hold:*

$$W^L(L_w^p) * W^R(L_w^p) \hookrightarrow L_w^p \qquad and \qquad W(L_w^p) * W(L_w^p) \hookrightarrow W_C(L_w^p).$$

(ii) *If w is a strong control weight for a solid, translation-invariant quasi-Banach function space Y in the sense of Definition 3.1, then the following convolution relations hold*

$$W^L(Y) * W^R(L_w^p) \hookrightarrow Y \qquad and \qquad W^L(Y) * W(L_w^p) \hookrightarrow W^L(Y).$$

In particular, Y is L_w^p-compatible.

Proof. (i) We apply Theorem 3.7 with $Y = L_w^p$ and $v \equiv 1$. For this, note that applying the identity $\int R_y F \, d\mu_G = \Delta(y^{-1}) \int F \, d\mu_G$ for measurable $F : G \to [0, \infty]$ implies for $f \in L_w^p$ that

$$
\begin{aligned}
\|R_y f\|_{L_w^p}^p &= \int_G |w(xyy^{-1})f(xy)|^p d\mu_G(x) \leq [w(y^{-1})]^p \int_G R_y[w \cdot f|^p] \, d\mu_G \\
&= [w(y^{-1}) \Delta^{1/p}(y^{-1})]^p \cdot \|f\|_{L_w^p}^p,
\end{aligned}
$$

and hence $\|R_y\|_{Y \to Y} \leq w(y^{-1}) \cdot \Delta^{1/p}(y^{-1})$. Since $w(y^{-1}) \cdot \Delta^{1/p}(y^{-1}) = w(y)$ by condition (w3) of Definition 3.1, this shows that w is a control weight for Y. Furthermore, Lemma 3.3

shows because of $w \geq 1 \equiv v = \frac{1}{v}$ and $p \leq 1$ that $W^L(L_w^p) \hookrightarrow L_w^\infty \hookrightarrow L_v^\infty = L_{1/v}^\infty$. Similarly, Lemma 3.3 shows that $W^R(L_w^p) \hookrightarrow [L_w^1]^\vee \hookrightarrow [L_v^1]^\vee$. This shows that all assumptions of Theorem 3.7 are satisfied, and thus $W^L(L_w^p) * W^R(L_w^p) \hookrightarrow L_w^p$.

Finally, for $F_1, F_2 \in W(L_w^p)$, an application of Equation (2.8) yields

$$\|F_1 * F_2\|_{W(L_w^p)} = \|M^L M^R (F_1 * F_2)\|_{L_w^p} \leq \|(M^R F_1) * (M^L F_2)\|_{L_w^p}$$
$$\lesssim \|M^R F_1\|_{W^L(L_w^p)} \|M^L F_2\|_{W^R(L_w^p)} \leq \|F_1\|_{W(L_w^p)} \|F_2\|_{W(L_w^p)},$$

as asserted, where Theorem 3.7 shows that $F_1 * F_2$ is continuous for $F_1 \in W(L_w^p) \hookrightarrow W^L(L_w^p)$ and $F_2 \in W(L_w^p) \hookrightarrow W^R(L_w^p)$.

(ii) By assumption, $\|L_{y^{-1}}\|_{Y \to Y} \leq w(y)$ for all $y \in G$, and thus Lemma 3.3 shows that $W^L(Y) \hookrightarrow L_{1/w}^\infty$. Furthermore, Lemma 3.3 also shows that $W^R(L_w^p) \hookrightarrow [L_w^1]^\vee$. Thus, Theorem 3.7 is applicable and shows $W^L(Y) * W^R(L_w^p) \hookrightarrow Y$. Combining this convolution relation with Equation (2.8) and the solidity of Y, we see for $F_1 \in W^L(Y)$ and $F_2 \in W(L_w^p)$ that

$$\|F_1 * F_2\|_{W^L(Y)} = \|M^L (F_1 * F_2)\|_Y \leq \||F_1| * M^L F_2\|_Y$$
$$\lesssim \|F_1\|_{W^L(Y)} \|M^L F_2\|_{W^R(L_w^p)} = \|F_1\|_{W^L(Y)} \|F_2\|_{W(L_w^p)},$$

completing the proof. $\qquad\square$

Corollary 3.10. *Let $p \in (0,1]$ and let $w : G \to (0,\infty)$ be a p-weight. Then each of the spaces $W^L(L_w^p), W^R(L_w^p)$ and $W(L_w^p)$ is L_w^p-compatible.*

Proof. This easily follows by combining Corollary 3.9 and Lemma 3.3. $\qquad\square$

The last result of this subsection concerns genuine *Banach* function spaces satisfying the so-called *Fatou property*, see, e.g., [56, Chapter 15, § 65]. Among others, it applies to weighted Lebesgue spaces.

Lemma 3.11. *Let Y be a solid Banach function space that is right-translation-invariant and satisfies the* weak Fatou property*, meaning that there exists a constant $C > 0$ such that*

$$\left\| \liminf_{n \to \infty} F_n \right\|_Y \leq C \cdot \liminf_{n \to \infty} \|F_n\|_Y$$

for every sequence $(F_n)_{n \in \mathbb{N}}$ of non-negative functions $F_n \in Y$ with $\liminf_{n \to \infty} \|F_n\|_Y < \infty$.

If $w : G \to (0,\infty)$ is measurable with $w(x) \geq \|R_x\|_{Y \to Y}$ for all $x \in G$, then $F_1 * F_2$ is almost-everywhere well-defined for every $F_1 \in Y$ and $F_2 : G \to \mathbb{C}$ with $F_2^\vee \in L_w^1$, and it holds

$$\|F_1 * F_2\|_Y \lesssim \|F_1\|_Y \cdot \|F_2^\vee\|_{L_w^1}.$$

In particular, if w is a 1-weight, then the following convolution relations hold:

$$Y * L_w^1 \hookrightarrow Y \qquad and \qquad W^L(Y) * W(L_w^1) \hookrightarrow Y * W^L(L_w^1) \hookrightarrow W^L(Y).$$

Proof. Let M^+ denote the set of all non-negative measurable functions $F : G \to [0,\infty]$, where functions are identified whenever they agree almost everywhere. The norm $\|\cdot\|_Y$ will be extended to a map $M^+ \to [0,\infty]$ by setting $\|F\|_Y := \infty$ if $F \notin Y$. It is then straightforward to verify that $\|\cdot\|_Y : M^+ \to [0,\infty]$ is a function norm in the sense of [56, Section 63]. By our assumptions, it also follows that $\|\cdot\|_Y$ satisfies the weak Fatou property as defined in [56, Section 65]; see [56, Theorem 3 in Section 65].

In accordance with [56, Section 68], the *associated seminorms* $\|\cdot\|_Y^{(n)} : M^+ \to [0,\infty]$ are inductively defined by $\|\cdot\|_Y^{(0)} := \|\cdot\|_Y$ and

$$\|F\|_Y^{(n)} := \sup \left\{ \int_G F \cdot H \, d\mu_G \ : \ H \in M^+ \text{ with } \|H\|_Y^{(n-1)} \leq 1 \right\} \quad \text{for } n \geq 1.$$

Since $\|\cdot\|_Y$ satisfies the weak Fatou property, an application of [56, Theorem 3 in Section 71] shows that $\|\cdot\|_Y \asymp \|\cdot\|_Y^{(2)}$.

Let $F_1, F_2, H \in M^+$ with $F_1 \in Y$, $F_2^\vee \in L_w^1$, and $\|H\|_Y^{(1)} \leq 1$. Tonelli's theorem shows $F_1 * F_2 \in M^+$. Further, $F_1 * F_2(x) = \int_G F_1(y)F_2(y^{-1}x)d\mu_G(y) = \int_G (R_z F_1)(x) F_2(z^{-1})d\mu_G(z)$. Based on this identity, another application of Tonelli's theorem yields that

$$\int_G (F_1 * F_2)(x)H(x) \, d\mu_G(x) = \int_G F_2(z^{-1}) \int_G (R_z F_1)(x) \, H(x) \, d\mu_G(x)d\mu_G(z).$$

By definition of $\|\cdot\|_Y^{(1)}$, it follows $\int_G (R_z F_1)(x) \, H(x)d\mu_G(x) \leq \|H\|_Y^{(1)} \cdot \|R_z F_1\|_Y \leq w(z) \cdot \|F_1\|_Y$. Hence,

$$\int_G (F_1 * F_2)(x)H(x)d\mu_G(x) \leq \|F_1\|_Y \int_G w(z) \cdot F_2^\vee(z)d\mu_G(z) = \|F_1\|_Y \|F_2^\vee\|_{L_w^1}.$$

Since this holds for every $H \in M^+$ with $\|H\|_Y^{(1)} \leq 1$, this implies

$$\|F_1 * F_2\|_Y \lesssim \|F_1 * F_2\|_Y^{(2)} \leq \|F_1\|_Y \cdot \|F_2^\vee\|_{L_w^1}.$$

Thus, the convolution relation $Y * [L_w^1]^\vee \hookrightarrow Y$ holds for non-negative functions; by solidity, this easily implies the general case.

Finally, if w is a 1-weight, it follows that $L_w^1 = [L_w^1]^\vee$ with identical norms, so that the preceding convolution relation implies that $Y * L_w^1 \hookrightarrow Y$. Moreover, using the estimate (2.8), it follows for $F_1 \in Y$ and $F_2 \in W^L(L_w^1)$ that

$$\|F_1 * F_2\|_{W^L(Y)} = \|M^L(F_1 * F_2)\|_Y \leq \||F_1| * M^L F_2\|_Y$$
$$\lesssim \||F_1|\|_Y \cdot \|M^L F_2\|_{L_w^1} = \|F_1\|_Y \cdot \|F_2\|_{W^L(L_w^1)},$$

as asserted. □

3.3. **Counterexamples.** In this section two examples (both taken from [54, Section 2.3]) are provided, showing that in general

$$W^L(L_v^p) * [W^L(Y^\vee)]^\vee \nsubseteq W^L(Y)$$

with $v(x) = \|L_{x^{-1}}\|_{W^L(Y) \to W^L(L)}$ and/or $v(x) = \|L_x\|_{W^L(Y) \to W^L(Y)}$. These examples show that the convolution relation asserted in [47, Theorem 5.2] fails in general. This (incorrect) convolution relation is used several times in the coorbit theory developed in [45,46].

The first example shows that one cannot use the weight $v(x) = \|L_{x^{-1}}\|_{W^L(Y) \to W^L(Y)}$.

Example 3.12. Let $G = \mathbb{R}$ and consider the weight $w : \mathbb{R} \to (0, \infty)$, $x \mapsto e^x$. Let $Y := L_w^1(\mathbb{R})$. For $f \in Y$ and $x \in \mathbb{R}$, it follows that

$$\|L_x f\|_Y = \int_\mathbb{R} |f(y - x)| \, e^y \, dy = \int_\mathbb{R} |f(z)| \, e^{z+x} \, dz = e^x \cdot \|f\|_Y,$$

and hence $\|L_x\|_{Y \to Y} = e^x$. As seen in Equation (2.9), this implies

$$v(x) := \|L_{x^{-1}}\|_{W^L(Y) \to W^L(Y)} \leq \|L_{x^{-1}}\|_{Y \to Y} = e^{-x},$$

where we have written $G = \mathbb{R}$ multiplicatively, as well as additively, i.e., $x^{-1} = -x$. Moreover note, for any measurable $f : \mathbb{R} \to \mathbb{C}$ that

$$\|f^\vee\|_Y = \int_\mathbb{R} |f(y)| \, e^{-y} \, dy = \|f\|_{L_u^1}$$

for $u : \mathbb{R} \to (0, \infty)$, $x \mapsto e^{-x}$. Thus, $Y^\vee = L_u^1(\mathbb{R})$.

Let $T > 0$ be arbitrary and define $f := \mathbb{1}_{(T,T+1)}$ as well as $g := \mathbb{1}_{(-T-1,-T)}$. Using the (open, relatively compact, symmetric) unit neighborhood $Q := (-1, 1)$, it follows that

$$\left(M^L \mathbb{1}_A\right)(x) = \operatorname*{ess\,sup}_{y \in x+Q} \mathbb{1}_A(y) \leq \mathbb{1}_{A+Q}(x)$$

for each measurable $A \subset \mathbb{R}$, since $\mathbb{1}_A(y) \neq 0$ for some $y \in x + Q$ implies $x + q = y \in A$ for a suitable $q \in Q$, which in turn yields $x = y - q \in A - Q = A + Q$. Note that $\| \cdot \|_{L^1_w} = \| \cdot \|_Y$ is a p-norm with $p = 1$. Thus,

$$\|f\|_{W^L(L^p_v)} = \|M^L \mathbb{1}_{(T,T+1)}\|_{L^1_v} \leq \left\|\mathbb{1}_{(T-1,T+2)}\right\|_{L^1_v} \leq \int_{T-1}^{T+2} e^{-x} \, dx = e^{-T} \cdot (e^1 - e^{-2}) \leq e \cdot e^{-T}$$

and $\|g\|_{[W^L(Y^\vee)]^\vee} = \|g^\vee\|_{W^L(L^1_u)} = \|M^L \mathbb{1}_{(T,T+1)}\|_{L^1_u} \leq \|\mathbb{1}_{(T-1,T+2)}\|_{L^1_u} \leq e \cdot e^{-T}$.

Since $f, g \in L^2(\mathbb{R})$, it is easy to see that $f * g$ is continuous with

$$f * g(0) = \int_{\mathbb{R}} \mathbb{1}_{(T,T+1)}(x) \cdot \mathbb{1}_{(-T-1,-T)}(0 - x) \, dx = 1.$$

Hence, for arbitrary $x \in (-1,1)$, we have $M^L[f * g](x) \geq f * g(0) = 1$, which yields that $\|f * g\|_{W^L(Y)} \geq \int_{(-1,1)} e^x \, dx \geq 1$.

Assume towards a contradiction that $W^L(L^p_v) * [W^L(Y^\vee)]^\vee \subset W^L(Y)$. As an easy consequence of the closed graph theorem, this implies existence of a constant $C = C(w, Q) > 0$ satisfying $\|f * g\|_{W^L(Y)} \leq C \|f\|_{W^L(L^p_v)} \|g\|_{[W^L(Y^\vee)]^\vee}$. Then the above estimates yield that

$$1 \leq \|f * g\|_{W^L(Y)} \leq C \|f\|_{W^L(L^p_v)} \|g\|_{[W^L(Y^\vee)]^\vee} \leq C \cdot e^2 \cdot e^{-2T} \to 0 \quad \text{as} \quad T \to \infty,$$

a contradiction. Thus, $W^L(L^p_v) * [W^L(Y^\vee)]^\vee \nsubseteq W^L(Y)$. ∎

The next example shows that [47, Theorem 5.2] also fails[1] for $v(x) = \|L_x\|_{W^L(Y) \to W^L(Y)}$.

Example 3.13. Let G denote the affine group, i.e., $G := \mathbb{R} \times (0, \infty)$ with multiplication given by

$$(x, a) \cdot (y, b) = (x + ay, ab).$$

Neutral element and inverse in G are given by $e_G = (0, 1) \in G$ and $(x, a)^{-1} = \left(-\frac{x}{a}, a^{-1}\right)$, respectively. The left Haar integral on G is given by

$$\int_G f(g) \, d\mu_G(g) = \int_0^\infty \int_{\mathbb{R}} f(x, a) dx \, \frac{da}{a^2}$$

with modular function $\Delta((x, a)) = a^{-1}$.

Define the weight

$$w : \quad G \to (1, \infty), \quad (x, a) \mapsto 1 + a = 1 + \Delta\left((x, a)^{-1}\right).$$

Then w is clearly continuous and submultiplicative. Set $Y := L^1_w(G)$. Since $L^1(G)$ is right invariant and isometrically left invariant, it follows that $W^L(Y)$ is left and right invariant (cf. Section 2.6) with

$$v(x, a) := \|L_{(x,a)}\|_{W^L(Y) \to W^L(Y)} \leq \|L_{(x,a)}\|_{Y \to Y} \leq w(x, a) = 1 + a.$$

Moreover, as seen in Section 2.6, $W^L_{Q'}(Y)$ is independent of the choice of the (open, relatively compact) unit neighborhood $Q' \subset G$. In addition, a direct calculation using the identity $\int_G F(z) \, d\mu_G(z) = \int_G F(z^{-1}) \Delta(z^{-1}) d\mu_G(z)$ shows that

$$\|f^\vee\|_{L^1_w} = \int_G |f^\vee(z)| \cdot w(z) \, d\mu_G(z) = \int_G \left|f(z^{-1})\right| d\mu_G(z) + \int_G \left|f(z^{-1})\right| \cdot \Delta(z^{-1}) \, d\mu_G(z)$$

$$= \int_G |f(z)| \cdot \Delta(z^{-1}) \, d\mu_G(z) + \int_G |f(z)| \, d\mu_G(z) = \|f\|_{L^1_w},$$

so that $Y = Y^\vee$.

Let $\alpha \in (1, \infty)$ and $\beta \in (0, 1)$ and set $\delta := \max\{\alpha, \beta\} = \alpha$. Define $f : G \to (0, \infty)$ by $f((x, a)) = e^{-|x|} \cdot \min\left\{a^\alpha, a^{-\beta}\right\}$. Then f is clearly continuous and is easily seen to

---

[1]The weight $v(x) = \|L_x\|_{W^L(Y) \to W^L(Y)}$ does not occur in the statement of [47, Theorem 5.2], but it is this weight that is used in its proposed proof.

satisfy $\|f\|_{\sup} \leq 1$. In the following, we use the (open, relatively compact) unit neighborhoods $Q_0 := (-1,1) \times (\frac{1}{2}, 2)$ and $Q' := Q_0^{-1}$. Then the substitutions $(z,c) = (y,b)^{-1}$ and $(\mu, \nu) = (z,c) \cdot (x,a)$ yield

$$
\begin{aligned}
\left(M_{Q'}^L f^\vee \right)(x,a) &\leq \sup_{(y,b) \in (x,a)Q'} |f^\vee(y,b)| = \sup_{(z,c) \in Q_0(x,a)^{-1}} |f(z,c)| \\
&= \sup_{(\mu,\nu) \in Q_0} \left| f\left((\mu,\nu) \cdot (x,a)^{-1} \right) \right| = \sup_{(\mu,\nu) \in Q_0} \left| f\left((\mu,\nu) \cdot \left(-\frac{x}{a}, a^{-1} \right) \right) \right| \\
&= \sup_{(\mu,\nu) \in Q_0} \left| f\left(\mu - \frac{\nu}{a} x, \frac{\nu}{a} \right) \right| = \sup_{(\mu,\nu) \in Q_0} e^{-|\mu - \frac{\nu}{a} x|} \cdot \min\left\{ (\nu/a)^\alpha, (\nu/a)^{-\beta} \right\}.
\end{aligned}
$$

Using that $(\mu, \nu) \in Q_0 = (-1,1) \times (\frac{1}{2}, 2)$, we see

$$
\left| \mu - \frac{\nu}{a} x \right| \geq \frac{\nu}{a} |x| - |\mu| \geq \frac{\nu}{a} |x| - 1 \geq \frac{|x|}{2a} - 1
$$

and thus $e^{-|\mu - \frac{\nu}{a} x|} \leq e^{-\left(\frac{|x|}{2a} - 1 \right)} = e \cdot e^{-|x|/2a}$. Furthermore, $\frac{1}{2a} \leq \frac{\nu}{a} \leq \frac{2}{a}$, which entails $(\frac{\nu}{a})^\alpha \leq (\frac{2}{a})^\alpha \leq 2^\delta \cdot a^{-\alpha}$ and $(\frac{\nu}{a})^{-\beta} = (\frac{a}{\nu})^\beta \leq 2^\beta \cdot a^\beta \leq 2^\delta \cdot a^\beta$. Combining these estimates yields

$$
\left(M_{Q'}^L f^\vee \right)(x,a) \leq 2^\delta e \cdot e^{-|x|/2a} \cdot \min\left\{ a^{-\alpha}, a^\beta \right\} \quad \text{for all } (x,a) \in G. \tag{3.4}
$$

For $a \in (0, \infty)$, using the identity $C_1 := \int_{\mathbb{R}} e^{-|y|} \, dy = \int_{\mathbb{R}} e^{-|x|/2a} \frac{dx}{2a}$, it follows that

$$
\begin{aligned}
\|f^\vee\|_{W_{Q'}^L(Y)} &\leq 2^\delta e \int_G e^{-|x|/2a} \cdot \min\left\{ a^{-\alpha}, a^\beta \right\} \cdot w(x,a) \, d\mu_G(x,a) \\
&= 2^\delta e \int_0^\infty \frac{\min\left\{ a^{-\alpha}, a^\beta \right\} \cdot (1+a)}{a^2} \cdot 2a \int_{\mathbb{R}} e^{-|x|/2a} \frac{dx}{2a} \, da \\
&= 2^{\delta+1} e C_1 \int_0^\infty \frac{\min\left\{ a^{-\alpha}, a^\beta \right\} \cdot (1+a)}{a} \, da \\
&\leq 2^{\delta+1} e C_1 \cdot \left[\int_0^1 a^\beta \cdot (1+a) \frac{da}{a} + \int_1^\infty a^{-\alpha} \cdot (1+a) \frac{da}{a} \right].
\end{aligned}
$$

Using $1 + a \leq 2$ for $a \in (0,1)$ and $1 + a \leq a + a = 2a$ for $a \in [1, \infty)$, this gives

$$
\|f^\vee\|_{W_{Q'}^L(Y)} \leq 2^{\delta+2} e C_1 \cdot \left[\int_0^1 a^\beta \frac{da}{a} + \int_1^\infty a^{1-\alpha} \frac{da}{a} \right] < \infty,
$$

because of $\beta > 0$ and $\alpha > 1$. This implies $f^\vee \in W^L(Y) = W^L(Y^\vee)$ and thus $f \in \left[W^L(Y^\vee) \right]^\vee$, since $f^\vee \in W^L(Y) = W^L(L_w^1) \hookrightarrow W^L(L_v^1)$, since $v \leq w$.

Note that $Y = L_w^1(G)$ is a Banach space, so that $\|\cdot\|_Y$ is a p-norm for $p = 1$. Now, if the convolution relation $W^L(L_v^p) * [W^L(Y^\vee)]^\vee \subseteq W^L(Y)$ would hold, then the above would yield $f^\vee * f \in W^L(Y)$. We will now show that in fact $f^\vee * f \notin W^L(Y)$. For this, note first that $f^\vee \in W^L(L_w^1) \hookrightarrow L^1$ since $w \geq 1$. Furthermore, f is bounded, so that $f^\vee * f : G \to \mathbb{C}$ is a well-defined, continuous, bounded function. A direct calculation gives

$$
\begin{aligned}
(f^\vee * f)(x,a) &= \int_G f^\vee(y,b) \cdot f\left((y,b)^{-1} \cdot (x,a) \right) d\mu_G(y,b) \\
&= \int_0^\infty \int_{\mathbb{R}} f\left(-\frac{y}{b}, \frac{1}{b} \right) \cdot f\left(\frac{x-y}{b}, \frac{a}{b} \right) dy \frac{db}{b^2} \\
&= \int_0^\infty \min\left\{ b^{-\alpha}, b^\beta \right\} \min\left\{ \left(\frac{a}{b} \right)^\alpha, \left(\frac{a}{b} \right)^{-\beta} \right\} \cdot \frac{b}{b^2} \int_{\mathbb{R}} e^{-|-\frac{y}{b}|} e^{-|\frac{x-y}{b}|} \frac{dy}{b} \, db \\
&= \int_0^\infty \min\left\{ b^{-\alpha}, b^\beta \right\} \cdot \min\left\{ \left(\frac{a}{b} \right)^\alpha, \left(\frac{b}{a} \right)^\beta \right\} \cdot \int_{\mathbb{R}} e^{-|z|} e^{-|z - \frac{x}{b}|} \, dz \frac{db}{b}.
\end{aligned}
$$

Let $x = 0$ and $a \geq 1$. Then, for each $b \in (0, 1)$, it follows that $b^\beta \leq 1 \leq b^{-\alpha}$ as well as $b < 1 \leq a$, and thus $(b/a)^\beta \leq 1^\beta = 1 \leq (a/b)^\alpha$. With the abbreviation $C_2 := \int_{\mathbb{R}} (e^{-|z|})^2 dz \in (0, \infty)$, these considerations and the above calculation imply

$$
\begin{aligned}
(f^\vee * f)(0, a) &\geq C_2 \cdot \int_0^1 b^\beta \cdot (b/a)^\beta \, \frac{db}{b} = C_2 \cdot a^{-\beta} \cdot \int_0^1 b^{2\beta - 1} db = C_2 \cdot a^{-\beta} \cdot \frac{b^{2\beta}}{2\beta} \Big|_0^1 \\
&= \frac{C_2}{2\beta} \cdot a^{-\beta}
\end{aligned}
\tag{3.5}
$$

for all $a \geq 1$. For using this to obtain a lower bound on $M^L(f^\vee * f)$, first note for $(y, b) \in G$ that

$$
(y, b) Q_0 = [y + (-b, b)] \times \left(\frac{b}{2}, 2b \right) =: B_b(y) \times \left(\frac{b}{2}, 2b \right).
$$

For $b \in (1, \infty)$ and $y \in (-b, b)$, we thus have $(0, b) \in (y, b) Q_0$. Since $f^\vee * f$ is continuous and $(y, b) Q_0$ is open, it follows that, for $b \in (1, \infty)$ and $y \in (-b, b)$,

$$
M_{Q_0}^L[f^\vee * f](y, b) \geq (f^\vee * f)(0, b) \geq \frac{C_2}{2\beta} \cdot b^{-\beta},
$$

where the estimate (3.5) was used in the last step. Combining the obtained estimates gives

$$
\begin{aligned}
\|f^\vee * f\|_{W_{Q_0}^L(Y)} &\geq \frac{C_2}{2\beta} \cdot \int_G \mathbb{1}_{(1,\infty)}(b) \cdot \mathbb{1}_{(-b,b)}(y) \cdot b^{-\beta} \cdot w(y, b) \, d\mu_G(y, b) \\
&= \frac{C_2}{2\beta} \cdot \int_1^\infty \frac{b^{-\beta}}{b^2} \cdot (1 + b) \cdot \int_{-b}^b dy \, db \\
&\geq \frac{C_2}{2\beta} \cdot \int_1^\infty b^{-\beta} \, db = \infty,
\end{aligned}
$$

because of $\beta \in (0, 1)$. Thus, $f^\vee * f \notin W^L(Y)$.   ∎

Remark 3.14. The above example also shows that

$$
W^L(L_w^p) * \left[W^L(L_{w^*}^p) \right]^\vee \nsubseteq W^L(L_w^p)
\tag{3.6}
$$

for the weight $w^*(x) = \Delta(x^{-1}) \cdot w(x^{-1})$. Hence, the convolution relation that is stated in [47, Corollary 5.4] is also false in general (even for $p = 1$). To see that the above example indeed implies Equation (3.6), note that the weight w in the example satisfies $w^* = w$ and that $f^\vee \in W^L(L_w^1)$ as well as $f \in \left[W^L(L_{w^*}^1) \right]^\vee$, but $f^\vee * f \notin W^L(Y) = W^L(L_w^1)$.

4. COORBIT SPACES ASSOCIATED WITH INTEGRABLE GROUP REPRESENTATIONS

This section develops the basic theory of coorbit spaces associated with (possibly projective) integrable representations and quasi-Banach function spaces.

4.1. **Admissible vectors and reproducing formulae.** Let $(\pi, \mathcal{H}_\pi)$ be a unitary σ-representation of G on the separable Hilbert space $\mathcal{H}_\pi \neq \{0\}$, i.e., a strongly measurable[2] map $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ into the set $\mathcal{U}(\mathcal{H}_\pi)$ of unitary operators on $\mathcal{H}_\pi$ satisfying $\pi(e_G) = I_{\mathcal{H}_\pi}$ and

$$
\pi(x)\pi(y) = \sigma(x, y)\pi(xy), \quad \text{for all } x, y \in G,
\tag{4.1}
$$

for some function $\sigma : G \times G \to \mathbb{T}$. Any such function σ is a cocycle in the sense of Section 2.5. One also says that $(\pi, \mathcal{H}_\pi)$ is a unitary *projective representation*. The adjoint operator $\pi(x)^*$ of $\pi(x)$ for $x \in G$ is given by

$$
[\pi(x)]^* = [\pi(x)]^{-1} = \overline{\sigma(x, x^{-1})} \, \pi(x^{-1}) = \overline{\sigma(x^{-1}, x)} \, \pi(x^{-1}).
\tag{4.2}
$$

---

[2]This means that for each $f \in \mathcal{H}_\pi$, the map $G \to \mathcal{H}_\pi, x \mapsto \pi(x)f$ is measurable.

In particular, $\sigma(x, x^{-1}) = \sigma(x^{-1}, x)$. The map π is called *irreducible* if $\{0\}$ and $\mathcal{H}_\pi$ are the only closed π-invariant subspaces of $\mathcal{H}_\pi$, i.e., if $V \subset \mathcal{H}_\pi$ and $\pi(x)v \in V$ for all $v \in V$ and $x \in G$, then $V = \{0\}$ or $V = \mathcal{H}_\pi$.

For $g \in \mathcal{H}_\pi$, define its associated *coefficient transform* $V_g : \mathcal{H}_\pi \to L^\infty(G)$ by

$$V_g f(x) = \langle f, \pi(x)g \rangle, \quad x \in G.$$

By [53, Lemma 7.1, Theorem 7.5], it follows that

$$|V_g f| : G \to \mathbb{C} \quad \text{is continuous for all } f, g \in \mathcal{H}_\pi. \tag{4.3}$$

A vector $g \in \mathcal{H}_\pi$ is called *admissible* if $V_g : \mathcal{H}_\pi \to L^2(G)$ is well-defined and an isometry.

The following simple lemma collects several identities that will be used repeatedly throughout this article. Here, the twisted translation operators L_x^σ and R_x^σ and twisted convolution operator $*_\sigma$ are as defined in Section 2.5.

Lemma 4.1. *Let* $f, g, h \in \mathcal{H}_\pi$.

(i) *For* $x \in G$, *the following* intertwining property *holds:*

$$V_g f(x^{-1}) = \overline{\sigma(x, x^{-1})} \, \overline{V_f g(x)}, \quad V_g[\pi(x)f] = L_x^\sigma[V_g f] \quad and \quad V_{\pi(x)g} f = R_x^\sigma[V_g f].$$

(ii) *If* $h \in \mathcal{H}_\pi$ *is admissible, then the following* reproducing formula *holds:*

$$V_g f = V_h f *_\sigma V_g h. \tag{4.4}$$

Proof. (i) If $x, y \in G$, then

$$V_g[\pi(x)f](y) = \langle f, \pi(x)^* \pi(y)g \rangle = \sigma(x, x^{-1}) \langle f, \pi(x^{-1})\pi(y)g \rangle$$
$$= \sigma(x, x^{-1})\overline{\sigma(x^{-1}, y)}V_g f(x^{-1}y).$$

Using that $\sigma(x, x^{-1}y) = \overline{\sigma(x^{-1}, y)}\,\sigma(xx^{-1}, y)\,\sigma(x, x^{-1}) = \sigma(x, x^{-1})\,\overline{\sigma(x^{-1}, y)}$, it follows that $V_g[\pi(x)f] = L_x^\sigma[V_g f]$. The other two identities of part (i) are immediate consequences of the definitions and of Equations (4.1) and (4.2).

(ii) If $x \in G$, then using that $V_h : \mathcal{H}_\pi \to L^2(G)$ is an isometry gives

$$V_g f(x) = \langle f, \pi(x)g \rangle = \langle V_h f, V_h[\pi(x)g] \rangle_{L^2} = \int_G V_h f(y) \, \langle \pi(y)h, \pi(x)g \rangle_{\mathcal{H}_\pi} \, d\mu_G(y)$$

$$= \int_G V_h f(y) L_y^\sigma[V_g h](x) \, d\mu_G(y) = (V_h f *_\sigma V_g h)(x),$$

where the penultimate equality marked follows from part (i). $\square$

If $g \in \mathcal{H}_\pi$ is admissible, then the image space $\mathcal{K}_g := V_g(\mathcal{H}_\pi)$ is a closed subspace of $L^2(G)$, since it is the isometric image of the Hilbert space $\mathcal{H}_\pi$. For arbitrary $F \in \mathcal{K}_g$, say $F = V_g f$, it follows from Lemma 4.1 that for any $x \in G$,

$$F(x) = \langle f, \pi(x)g \rangle = \langle V_g f, V_g[\pi(x)g] \rangle = \langle V_g f, L_x^\sigma[V_g g] \rangle = \int_G F(y)\overline{L_x^\sigma[V_g g](y)} \, d\mu_G(y), \tag{4.5}$$

which shows that $\mathcal{K}_g$ is a *reproducing kernel Hilbert space*, i.e., for any $x \in G$, the point evaluation $\mathcal{K}_g \ni F \mapsto F(x) \in \mathbb{C}$ is a well-defined, continuous linear functional. The *reproducing kernel* of $\mathcal{K}_g$ is the function

$$K : \quad G \times G \to \mathbb{C}, \quad (x, y) \mapsto V_g[\pi(y)g](x) = \overline{L_x^\sigma[V_g g](y)} \tag{4.6}$$

satisfying $F(x) = \langle F, K(\cdot, x) \rangle_{L^2}$ for any $F \in \mathcal{K}_g$ and $x \in G$.

For an *irreducible* σ-representation $(\pi, \mathcal{H}_\pi)$, a convenient criterion for the existence of admissible vectors is provided by the *orthogonality relations* for square-integrable representations, cf. [16, Theorem 3] or [6, Theorem 4.3] for genuine representations, and [2, Theorem 3] for projective representations.

Theorem 4.2 ([2,6,16]). *Let $(\pi, \mathcal{H}_\pi)$ be an irreducible σ-representation of G. Suppose there exists $g \in \mathcal{H}_\pi \setminus \{0\}$ satisfying $V_g g \in L^2(G)$. Then there exists a unique, self-adjoint, positive operator $C_\pi : \mathrm{dom}(C_\pi) \to \mathcal{H}_\pi$ such that*

$$\langle V_{g_1} f_1, V_{g_2} f_2 \rangle_{L^2} = \langle C_\pi g_2, C_\pi g_1 \rangle_{\mathcal{H}_\pi} \langle f_1, f_2 \rangle_{\mathcal{H}_\pi} \tag{4.7}$$

for all $f_1, f_2 \in \mathcal{H}_\pi$ and $g_1, g_2 \in \mathrm{dom}(C_\pi)$, with $\mathrm{dom}(C_\pi) = \{g \in \mathcal{H}_\pi : V_g g \in L^2(G)\}$.

Remark. The positivity of C_π implies, in particular, that C_π is injective. This can also be derived from Equation (4.7) by noting that

$$\|C_\pi h\|_{\mathcal{H}_\pi}^2 = \|V_h h\|_{L^2}^2 / \|h\|_{\mathcal{H}_\pi}^2 > 0 \quad \text{for} \quad h \in \mathrm{dom}(C_\pi) \setminus \{0\}. \tag{4.8}$$

Here, the norm $\|V_h h\|_{L^2}$ is positive since $|V_h h|$ is continuous (see Equation (4.3)) and satisfies $|V_h h(e_G)| = \|h\|_{\mathcal{H}_\pi}^2 > 0$.

Theorem 4.2 yields that if π is irreducible and $\mathrm{dom}(C_\pi) \neq \{0\}$, then any $g \in \mathrm{dom}(C_\pi)$ is (a scalar multiple of) an admissible vector. In addition, the orthogonality relations (4.7) yield that

$$\langle C_\pi g_1, C_\pi g_2 \rangle_{\mathcal{H}_\pi} V_{f_2} f_1 = V_{g_2} f_1 *_\sigma V_{f_2} g_1 \tag{4.9}$$

for $f_1, f_2 \in \mathcal{H}_\pi$ and $g_1, g_2 \in \mathrm{dom}(C_\pi)$; the argument is similar to the proof of Part (ii) of Lemma 4.1.

4.2. **Integrable vectors.** Let $w : G \to [1, \infty)$ be a measurable submultiplicative weight, i.e., such that $w(xy) \leq w(x)w(y)$ for all $x, y \in G$. Henceforth, the σ-representation π is assumed to be w-integrable in the sense that the set

$$\mathcal{A}_w := \{g \in \mathcal{H}_\pi : V_g g \in L^1_w(G)\}$$

is non-trivial; that is, $\mathcal{A}_w \neq \{0\}$. In addition, it is assumed that there exists an *admissible* vector $g \in \mathcal{A}_w$. For any such admissible $g \in \mathcal{A}_w$, define the space

$$\mathcal{H}_w^1 := \mathcal{H}_w^1(g) := \{f \in \mathcal{H}_\pi : V_g f \in L^1_w(G)\}$$

and equip it with the norm $\|f\|_{\mathcal{H}_w^1} := \|f\|_{\mathcal{H}_w^1(g)} := \|V_g f\|_{L^1_w}$.

The following lemma collects several basic properties of the space $\mathcal{H}_w^1$.

Lemma 4.3. *Fix an admissible vector $g \in \mathcal{A}_w$ and write $\mathcal{H}_w^1 := \mathcal{H}_w^1(g)$.*

  (i) *If $h \in \mathcal{A}_w$ is an admissible vector satisfying $V_g h, V_h g \in L^1_w(G)$, then $\mathcal{H}_w^1(g) = \mathcal{H}_w^1(h)$ with norm equivalence $\| \cdot \|_{\mathcal{H}_w^1(g)} \asymp \| \cdot \|_{\mathcal{H}_w^1(h)}$.*
 (ii) *The pair $(\mathcal{H}_w^1, \| \cdot \|_{\mathcal{H}_w^1})$ is a π-invariant separable Banach space satisfying $\mathcal{H}_w^1 \hookrightarrow \mathcal{H}_\pi$ and $\pi(x)g \in \mathcal{H}_w^1$ for all $x \in G$. Moreover, $\mathcal{H}_w^1$ is norm dense in $\mathcal{H}_\pi$ and π acts on $\mathcal{H}_w^1$, with operator norm bounded by*

$$\|\pi(x)\|_{\mathcal{H}_w^1 \to \mathcal{H}_w^1} \leq w(x), \quad x \in G.$$

(iii) *For each $f \in \mathcal{H}_w^1$, the vector-valued maps*

$$\Xi_f : \quad G \to \mathcal{H}_w^1, \quad x \mapsto \pi(x)f$$

$$and \quad F_{f,g} : \quad G \to \mathcal{H}_w^1, \quad x \mapsto V_g f(x) \cdot \pi(x)g$$

  are Bochner measurable. Furthermore, $F_{f,g}$ is Bochner integrable with $f = \int_G F_{f,g} d\mu_G$.
 (iv) *The orbit $\pi(G)g$ is complete in $\mathcal{H}_w^1$, i.e., $\overline{\mathrm{span}}\{\pi(x)g : x \in G\} = \mathcal{H}_w^1$.*

Proof. (i) If $f \in \mathcal{H}_w^1(g)$, then (4.4) yields that $V_h f = V_g f *_\sigma V_h g$. Since $V_g f, V_h g \in L^1_w$ by assumption and since $f \in \mathcal{H}_w^1(g)$, and because of $L^1_w * L^1_w \hookrightarrow L^1_w$ (see, e.g., [49, §3.7]), this implies that $V_h f \in L^1_w(G)$ (and hence $f \in \mathcal{H}_w^1(h)$), with

$$\|f\|_{\mathcal{H}_w^1(h)} \leq \||V_g f| * |V_h g|\|_{L^1_w} \lesssim \|V_g f\|_{L^1_w} = \|f\|_{\mathcal{H}_w^1(g)}.$$

By symmetry, this yields that $\| \cdot \|_{\mathcal{H}_w^1(h)} \asymp \| \cdot \|_{\mathcal{H}_w^1(g)}$. In the sequel, simply set $\mathcal{H}_w^1 := \mathcal{H}_w^1(g)$.

(ii) If $f \in \mathcal{H}_w^1$ and $x \in G$, then

$$\|\pi(x)f\|_{\mathcal{H}_w^1} = \|L_x^\sigma[V_g f]\|_{L_w^1} = \|L_x[V_g f]\|_{L_w^1} \leq w(x) \cdot \|V_g f\|_{L_w^1} \leq w(x)\|f\|_{\mathcal{H}_w^1}$$

since $\|L_x\|_{L_w^1 \to L_w^1} \leq w(x)$; see, e.g., [49, Proposition 3.7.6]. Thus, $\mathcal{H}_w^1$ is π-invariant and $\|\pi(x)\|_{\mathcal{H}_w^1 \to \mathcal{H}_w^1} \leq w(x)$. The orbit $\pi(G)g$ is complete in $\mathcal{H}_\pi$ if and only if $V_g : \mathcal{H}_\pi \to L^2(G)$ is injective. Therefore, since V_g is an isometry and since $g \in \mathcal{A}_w$ and thus $g \in \mathcal{H}_w^1$, it follows that $\mathcal{H}_w^1 \supset \pi(G)g$ is norm dense in $\mathcal{H}_\pi$. The reproducing formula (4.4), combined with the convolution relation $L^1 * L^2 \hookrightarrow L^2$ (see e.g. [22, Proposition 2.39]) and with $w \geq 1$ yields that

$$\|f\|_{\mathcal{H}_\pi}^2 = \|V_g f\|_{L^2}^2 \leq \||V_g f| * |V_g g|\|_{L^2}^2 \leq \|V_g g\|_{L^2}^2 \|V_g f\|_{L^1}^2 \lesssim \|f\|_{\mathcal{H}_w^1}^2. \tag{4.10}$$

This shows that $\mathcal{H}_w^1 \hookrightarrow \mathcal{H}_\pi$ and that $\|\cdot\|_{\mathcal{H}_w^1}$ is positive definite, hence defines a norm. In addition, if $(f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}_w^1$, then (4.10) yields that $(f_n)_{n \in \mathbb{N}}$ is Cauchy in $\mathcal{H}_\pi$, and hence converges to some $f' \in \mathcal{H}_\pi$. The sequence $F_n := V_g f_n$ being Cauchy in $L_w^1(G)$ yields convergence to some $F \in L_w^1(G)$. Since $F_n(x) = V_g f_n(x) \to V_g f'(x)$ as $n \to \infty$ for all $x \in G$, it follows that $V_g f' = F$, and thus $f' \in \mathcal{H}_w^1$ with $\|f' - f_n\|_{\mathcal{H}_w^1} = \|F_n - F\|_{L_w^1} \to 0$. This shows that $(\mathcal{H}_w^1, \|\cdot\|_{\mathcal{H}_w^1})$ is a Banach space.

(iii) Part (ii) shows that Ξ_f and $F_{f,g}$ are well-defined for $f \in \mathcal{H}_w^1$. We first show that Ξ_f is Bochner measurable. Since $V_g f$ is measurable and $g \in \mathcal{H}_w^1$, this then easily implies the measurability of $F_{f,g} = V_g f \cdot \Xi_g$.

To show that Ξ_f is measurable, first note that the group G is second countable, so that the space $L_w^1(G)$ is separable; see, e.g., [13, Proposition 3.4.5]. As $V_g : \mathcal{H}_w^1 \to L_w^1(G)$ is an isometry, also $\mathcal{H}_w^1$ is separable, hence Pettis' measurability theorem (cf. [13, Theorem E.9]) implies that $\Xi_f : G \to \mathcal{H}_w^1$ is strongly measurable whenever $\varphi \circ \Xi_f$ is Borel measurable for each continuous linear functional $\varphi \in (\mathcal{H}_w^1)'$. To show the latter, given $\varphi \in (\mathcal{H}_w^1)'$, define $\psi_0 : V_g(\mathcal{H}_w^1) \to \mathbb{C}$ by $\psi_0(V_g h) = \varphi(h)$ for $h \in \mathcal{H}_w^1$. Since $V_g : \mathcal{H}_w^1 \to L_w^1(G)$ is an isometry, the functional ψ_0 is well-defined, linear, and bounded with respect to $\|\cdot\|_{L_w^1}$. Hence, by the Hahn-Banach theorem, ψ_0 extends to a linear functional $\psi \in (L_w^1(G))'$, which is then given by integration against some $H \in L_{1/w}^\infty(G)$. Thus,

$$\varphi(\Xi_f(x)) = \psi_0(V_g[\pi(x)f]) = \int_G H(y) \cdot V_g[\pi(x)f](y) \, d\mu_G(y)$$

$$= \int_G H(y) \cdot L_x^\sigma[V_g f](y) \, d\mu_G(y),$$

and hence Fubini's theorem implies the Borel measurability of $x \mapsto \varphi(\Xi_f(x))$. Thus, by Pettis' measurability theorem, the map Ξ_f is strongly measurable, and hence so is the map $F_{f,g}$. In addition, the estimate $\|\pi(x)g\|_{\mathcal{H}_w^1} \leq w(x)\|g\|_{\mathcal{H}_w^1}$ of part (ii) implies directly that

$$\|F_{f,g}(\cdot)\|_{\mathcal{H}_w^1} \leq w(\cdot)\|g\|_{\mathcal{H}_w^1}|V_g f(\cdot)| \in L^1(G),$$

whence $F_{f,g}$ is Bochner integrable. Thus $f' := \int_G F_{f,g} \, d\mu_G \in \mathcal{H}_w^1 \hookrightarrow \mathcal{H}_\pi$ is well-defined. For showing $f = f'$, it suffices to show that $V_g f' = V_g f$, since $V_g : \mathcal{H}_\pi \to L^2(G)$ is an isometry. Let $\iota : L_w^1(G) \hookrightarrow L^1(G)$ be the canonical embedding. Since the Bochner integral commutes with bounded linear operators (cf. [42, VI, Theorem 4.1]), a direct calculation using Equation (4.4) entails

$$\iota(V_g f') = \int_G [(\iota \circ V_g)F_{f,g}](x) \, d\mu_G(x) = \int_G V_g f(x) L_x^\sigma[V_g g](\cdot) \, d\mu_G(x)$$

$$= V_g f *_\sigma V_g g = V_g f = \iota(V_g f),$$

which yields that $V_g f' = V_g f$.

(iv) If $f \in \mathcal{H}_w^1$, then clearly $V_g f(x) \cdot \pi(x)g \in \overline{\mathrm{span}\, \pi(G)g} \leq \mathcal{H}_w^1$ for each $x \in G$. Hence, by part (iii), also $f = \int_G V_g f(x)\pi(x)g \, d\mu_G(x) \in \overline{\mathrm{span}\, \pi(G)g}$. Thus, $\mathcal{H}_w^1 = \overline{\mathrm{span}\, \pi(G)g}$. $\square$

The proofs of parts (i) and (ii) of Lemma 4.3 closely follow [18, Lemma 4.2], but the direct proof of part (iv) using the Bochner integral (see part (iii)) appears to be new.

Remark 4.4. In Part (i) of Lemma 4.3, the assumption that the admissible vectors $g, h \in \mathcal{A}_w$ satisfy $V_g h, V_h g \in L^1_w(G)$ is essential for the independence claim—at least, when π is reducible. Indeed, in [28, §2.1], an example is given of a reducible representation π admitting admissible vectors $g, h \in \mathcal{A}_w$, but for which $V_g h \notin L^1_w(G)$, showing that $h \in \mathcal{H}^1_w(h)$ but $h \notin \mathcal{H}^1_w(g)$.

The following result shows that the behavior discussed in the above remark cannot occur for *irreducible* representations; in particular, it shows that the additional assumption $V_g h, V_h g \in L^1_w(G)$ in Part (i) of Lemma 4.3 is automatic in the irreducible case.

Lemma 4.5. *Let $(\pi, \mathcal{H}_\pi)$ be an* irreducible *unitary σ-representation. Let $Y \hookrightarrow L^1(G)$ be a solid quasi-Banach function space that is left- and right translation invariant and also satisfies $Y * Y \hookrightarrow Y$ (or, $Y * Y^\vee \hookrightarrow Y$). Then the space*

$$\mathcal{C}_Y := \{g \in \mathcal{H}_\pi : V_g g \in Y\}$$

is a π-invariant vector space with $V_h g \in Y$ for all $g, h \in \mathcal{C}_Y$ and with $\mathcal{C}_Y \subset \mathrm{dom}(C^2_\pi)$, where $C_\pi : \mathrm{dom}(C_\pi) \to \mathcal{H}_\pi$ is the operator given by Theorem 4.2. If $\mathcal{C}_Y$ is non-trivial, then $\mathcal{C}_Y$ is norm dense in $\mathcal{H}_\pi$.

Proof. If $g \in \mathcal{C}_Y \subset \mathcal{H}_\pi$, then $V_g g \in L^\infty(G)$ and $V_g g \in Y \hookrightarrow L^1(G)$ and thus $V_g g \in L^2(G)$. Therefore, $g \in \mathrm{dom}(C_\pi)$, and $C_\pi g \in \mathcal{H}_\pi$ is well-defined. We next show that $C_\pi g \in \mathrm{dom}(C^*_\pi)$, for which we can clearly assume that $g \neq 0$. For $f \in \mathrm{dom}(C_\pi)$, the orthogonality relations (4.7) gives

$$|\langle C_\pi f, C_\pi g \rangle_{\mathcal{H}_\pi}| = |\langle V_g g, V_f g \rangle_{L^2}| \cdot \|g\|^{-2}_{\mathcal{H}_\pi} \leq \|g\|^{-2}_{\mathcal{H}_\pi} \|V_g g\|_{L^1} \|V_f g\|_{L^\infty} \lesssim \|g\|^{-1}_{\mathcal{H}_\pi} \|f\|_{\mathcal{H}_\pi} \|V_g g\|_Y.$$

In particular, this shows that the linear functional $f \mapsto \langle C_\pi f, C_\pi g \rangle_{\mathcal{H}_\pi}$ is bounded from $\mathrm{dom}(C_\pi)$ into $\mathbb{C}$. Thus $C_\pi g \in \mathrm{dom}(C^*_\pi) = \mathrm{dom}(C_\pi)$ and therefore $g \in \mathrm{dom}(C^2_\pi)$.

Let $g, h \in \mathcal{C}_Y \setminus \{0\}$ be arbitrary. For showing that $V_h g \in Y$, it will first be shown that there exists $f \in \mathcal{C}_Y$ with $\langle C_\pi g, C_\pi f \rangle_{\mathcal{H}_\pi} \neq 0 \neq \langle C_\pi h, C_\pi f \rangle_{\mathcal{H}_\pi}$. Note that if $\langle C_\pi g, C_\pi h \rangle_{\mathcal{H}_\pi} \neq 0$, then choosing $f = g$ gives $\langle C_\pi h, C_\pi f \rangle_{\mathcal{H}_\pi} = \langle C_\pi h, C_\pi g \rangle_{\mathcal{H}_\pi} \neq 0$ and $\langle C_\pi g, C_\pi f \rangle_{\mathcal{H}_\pi} = \|C_\pi g\|^2_{\mathcal{H}_\pi} > 0$; see Equation (4.8). Therefore, it remains to consider the case $\langle C_\pi g, C_\pi h \rangle_{\mathcal{H}_\pi} = 0$. Since $h \neq 0$ and C_π is injective, it follows that $C_\pi h \neq 0 \neq C^2_\pi h$. Since π is irreducible, the orbit $\pi(G)g$ is complete in $\mathcal{H}_\pi$, and hence $0 \neq \langle \pi(x)g, C^2_\pi h \rangle_{\mathcal{H}_\pi} = \langle C_\pi[\pi(x)g], C_\pi h \rangle_{\mathcal{H}_\pi}$ for a suitable $x \in G$. For $\varepsilon > 0$, define $f_\varepsilon := g + \varepsilon \cdot \pi(x)g$. Then

$$\langle C_\pi f_\varepsilon, C_\pi h \rangle_{\mathcal{H}_\pi} = \langle C_\pi g, C_\pi h \rangle_{\mathcal{H}_\pi} + \varepsilon \langle C_\pi[\pi(x)g], C_\pi h \rangle_{\mathcal{H}_\pi} = \varepsilon \cdot \langle C_\pi[\pi(x)g], C_\pi h \rangle_{\mathcal{H}_\pi} \neq 0$$

and

$$\langle C_\pi f_\varepsilon, C_\pi g \rangle_{\mathcal{H}_\pi} = \|C_\pi g\|^2_{\mathcal{H}_\pi} + \varepsilon \cdot \langle C_\pi[\pi(x)g], C_\pi g \rangle_{\mathcal{H}_\pi} \neq 0,$$

whenever $\varepsilon > 0$ is chosen sufficiently small. For such ε, it remains to show that $f := f_\varepsilon \in \mathcal{C}_Y$. Using the intertwining properties (Part (i) of Lemma 4.1) and the left- and right invariance of Y, it follows that

$$V_f f = V_g g + \varepsilon V_{\pi(x)g} g + \varepsilon V_g[\pi(x)g] + \varepsilon^2 V_{\pi(x)g}[\pi(x)g]$$
$$= V_g g + \varepsilon R^\sigma_x[V_g g] + \varepsilon L^\sigma_x[V_g g] + \varepsilon^2 L^\sigma_x R^\sigma_x[V_g g] \in Y,$$

so $f \in \mathcal{C}_Y$. Applying the reproducing formula (4.9) yields

$$V_h g = \frac{1}{\langle C_\pi h, C_\pi f \rangle_{\mathcal{H}_\pi}} V_f g *_\sigma V_h h = \frac{1}{\langle C_\pi h, C_\pi f \rangle_{\mathcal{H}_\pi} \langle C_\pi f, C_\pi g \rangle_{\mathcal{H}_\pi}} V_g g *_\sigma V_f f *_\sigma V_h h.$$

Therefore, if $Y * Y \hookrightarrow Y$, then $V_h g \in Y$. On the other hand, if $Y * Y^\vee \hookrightarrow Y$ holds, then using that $V_g g, V_f f, V_h h \in Y \cap Y^\vee$, also easily yields $V_h g \in Y$.

Next, we show that $\mathcal{C}_Y$ is a π-invariant vector space. The invariance follows directly from the identity $V_{\pi(x)f}[\pi(x)f] = L_x^\sigma R_x^\sigma[V_f f]$, where $x \in G$, combined with the left- and right invariance of Y. For $g, h \in \mathcal{C}_Y$, the identity

$$V_{g+h}(g + h) = V_g g + V_g h + V_h g + V_h h,$$

and $V_g h, V_h g \in Y$, imply that $\mathcal{C}_Y$ is a vector space.

Finally, if $\mathcal{C}_Y \neq \{0\}$, then it is norm dense in $\mathcal{H}_\pi$ by irreducibility of π and since $\mathcal{C}_Y$ is π-invariant. $\square$

Lemma 4.3 implies, in particular, that the *anti*-dual space

$$\mathcal{R}_w := \mathcal{R}_w(g) := \left(\mathcal{H}_w^1(g)\right)^*$$

is a well-defined Banach space. The associated conjugate-linear pairing will be denoted by

$$\langle f, h \rangle := f(h), \quad f \in \mathcal{R}_w, \ h \in \mathcal{H}_w^1.$$

The representation π on $\mathcal{H}_\pi$ can be extended to act on $\mathcal{R}_w$ via

$$\pi(x)f : \quad \mathcal{H}_w^1 \to \mathbb{C}, \quad h \mapsto f([\pi(x)]^*h) = \sigma(x^{-1}, x)f(\pi(x^{-1})h), \tag{4.11}$$

for $x \in G$ and $f \in \mathcal{R}_w$. This is well-defined, since $\pi(x^{-1}) : \mathcal{H}_w^1 \to \mathcal{H}_w^1$ is well-defined, linear, and bounded. The associated (extended) matrix coefficients are defined by

$$V_h f(x) = \langle f, \pi(x)h \rangle, \quad f \in \mathcal{R}_w, h \in \mathcal{H}_w^1$$

for $x \in G$.

The next lemma collects the most important properties of these objects for the purposes of this article.

Lemma 4.6. *Let $g \in \mathcal{A}_w$ be admissible and write $\mathcal{H}_w^1 = \mathcal{H}_w^1(g)$. Then the following hold:*

(i) *The pairing $\langle \cdot, \cdot \rangle : \mathcal{R}_w \times \mathcal{H}_w^1 \to \mathbb{C}$ is an extension of the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_\pi}$; that is, $\mathcal{H}_\pi \hookrightarrow \mathcal{R}_w$ and $\langle f, h \rangle = \langle f, h \rangle_{\mathcal{H}_\pi}$ for $f \in \mathcal{H}_\pi$ and $h \in \mathcal{H}_w^1$. Moreover, if $f \in \mathcal{H}_\pi \subset \mathcal{R}_w$, then the definition of $\pi(x)f$ in Equation (4.11) agrees with the original definition. Finally, for any $f \in \mathcal{R}_w$, the extended mapping*

$$V_h : \quad \mathcal{R}_w \to L_{1/w}^\infty(G)$$

is well-defined, linear, and bounded.

(ii) *If $f \in \mathcal{R}_w$ and $h \in \mathcal{H}_w^1$, then*

$$V_h[\pi(x)f] = L_x^\sigma[V_h f],$$

for $x \in G$.

(iii) *The map $V_g : \mathcal{R}_w \to L_{1/w}^\infty(G)$ is injective. In addition,*

$$V_h f = V_g f *_\sigma V_h g \tag{4.12}$$

for any $f \in \mathcal{R}_w$ and $h \in \mathcal{H}_w^1$. Moreover, $\langle f, h \rangle = \langle V_g f, V_g h \rangle_{L_{1/w}^\infty, L_w^1}$.

(iv) *There exists a bounded linear operator*

$$V_g^* : \quad L_{1/w}^\infty(G) \to \mathcal{R}_w$$

satisfying $V_g(V_g^*F) = F$ for every $F \in L_{1/w}^\infty(G)$ that satisfies $F = F *_\sigma V_g g$.

(v) *The function $f \mapsto \|V_g f\|_{L_{1/w}^\infty}$ defines an equivalent norm on $\mathcal{R}_w$.*

Proof. (i) Using the embedding $\mathcal{H}_w^1 \hookrightarrow \mathcal{H}_\pi$ of Lemma 4.3, it follows that $\iota : \mathcal{H}_\pi \to \mathcal{R}_w$ given by $(\iota f)(h) = \langle f, h \rangle_{\mathcal{H}_\pi}$ for $h \in \mathcal{H}_w^1$ is well-defined, and $|(\iota f)(h)| = |\langle f, h \rangle_{\mathcal{H}_\pi}| \lesssim \|f\|_{\mathcal{H}_\pi}\|h\|_{\mathcal{H}_w^1}$ for $f \in \mathcal{H}_\pi$ and $h \in \mathcal{H}_w^1$. Thus $\iota f \in \mathcal{R}_w$ and $\iota : \mathcal{H}_\pi \to \mathcal{R}_w$ is bounded. Since $\mathcal{H}_w^1$ is norm dense in $\mathcal{H}_\pi$ by Lemma 4.3, the map ι is injective, and thus a continuous embedding. In particular, $\langle \iota f, h \rangle = (\iota f)(h) = \langle f, h \rangle_{\mathcal{H}_\pi}$, so that $\langle \cdot, \cdot \rangle$ extends $\langle \cdot, \cdot \rangle_{\mathcal{H}_\pi}$.

Next, if $f \in \mathcal{H}_\pi$ and $h \in \mathcal{H}_w^1 \subset \mathcal{H}_\pi$, then we have with $\pi(x)f \in \mathcal{H}_\pi$ as defined before and $\pi(x)\iota f \in \mathcal{R}_w$ as defined in Equation (4.11) that

$$\iota[\pi(x)f](h) = \langle \pi(x)f, h \rangle_{\mathcal{H}_\pi} = \langle f, [\pi(x)]^* h \rangle_{\mathcal{H}_\pi} = \iota(f)([\pi(x)]^* h) = \pi(x)[\iota(f)](h)$$

and hence $\iota[\pi(x)f] = \pi(x)[\iota(f)]$, showing that the definition of $\pi(x)f$ in Equation (4.11) agrees with the action of π on $\mathcal{H}_\pi \subset \mathcal{R}_w$.

Next, let $f \in \mathcal{R}_w = (\mathcal{H}_w^1)^*$ and $h \in \mathcal{H}_w^1$ and recall from Part (iii) of Lemma 4.3 that the map $\Xi_h : G \to \mathcal{H}_w^1$, $x \mapsto \pi(x)h$ is measurable. Since $f : \mathcal{H}_w^1 \to \mathbb{C}$ is continuous, this implies that $V_h f : G \to \mathbb{C}, x \mapsto V_h f(x) = f(\pi(x)h)$ is Borel measurable. In addition, the norm estimate of Part *(ii)* of Lemma 4.3 shows that

$$|V_h f(x)| \leq \|f\|_{\mathcal{R}_w} \|\pi(x)h\|_{\mathcal{H}_w^1} \leq w(x) \|f\|_{\mathcal{R}_w} \cdot \|h\|_{\mathcal{H}_w^1},$$

whence $V_h f \in L_{1/w}^\infty(G)$.

(ii) If $x, y \in G$, then

$$(V_h \pi(x)f)(y) = \sigma(x, x^{-1}) \langle f, \pi(x^{-1})\pi(y)h \rangle = \sigma(x, x^{-1})\overline{\sigma(x^{-1}, y)} V_h f(x^{-1}y) = L_x^\sigma[V_h f](y),$$

since $\sigma(x, x^{-1}y) = \overline{\sigma(x^{-1}, y)}\sigma(xx^{-1}, y)\sigma(x, x^{-1}) = \sigma(x, x^{-1})\overline{\sigma(x^{-1}, y)}$.

(iii) By Lemma 4.3, $\pi(G)g$ is complete in $\mathcal{H}_w^1 = \mathcal{H}_w^1(g)$; thus, the mapping $V_g : \mathcal{R}_w \to L_{1/w}^\infty$ is injective. Let $f \in \mathcal{R}_w$, $h \in \mathcal{H}_w^1$ and $x \in G$. Let $(\mathcal{H}_w^1)'$ denote the dual space of $\mathcal{H}_w^1$ and define $f' \in (\mathcal{H}_w^1)'$ by $f'(k) = \overline{f(\pi(x)k)}$ for $k \in \mathcal{H}_w^1$. Using the Bochner integrable map $F_{h,g} : G \to \mathcal{H}_w^1$ defined in Lemma 4.3, it follows that $h = \int_G F_{h,g} \, d\mu_G$ with convergence in $\mathcal{H}_w^1$. Since Bochner integration commutes with bounded linear functionals (cf. [13, Proposition E.11]), a direct calculation using Lemma 4.1 entails

$$
\begin{aligned}
V_h f(x) = \overline{f'(h)} &= \overline{\int_G f'(F_{h,g}(y)) \, d\mu_G(y)} = \overline{\int_G V_g h(y) f'(\pi(y)g) \, d\mu_G(y)} \\
&= \int_G \overline{V_g h(y)} f(\pi(x)\pi(y)g) \, d\mu_G(y) = \int_G \sigma(y, y^{-1}) V_h g(y^{-1}) \overline{\sigma(x, y)} f(\pi(xy)g) \, d\mu_G(y) \\
&= \int_G \sigma(xy, y^{-1}) V_h g(y^{-1}) V_g f(xy) \, d\mu_G(y) = \int_G \sigma(z, z^{-1}x) V_h g(z^{-1}x) V_g f(z) \, d\mu_G(z) \\
&= V_g f *_\sigma V_h g(x),
\end{aligned}
$$

where the sixth step used that $\sigma(xy, y^{-1}) = \overline{\sigma(x, y)}\sigma(x, yy^{-1})\sigma(y, y^{-1}) = \sigma(y, y^{-1})\overline{\sigma(x, y)}$ for $x, y \in G$. Setting $x = e_G$, the above calculations also show that $\langle f, h \rangle = \langle V_g f, V_g h \rangle_{L_{1/w}^\infty, L_w^1}$.

(iv) For $F \in L_{1/w}^\infty(G)$, define

$$V_g^* F : \mathcal{R}_w \to \mathbb{C}, \quad h \mapsto \int_G F(y) \langle \pi(y)g, h \rangle_{\mathcal{H}_\pi} \, d\mu_G(y),$$

which is well-defined since

$$
\begin{aligned}
\int_G |F(y)| |\langle \pi(y)g, h \rangle_{\mathcal{H}_\pi}| \, d\mu_G(y) &\leq \int_G \frac{1}{w(y)} |F(y)| w(y) |V_g h(y)| \, d\mu_G(y) \\
&\leq \|F\|_{L_{1/w}^\infty} \|V_g h\|_{L_w^1} = \|F\|_{L_{1/w}^\infty} \|h\|_{\mathcal{H}_w^1} < \infty.
\end{aligned}
$$

In particular, this shows that $V_g^* : L_{1/w}^\infty(G) \to \mathcal{R}_w$ is a bounded linear map.

If $F \in L_{1/w}^\infty(G)$ satisfies $F = F *_\sigma V_g g$, then Lemma 4.1 shows

$$\langle \pi(y)g, \pi(x)g \rangle_{\mathcal{H}_\pi} = V_g[\pi(y)g](x) = L_y^\sigma[V_g g](x)$$

and hence

$$V_g(V_g^*F)(x) = \int_G F(y)\langle \pi(y)g, \pi(x)g\rangle_{\mathcal{H}_\pi} \, d\mu_G(y)$$
$$= \int_G F(y)L_y^\sigma[V_gg](x) \, d\mu_G(y)$$
$$= F *_\sigma V_gg(x) = F(x)$$

for any $x \in G$.

(v) Let $f \in \mathcal{R}_w$. The estimate $\|V_gf\|_{L_{1/w}^\infty} \lesssim \|f\|_{\mathcal{R}_w}$ follows from Part (i). For the converse, set $F := V_gf \in L_{1/w}^\infty(G)$, so that $F = F *_\sigma V_gg$ by Part (ii). Then, by Part (iv), $h := V_g^*F \in \mathcal{R}_w$ with $\|h\|_{\mathcal{R}_w} \lesssim \|F\|_{L_{1/w}^\infty}$, and $V_gh = F = V_gf$. Since $V_g : \mathcal{R}_w \to L_{1/w}^\infty(G)$ is injective by Part (iii), it follows that $f = h$, and thus $\|f\|_{\mathcal{R}_w} \lesssim \|F\|_{L_{1/w}^\infty} = \|V_gf\|_{L_{1/w}^\infty}$, where the implied constant is independent of $f \in \mathcal{R}_w$. $\qquad\square$

4.3. **Coorbit spaces.** Throughout this section, $Y \subset L^0(G)$ will be a solid quasi-Banach function space on G and w will be a p-weight for some $p \in (0,1]$. We also assume that Y is translation-invariant and that Y is L_w^p-compatible in the sense of Definition 3.5.

The σ-representation $(\pi, \mathcal{H}_\pi)$ will be assumed to be p-integrable in the sense that for

$$\mathcal{B}_w^p = \{g \in \mathcal{H}_\pi : V_gg \in W(L_w^p)\},$$

there exists an *admissible* $g \in \mathcal{B}_w^p$. Note that $\mathcal{B}_w^p \subseteq \mathcal{A}_w$, since $W(L_w^p) \hookrightarrow W^L(L_w^p) \hookrightarrow L_w^1$ by Part (ii) of Lemma 3.3.

Definition 4.7. Let Y be L_w^p-compatible and let $g \in \mathcal{B}_w^p$ be an admissible vector. The associated coorbit space $\mathrm{Co}(Y) = \mathrm{Co}_g(Y)$ is defined as the collection

$$\mathrm{Co}_g(Y) := \{f \in \mathcal{R}_w : V_gf \in W^L(Y)\} \tag{4.13}$$

and equipped with the quasi-norm $\|f\|_{\mathrm{Co}(Y)} := \|f\|_{\mathrm{Co}_g(Y)} := \|V_gf\|_{W^L(Y)}$.

Proposition 4.8. *Let Y be L_w^p-compatible and let $g \in \mathcal{B}_w^p$ be an admissible vector.*

(i) *If $h \in \mathcal{B}_w^p$ is an admissible vector such that $V_hg \in W(L_w^p)$, then $\mathrm{Co}_g(Y) = \mathrm{Co}_h(Y)$ with $\|\cdot\|_{\mathrm{Co}_g(Y)} \asymp \|\cdot\|_{\mathrm{Co}_h(Y)}$.*

(ii) *The space $\mathrm{Co}(Y)$ is a π-invariant quasi-Banach space with $\mathrm{Co}(Y) \hookrightarrow \mathcal{R}_w$, and $\|\cdot\|_{\mathrm{Co}(Y)}$ is a p-norm.*

Proof. (i) For $f \in \mathrm{Co}_g(Y)$, the reproducing formula (4.12) yields that $V_hf = V_gf *_\sigma V_hg$. The space Y is assumed to be L_w^p-compatible, and therefore $W^L(Y) * W(L_w^p) \hookrightarrow W^L(Y)$, which yields that

$$\|f\|_{\mathrm{Co}_h(Y)} \leq \||V_gf| * |V_hg|\|_{W^L(Y)} \lesssim \|V_gf\|_{W^L(Y)}\|V_hg\|_{W(L_w^p)} \lesssim \|f\|_{\mathrm{Co}_g(Y)}.$$

Since $|V_gh| = |V_hg|^\vee$ and w is a p-weight, it follows that also $V_gh \in W(L_w^p)$. Therefore, interchanging the role of g and h also yields $\|\cdot\|_{\mathrm{Co}_g(Y)} \lesssim \|\cdot\|_{\mathrm{Co}_h(Y)}$.

(ii) The π-invariance follows directly from Part (ii) of Lemma 4.6 and the (left) translation-invariance of Y, which implies the (left) translation-invariance of $W^L(Y)$. To show that $\mathrm{Co}_g(Y) \hookrightarrow \mathcal{R}_w$, let $f \in \mathrm{Co}(Y) = \mathrm{Co}_g(Y)$. Since Y is assumed to be L_w^p-compatible, we have, in particular, $W^L(Y) \hookrightarrow L_{1/w}^\infty$, and thus $\|V_gf\|_{L_{1/w}^\infty} \lesssim \|V_gf\|_{W^L(Y)} = \|f\|_{\mathrm{Co}(Y)}$. Since the mapping $f \mapsto \|V_gf\|_{L_{1/w}^\infty}$ defines an equivalent norm on $\mathcal{R}_w$ by Lemma 4.6, it follows immediately that $\|f\|_{\mathcal{R}_w} \lesssim \|f\|_{\mathrm{Co}_g(Y)}$.

To prove the completeness of $\mathrm{Co}(Y)$, let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathrm{Co}(Y)$. Then the embedding $\mathrm{Co}(Y) \hookrightarrow \mathcal{R}_w$ yields that $(f_n)_{n\in\mathbb{N}}$ is Cauchy in $\mathcal{R}_w$, whence convergent to some $f \in \mathcal{R}_w$. In particular, this implies $V_gf_n(x) \to V_gf(x)$ for $x \in G$ as $n \to \infty$. On the other hand, the sequence $(V_gf_n)_{n\in\mathbb{N}}$ is Cauchy in $W^L(Y)$, hence converging to some $F \in W^L(Y)$.

Since $W^L(Y) \hookrightarrow L_{1/w}^\infty(G)$ as Y is L_w^p-compatible, it also follows that $V_g f_n \to F$ in $L_{1/w}^\infty(G)$. Thus $F = V_g f$. Since $F \in W^L(Y)$, this means that $f \in \mathrm{Co}(Y)$, and

$$\|f - f_n\|_{\mathrm{Co}(Y)} = \|V_g f_n - F\|_{W^L(Y)} \to 0, \quad \text{as} \quad n \to \infty.$$

The p-norm properties of $\|\cdot\|_{\mathrm{Co}(Y)}$ follow easily from those of $\|\cdot\|_{W^L(Y)}$. $\qquad\square$

The following simple consequence of Lemma 4.6 is often helpful in concrete settings.

Corollary 4.9. *Let Y be L_w^p-compatible, let $\mathcal{S} \hookrightarrow \mathcal{H}_\pi$ be a π-invariant topological vector space and let $g \in \mathcal{S} \cap \mathcal{B}_w^p$ be admissible. Suppose that $\mathcal{S} \hookrightarrow \mathcal{H}_w^1(g)$ and that the reproducing formula*

$$\langle f, h \rangle = \int_G \langle f, \pi(x)g \rangle_{\mathcal{S}^*, \mathcal{S}} \cdot \langle \pi(x)g, h \rangle_{\mathcal{H}_\pi} \, d\mu_G(x) \tag{4.14}$$

holds for all $f \in \mathcal{S}^*$ (the anti-dual space of $\mathcal{S}$) and $h \in \mathcal{S}$, where $\langle \cdot, \cdot \rangle_{\mathcal{S}^*, \mathcal{S}} = \langle \cdot, \cdot \rangle : \mathcal{S}^* \times \mathcal{S} \to \mathbb{C}$ denotes the anti-dual pairing.

Then

$$\mathrm{Co}_g(Y) = \{ f \in \mathcal{R}_w : V_g f \in W^L(Y) \} = \{ f \in \mathcal{S}^* : V_g f \in W^L(Y) \}.$$

in the sense that the restriction map

$$\mathrm{Co}_g(Y) \to \{ f \in \mathcal{S}^* : V_g f \in W^L(Y) \}, \quad f \mapsto f|_{\mathcal{S}}$$

is a bijection. Here, the coefficient transform $V_g f$ for $f \in \mathcal{S}^*$ is defined by $V_g f(x) = \langle f, \pi(x)g \rangle$.

Proof. Since $\mathcal{S} \hookrightarrow \mathcal{H}_w^1$, it follows that the restriction map $f \mapsto f|_{\mathcal{S}}$ is well-defined from $\mathcal{R}_w$ into $\mathcal{S}^*$ and hence also from $\mathrm{Co}_g(Y)$ into $\{ f \in \mathcal{S}^* : V_g f \in W^L(Y) \}$. Furthermore, since we have $\pi(G)g \subset \mathcal{S}$ and since $\pi(G)g \subset \mathcal{H}_w^1$ is norm-dense by Lemma 4.3, this restriction map is injective.

To show surjectivity, let $f \in \mathcal{S}^*$ be such that $F := V_g f \in W^L(Y)$. Then also $F \in L_{1/w}^\infty(G)$ since Y is L_w^p-compatible and hence $W^L(Y) \hookrightarrow L_{1/w}^\infty(G)$. The assumption (4.14) yields, in particular, that $F = F *_\sigma V_g g$. Hence, by Part (iv) of Lemma 4.6, there exists $\widetilde{f} := V_g^* F \in \mathcal{R}_w$ such that $V_g \widetilde{f} = F = V_g f \in W^L(Y)$. Therefore, $\widetilde{f} \in \mathrm{Co}_g(Y)$, and $\widetilde{f}|_{\mathcal{S}} = f$ by (4.14) and Part (iii) of Lemma 4.6. $\qquad\square$

4.4. **Analyzing and better vectors.** This section provides (somewhat more) explicit descriptions of the coorbit spaces $\mathrm{Co}(Y)$ for the specific choices Y being $L_w^1(G), L^2(G)$ or $L_{1/w}^\infty(G)$. In particular, this section completely resolves the question of the relation between the sets of so-called "analyzing vectors" and "better vectors"; see Proposition 4.13. To the best of our knowledge, this question was an open problem in the literature even for Banach spaces; see for instance [34, Section 6.3].

As defined in Equation (4.13), the coorbit space $\mathrm{Co}_g(Y)$ consists of those $f \in \mathcal{R}_w$ for which $V_g f \in W^L(Y)$. However, in the literature considering coorbit spaces associated with genuine Banach function spaces Y, one usually defines $\mathrm{Co}_g(Y) = \{ f \in \mathcal{R}_w : V_g f \in Y \}$. The following proposition identifies a sufficient condition regarding Y under which both definitions coincide; in particular, it is applicable to all Banach function spaces.

Proposition 4.10. *Let Y be a solid, translation-invariant quasi-Banach function space with p-norm $\|\cdot\|_Y$ (for some $p \in (0,1]$) and such that $Y * W(L_w^p) \hookrightarrow Y$ and $W^L(Y) \hookrightarrow L_{1/w}^\infty(G)$ for a p-weight w. Then Y is L_w^p-compatible, and if $g \in \mathcal{B}_w^p$ is an admissible vector, then*

$$\mathrm{Co}_g(Y) := \{ f \in \mathcal{R}_w : V_g f \in W^L(Y) \} = \{ f \in \mathcal{R}_w : V_g f \in Y \}, \tag{4.15}$$

with $\|f\|_{\mathrm{Co}_g(Y)} \asymp \|V_g f\|_Y$.

In addition, if $W^L(Y) \hookrightarrow L^2(G)$, then (up to canonical identifications), it holds that

$$\mathrm{Co}_g(Y) = \{ f \in \mathcal{H}_\pi : V_g f \in W^L(Y) \} = \{ f \in \mathcal{H}_\pi : V_g f \in Y \}. \tag{4.16}$$

Proof. Using the estimate (2.8), it holds for $F \in W^L(Y)$ and $H \in W(L_w^p)$ that

$$\|F * H\|_{W^L(Y)} = \|M^L[F * H]\|_{W^L(Y)} \leq \||F| * M^L H\|_Y$$
$$\lesssim \||F|\|_Y \cdot \|M^L H\|_{W(L_w^p)} \lesssim \|F\|_{W^L(Y)} \cdot \|H\|_{W(L_w^p)}.$$

Hence, Y is L_w^p-compatible.

Assume $g \in \mathcal{H}_\pi$ is admissible and $V_g g \in W(L_w^p) \subseteq L_w^1$. An application of Lemma 4.6 yields that $V_g f = V_g f *_\sigma V_g g$ for $f \in \mathcal{R}_w$. Therefore, if $V_g f \in Y$, then the estimate (2.8) shows that

$$\|V_g f\|_{W^L(Y)} \leq \|M^L(|V_g f| * |V_g g|)\|_Y \lesssim \|V_g f\|_Y \|M^L[V_g g]\|_{W(L_w^p)} \lesssim \|V_g f\|_Y \|V_g g\|_{W(L_w^p)},$$

wand hence $\|f\|_{\mathrm{Co}_g(Y)} \lesssim \|V_g f\|_Y$. The estimate $\|V_g f\|_Y \leq \|f\|_{\mathrm{Co}_g(Y)}$ is immediate.

To prove Equation (4.16), it suffices because of $\mathcal{H}_\pi \hookrightarrow \mathcal{R}_w$ to show that each $f \in \mathrm{Co}_g(Y)$ satisfies $f \in \iota(\mathcal{H}_\pi)$, with the canonical embedding $\iota : \mathcal{H}_\pi \to \mathcal{R}_w$. For clarity, we denote by V_g^e (resp. V_g) the coefficient transform on $\mathcal{R}_w$ (resp. $\mathcal{H}_\pi$). Let $f \in \mathrm{Co}_g(Y)$ be arbitrary and note that $F := V_g^e f \in W^L(Y) \hookrightarrow L^2(G)$ satisfies $F = F *_\sigma V_g g$; see Lemma 4.6. With the usual Hilbert-space adjoint $V_g^* : L^2(G) \to \mathcal{H}_\pi$ of $V_g : \mathcal{H}_\pi \to L^2(G)$, let $\widetilde{f} := V_g^* F \in \mathcal{H}_\pi$. Then, for any $x \in G$,

$$V_g^e[\iota \widetilde{f}](x) = \langle \iota \widetilde{f}, \pi(x)g \rangle_{\mathcal{R}_w, \mathcal{H}_w^1} = \langle \widetilde{f}, \pi(x)g \rangle_{\mathcal{H}_\pi} = \langle V_g^* F, \pi(x)g \rangle_{\mathcal{H}_\pi} = \langle F, V_g[\pi(x)g] \rangle_{L^2}$$
$$= \int_G F(y) \overline{\langle \pi(x)g, \pi(y)g \rangle} \, d\mu_G(y) = \int_G F(y) V_g[\pi(y)g](x) \, d\mu_G(y)$$
$$= \int_G F(y) L_y^\sigma[V_g g](x) \, d\mu_G(y) = (F *_\sigma V_g g)(x) = F(x) = V_g^e f(x).$$

Hence, $V_g^e(\iota \widetilde{f}) = V_g^e f$. Since $V_g^e : \mathcal{R}_w \to L_{1/w}^\infty(G)$ is injective by Lemma 4.6, this implies $f = \iota \widetilde{f} \in \iota(\mathcal{H}_\pi)$, as required. $\qquad\square$

Remark 4.11. As stated above, Proposition 4.10 could also apply for *quasi*-Banach spaces, but the issue is that when Y is not a Banach space but rather a quasi-Banach space, then the convolution relation $Y * W(L_w^p) \hookrightarrow Y$ is rarely valid. Nevertheless, the identity (4.15) is known to hold for certain quasi-Banach spaces in some settings; see, e.g., [31,43,46]. It is an open problem whether (4.15) holds in general for quasi-Banach spaces.

Lemma 4.12. *Let* $w : G \to [1, \infty)$ *be a p-weight for some $p \in (0, 1]$ and let $g \in \mathcal{B}_w^p$ be admissible. Then each of the spaces $L_w^1(G), L^2(G), L_{1/w}^\infty(G)$ and $W^R(L_w^1)$ is L_w^p-compatible. Furthermore, the following identifications hold:*

(i) $\mathrm{Co}_g(L_w^1(G)) = \mathcal{H}_w^1(g)$.
(ii) $\mathrm{Co}_g(L^2(G)) = \mathcal{H}_\pi$.
(iii) $\mathrm{Co}_g(W^R(L_w^1)) = \{f \in \mathcal{H}_\pi : V_g f \in W(L_w^1)\}$.
(iv) $\mathrm{Co}_g(L_{1/w}^\infty(G)) = \mathcal{R}_w$.

Proof. As a preparation, suppose that $v, v_0 : G \to (0, \infty)$ are measurable and such that $v(xy) \leq v(x)v_0(y)$ and $v(xy) \leq v_0(x)v(y)$ for all $x, y \in G$. Using the well-known identity $\int_G f(ts) \, d\mu_G(t) = \Delta(s^{-1}) \int_G f(t) \, d\mu_G(t)$ (see, e.g., [22, Proposition 2.24]), it then follows that

$$\|R_x f\|_{L_v^p}^p = \int_G |f(yx)|^p \cdot [v(yxx^{-1})]^p \, d\mu_G(y)$$
$$= \Delta(x^{-1}) \int_G |f(z)|^p \cdot [v(zx^{-1})]^p \, d\mu_G(z) \leq [v_0(x^{-1})]^p \cdot \Delta(x^{-1}) \cdot \|f\|_{L_v^p}^p.$$

A similar computation for the left-translation shows

$$\|L_x f\|_{L_v^p} \leq v_0(x) \cdot \|f\|_{L_v^p} \quad \text{and} \quad \|R_x f\|_{L_v^p} \leq v_0(x^{-1}) \cdot [\Delta(x^{-1})]^{1/p} \cdot \|f\|_{L_v^p} \qquad (4.17)$$

for all $x \in G$ and $f \in L_v^p(G)$, for arbitrary $p \in (0, \infty)$; these estimates also remain valid for $p = \infty$. Next, note $w(x) = w(xyy^{-1}) \leq w(xy)w(y^{-1})$ and hence $\frac{1}{w(xy)} \leq \frac{1}{w(x)}w^\vee(y)$;

similar arguments show that also $\frac{1}{w(xy)} \leq w^\vee(x)\frac{1}{w(y)}$. Therefore, applying Equation (4.17) with $(v, v_0) = (w, w)$ or $(v, v_0) = (1, 1)$ or $(v, v_0) = (\frac{1}{w}, w^\vee)$, respectively, shows that each space $Y \in \{L^1_w, L^2, L^\infty_{1/w}\}$ is invariant under left- and right-translations. More precisely, we see because of $p \in (0, 1]$ and by conditions (w1) and (w3) that

$$\|R_x\|_{L^2 \to L^2} \leq [\Delta(x^{-1})]^{1/2} \leq \begin{cases} [\Delta(x^{-1})]^{1/p} \leq w(x^{-1})\Delta^{1/p}(x^{-1}) = w(x), & \text{if } \Delta(x^{-1}) \geq 1, \\ 1 \leq w(x), & \text{if } \Delta(x^{-1}) < 1. \end{cases}$$

and $\|R_x\|_{L^\infty_{1/w} \to L^\infty_{1/w}} \leq w^\vee(x^{-1}) = w(x)$.

We now verify that Proposition 4.10 is applicable for each of the spaces in question.

First, note that any $Y \in \{L^1_w(G), L^2(G), L^\infty_{1/w}(G), W^R(L^1_w)\}$ is a solid Banach space and thus, in particular, a p-normed solid quasi-Banach function space. We saw above that each of the spaces $Y \in \{L^1_w(G), L^2(G), L^\infty_{1/w}(G)\}$ is translation-invariant. By the properties of Wiener amalgam spaces collected in Section 2.6, the same then also holds for $Y = W^R(L^1_w)$.

Second, it will be shown that $Y * W(L^p_w) \hookrightarrow Y$. To handle the case $Y = L^1_w$, we recall from [49, §3.7] that the convolution relation $L^1_w * L^1_w \hookrightarrow L^1_w$ holds, since w is submultiplicative. Next, note that $W(L^p_w) = W^R(W^L(L^p_w)) \hookrightarrow W^L(L^p_w) \hookrightarrow L^1_w$, where the last step follows from Lemma 3.3. Overall, we thus see that $L^1_w * W(L^p_w) \hookrightarrow L^1_w * L^1_w \hookrightarrow L^1_w$, as required. For $F_1 \in W^R(L^1_w)$ and $F_2 \in W(L^p_w)$, Equation (2.8) shows that

$$\|F_1 * F_2\|_{W^R(L^1_w)} = \|M^R[F_1 * F_2]\|_{L^1_w} \leq \|M^R F_1 * |F_2|\|_{L^1_w}$$
$$\lesssim \|M^R F_1\|_{L^1_w} \cdot \|F_2\|_{W(L^p_w)} = \|F_1\|_{W^R(L^1_w)} \cdot \|F_2\|_{W(L^p_w)},$$

which proves that $W^R(L^1_w) * W(L^p_w) \hookrightarrow W^R(L^1_w)$, thereby settling the case $Y = W^R(L^1_w)$. If $Y \in \{L^2, L^\infty_{1/w}\}$ recall from above that $\|R_x\|_{Y \to Y} \leq w(x)$; hence, Lemma 3.11 shows that $Y * [L^1_w]^\vee \hookrightarrow Y$. Since $W(L^p_w) \hookrightarrow W^R(L^p_w) \hookrightarrow [L^1_w]^\vee$ by Lemma 3.3, this implies that $Y * W(L^p_w) \hookrightarrow Y * [L^1_w]^\vee \hookrightarrow Y$ for $Y \in \{L^2, L^\infty_{1/w}\}$.

Lastly, it will be shown that $W^L(Y) \hookrightarrow L^\infty_{1/w}(G)$ Since $w \geq 1$, we have $L^\infty_w \hookrightarrow L^\infty_{1/w}$. Thus, Lemma 3.3 implies that $W^L(W^R(L^1_w)) \hookrightarrow W^L(L^1_w) \hookrightarrow L^\infty_w \hookrightarrow L^\infty_{1/w}$, which proves the conclusion for $Y \in \{L^1_w, W^R(L^1_w)\}$. Lemma 3.3 also shows $W^L(L^2) \hookrightarrow L^\infty \hookrightarrow L^\infty_{1/w}$, where the last step again used that $w \geq 1$. We trivially have $W^L(L^\infty_{1/w}) \hookrightarrow L^\infty_{1/w}$.

Overall, Proposition 4.10 shows that each $Y \in \{L^1_w, L^2, L^\infty_{1/w}, W^R(L^1_w)\}$ is L^p_w-compatible. We now prove the remaining parts of the lemma.

(i) Lemma 3.3 shows that $W^L(L^1_w) \hookrightarrow L^2_w \hookrightarrow L^2$, since $w \geq 1$. Therefore, Proposition 4.10 shows that $\mathrm{Co}_g(L^1_w) = \{f \in \mathcal{H}_\pi \colon V_g f \in L^1_w\} = \mathcal{H}^1_w$.

(ii) We trivially have $W^L(L^2) \hookrightarrow L^2$, and by admissibility of g it holds that $V_g f \in L^2(G)$ for every $f \in \mathcal{H}_\pi$. Therefore, Proposition 4.10 shows $\mathrm{Co}_g(L^2) = \{f \in \mathcal{H}_\pi \colon V_g f \in L^2(G)\} = \mathcal{H}_\pi$.

(iii) We have $W^R(L^1_w) \hookrightarrow L^1_w \hookrightarrow L^2$ (see the proof of (i)). Therefore, Proposition 4.10 shows that $\mathrm{Co}_g(W^R(L^1_w)) = \{f \in \mathcal{H}_\pi \colon V_g f \in W^L(W^R(L^1_w))\} = \{f \in \mathcal{H}_\pi \colon V_g f \in W(L^1_w)\}$, as claimed.

(iv) By Lemma 4.6, we have $V_g f \in L^\infty_{1/w}(G)$ for all $f \in \mathcal{R}_w$. Therefore, Proposition 4.10 shows $\mathrm{Co}_g(L^\infty_{1/w}) = \{f \in \mathcal{R}_w \colon V_g f \in L^\infty_{1/w}\} = \mathcal{R}_w$, as claimed. $\qquad\square$

Lastly, it will be shown that the important auxiliary spaces of so-called "analyzing vectors" and "better vectors" (cf. [19,20]) coincide precisely if the group G is an IN group. That G being IN is sufficient is simple and well-known (see, e.g., [20, Lemma 7.2]), but its necessity remained open; see also [34, Section 6.3].

Proposition 4.13. *Let $w : G \to [1, \infty)$ be a p-weight and let $g \in \mathcal{B}^p_w$ be admissible. The following assertions are equivalent:*

(i) *The spaces* $\mathrm{Co}_g(L_w^1)$ *and* $\mathrm{Co}_g(W^R(L_w^1))$ *coincide, i.e.,* $\mathrm{Co}_g(L_w^1) = \mathrm{Co}_g(W^R(L_w^1))$.

(ii) *The group* G *is an* IN *group, i.e., there exists a relatively compact unit neighborhood* $U \subseteq G$ *such that* $x^{-1}Ux = U$ *for all* $x \in G$.

The following result is an essential ingredient in the proof of Proposition 4.13. Its proof was provided to us by T. Tao [52]; hence, no originality is claimed. The complete argument is included here, with a few added details.

Proposition 4.14. *If there exists an open, relatively compact unit neighborhood* $Q \subseteq G$ *satisfying*

$$\mu_G(QxQ) \lesssim 1 \qquad \text{for all } x \in G,$$

then G *is an* IN *group.*

Proof. Throughout the proof, let $C > 0$ be a constant satisfying $\mu_G(QxQ) \leq C$ for all $x \in G$.

Step 1. This step shows that G is unimodular. If not, then $\Delta(x_0) \neq 1$ for some $x_0 \in G$. Replacing x_0 by x_0^{-1} if necessary, it may be assumed that $\Delta(x_0) > 1$. Then, by definition of the modular function,

$$C \geq \mu_G(Qx_0^nQ) \geq \mu_G(Qx_0^n) = \Delta(x_0^n)\,\mu_G(Q) = (\Delta(x_0))^n\,\mu_G(Q) \to \infty \quad \text{as} \quad n \to \infty,$$

which is a contradiction.

Step 2. For the remainder of the proof, the space $L^2(G) = L^2(G;\mathbb{R})$ will be considered as a vector space over $\mathbb{R}$. For $U := Q \cap Q^{-1}$, this step will show that there exists $\delta > 0$ such that

$$\langle \mathbb{1}_{xU^2x^{-1}}, \mathbb{1}_{U^2} \rangle_{L^2} \geq \delta > 0, \quad x \in G. \tag{4.18}$$

For this, note by unimodularity of G that $\mu_G(U \cdot xUx^{-1}) = \mu_G(UxU) \leq \mu_G(QxQ) \leq C$ for all $x \in G$. Furthermore, $(\mathbb{1}_U * \mathbb{1}_{xUx^{-1}})^{-1}(\mathbb{C} \setminus \{0\}) \subset U \cdot xUx^{-1}$. Hence, using Tonelli's theorem, it follows

$$\left(\mu_G(U)\right)^2 = \mu_G(U) \cdot \mu_G(xUx^{-1}) = \int_G \mathbb{1}_U(z) \int_G \mathbb{1}_{xUx^{-1}}(z^{-1}y)\,d\mu_G(y)\,d\mu_G(z)$$

$$= \int_G (\mathbb{1}_U * \mathbb{1}_{xUx^{-1}})(y)\,d\mu_G(y) \leq \mu_G(U \cdot xUx^{-1}) \cdot \|\mathbb{1}_U * \mathbb{1}_{xUx^{-1}}\|_{L^\infty}$$

$$\leq C \cdot \|\mathbb{1}_U * \mathbb{1}_{xUx^{-1}}\|_{L^\infty}.$$

Thus, there exists $y = y(x) \in G$ satisfying

$$0 < \delta := \frac{(\mu_G(U))^2}{C} \leq (\mathbb{1}_U * \mathbb{1}_{xUx^{-1}})(y^{-1}) = \int_G \mathbb{1}_U(z)\mathbb{1}_{xUx^{-1}}(z^{-1}y^{-1})\,d\mu_G(z)$$

$$= \int_G \mathbb{1}_U(y^{-1}w)\mathbb{1}_{xUx^{-1}}(w^{-1})\,d\mu_G(w) = \mu_G(yU \cap xUx^{-1}),$$

where the change-of-variables $w = yz$ and the identity $(xUx^{-1})^{-1} = xUx^{-1}$ were used.

If $W \subset G$ is open and satisfies $\mu_G(W) \geq \delta$, then there exists some $w \in W$, which implies that

$$\mu_G(W^{-1}W) \geq \mu_G(w^{-1}W) = \mu_G(W) \geq \delta.$$

Applying this observation to $W = W_y := yU \cap xUx^{-1}$ and using that $W_y^{-1}W_y \subset U^2 \cap xU^2x^{-1}$, it follows that

$$\langle \mathbb{1}_{xU^2x^{-1}}, \mathbb{1}_{U^2} \rangle_{L^2} = \mu_G(U^2 \cap xU^2x^{-1}) \geq \mu_G(W_y^{-1}W_y) \geq \delta, \qquad x \in G,$$

which establishes the asserted claim (4.18).

Step 3. Set $\Omega := \{\mathbb{1}_{xU^2x^{-1}} : x \in G\} \subset L^2(G)$. Since $\mathbb{1}_{xU^2x^{-1}} = L_x R_x \mathbb{1}_{U^2}$, it is easy to see $L_x R_x \Omega \subset \Omega$ for all $x \in G$. Furthermore, since $U = U^{-1}$, each $F \in \Omega$ satisfies $F^\vee = F$, $0 \leq F \leq 1$ almost everywhere, and $\langle F, \mathbb{1}_{U^2} \rangle_{L^2} \geq \delta > 0$ (cf. Step 2). Note that all of these properties are preserved under convex combinations and under limits in L^2; this uses that G is unimodular so that $F \mapsto F^\vee$ is a bounded linear map on L^2 and that if $F_n \to F$ in L^2, then

$F_{n_\ell} \to F$ almost everywhere for a suitable subsequence. Hence, letting $\Sigma := \overline{\operatorname{conv} \Omega} \subset L^2$ denote the closed convex hull of Ω, it follows that $L_x R_x \Sigma \subset \Sigma$ for all $x \in G$ and that each $F \in \Sigma$ satisfies $F = F^\vee$ and $0 \le F \le 1$ almost everywhere, and finally $\langle F, \mathbb{1}_{U^2} \rangle_{L^2} \ge \delta > 0$.

By the Hilbert projection theorem (see, e.g., [51, Theorem 12.3]), there exists a unique $F_0 \in \Sigma$ satisfying $\|F_0\|_{L^2} \le \|H\|_{L^2}$ for all $H \in \Sigma$. Since G is unimodular, the operator $L_x R_x : L^2 \to L^2$ is unitary; hence, $L_x R_x F_0 \in \Sigma$ with $\|L_x R_x F_0\|_{L^2} = \|F_0\|_{L^2} \le \|H\|_{L^2}$ for all $H \in \Sigma$. By the uniqueness of F_0, this implies $L_x R_x F_0 = F_0$ for all $x \in G$. Note that F_0 is non-trivial since $\langle F_0, \mathbb{1}_{U^2} \rangle_{L^2} \ge \delta > 0$.

Step 4. Since G is unimodular, the identity $F * H(x) = \langle F, L_x H^\vee \rangle$ shows that the bilinear map

$$L^2(G) \times L^2(G) \to C_b(G), \quad (F, H) \mapsto F * H$$

is well-defined and continuous. Since $F * H \in C_c(G)$ for $F, H \in C_c(G)$, this implies by density that $F * H \in C_0(G)$ for all $F, H \in L^2(G)$. In particular, $H_0 := F_0 * F_0 \in C_0(G)$, where F_0 is as in Step 3. Furthermore, since F_0 satisfies $F_0 = F_0^\vee$ and $F_0 = L_x R_x F_0$, it follows that for all $x, y \in G$, it holds that $F_0(z^{-1} x^{-1} y x) = F_0(x^{-1} y^{-1} x z) = F_0(y^{-1} x z x^{-1})$ for almost all $z \in G$. Since G is unimodular, the change-of-variables $w = x z x^{-1}$ therefore shows

$$
\begin{aligned}
H_0(x^{-1} y x) &= \int_G F_0(z) F_0(z^{-1} x^{-1} y x) \, d\mu_G(z) = \int_G F_0(z) F_0(y^{-1} x z x^{-1}) \, d\mu_G(z) \\
&= \int_G F_0(x^{-1} w x) F_0(y^{-1} w) \, d\mu_G(w) = \int_G F_0(w) F_0(w^{-1} y) \, d\mu_G(w) \\
&= (F_0 * F_0)(y) = H_0(y).
\end{aligned}
\tag{4.19}
$$

where the fourth step used again that $F_0 = L_x R_x F_0$ and $F_0 = F_0^\vee$.

Lastly, note because of $F_0 \ge 0$ and $F_0 = F_0^\vee$ that

$$H_0(e_G) = \int_G F_0(x) F_0(x^{-1}) \, d\mu_G(x) = \|F_0\|_{L^2}^2 > 0.$$

Since $H_0 \in C_0(G)$, this implies that $V := \{x \in G : H_0(x) > \|F_0\|_{L^2}^2 / 2\}$ is an open, relatively compact unit neighborhood. In view of Equation (4.19), it follows that $x V x^{-1} = V$ for all $x \in G$. Hence, G is an IN group. $\qquad \square$

Proof of Proposition 4.13. Throughout the proof, the identifications

$$\operatorname{Co}_g(L_w^1) = \{f \in \mathcal{H}_\pi : V_g f \in W^L(L_w^1)\} \quad \text{and} \quad \operatorname{Co}_g(W^R(L_w^1)) = \{f \in \mathcal{H}_\pi : V_g f \in W(L_w^1)\}$$

provided by Lemma 4.12 will be used.

First, suppose that G is an IN group. Since the spaces $W^L(L_w^1)$ and $W(L_w^1)$ are independent of the choice of the neighborhood Q (as L_w^1 is left- and right invariant), and since G is an IN group, it may be assumed that $x Q x^{-1} = Q$ for all $x \in G$, that is, $x Q = Q x$. This easily implies $M^L F = M^R F$ for any measurable $F : G \to \mathbb{C}$ and hence $W^L(Y) = W^R(Y)$ for any solid quasi-Banach function space Y on G. Therefore,

$$W(L_w^1) = W^R(W^L(L_w^1)) = W^L(W^L(L_w^1)) = W^L(L_w^1),$$

which easily implies $\operatorname{Co}_g(L_w^1) = \operatorname{Co}_g(W^R(L_w^1))$.

Second, suppose $\operatorname{Co}_g(W^R(L_w^1)) = \operatorname{Co}_g(L_w^1)$. The inclusion $\iota : \operatorname{Co}_g(W^R(L_w^1)) \hookrightarrow \operatorname{Co}_g(L_w^1)$ is clearly bounded and linear. Since $\operatorname{Co}_g(L_w^1) = \operatorname{Co}_g(W^R(L_w^1))$, it follows that ι is bijective. By the bounded inverse theorem, this implies that ι is boundedly invertible, i.e., $\|\cdot\|_{\operatorname{Co}_g(W^R(L_w^1))} \asymp \|\cdot\|_{\operatorname{Co}_g(L_w^1)}$. In particular, this implies existence of a constant $C_1 > 0$

satisfying

$$\|\pi(x)g\|_{\mathrm{Co}_g(W^R(L_w^1))} \le C_1 \cdot \|\pi(x)g\|_{\mathrm{Co}_g(L_w^1)} = C_1 \cdot \|V_g[\pi(x)g]\|_{W^L(L_w^1)} = C_1 \cdot \|L_x^\sigma[V_g g]\|_{W^L(L_w^1)}$$

$$= C_1 \cdot \|L_x[V_g g]\|_{W^L(L_w^1)} = C_1 \cdot \|M^L[L_x(V_g g)]\|_{L_w^1} = C_1 \cdot \|L_x[M^L(V_g g)]\|_{L_w^1}$$

$$\le C_1 \cdot w(x) \cdot \|M^L[V_g g]\|_{L_w^1} =: C_2 \cdot w(x) \tag{4.20}$$

for all $x \in G$. Since $|V_g g|$ is continuous with $|V_g g(e_G)| = \|g\|_{\mathcal{H}_\pi}^2 > 0$, there exist $\delta > 0$ and an open, symmetric, relatively compact unit neighborhood $U \subset Q$ satisfying $|V_g g(x)| \ge \delta$ for all $x \in U$. In particular, this implies that

$$M(V_g[\pi(x)g])(y) = \operatorname*{ess\,sup}_{q_1,q_2 \in Q} |V_g[\pi(x)g](q_1 y q_2)| = \operatorname*{ess\,sup}_{q_1,q_2 \in Q} |L_x^\sigma[V_g g](q_1 y q_2)|$$

$$= \sup_{q_1,q_2 \in Q} |L_x[V_g g](q_1 y q_2)| \ge \delta \cdot \sup_{q_1,q_2 \in Q} \mathbb{1}_U(x^{-1} q_1 y q_2)$$

$$= \delta \cdot \mathbb{1}_{QxUQ}(y) \ge \delta \cdot \mathbb{1}_{QxQ}(y)$$

for all $x, y \in G$. Since w is locally bounded (cf. Remark 3.2), there exists a constant $C_3 = C_3(w, Q) > 0$ satisfying $w(q) \le C_3$ for all $q \in Q = Q^{-1}$. Hence, if $\mathbb{1}_{QxQ}(y) \ne 0$, then $x = q_1^{-1} y q_2^{-1}$ for certain $q_1, q_2 \in Q$ and hence $w(x) \le w(q_1^{-1}) w(y) w(q_2^{-1}) \le C_3^2 w(y)$. Combining these observations with (4.20) gives

$$C_2 \cdot w(x) \ge \|\pi(x)g\|_{\mathrm{Co}_g(W^R(L_w^1))} = \|V_g[\pi(x)g]\|_{W(L_w^1)} = \|M(V_g[\pi(x)g])\|_{L_w^1} \ge \delta \cdot \|\mathbb{1}_{QxQ}\|_{L_w^1}$$

$$\ge C_3^{-2}\delta \cdot w(x) \cdot \|\mathbb{1}_{QxQ}\|_{L^1} = C_3^{-2}\delta \cdot w(x) \cdot \mu_G(QxQ).$$

It follows therefore that $\mu_G(QxQ) \le C_2 C_3^2 / \delta =: C$ for all $x \in G$. An application of Proposition 4.14 shows that G is an IN group. $\qquad\square$

5. Convolution-dominated operators and local spectral invariance

This section considers classes of convolution-dominated operators. The first subsection is devoted to *integral operators*, whereas the second concerns convolution-dominated *matrices*.

Throughout the section, the weight $w : G \to [1, \infty)$ will always be assumed to be a p-weight for some $p \in (0, 1]$.

5.1. **Integral operators.** Throughout, let $g \in \mathcal{B}_w^p$ be a fixed admissible vector (see Section 4.3). Then the image space

$$\mathcal{K}_g := V_g(\mathcal{H}_\pi) \le L^2(G)$$

is a closed subspace forming a reproducing kernel Hilbert space, with reproducing kernel

$$K : \quad G \times G \to \mathbb{C}, \quad (x, y) \mapsto V_g[\pi(y)g](x) = L_y^\sigma[V_g g](x),$$

see Section 4.1. Since $g \in \mathcal{B}_w^p$ by assumption, it follows that $|V_g g| \in W_C(L_w^p)$, and

$$|K(x, y)| = |K(y, x)| \le |V_g g|(y^{-1} x), \quad x, y \in G.$$

The following general class of localized kernels will be the central object of study.

Definition 5.1. Let $p \in (0, 1]$, let $w : G \to [1, \infty)$ be a p-weight, and let $g \in \mathcal{B}_w^p$ be admissible. A measurable function $H : G \times G \to \mathbb{C}$ is called L_w^p-*localized in* $\mathcal{K}_g$ if it satisfies the following properties:

(i1) $H(\cdot, y) \in \mathcal{K}_g$ for all $y \in G$,
(i2) $\overline{H(x, \cdot)} \in \mathcal{K}_g$ for all $x \in G$,
(i3) There exists a non-negative *envelope* $\Phi \in W_C(L_w^p)$ satisfying

$$\max\{|H(x, y)|, |H(y, x)|\} \le \Phi(y^{-1} x), \qquad x, y \in G. \tag{5.1}$$

For a measurable function H satisfying condition (i3), the notation $H \prec \Phi$ will be used.

Remark 5.2. Note that if Equation (5.1) holds, then it also holds for $\Phi_0 = \min\{\Phi, \Phi^\vee\}$ instead of Φ. Hence, one can always assume Φ to be symmetric.

The following lemma summarizes the basic elementary properties of L_w^p-localized kernels and of the associated integral operators.

Lemma 5.3. *If H is L_w^p-localized in $\mathcal{K}_g$, then the associated integral operator*

$$T_H: \quad L^r(G) \to L^r(G), \quad T_H F(x) = \int_G H(x,y) F(y) \, d\mu_G(y)$$

is well-defined and bounded for arbitrary $r \in [1, \infty]$, with absolute convergence of the defining integral for all $x \in G$. Moreover, the following properties hold:

(i) *The map $T_H : L^2(G) \to \mathcal{K}_g$ is well-defined.*
(ii) *For all $x, y \in G$,*

$$H(x,y) = \langle T_H[V_g(\pi(y)g)], V_g(\pi(x)g) \rangle_{L^2} \tag{5.2}$$

and $|H(x,y)| \leq \|T_H\|_{\mathcal{K}_g \to L^2} \cdot \|g\|_{\mathcal{H}_\pi}^2$.
(iii) *The* adjoint kernel *$\widetilde{H} : G \times G \to \mathbb{C}$ defined by $\widetilde{H}(x,y) = \overline{H(y,x)}$ is also L_w^p-localized in $\mathcal{K}_g$. In fact, if $H \prec \Phi$, then $\widetilde{H} \prec \Phi$.*
(iv) *If $L : G \times G \to \mathbb{C}$ is also L_w^p-localized in $\mathcal{K}_g$, then so is the product $H \odot L$ defined by*

$$H \odot L(x,y) := \int_G H(x,z) \cdot L(z,y) d\mu_G(z) = T_H[L(\cdot, y)](x).$$

In addition, if $H \prec \Phi$ and $L \prec \Theta$ with Φ, Θ symmetric, then $H \odot L \prec \max\{\Phi * \Theta, \Theta * \Phi\}$. Moreover, the identity $(T_H \circ T_L)F = T_{H \odot L}F$ holds for all $F \in L^2(G)$.

Proof. Throughout the proof, let $\Phi \in W_C(L_w^p)$ be symmetric with $H \prec \Phi$, so that

$$|H(x,y)| \leq \Phi(y^{-1}x) = \Phi(x^{-1}y) \tag{5.3}$$

for all $x, y \in G$. Before proving the individual statements of the lemma, we collect a few auxiliary observations and prove the boundedness of $T_H : L^r \to L^r$.

By Lemma 3.3 and because of $w \geq 1$, it follows that $W(L_w^p) \hookrightarrow W^L(L_w^p) \hookrightarrow L^s$ for all $s \in [1, \infty]$. The estimate $|H(x,y)| \leq \Phi(x^{-1}y)$ shows that $H(x, \cdot) \in L^s$ for all $x \in G$ and $s \in [1, \infty]$, which implies that the integral defining $T_H F(x)$ exists for any $F \in L^r$ and $x \in G$. Moreover, the estimate (5.3) yields

$$\int_G |H(x,y)| \, d\mu_G(y) \leq \|\Phi\|_{L^1} \qquad \text{and} \qquad \int_G |H(x,y)| \, d\mu_G(x) \leq \|\Phi\|_{L^1}$$

for all $x, y \in G$. An application of Schur's test (see, e.g., [23, Theorem 6.18]) therefore yields that $T_H : L^r(G) \to L^r(G)$ is bounded for arbitrary $r \in [1, \infty]$.

Let $F_1, F_2 \in L^2(G)$. Then, using the Cauchy-Schwarz inequality, the pointwise estimate $|H(x,y)| \leq \sqrt{\Phi(x^{-1}y)\Phi(y^{-1}x)}$ and Tonelli's theorem, it follows that

$$\int_G \int_G |F_2(x)| \cdot |H(x,y)| \cdot |F_1(y)| \, d\mu_G(y) d\mu_G(x)$$

$$\leq \int_G |F_2(x)| \left(\int_G \Phi(x^{-1}y) \, d\mu_G(y) \right)^{1/2} \left(\int_G \Phi(y^{-1}x)|F_1(y)|^2 \, d\mu_G(y) \right)^{1/2} d\mu_G(x)$$

$$\leq \|\Phi\|_{L^1}^{1/2} \left(\int_G |F_2(x)|^2 \, d\mu_G(x) \right)^{1/2} \left(\int_G \int_G \Phi(y^{-1}x)|F_1(y)|^2 \, d\mu_G(y) d\mu_G(x) \right)^{1/2}$$

$$\leq \|\Phi\|_{L^1}^{1/2} \cdot \|F_2\|_{L^2} \cdot \|\Phi\|_{L^1}^{1/2} \cdot \|F_1\|_{L^2}.$$

Therefore, Fubini's theorem is applicable and justifies the calculation

$$
\begin{aligned}
\langle T_H F_1, F_2 \rangle &= \int_G \int_G \overline{F_2(x)} \cdot H(x,y) \cdot F_1(y) \, d\mu_G(y) d\mu_G(x) \\
&= \int_G F_1(y) \int_G H(x,y) \cdot \overline{F_2(x)} \, d\mu_G(x) d\mu_G(y) \\
&= \int_G F_1(y) \cdot \langle H(\cdot,y), F_2 \rangle \, d\mu_G(y)
\end{aligned}
\tag{5.4}
$$

for arbitrary $F_1, F_2 \in L^2(G)$.

(i) Let $F_1 \in L^2(G)$. Since $\mathcal{K}_g \leq L^2(G)$ is closed, to show $T_H F_1 \in \mathcal{K}_g$, it suffices to show $\langle T_H F_1, F_2 \rangle = 0$ for all $F_2 \in \mathcal{K}_g^\perp$. For this, simply note that $H(\cdot,y) \in \mathcal{K}_g$ for all $y \in G$, so that Equation (5.4) shows $\langle T_H F_1, F_2 \rangle = \int_G F_1(y) \cdot \langle H(\cdot,y), F_2 \rangle \, d\mu_G(y) = 0$.

(ii) Note that the reproducing formula (4.5) gives $F(x) = \langle F, V_g(\pi(x)g) \rangle_{L^2}$ for all $F \in \mathcal{K}_g$ and $x \in G$. This, combined with the identity (5.4), and $H(\cdot,z) \in \mathcal{K}_g$ and $\overline{H(x,\cdot)} \in \mathcal{K}_g$, gives

$$
\begin{aligned}
\langle T_H[V_g(\pi(y)g)], V_g(\pi(x)g) \rangle_{L^2} &= \int_G V_g(\pi(y)g)(z) \cdot \langle H(\cdot,z), V_g(\pi(x)g) \rangle \, d\mu_G(z) \\
&= \int_G V_g(\pi(y)g)(z) \cdot H(x,z) \, d\mu_G(z) \\
&= \overline{\langle \overline{H(x,\cdot)}, V_g(\pi(y)g) \rangle} = H(x,y),
\end{aligned}
$$

which shows (5.2). In particular, using the isometry of V_g and because of $\|\pi(x)g\|_{\mathcal{H}_\pi} = \|g\|_{\mathcal{H}_\pi}$ and $V_g(\pi(y)g) \in \mathcal{K}_g$, this implies that

$$
|H(x,y)| \leq \|T_H\|_{\mathcal{K}_g \to L^2} \|V_g(\pi(y)g)\|_{L^2} \|V_g(\pi(x)g)\|_{L^2} = \|T_H\|_{\mathcal{K}_g \to L^2} \|g\|_{\mathcal{H}_\pi}^2,
$$

which proves part (ii).

(iii) Simply note that $\widetilde{H}(\cdot,y) = \overline{H(y,\cdot)} \in \mathcal{K}_g$ and $\overline{\widetilde{H}(x,\cdot)} = H(\cdot,x) \in \mathcal{K}_g$ for all $x,y \in G$ and that $\max\{|\widetilde{H}(x,y)|, |\widetilde{H}(y,x)|\} = \max\{|H(x,y)|, |H(y,x)|\} \leq \Phi(y^{-1}x)$, even without assuming that Φ is symmetric.

(iv) By part (i) and because of $L(\cdot,y) \in \mathcal{K}_g$, it follows that $(H \odot L)(\cdot,y) = T_H[L(\cdot,y)] \in \mathcal{K}_g$. Another direct calculation shows that $\overline{(H \odot L)(x,\cdot)} = T_{\widetilde{L}}[\widetilde{H}(\cdot,x)] \in \mathcal{K}_g$, by Part *(i)* applied to $\widetilde{L}$ (which is L_w^p-localized in $\mathcal{K}_g$ by Part (iii) and since $\widetilde{H}(\cdot,x) \in \mathcal{K}_g$.

If $H \prec \Phi$ and $L \prec \Theta$ with symmetric $\Phi, \Theta \in W_C(L_w^p)$, then

$$
\begin{aligned}
|H \odot L(x,y)| &\leq \int_G |H(x,z)| \cdot |L(z,y)| \, d\mu_G(z) \leq \int_G \Phi(x^{-1}z)\Theta(z^{-1}y) \, d\mu_G(z) \\
&= \int_G \Phi(w)\Theta(w^{-1}x^{-1}y) \, d\mu_G(w) = (\Phi * \Theta)(x^{-1}y) = (\Theta * \Phi)(y^{-1}x),
\end{aligned}
$$

where the last step used the elementary identity $(F * H)^\vee = H^\vee * F^\vee$. The calculation from above also shows $|H \odot L(y,x)| \leq (\Phi * \Theta)(y^{-1}x)$. Hence, $H \odot L \prec \max\{\Phi * \Theta, \Theta * \Phi\}$. Since $\Phi * \Theta, \Theta * \Phi \in W_C(L_w^p)$ by Corollary 3.9, it follows that $H \odot L$ is L_w^p-localized in $\mathcal{K}_g$.

Lastly, note that Lemma 3.3 shows $\Phi * \Theta \in W(L_w^p) \hookrightarrow W^L(L_w^p) \hookrightarrow L^2$. Hence, it follows for arbitrary $F \in L^2(G)$ that

$$
\begin{aligned}
\int_G \int_G |H(x,y) \cdot L(y,z)||F(z)| \, d\mu_G(y) d\mu_G(z) &\leq \int_G \int_G \Phi(x^{-1}y)\Theta(y^{-1}z)|F(z)| \, d\mu_G(y) d\mu_G(z) \\
&= \int_G |F(z)| \cdot (L_x(\Phi * \Theta))(z) \, d\mu_G(z) \\
&\leq \|F\|_{L^2} \|L_x(\Phi * \Theta)\|_{L^2} < \infty
\end{aligned}
$$

for any $x \in G$. Therefore, Fubini's theorem is applicable in the calculation

$$[T_H(T_L F)](x) = \int_G H(x,y) \cdot T_L F(y) \, d\mu_G(y) = \int_G \int_G H(x,y) \, L(y,z) \, F(z) \, d\mu_G(z) d\mu_G(y)$$

$$= \int_G (H \odot L)(x,z) \cdot F(z) \, d\mu_G(z) = (T_{H \odot L} F)(x),$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The next result establishes a form of *local spectral invariance* of L_w^p-localized integral operators. Concerning the *holomorphic spectral calculus* appearing in the statement of the theorem, see, e.g., [51, Sections 10.21–10.29].

Theorem 5.4. *Let $p \in (0,1]$ and let $w : G \to [1, \infty)$ be a p-weight. Let $\Phi := |V_g g| \in W_C(L_w^p)$ for an admissible $g \in \mathcal{B}_w^p$ and let $\Theta \in W_C(L_w^p)$ be non-negative.*

For arbitrary $\delta > 0$, there exists $\varepsilon = \varepsilon(\Theta, g, Q, w, p, \delta) \in (0, \delta)$ with the following property: If $\phi : B_\delta(1) \subset \mathbb{C} \to \mathbb{C}$ is holomorphic and if $H : G \times G \to \mathbb{C}$ is L_w^p-localized in $\mathcal{K}_g$ satisfying

(1) $H \prec \Theta$,
(2) $\|T_H - \mathrm{id}_{\mathcal{K}_g}\|_{\mathcal{K}_g \to L^2} \leq \varepsilon$,

then there exists an L_w^p-localized $H_\phi : G \times G \to \mathbb{C}$ such that the operator $\phi(T_H) : \mathcal{K}_g \to \mathcal{K}_g$ defined through the holomorphic functional calculus satisfies $\phi(T_H) = T_{H_\phi}|_{\mathcal{K}_g}$.

Proof. The proof is divided into several steps and closely follows the proof of [50, Theorem 4.3].

Step 1. *(Choice of ε).* Let $\delta > 0$ be given and set $\beta := \|g\|_{\mathcal{H}_\pi}^2$ and $\Theta_0 := \min\{\Theta, \Theta^\vee\}$, so that $H \prec \Theta_0$. Using Corollary 3.9, choose a constant $C_1 = C_1(w, p, Q) \geq 1$ satisfying $\|F_1 * F_2\|_{W(L_w^p)} \leq \||F_1| * |F_2|\|_{W(L_w^p)} \leq C_1 \cdot \|F_1\|_{W(L_w^p)} \cdot \|F_2\|_{W(L_w^p)}$ for all $F_1, F_2 \in W(L_w^p)$.

For $\varepsilon > 0$, set $\Psi_\varepsilon := \min\{\varepsilon\beta, \Phi + \Theta_0\}$. Since $M\Psi_\varepsilon \leq \min\{\varepsilon\beta, M\Phi + M\Theta\} \in L_w^p$, it follows from the dominated convergence theorem (applied along an arbitrary null-sequence $\varepsilon_n \to 0$) that $\|\Psi_\varepsilon\|_{W(L_w^p)} \to 0$ as $\varepsilon \downarrow 0$. Hence, there exists $\varepsilon = \varepsilon(\Theta, g, \delta, w, p, Q) \in (0, \frac{\delta}{2})$ such that $\|\Psi_\varepsilon\|_{W(L_w^p)} \leq \frac{\delta}{4C_1}$.

Step 2. *(Series representation of $\phi(T_H)$).* Let $\phi : B_\delta(1) \to \mathbb{C}$ be holomorphic. By assumption, $\|\mathrm{id}_{\mathcal{K}_g} - T_H\|_{\mathcal{K}_g \to \mathcal{K}_g} \leq \varepsilon < \frac{\delta}{2}$, and hence $\sigma(T_H) \subset B_{\delta/2}(1)$. This implies that $\phi(T_H) : \mathcal{K}_g \to \mathcal{K}_g$ is a well-defined bounded linear operator. By expanding ϕ into a power series, we can write $\phi(z) = \sum_{n=0}^\infty a_n (z-1)^n$ for all $z \in B_\delta(1)$, for a suitable sequence $(a_n)_{n \in \mathbb{N}_0} \subset \mathbb{C}$. The series representing ϕ convergences locally uniformly on $B_\delta(1)$. Therefore, elementary properties of the holomorphic functional calculus (see, e.g., [51, Theorem 10.27]) show that

$$\phi(T_H) = \sum_{n=0}^\infty a_n (T_H - \mathrm{id}_{\mathcal{K}_g})^n, \qquad\qquad (5.5)$$

with convergence in the operator norm. An application of the Cauchy-Hadamard formula gives $\delta \leq \left[\limsup_{n \to \infty} |a_n|^{1/n} \right]^{-1}$. Thus, there exists some $N = N(\phi, \delta) \in \mathbb{N}$ such that $|a_n|^{1/n} \leq \frac{2}{\delta}$ for all $n \geq N$. Consequently, there exists $C_\phi = C_\phi(\delta) > 0$ such that

$$|a_n| \leq C_\phi \cdot (2/\delta)^n \qquad\qquad (5.6)$$

for all $n \in \mathbb{N}_0$.

Step 3. *(Integral representation of $T_H - \mathrm{id}_{\mathcal{K}_g}$).* For $F \in \mathcal{K}_g$ and $x \in G$, the reproducing formula (4.5) yields that $F(x) = T_K F(x)$, where K is the reproducing kernel given by (4.6). In other words,

$$T_K|_{\mathcal{K}_g} = \mathrm{id}_{\mathcal{K}_g} \qquad \text{and} \qquad T_H - \mathrm{id}_{\mathcal{K}_g} = T_L|_{\mathcal{K}_g} \quad \text{for} \quad L := H - K. \qquad (5.7)$$

Since $\overline{K(x,\cdot)} = V_g[\pi(x)g] \in \mathcal{K}_g$ and $K(\cdot,y) = V_g[\pi(y)g] \in \mathcal{K}_g$ for arbitrary $x, y \in G$, it follows that $L(\cdot,y) \in \mathcal{K}_g$ and $\overline{L(x,\cdot)} \in \mathcal{K}_g$ for all $x, y \in G$. Since also

$$|L(y,x)| \leq \Phi(x^{-1}y) + \Theta_0(x^{-1}y) = \Phi(y^{-1}x) + \Theta_0(y^{-1}x)$$

and $|L(x,y)| \leq \Phi(y^{-1}x) + \Theta_0(y^{-1}x)$ and hence $L \prec \Phi + \Theta_0$ with $\Phi + \Theta_0 \in W_C(L_w^p)$, it follows that L is L_w^p-localized in $\mathcal{K}_g$.

Step 4. *(Refined L_w^p-localization of L).* By Step 3 and the assumptions of the theorem, it holds that

$$\|T_L\|_{\mathcal{K}_g \to L^2} = \|T_H - \mathrm{id}_{\mathcal{K}_g}\|_{\mathcal{K}_g \to L^2} \leq \varepsilon.$$

Therefore, the pointwise estimate following from the identity (5.2) shows that

$$|L(x,y)| \leq \|T_L\|_{\mathcal{K}_g \to L^2} \cdot \|g\|_{\mathcal{H}_\pi}^2 \leq \varepsilon\beta, \qquad x, y \in G.$$

Combined with the estimate from the end of Step 3, this shows $L \prec \Psi_\varepsilon$.

Step 5. *(Powers of Ψ_ε and L):* For $n \in \mathbb{N}$, define $\Psi_\varepsilon^{*(1)} := \Psi_\varepsilon$ and $\Psi_\varepsilon^{*(n+1)} := \Psi_\varepsilon * \Psi_\varepsilon^{*(n)}$ inductively. Similarly, let $L^{\circ(1)} := L$ and $L^{\circ(n+1)} := L^{\circ(n)} \odot L$ for $n \in \mathbb{N}$, where $\odot$ is the product defined in Lemma 5.3. By a straightforward induction, that lemma shows that $(T_H - \mathrm{id}_{\mathcal{K}_g})^n = T_L^n|_{\mathcal{K}_g} = T_{L^{\circ(n)}}|_{\mathcal{K}_g}$ and that $L^{\circ(n)}$ is L_w^p-localized in $\mathcal{K}_g$ for each $n \in \mathbb{N}$. In particular, this implies

$$L^{\circ(n)}(\cdot,y) \in \mathcal{K}_g \quad \text{and} \quad \overline{L^{\circ(n)}(x,\cdot)} \in \mathcal{K}_g, \qquad x, y \in G. \tag{5.8}$$

Furthermore, a straightforward induction shows $\Psi_\varepsilon^{*(n+1)} = \Psi_\varepsilon^{*(n)} * \Psi_\varepsilon$ and $(\Psi_\varepsilon^{*(n)})^\vee = \Psi_\varepsilon^{*(n)}$ for all $n \in \mathbb{N}$.

Another induction argument shows that $L^{\circ(n)} \prec \Psi_\varepsilon^{*(n)}$: For $n = 1$, this was shown in Step 4. For the induction step, Lemma 5.3 shows by symmetry of Ψ_ε and $\Psi_\varepsilon^{*(n)}$ and because of $L^{\circ(n)} \prec \Psi_\varepsilon^{*(n)}$ and $L \prec \Psi_\varepsilon$ that

$$L^{\circ(n+1)} = L^{\circ(n)} \odot L \prec \max\left\{\Psi_\varepsilon * \Psi_\varepsilon^{*(n)}, \Psi_\varepsilon^{*(n)} * \Psi_\varepsilon\right\} = \Psi_\varepsilon^{*(n+1)},$$

as required. Lastly, it holds that

$$\|\Psi_\varepsilon^{*(n)}\|_{W(L_w^p)} \leq (\delta/4)^n, \qquad n \in \mathbb{N}. \tag{5.9}$$

Indeed, for $n = 1$ this follows since $C_1 \geq 1$ and hence $\|\Psi_\varepsilon\|_{W(L_w^p)} \leq \frac{\delta}{4C_1} \leq \frac{\delta}{4}$ by the choice of ε in Step 1. Next, for the induction step note by choice of C_1 that

$$\|\Psi_\varepsilon^{*(n+1)}\|_{W(L_w^p)} = \|\Psi_\varepsilon * \Psi_\varepsilon^{*(n)}\|_{W(L_w^p)} \leq C_1 \cdot \|\Psi_\varepsilon\|_{W(L_w^p)} \cdot \|\Psi_\varepsilon^{*(n)}\|_{W(L_w^p)} \leq \delta/4 \cdot (\delta/4)^n = (\delta/4)^{n+1},$$

which establishes the claim (5.9).

Step 6. *(Construction of H_ϕ).* Combining Equations (5.6) and (5.9) gives

$$\sum_{n=1}^\infty \left(|a_n| \cdot \|\Psi_\varepsilon^{*(n)}\|_{W(L_w^p)}\right)^p \leq \sum_{n=1}^\infty \left[C_\phi \cdot (2/\delta)^n \cdot (\delta/4)^n\right]^p = C_\phi^p \cdot \sum_{n=1}^\infty (1/2^p)^n < \infty.$$

Since the norm on $W_C(L_w^p)$ is a p-norm, the preceding estimate implies by Lemma A.1 that the series $\sum_{n=1}^\infty |a_n| \Psi_\varepsilon^{*(n)}$ is unconditionally convergent in $W_C(L_w^p)$. Since $W_C(L_w^p) \hookrightarrow C_b(G)$ as a consequence of Lemma 3.3, the series in particular converges uniformly.

Define $\Psi := |a_0| \Phi + \sum_{n=1}^\infty |a_n| \Psi_\varepsilon^{*(n)} \in W_C(L_w^p)$. Then the kernel $H_\phi : G \times G \to \mathbb{C}$ defined by

$$H_\phi(x,y) = a_0 \cdot K(x,y) + \sum_{n=1}^\infty a_n L^{\circ(n)}(x,y)$$

is well-defined with the series converging absolutely, and

$$|H_\phi(x,y)| \leq |a_0| \cdot |K(x,y)| + \sum_{n=1}^{\infty} |a_n| \cdot |L^{\circ(n)}(x,y)|$$

$$\leq |a_0| \cdot \Phi(y^{-1}x) + \sum_{n=1}^{\infty} |a_n| \cdot \Psi_\varepsilon^{*(n)}(y^{-1}x) = \Psi(y^{-1}x) < \infty, \tag{5.10}$$

since $L^{\circ(n)} \prec \Psi_\varepsilon^{*(n)}$ (cf. Step 5.) Similar arguments show that also $|H_\phi(y,x)| \leq \Psi(y^{-1}x)$ and thus $H_\phi \prec \Psi$.

To prove that H_ϕ is L_w^p-localized in $\mathcal{K}_g$, it remains to show $H_\phi(\cdot,y) \in \mathcal{K}_g$ and $\overline{H_\phi(x,\cdot)} \in \mathcal{K}_g$ for all $x,y \in G$. To see this, note that Lemma 3.3 shows $\Psi \in W(L_w^p) \hookrightarrow W^L(L_w^p) \hookrightarrow L^2$. In combination with Equation (5.10) and the dominated convergence theorem, this implies that the series defining $H_\phi(\cdot,y)$ converges in $L^2(G)$. Since $\mathcal{K}_g \subset L^2(G)$ is closed, since $L^{\circ(n)}(\cdot,y) \in \mathcal{K}_g$ by Equation (5.8), and since $K(\cdot,y) \in \mathcal{K}_g$, this implies $H_\phi(\cdot,y) \in \mathcal{K}_g$, as required. The proof of $\overline{H_\phi(x,\cdot)} \in \mathcal{K}_g$ is similar, using that $\big(y \mapsto \Psi(y^{-1}x) = \Psi(x^{-1}y)\big) \in L^2(G)$.

Step 7. *($\phi(T_H) = T_{H_\phi}|_{\mathcal{K}_g}$).* Let $F \in \mathcal{K}_g$ and $x \in G$. Note that Step 6 shows

$$\sum_{n=1}^{\infty} |a_n| |L^{\circ(n)}(x,\cdot)| \leq \Psi(x^{-1}\cdot),$$

where $\Psi \in L^2(G)$. Hence, the dominated convergence theorem justifies the following calculation:

$$T_{H_\phi} F(x) = a_0 \cdot T_K F(x) + \sum_{n=1}^{\infty} [a_n \cdot T_{L^{\circ(n)}} F(x)] = a_0 \cdot F(x) + \sum_{n=1}^{\infty} \big(a_n \cdot [(T_H - \mathrm{id}_{\mathcal{K}_g})^n F](x)\big)$$

$$= \sum_{n=0}^{\infty} \big(a_n \cdot [(T_H - \mathrm{id}_{\mathcal{K}_g})^n F](x)\big) = [\phi(T_H)F](x),$$

where the second (resp. fourth) equality used (5.7) (resp. (5.5)). Thus, $\phi(T_H) = T_{H_\phi}|_{\mathcal{K}_g}$. $\quad\square$

Theorem 5.4 provides an extension of [50, Theorem 4.3] from L_w^1-localized kernels to general L_w^p-localized kernels for $p \in (0,1]$. This extension will be crucial for developing the theory of molecules for quasi-Banach coorbit spaces in Section 6.

5.2. **Matrices.** This section concerns matrices that are indexed by discrete subsets of G (that do not need to be subgroups) and that possess a certain off-diagonal decay. The precise notion is as follows.

Definition 5.5. Let $p \in (0,1]$ and let $w : G \to [1,\infty)$ be a p-weight. Let $\Lambda = (\lambda_i)_{i \in I}$ and $\Gamma = (\gamma_j)_{j \in J}$ be relatively separated families in G.

A matrix $A = (A_{i,j})_{(i,j) \in I \times J} \in \mathbb{C}^{I \times J}$ is called L_w^p-*localized* if there exists an *envelope* $\Phi \in W_C(L_w^p)$ such that

$$|A_{i,j}| \leq \min\{\Phi(\gamma_j^{-1}\lambda_i),\ \Phi(\lambda_i^{-1}\gamma_j)\}, \quad i \in I,\ j \in J. \tag{5.11}$$

If condition (5.11) holds, the notation $A \prec \Phi$ will be used. The space of all L_w^p-localized matrices in $\mathbb{C}^{\Lambda \times \Gamma}$ is the collection

$$\mathcal{C}_w^p(\Lambda,\Gamma) := \Big\{ A \in \mathbb{C}^{I \times J} \ :\ A \prec \Phi \text{ for some } \Phi \in W_C(L_w^p) \Big\},$$

and is equipped with the mapping $\|\cdot\|_{\mathcal{C}_w^p} : \mathcal{C}_w^p \to [0,\infty)$ defined by

$$\|A\|_{\mathcal{C}_w^p} := \inf \{\|\Phi\|_{W(L_w^p)} \ :\ A \prec \Phi \in W_C(L_w^p)\}.$$

In case of $\Lambda = \Gamma$, the notation $\mathcal{C}_w^p(\Lambda) := \mathcal{C}_w^p(\Lambda,\Lambda)$ will be used.

The basic properties of L_w^p-localized matrices are collected in the following lemma.

Lemma 5.6. *Let* $\Lambda = (\lambda_i)_{i \in I}$, $\Gamma = (\gamma_j)_{j \in J}$, *and* $\Upsilon = (v_k)_{k \in K}$ *be relatively separated families in* G. *Then the following hold:*

(i) The space $\mathcal{C}_w^p(\Lambda, \Gamma)$ *forms a quasi-Banach space with p-norm* $\| \cdot \|_{\mathcal{C}_w^p}$.

(ii) Any $A = (A_{i,j})_{(i,j) \in I \times J} \in \mathcal{C}_w^p(\Lambda, \Gamma)$ *satisfies*

$$\sum_{i \in I} |A_{i,j}| \lesssim \mathrm{Rel}(\Lambda) \cdot \|A\|_{\mathcal{C}_w^p} \quad and \quad \sum_{j \in J} |A_{i,j}| \lesssim \mathrm{Rel}(\Gamma) \cdot \|A\|_{\mathcal{C}_w^p}, \tag{5.12}$$

with an implicit constant depending only on p, w *and* Q. *In particular, the embedding* $\mathcal{C}_w^p(\Lambda, \Gamma) \hookrightarrow \mathcal{B}(\ell^r(J), \ell^r(I))$ *holds for all* $r \in [1, \infty]$, *with*

$$\|A\|_{\ell^r(J) \to \ell^r(I)} \lesssim \max\{\mathrm{Rel}(\Lambda), \mathrm{Rel}(\Gamma)\} \cdot \|A\|_{\mathcal{C}_w^p}. \tag{5.13}$$

(iii) If $A \in \mathcal{C}_w^p(\Upsilon, \Lambda)$ *and* $B \in \mathcal{C}_w^p(\Lambda, \Gamma)$, *then the product* AB *(which is defined as usual by* $(AB)_{k,j} = \sum_{i \in I} A_{k,i} B_{i,j}$) *satisfies* $AB \in \mathcal{C}_w^p(\Upsilon, \Gamma)$, *with*

$$\|AB\|_{\mathcal{C}_w^p} \lesssim \mathrm{Rel}(\Lambda) \cdot \|A\|_{\mathcal{C}_w^p} \|B\|_{\mathcal{C}_w^p}. \tag{5.14}$$

for an implicit constant depending only on p, w *and* Q.

Proof. All implied constants in this proof only depend on p, w, Q.

(i) The absolute homogeneity and the p-norm property $\|A + B\|_{\mathcal{C}_w^p}^p \leq \|A\|_{\mathcal{C}_w^p}^p + \|B\|_{\mathcal{C}_w^p}^p$ of $\| \cdot \|_{\mathcal{C}_w^p}$ follow directly from the corresponding properties of $\| \cdot \|_{W(L_w^p)}$. For $A \in \mathcal{C}_w^p(\Lambda, \Gamma)$ satisfying $A \prec \Phi$ with $\Phi \in W_C(L_w^p)$, the inequality (cf. Lemma 3.3)

$$|A_{i,j}| \leq \Phi(\gamma_j^{-1} \lambda_i) \leq \|\Phi\|_{C_b} = \|\Phi\|_{L^\infty} \lesssim \|\Phi\|_{W(L_w^p)}, \tag{5.15}$$

directly shows that $|A_{i,j}| \lesssim \|A\|_{\mathcal{C}_w^p}$, and hence that $\| \cdot \|_{\mathcal{C}_w^p}$ is positive definite.

It remains to show that $\mathcal{C}_w^p(\Lambda, \Gamma)$ is complete. For this, by Lemma A.1, it suffices to show that if $(A^{(n)})_{n \in \mathbb{N}}$ is a sequence in $\mathcal{C}_w^p(\Lambda, \Gamma)$ satisfying $\sum_{n \in \mathbb{N}} \|A^{(n)}\|_{\mathcal{C}_w^p}^p < \infty$, then $\sum_{n \in \mathbb{N}} A^{(n)}$ converges in $\mathcal{C}_w^p(\Lambda, \Gamma)$. Given such a sequence $(A^{(n)})_{n \in \mathbb{N}}$, define $A \in \mathbb{C}^{I \times J}$ by $A_{i,j} := \sum_{n \in \mathbb{N}} A_{i,j}^{(n)}$. That A is well-defined follows from the observation after Equation (5.15) by noting

$$\sum_{n \in \mathbb{N}} |A_{i,j}^{(n)}| \leq \sum_{n \in \mathbb{N}} |A_{i,j}^{(n)}|^p \lesssim \sum_{n \in \mathbb{N}} \|A^{(n)}\|_{\mathcal{C}_w^p}^p < \infty.$$

For $n \in \mathbb{N}$, let $\Phi_n \in W_C(L_w^p)$ be such that $A^{(n)} \prec \Phi_n$ and $\|\Phi_n\|_{W(L_w^p)} \leq 2\|A^{(n)}\|_{\mathcal{C}_w^p}$, and define $\Theta_N := \sum_{n=N+1}^{\infty} \Phi_n$ for $N \in \mathbb{N}_0$. The choice of Φ_n implies that

$$\sum_{n \in \mathbb{N}} \|\Phi_n\|_{W(L_w^p)}^p \leq 2^p \sum_{n \in \mathbb{N}} \|A^{(n)}\|_{\mathcal{C}_w^p}^p < \infty,$$

so that $\Theta_N \in W_C(L_w^p)$ by Lemma A.1. In addition, each Θ_N satisfies the envelope property

$$\left| \left(A - \sum_{n=1}^{N} A^{(n)} \right)_{i,j} \right| \leq \sum_{n=N+1}^{\infty} \min\{\Phi_n(\gamma_j^{-1} \lambda_i), \ \Phi_n(\lambda_i^{-1} \gamma_j)\} \leq \min\{\Theta_N(\gamma_j^{-1} \lambda_i), \ \Theta_N(\lambda_i^{-1} \gamma_j)\}$$

for all $i \in I$ and $j \in J$. In particular, this implies that $A - \sum_{n=1}^{N} A^{(n)} \in \mathcal{C}_w^p(\Lambda, \Gamma)$ with

$$\left\| A - \sum_{n=1}^{N} A^{(n)} \right\|_{\mathcal{C}_w^p}^p \leq \|\Theta_N\|_{W(L_w^p)}^p \leq \sum_{n=N+1}^{\infty} \|\Phi_n\|_{W(L_w^p)}^p \to 0 \quad as \quad N \to \infty.$$

Thus, $\sum_{n \in \mathbb{N}} A^{(n)}$ converges in $\mathcal{C}_w^p(\Lambda, \Gamma)$. Overall, this shows that $\mathcal{C}_w^p(\Lambda, \Gamma)$ is complete.

(ii) Let $A \in \mathcal{C}_w^p(\Lambda, \Gamma)$ and let $\Phi \in W_C(L_w^p)$ be such that $A \prec \Phi$. Because of $w \geq 1$, an application of Lemma 2.1 yields, for all $j \in J$, that

$$\sum_{i \in I} |A_{i,j}| \leq \sum_{i \in I} \Phi(\gamma_j^{-1} \lambda_i) \leq \frac{\mathrm{Rel}(\Lambda)}{\mu_G(Q)} \|\Phi\|_{W^L(L_w^1)} \lesssim \frac{\mathrm{Rel}(\Lambda)}{\mu_G(Q)} \|\Phi\|_{W_C(L_w^p)},$$

where the last inequality follows from Lemma 3.3. Similarly, $\sum_{j \in J} |A_{i,j}| \lesssim \frac{\mathrm{Rel}(\Gamma)}{\mu_G(Q)} \|\Phi\|_{W_C(L_w^p)}$. This implies the estimates (5.12). The embedding (5.13) is a direct consequence of Schur's test, see, e.g., [23, Theorem 6.18].

(iii) Let $\Phi, \Theta \in W_C(L_w^p)$ be such that $A \prec \Phi$ and $B \prec \Theta$. By Lemma 2.1, it follows that, for arbitrary $\upsilon_k \in \Upsilon$ and $\gamma_j \in \Gamma$,

$$|(AB)_{k,j}| \leq \sum_{i \in I} |A_{k,i}||B_{i,j}| \leq \sum_{i \in I} \Phi(\lambda_i^{-1} \upsilon_k) \Theta(\gamma_j^{-1} \lambda_i) \leq \frac{\mathrm{Rel}(\Lambda)}{\mu_G(Q)} \cdot \left(M^L \Theta * M^R \Phi \right)(\gamma_j^{-1} \upsilon_k).$$

A similar calculation also gives $|(AB)_{k,j}| \leq \frac{\mathrm{Rel}(\Lambda)}{\mu_G(Q)} \cdot (M^L \Phi * M^R \Theta)(\upsilon_k^{-1} \gamma_j)$. Define the function $H : G \to [0, \infty)$ by

$$H := \frac{\mathrm{Rel}(\Lambda)}{\mu_G(Q)} \cdot \left((M^L \Theta * M^R \Phi) + (M^L \Phi * M^R \Theta) \right),$$

so that $|(AB)_{k,j}| \leq \min\{H(\gamma_j^{-1} \upsilon_k), H(\upsilon_k^{-1} \gamma_j)\}$ for all $j \in J$ and $k \in K$. By Corollary 3.9, it follows that $M^L \Theta * M^R \Phi \in W_C(L_w^p)$ and $M^L \Phi * M^R \Theta \in W_C(L_w^p)$, and thus $H \in W_C(L_w^p)$, with

$$\|H\|_{W_C(L_w^p)} \lesssim \mathrm{Rel}(\Lambda) \cdot \|\Theta\|_{W_C(L_w^p)} \|\Phi\|_{W_C(L_w^p)}.$$

Overall, $AB \prec H$ with $\|AB\|_{\mathcal{C}_w^p} \leq \|H\|_{W_C(L_w^p)}$, which easily yields the desired claim. $\square$

The proof of the following theorem resembles the proof of Theorem 5.4, but for matrices instead of integral operators. For L_w^1-localized matrices, the result can already be found in [50].

Theorem 5.7. *Let $w : G \to [1, \infty)$ be a p-weight for some $p \in (0, 1]$ and let $\Theta \in W_C(L_w^p)$. Let $R > 0$, and let $\Lambda = (\lambda_i)_{i \in I}$ be a relatively separated family in G with $\mathrm{Rel}(\Lambda) \leq R$.*

For arbitrary $\delta > 0$, there exists $\varepsilon = \varepsilon(\Theta, R, \delta, w, p, Q) \in (0, \delta)$ with the following property: If $\phi : B_\delta(1) \subset \mathbb{C} \to \mathbb{C}$ is holomorphic and if $A \in \mathcal{C}_w^p(\Lambda)$ satisfies

(1) $A \prec \Theta$,
(2) $\|A - \mathrm{id}_{\ell^2(I)}\|_{\ell^2(I) \to \ell^2(I)} \leq \varepsilon$,

then the operator $\phi(A) : \ell^2(I) \to \ell^2(I)$ defined through the holomorphic functional calculus is well-defined and its associated matrix is an element of $\mathcal{C}_w^p(\Lambda)$.

Proof. Throughout, let $\Phi \in W_C(L_w^p)$ be such that $A \prec \Phi$. The proof is split into four steps.

Step 1. *(Choice of ε).* For a symmetric function $\varphi \in C_c(G) \subseteq W_C(L_w^p)$ such that $\varphi \geq 0$ and $\varphi(e_G) = 1$, define $\Theta := \varphi + \Phi$. For $k \in \mathbb{N}$, let $\Theta_k := \min\{k^{-1}, \Theta\}$ and note the pointwise estimate and convergence

$$M\Theta_k(x) \leq \min\{k^{-1}, M\Theta(x)\} \to 0 \quad \text{as} \quad k \to \infty.$$

Since $M\Theta \in L_w^p(G)$, an application of Lebesgue's dominated convergence theorem implies $\|\Theta_k\|_{W(L_w^p)} = \|M\Theta_k\|_{L_w^p} \to 0$ as $k \to \infty$.

Let $C = C(p, w, Q) > 0$ be the implicit constant appearing in Equation (5.14) and define $C_1 := \max\{1, CR\}$ and $C_2 := (4/\delta) \cdot C_1 > 0$. By the previous paragraph, there exists $k_0 \in \mathbb{N}$ satisfying $k_0 \geq 2/\delta$ and $\|\Theta_{k_0}\|_{W_C(L_w^p)} \leq 1/C_2$. For such a fixed k_0, set $\varepsilon := k_0^{-1}$ throughout.

Step 2. *(Series representation of $\phi(A)$).* Let $(a_n)_{n \in \mathbb{N}_0}$ with $\phi(z) = \sum_{n \in \mathbb{N}_0} a_n \cdot (z - 1)^n$ for all $z \in B_\delta(1)$ with uniform convergence on compact subsets of $B_\delta(1)$. By assumption, $\|A - \mathrm{id}_{\ell^2(I)}\|_{\ell^2 \to \ell^2} \leq \varepsilon$, and $\varepsilon \leq \delta/2$ by Step 1, and thus $\sigma(A) \subset \overline{B_{\delta/2}(1)} \subset B_\delta(1)$. Therefore, basic properties of the holomorphic functional calculus (see, e.g., [51, Theorem 10.27]) yield that

$$\phi(A) = \sum_{n \in \mathbb{N}_0} a_n \cdot (A - \mathrm{id}_{\ell^2(I)})^n \tag{5.16}$$

with convergence in the operator norm topology. Since $\delta \leq \left[\limsup_{n\to\infty} |a_n|^{1/n}\right]^{-1}$ by the Cauchy-Hadamard formula, there exists $N = N(\phi, \delta) \in \mathbb{N}$ such that $|a_n|^{1/n} \leq 2/\delta$ for all $n \geq N$. In particular, this implies that there exists $C_\phi = C_\phi(\delta) > 0$ satisfying

$$|a_n| \leq C_\phi \cdot \left(2/\delta\right)^n \tag{5.17}$$

for all $n \in \mathbb{N}_0$.

Step 3. *(L_w^p-localization of $A - \mathrm{id}_{\ell^2(I)}$).* Identify the operator $A - \mathrm{id}_{\ell^2(I)}$ with the matrix $B \in \mathbb{C}^{I \times I}$ given by $B_{i,i'} = A_{i,i'} - \delta_{i,i'}$ for $i, i' \in I$. Since $\|A - \mathrm{id}_{\ell^2(I)}\|_{\ell^2 \to \ell^2} \leq \varepsilon = k_0^{-1}$, it follows that $|B_{i,i'}| \leq k_0^{-1}$. In addition, a direct calculation gives

$$|B_{i,i'}| = |A_{i,i'} - \delta_{i,i'}| \leq \Phi(\lambda_{i'}^{-1}\lambda_i) + \varphi(\lambda_{i'}^{-1}\lambda_i) = \Theta(\lambda_{i'}^{-1}\lambda_i), \quad i, i' \in I.$$

Similarly, it follows that $|B_{i,i'}| \leq \Theta(\lambda_i^{-1}\lambda_{i'})$. Thus, $B \prec \Theta_{k_0}$.

Step 4. *(Norm convergence of $\phi(A)$).* By the choice of $C_1 \geq 1$ in Step 1, it follows by an induction argument and Equation (5.14) that $B^n \in \mathcal{C}_w^p(\Lambda)$, with

$$\|B^n\|_{\mathcal{C}_w^p} \leq C_1^n \cdot \|B\|_{\mathcal{C}_w^p}^n \leq \left(\frac{C_1}{C_2}\right)^n = \left(\frac{\delta}{4}\right)^n, \quad n \in \mathbb{N},$$

where the second equality used that $\|B\|_{\mathcal{C}_w^p} \leq \|\Theta_{k_0}\|_{W(L_w^p)} \leq C_2^{-1}$; see Step 3. Combining this, together with Equation (5.17), yields

$$\sum_{n \in \mathbb{N}} \left(|a_n| \cdot \|(A - \mathrm{id}_{\ell^2(I)})^n\|_{\mathcal{C}_w^p}\right)^p \leq \sum_{n \in \mathbb{N}} \left(C_\phi \cdot (2/\delta)^n \cdot (\delta/4)^n\right)^p \leq C_\phi^p \sum_{n \in \mathbb{N}} (2^{-p})^n < \infty.$$

Since Lemma 5.6 shows that $\mathcal{C}_w^p(\Lambda)$ is a quasi-Banach space with p-norm $\|\cdot\|_{\mathcal{C}_w^p}$, it follows by Lemma A.1 that the series $\sum_{n \in \mathbb{N}} a_n (A - \mathrm{id}_{\ell^2(I)})^n$ converges in $\mathcal{C}_w^p(\Lambda)$. Since $\mathrm{id}_{\ell^2(I)} \prec \varphi$, it follows that $(A - \mathrm{id}_{\ell^2(I)})^0 = \mathrm{id}_{\ell^2(I)} \in \mathcal{C}_w^p(\Lambda)$. Therefore,

$$\phi(A) = a_0(A - \mathrm{id}_{\ell^2(I)})^0 + \sum_{n \in \mathbb{N}} a_n (A - \mathrm{id}_{\ell^2(I)})^n \in \mathcal{C}_w^p,$$

which completes the proof. $\qquad\square$

6. Dual molecules for coorbit spaces

This section is devoted to the notion of molecules in coorbit spaces. The main results obtained show that coorbit spaces and associated sequence spaces can be decomposed in terms of dual molecules of frames and Riesz sequences. These results will be used in Section 7 to provide criteria for boundedness of operators on coorbit spaces.

6.1. **Frames and Riesz sequences.** Let $\mathcal{H}$ be a separable Hilbert space. A countable family $(g_i)_{i \in I}$ in $\mathcal{H}$ is called a *Bessel sequence* in $\mathcal{H}$ if there exists $B > 0$ such that

$$\sum_{i \in I} |\langle f, g_i \rangle|^2 \leq B\|f\|_{\mathcal{H}}^2 \quad \text{for all} \quad f \in \mathcal{H}.$$

Equivalently, $(g_i)_{i \in I}$ is a Bessel sequence if the *coefficient operator* associated to $(g_i)_{i \in I}$,

$$\mathscr{C}: \quad \mathcal{H} \to \ell^2(I), \quad f \mapsto (\langle f, g_i \rangle)_{i \in I}$$

is well-defined and bounded. The *reconstruction operator* $\mathscr{D} := \mathscr{C}^* : \ell^2(I) \to \mathcal{H}$ associated to $(g_i)_{i \in I}$ is given by $\mathscr{D}(c_i)_{i \in I} = \sum_{i \in I} c_i g_i$. The *frame* and *Gramian operator* associated to $(g_i)_{i \in I}$ are defined by $\mathscr{S} := \mathscr{D}\mathscr{C} : \mathcal{H} \to \mathcal{H}$ and $\mathscr{G} := \mathscr{C}\mathscr{D} : \ell^2(I) \to \ell^2(I)$, respectively.

A Bessel sequence $(g_i)_{i \in I}$ is called a *frame* for $\mathcal{H}$, if there exist $A, B > 0$ (called *frame bounds*) satisfying

$$A\|f\|_{\mathcal{H}}^2 \leq \sum_{i \in I} |\langle f, g_i \rangle|^2 \leq B\|f\|_{\mathcal{H}}^2 \quad \text{for all} \quad f \in \mathcal{H}; \tag{6.1}$$

this holds if and only if the frame operator $\mathscr{S} : \mathcal{H} \to \mathcal{H}$ is bounded and invertible. Two Bessel sequences $(g_i)_{i \in I}$ and $(h_i)_{i \in I}$ are said to be *dual frames* for $\mathcal{H}$ if

$$f = \sum_{i \in I} \langle f, g_i \rangle h_i = \sum_{i \in I} \langle f, h_i \rangle g_i \quad \text{for all} \quad f \in \mathcal{H}.$$

If $(g_i)_{i \in I}$ is a frame for $\mathcal{H}$, then $(\mathscr{S}^{-1} g_i)_{i \in I}$ is a dual frame of $(g_i)_{i \in I}$, called the *canonical dual frame*. A frame is called *Parseval* if (6.1) holds with equality, i.e., if $\mathscr{S} = \mathrm{id}_{\mathcal{H}}$. If $(g_i)_{i \in I}$ is a frame for $\mathcal{H}$, then the system $(\mathscr{S}^{-1/2} g_i)_{i \in I}$ is a Parseval frame for $\mathcal{H}$.

A Bessel sequence $(g_i)_{i \in I}$ is called a *Riesz sequence* in $\mathcal{H}$, if there exist $A, B > 0$ (called *Riesz bounds*) satisfying

$$A \|c\|_{\ell^2}^2 \leq \left\| \sum_{i \in I} c_i g_i \right\|_{\mathcal{H}}^2 \leq B \|c\|_{\ell^2}^2 \quad \text{for all} \quad c = (c_i)_{i \in I} \in \ell^2(I). \tag{6.2}$$

Equivalently, $(g_i)_{i \in I}$ is a Riesz sequence if and only if the Gramian operator $\mathscr{G} : \ell^2(I) \to \ell^2(I)$ is bounded and invertible. If $(g_i)_{i \in I}$ is a Riesz sequence in $\mathcal{H}$, then it admits a unique biorthogonal system $(h_i)_{i \in I}$ in $\overline{\mathrm{span}}\{g_i : i \in I\}$. A Riesz sequence $(g_i)_{i \in I}$ is a frame for $\overline{\mathrm{span}}\{g_i : i \in I\}$, and $(\mathscr{S}^{-1/2} g_i)_{i \in I}$ is an orthonormal sequence in $\mathcal{H}$, where $\mathscr{S}$ is the frame operator considered as an operator on $\overline{\mathrm{span}}\{g_i : i \in I\}$.

For background, proofs, and further properties, see, e.g., the books [12,55].

6.2. **Molecules and their basic properties.** Throughout this section, $g \in \mathcal{B}_w^p$ denotes an admissible vector and $\mathcal{K}_g := V_g(\mathcal{H}_\pi)$ the associated reproducing kernel Hilbert space.

The following definition introduces the central notion of this section.

Definition 6.1. Let $w : G \to [1, \infty)$ be a p-weight for some $p \in (0, 1]$. Let $\Lambda = (\lambda_i)_{i \in I}$ be relatively separated in G.

   (a) A family $(h_i)_{i \in I}$ in $\mathcal{H}_\pi$ is a system of (L_w^p, g)-*molecules* in $\mathcal{H}_\pi$ if there exists a symmetric *envelope* $\Phi \in W_C(L_w^p)$ such that

$$|V_g h_i(x)| \leq \Phi(\lambda_i^{-1} x) \tag{6.3}$$

     for all $i \in I$ and $x \in G$.

   (b) A family $(H_i)_{i \in I}$ in $\mathcal{K}_g$ is a system of L_w^p-*molecules* in $\mathcal{K}_g$ if there exists a symmetric *envelope* $\Phi \in W_C(L_w^p)$ such that

$$|H_i(x)| \leq \Phi(\lambda_i^{-1} x) \tag{6.4}$$

     for all $i \in I$ and $x \in G$.

If condition (6.3) (resp. (6.4)) holds, this will be indicated using the notation $(h_i)_{i \in I} \prec \Phi$ (resp. $(H_i)_{i \in I} \prec \Phi$).

Remark 6.2. A family $(H_i)_{i \in I}$ is a system of L_w^p-molecules in $\mathcal{K}_g$ if and only if $(V_g^{-1} H_i)_{i \in I}$ is a system of (L_w^p, g)-molecules in $\mathcal{H}_\pi$.

The molecule condition (6.3) is independent of the choice of the admissible vectors $g, h \in \mathcal{B}_w^p$, as long as the matrix coefficient $V_g h$ is well-localized.

Lemma 6.3. *If $g, h \in \mathcal{B}_w^p$ are admissible with $V_g h \in W(L_w^p)$, and if $(h_i)_{i \in I} \subset \mathcal{H}_\pi$ is a system of (L_w^p, g) molecules, then it is also a system of (L_w^p, h) molecules.*

Proof. Since $(h_i)_{i \in I}$ is a system of (L_w^p, g) molecules, there exists a symmetric envelope $\Phi \in W_C(L_w^p)$ satisfying $|V_g h_i| \leq L_{\lambda_i} \Phi$ for all $i \in I$. Since w is a p-weight, $[L_w^p]^\vee = L_w^p$, whence the assumption $V_g h \in W(L_w^p)$ implies by Lemma 4.1 that $\Psi_0 := |V_g h| + |V_h g| + \Phi \in W(L_w^p)$ is symmetric. Therefore, Corollary 3.9 and the elementary identity $(F_1 * F_2)^\vee = F_2^\vee * F_1^\vee$ imply

that $\Psi := \Psi_0 * \Psi_0 \in W_C(L_w^p)$ is symmetric. Using the reproducing formula $V_h h_i = V_g h_i *_\sigma V_h g$ (cf. Lemma 4.1), it follows that

$$|V_h h_i(x)| \le (|V_g h_i| * |V_h g|)(x) \le (L_{\lambda_i} \Phi * |V_h g|)(x) \le L_{\lambda_i}[\Phi * |V_h g|](x)$$
$$\le L_{\lambda_i}[\Psi_0 * \Psi_0](x) = \Psi(\lambda_i^{-1} x),$$

where it is used that $(L_\lambda F_1) * F_2 = L_\lambda[F_1 * F_2]$. This shows that $(h_i)_{i \in I}$ is a family of (L_w^p, h) molecules. $\qquad\square$

In light of Lemma 6.3, a system $(h_i)_{i \in I}$ satisfying the molecule condition (6.3) will simply be referred to as a system of L_w^p-molecules in $\mathcal{H}_\pi$.

The following lemma shows that the molecule property is preserved under the action of convolution-dominated matrices and integral operators.

Lemma 6.4. *Let $(H_i)_{i \in I}$ be a system of L_w^p-molecules in $\mathcal{K}_g$ indexed by the relatively separated family $\Lambda = (\lambda_i)_{i \in I}$. Then the following hold:*

(i) If $H : G \times G \to \mathbb{C}$ is L_w^p-localized in $\mathcal{K}_g$ (see Definition 5.1), then the family $(\widetilde{H}_i)_{i \in I} \subset \mathcal{K}_g$ defined by

$$\widetilde{H}_i(x) := (T_H H_i)(x) = \int_G H(x, y) H_i(x) \, d\mu_G(y)$$

is also a system of L_w^p-molecules in $\mathcal{K}_g$.

(ii) If $\Lambda = (\lambda_i)_{i \in I}$ and $\Gamma = (\gamma_j)_{j \in J}$ are relatively separated families in G and if the matrix $A = (A_{j,i})_{(j,i) \in J \times I} \in \mathcal{C}_w^p(\Gamma, \Lambda)$, then the family $(H_j')_{j \in J} \subset \mathcal{K}_g$ defined by

$$H_j' := A(H_i)_{i \in I} := \sum_{i \in I} A_{j,i} H_i$$

is also a system of L_w^p-localized molecules in $\mathcal{K}_g$.

Proof. Let $\Phi \in W_C(L_w^p)$ be a symmetric envelope such that $(H_i)_{i \in I} \prec \Phi$.

(i) Let $\Theta \in W_C(L_w^p)$ be a symmetric envelope for H. An application of Lemma 5.3 shows that $\widetilde{H}_i = T_H H_i \in \mathcal{K}_g$ for all $i \in I$. Moreover, it holds that

$$|\widetilde{H}_i(x)| \le \int_G \Theta(y^{-1} x) \Phi(\lambda_i^{-1} y) d\mu_G(y) = (\Phi * \Theta)(\lambda_i^{-1} x)$$

and similarly $|\widetilde{H}_i(x)| \le (\Theta * \Phi)(x^{-1}\lambda_i)$. Therefore, using the convolution relation of Corollary 3.9, it follows that $\Phi * \Theta + \Theta * \Phi \in W_C(L_w^p)$ is a (symmetric) envelope for $(\widetilde{H}_i)_{i \in I}$.

(ii) Let $\Theta \in W_C(L_w^p)$ be such that $A \prec \Theta$. For $x \in G$ and $j \in J$, an application of Lemma 2.1 gives

$$|H_j'(x)| \le \sum_{i \in I} |\Theta(\gamma_j^{-1}\lambda_i)||\Phi(\lambda_i^{-1}x)| \le \frac{\mathrm{Rel}(\Lambda)}{\mu_G(Q)}(M^L\Theta * M^R\Phi)(\gamma_j^{-1}x).$$

Since $\Phi, \Theta \in W_C(L_w^p)$, it follows that also $M^L\Theta, M^R\Phi \in W_C(L_w^p)$ (cf. Section 2.6), and hence $\Psi := M^L\Theta * M^R\Phi \in W_C(L_w^p) + (M^L\Theta + M^R\Phi)^\vee$ by Corollary 3.9 and because of $[W_C(L_w^p)]^\vee = W_C(L_w^p)$, which holds since w is a p-weight. Thus, Ψ is a symmetric envelope for $(H_j')_{j \in J}$. $\qquad\square$

The frame operator and Gramian associated to a system of molecules are convolution-dominated operators, as shown next.

Lemma 6.5. *Let $(H_i)_{i \in I}$ be a system of L_w^p-molecules in $\mathcal{K}_g$, indexed by the relatively separated family $\Lambda = (\lambda_i)_{i \in I}$. Then the following hold:*

(i) The kernel

$$H: \quad G \times G \to \mathbb{C}, \quad (x, y) \mapsto \sum_{i \in I} H_i(x)\overline{H_i(y)} \tag{6.5}$$

is well-defined (with absolute convergence of the series) and L_w^p-localized in $\mathcal{K}_g$.

(ii) The family $(H_i)_{i \in I} \subset \mathcal{K}_g$ is Bessel and the frame operator $\mathscr{S} : \mathcal{K}_g \to \mathcal{K}_g$ is given by $\mathscr{S} = T_H|_{\mathcal{K}_g}$.

(iii) If $\Phi \in W_C(L_w^p)$ is symmetric and such that $(H_i)_{i \in I} \prec \Phi$, then the Gramian matrix $\mathscr{G} = (\langle H_{i'}, H_i \rangle)_{i,i' \in I}$ is an element of $\mathcal{C}_w^p(\Lambda)$ with envelope $\Phi * \Phi \in W_C(L_w^p)$.

Proof. Let $\Phi \in W_C(L_w^p)$ be a symmetric envelope for $(H_i)_{i \in I}$.

(i) For $x, y \in G$, an application of Equation (2.10) gives

$$|H(x, y)| \le \sum_{i \in I} |H_i(x)| \, |H_i(y)| \le \sum_{i \in I} \Phi(\lambda_i^{-1}x)\Phi(y^{-1}\lambda_i) \le \frac{\mathrm{Rel}(\Lambda)}{\mu_G(Q)}(M^L\Phi * M^R\Phi)(y^{-1}x).$$

Since $\Phi \in W_C(L_w^p)$, also $M^L\Phi, M^R\Phi \in W_C(L_w^p)$; see Section 2.6. Thus, Corollary 3.9 shows that $\Theta := M^L\Phi * M^R\Phi \in W_C(L_w^p)$ as well and in particular that $(M^L\Phi * M^R\Phi)(y^{-1}x) < \infty$. Thus, the series defining $H(x, y)$ converges absolutely. Since $|H(x, y)| = |H(y, x)|$, the above estimate also shows that $H \prec M^L\Phi * M^R\Phi$. In addition, since $\overline{H(x, \cdot)} = H(\cdot, x)$, to show that H is L_w^p-localized in $\mathcal{K}_g$, it remains to show that $H(\cdot, y) \in \mathcal{K}_g$ for all $y \in G$. For this, first note as a consequence of Equation (2.11) and Lemma 3.3 that

$$\sum_{i \in I} |H_i(y)| \le \sum_{i \in I} \Phi(y^{-1}\lambda_i) \le \frac{\mathrm{Rel}(\Lambda)}{\mu_G(Q)}\|\Phi\|_{W^L(L^1)} \lesssim \frac{\mathrm{Rel}(\Lambda)}{\mu_G(Q)}\|\Phi\|_{W(L_w^p)} < \infty.$$

Furthermore, Lemma 3.3 also shows $\|H_i\|_{L^2} \le \|\Phi(\lambda_i^{-1}\cdot)\|_{L^2} \le \|\Phi\|_{L^2} \lesssim \|\Phi\|_{W(L_w^p)}$, so that it follows that $\sum_{i \in I} |H_i(y)| \, \|H_i\|_{L^2} < \infty$. This shows that the series defining $H(\cdot, y)$ converges in $L^2(G)$. Since $H_i \in \mathcal{K}_g$ for all $i \in I$ and since $\mathcal{K}_g \subset L^2(G)$ is closed, it follows that $H(\cdot, y) \in \mathcal{K}_g$.

(ii) For showing that $(H_i)_{i \in I}$ is a Bessel sequence in $L^2(G)$, let $c = (c_i)_{i \in I} \in \ell^2(I)$ be arbitrary. Recall from Equation (2.11) that $\sum_{i \in I} \Phi(x^{-1}\lambda_i) \lesssim 1$. Using the Cauchy-Schwarz inequality and the estimate $|H_i(x)| \le \Phi(\lambda_i^{-1}x) = \Phi(x^{-1}\lambda_i)$, it follows therefore that

$$\int_G \left(\sum_{i \in I} |c_i| \cdot |H_i(x)| \right)^2 d\mu_G(x) \le \int_G \left(\sum_{i \in I} |c_i|^2 \, \Phi(\lambda_i^{-1}x) \right) \left(\sum_{i \in I} \Phi(x^{-1}\lambda_i) \right) d\mu_G(x)$$

$$\lesssim \sum_{i \in I} |c_i|^2 \|\Phi\|_{L^1} \lesssim \|c\|_{\ell^2}^2.$$

This shows that $(H_i)_{i \in I} \subset L^2(G)$ is a Bessel sequence. Lastly, by the first part of the proof and Lemma 5.3, it follows that $T_H|_{\mathcal{K}_g} : \mathcal{K}_g \to \mathcal{K}_g$ is well-defined and bounded. Let $F \in \mathcal{K}_g$ and $x \in G$. As shown in the proof of (i), the series $\overline{H(x, \cdot)} = \sum_{i \in I} \overline{H_i(x)} \, H_i$ converges in $L^2(G)$, and thus

$$T_H F(x) = \langle F, \overline{H(x, \cdot)} \rangle_{L^2} = \sum_{i \in I} \langle F, \overline{H_i(x)} \, H_i \rangle_{L^2} = \sum_{i \in I} \langle F, H_i \rangle_{L^2} \, H_i(x) = \mathscr{S}F(x),$$

so that $\mathscr{S} = T_H|_{\mathcal{K}_g}$, as claimed.

(iii) For $i, i' \in I$, a direct calculation gives

$$|\langle H_{i'}, H_i \rangle| = |\langle H_i, H_{i'} \rangle| \le \int_G \Phi(\lambda_i^{-1}x)\Phi(\lambda_{i'}^{-1}x) \, d\mu_G(x) = \int_G \Phi(x^{-1}\lambda_i)\Phi(\lambda_{i'}^{-1}x) \, d\mu_G(x)$$

$$= (\Phi * \Phi)(\lambda_{i'}^{-1}\lambda_i),$$

where the penultimate equality used the symmetry of Φ. By Corollary 3.9, $\Phi * \Phi \in W_C(L_w^p)$, and $(\Phi * \Phi)^\vee = \Phi^\vee * \Phi^\vee = \Phi * \Phi$. Thus, $(\langle H_{i'}, H_i \rangle)_{i,i' \in I} \in \mathcal{C}_w^p(\Lambda)$. $\qquad\square$

6.3. **Dual frames of molecules.** Henceforth, let $w : G \to [1, \infty)$ be a p-weight for some $p \in (0, 1]$ and let $g \in \mathcal{B}_w^p$ be an admissible vector. Recall that $\mathcal{K}_g := V_g(\mathcal{H}_\pi)$ is a reproducing kernel Hilbert space (cf. Equation (4.5)) with reproducing kernel given by

$$K(x, y) = V_g[\pi(y)g](x) = \overline{L_x^\sigma[V_g g](y)},$$

see Equation (4.6). For fixed $x \in G$, the notation

$$K_x := \overline{K(x, \cdot)} = L_x^\sigma[V_g g] = V_g[\pi(x)g]$$

will be used.

The following result proves the existence of *almost tight* frames of (weighted) reproducing kernels. For the notion of a *disjoint cover* that appears in the statement, cf. Section 2.3.

Proposition 6.6. *For every $\varepsilon \in (0, 1)$, there exists a compact, symmetric unit neighborhood $U \subset Q$ with the following property:*

If $\Lambda = (\lambda_i)_{i \in I}$ is a relatively separated, U-dense family in G and $(U_i)_{i \in I}$ is a disjoint cover of G associated to U and Λ, then the family $\left(\sqrt{\mu_G(U_i)} \cdot K_{\lambda_i} \right)_{i \in I}$ is a frame for $\mathcal{K}_g$ with lower frame bound $1 - \varepsilon$ and upper frame bound $1 + \varepsilon$.

Proof. The proof is divided into three steps.

Step 1. In this step, it will be shown that there exists a measurable map $\tau : G \times G \to \mathbb{T}$ such that

$$\|K_x - \tau(x, y)K_y\|_{L^2} \to 0 \quad \text{as} \quad y^{-1}x \to e_G. \tag{6.6}$$

For showing (6.6), consider the map $((x, y), \tau) \mapsto -\|K_x - \tau K_y\|_{L^2}$ from $(G \times G) \times \mathbb{T}$ into $\mathbb{R}$, which is clearly measurable with respect to (x, y) and continuous with respect to τ. A straightforward application of the measurable maximum theorem (see, e.g., [1, Theorem 18.19]) yields a measurable map $\tau : G \times G \to \mathbb{T}$ satisfying

$$\|K_x - \tau(x, y)K_y\|_{L^2} = \min_{z \in \mathbb{T}} \|K_x - z K_y\|_{L^2} \qquad \text{for all} \quad x, y \in G.$$

Since G is first countable, it suffices to show $\|K_{x_n} - \tau(x_n, y_n)K_{y_n}\|_{L^2} \to 0$ as $n \to \infty$, for arbitrary sequences $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \subset G$ satisfying $y_n^{-1}x_n \to e_G$.

Let $\mathcal{U} := \mathcal{U}(\mathcal{H}_\pi)$ denote the group of unitary operators on $\mathcal{H}_\pi$, equipped with the strong topology. Furthermore, let $\mathcal{P} := \mathcal{U}/(\mathbb{T} \, \mathrm{id}_{\mathcal{H}_\pi})$ and let $\varrho : \mathcal{U} \to \mathcal{P}$ denote the canonical projection. Since π is a σ-representation, an application of [53, Theorem 7.5] implies that the quotient map $[\pi] : G \to \mathcal{P}, x \mapsto \varrho(\pi(x))$ is continuous. Thus, $\varrho(\pi(x_n^{-1}y_n)) \to \varrho(\pi(e_G)) = \varrho(\mathrm{id}_{\mathcal{H}_\pi})$, and an application of [53, Lemma 7.1] yields a sequence $(z_n)_{n \in \mathbb{N}}$ of numbers $z_n \in \mathbb{T}$ satisfying $z_n \cdot \pi(x_n^{-1}y_n) \to \mathrm{id}_{\mathcal{H}_\pi}$ in the strong topology. Since $V_g : \mathcal{H}_\pi \to L^2(G)$ is an isometry and $K_x = V_g[\pi(x)g]$, it follows that

$$\begin{aligned}
\left\|K_{x_n} - \tau(x_n, y_n)K_{y_n}\right\|_{L^2} &= \inf_{z \in \mathbb{T}} \left\|K_{x_n} - z \cdot K_{y_n}\right\|_{L^2} = \inf_{z \in \mathbb{T}} \left\|\pi(x_n)g - z \cdot \pi(y_n)g\right\|_{\mathcal{H}_\pi} \\
&= \inf_{z \in \mathbb{T}} \left\|g - z \, \overline{\sigma(x_n^{-1}, x_n)}\sigma(x_n^{-1}, y_n)\pi(x_n^{-1}y_n)g\right\|_{\mathcal{H}_\pi} \\
&\leq \left\|g - z_n \cdot \pi(x_n^{-1}y_n)g\right\|_{\mathcal{H}_\pi} \to 0,
\end{aligned}$$

where the penultimate step used that $\pi(x_n)$ is unitary and that

$$\pi(x_n)^{-1}\pi(y_n) = \overline{\sigma(x_n^{-1}, x_n)}\sigma(x_n^{-1}, y_n)\pi(x_n^{-1}y_n).$$

Therefore, $\|K_{x_n} - \tau(x_n, y_n)K_{y_n}\|_{L^2} \leq \|g - z_n \cdot \pi(x_n^{-1}y_n)g\|_{\mathcal{H}_\pi} \to 0$ as $n \to \infty$.

Step 2. This step shows that also $\|K_x - \tau(x, y)K_y\|_{L^1} \to 0$ as $y^{-1}x \to e_G$. For this, set $\Theta := |V_g g|$ and note by Lemma 3.3 and because of $V_g g \in W(L_w^p)$ and $w \geq 1$ that $\Theta \in W_C(L_w^p) \hookrightarrow W^L(L_w^1) \hookrightarrow L^1$; see also Equation (4.3). Let $\delta > 0$ be arbitrary and choose a

compact set $\Omega \subset G$ satisfying $\int_{G\setminus\Omega} \Theta d\mu_G \leq \frac{\delta}{4}$. By Step 1, there exists a compact, symmetric unit neighborhood $V \subset Q$ such that for all $x, y \in G$ satisfying $y^{-1}x \in V$,

$$\|K_x - \tau(x,y)K_y\|_{L^2} \leq \frac{\delta/2}{1 + \sqrt{\mu_G(Q\Omega)}} \leq \frac{\delta/2}{1 + \sqrt{\mu_G(V\Omega)}}.$$

Using the estimate $|K_x(y)| \leq \Theta(x^{-1}y)$ and the Cauchy-Schwarz inequality, it follows therefore that, for $x, y \in G$ with $y^{-1}x \in V$,

$$\|K_x - \tau(x,y)K_y\|_{L^1}$$
$$\leq \int_{G\setminus xV\Omega} \Theta(x^{-1}z) + \Theta(y^{-1}z)d\mu_G(z) + \int_G \mathbb{1}_{xV\Omega}(z) \cdot |K_x(z) - \tau(x,y)K_y(z)|d\mu_G(z)$$
$$\leq 2\int_{G\setminus\Omega} \Theta(w)d\mu_G(w) + \|\mathbb{1}_{xV\Omega}\|_{L^2} \cdot \|K_x - \tau(x,y)K_y\|_{L^2}$$
$$\leq \frac{\delta}{2} + \sqrt{\mu_G(V\Omega)} \cdot \frac{\delta/2}{1 + \sqrt{\mu_G(V\Omega)}} \leq \delta,$$

where the second inequality used the change of variables $w = x^{-1}z$, respectively $w = y^{-1}z$, and the inclusions $G \setminus V\Omega \subset G \setminus \Omega$ and $G \setminus y^{-1}xV\Omega \subset G \setminus \Omega$, which holds since $x^{-1}y \in V$.

Step 3. Let $\varepsilon > 0$, and choose $\delta > 0$ so small that $1 + \sqrt{\delta} \leq \sqrt{1 + \varepsilon}$ and $1 - \sqrt{\delta} \geq \sqrt{1 - \varepsilon}$. By Step 2, there exists a compact symmetric unit neighborhood $U \subset Q$ satisfying

$$\|K_x - \tau(x,y)K_y\|_{L^1} \leq \delta/(1 + 2\|M^L\Theta\|_{L^1})$$

for all $x, y \in G$ satisfying $y^{-1}x \in U$. Given this choice of U, let $\Lambda = (\lambda_i)_{i\in I}$ and $(U_i)_{i\in I}$ as in the statement of the proposition. Let $F \in \mathcal{K}_g$ be arbitrary. Since the family $(U_i)_{i\in I}$ is pairwise disjoint and satisfies $G = \bigcup_{i\in I} U_i$, a direct calculation entails

$$\left| \|F\|_{L^2} - \left(\sum_{i\in I} |\langle F, \sqrt{\mu_G(U_i)}\, K_{\lambda_i}\rangle|^2\right)^{\frac{1}{2}} \right| = \left| \big\| \,|F|\, \big\|_{L^2} - \left\|\left(\sum_{i\in I} |\langle F, \tau(x, \lambda_i)\, K_{\lambda_i}\rangle|^2 \cdot \mathbb{1}_{U_i}(x)\right)^{\frac{1}{2}}\right\|_{L_x^2} \right|$$
$$\leq \left\| |F(x)| - \Big|\sum_{i\in I} \langle F, \tau(x, \lambda_i)\, K_{\lambda_i}\rangle \cdot \mathbb{1}_{U_i}(x)\Big| \right\|_{L_x^2}$$
$$\leq \left\| \sum_{i\in I} \mathbb{1}_{U_i}(x) \cdot \big(F(x) - \overline{\tau(x,\lambda_i)}\, F(\lambda_i)\big) \right\|_{L_x^2},$$

where the notation $\|\cdot\|_{L_x^2}$ is used to indicate that the L^2-norm is taken with respect to $x \in G$. Thus, setting

$$H(x) := \left| \sum_{i\in I} \mathbb{1}_{U_i}(x) \int_G F(y) \cdot \overline{(K_x(y) - \tau(x,\lambda_i)K_{\lambda_i}(y))}\, d\mu_G(y) \right|,$$

it holds that $\left|\|F\|_{L^2} - \left(\sum_{i\in I} |\langle F, \sqrt{\mu_G(U_i)}\, K_{\lambda_i}\rangle|^2\right)^{1/2}\right| \leq \|H\|_{L^2}$.

Note that if $\mathbb{1}_{U_i}(x) \neq 0$, then $x \in U_i \subset \lambda_i U \subset \lambda_i Q$. On the one hand, this implies $\lambda_i^{-1}x \in U$ and hence $\|K_x - \tau(x,\lambda_i)\, K_{\lambda_i}\|_{L^1} \leq \delta/(1 + 2\|M^L\Theta\|_{L^1})$. On the other hand, the above considerations show for $\mathbb{1}_{U_i}(x) \neq 0$ that $y^{-1}\lambda_i = y^{-1}xx^{-1}\lambda_i \in y^{-1}xQ^{-1} = y^{-1}xQ$ and hence $\Theta(y^{-1}\lambda_i) \leq M^L\Theta(y^{-1}x)$, since Θ is continuous and Q is open. Combining this with

the estimate $|K_x(y)| \leq \Theta(y^{-1}x)$ and the Cauchy-Schwarz inequality, it follows that

$$H(x) \leq \sum_{i \in I} \mathbb{1}_{U_i}(x) \int_G |F(y)| \cdot |K_x(y) - \tau(x, \lambda_i) K_{\lambda_i}(y)|^{1/2} \cdot (\Theta(y^{-1}x) + \Theta(y^{-1}\lambda_i))^{1/2} \, d\mu_G(y)$$

$$\leq \left(\sum_{i \in I} \int_G \mathbb{1}_{U_i}(x) \cdot 2\, M^L \Theta(y^{-1}x) \cdot |F(y)|^2 \, d\mu_G(y) \right)^{\frac{1}{2}} \left(\sum_{i \in I} \mathbb{1}_{U_i}(x) \, \|K_x - \tau(x, \lambda_i)\, K_{\lambda_i}\|_{L^1} \right)^{\frac{1}{2}}$$

$$\leq \left(\frac{\delta}{1 + 2\, \|M^L \Theta\|_{L^1}} \right)^{1/2} \left(2 \int_G |F(y)|^2 \cdot M^L \Theta(y^{-1}x) \, d\mu_G(y) \right)^{1/2}.$$

Therefore,

$$\|H\|_{L^2}^2 \leq \frac{2\delta}{1 + 2\, \|M^L \Theta\|_{L^1}} \int_G \int_G |F(y)|^2 \cdot M^L \Theta(y^{-1}x) \, d\mu_G(y) d\mu_G(x)$$

$$\leq \frac{2\delta}{1 + 2\, \|M^L \Theta\|_{L^1}} \|F\|_{L^2}^2 \|M^L \Theta\|_{L^1} \leq \delta \, \|F\|_{L^2}^2,$$

and hence

$$\left| \|F\|_{L^2} - \left(\sum_{i \in I} |\langle F, \sqrt{\mu_G(U_i)}\, K_{\lambda_i}\rangle|^2 \right)^{1/2} \right| \leq \|H\|_{L^2} \leq \sqrt{\delta}\, \|F\|_{L^2}.$$

By the choice of δ, this easily implies

$$(1 - \varepsilon) \cdot \|F\|_{L^2}^2 \leq \sum_{i \in I} |\langle F, \sqrt{\mu_G(U_i)}\, K_{\lambda_i}\rangle|^2 \leq (1 + \varepsilon) \cdot \|F\|_{L^2}^2$$

for all $F \in \mathcal{K}_g$, as required. $\qquad\square$

We aim to apply Theorem 5.4 to prove the existence of a dual frame for the family $(K_{\lambda_i})_{i \in I}$ that forms a system of molecules. To this end, we need to construct a frame of molecules $(\tau_i K_{\lambda_i})_{i \in I}$ with a fixed envelope Φ but such that the associated frame operator $\mathscr{S}$ satisfies $\|\mathrm{id}_{\mathcal{K}_g} - \mathscr{S}\|_{\mathcal{K}_g \to \mathcal{K}_g} \leq \varepsilon = \varepsilon(\Phi)$. In combination with Proposition 6.6, the following lemma shows that this can indeed be done.

Lemma 6.7. *Let $U \subset Q$ be a unit neighborhood and assume that $\Lambda = (\lambda_i)_{i \in I} \subset G$ is relatively separated and that $(U_i)_{i \in I}$ is a family of measurable sets $U_i \subset \lambda_i U$ satisfying $G = \biguplus_{i \in I} U_i$.*

If $(\tau_i)_{i \in I} \subset [0, \infty)$ satisfies $\tau_i \leq C \cdot \mu_G(U_i)$ for all $i \in I$ and some $C > 0$, then the family $\left(\tau_i^{1/2} \cdot K_{\lambda_i} \right)_{i \in I}$ is a system of L_w^p-molecules in $\mathcal{K}_g$ (indexed by Λ) and the kernel H in Equation (6.5) associated to $\left(\tau_i^{1/2} \cdot K_{\lambda_i} \right)_{i \in I}$ satisfies

$$|H(y, x)| = |H(x, y)| \leq C \cdot \left[M^L(V_g g) * M^R(V_g g) \right](y^{-1}x)$$

for all $x, y \in G$.

Proof. Set $\Phi := |V_g g| \in W_C(L_w^p)$ and note that Φ satisfies $|K_x(y)| \leq \Phi(x^{-1}y) = \Phi(y^{-1}x)$ for all $x, y \in G$. Since $\tau_i \leq C \cdot \mu_G(U_i) \leq C \cdot \mu_G(U)$, this implies that $\left(\tau_i^{1/2} \cdot K_{\lambda_i} \right)_{i \in I}$ is a system of L_w^p-molecules in $\mathcal{K}_g$ with envelope $\sqrt{C \cdot \mu_G(U)} \cdot \Phi$.

Let $x, y \in G$. For $i \in I$ and $z \in U_i \subset \lambda_i Q$, we have $\lambda_i^{-1}x = \lambda_i^{-1}zz^{-1}x \in Qz^{-1}x$ and $y^{-1}\lambda_i = y^{-1}z(\lambda_i^{-1}z)^{-1} \in y^{-1}zQ^{-1} = y^{-1}zQ$. Since Q is open and Φ is continuous, this implies $|K_{\lambda_i}(x)| \leq \Phi(\lambda_i^{-1}x) \leq M^R \Phi(z^{-1}x)$ and $|K_{\lambda_i}(y)| \leq \Phi(y^{-1}\lambda_i) \leq M^L \Phi(y^{-1}z)$ for all $i \in I$ and $z \in U_i$. Hence, by definition of H in Equation (6.5),

$$|H(x, y)| \leq C \sum_{i \in I} \left[\mu_G(U_i) \cdot |K_{\lambda_i}(x)| \cdot |K_{\lambda_i}(y)| \right] \leq C \sum_{i \in I} \int_{U_i} M^R \Phi(z^{-1}x) M^L \Phi(y^{-1}z) d\mu_G(z)$$

$$= C \int_G M^L \Phi(w) M^R \Phi(w^{-1}y^{-1}x) d\mu_G(w) = C \cdot (M^L \Phi * M^R \Phi)(y^{-1}x).$$

Since $H(y, x) = \overline{H(x, y)}$, this completes the proof. $\qquad\square$

Using the previous two results, we can prove the main result of this subsection.

Theorem 6.8. *There exists a compact unit neighborhood $U \subset Q$ with the following property: If $\Lambda = (\lambda_i)_{i \in I}$ is relatively separated and U-dense in G, the following assertions hold:*

    (i) *The family $(K_{\lambda_i})_{i \in I}$ is a frame for $\mathcal{K}_g$ and admits a dual frame $(H_i)_{i \in I}$ of L_w^p-molecules in $\mathcal{K}_g$.*

    (ii) *There exists a Parseval frame $(F_i)_{i \in I}$ for $\mathcal{K}_g$ that is a system of L_w^p-molecules in $\mathcal{K}_g$.*

Proof. Set $C := 1$ and $\Theta := C \cdot M^L(V_g g) * M^R(V_g g)$. Since $|V_g g| \in W_C(L_w^p)$ and thus also $M^L(V_g g), M^R(V_g g) \in W_C(L_w^p)$, Corollary 3.9 shows that $\Theta \in W_C(L_w^p)$ as well. An application of Theorem 5.4 yields $\varepsilon = \varepsilon(g, Q, w, p) \in (0, 1)$ such that whenever $H : G \times G \to \mathbb{C}$ is L_w^p-localized in $\mathcal{K}_g$ with $H \prec \Theta$ and with $\|T_H - \mathrm{id}_{\mathcal{K}_g}\|_{\mathcal{K}_g \to L^2} \leq \varepsilon$, then there exist kernels $H_1, H_2 : G \times G \to \mathbb{C}$ that are L_w^p-localized in $\mathcal{K}_g$ and such that $(T_H|_{\mathcal{K}_g})^{-1} = T_{H_1}|_{\mathcal{K}_g}$ and $(T_H|_{\mathcal{K}_g})^{-1/2} = T_{H_2}|_{\mathcal{K}_g}$, where the operators on the left-hand side are defined by the holomorphic functional calculus.

For this choice of ε, let the compact unit neighborhood $U \subset Q$ be as provided by Proposition 6.6. Let $\Lambda = (\lambda_i)_{i \in I} \subset G$ be any relatively separated and U-dense family. Then I is countable and there exists a disjoint cover $(U_i)_{i \in I}$ associated to Λ and U (cf. Section 2.3), i.e., pairwise disjoint Borel sets $U_i \subset \lambda_i U$ satisfying $G = \biguplus_{i \in I} U_i$. Set $\tau_i := \mu_G(U_i)$ and note $\tau_i \leq \mu_G(U_i) \leq C \cdot \mu_G(U_i)$. Therefore, Lemma 6.7 shows that the family $(\tau_i^{1/2} K_{\lambda_i})_{i \in I}$ is a system of L_w^p-molecules in $\mathcal{K}_g$ and that the kernel H in Equation (6.5) is L_w^p-localized in $\mathcal{K}_g$ with $H \prec \Theta$. Furthermore, Lemma 6.5 shows that $(\tau_i^{1/2} K_{\lambda_i})_{i \in I}$ is a Bessel sequence and that the associated frame operator $\mathscr{S} : \mathcal{K}_g \to \mathcal{K}_g$ satisfies $\mathscr{S} = T_H|_{\mathcal{K}_g}$. Moreover, the choice of U (cf. Proposition 6.6) ensures that

$$-\varepsilon \|F\|_{L^2}^2 \leq \langle (\mathscr{S} - \mathrm{id}_{\mathcal{K}_g})F, F \rangle \leq \varepsilon \|F\|_{L^2}^2 \quad \text{for all} \quad F \in \mathcal{K}_g.$$

Since $\mathscr{S}$ is self-adjoint and $\mathscr{S} = T_H|_{\mathcal{K}_g}$, this yields $\|T_H - \mathrm{id}_{\mathcal{K}_g}\|_{\mathcal{K}_g \to L^2} = \|\mathscr{S} - \mathrm{id}_{\mathcal{K}_g}\|_{\mathcal{K}_g \to L^2} \leq \varepsilon$. By the choice of ε, this implies that

$$\mathscr{S}^{-1} = (T_H|_{\mathcal{K}_g})^{-1} = T_{H_1}|_{\mathcal{K}_g} \qquad \text{and} \qquad \mathscr{S}^{-1/2} = (T_H|_{\mathcal{K}_g})^{-1/2} = T_{H_2}|_{\mathcal{K}_g}$$

for suitable kernels $H_1, H_2 : G \times G \to \mathbb{C}$ that are L_w^p-localized in $\mathcal{K}_g$.

(i) Since $|V_g g| \in W_C(L_w^p)$ by assumption, it follows directly from the definitions that $(K_{\lambda_i})_{i \in I}$ is a system of L_w^p-molecules in $\mathcal{K}_g$. By Lemma 6.5, this implies that $(K_{\lambda_i})_{i \in I}$ is a Bessel family. Similarly, since $0 \leq \tau_i \leq \mu_G(\lambda_i U) = \mu_G(U)$ for all $i \in I$, it follows that $(\tau_i K_{\lambda_i})_{i \in I}$ is also a system of L_w^p-molecules in $\mathcal{K}_g$. By the above, $\mathscr{S}^{-1} = T_{H_1}|_{\mathcal{K}_g}$, and thus Lemma 6.4 shows that $(H_i)_{i \in I} := (\mathscr{S}^{-1}[\tau_i K_{\lambda_i}])_{i \in I}$ is also a system of L_w^p-molecules in $\mathcal{K}_g$, and hence a Bessel sequence by Lemma 6.5. Moreover, for arbitrary $F \in \mathcal{K}_g$,

$$F = \mathscr{S}^{-1} \left(\sum_{i \in I} \langle F, \tau_i^{1/2} K_{\lambda_i} \rangle \, \tau_i^{1/2} K_{\lambda_i} \right) = \sum_{i \in I} \langle F, K_{\lambda_i} \rangle \mathscr{S}^{-1}(\tau_i K_{\lambda_i}) = \sum_{i \in I} \langle F, K_{\lambda_i} \rangle H_i.$$

Since $(K_{\lambda_i})_{i \in I}$ and $(H_i)_{i \in I}$ are both Bessel families, this implies that they form a pair of dual frames.

(ii) Since $(\tau_i^{1/2} K_{\lambda_i})_{i \in I}$ is a system of L_w^p-molecules in $\mathcal{K}_g$, an application of Lemma 6.4 shows that the same is true for $(F_i)_{i \in I} := (\mathscr{S}^{-1/2}[\tau_i^{1/2} K_{\lambda_i}])_{i \in I} = (T_{H_2}[\tau_i^{1/2} K_{\lambda_i}])_{i \in I}$. The system $(F_i)_{i \in I}$ is a Parseval frame for $\mathcal{K}_g$. $\qquad \square$

6.4. **Dual Riesz sequences of molecules.** With notation as in Section 6.3, the aim of this section is to establish the existence of a Riesz sequence $(K_{\lambda_i})_{i \in I}$ of reproducing kernels

$$K_{\lambda_i} = L_{\lambda_i}^{\sigma}[V_g g]$$

whose biorthogonal system also forms a family of L_w^p-molecules in $\mathcal{K}_g := V_g(\mathcal{H}_\pi)$.

The first result shows the existence of almost tight Riesz sequences of reproducing kernels. This fact could be cited from the general result [50, Lemma 6.1], but the proof is included for the sake of completeness.

Proposition 6.9. *For every $\varepsilon \in (0,1)$, there exists a compact unit neighborhood $U \supset Q$ in G with the following property:*

If $\Lambda = (\lambda_i)_{i \in I}$ is any countable family in G that is U-separated, then the system $(K_{\lambda_i})_{i \in I}$ forms a Riesz sequence in $\mathcal{K}_g$ with lower Riesz bound $(1-\varepsilon)^2 \|V_g g\|_{L^2}^2$ and upper Riesz bound $(1+\varepsilon)^2 \|V_g g\|_{L^2}^2$.

Proof. Throughout the proof, let $\Phi := |V_g g| \in W_C(L_w^p)$ and define the normalized reproducing kernels $\widetilde{K}_y := K_y / \|V_g g\|_{L^2}$ for $y \in G$. We note that

$$|K_y(x)| = |L_y^\sigma[V_g g]| = L_y |V_g g|(x) = (L_y \Phi)(x) \quad \text{and hence} \quad \|\widetilde{K}_y\|_{L^2} = \frac{\|L_y \Phi\|_{L^2}}{\|V_g g\|_{L^2}} = 1. \quad (6.7)$$

The proof is split into three steps.

Step 1. *(Localized norm estimate).* The group G being second-countable, there exists an increasing sequence $(U_n)_{n \in \mathbb{N}}$ of compact sets $U_n \subset G$ such that $G = \bigcup_{n \in \mathbb{N}} U_n$. By Lemma 3.3, it follows that $\Phi \in W(L_w^p) \hookrightarrow L_w^2 \hookrightarrow L^2$. In addition, $\mathbb{1}_{U_n^c} \cdot \Phi \to 0$ as $n \to \infty$, with pointwise convergence, and $|\mathbb{1}_{U_n^c} \cdot \Phi|^2 \le |\Phi|^2 \in L^1$. Therefore, an application of Lebesgue's dominated convergence theorem yields the existence of $n_0 \in \mathbb{N}$ such that $\|\mathbb{1}_{U_{n_0}^c} \cdot \Phi\|_{L^2} \le \frac{\varepsilon}{2} \cdot \|V_g g\|_{L^2} = \frac{\varepsilon}{2} \cdot \|\Phi\|_{L^2}$.

Using Equation (6.7), we see for any measurable $U \supset U_{n_0}$ in G and any $y \in G$ that

$$\left| \|K_y\|_{L^2}^2 - \|K_y \cdot \mathbb{1}_{yU}\|_{L^2}^2 \right| \le \|K_y \cdot \mathbb{1}_{(yU_{n_0})^c}\|_{L^2}^2 \le \|(L_y \Phi) \cdot (L_y \mathbb{1}_{U_{n_0}^c})\|_{L^2}^2 = \|\Phi \cdot \mathbb{1}_{U_{n_0}^c}\|_{L^2}^2.$$

This, combined with the estimate $\|\mathbb{1}_{U_{n_0}^c} \cdot \Phi\|_{L^2} \le \frac{\varepsilon}{2} \|V_g g\|_{L^2}$, gives

$$\left| \|\widetilde{K}_y \cdot \mathbb{1}_{yU}\|_{L^2}^2 - 1 \right| = \|V_g g\|_{L^2}^{-2} \cdot \left| \|K_y\|_{L^2}^2 - \|K_y \cdot \mathbb{1}_{yU}\|_{L^2}^2 \right| \le \left(\frac{\varepsilon}{2} \right)^2.$$

Using that $\varepsilon \in (0,1)$, this easily implies

$$\left(1 - \frac{\varepsilon}{2}\right)^2 \le 1 - \left(\frac{\varepsilon}{2}\right)^2 \le \|\widetilde{K}_y \cdot \mathbb{1}_{yU}\|_{L^2}^2 \le 1 + \left(\frac{\varepsilon}{2}\right)^2 \le \left(1 + \frac{\varepsilon}{2}\right)^2 \quad (6.8)$$

for arbitrary $y \in G$ and any Borel set $U \supset U_{n_0}$.

Step 2. *(Construction of compact set).* For the sequence $(U_n)_{n \in \mathbb{N}}$ of Step 1, let $(\widetilde{U}_n)_{n \in \mathbb{N}}$ be defined by $\widetilde{U}_n := (U_n \overline{Q})^{-1}$. Since Φ is symmetric, it follows that if $x \in G$ is such that

$$0 \ne M^L((\Phi \cdot \mathbb{1}_{\widetilde{U}_n^c})^\vee)(x) = \|(\Phi \cdot \mathbb{1}_{\widetilde{U}_n^c})^\vee\|_{L^\infty(xQ)} = \|\Phi \cdot \mathbb{1}_{(U_n \overline{Q})^c}\|_{L^\infty(xQ)},$$

then $(U_n \overline{Q})^c \cap xQ \ne \varnothing$. Therefore, there exists $q \in Q$ such that $xq \in (U_n \overline{Q})^c$, which implies that $x \notin U_n$. Hence,

$$0 \le M^L((\Phi \cdot \mathbb{1}_{\widetilde{U}_n^c})^\vee) \le \mathbb{1}_{U_n^c} \cdot M^L(\Phi^\vee) = \mathbb{1}_{U_n^c} \cdot M^L \Phi \to 0 \quad \text{as} \quad n \to \infty,$$

with pointwise convergence. An application of Lebesgue's dominated convergence theorem yields therefore some $m \in \mathbb{N}$ such that $\|\Phi\|_{L^1} \|(\Phi \cdot \mathbb{1}_{\widetilde{U}_m^c})^\vee\|_{W^L(L^1)} < \|\Phi\|_{L^2} \cdot \mu_G(Q) \cdot (\varepsilon/2)^2$. Choose a symmetric $\varphi \in C_c(G)$ satisfying $0 \le \varphi \le 1$ and $\varphi \equiv 1$ on $\widetilde{U}_m$. Set $U' := \operatorname{supp} \varphi$. For U_{n_0} as in Step 1, define $U := \overline{Q} \cup U_{n_0} \cup U'$ and $\Theta := \Phi \cdot (1 - \varphi)$. Then $0 \le \Theta \le \Phi \cdot \mathbb{1}_{\widetilde{U}_m^c}$, and thus

$$\|\Theta\|_{L^1} \|\Theta\|_{W^L(L^1)} < \|\Phi\|_{L^2} \cdot \mu_G(Q) \cdot \left(\frac{\varepsilon}{2}\right)^2 \quad (6.9)$$

by construction.

Step 3. *(ε-Riesz sequences).* Let $\Lambda = (\lambda_i)_{i \in I}$ be a U-separated family in G. To ease notation, set $H_{\lambda_i} := \widetilde{K}_{\lambda_i} \cdot \mathbb{1}_{\lambda_i U}$ for $i \in I$. By construction of U, if $x \in (\lambda_i U)^c$, then $\lambda_i^{-1} x \notin U \supset \operatorname{supp} \varphi$, so that $1 - \varphi(\lambda_i^{-1} x) = 1$. Hence,

$$\mathbb{1}_{(\lambda_i U)^c} \leq L_{\lambda_i}(1 - \varphi) \quad \text{and} \quad |\widetilde{K}_{\lambda_i} - H_{\lambda_i}| = \|V_g g\|_{L^2}^{-1} \cdot |K_{\lambda_i}| \cdot \mathbb{1}_{(\lambda_i U)^c} \leq \|V_g g\|_{L^2}^{-1} \cdot L_{\lambda_i} \Theta.$$

For showing the Riesz inequalities, let $c = (c_i)_{i \in I} \in c_{00}(I)$ be arbitrary. Then the above estimate yields

$$\left| \left\| \sum_{i \in I} c_i \widetilde{K}_{\lambda_i} \right\|_{L^2} - \left\| \sum_{i \in I} c_i H_{\lambda_i} \right\|_{L^2} \right| \leq \left\| \sum_{i \in I} c_i \cdot \left(\widetilde{K}_{\lambda_i} - H_{\lambda_i} \right) \right\|_{L^2} \leq \|V_g g\|_{L^2}^{-1} \left\| \sum_{i \in I} |c_i| L_{\lambda_i} \Theta \right\|_{L^2}.$$

Since Λ is U-separated and $U \supset Q$, it follows that $\operatorname{Rel}(\Lambda) \leq 1$. Therefore, by Lemma 2.1 and Equation (6.9) and since $\Theta^{\vee} = \Theta$, it follows that

$$\left\| \sum_{i \in I} |c_i| L_{\lambda_i} \Theta \right\|_{L^2}^2 \leq (\mu_G(Q))^{-1} \cdot \|\Theta^{\vee}\|_{W^L(L^1)} \cdot \|\Theta\|_{L^1} \cdot \|c\|_{\ell^2}^2 \leq \|\Phi\|_{L^2} \cdot \left(\frac{\varepsilon}{2} \right)^2 \cdot \|c\|_{\ell^2}^2.$$

Combining the obtained inequalities thus gives

$$\left| \left\| \sum_{i \in I} c_i \widetilde{K}_{\lambda_i} \right\|_{L^2} - \left\| \sum_{i \in I} c_i H_{\lambda_i} \right\|_{L^2} \right| \leq \frac{\varepsilon}{2} \cdot \|c\|_{\ell^2}.$$

The family $(H_{\lambda_i})_{i \in I}$ of vectors $H_{\lambda_i} = \widetilde{K}_{\lambda_i} \cdot \mathbb{1}_{\lambda_i U}$ is orthogonal, since $(\lambda_i U)_{i \in I}$ is pairwise disjoint. Therefore, Equation (6.8) implies that

$$\left(1 - \frac{\varepsilon}{2} \right) \cdot \|c\|_{\ell^2} \leq \left\| \sum_{i \in I} c_i H_{\lambda_i} \right\|_{L^2} \leq \left(1 + \frac{\varepsilon}{2} \right) \cdot \|c\|_{\ell^2}.$$

Therefore, an application of the triangle inequality easily yields that

$$(1 - \varepsilon) \cdot \|c\|_{\ell^2} \leq \left\| \sum_{i \in I} c_i \widetilde{K}_{\lambda_i} \right\|_{L^2} \leq (1 + \varepsilon) \cdot \|c\|_{\ell^2},$$

which easily completes the proof. $\qquad \square$

The following theorem establishes the existence of Riesz sequences $(K_{\lambda_i})_{i \in I}$ that admit a biorthogonal system of L_w^p molecules.

Theorem 6.10. *There exists a compact unit neighborhood $U \supset Q$ with the following property: If $\Lambda = (\lambda_i)_{i \in I}$ is U-separated in G, the following assertions hold:*

   (i) *The family $(K_{\lambda_i})_{i \in I}$ is a Riesz sequence in $\mathcal{K}_g$ whose unique biorthogonal system $(H_i)_{i \in I}$ in $\overline{\operatorname{span}}\{K_{\lambda_i} : i \in I\}$ is a family of L_w^p-molecules in $\mathcal{K}_g$.*
   (ii) *There exists an orthonormal sequence $(F_i)_{i \in I}$ in $\overline{\operatorname{span}}\{K_{\lambda_i} : i \in I\}$ that is a system of L_w^p-molecules in $\mathcal{K}_g$.*

Proof. Throughout, let $\Theta := \|V_g g\|_{L^2}^{-2} \cdot (|V_g g| * |V_g g|)$. Then $\Theta \in W_C(L_w^p)$ by Corollary 3.9.

By Theorem 5.7, there exists $\varepsilon = \varepsilon(g, p, w, Q) \in (0, 1)$ such that, for any Q-separated family $\Lambda = (\lambda_i)_{i \in I}$ in G and any $A \in \mathcal{C}_w^p(\Lambda)$ satisfying $A \prec \Theta$ and $\|A - \operatorname{id}_{\ell^2(I)}\|_{\ell^2 \to \ell^2} \leq \varepsilon$, it holds that $A^{-1} \in \mathcal{C}_w^p(\Lambda)$ and $A^{-1/2} \in \mathcal{C}_w^p(\Lambda)$. Using Proposition 6.9, let $U \subset G$ be a compact set such that $Q \subset U$ and such that for every U-separated family $\Lambda = (\lambda_i)_{i \in I}$ in G, the family $(\widetilde{K}_{\lambda_i})_{i \in I}$ of normalized kernels $\widetilde{K}_{\lambda_i} := \|V_g g\|_{L^2}^{-1} \cdot K_{\lambda_i}$ satisfies the Riesz inequalities

$$\left(1 - \frac{\varepsilon}{3} \right)^2 \|c\|_{\ell^2}^2 \leq \left\| \sum_{i \in I} c_i \widetilde{K}_{\lambda_i} \right\|_{L^2}^2 \leq \left(1 + \frac{\varepsilon}{3} \right)^2 \|c\|_{\ell^2}^2 \tag{6.10}$$

for all $c \in \ell^2(I)$.

Let $\Lambda = (\lambda_i)_{i \in I}$ be an arbitrary U-separated family in G. Clearly, Λ is Q-separated. An application of Lemma 6.5 yields that the Gramian matrix $\widetilde{\mathscr{G}} = (\langle \widetilde{K}_{\lambda_{i'}}, \widetilde{K}_{\lambda_i} \rangle)_{i,i' \in I}$ of $(\widetilde{K}_{\lambda_i})_{i \in I}$ satisfies $\widetilde{\mathscr{G}} \in \mathcal{C}_w^p(\Lambda)$ with $\widetilde{\mathscr{G}} \prec \Theta$. Note that $\widetilde{\mathscr{G}} = \widetilde{\mathscr{C}} \circ \widetilde{\mathscr{D}}$, where $\widetilde{\mathscr{C}} = \widetilde{\mathscr{D}}^*$ and $\widetilde{\mathscr{D}}$ are, respectively, the analysis and synthesis operators associated to $(\widetilde{K}_{\lambda_i})_{i \in I}$. Using Equation (6.10), we see

$$|\langle (\widetilde{\mathscr{G}} - \mathrm{id}_{\ell^2(I)})c, c\rangle| = |\langle \widetilde{\mathscr{D}}c, \widetilde{\mathscr{D}}c \rangle - \|c\|_{\ell^2}^2| \le \max\left\{ (1 + \tfrac{\varepsilon}{3})^2 - 1, \quad 1 - (1 - \tfrac{\varepsilon}{3})^2 \right\} \cdot \|c\|_{\ell^2}^2$$
$$\le \varepsilon \cdot \|c\|_{\ell^2}^2$$

and hence $\|\mathrm{id}_{\ell^2(I)} - \widetilde{\mathscr{G}}\|_{\ell^2 \to \ell^2} \le \varepsilon$; here, we used that $\mathrm{id}_{\ell^2(I)} - \widetilde{\mathscr{G}}$ is self-adjoint. Thus, the choice of ε via Theorem 5.7 yields that $\widetilde{\mathscr{G}}^{-1} \in \mathcal{C}_w^p(\Lambda)$ and $\widetilde{\mathscr{G}}^{-1/2} \in \mathcal{C}_w^p(\Lambda)$.

(i) Let $\overline{\widetilde{\mathscr{G}}^{-1}} \in \mathcal{C}_w^p(\Lambda)$ be the element-wise conjugate matrix of $\widetilde{\mathscr{G}}^{-1}$. By Lemma 6.4, the family $(H_i)_{i \in I} := \overline{\widetilde{\mathscr{G}}^{-1}}(\widetilde{K}_{\lambda_{i'}})_{i' \in I}$ is a system of L_w^p-molecules. Moreover, Lemma 5.6 yields that the series $\widetilde{H}_i := \sum_{i' \in I} (\overline{\widetilde{\mathscr{G}}^{-1}})_{i,i'} \widetilde{K}_{\lambda_{i'}}$ is norm convergent in L^2, and thus $\widetilde{H}_i \in \overline{\mathrm{span}}\{K_{\lambda_{i'}} : i' \in I\}$. For $i \in I$, let $H_i := \|V_g g\|_{L^2}^{-1} \cdot \widetilde{H}_i$. Then also $H_i \in \overline{\mathrm{span}}\{K_{\lambda_{i'}} : i' \in I\}$, and a direct calculation entails

$$\langle H_{i'}, K_{\lambda_i} \rangle = \|V_g g\|_{L^2}^{-1} \Big\langle \sum_{\ell \in I} (\overline{\widetilde{\mathscr{G}}^{-1}})_{i',\ell} \widetilde{K}_{\lambda_\ell}, K_{\lambda_i} \Big\rangle = \sum_{i \in I} (\overline{\widetilde{\mathscr{G}}^{-1}})_{i',\ell} \langle \widetilde{K}_{\lambda_\ell}, \widetilde{K}_{\lambda_i} \rangle \qquad (6.11)$$

$$= \overline{\sum_{\ell \in I} (\widetilde{\mathscr{G}}^{-1})_{i',\ell} (\widetilde{\mathscr{G}})_{\ell,i}} = \overline{(\mathrm{id}_{\ell^2(I)})_{i',i}} = \delta_{i',i}, \qquad (6.12)$$

which shows the desired biorthogonality.

(ii) By similar arguments as in (i), it follows that the system $(F_i)_{i \in I} := \overline{\widetilde{\mathscr{G}}^{-1/2}}(\widetilde{K}_{\lambda_{i'}})_{i' \in I}$ is L_w^p-localized and that $F_i \in \overline{\mathrm{span}}\{K_{\lambda_{i'}} : i' \in I\}$. Since $\widetilde{\mathscr{G}} = \widetilde{\mathscr{G}}^*$, it follows that also $(\widetilde{\mathscr{G}}^{-1/2})^* = \widetilde{\mathscr{G}}^{-1/2}$, and thus $(\overline{\widetilde{\mathscr{G}}^{-1/2}})_{i,i'} = (\widetilde{\mathscr{G}}^{-1/2})_{i',i}$ for all $i, i' \in I$. Using this, together with $\widetilde{\mathscr{G}}\widetilde{\mathscr{G}}^{-1/2} = \widetilde{\mathscr{G}}^{-1/2}\widetilde{\mathscr{G}}$, it follows that

$$\langle F_{i'}, F_i \rangle = \sum_{j,j' \in I} \Big\langle (\overline{\widetilde{\mathscr{G}}^{-1/2}})_{i',j} \widetilde{K}_{\lambda_j}, (\overline{\widetilde{\mathscr{G}}^{-1/2}})_{i,j'} \widetilde{K}_{\lambda_{j'}} \Big\rangle = \sum_{j,j' \in I} (\widetilde{\mathscr{G}}^{-1/2})_{j,i'} (\widetilde{\mathscr{G}}^{-1/2})_{i,j'} (\widetilde{\mathscr{G}})_{j',j}$$

$$= \sum_{j \in I} (\widetilde{\mathscr{G}}^{-1/2}\widetilde{\mathscr{G}})_{i,j} (\widetilde{\mathscr{G}}^{-1/2})_{j,i'} = (\widetilde{\mathscr{G}}^{-1/2}\widetilde{\mathscr{G}}\widetilde{\mathscr{G}}^{-1/2})_{i,i'} = \delta_{i,i'}$$

for all $i, i' \in I$. Hence, the sequence $(F_i)_{i \in I}$ is orthonormal. $\square$

6.5. **Dual molecules for coorbit spaces.** The purpose of this section is to show that the canonical reproducing properties of molecular frames and Riesz sequences on $\mathcal{H}_\pi$ and $\ell^2(\Lambda)$ extend to coorbit spaces $\mathrm{Co}(Y)$ and associated sequence spaces $Y_d(\Lambda)$.

The first result shows that the analysis (resp. synthesis) operator associated to a system of molecules acts boundedly from (resp. into) coorbit spaces.

Proposition 6.11. *Let $w : G \to [1, \infty)$ be a p-weight for some $p \in (0, 1]$ and let $g \in \mathcal{B}_w^p$ be admissible. Let Y be an L_w^p-compatible quasi-Banach function space on G. Suppose that $(g_i)_{i \in I}$ is a family of L_w^p-molecules in $\mathcal{H}_\pi$ indexed by the relatively separated family $\Lambda = (\lambda_i)_{i \in I}$, with $(g_i)_{i \in I} \prec_\Lambda \Phi$ for a symmetric $\Phi \in W_C(L_w^p)$. Then the following hold:*

(i) The coefficient operator

$$\mathscr{C} : \quad \mathrm{Co}_g(Y) \to Y_d(\Lambda), \quad f \mapsto (\langle f, g_i \rangle)_{i \in I}$$

is well-defined and bounded with $\|\mathscr{C}\| \lesssim \|\Phi\|_{W(L_w^p)}$, with implied constant depending only on $g, Q, w, p, \mathrm{Rel}(\Lambda)$.

(ii) It holds $Y_d(\Lambda) \hookrightarrow (W^L(Y))_d(\Lambda) \hookrightarrow \ell^\infty_{1/w}(\Lambda)$, and the reconstruction operator

$$\mathscr{D}: \quad Y_d(\Lambda) \to \mathrm{Co}_g(Y), \quad c = (c_i)_{i\in I} \mapsto \sum_{i\in I} c_i g_i \qquad (6.13)$$

is well-defined and bounded, with the series (6.13) converging unconditionally in the weak-$*$-topology on $\mathcal{R}_w = \mathcal{R}_w(g)$ and with $\|\mathscr{D}\| \lesssim \|\Phi\|_{W(L^p_w)}$, where the implied constant only depends on g, Q, w, p, Y.

Proof. (i) Since $W(L^p_w) \hookrightarrow W^L(L^p_w) \hookrightarrow L^1_w(G)$ by Lemma 3.3, it follows from (6.3) that $g_i \in \mathcal{H}^1_w(g)$. Therefore, for every $f \in \mathcal{R}_w$ and $i \in I$, the pairing $\langle f, g_i \rangle_{\mathcal{R}_w, \mathcal{H}^1_w} \in \mathbb{C}$ is well-defined. Using Part (iii) of Lemma 4.6 and noting that Φ is symmetric, it follows from (6.3) that

$$|\langle f, g_i \rangle| = |\langle V_g f, V_g g_i \rangle_{L^\infty_{1/w}, L^1_w}| \le \int_G |V_g f(x)| \cdot \Phi(\lambda_i^{-1} x) \, d\mu_G(x)$$
$$= (|V_g f| * \Phi^\vee)(\lambda_i) = (|V_g f| * \Phi)(\lambda_i)$$

for every $i \in I$. If $x \in \lambda_i Q$, then $\lambda_i = xq$ for some $q \in Q$. Since Q is open and Φ is continuous, this implies $\Phi(y^{-1}\lambda_i) = \Phi(y^{-1}xq) \le M^L \Phi(y^{-1}x)$ for all $y \in G$. Thus,

$$|\langle f, g_i \rangle| \le [|V_g f| * \Phi](\lambda_i) = \int_G |V_g f(y)| \cdot \Phi(y^{-1}\lambda_i) \, d\mu_G(y)$$
$$\le \int_G |V_g f(y)| M^L \Phi(y^{-1}x) \, d\mu_G(y) = [|V_g f| * M^L \Phi](x)$$

for every $x \in \lambda_i Q$. Therefore,

$$\left\| (\langle f, g_i \rangle)_{i\in I} \right\|_{Y_d} = \left\| \sum_{i\in I} |\langle f, g_i \rangle| \mathbb{1}_{\lambda_i Q} \right\|_Y \le \left\| (|V_g f| * M^L \Phi) \sum_{i\in I} \mathbb{1}_{\lambda_i Q} \right\|_Y \le \mathrm{Rel}(\Lambda) \, \| \, |V_g f| * M^L \Phi \|_Y.$$

Since $\Phi \in W(L^p_w)$, it also holds $M^L \Phi \in W(L^p_w)$, with $\|M^L \Phi\|_{W(L^p_w)} \lesssim \|\Phi\|_{W(L^p_w)}$, where the implied constant only depends on p, w, Q. The convolution relation $W^L(Y) * W(L^p_w) \hookrightarrow W^L(Y)$ from condition (c2) of Definition 3.5 therefore yields that

$$\left\| (\langle f, g_i \rangle)_{i\in I} \right\|_{Y_d} \le \mathrm{Rel}(\Lambda) \cdot \| \, |V_g f| * M^L \Phi \|_Y$$
$$\lesssim_{Y, w, p, Q} \mathrm{Rel}(\Lambda) \cdot \|V_g f\|_{W^L(Y)} \cdot \|M^L \Phi\|_{W(L^p_w)}$$
$$\lesssim_{w, p, Q} \mathrm{Rel}(\Lambda) \cdot \|\Phi\|_{W(L^p_w)} \cdot \|f\|_{\mathrm{Co}_g(Y)},$$

and thus $\mathscr{C}: \mathrm{Co}_g(Y) \to Y_d(\Lambda)$ is well-defined and bounded, with $\|\mathscr{C}\| \lesssim \mathrm{Rel}(\Lambda) \cdot \|\Phi\|_{W(L^p_w)}$.

(ii) To verify the embedding $Y_d(\Lambda) \hookrightarrow (W^L(Y))_d(\Lambda)$, let $c = (c_i)_{i\in I} \in Y_d(\Lambda)$. Note for $x, y \in G$ and $i \in I$ satisfying $\mathbb{1}_{xQ}(y) \ne 0 \ne \mathbb{1}_{\lambda_i Q}(y)$ that $x \in yQ^{-1} \subset \lambda_i QQ$. Thus,

$$M^L \Big(\sum_{i\in I} |c_i| \mathbb{1}_{\lambda_i Q} \Big)(x) = \left\| \mathbb{1}_{xQ} \sum_{i\in I} |c_i| \mathbb{1}_{\lambda_i Q} \right\|_{L^\infty} \le \sum_{i\in I} |c_i| \mathbb{1}_{\lambda_i QQ}(x).$$

Since $Y_d(\Lambda, Q) = Y_d(\Lambda, QQ)$ with equivalent quasi-norms (where the implied constant only depends on Y and Q, but not on Λ; see Equation (2.3)), this shows that $\sum_{i\in I} |c_i| \mathbb{1}_{\lambda_i Q} \in W^L(Y)$ with $\| \sum_{i\in I} |c_i| \mathbb{1}_{\lambda_i Q} \|_{W^L(Y)} \le \| \sum_{i\in I} |c_i| \mathbb{1}_{\lambda_i QQ} \|_Y \lesssim \|c\|_{Y_d(\Lambda)}$. In addition, since Y is L^p_w-compatible, condition (c3) of Definition 3.5 implies that $W^L(Y) \hookrightarrow L^\infty_{1/w}$, which then implies the embedding $(W^L(Y))_d(\Lambda) \hookrightarrow (L^\infty_{1/w})_d(\Lambda) = \ell^\infty_{1/w}(\Lambda)$.

To prove the claim regarding the reconstruction operator $\mathscr{D}$, note that if $F: G \to \mathbb{C}$ is continuous, then since Q is open it holds that $|F(x)| = |F(qq^{-1}x)| \le (M^R F)(q^{-1}x)$ for all $q \in Q$.

Since $(g_i)_{i \in I} \prec \Phi$, we thus see $|V_g g_i(x)| \leq \Phi(\lambda_i^{-1} x) \leq (M^R \Phi)(q^{-1} \lambda_i^{-1} x) = (M^R \Phi)((\lambda_i q)^{-1} x)$ for all $q \in Q$. Integrating this estimate over $q \in Q$, we obtain

$$\mu_G(Q) \cdot |V_g g_i(x)| \leq \int_G \mathbb{1}_Q(q)(M^R\Phi)((\lambda_i q)^{-1}x) \, d\mu_G(q) = \int_G \mathbb{1}_Q(\lambda_i^{-1}\widetilde{q})(M^R\Phi)(\widetilde{q}^{-1}x) \, d\mu_G(\widetilde{q})$$
$$= (\mathbb{1}_{\lambda_i Q} * M^R\Phi)(x)$$

for every $x \in G$. For $c = (c_i)_{i \in I} \in Y_d(\Lambda)$, this implies

$$\mu_G(Q) \cdot \sum_{i \in I} |c_i| \, |V_g g_i| \leq \left(\sum_{i \in I} |c_i| \mathbb{1}_{\lambda_i Q} \right) * M^R\Phi.$$

An application of the convolution relation $W^L(Y) * W(L_w^p) \hookrightarrow W^L(Y)$ from condition (c1) of Definition 3.5 and of the embedding $Y_d(\Lambda) \hookrightarrow (W^L(Y))_d(\Lambda)$ from above shows that

$$\left\| \sum_{i \in I} |c_i| \, |V_g g_i| \right\|_{W^L(Y)} \lesssim_Q \left\| \left(\sum_{i \in I} |c_i| \, \mathbb{1}_{\lambda_i Q} \right) * M^R\Phi \right\|_{W^L(Y)}$$
$$\lesssim_{Y,p,w,Q} \left\| \sum_{i \in I} |c_i| \mathbb{1}_{\lambda_i Q} \right\|_{W^L(Y)} \|M^R\Phi\|_{W(L_w^p)}$$
$$\lesssim_{p,w,Q} \|c\|_{(W^L(Y))_d} \cdot \|\Phi\|_{W(L_w^p)}$$
$$\lesssim_{Q,Y} \|c\|_{Y_d(\Lambda)} \|\Phi\|_{W(L_w^p)}. \tag{6.14}$$

By the solidity of $W^L(Y)$, it follows that the map $\mathscr{D}_0 : Y_d(\Lambda) \to W^L(Y)$, $(c_i)_{i \in I} \mapsto \sum_{i \in I} c_i V_g g_i$ is well-defined and bounded.

For showing that $\mathscr{D}$ is well-defined and bounded, let $f \in \mathcal{H}_w^1 \subset \mathcal{H}_\pi$. Since $V_g : \mathcal{H}_\pi \to L^2(G)$ is an isometry,

$$|\langle g_i, f \rangle_{\mathcal{H}_\pi}| = |\langle V_g g_i, V_g f \rangle_{L^2(G)}| \leq \int_G |V_g g_i(x)| \cdot |V_g f(x)| \, d\mu_G(x).$$

This, combined with the embedding $W^L(Y) \hookrightarrow L_{1/w}^\infty(G)$ (see condition (c2) of Definition 3.5) and the estimate (6.14), gives

$$\sum_{i \in I} |c_i| \, |\langle g_i, f \rangle_{\mathcal{H}_\pi}| \leq \|V_g f\|_{L_w^1} \left\| \sum_{i \in I} |c_i| \cdot |V_g g_i| \right\|_{L_{1/w}^\infty} \lesssim \|f\|_{\mathcal{H}_w^1} \left\| \sum_{i \in I} |c_i| \cdot |V_g g_i| \right\|_{W^L(Y)}$$
$$\lesssim \|c\|_{Y_d(\Lambda)} \|f\|_{\mathcal{H}_w^1}.$$

The absolute convergence of $\sum_{i \in I} c_i \langle g_i, f \rangle$ for $f \in \mathcal{H}_\pi$ implies that $\mathscr{D}c = \sum_{i \in I} c_i g_i \in \mathcal{R}_w$ is well-defined and that the series defining $\mathscr{D}c$ converges unconditionally in the weak-$*$-topology on $\mathcal{R}_w$. The identity $V_g(\mathscr{D}c) = \sum_{i \in I} c_i V_g g_i = \mathscr{D}_0 c$ and the estimate (6.14) yield that $\|\mathscr{D}c\|_{\mathrm{Co}_g(Y)} = \|V_g(\mathscr{D}c)\|_{W^L(Y)} = \|\mathscr{D}_0 c\|_{W^L(Y)} \lesssim \|\Phi\|_{W(L_w^p)} \|c\|_{Y_d(\Lambda)}$, which completes the proof. $\qquad\square$

The method of proof used for Proposition 6.11 resembles the one of [50, Lemma A.3], with several modifications to apply the convolution relation (c1) of Definition 3.5.

The next result will be helpful for extending certain identities from $\mathcal{H}_\pi$ to $\mathcal{R}_w$ by density.

Lemma 6.12. *Let $(g_i)_{i \in I} \subset \mathcal{H}_\pi$ be a system of L_w^p-molecules indexed by the relatively separated family $\Lambda = (\lambda_i)_{i \in I}$. Then the associated reconstruction and coefficient operators $\mathscr{D} : \ell_{1/w}^\infty(I) \to \mathcal{R}_w(g)$ and $\mathscr{C} : \mathcal{R}_w(g) \to \ell_{1/w}^\infty(I)$ are well-defined, bounded, and weak-$*$-continuous.*

Proof. Throughout the proof, it will be used that L_w^1 and $L_{1/w}^\infty$ are L_w^p-compatible and that $\mathrm{Co}(L_w^1) = \mathcal{H}_w^1$ and $\mathrm{Co}(L_{1/w}^\infty) = \mathcal{R}_w$ (cf. Lemma 4.12) and that $(L_w^1)_d(\Lambda) = \ell_w^1(I)$ and $(L_{1/w}^\infty)_d(\Lambda) = \ell_{1/w}^\infty(I)$, with the interpretation $w(i) = w(\lambda_i)$ for $i \in I$ and similarly for $1/w$. (cf. Section 2.3).

An application of Proposition 6.11 with $Y = L_{1/w}^\infty$ (resp. $Y = L_w^1$) yields that the synthesis operator $\mathscr{D} : \ell_{1/w}^\infty(I) \to \mathcal{R}_w$ and the coefficient operator $\mathscr{C}_0 : \mathcal{H}_w^1 \to \ell_w^1(I), f \mapsto (\langle f, g_i \rangle)_{i \in I}$ are well-defined and bounded. Therefore, if $f \in \mathcal{H}_w^1$ and $c \in \ell_{1/w}^\infty(I)$, then

$$\langle \mathscr{D}c, f \rangle_{\mathcal{R}_w, \mathcal{H}_w^1} = \sum_{i \in I} c_i \langle g_i, f \rangle_{\mathcal{H}_\pi} = \sum_{i \in I} c_i \overline{\langle f, g_i \rangle_{\mathcal{R}_w, \mathcal{H}_w^1}} = \langle c, \mathscr{C}_0 f \rangle.$$

This easily implies that $\mathscr{D} : \ell_{1/w}^\infty \to \mathcal{R}_w$ is continuous if the domain and co-domain are equipped with the weak-$*$-topology.

Similarly, Proposition 6.11 implies that $\mathscr{C} : \mathcal{R}_w \to \ell_{1/w}^\infty(I)$ and the reconstruction operator $\mathscr{D}_0 : \ell_w^1(I) \to \mathcal{H}_w^1, (c_i)_{i \in I} \mapsto \sum_{i \in I} c_i\, g_i$ are well-defined and bounded. Since $c = \sum_{i \in I} c_i \delta_i$ with unconditional convergence in ℓ_w^1 for $c = (c_i)_{i \in I} \in \ell_w^1$, this implies that $\mathscr{D}_0 c = \sum_{i \in I} c_i g_i$ converges unconditionally in $\mathcal{H}_w^1$. Therefore, if $c \in \ell_w^1(I)$ and $f \in \mathcal{R}_w$, then

$$\langle f, \mathscr{D}_0 c \rangle = \sum_{i \in I} \langle f, g_i \rangle \, \overline{c_i} = \langle \mathscr{C}f, c \rangle.$$

This easily implies that the map $\mathscr{C} : \mathcal{R}_w \to \ell_{1/w}^\infty(I)$ is weak-$*$-continuous. $\qquad\square$

As a consequence, we can now show that certain reproducing formulas involving families of molecules that are valid on $\mathcal{H}_\pi$ (or even only on $\mathcal{H}_w^1$) extend to $\mathcal{R}_w$ (and hence to all coorbit spaces $\mathrm{Co}(Y)$).

Corollary 6.13. *Let $(g_i)_{i \in I}, (h_i)_{i \in I} \subset \mathcal{H}_\pi$ be two systems of L_w^p-molecules index by the relatively separated family $\Lambda = (\lambda_i)_{i \in I}$. Then the following hold:*

(i) If $f = \sum_{i \in I} \langle f, h_i \rangle g_i$ for all $f \in \mathcal{H}_w^1$, then the same holds for all $f \in \mathcal{R}_w$.

(ii) If $c_i = \langle \sum_{\ell \in I} c_\ell\, g_\ell, h_i \rangle_{\mathcal{H}_\pi}$ holds for all finitely supported sequences $c \in c_{00}(I)$ and all $i \in I$, then

$$c_i = \left\langle \sum_{\ell \in I} c_\ell\, g_\ell, h_i \right\rangle_{\mathcal{R}_w, \mathcal{H}_w^1}$$

holds for all $c \in \ell_{1/w}^\infty(I)$ and all $i \in I$.

Proof. (i) Lemma 4.3 shows that $\pi(G)g \subset \mathcal{H}_w^1$ is complete in $\mathcal{H}_w^1$; therefore, $\mathcal{H}_w^1 \hookrightarrow \mathcal{H}_\pi \hookrightarrow \mathcal{R}_w$ separates the points of $\mathcal{H}_w^1$; i.e., if $g \in \mathcal{H}_w^1$ satisfies $f(g) = \langle f, g \rangle_{\mathcal{H}_\pi} = 0$ for all $f \in \mathcal{H}_w^1$, then $g = 0$. In particular, this implies that $\mathcal{H}_w^1$ is weak-$*$-dense in $\mathcal{R}_w$; see, e.g., [1, Corollary 5.108].

Let $\mathscr{C}_g$ (resp. $\mathscr{C}_h$) and $\mathscr{D}_g$ (resp. $\mathscr{D}_h$) denote the coefficient and reconstruction operators associated to $(g_i)_{i \in I}$ (resp. $(h_i)_{i \in I}$). By assumption, it holds $f = \mathscr{D}_g \mathscr{C}_h f$ for all $f \in \mathcal{H}_w^1$. Since these operators are bounded between $\mathcal{R}_w$ and $\ell_{1/w}^\infty(I)$ and continuous with respect to the weak-$*$-topologies on $\mathcal{R}_w$ and $\ell_{1/w}^\infty$ (cf. Lemma 6.12), it follows that $f = \mathscr{D}_g \mathscr{C}_h f$ for all $f \in \mathcal{R}_w$.

(ii) Let $\mathscr{C}_g$ (resp. $\mathscr{C}_h$) and $\mathscr{D}_g$ (resp. $\mathscr{D}_h$) denote the coefficient and reconstruction operators associated to $(g_i)_{i \in I}$ (resp. $(h_i)_{i \in I}$). By Lemma 6.12, these operators are bounded between $\mathcal{R}_w$ and $\ell_{1/w}^\infty(I)$ and continuous with respect to the weak-$*$-topologies on $\mathcal{R}_w$ and $\ell_{1/w}^\infty$. By assumption, we have $c = \mathscr{C}_h \mathscr{D}_g$ for all $c \in c_{00}(I)$. Since $c_{00}(I)$ is weak-$*$-dense in $\ell_{1/w}^\infty(I)$, this implies the claim. $\qquad\square$

The following result establishes an atomic decomposition of the coorbit space $\mathrm{Co}(Y)$ in terms of two families of molecules.

Theorem 6.14. *Let w be a p-weight for some $p \in (0, 1]$ and let Y be a solid, translation-invariant quasi-Banach function space such that Y is L_w^p-compatible. Suppose $g \in \mathcal{B}_w^p$ is admissible.*

There exists a compact unit neighborhood $U \subset G$ such that for every relatively separated, U-dense family $\Lambda = (\lambda_i)_{i \in I}$ in G, the following properties hold:

(i) *There exists a family $(h_i)_{i \in I}$ of L_w^p-localized molecules in $\mathcal{H}_\pi$ indexed by Λ such that any $f \in \mathrm{Co}_g(Y)$ can be represented as*

$$f = \sum_{i \in I} \langle f, \pi(\lambda_i)g \rangle h_i = \sum_{i \in I} \langle f, h_i \rangle \pi(\lambda_i)g$$

with unconditional convergence of the series in the weak-$*$-topology on $\mathcal{R}_w(g)$.

(ii) *There exists a family $(g_i)_{i \in I}$ of L_w^p-localized molecules in $\mathcal{H}_\pi$ such that any $f \in \mathrm{Co}_g(Y)$ can be represented as*

$$f = \sum_{i \in I} \langle f, g_i \rangle g_i$$

with unconditional convergence of the series in the weak-$*$-topology on $\mathcal{R}_w(g)$.

Proof. By Theorem 6.8 and since $V_g : \mathcal{H}_\pi \to \mathcal{K}_g \subset L^2(G)$ is unitary and $K_\lambda = V_g[\pi(\lambda)g]$, there exists a compact unit neighborhood $U \subset G$ such that for every relatively separated, U-dense family Λ in G, there exists a system of (L_w^p, g)-localized molecules $(h_i)_{i \in I}$ (namely, $h_i = V_g^{-1}[H_i]$ with $(H_i)_{i \in I}$ as in Theorem 6.8) that forms a dual frame of $(\pi(\lambda_i)g)_{i \in I}$. The claim then follows by recalling that $\mathrm{Co}_g(Y) \subset \mathcal{R}_w$ and by combining Part (i) of Corollary 6.13 with Proposition 6.11. This shows (i).

The proof of (ii) is similar, using Part (ii) of Theorem 6.8 and Part (ii) of Corollary 6.13. $\square$

Dual to Theorem 6.14, we show the existence of dual Riesz sequences of molecules for coorbit spaces.

Theorem 6.15. *Let w be a p-weight for some $p \in (0, 1]$ and let Y be a solid, translation-invariant quasi-Banach function space such that Y is L_w^p-compatible. Suppose $g \in \mathcal{B}_w^p$ is admissible.*

There exists a compact unit neighborhood $U \subset G$ such that for every U-separated family $\Lambda = (\lambda_i)_{i \in I}$ in G, there exists a family $(h_i)_{i \in I} \subset \overline{\mathrm{span}}\{\pi(\lambda_i)g : i \in I\} \subset \mathcal{H}_\pi$ which forms a system of L_w^p-localized molecules and such that the moment problem

$$\langle f, \pi(\lambda_i)g \rangle = c_i, \quad i \in I,$$

admits the solution $f = \sum_{i \in I} c_i h_i \in \mathrm{Co}_g(Y)$ for any given sequence $(c_i)_{i \in I} \in Y_d(\Lambda)$.

Proof. By Theorem 6.10 and since $V_g : \mathcal{H}_\pi \to \mathcal{K}_g \subset L^2(G)$ is unitary and $K_\lambda = V_g[\pi(\lambda)g]$, there exists a compact unit neighborhood $U \subset G$ such that for any U-separated family $\Lambda = (\lambda_i)_{i \in I}$ in G, the family $(\pi(\lambda_i)g)_{i \in I}$ is a Riesz sequence in $\mathcal{H}_\pi$ with unique biorthogonal sequence $(h_i)_{i \in I}$ in $\overline{\mathrm{span}}\{\pi(\lambda_i)g : i \in I\}$ forming a family of (L_w^p, g) molecules in $\mathcal{H}_\pi$ indexed by Λ. Let $\mathscr{C}_g$ (resp. $\mathscr{C}_h$) and $\mathscr{D}_g$ (resp. $\mathscr{D}_h$) denote the coefficient and reconstruction operators associated to $(\pi(\lambda_i)g)_{i \in I}$ (resp. $(h_i)_{i \in I}$). Then $c = \mathscr{C}_g \mathscr{D}_h c$ for all $c \in c_{00}(\Lambda)$. By Part (ii) of Corollary 6.13, the same then holds for all $c \in \ell_{1/w}^\infty(I)$. Since $Y_d(\Lambda) \hookrightarrow \ell_{1/w}^\infty(I)$ by Proposition 6.11, this implies the claim. $\square$

7. APPLICATIONS

This section provides two applications of the existence of dual frames and Riesz sequences of molecules.

7.1. **Boundedness of operators.** The first result of this section is an extension of [36, Theorem 3.5] to quasi-Banach spaces.

Theorem 7.1. *Let w be a p-weight for some $p \in (0, 1]$ and let Y be a solid, translation-invariant quasi-Banach function space such that Y is L_w^p-compatible. For an admissible $g \in \mathcal{B}_w^p$, let $\Lambda = (\lambda_i)_{i \in I}$ be a relatively separated family in G such that $(\pi(\lambda_i)g)_{i \in I}$ is a frame for $\mathcal{H}_\pi$ with a dual frame $(h_i)_{i \in I}$ forming a family of L_w^p-molecules in $\mathcal{H}_\pi$.*

If $T : \mathcal{H}_w^1 \to \mathcal{R}_w$ is a bounded linear operator such that the vectors

$$m_i := T[\pi(\lambda_i)g], \quad i \in I,$$

define a system of L_w^p-molecules with envelope $\Phi \in W_C(L_w^p)$, then T extends to a well-defined bounded linear operator $T : \mathrm{Co}_g(Y) \to \mathrm{Co}_g(Y)$ with

$$\|T\|_{\mathrm{Co}(Y) \to \mathrm{Co}(Y)} \lesssim \|\Phi\|_{W(L_w^p)}, \tag{7.1}$$

where the implicit constant only depends on g, Q, w, p, Y, Λ, and $(h_i)_{i \in I}$.

Proof. Since $(\pi(\lambda_i)g)_{i \in I}$ and $(h_i)_{i \in I}$ are dual frames of molecules, it follows by Corollary 6.13 and because of $\mathrm{Co}(Y) \subset \mathcal{R}_w$ that any $f \in \mathrm{Co}(Y)$ can be represented as $f = \sum_{i \in I} \langle f, h_i \rangle \pi(\lambda_i)g$. In addition, Proposition 6.11 provides the estimate $\|\mathscr{C}_h\|_{Y_d(\Lambda)} = \|(\langle f, h_i \rangle)_{i \in I}\|_{Y_d(\Lambda)} \lesssim \|f\|_{\mathrm{Co}(Y)}$, where the implicit constant is independent of f (in fact, only depending on $Y, Q, \Lambda, (h_i)_{i \in I}$). With the family $(m_i)_{i \in I} = (T[\pi(\lambda_i)g])_{i \in I}$ from the statement of the theorem, Proposition 6.11 also shows that the reconstruction operator $\mathscr{D}_m : Y_d(\Lambda) \to \mathrm{Co}_g(Y), (c_i)_{i \in I} \mapsto \sum_{i \in I} c_i \, m_i$ is well-defined and bounded, with unconditional convergence of the series in the weak-$*$-topology of $\mathcal{R}_w \supset \mathrm{Co}_g(Y)$ and with $\|\mathscr{D}_m\|_{Y_d \to \mathrm{Co}(Y)} \lesssim \|\Phi\|_{W(L_w^p)}$, where the implied constant only depends on g, Q, w, p, Y.

Now, define $\widetilde{T} : \mathrm{Co}_g(Y) \to \mathrm{Co}_g(Y)$ by

$$\widetilde{T} := \mathscr{D}_m \circ \mathscr{C}_h, \quad \text{that is,} \quad \widetilde{T}f = \sum_{i \in I} \langle f, h_i \rangle m_i = \sum_{i \in I} \langle f, h_i \rangle T[\pi(\lambda_i)g].$$

As a composition of bounded linear operators, $\widetilde{T} : \mathrm{Co}(Y) \to \mathrm{Co}(Y)$ is itself a well-defined bounded linear operator, with

$$\|\widetilde{T}\|_{\mathrm{Co}(Y) \to \mathrm{Co}(Y)} \leq \|\mathscr{C}_h\| \cdot \|\mathscr{D}_m\| \lesssim \|\Phi\|_{W(L_w^p)},$$

where the implied constant only depends on g, Q, w, p, Y, Λ, and $(h_i)_{i \in I}$. This shows (7.1).

To prove that $\widetilde{T}$ is an extension of T, let $\mathscr{D}_g$ denote the reconstruction operator associated to the family $(\pi(\lambda_i)g)_{i \in I}$. Since $\mathcal{H}_w^1 = \mathrm{Co}_g(L_w^1)$ by Lemma 4.12 and since $(L_w^1)_d(\Lambda) = \ell_w^1(I)$, Proposition 6.11 shows that $\mathscr{D}_g : \ell_w^1(I) \to \mathcal{H}_w^1$ is well-defined and bounded. Since we have $c = \sum_{i \in I} c_i \, \delta_i$ with unconditional convergence in $\ell_w^1(I)$ for all $c \in \ell_w^1(I)$, it follows that the series defining $\mathscr{D}_g c = \sum_{i \in I} c_i \, \pi(\lambda_i)g$ converges unconditionally in $\mathcal{H}_w^1$ for any $c \in \ell_w^1(I)$. Since Proposition 6.11 also implies that $(\langle f, h_i \rangle)_{i \in I} \in \ell_w^1(I)$ for any $f \in \mathcal{H}_w^1 = \mathrm{Co}_g(L_w^1)$, and since $(h_i)_{i \in I}$ is a dual frame for $(\pi(\lambda_i)g)_{i \in I}$, we thus see $f = \sum_{i \in I} \langle f, h_i \rangle \pi(\lambda_i)g$ for all $f \in \mathcal{H}_w^1 \subset \mathcal{H}_\pi$, where the series converges unconditionally in $\mathcal{H}_w^1$. Because $T : \mathcal{H}_w^1 \to \mathcal{R}_w$ is a bounded linear operator, this implies

$$Tf = T\left(\sum_{i \in I} \langle f, h_i \rangle \pi(\lambda_i)g\right) = \sum_{i \in I} \langle f, h_i \rangle T[\pi(\lambda_i)g] = \mathscr{D}_m \mathscr{C}_h f = \widetilde{T}f$$

for all $f \in \mathcal{H}_w^1$. Thus, $\widetilde{T} : \mathrm{Co}_g(Y) \to \mathrm{Co}_g(Y)$ is indeed a bounded extension of T. $\qquad \square$

The proof of Theorem 7.1 follows the arguments in [36] closely.

Remark 7.2. With notation as in Theorem 7.1, the following hold:

(a) The existence of a frame $(\pi(\lambda_i)g)_{i \in I}$ for $\mathcal{H}_\pi$ with a dual frame $(h_i)_{i \in I}$ forming a family of (L_w^p, g)-molecules is guaranteed by Theorem 6.8, for Λ "sufficiently dense".

(b) The assumption that $T : \mathcal{H}_w^1 \to \mathcal{R}_w$ is bounded is satisfied, in particular, whenever $T : \mathcal{H}_\pi \to \mathcal{H}_\pi$ is bounded, since $\mathcal{H}_w^1 \hookrightarrow \mathcal{H}_\pi \hookrightarrow \mathcal{R}_w$ by Section 4.2.

7.2. Embeddings. In this section, we show that the embedding property $\mathrm{Co}_g(Y) \hookrightarrow \mathrm{Co}_g(Z)$ can be decided at the level of sequence spaces; namely, this embedding is valid if and only if $Y_d(\Lambda) \hookrightarrow Z_d(\Lambda)$. For Banach spaces, this result is already known; see [20, Theorem 8.4].

We start by showing that the boundedness of the embedding $Y_d(\Lambda) \hookrightarrow Z_d(\Lambda)$ does not depend on the choice of the relatively separated and relatively dense family Λ.

Lemma 7.3. *Let Y, Z be translation-invariant, solid quasi-Banach function spaces on G. Let $\Lambda = (\lambda_i)_{i \in I} \subset G$ be relatively separated and relatively dense and assume that $Y_d(\Lambda) \hookrightarrow Z_d(\Lambda)$. Then, if $\Gamma = (\gamma_j)_{j \in J} \subset G$ is any relatively separated family, the embedding $Y_d(\Gamma) \hookrightarrow Z_d(\Gamma)$ holds.*

Proof. Since Λ is relatively dense, there exists a relatively compact unit neighborhood $U \subset G$ satisfying $\sum_{i \in I} \mathbb{1}_{\lambda_i U}(x) \geq 1$ for all $x \in G$. Let $c = (c_j)_{j \in J} \in Y_d(\Gamma)$ be arbitrary and note for $e_i := \sum_{j \in J} |c_j| \mathbb{1}_{\gamma_j Q \cap \lambda_i U \neq \varnothing}$ that

$$0 \leq \sum_{j \in J} |c_j| \, \mathbb{1}_{\gamma_j Q}(x) \leq \sum_{i \in I} \left[\mathbb{1}_{\lambda_i U}(x) \sum_{j \in J} |c_j| \, \mathbb{1}_{\gamma_j Q}(x) \, \mathbb{1}_{\lambda_i U}(x) \right] \leq \sum_{i \in I} e_i \, \mathbb{1}_{\lambda_i U}(x). \tag{7.2}$$

Let $V := QU^{-1}Q$; we claim for arbitrary $x \in G$, $i \in I$, and $j \in J$ that

$$\mathbb{1}_{\lambda_i Q}(x) \mathbb{1}_{\gamma_j Q \cap \lambda_i U \neq \varnothing} \leq \mathbb{1}_{\lambda_i Q}(x) \mathbb{1}_{\gamma_j V}(x).$$

Indeed, this is trivial if the left-hand side vanishes. If not, then $x = \lambda_i q$ for some $q \in Q$ and there exists some $y = \lambda_i u \in \gamma_j Q \cap \lambda_i U$. This implies

$$x = \lambda_i q = \lambda_i u u^{-1} q = y u^{-1} q \in \gamma_j Q U^{-1} Q = \gamma_j V,$$

from which the claimed estimate follows easily. Combining this estimate with the definition of e_i, it follows that

$$0 \leq \sum_{i \in I} e_i \mathbb{1}_{\lambda_i Q}(x) = \sum_{j \in J} \left[|c_j| \sum_{i \in I} \mathbb{1}_{\lambda_i Q}(x) \mathbb{1}_{\gamma_j Q \cap \lambda_i U \neq \varnothing} \right]$$

$$\leq \sup_{x \in G} \left[\sum_{i \in I} \mathbb{1}_{\lambda_i Q}(x) \right] \cdot \sum_{j \in J} |c_j| \mathbb{1}_{\gamma_j V}(x) = \mathrm{Rel}(\Lambda) \cdot \sum_{j \in J} |c_j| \mathbb{1}_{\gamma_j V}(x);$$

for arbitrary $x \in G$.

Lastly, let us write $\| \cdot \|_{Y_d(\Lambda,U)}$ to denote the (quasi)-norm on the discrete sequence space $Y_d(\Lambda)$, but using the set U instead of Q in Equation (2.2); similar notation will also be used for Z_d. As noted in Section 2.4, since Y, Z are translation-invariant, different choices of relatively compact unit neighborhoods for the set Q yield equivalent (quasi)-norms. By combining Equation (7.2) with the solidity of Z, then using the embedding $Y_d(\Lambda) \hookrightarrow Z_d(\Lambda)$ and finally combining the previous estimate with the solidity of Y, we thus see

$$\|c\|_{Z_d(\Gamma)} = \left\| \sum_{j \in J} |c_j| \, \mathbb{1}_{\gamma_j Q} \right\|_Z \leq \left\| \sum_{i \in I} e_i \mathbb{1}_{\lambda_i U} \right\|_Z$$

$$= \|e\|_{Z_d(\Lambda,U)} \lesssim \|e\|_{Z_d(\Lambda)} \lesssim \|e\|_{Y_d(\Lambda)} = \left\| \sum_{i \in I} e_i \, \mathbb{1}_{\lambda_i Q} \right\|_Y$$

$$\leq \mathrm{Rel}(\Lambda) \left\| \sum_{j \in J} |c_j| \, \mathbb{1}_{\gamma_j V} \right\|_Y = \mathrm{Rel}(\Lambda) \, \|c\|_{Y_d(\Gamma,V)} \lesssim \|c\|_{Y_d(\Gamma)}.$$

This shows that $Y_d(\Gamma) \hookrightarrow Z_d(\Gamma)$ and thus completes the proof. $\square$

Using the previous lemma, we can now state and prove the main result of this subsection.

Theorem 7.4. *Let $w : G \to [1, \infty)$ be a p-weight for some $p \in (0, 1]$ and let $g \in \mathcal{B}_w^p$ be admissible. Let Y, Z be solid, translation-invariant quasi-Banach spaces on G that are both L_w^p-compatible.*

Then the embedding $\mathrm{Co}_g(Y) \hookrightarrow \mathrm{Co}_g(Z)$ *holds if and only if* $Y_d(\Lambda) \hookrightarrow Z_d(\Lambda)$ *holds for some (every) relatively separated and relatively dense family* $\Lambda = (\lambda_i)_{i \in I} \subset G$.

Proof. First, suppose that the identity map $\iota : \mathrm{Co}_g(Y) \to \mathrm{Co}_g(Z), f \mapsto f$ is well-defined and bounded. Choose a compact unit neighborhood $U \subset G$ as provided by Theorem 6.15. Let $\Lambda = (\lambda_i)_{i \in I}$ be a maximal U-separated family in G. For every $x \in G$ we then either have $x = \lambda_i$ for some $i \in I$ (and hence $x \in \bigcup_{i \in I} \lambda_i U U^{-1}$) or $x \notin \{\lambda_i : i \in I\}$ and then $xU \cap \lambda_i U \neq \varnothing$ for some $i \in I$, by maximality; hence, $x \in \lambda_i U U^{-1}$. Overall, this shows that $G = \bigcup_{i \in I} \lambda_i U U^{-1}$, which easily implies that Λ is relatively separated and relatively dense in G.

Theorem 6.15 yields a family $(h_i)_{i \in I} \subset \mathcal{H}_\pi$ that forms a system of L_w^p-localized molecules and such that $c = \mathscr{C}_g \mathscr{D}_h c$ for every $c \in Y_d(\Lambda)$, where $\mathscr{C}_g$ and $\mathscr{D}_h$ are the coefficient and reconstruction operators associated to $(\pi(\lambda_i)g)_{i \in I}$ and $(h_i)_{i \in I}$, respectively. By Proposition 6.11, we know that $\mathscr{D}_h : Y_d(\Lambda) \to \mathrm{Co}_g(Y)$ and $\mathscr{C}_g : \mathrm{Co}_g(Z) \to Z_d(\Lambda)$ are bounded. Hence, for every $c \in Y_d(\Lambda)$,

$$\|c\|_{Z_d(\Lambda)} = \|\mathscr{C}_g \iota \mathscr{D}_h c\|_{Z_d(\Lambda)} \leq \|\mathscr{C}_g\|_{\mathrm{Co}_g(Z) \to Z_d} \cdot \|\iota\|_{\mathrm{Co}_g(Y) \to \mathrm{Co}_g(Z)} \cdot \|\mathscr{D}_h\|_{Y_d(\Lambda) \to \mathrm{Co}_g(Y)} \cdot \|c\|_{Y_d(\Lambda)} < \infty.$$

This proves that $Y_d(\Lambda) \hookrightarrow Z_d(\Lambda)$. Since Λ is relatively separated and relatively dense in G, Lemma 7.3 shows that in fact $Y_d(\Gamma) \hookrightarrow Z_d(\Gamma)$ for every relatively separated family Γ in G.

Conversely, suppose that the identity map $\iota : Y_d(\Lambda) \to Z_d(\Lambda)$ is well-defined and bounded for some (hence, every) relatively separated and relatively dense family Λ in G by Lemma 7.3. Theorem 6.14 shows that there exists a relatively separated and relatively dense set $\Lambda = (\lambda_i)_{i \in I}$ in G and a family $(g_i)_{i \in I}$ of L_w^p-localized molecules in $\mathcal{H}_\pi$ indexed by Λ such that $f = \mathscr{D}_g \mathscr{C}_g f$ for all $f \in \mathrm{Co}_g(Y)$. By Proposition 6.11, it follows that that $\mathscr{C}_g : \mathrm{Co}_g(Y) \to Y_d(\Lambda)$ and $\mathscr{D}_g : Z_d(\Lambda) \to \mathrm{Co}_g(Z)$ are well-defined and bounded. Hence, we see for any $f \in \mathrm{Co}_g(Y)$ that $f = \mathscr{D}_g \iota \mathscr{C}_g f \in \mathrm{Co}_g(Z)$ with

$$\|f\|_{\mathrm{Co}_g(Z)} = \|\mathscr{D}_g \iota \mathscr{C}_g f\|_{\mathrm{Co}_g(Z)} \leq \|\mathscr{D}_g\|_{Z_d(\Lambda) \to \mathrm{Co}_g(Z)} \cdot \|\iota\|_{Y_d(\Lambda) \to Z_d(\Lambda)} \cdot \|\mathscr{C}_g\|_{\mathrm{Co}_g(y) \to Y_d(\Lambda)}.$$

This shows that $\mathrm{Co}_g(Y) \hookrightarrow \mathrm{Co}_g(Z)$. $\square$

8. Main results for irreducible, square-integrable representations

This section provides an overview of the key results of this paper for the case of irreducible, square-integrable representations. Several of the results obtained in the main text have simplified statements for such representations, which recover known results for Banach spaces. We include these statements here to allow for an easy reference.

In what follows, we will always assume that the following assumptions are met.

Assumption 8.1. Assume that the following hold:

(a1) G is a second countable, locally compact group.

(a2) $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ is an irreducible unitary representation of G on a separable Hilbert space $\mathcal{H}_\pi \neq \{0\}$.

(a3) $w : G \to [1, \infty)$ is a p-weight for $p \in (0, 1]$ (see Definition 3.1).

(a4) The Wiener amalgam spaces W^L, W^R, W (see Section 2.6) are defined relatively to a fixed open, symmetric, relatively compact unit neighborhood $Q \subset G$.

(a5) With the coefficient transform $V_g f$ as defined in Section 4.1, it is assumed that $\mathcal{B}_w^p \neq \{0\}$, where
$$\mathcal{B}_w^p := \{g \in \mathcal{H}_\pi : V_g g \in W(L_w^p)\},$$
and we fix some $g \in \mathcal{B}_w^p \setminus \{0\}$.

(a6) The space $Y \subset L^0(G)$ is assumed to be a solid, translation-invariant quasi-Banach function space with p-norm $\|\cdot\|_Y$. In addition, Y is assumed to be L_w^p-compatible (see Definition 3.5).

8.1. **Admissibility.** Since w is a p-weight (in particular, $w \geq 1$), it follows from an application of Lemma 3.3 that $W(L_w^p) \hookrightarrow W^L(L_w^p) \hookrightarrow L_w^1 \cap L_w^\infty \hookrightarrow L^2$, so that every $h \in \mathcal{B}_w^p$ satisfies $\int_G |\langle h, \pi(x)h \rangle|^2 \, d\mu_G(x) < \infty$, i.e., the representation π is *square-integrable*. Then, by Theorem 4.2, there exists a unique, self-adjoint, positive operator $C_\pi : \mathrm{dom}(C_\pi) \to \mathcal{H}_\pi$ such that

$$\int_G |\langle f, \pi(x)h \rangle|^2 \, d\mu_G(x) = \|C_\pi h\|_{\mathcal{H}_\pi}^2 \|f\|_{\mathcal{H}_\pi}^2 \tag{8.1}$$

for all vectors $f, h \in \mathcal{H}_\pi$. The domain of C_π is given by $\mathrm{dom}(C_\pi) = \{h \in \mathcal{H}_\pi : V_h h \in L^2(G)\}$. This implies that $\mathcal{B}_w^p \subset \mathrm{dom}(C_\pi)$. Furthermore, Equation (8.1) implies that any vector in $\mathrm{dom}(C_\pi) \setminus \{0\}$ can be normalized to obtain an admissible vector h for π, i.e., such that $V_h : \mathcal{H}_\pi \to L^2(G)$ is an isometry.

8.2. **Coorbit spaces.** With the fixed vector $g \in \mathcal{B}_w^p \setminus \{0\}$ from Assumption 8.1, define

$$\mathcal{H}_w^1 := \mathcal{H}_w^1(g) := \{f \in \mathcal{H}_\pi : V_g f \in L_w^1(G)\}$$

and equip it with the norm $\|f\|_{\mathcal{H}_w^1} = \|f\|_{\mathcal{H}_w^1(g)} = \|V_g f\|_{L_w^1}$. Note that $W(L_w^p) \hookrightarrow L_w^1$ by Lemma 3.3, so that $\mathcal{B}_w^p \subset \mathcal{H}_w^1$; in particular, $g \in \mathcal{H}_w^1$.

Although the space $\mathcal{H}_w^1(g)$ is defined relative to the fixed vector $g \in \mathcal{B}_w^p \setminus \{0\}$, the following lemma shows that the space $\mathcal{H}_w^1$ is actually independent of the choice of $g \in \mathcal{B}_w^p \setminus \{0\}$. This crucially uses that π is an *irreducible* representation; see also Remark 4.4.

Lemma 8.2. *Under Assumption 8.1, the following hold:*

    (i) If $h \in \mathcal{B}_w^p \setminus \{0\}$, then $\mathcal{H}_w^1(g) = \mathcal{H}_w^1(h)$ with $\|\cdot\|_{\mathcal{H}_w^1(g)} \asymp \|\cdot\|_{\mathcal{H}_w^1(h)}$.
    (ii) The space $\mathcal{H}_w^1$ is a π-invariant Banach space satisfying $\mathcal{H}_w^1 \hookrightarrow \mathcal{H}_\pi$.

Proof. As explained in Section 8.1, we have $\mathcal{B}_w^p \subset \mathrm{dom}(C_\pi)$, and the normalized vector $\widetilde{g} = g/\|C_\pi g\|_{\mathcal{H}_\pi}$ is admissible for π. Since clearly $\mathcal{H}_w^1(g) = \mathcal{H}_w^1(\widetilde{g})$, we can assume that g is admissible and similarly (for proving Part (i)) also that h is admissible.

(i) Since $L_w^1 * L_w^1 \hookrightarrow L_w^1$ (see, e.g., [49, Section 3.7]), it follows by an application of Lemma 4.5 that the collection $\mathcal{C}_{L_w^1} = \{f \in \mathcal{H}_\pi : V_f f \in L_w^1\}$ is a vector space and that $V_{f_2} f_1 \in L_w^1$ for all $f_1, f_2 \in \mathcal{C}_{L_w^1}$. In particular, this implies that $V_g h, V_h g \in L_w^1(G)$, so that the conclusion follows from Lemma 4.3.

Assertion (ii) follows directly from Lemma 4.3. $\qquad\square$

Let $\mathcal{R}_w = (\mathcal{H}_w^1)^*$ be the *anti*-dual space of $\mathcal{H}_w^1(g)$. Denote the associated conjugate-linear pairing by

$$\langle f, h \rangle := f(h), \quad f \in \mathcal{R}_w, \ h \in \mathcal{H}_w^1.$$

The action π of G on $\mathcal{H}_w^1$ can be naturally extended to act on an element $f \in \mathcal{R}_w$ by

$$\pi(x)f : \quad \mathcal{H}_w^1 \to \mathbb{C}, \quad x \mapsto f([\pi(x)]^* h), \quad h \in \mathcal{H}_w^1,$$

for $x \in G$. The associated matrix coefficients are defined as $V_h f : G \to \mathbb{C}, \ x \mapsto \langle f, \pi(x)h \rangle$.

For $g \in \mathcal{B}_w^p \setminus \{0\}$, the *coorbit space* of Y is the collection

$$\mathrm{Co}(Y) = \mathrm{Co}_g(Y) = \{f \in \mathcal{R}_w : V_g f \in W^L(Y)\},$$

where $W^L(Y)$ is the left-sided Wiener amalgam space of Y (cf. Section 2.6). The space is equipped with the norm $\|f\|_{\mathrm{Co}(Y)} = \|V_g f\|_{W^L(Y)}$.

As for the space $\mathcal{H}_w^1$ (cf. Lemma 8.2), also the coorbit spaces are independent of the choice of the analyzing vector $g \in \mathcal{B}_w^p \setminus \{0\}$, whenever π is irreducible.

Proposition 8.3. *Under Assumption 8.1, the following assertions hold:*

    (i) If $h \in \mathcal{B}_w^p \setminus \{0\}$, then $\mathrm{Co}_g(Y) = \mathrm{Co}_h(Y)$ with $\|\cdot\|_{\mathrm{Co}_g(Y)} \asymp \|\cdot\|_{\mathrm{Co}_h(Y)}$.
    (ii) The space $\mathrm{Co}(Y)$ is a π-invariant quasi-Banach space with p-norm $\|\cdot\|_{\mathrm{Co}(Y)}$ and satisfies $\mathrm{Co}(Y) \hookrightarrow \mathcal{R}_w$.

Proof. As in the proof of Lemma 8.2, the normalized vector $\widetilde{g} := g/\|C_\pi g\|_{\mathcal{H}_\pi}$ is admissible for π, and (for proving Part (i)) the same holds for the normalized version of h. Furthermore, we clearly have $\mathrm{Co}_g(Y) = \mathrm{Co}_{\widetilde{g}}(Y)$ and similarly for h. Hence, we can assume that g and h are admissible.

(i) By Corollary 3.9, it holds that $W(L_w^p) * W(L_w^p) \hookrightarrow W(L_w^p)$. Since $\mathcal{B}_w^p = \mathcal{C}_{W(L_w^p)}$, where $\mathcal{C}_{W(L_w^p)}$ is as defined in Lemma 4.5, an application of that result yields that $V_h g \in W(L_w^p)$. The equivalence $\| \cdot \|_{\mathrm{Co}_g(Y)} \asymp \| \cdot \|_{\mathrm{Co}_h(Y)}$ follows therefore from Proposition 4.8.

Assertion (ii) is a direct consequence of Proposition 4.8. $\qquad\square$

8.3. Molecular decompositions.
This subsection provides self-contained statements of the main results on molecular decompositions.

A family $(h_i)_{i \in I}$ of vector $h_i \in \mathcal{H}_\pi$, indexed by a relatively separated $\Lambda = (\lambda_i)_{i \in I} \subset G$, is called a *system of L_w^p-molecules*, if there exists an envelope $\Phi \in W_C(L_w^p)$ satisfying

$$|V_g h_i(x)| \leq \Phi(\lambda_i^{-1} x) \tag{8.2}$$

for all $i \in I$ and $x \in G$.

The following result summarizes the main properties of a system of molecules.

Lemma 8.4. *Under Assumption 8.1, the following hold:*

(i) *The notion of a system of L_w^p-molecules is independent of the choice of $g \in \mathcal{B}_w^p \setminus \{0\}$, i.e., if $(h_i)_{i \in I} \subset \mathcal{H}_\pi$ satisfies condition (8.2) for some $\Phi \in W_C(L_w^p)$ and if $h \in \mathcal{B}_w^p \setminus \{0\}$ is arbitrary, then there exists $\Theta \in W_C(L_w^p)$ satisfying*

$$|V_h h_i(x)| \leq \Theta(\lambda_i^{-1} x)$$

for all $i \in I$ and $x \in G$.

(ii) *If $(h_i)_{i \in I} \subset \mathcal{H}_\pi$ is a system of L_w^p-molecules indexed by the relatively separated family $\Lambda = (\lambda_i)_{i \in I}$, then $(h_i)_{i \in I} \subset \mathcal{H}_w^1$ and the associated coefficient operator*

$$\mathscr{C} : \ \mathcal{R}_w \to \ell_{1/w}^\infty(I), \ \ f \mapsto \left(\langle f, h_i \rangle_{\mathcal{R}_w, \mathcal{H}_w^1} \right)_{i \in I}$$

and reconstruction operator

$$\mathscr{D} : \ \ell_{1/w}^\infty(I) \to \mathcal{R}_w, \ \ (c_i)_{i \in I} \mapsto \sum_{i \in I} c_i h_i$$

are well-defined and bounded, with unconditional convergence of the series in the weak-$*$-topology on $\mathcal{R}_w$.

(iii) *The following restrictions of the operators $\mathscr{C}$ and $\mathscr{D}$ are well-defined and bounded:*
  (a) *$\mathscr{C} : \mathcal{H}_\pi \to \ell^2(I)$, $\mathscr{C} : \mathcal{H}_w^1 \to \ell_w^1(I)$, and $\mathscr{C} : \mathrm{Co}(Y) \to Y_d(\Lambda)$.*
  (b) *$\mathscr{D} : \ell^2(I) \to \mathcal{H}_\pi$, $\mathscr{D} : \ell_w^1(I) \to \mathcal{H}_w^1$, and $\mathscr{D} : Y_d(\Lambda) \to \mathrm{Co}(Y)$.*
  Here, the space $Y_d(\Lambda) \hookrightarrow \ell_{1/w}^\infty(I)$ is as defined in Section 2.4.

Proof. (i) If $(h_i)_{i \in I}$ is a family of L_w^p-molecules with respect to the window g, then the same holds with respect to the normalized (admissible) window $\widetilde{g} = g/\|C_\pi g\|_{\mathcal{H}_\pi}$, and vice versa. Therefore, we can assume that both $g, h \in \mathcal{B}_w^p \setminus \{0\}$ are admissible. By Corollary 3.9, it holds that $W(L_w^p) * W(L_w^p) \hookrightarrow W(L_w^p)$. Since $\mathcal{B}_w^p = \mathcal{C}_{W(L_w^p)}$, where $\mathcal{C}_{W(L_w^p)}$ is as defined in Lemma 4.5, an application of that result yields that $V_h g, V_g h \in W(L_w^p)$. Therefore, Lemma 6.3 yields the claimed independence.

Assertions (ii) and (iii) follow from Proposition 6.11. For this, note that Lemma 3.3 shows that $\Phi \in W(L_w^p) \hookrightarrow W^L(L_w^p) \hookrightarrow L_w^1$. Because of Equation (8.2) and the translation-invariance of L_w^1, this easily implies $V_g h_i \in L_w^1$ and hence $h_i \in \mathcal{H}_w^1$ for all $i \in I$. Lemma 4.12 shows that $L^2(G)$, $L_w^1(G)$ and $L_{1/w}^\infty$ are all L_w^p-compatible and that $\mathrm{Co}(L^2) = \mathcal{H}_\pi$, $\mathrm{Co}(L_w^1) = \mathcal{H}_w^1$, and $\mathrm{Co}(L_{1/w}^\infty) = \mathcal{R}_w$. Therefore, Proposition 6.11 implies all the stated properties. $\qquad\square$

The next result shows that the orbit $\pi(G)g$ always contains discrete subsystems that form a frame for $\mathcal{H}_\pi$ and admit a dual system consisting of L_w^p-molecules.

Theorem 8.5. *Under Assumption 8.1, there exists a compact unit neighborhood $U \subset G$ such that for every relatively separated, U-dense family $\Lambda = (\lambda_i)_{i \in I}$ in G there exists a family $(h_i)_{i \in I}$ of vectors $h_i \in \mathcal{H}_w^1$ with the following properties:*

(i) *The system $(h_i)_{i \in I}$ forms a system of L_w^p-molecules.*
(ii) *Any $f \in \mathrm{Co}(Y)$ can be represented as*

$$f = \sum_{i \in I} \langle f, h_i \rangle \, \pi(\lambda_i)g = \sum_{i \in I} \langle f, \pi(\lambda_i)g \rangle \, h_i,$$

with unconditional convergence of the series in the weak-*-topology on $\mathcal{R}_w$.

Proof. Both claims follow from Theorem 6.14, strictly speaking after replacing g by the admissible vector $\widetilde{g} = g/\|C_\pi g\|_{\mathcal{H}_\pi}$. $\square$

A complementing result on dual Riesz sequences of molecules in $\mathcal{H}_\pi$ is provided by the following.

Theorem 8.6. *Under Assumption 8.1, there exists a compact unit neighborhood $U \subset G$ such that for every U-separated family $\Lambda = (\lambda_i)_{i \in I}$ in G there exists a family $(h_i)_{i \in I} \subset \mathcal{H}_w^1$ of vectors $h_i \in \overline{\mathrm{span}}\{\pi(\lambda_i)g : i \in I\} \subset \mathcal{H}_\pi$ with the following properties:*

(i) *The system $(h_i)_{i \in I}$ forms a system of L_w^p-molecules.*
(ii) *For any sequence $(c_i)_{i \in I} \in Y_d(\Lambda)$, the vector $f = \sum_{i \in I} c_i h_i$ is an element of $\mathrm{Co}(Y)$ that solves the moment problem*

$$\langle f, \pi(\lambda_i)g \rangle = c_i, \quad i \in I.$$

Proof. Both claims follow from Theorem 6.15, strictly speaking after replacing g by the admissible vector $\widetilde{g} = g/\|C_\pi g\|_{\mathcal{H}_\pi}$. $\square$

Appendix A. Miscellany on quasi-Banach spaces

This appendix contains two auxiliary results on quasi-Banach spaces used throughout the main text. As no reference could be found in the literature, their proofs are provided.

Lemma A.1. *Let Y be a quasi-normed vector space with p-norm $\|\cdot\|$ for some $p \in (0, 1]$. If Y is complete, then for every countable family $(f_i)_{i \in I}$ in Y satisfying $\sum_{i \in I} \|f_i\|^p < \infty$, the series $\sum_{i \in I} f_i$ is unconditionally convergent in Y.*

Conversely, if for every sequence $(f_i)_{i \in \mathbb{N}}$ with $\sum_{i \in \mathbb{N}} \|f_i\|^p < \infty$ the series $\sum_{i=1}^\infty f_i$ converges in Y, then Y is complete.

Proof. Suppose first that Y is complete. It clearly suffices to consider the case $I = \mathbb{N}$. Thus, assume that $\sum_{i \in \mathbb{N}} \|f_i\|^p < \infty$. Let $F_n := \sum_{i=1}^n f_i$. For $n \geq m \geq n_0$, the p-norm property gives

$$\|F_n - F_m\|^p = \left\| \sum_{i=m+1}^n f_i \right\|^p \leq \sum_{i=m+1}^n \|f_i\|^p \leq \sum_{i=n_0+1}^\infty \|f_i\|^p \to 0 \quad \text{as} \quad n_0 \to \infty..$$

Hence, $(F_n)_{n \in \mathbb{N}} \subset Y$ is Cauchy. Since Y is complete, it follows that $(F_n)_{n \in \mathbb{N}}$ converges to some $F \in Y$; i.e., $F = \sum_{i=1}^\infty f_i$. It remains to show that unconditional convergence of the defining series. For this, let $\varepsilon > 0$ be arbitrary and choose $N_0 \in \mathbb{N}$ such that $\sum_{i=N_0+1}^\infty \|f_i\|^p < \varepsilon^p$. Let $J \subset \mathbb{N}$ be any finite set with $J \supset \{1, \ldots, N_0\}$. Since $\|\cdot\|_Y$ is a p-norm, $\|\cdot\|^p$ is a metric and

hence continuous; therefore,

$$\left\| F - \sum_{j \in J} f_j \right\|^p = \lim_{n \to \infty} \left\| F_n - \sum_{j \in J} f_j \right\|^p = \lim_{\substack{n \to \infty, \\ n \geq N_0}} \left\| \sum_{i \in \{1,\dots,n\} \setminus J} f_i \right\|^p$$

$$\leq \lim_{\substack{n \to \infty, \\ n \geq N_0}} \sum_{i \in \{1,\dots,n\} \setminus J} \|f_i\|^p \leq \sum_{i=N_0+1}^{\infty} \|f_i\|^p < \varepsilon^p.$$

Since this holds for any finite set $J \subset \mathbb{N}$ with $J \supset \{1, \dots, N_0\}$, the series converges unconditionally.

For the remaining implication, let $(g_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in Y. Choose a strictly increasing sequence $(n_i)_{i \in \mathbb{N}}$ such that $\|g_n - g_m\| \leq 2^{-i}$ for all $n, m \geq n_i$. Note that $n_i \geq i$ for all $i \in \mathbb{N}$. Define $f_i := g_{n_{i+1}} - g_{n_i}$ and note $\|f_i\| \leq 2^{-i}$, so that $\sum_{i \in \mathbb{N}} \|f_i\|^p < \infty$. By assumption, $F := \sum_{i=1}^{\infty} f_i \in Y$, and

$$F = \lim_{N \to \infty} \sum_{i=1}^{N} f_i = \lim_{N \to \infty} \sum_{i=1}^{N} (g_{n_{i+1}} - g_{n_i}) = \lim_{N \to \infty} (g_{n_{N+1}} - g_{n_1}).$$

For fixed, but arbitrary $\varepsilon > 0$, choose $N_0 \in \mathbb{N}$ such that $\|g_n - g_m\|^p < \frac{\varepsilon^p}{2}$ for all $n, m \geq N_0$ and such that $\|F - (g_{n_{N_0+1}} - g_{n_1})\|^p < \frac{\varepsilon^p}{2}$. Then, it holds for all $n \geq N_0$ that

$$\|g_n - (F + g_{n_1})\|^p \leq \|g_n - g_{n_{N_0+1}}\|^p + \|F - (g_{n_{N_0+1}} - g_{n_1})\|^p < \varepsilon^p.$$

Thus, $(g_n)_{n \in \mathbb{N}}$ has the limit $\lim_{n \to \infty} g_n = F + g_{n_1} \in Y$, showing that Y is complete. □

Lemma A.2. *Let $(Y, \|\cdot\|_Y)$ be a solid quasi-Banach function space on a locally compact group G. Suppose that $\|\cdot\|_Y$ is a p-norm, with $p \in (0, 1]$.*

Let $I \neq \varnothing$ be countable and let $(F_i)_{i \in I} \subset Y$ with $\sum_{i \in I} \|F_i\|_Y^p < \infty$. Then the series defining $F := \sum_{i \in I} F_i$ is almost everywhere absolutely convergent, and $F \in Y$ with

$$\|F\|_Y \leq \left(\sum_{i \in I} \|F_i\|_Y^p \right)^{1/p}.$$

Proof. The claim is trivial if I is finite; hence, we can assume that I is countably infinite and then without loss of generality that $I = \mathbb{N}$. The proof consists of three steps.

Step 1. Let $H_n := \sum_{i=1}^{n} |F_i|$ for $n \in \mathbb{N}$. In this step, we show that $(H_n)_{n \in \mathbb{N}}$ converges to some $H \in Y$. For this, note for $N \in \mathbb{N}$ and $n \geq m \geq N$ that

$$\|H_n - H_m\|_Y^p = \left\| \sum_{i=m+1}^{n} |F_i| \right\|_Y^p \leq \sum_{i=m+1}^{n} \| |F_i| \|_Y^p \leq \sum_{i=N}^{\infty} \|F_i\|_Y^p$$

and hence

$$\sup_{n,m \geq N} \|H_n - H_m\|_Y \leq \left(\sum_{i=N}^{\infty} \|F_i\|_Y^p \right)^{1/p} \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, $(H_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in Y. Since Y is complete, the claim follows.

Step 2. We show that the function H from Step 1 satisfies $H = \sum_{i=1}^{\infty} |F_i|$ almost everywhere. For $n \in \mathbb{N}$, define

$$\Xi_n := \left| H - \sum_{i=1}^{n} |F_i| \right| = |H - H_n|$$

Then $\|\Xi_n\|_Y = \|H - H_n\|_Y \to 0$ as $n \to \infty$. Setting $\Xi^{(n)} := \inf_{i \geq n} \Xi_i$ yields $0 \leq \Xi^{(n)} \leq \Xi_i$ for all $i \geq n$, and hence $\|\Xi^{(n)}\|_Y \leq \|\Xi_i\|_Y \to 0$ as $n \to \infty$ by solidity of Y. Therefore,

$\Xi^{(n)} = 0$ almost everywhere for all $n \in \mathbb{N}$, showing that there exists a null-set $N \subset G$ satisfying $\Xi^{(n)}(x) = 0$ for all $n \in \mathbb{N}$ and $x \in G \setminus N$. For $x \in G \setminus N$, it follows therefore that

$$0 = \lim_{n \to \infty} \Xi^{(n)}(x) = \liminf_{n \to \infty} \Xi_n(x) = \liminf_{n \to \infty} \left| H(x) - \sum_{i=1}^{n} |F_i(x)| \right|,$$

so that there exists a subsequence $(n_\ell)_{\ell \in \mathbb{N}}$ (possibly dependent on x) such that

$$0 = \lim_{\ell \to \infty} \left| H(x) - \sum_{i=1}^{n_\ell} |F_i(x)| \right|.$$

Hence, $H(x) = \lim_{\ell \to \infty} \sum_{i=1}^{n_\ell} |F_i(x)| = \sum_{i=1}^{\infty} |F_i(x)|$. Since this holds for all $x \in G \setminus N$, we conclude $H = \sum_{i=1}^{\infty} |F_i|$ almost everywhere.

Step 3. First, note for $\Omega := \{x \in G : |H(x)| = \infty\}$ that $n \cdot \mathbb{1}_\Omega \leq |H|$ for arbitrary $n \in \mathbb{N}$. Hence $\|\mathbb{1}_\Omega\|_Y \leq \frac{1}{n}\|H\|_Y \to 0$ as $n \to \infty$, so that $\mathbb{1}_\Omega = 0$ almost everywhere and thus $|H(x)| < \infty$ almost everywhere. Combined with Step 2, this shows $\sum_{n=1}^{\infty} |F_n(x)| < \infty$ almost everywhere. Hence, the series defining $F := \sum_{n=1}^{\infty} F_n$ converges absolutely almost everywhere, with $|F(x)| \leq \sum_{n=1}^{\infty} |F_n(x)| \leq |H(x)|$ for a.e. $x \in G$.

Second, note that $d(f, g) := \|f - g\|_Y^p$ is a metric and that $d(H_n, H) \to 0$ as $n \to \infty$. Therefore, $\|H_n\|_Y^p = d(H_n, 0) \to d(H, 0) = \|H\|_Y^p$. This means

$$\|F\|_Y^p \leq \|H\|_Y^p = \lim_{n \to \infty} \|H_n\|_Y^p = \lim_{n \to \infty} \left\| \sum_{i=1}^{n} |F_i| \right\|_Y^p \leq \lim_{n \to \infty} \sum_{i=1}^{n} \| |F_i| \|_Y^p = \sum_{i=1}^{\infty} \|F_i\|_Y^p,$$

which implies $\|F\|_Y \leq \left(\sum_{i=1}^{\infty} \|F_i\|_Y^p \right)^{1/p}$, as claimed. $\qquad\square$

Acknowledgements

References

[1] C. D. Aliprantis and K. C. Border. *Infinite dimensional analysis*. Springer, Berlin, third edition, 2006.

[2] P. Aniello. Square integrable projective representations and square integrable representations modulo a relatively central subgroup. *Int. J. Geom. Methods Mod. Phys.*, 3(2):233–267, 2006.

[3] J.-P. Anker. A short proof of a classical covering lemma. *Monatsh. Math.*, 107(1):5–7, 1989.

[4] A. Burtscheidt. *Vanishing moments conditions for atomic decompositions of coorbit spaces on quasi-Banach spaces*. PhD thesis, RWTH Aachen University, 2020. https://publications.rwth-aachen.de/record/808437.

[5] R. C. Busby and H. A. Smith. Product-convolution operators and mixed-norm spaces. *Trans. Am. Math. Soc.*, 263:309–341, 1981.

[6] A. L. Carey. Square-integrable representations of non-unimodular groups. *Bull. Austral. Math. Soc.*, 15(1):1–12, 1976.

[7] J. G. Christensen, A. H. Darweesh, and G. Ólafsson. Coorbits for projective representations with an application to Bergman spaces. *Monatsh. Math.*, 189(3):385–420, 2019.

[8] J. G. Christensen, K. Gröchenig, and G. Ólafsson. New atomic decompositions for Bergman spaces on the unit ball. *Indiana Univ. Math. J.*, 66(1):205–235, 2017.

[9] J. G. Christensen and G. Ólafsson. Coorbit spaces for dual pairs. *Appl. Comput. Harmon. Anal.*, 31(2):303–324, 2011.

[10] J. G. Christensen and G. Ólafsson. New atomic decompositions of Bergman spaces on bounded symmetric domains. *J. Geom. Anal.*, 31(7):7513–7537, 2021.

[11] O. Christensen. Atomic decomposition via projective group representations. *Rocky Mountain J. Math.*, 26(4):1289–1312, 1996.

[12] O. Christensen. *An introduction to frames and Riesz bases*. Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, [Cham], second edition, 2016.

[13] D. L. Cohn. *Measure theory*. Birkhäuser Advanced Texts. Birkhäuser/Springer, New York, second edition, 2013.

[14] S. Dahlke, F. De Mari, E. De Vito, D. Labate, G. Steidl, G. Teschke, and S. Vigogna. Coorbit spaces with voice in a Fréchet space. *J. Fourier Anal. Appl.*, 23(1):141–206, 2017.

[15] R. A. DeVore and G. G. Lorentz. *Constructive approximation*, volume 303. Berlin: Springer-Verlag, 1993.

[16] M. Duflo and C. C. Moore. On the regular representation of a nonunimodular locally compact group. *J. Functional Analysis*, 21(2):209–243, 1976.

[17] H. G. Feichtinger. Banach convolution algebras of Wiener type. In *Functions, series, operators, Vol. I, II (Budapest, 1980)*, volume 35 of *Colloq. Math. Soc. János Bolyai*, pages 509–524. North-Holland, Amsterdam, 1983.

[18] H. G. Feichtinger and K. Gröchenig. A unified approach to atomic decompositions via integrable group representations. In *Function spaces and applications (Lund, 1986)*, volume 1302 of *Lecture Notes in Math.*, pages 52–73. Springer, Berlin, 1988.

[19] H. G. Feichtinger and K. H. Gröchenig. Banach spaces related to integrable group representations and their atomic decompositions. I. *J. Funct. Anal.*, 86(2):307–340, 1989.

[20] H. G. Feichtinger and K. H. Gröchenig. Banach spaces related to integrable group representations and their atomic decompositions. II. *Monatsh. Math.*, 108(2-3):129–148, 1989.

[21] V. Fischer, D. Rottensteiner, and M. Ruzhansky. Heisenberg-modulation spaces at the crossroads of coorbit theory and decomposition space theory. *Preprint. arXiv: 1812.07876*, 2018.

[22] G. Folland. *A Course in Abstract Harmonic Analysis*. CRC PressINC, 1995.

[23] G. Folland. *Real Analysis: Modern Techniques and Their Applications*. Pure and applied mathematics. Wiley, second edition, 1999.

[24] M. Fornasier and H. Rauhut. Continuous frames, function spaces, and the discretization problem. *J. Fourier Anal. Appl.*, 11(3):245–287, 2005.

[25] S. Foucart and H. Rauhut. *A mathematical introduction to compressive sensing*. New York, NY: Birkhäuser/Springer, 2013.

[26] J. J. F. Fournier and J. Stewart. Amalgams of L^p and l^q. *Bull. Amer. Math. Soc. (N.S.)*, 13(1):1–21, 1985.

[27] M. Frazier and B. Jawerth. A discrete transform and decompositions of distribution spaces. *J. Funct. Anal.*, 93(1):34–170, 1990.

[28] H. Führ. Coorbit spaces and wavelet coefficient decay over general dilation groups. *Trans. Am. Math. Soc.*, 367(10):7373–7401, 2015.

[29] H. Führ. Vanishing moment conditions for wavelet atoms in higher dimensions. *Adv. Comput. Math.*, 42(1):127–153, 2016.

[30] H. Führ and J. T. van Velthoven. Coorbit spaces associated to integrably admissible dilation groups. *J. Anal. Math.*, 144(1):351–395, 2021.

[31] Y. V. Galperin and S. Samarah. Time-frequency analysis on modulation spaces $M_m^{p,q}$, $0 < p, q \leqslant \infty$. *Appl. Comput. Harmon. Anal.*, 16(1):1–18, 2004.

[32] J. E. Gilbert, Y. S. Han, J. A. Hogan, J. D. Lakey, D. Weiland, and G. Weiss. Smooth molecular decompositions of functions and singular integral operators. *Mem. Amer. Math. Soc.*, 156(742):viii+74, 2002.

[33] L. Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2014.

[34] K. Gröchenig. Describing functions: atomic decompositions versus frames. *Monatsh. Math.*, 112(1):1–42, 1991.

[35] K. Gröchenig. New function spaces associated to representations of nilpotent Lie groups and generalized time-frequency analysis. *J. Lie Theory*, 31(3):659–680, 2021.

[36] K. Gröchenig and M. Piotrowski. Molecules in coorbit spaces and boundedness of operators. *Studia Math.*, 192(1):61–77, 2009.

[37] F. Holland. Harmonic analysis on amalgams of L^p and l^q. *J. London Math. Soc. (2)*, 10:295–305, 1975.

[38] K. Iwasawa. Topological groups with invariant compact neighbourhoods of the identity. *Ann. Math. (2)*, 54:345–348, 1951.

[39] E. Kaniuth. *A course in commutative Banach algebras*, volume 246 of *Graduate Texts in Mathematics*. Springer, New York, 2009.

[40] H. Kempka, M. Schäfer, and T. Ullrich. General coorbit space theory for quasi-Banach spaces and inhomogeneous function spaces with variable smoothness and integrability. *J. Fourier Anal. Appl.*, 23(6):1348–1407, 2017.

[41] S. Koppensteiner, J. T. Van Velthoven, and F. Voigtlaender. Anisotropic Triebel-Lizorkin spaces and wavelet coefficient decay over one-parameter dilation groups, I. *Preprint. arXiv: 2104.14361*, 2021.

[42] S. Lang. *Real and functional analysis*, volume 142 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 1993.

[43] Y. Liang, Y. Sawano, T. Ullrich, D. Yang, and W. Yuan. New characterizations of Besov-Triebel-Lizorkin-Hausdorff spaces including coorbits and wavelets. *J. Fourier Anal. Appl.*, 18(5):1067–1111, 2012.

[44] T. W. Palmer. Classes of nonabelian, noncompact, locally compact groups. *Rocky Mt. J. Math.*, 8:683–741, 1978.

[45] H. Rauhut. Coorbit space theory for quasi-Banach spaces. *Preprint. arXiv: 0507466*, 2005.

[46] H. Rauhut. Coorbit space theory for quasi-Banach spaces. *Stud. Math.*, 180(3):237–253, 2007.

[47] H. Rauhut. Wiener amalgam spaces with respect to quasi-Banach spaces. *Colloq. Math.*, 109(2):345–362, 2007.

[48] H. Rauhut and T. Ullrich. Generalized coorbit space theory and inhomogeneous function spaces of Besov-Lizorkin-Triebel type. *J. Funct. Anal.*, 260(11):3299–3362, 2011.

[49] H. Reiter and J. D. Stegeman. *Classical harmonic analysis and locally compact groups*, volume 22 of *London Mathematical Society Monographs. New Series*. The Clarendon Press, Oxford University Press, New York, second edition, 2000.

[50] J. L. Romero, J. T. van Velthoven, and F. Voigtlaender. On dual molecules and convolution-dominated operators. *J. Funct. Anal.*, 280(10):Paper No. 108963, 56, 2021.

[51] W. Rudin. *Functional analysis.* International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, second edition, 1991.

[52] T. Tao. For what LCH groups is the Haar measure $\mu(UxU)$ bounded? MathOverflow. URL: `https://mathoverflow.net/q/401943` (version: 2021-08-17).

[53] V. S. Varadarajan. Geometry of quantum theory. 2nd ed. New York etc.: Springer-Verlag. XVIII, 412 p. DM 178.00 (1985)., 1985.

[54] F. Voigtlaender. *Embedding Theorems for Decomposition Spaces with Applications to Wavelet Coorbit Spaces.* PhD thesis, RWTH Aachen University, 2015. `https://publications.rwth-aachen.de/record/564979`.

[55] R. M. Young. *An introduction to nonharmonic Fourier series.* Academic Press, Inc., San Diego, CA, first edition, 2001.

[56] A. Zaanen. *Integration.* North-Holland Publishing Co., Amsterdam; Interscience Publishers John Wiley & Sons, Inc., New York, 1967.

Delft University of Technology, Mekelweg 4, Building 36, 2628 CD Delft, The Netherlands.

Email address: `j.t.vanvelthoven@tudelft.nl`

Department of Mathematics, Technical University of Munich, Boltzmannstr. 3, 85748 Garching bei München, Germany

Catholic University of Eichstätt-Ingolstadt, Faculty of Mathematics and Geography, Mathematical Institute for Machine Learning and Data Science (MIDS), Lehrstuhl Reliable Machine Learning, 85071 Eichstätt, Germany

Email address: `felix.voigtlaender@ku.de, felix@voigtlaender.xyz`