Mathematisches Forschungsinstitut Oberwolfach

Report No. 23/2022

DOI: 10.4171/OWR/2022/23

Interactions between Algebraic Geometry and Noncommutative Algebra

Organized by
Bill Crawley-Boevey, Bielefeld
Markus Reineke, Bochum
Catharina Stroppel, Bonn
Michel Van den Bergh, Diepenbeek

1 May – 7 May 2022

Abstract. This workshop was on the interactions between noncommutative algebra, representation theory and algebraic geometry. The major objective was to bring together researchers from those areas with the focus on topics and problems where geometric methods are prevalent.

Mathematics Subject Classification (2020): 17A36, 14D20, 16E35.

Introduction by the Organizers

The workshop *Interactions between Algebraic Geometry and Noncommutative Algebra* organised by W. Crawley-Boevey, M. Reineke, C. Stroppel, and M. Van den Bergh was well-attended with more than 40 participants. The group of participants consisted of a broad mix of researchers with respect to age, career status, gender and geographic diversity. Unfortunately, the COVID and political situation did not allow the personal attendance of several researchers, in particular of those located in Japan and Russia. They participated online and were included into the conference via a zoom hybrid format. The format of the talks was the usual 50 minutes talks. The organisers decided to keep the amount of online talks to a minimum which meant one online talk per day. In this way many discussions emerged during and after the talks and created a lively open atmosphere amongst the participants. Younger participants were included substantially, in particular also as speakers. The mathematical spectrum was rather broad ranging from classical ring theory, over derived noncommutative geometry, algebraic geometry and higher Auslander-Reiten theory to geometric representation theory. Derived categories and Hall algebras of different types were some of the main linking themes.

An introduction into dg techniques and derived commutative algebra summarizing well established results which then could be viewed as granted in the talks during the week was given by Amnon Yekutieli. By putting dg structures on cluster categories and taking dg Drinfeld quotients, Bernhard Keller recovered Auslander–Reiten theory at a very high level.

Michael Wemyss talked about Gromov–Witten and Gopakumar–Vafa invariants for 3-fold flops and described them in terms of a beautiful combinatorics. Using the stable $\mathbb{Z}$-graded category of CM modules on the non-isolated A_∞-curve singularity, Jenny August constructed an additive category that realises the combinatorics of infinite-type cluster algebras. Will Donovan used a conjectural topological description of the Stringy Kaehler Moduli space to both find and prove relationships between natural functors on higher dimensional flopping contractions.

Ben Davison explained the construction of a BPS Lie algebra resembling a current Lie Algebra, whose Euler characteristic of a certain graded piece recovers the Kac polynomial. Its 0th-cohomologically graded piece is the positive part of an associated Kac-Moody Lie algebra, whereas the full object is a generalised KM Lie algebra. Daniel Kaplan used Davison's result that moduli in a 2-CY category are locally a quiver variety as motivation to prove fundamental results for the multiplicative preprojective algebras.

Matthew Young showed that Knörrer periodicity lifts to a new equivariant quasi-equivalence of matrix factorisations, motivated by Atiyah's real vector bundles. The construction arises as some fixed point construction under an involution. An open question hereby is whether this could shed some light on orthogonal Khovanov-Rozansky knot homologies. Given a d-dimensional connected $\mathbb{N}$-graded ring (with a fixed α-invariant a), such that the singularity category is the perfect category of a finite dimensional algebra A, Osamu Iyama constructed a very general categorical equivalence between the $\mathbb{Z}/a\mathbb{Z}$-singularity category and the d-1-cluster category of A. This can be applied to a striking number of examples, including Geigle–Lenzing spaces and Grassmannian cluster categories.

Travis Schedler talked about how to find crepant resolutions of Nakajima quiver varieties. This was done by embedding the movable cone as a certain region of a hyperplane arrangement, generalising previous work of Bellamy–Craw. Magdalena Boos considered quiver representations and the corresponding theory of fixed points under diagram involutions. A general framework of quiver representations equipped with orthogonal or symplectic forms was developed. The behaviour under the usual constructions (like degenerations) is surprisingly difficult. Hans Franzen talked about attracting cells of torus moduli spaces with the goal of explicit and concrete descriptions. He gave a concrete realisation of important modules whose existence was proved by Kac.

GIT constructions appeared as a crucial ingredient in several talks at the conference. Pieter Belmans however went the opposite direction and outlined a proof that quiver GIT spaces are projective without using GIT. This was done using the analogy between quiver GIT and moduli of vector bundles on curves.

Jens Eberhardt gave a beautiful overview on several known geometric representation theoretic constructions and then presented a general framework of Springer type resolutions and associated algebras using Chow groups and motives. It allowed to establish strong formality results and to incorporate naturally gradings and base chances. Shinnosuke Okawa used categorical polarisation data to reconstruct certain AS regular algebras A from categories of coherent sheaves.

In a different direction, Anya Nordskova gave a vast simplification and generalisations of both, a result of Seidel-Thomas and of Brav–Thomas, that there is a faithful ADE action given by any ADE configuration of spherical objects. The existing proofs are via Floer homology, intrigued combinatorics or seriously usage of normal forms in braid groups. Alice Rizzardo used twisted Hodge diamonds to construct a very large number of possible non-Fourrier-Mukai functors.

S. Paul Smith talked on Feigin and Odesskii's Elliptic Algebras and their properties. These form a very large family of noncommutative algebras which include Sklyanin algebras, twisted homogeneous rings and more. He could give a presentation, descriptions of point and line modules as well as shed some light on mysterious formulas involving theta functions.

Finally, based on Tamarkin's famous paper: *what do dg-categories form?* Dmitriy Kaledin talked about the question: *what do abelian categories form?*

Workshop: Interactions between Algebraic Geometry and Noncommutative Algebra

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Abstracts

GV and GW invariants via the enhanced movable cone

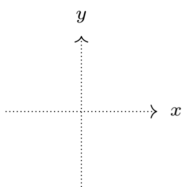
Michael Wemyss
(joint work with Navid Nabijou)

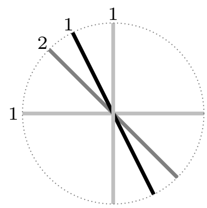
Curve invariants associated to smooth 3-fold flopping contractions turn out to be controlled by Dynkin data, via a subset I of an ADE Dynkin diagram Δ.

As a first example, consider the D_5 Dynkin diagram , where I corresponds to the two light gray vertices. There are twenty positive roots of Δ, five of which are illustrated below.

$$\begin{smallmatrix}&&0&\\0&1&0&0\end{smallmatrix}\quad\begin{smallmatrix}&&0&\\0&0&1&0\end{smallmatrix}\quad\begin{smallmatrix}&&1&\\1&1&1&0\end{smallmatrix}\quad\begin{smallmatrix}&&1&\\1&2&1&0\end{smallmatrix}\quad\begin{smallmatrix}&&1&\\1&2&2&1\end{smallmatrix}$$

Restricting all twenty positive roots to I, which are the dotted boxes above, and discarding zero gives the set of *restricted positive roots*. In the example above, these are $\{10, 01, 11, 21, 22\}$. Each of these gives rise to a dual hyperplane: the first gives $x = 0$, the second $y = 0$, the third $x + y = 0$, etc. This yields the following finite hyperplane arrangement.





Restricted Root	
01	
11, 22	
21	
10	

The hyperplane $x + y = 0$ is of course the same as the hyperplane $2x + 2y = 0$. The subtle point below is that for the curve counting, it is important to remember this multiplicity. This is the *enhanced* from the title. The number 2 is written next to the dark gray hyperplane, since that hyperplane appears twice.

In general, for any $I \subset \Delta$ with Δ ADE, repeating the above procedure always gives a finite hyperplane arrangement $\mathcal{H}_I$ together with multiplicities.

Now to a smooth 3-fold flopping contraction $f : \mathcal{X} \to \operatorname{Spec} \mathcal{R}$, Reid's general hyperplane section associates to f a subset $I \subset \Delta$. Furthermore, Katz associates Gopakumar–Vafa (GV) invariants via one-parameter deformation, the upshot of which is a number $n_\beta \in \mathbb{Z}_{\geq 0}$ for each curve class β. The first main result is that $n_\beta \neq 0$ if and only if β is a restricted positive root.

To describe the Gromov–Witten theory requires us to translate the finite hyperplanes in $\mathcal{H}_I$ over the integers, to obtain an infinite arrangement $\mathcal{H}_I^{\mathsf{aff}}$. The subtle point is that

$$\begin{aligned} x + y = 0 \quad &\longrightarrow \quad x + y \in \mathbb{Z} \\ 2(x + y) = 0 \quad &\longrightarrow \quad 2(x + y) \in \mathbb{Z} \end{aligned}$$

and so there are *more* hyperplanes in the infinite hyperplane arrangement than is typical in Coxeter theory. In the above example, $\mathcal{H}_I^{\mathsf{aff}}$ is drawn in [IW, §4].

The second main result is that the pole locus of the associated Gromov–Witten quantum potential is the (complexification of) the infinite arrangement $\mathcal{H}_I^{\mathsf{aff}}$. There are various corollaries, including a visual proof of the Crepant Resolution Conjecture in this context, and how the dimension of the contraction algebra changes under flop.

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Cluster structures for the A_∞ curve singularity

Jenny August
(joint work with M. W. Cheung, E. Faber, S. Gratz, S. Schroll)

Since their introduction by Fomin and Zelevinsky, *cluster algebras* have been a great source of interesting combinatorics. For example, some of the simplest cases, known as type A cluster algebras, have combinatorics which are completely described by triangulations of polygons and their mutations via flipping diagonals.

This has inspired many to search for similar combinatorics in other settings, such as in representation theory or geometry. One well known instance of this is the work of Jensen, King and Su [8] who demonstrated that the combinatorics of a *Grassmannian cluster algebra* are exhibited by the so called *cluster-tilting theory* of a certain curve singularity. As a special case (namely for the Grassmannians $\mathrm{Gr}(2,n)$), this provided a link between type A cluster algebras and type A curve singularities.

In particular, they showed the type A_n curve singularity given by $R_{2,n} = \mathbb{C}[x,y]/(x^2 - y^{n+1})$, has an associated category $\mathcal{C}_{2,n}$ of equivariant maximal Cohen-Macaulay $R_{2,n}$-modules where:

(1) the indecomposable objects of $\mathcal{C}_{2,n}$ are in bijection with the diagonals of an $(n+3)$-gon;
(2) under this bijection, the *cluster-tilting objects* (those with vanishing self-extensions + good mutation properties) precisely correspond to triangulations of the $(n+3)$-gon, and;
(3) mutation of cluster-tilting objects (as defined by Iyama and Yoshino [7]) coincides with flipping diagonals.

In other words, $\mathcal{C}_{2,n}$ has finite type A cluster combinatorics.

Our work starts by extending some of Jensen, King and Su's results to the the Grassmannian cluster algebras of infinite rank, as introduced by [6], which one should think of as corresponding to some kind of infinite Grassmannian "$\mathrm{Gr}(k,\infty)$". By naively sending $n \to \infty$ in the JKS construction, we associate the singularities $R_k = \mathbb{C}[x,y]/(x^k)$ and consider $\mathcal{C}_k$ to be the category of $\mathbb{Z}$-graded maximal Cohen-Macaulay modules over R_k, where x and y lie in degrees 1 and -1 respectively [1].

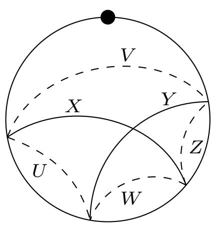
Some of our results hold for all k, but here we focus on the $k = 2$ case where R_2 is the well known A_∞ *curve singularity.*

One major advantage of the $k = 2$ case is that $\mathcal{C}_2$ is tame, and the indecomposable objects fall into two families [5]:

(1) the shifted ideals $(x, y^i)(j)$ for $i \geq 0, j \in \mathbb{Z}$;
(2) the shifted modules $\mathbb{C}[y](\ell)$ for $\ell \in \mathbb{Z}$.

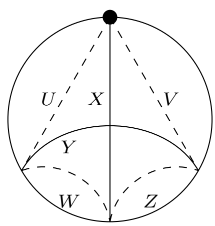
With this in mind, the objects in $\mathcal{C}_2$ are naturally in bijection with arcs in the *completed* ∞*-gon*. Indeed, one may think of the completed ∞-gon as a discrete set of points on the unit circle S^1 with one two-sided accumulation point. We can label the marked points by $\mathbb{Z}$ and the accumulation point as $-\infty$, and then an arc is simply an unordered pair $(a, b) \in \mathbb{Z} \cup \{-\infty\}$, which can be depicted by drawing a path between the corresponding points. The bijection is then given by $(x, y^i)(j) \leftrightarrow (-i - j, 1 - j)$ and $\mathbb{C}[y](\ell) \leftrightarrow (-\infty, -\ell)$.

The ideas of noncrossing diagonals and triangulations extend naturally to arcs in the completed ∞-gon, and the combinatorics of these is studied in [4]. Our goal was to show that, as in the finite setting, these triangulations control the cluster-tilting subcategories of $\mathcal{C}_2$ and their mutation. Keeping in mind that *rigidity* (the vanishing of extensions) is a key property of cluster-tilting, we show that the only nonzero extensions between indecomposable objects in $\mathcal{C}_2$ correspond to the following pictures:



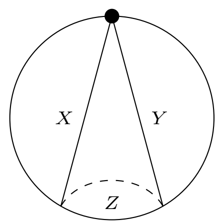
$0 \to X \to W \oplus V \to Y \to 0$

$0 \to Y \to U \oplus Z \to X \to 0$



$0 \to X \to W \oplus V \to Y \to 0$

$0 \to Y \to U \oplus Z \to X \to 0$



$0 \to X \to Z \to Y \to 0$

With this information, it is then straightforward to classify the cluster-tilting subcategories in $\mathcal{C}_2$ as certain triangulations of the completed ∞-gon [2]. Moreover, whenever mutation of a cluster-tilting subcategory is possible, we show it corresponds to flipping a diagonal exactly as in the finite case. In this way, we say that $\mathcal{C}_2$ (corresponding to the A_∞ singularity) has infinite type A cluster combinatorics.

However, not all triangulations of the completed ∞-gon correspond to cluster-tilting subcategories. In fact, any triangulation with more than one arc of the form $(-\infty, a)$ fails to even be rigid, as can be seen from the third picture above. Thus to characterise subcategories corresponding to triangulations, and so make use of the combinatorial results in [4], we need a more general notion.

For this, we were inspired by work of Barnard, Gunawan, Meehan and Schiffler [3] to say that a subcategory of $\mathcal{C}_2$ is *almost rigid* if any two indecomposbles either have no extensions between them, or the middle term of any nontrivial extension between them is indecomposable. It follows directly from the above description of extensions in $\mathcal{C}_2$ that triangulations of the completed ∞-gon correspond precisely to maximal almost rigid subcategories of $\mathcal{C}_2$. Moreover, we extend the mutation of cluster-tilting subcategories to this setting and show it corresponds to flipping diagonals [2].

By combining with [4], we conclude that we can pass between any two maximal almost rigid subcategories of $\mathcal{C}_2$ with a sequence of *transfinite* mutations, and we may say that $\mathcal{C}_2$ fully exhibits the combinatorics of the completed ∞-gon.

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The noncommutative conifold and fermionization of Yangians

Ben Davison

Let Q be a finite quiver, i.e. a pair of finite sets Q_1 (the arrows) and Q_0 (the vertices) and two morphisms $s, t\colon Q_1 \to Q_0$. We define the *double* $\overline{Q}$ by adding an arrow a^*, for each $a \in Q_1$, satisfying $s(a^*) = t(a)$ and $t(a^*) = s(a)$. We denote by $\mathbb{C}\overline{Q}$ the free path algebra of the doubled quiver, and by $\Pi_Q := \mathbb{C}\overline{Q}/\langle \sum_{a\in Q_1}[a, a^*]\rangle$ the preprojective algebra.

Via pullback and pushforward of Borel–Moore homology in the diagram

$$\mathfrak{M}(\Pi_Q) \times \mathfrak{M}(\Pi_Q) \xleftarrow{\pi_1\times\pi_3} \mathfrak{Exact}(\Pi_Q) \xrightarrow{\pi_2} \mathfrak{M}(\Pi_Q),$$

where $\mathfrak{Exact}(\Pi_Q)$ is the stack of short exact sequences of Π_Q-modules, Schiffmann and Vasserot defined [1] the cohomological Hall algebra structure on

$$\mathcal{A}_{\Pi_Q} = \bigoplus_{\mathbf{d}\in\mathbb{N}^{Q_0}} \mathrm{H}^{\mathrm{BM}}(\mathfrak{M}_{\mathbf{d}}(\Pi_Q), \mathbb{Q}_{\mathrm{vir}})$$

where the subscript vir denotes some system of cohomological shifts that we introduce so that the product respects cohomological degree, as well as the $\mathbb{N}^{Q_0}$-grading.

Some theorems [2, 3] on the cohomological Hall algebra $\mathcal{A}_{\Pi_Q}$:

(1) There is a $\mathbb{N}^{Q_0}$-graded and cohomologically graded subspace $\mathfrak{g}_{\Pi_Q} \subset \mathcal{A}_{\Pi_Q}$ which is closed under the commutator Lie bracket, and such that the PBW morphism $\mathrm{Sym}\left(\mathfrak{g}_{\Pi_Q} \otimes \mathrm{H}(\mathrm{pt}/\mathbb{C}^*, \mathbb{Q})\right) \to \mathcal{A}_{\Pi_Q}$ is an isomorphism, where we denote by $\mathrm{Sym}(V)$ the free supercommutative algebra generated by V.

(2) For $\gamma \in \mathbb{N}^{Q_0}$ a dimension vector there is an equality of Laurent polynomials

$$\chi_{q^{1/2}}(\mathfrak{g}_{\Pi_Q,\gamma}) := \sum_i \dim(\mathfrak{g}^i_{\Pi_Q}) q^{i/2} = \mathtt{a}_{Q,\gamma}(q^{-1})$$

where $\mathtt{a}_{Q,\gamma}(q)$ is the *Kac* polynomial, counting iso-classes of absolutely indecomposable γ-dimensional $\mathbb{F}_q Q$-modules.

(3) There is an isomorphism of Lie algebras $\mathfrak{g}^0_{\Pi_Q} \cong \mathfrak{n}^-_{Q^{\mathrm{re}}}$ where Q^{re} is the quiver obtained by removing all vertices supporting loops, and $\mathfrak{n}^-_{Q^{\mathrm{re}}}$ is one half of the associated Kac–Moody Lie algebra.

(4) (Work in progress, with L. Hennecart and S. Schlegel-Mejia): The Lie algebra $\mathfrak{g}_{\Pi_Q}$ is one half of a generalised Kac–Moody Lie algebra.

(1), (3) and (4) suggest that $\mathcal{A}_{\Pi_Q}$ is one half of some kind of generalized Yangian. Indeed in finite and affine type, this statement can be made precise. (2) tells us that the Lie algebra/algebra is concentrated entirely in *even* cohomological degrees. We are motivated by the

Question. Is there a "fermionized", or partially odd version of the above construction? I.e. can we recover Yangians of classical super-algebras like $\mathfrak{gl}(m|n)$?

We will answer this question by taking the word "fermionization" seriously: physicists teach us that particles acquire mass (i.e. become fermionic) by adding "mass terms" or quadratic terms, to superpotentials governing their gauge theory. So first we have to recast $\mathcal{A}_{\Pi_Q}$ in terms of potentials. A potential $W \in \mathbb{C}Q/[\mathbb{C}Q,\mathbb{C}Q]_{\mathrm{cyc}}$ is a linear combination of cyclic paths. If $W = a_1 \dots a_n$ and $a \in Q_1$ we define $\partial W/\partial a = \sum_{a_i = a} a_{i+1} \dots a_n a_1 \dots a_{i-1}$ and extend to arbitrary W by linearity. Then define $\mathrm{Jac}(Q,W) := \mathbb{C}Q/\langle \partial W/\partial a \mid a \in Q_1\rangle$.

An important class of examples come from the *tripling* construction. We denote by $\tilde{Q}$ the quiver obtained by adding a loop ω_i at every vertex of $\overline{Q}$, the doubled quiver. This carries the cubic potential $\tilde{W} = (\sum_{a\in Q_1}[a,a^*])(\sum_{i\in Q_0}\omega_i)$. It is easy to verify that there is a natural isomorphism $\Psi\colon \mathrm{Jac}(\tilde{Q},\tilde{W}) \cong \Pi_Q[\omega]$ with $\Psi^{-1}(\omega) = \sum_{i\in Q_0}\omega_i$.

Given a quiver Q with potential W we denote by $\mathrm{Tr}(W) \in \Gamma(\mathfrak{M}(\mathbb{C}Q))$ the resulting function on the stack of $\mathbb{C}Q$-modules. Then as substacks of $\mathfrak{M}(\mathbb{C}Q)$ there are equalities $\mathrm{crit}(\mathrm{Tr}(W)) = \mathfrak{M}(\mathrm{Jac}(Q,W)) = \mathrm{supp}(\phi_{\mathrm{Tr}(W)}\mathbb{Q}_{\mathrm{vir}})$ where $\phi_{\mathrm{Tr}(W)}\mathbb{Q}_{\mathrm{vir}}$ is a perverse sheaf on the stack of $\mathrm{Jac}(Q,W)$-modules that turns out to be the "right" one from the point of view of geometric representation theory. In particular, there is an isomorphism of algebras

$$\mathcal{A}_{\Pi_Q} \cong \mathcal{A}_{\tilde{Q},\tilde{W}} := \bigoplus_{\gamma\in\mathbb{N}^{Q_0}} \mathrm{H}(\mathfrak{M}_\gamma(\mathrm{Jac}(\tilde{Q},\tilde{W})), \phi_{\mathrm{Tr}(\tilde{W})}\mathbb{Q}_{\mathrm{vir}})$$

where the target carries the cohomological Hall algebra structure introduced by Kontsevich and Soibelman [4]. So this at least expresses our algebra in terms of something involving potentials, and now the question becomes:

Question. Is there a choice of quiver Q' and potential W' such that $\mathcal{A}_{Q',W'}$ "partially fermionizes" $\mathcal{A}_{\tilde{Q},\tilde{W}}$?

On the way to an affirmative answer to this question, we take some inspiration from the world of noncommutative geometry, where Jacobi algebras play an important role.

Let $G \subset \mathrm{SL}_2(\mathbb{C})$ be a finite group. We denote by $X_0 = \mathbb{C}^2/G$ the associated Kleinian singularity, and by $p\colon Y_0 \to X_0$ a minimal resolution. Then $p^{-1}(0)$ is a tree of rational curves, with incidence matrix Γ a graph of ADE type. Let $\mathfrak{h}$ denote the Cartan sub-algebra of the associated Lie algebra. It has a natural basis provided by Γ_0. There is a universal deformation given by the leftmost diagram:

$$\begin{array}{ccc} Y_0 & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ 0 & \hookrightarrow & \mathfrak{h} \end{array} \qquad\qquad \begin{array}{ccc} Y^\alpha & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \xrightarrow{t\mapsto t\cdot\alpha} & \mathfrak{h}. \end{array}$$

For $\alpha \in \mathfrak{h}$ (not necessarily non-zero) we define Y^α via the rightmost, Cartesian, diagram. Then Y^α is a smooth threefold. The curve C_i corresponding to the vertex $i \in \Gamma_0$ deforms along $\mathbb{A}^1$ if and only if $\alpha \in i^\perp$, so if we denote by $\mathcal{M}_{i,n}$ the moduli space of semistable coherent sheaves of class $\mathcal{O}_{C_i}(n)$, we find $\mathcal{M}_{i,n} \cong \mathbb{A}^1$ if $\alpha \in i^\perp$, and $\mathcal{M}_{i,n} \cong \mathrm{pt}$ otherwise. In general the "BPS cohomology" for a specific class in a specific 3-Calabi–Yau category is hard to claclulate, or even define, but for these smooth, simply connected, moduli spaces the definition is basically forced:

$$\mathfrak{g}_{i,n}^{\mathrm{BPS}} = \mathrm{H}(\mathcal{M}_{i,n}, \mathbb{Q}[\dim(\mathcal{M}_{i,n})])[-1]$$

In particular, this BPS cohomology is one-dimensional, with parity depending on which hyperplanes α avoids.

To connect with the noncommutative conifold, we set $G = \mathbb{Z}/2\mathbb{Z}$. Then $\mathfrak{g} = \mathfrak{sl}_2$ and $\mathfrak{h}$ is 1-dimensional. We consider the extended quiver for Γ. Then there is a derived equivalence between $\mathrm{Coh}(Y^0)$ and the category of $\Pi_Q[\omega]$-modules. Here, Y^0 is obtained by setting $\alpha = 0$ in the above construction. In particular, the BPS cohomology for the quiver $\tilde{Q}$ with potential $\tilde{W}$ is entirely even. If we pick any $\alpha \neq 0$, then Y^α is the resolved conifold singularity, and by results [5] of Morrison, Mozgovoy, Nagao and Szendrői (along with a purity result) the BPS cohomology is partially odd.

The resolved conifold also has a noncommutative model: if we set $W_{\mathtt{KW}} = aa^*b^*b - abb^*a^*$ and set $B = \mathrm{Jac}(\overline{Q}, W_{\mathtt{KW}})$, then there is a derived equivalence between coherent sheaves on Y^α and B-modules. So the answer to our question is "yes" for this particular $\tilde{Q}$ and $\tilde{W}$. For a more general answer, observe that if we set $\tilde{W}^{(1,-1)} = \tilde{W} - (\omega_0^2 - \omega_1^2)$ then via a simple noncommutative change of variables there is an isomorphism $\mathrm{Jac}(\overline{Q}, W_{\mathtt{KW}}) \cong \mathrm{Jac}(\tilde{Q}, \tilde{W}^{(1,-1)})$. Our main theorem generalises this story from the noncommutative conifold:

Theorem ([6]). *Given a quiver* Q *and* $\mu \in \mathbb{C}^{Q_0}$ *define* $\tilde{W}^{\mu} = \tilde{W} - (\sum_{i \in Q_0} \mu_i \omega_i^2) \in \mathbb{C}\tilde{Q}/[\mathbb{C}\tilde{Q}, \mathbb{C}\tilde{Q}]_{\mathrm{cyc}}$. *Then for* $\gamma \in \mathbb{N}^{Q_0}$ *there is an isomorphism*

$$\mathfrak{g}_{\tilde{Q},\tilde{W}^{\mu},\gamma} \cong \mathfrak{g}_{\tilde{Q},\tilde{W},\gamma}[\epsilon_\gamma]$$

where $\epsilon_\gamma = 0$ *if* $\mu \cdot \gamma = 0$ *and is* -1 *otherwise.*

We leave to future work the computation of the Lie algebra structure on these partially fermionized Yangians: for the case of the noncommutative resolved conifold, a conjecture of Kevin Costello tells us that we have indeed recovered a Yangian associated to $\mathfrak{gl}(1|1)$, while the general case remains somewhat mysterious.

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A Constructive Approach to Derived Algebra

Amnon Yekutieli

Let me begin by saying what is derived algebra, as used in this talk. It means replacing rings and modules by DG rings and DG modules, and then studying the corresponding derived categories. "DG" means "differential graded". DG rings come in two distinct flavors: commutative and noncommutative (NC). Here is the plan of my talk:

- Definition of the two kinds of DG rings.
- A panorama of the roles that DG rings (both kinds) have in algebra and geometry.
- The structure of the derived category of commutative DG rings.
- A conjectural formulation of the cotangent DG module of a commutative DG ring.

The emphasis is on working directly with DG rings, suitable resolutions of them, and certain homomorphisms among them. Hence the words "constructive approach" in the title of the talk. There is a lot of overlap between the derived algebra discussed in this talk and other, much more prominent, versions, e.g. those of J. Lurie and B. Toën. See the survey article [To]. A major contrast between our constructive approach and the other approaches is that we do not use Quillen model structures and simplicial methods at all. I wish to thank a few people who contributed to my understanding of derived algebra over the years: Vladimir Hinich, James Zhang, Bernhard Keller, Michel Van den Bergh, Liran Shaul, Mattia Ornaghi, and Asaf Yekutieli.

Here are some definitions and basic facts, for details see [Ye, Chapter 3]: A DG ring is a graded ring $A = \bigoplus_{i \in \mathbb{Z}} A^i$, with unit $1_A \in A^0$, together with an additive operator d_A of degree 1, called the differential. The conditions are:

▷ $\mathrm{d}_A \circ \mathrm{d}_A = 0$.
▷ The graded Leibniz rule: for all elements $a \in A^i$ and $b \in A^j$

$$\mathrm{d}_A(a \cdot b) = \mathrm{d}_A(a) \cdot b + (-1)^i \cdot a \cdot \mathrm{d}_A(b).$$

The traditional name for "DG ring" is "unital associative cochain DG algebra".

Let A and B be DG rings. A homomorphism of DG rings $f : A \to B$ is a homomorphism of rings that respects units, gradings and differentials. The category of DG rings is denoted by DGRng. Rings are viewed as DG rings concentrated in degree 0. Fix some DG ring A. A DG A-ring is a DG ring B equipped with a DG ring homomorphism $f : A \to B$. The DG A-rings form a category, which we denote by DGRng/A.

Let $A = \bigoplus_{i \in \mathbb{Z}} A^i$ be a DG ring.

▷ A is called nonpositive if $A^i = 0$ for all $i > 0$.
▷ A is called weakly commutative if $b \cdot a = (-1)^{i \cdot j} \cdot a \cdot b$ for all $a \in A^i$ and $b \in A^j$.
▷ A is called strongly commutative if it is weakly commutative, and also $a \cdot a = 0$ for all $a \in A^i$ such that i is odd.
▷ A is called a commutative DG ring if it is nonpositive and strongly commutative.

Weak commutativity is a manifestation of the Koszul sign rule; strong commutativity is subtle: it eliminates 2-torsion in semi-free commutative DG rings.

The cohomology $\mathrm{H}(A) = \bigoplus_{i \in \mathbb{Z}} \mathrm{H}^i(A)$ of a DG ring A is a graded ring. A DG ring homomorphism $f : A \to B$ is called a quasi-isomorphism if $\mathrm{H}(f) : \mathrm{H}(A) \to \mathrm{H}(A)$ is an isomorphism of graded rings.

The derived category $\mathsf{D}(\mathsf{CDGRng}/A)$ is the localization of CDGRng/A w.r.t. the quasi-isomorphisms in it. There is the localization functor

$$\mathrm{Q} : \mathsf{CDGRng}/A \to \mathsf{D}(\mathsf{CDGRng}/A).$$

We wish to describe the structure of this derived category. There is a notion of homotopy $\gamma : f_0 \Rightarrow f_1$ between homomorphisms $f_0, f_1 : B \to C$ in CDGRng/A. This is called a left homotopy in [Ho]. The homotopy relation requires a modification. A quasi-homotopy $(g, \gamma) : f_0 \Rightarrow f_1$ consists of a quasi-isomorphism $g : \tilde{B} \to B$, and a homotopy $\gamma : f_0 \circ g \Rightarrow f_1 \circ g$.

Theorem. *Quasi-homotopy is a congruence on the category* CDGRng/A.

This means that there is a category $\mathsf{K}(\mathsf{CDGRng}\,/A)$, called the homotopy category, with the same objects. The morphisms in $\mathsf{K}(\mathsf{CDGRng}\,/A)$ are the quasi-homotopy classes of homomorphisms in $\mathsf{CDGRng}\,/A$. There is a full functor

$$\mathrm{P} : \mathsf{CDGRng}\,/A \to \mathsf{K}(\mathsf{CDGRng}\,/A),$$

which is the identity on objects. The localization functor Q factors through P; there is a functor $\bar{\mathrm{Q}}$ s.t. $\mathrm{Q} = \bar{\mathrm{Q}} \circ \mathrm{P}$.

Theorem. *The functor $\bar{\mathrm{Q}}$ is a faithful right Ore localization.*

Thus, the derived category $\mathsf{D}(\mathsf{CDGRng}\,/A)$ admits a calculus of fractions.

A DG ring $B \in \mathsf{CDGRng}\,/A$ is called semi-free if the graded ring $B^{\natural}$, gotten by forgetting the differential, is a commutative polynomial ring over $A^{\natural}$ (in the strongly commutative graded sense). Every $B \in \mathsf{CDGRng}\,/A$ admits semi-free resolutions $\tilde{B} \to B$.

Theorem. *Let A be a commutative DG ring, let $B, C \in \mathsf{CDGRng}\,/A$, and assume that B is semi-free. Then the function*

$$\bar{\mathrm{Q}} : \mathrm{Hom}_{\mathsf{K}(\mathsf{CDGRng}\,/A)}(B, C) \to \mathrm{Hom}_{\mathsf{D}(\mathsf{CDGRng}\,/A)}(B, C)$$

is bijective.

We give a conjectural description of the cotangent DG module $\mathrm{L}_{B/A}$ associated to a homomorphism $A \to B$ of commutative DG rings. We believe that it coincides with the cotangent complex of Quillen, André and Illusie [Il] if A and B are rings.

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Calabi–Yau structures on Drinfeld quotients, after Junyang Liu

Bernhard Keller

In 2009, Amiot [1] constructed Calabi–Yau structures on certain Verdier quotients. Our aim is to lift her construction to the level of dg (=differential graded) enhancements.

Let k be a field and $\mathcal{N}$ a k-linear Hom-finite triangulated category. Let $d \in \mathbb{Z}$ and suppose that $\mathcal{N}$ is d-Calabi–Yau, i.e. endowed with bifunctorial isomorphisms

$$D\mathrm{Hom}(X, Y) \xrightarrow{\sim} \mathrm{Hom}(Y, \Sigma^d X)$$

where $X, Y \in \mathcal{N}$ and D denotes the dual over k. For example, let X be a smooth projective variety of dimension d over an algebraically closed ground field k and $\mathcal{N}$ the bounded derived category of coherent sheaves on X. Then, by one possible definition, the variety X is d-Calabi–Yau iff its canonical sheaf ω_X is trivialisable and that happens iff $\mathcal{N}$ is d-Calabi–Yau as a triangulated category. As a second example, suppose that A is a finite-dimensional basic algebra and $\mathcal{N}$ the bounded homotopy category of finitely generated projective A-modules. Then $\mathcal{N}$ is 0-Calabi–Yau iff A is symmetric, i.e. isomorphic to its dual DA as a bimodule. The examples we are most interested in are associated to quivers with potential (Q, W) which are Jacobi-finite, i.e. the associated (complete) Jacobi algebra is finite-dimensional, cf. [2] for the terminology. Let Γ be the Ginzburg dg algebra [4], cf. also [6], associated with such a quiver with potential. Let $\mathcal{D}\Gamma$ be its unbounded derived category, $\mathrm{per}\,(\Gamma)$ its perfect derived category (the full subcategory of compact objects in $\mathcal{D}\Gamma$) and $\mathrm{pvd}\,(\Gamma)$ its *perfectly valued derived category*, i.e. the full subcategory of objects M of $\mathcal{D}\Gamma$ such that the underlying complex $M|_k$ is perfect over k. By Amiot's definition [1] of the cluster category $\mathcal{C}_{Q,W}$, we have an exact sequence of triangulated categories

$$0 \longrightarrow \mathrm{pvd}\,(\Gamma) \longrightarrow \mathrm{per}\,(\Gamma) \longrightarrow \mathcal{C}_{Q,W} \longrightarrow 0.$$

Here the perfectly valued derived category is 3-Calabi–Yau, the perfect derived category is not Calabi–Yau and the cluster-category is 2-Calabi–Yau, cf. [1] and the references given there.

The last example can be generalized as follows: Suppose that the category $\mathcal{N}$ is contained as a thick subcategory in a k-linear triangulated category $\mathcal{T}$. As shown by Amiot [1], in the exact sequence

$$0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{N} \longrightarrow 0,$$

the Verdier quotient $\mathcal{T}/\mathcal{N}$ is often $(d-1)$-Calabi–Yau. Let us recall the construction of the Calabi–Yau structure. For $N \in \mathcal{N}$, let us denote by

$$t_N : \mathrm{Hom}(N, \Sigma^d N) \to k$$

the *trace form*, which by definition corresponds to the identity of $\Sigma^d N$ under the given isomorphism

$$D\mathrm{Hom}(N, \Sigma^d N) \xrightarrow{\sim} \mathrm{Hom}(\Sigma^d N, \Sigma^d N).$$

By bifunctoriality, these trace forms determine the Calabi–Yau structure so that it suffices to construct the corresponding trace forms for the objects of $\mathcal{T}/\mathcal{N}$.

Theorem 1 (Amiot [1])**.** *Under suitable non degeneracy hypotheses, the Verdier quotient $\mathcal{T}/\mathcal{N}$ becomes $(d-1)$-Calabi–Yau for the trace forms $t_X^{\mathcal{T}/\mathcal{N}}$ defined by*

$$t_X^{\mathcal{T}/\mathcal{N}}(\text{“}f \circ s^{-1}\text{”}) = t_N^{\mathcal{N}}((\Sigma^d b) \circ f \circ a),$$

where “$f \circ s^{-1}$” is a fraction given by morphisms f and s (with cone $\Sigma N \in \mathcal{N}$) as in the following diagram

$$\begin{array}{ccccccc} N & \xrightarrow{a} & X' & \xrightarrow{s} & X & \xrightarrow{b} & \Sigma N \\ & & \downarrow{\scriptstyle f} & & & & \\ & & \Sigma^{d-1}X & \xrightarrow{\Sigma^d b} & \Sigma^d N. & & \end{array}$$

For example, if A is a finite-dimensional symmetric algebra, we have the short exact sequence of triangulated categories

$$0 \longrightarrow \mathcal{H}^b(\mathrm{proj}A) \longrightarrow \mathcal{D}^b(\mathrm{mod}\, A) \longrightarrow \mathrm{sg}(A) \longrightarrow 0$$

and we find that the stable category $\mathrm{sg}(A) \xrightarrow{\sim} \underline{\mathrm{mod}}\, A$ is (-1)-Calabi–Yau, which corresponds to the well-known fact that the Auslander–Reiten translation is given by the square of the syzygy functor $\Omega^2 = \Sigma^{-2}$.

Our aim is to lift this construction to the level of dg (=differential graded) categories. Let $\mathcal{A}$ be a (small) dg category and $\mathcal{DA}$ its derived category. Its objects are the dg functors $M : \mathcal{A}^{op} \to \mathcal{C}_{dg}(\mathrm{Mod}\, k)$ from $\mathcal{A}^{op}$ to the dg category of k-vector spaces. Suppose that $\mathcal{A}$ is *pretriangulated*, i.e. the Yoneda functor

$$H^0(\mathcal{A}) \to \mathcal{DA}\,,\; X \mapsto \mathcal{A}(?, X)$$

is an equivalence onto a full triangulated subcategory. Let $\mathcal{B} \subseteq \mathcal{A}$ be a full pretriangulated dg subcategory. By definition [3], the *Drinfeld quotient* $\mathcal{A}/_{Dr}\mathcal{B} = \mathcal{A}/\mathcal{B}$ is obtained from $\mathcal{A}$ by formally adjoining a contracting homotopy

$$h_N : N \to N, |h_N| = -1, d(h_N) = \mathbf{1}_N$$

for each object $N \in \mathcal{B}$. Thus, by definition, the canonical functor $\mathcal{A} \to \mathcal{A}/\mathcal{B}$ is *strictly universal* among the dg functors $F : \mathcal{A} \to \mathcal{F}$ to a dg category where all the objects FN, $N \in \mathcal{B}$, are endowed with a contracting homotopy. As shown in [3], we have an induced short exact sequence of triangulated categories

$$0 \longrightarrow H^0\mathcal{B} \longrightarrow H^0\mathcal{A} \longrightarrow H^0(\mathcal{A}/\mathcal{B}) \longrightarrow 0.$$

It follows that the sequence of dg categories

$$0 \longrightarrow \mathcal{B} \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{B} \longrightarrow 0$$

is *homotopy exact*, i.e. the sequence of triangulated categories

$$0 \longrightarrow \mathcal{DB} \longrightarrow \mathcal{DA} \longrightarrow \mathcal{D}(\mathcal{A}/\mathcal{B}) \longrightarrow 0$$

is exact. Therefore, by the main result of [5], we have long exact sequences in Hochschild and cyclic homology

$$\ldots \longrightarrow HC_*(\mathcal{B}) \longrightarrow HC_*(\mathcal{A}) \longrightarrow HC_*(\mathcal{A}/\mathcal{B}) \longrightarrow HC_{*-1}(\mathcal{B}) \longrightarrow \ldots$$

Now suppose that $\mathcal{B}$ is an H^*-finite dg category, i.e. the spaces $H^p(\mathcal{B})(X, Y)$ are finite-dimensional for all X, Y in $\mathcal{B}$ and all $p \in \mathbb{Z}$. Let d be an integer. Following Kontsevich, a (right) *d-Calabi–Yau structure* on $\mathcal{B}$ is a class $c \in DHC_{-d}(\mathcal{B})$ which is *non degenerate*, i.e. its image under the composition of canonical maps

$$DHC_{-d}(\mathcal{B}) \to DHH_{-d}(\mathcal{B}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}(\mathcal{B}^e)}(\mathcal{D}, \Sigma^{-d} D\mathcal{B})$$

is invertible. Here, by abuse of notation, we denote by $\mathcal{B}$ the $\mathcal{B}$-bimodule $(X, Y) \mapsto \mathcal{B}(X, Y)$ and by $D\mathcal{B}$ the $\mathcal{B}$-bimodule $(X, Y) \mapsto D\mathcal{B}(Y, X)$. If $\mathcal{B}$ is pretriangulated and carries a d-Calabi–Yau structure, then $H^0\mathcal{B}$ becomes d-Calabi–Yau as a triangulated category since we have

$$(H^0\mathcal{B})(X,Y) \xrightarrow{\sim} H^0\Sigma^{-d}D\mathcal{B}(Y,X) = D(H^0\mathcal{B})(X,\Sigma^d Y).$$

Thus, we obtain a canonical map

$$DHC^{nd}_{-d}(\mathcal{B}) \longrightarrow \{d\text{-Calabi–Yau structures on } H^0\mathcal{B}\},$$

where $DHC^{nd}_{-d}(\mathcal{B})$ denotes the space of non degenerate classes in $DHC_{-d}(\mathcal{B})$.

Now let $\mathcal{B} \subseteq \mathcal{A}$ be a full pretriangulated subcategory of a pretriangulated dg category $\mathcal{A}$. Thus, we have a long exact sequence

$$\ldots \longrightarrow HC_{-d+1}\mathcal{B} \longrightarrow HC_{-d+1}\mathcal{A} \longrightarrow HC_{-d+1}(\mathcal{A}/\mathcal{B}) \xrightarrow{\delta} HC_{-d}\mathcal{B} \longrightarrow \ldots$$

Theorem 2 (Liu 2022 [7])**.** *The square*

$$\begin{array}{ccc} DHC^{nd}_{-d}(\mathcal{B}) & \xrightarrow{D\delta} & DHC^{nd}_{-d+1}(\mathcal{A}/\mathcal{B}) \\ \downarrow & & \downarrow \\ \{d\text{-}CY \text{ structures on } H^0\mathcal{B}\} & \longrightarrow & \{(d-1)\text{-}CY \text{ structures on } H^0(\mathcal{A}/\mathcal{B})\}, \end{array}$$

where the bottom horizontal arrow is given by Amiot's construction, is commutative.

Let us sketch the proof: Let $HH(\mathcal{A})$ be the Hochschild complex of $\mathcal{A}$. It is the sum total complex of the double complex

$$\textstyle\coprod_{A_0} \mathcal{A}(A_0, A_0) \longleftarrow \coprod_{A_0,A_1} \mathcal{A}(A_0, A_1) \otimes \mathcal{A}(A_1, A_0) \longleftarrow \ldots$$

where the sums are taken over the objects of $\mathcal{A}$ and, for example, the leftmost differential sends $f_0 \otimes f_1$ to $f_1 \circ f_0 - (-1)^{|f_0||f_1|} f_1 \circ f_0$. Then the sequence of dg categories

$$0 \longrightarrow \mathcal{D}\mathcal{B} \longrightarrow \mathcal{D}\mathcal{A} \longrightarrow \mathcal{D}(\mathcal{A}/\mathcal{B}) \longrightarrow 0$$

with the given nullhomotopy for the composition $\mathcal{B} \to \mathcal{A}/\mathcal{B}$ yields a *homotopy short exact sequence of complexes* (defined below)

$$HH(\mathcal{B}) \xrightarrow{i} HH(\mathcal{A}) \xrightarrow{p} HH(\mathcal{A}/\mathcal{B}), \qquad h: HH(\mathcal{B}) \to HH(\mathcal{A}/\mathcal{B})$$

where the map h sends $f_p \otimes \cdots f_0$ to $h_{B_0} f_p \otimes \cdots \otimes f_0$ (B_0 is the source of f_0 and the target of f_p). The *homotopy snake lemma* (stated below) then allows us to compute the connecting morphism

$$\delta : HH_{-d+1}(\mathcal{A}/\mathcal{B}) \longrightarrow HH_{-d}(\mathcal{B})$$

and to check the commutativity claimed in the theorem.

We define a *homotopy short exact sequence of complexes* to be a diagram of complexes of abelian groups

$$\overset{h}{\overbrace{B \xrightarrow{\;i\;} A \xrightarrow{\;p\;} C}}$$

such that i and p are morphisms of complexes and h is a homogeneous morphism of degree -1 such that $p \circ i = d(h)$ and that the graded object $C \oplus \Sigma A \oplus \Sigma^2 B$ endowed with the differential

$$\begin{bmatrix} d & p & h \\ 0 & -d & i \\ 0 & 0 & d \end{bmatrix}$$

is acyclic. Such a diagram yields a triangle

$$B \longrightarrow A \longrightarrow C \longrightarrow \Sigma B$$

in the derived category of abelian groups and we can compute the connecting morphism

$$\delta : H^p C \to H^{p+1} B$$

thanks to the following homotopy snake lemma, where $b \in B^{p+1}$, $c \in C^p$, $a \in A^p$.

Lemma 1. *We have $\delta(\overline{c}) = \overline{b}$ if and only if there is an $a \in A$ such that we have $i(b) = -d(a)$ and $p(a) + h(b) = c$.*

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Matrix factorizations, Reality and Knörrer periodicity

Matthew B. Young

(joint work with Jan-Luca Spellmann)

Let $w \in R = \mathbb{C}[\![x_1, \ldots, x_n]\!]$ be a non-zero polynomial without constant term. A *matrix factorization* of w is a $\mathbb{Z}/2\mathbb{Z}$-graded finite rank free R-module M with an odd R-linear endomorphism d_M which satisfies $d_M^2 = w \cdot \mathrm{id}_M$. The 2-periodic dg category of matrix factorizations $\mathrm{MF}(R, w)$ is a categorical invariant of (R, w)

introduced by Eisenbud. Much recent interest in $\mathrm{MF}(R, w)$ stems from its appearance in high energy physics as the category of D-branes in the Landau–Ginzburg model defined by (R, w).

A fundamental property of matrix factorizations is their *Knörrer periodicity* [3], which is a quasi-equivalence

$$\mathrm{MF}(R, w) \xrightarrow{\sim} \mathrm{MF}(R[\![y, z]\!], w + y^2 + z^2).$$

The goal of this talk is to explain generalizations of Knörrer periodicity to categories of *Real matrix factorizations.* The word Real denotes two distinct, but related, structures on matrix factorizations. The first is a matrix factorization theoretic analogue of Atiyah's Real structures on complex vector bundles. Recall that Real vector bundles are the fundamental geometric objects of KR-theory [1]. The second is a matrix factorization theoretic analogue of the fundamental geometric objects of the Grothendieck–Witt theory of a scheme, the algebraic counterpart of KR-theory. Accordingly, we prove two distinct, but related, Real generalizations of Knörrer periodicity, which we argue are structurally similar to (1,1)-periodicity for KR-theory and 4-periodicity for Grothendieck–Witt theory.

To state our results, let C_2 be the multiplicative group $\{1, -1\}$ and $\pi : \hat{G} \to C_2$ a C_2-graded finite group. Set $G = \ker \pi$. Suppose first that $\hat{G}$ acts on R by ring automorphisms $\{\sigma : R \to R\}_{\sigma \in \hat{G}}$ which are $\mathbb{C}$-linear if $\pi(\sigma) = 1$ and $\mathbb{C}$-antilinear if $\pi(\sigma) = -1$ and such that w is $\hat{G}$-invariant, i.e $\sigma(w) = w$, for $\sigma \in \hat{G}$.

In particular, only G is a symmetry of (R, w) in the sense of Landau–Ginzburg orbifolds. A *Real G-equivariant structure* on a matrix factorization (M, d_M) is a coherent family of d_M-linear graded abelian group isomorphisms $\{u_\sigma : M \to M\}_{\sigma \in \hat{G}}$ which satisfy $u_\sigma(rm) = \sigma^{-1}(r) u_\sigma(m)$ for all $r \in R$, $m \in M$. Let $\mathrm{MF}_{\hat{G}}(\mathbb{C}[\![x_1, \dots, x_n]\!], w)$ be the dg category of Real G-equivariant matrix factorizations. Extend the $\mathbb{C}$-antilinear action of $\hat{G}$ on R to $R[\![y, z]\!]$ by requiring G to act trivially and $\sigma(y) = -y$, $\sigma(z) = z$ for $\sigma \in \hat{G} \setminus G$. The following form of Real Knörrer periodicity is an analogue of $(1, 1)$-periodicity:

Theorem 1. *There is a quasi-equivalence of $\mathbb{R}$-linear dg categories*

$$\mathrm{Perf}(\mathrm{MF}_{\hat{G}}(R, w)) \xrightarrow{\sim} \mathrm{Perf}(\mathrm{MF}_{\hat{G}}(R[\![y, z]\!], w + y^2 + z^2)),$$

where $\mathrm{Perf}(\mathcal{C})$ *denotes the triangulated hull of a dg category* $\mathcal{C}$.

In the second setting, suppose that $\hat{G}$ acts on R by $\mathbb{C}$-algebra automorphisms $\{\sigma : R \to R\}_{\sigma \in \hat{G}}$ such that w is π-semi-invariant, i.e. $\sigma(w) = \pi(\sigma) w$ for $\sigma \in \hat{G}$. Again, only G is an orbifold symmetry of (R, w). However, as explained by Hori–Walcher [5], the entire group $\hat{G}$ is an *orientifold* symmetry of (R, w). Orientifolding is a physical construction which produces an unoriented string theory from an oriented one, in contrast to the standard orbifold construction, which preserves orientability. As such, the second setting can be seen as a precise mathematical approach to Landau–Ginzburg orientifolds. From the above data, we construct a duality structure on G-equivariant matrix factorizations $\mathrm{MF}_G(R, w)$, that is, a dg functor $\mathrm{MF}_G(R, w)^{\mathrm{op}} \to \mathrm{MF}_G(R, w)$ with coherence data asserting that it is an involution. Extend the $\mathbb{C}$-linear action of $\hat{G}$ on R to $R[\![y,z]\!]$ by requiring G to act trivially and $\sigma(y) = -iz$, $\sigma(z) = iy$ for $\sigma \in \hat{G} \setminus G$. The second form of Real Knörrer periodicity, the second part of which is an analogue of 4-periodicity of Grothendieck–Witt theory, is as follows.

Theorem 2. *There is a quasi-equivalence of $\mathbb{C}$-linear dg categories with duality*

$$\mathrm{Perf}(\mathrm{MF}_G(R,w)) \xrightarrow{\sim} \mathrm{Perf}(\mathrm{MF}_G(R[\![y,z]\!], w + y^2 + z^2)),$$

where the duality structure of the codomain is a shifted and signed version of that of the domain. In particular, there is a quasi-equivalence of dg categories with duality

$$\mathrm{Perf}(\mathrm{MF}_G(R,w)) \xrightarrow{\sim} \mathrm{Perf}(\mathrm{MF}_G(R[\![y_1,z_1,y_2,z_2]\!], w + y_1^2 + z_1^2 + y_2^2 + z_2^2)),$$

where both dg categories are given the same duality structure.

Degenerate versions of Theorems 1 and 2 recover specializations of Hirano's equivariant Knörrer periodicity [4], Hori–Walcher's extended Knörrer periodicity [5] and Brown's 8-periodic version of Knörrer periodicity over $\mathbb{R}$ [2]. Theorems 1 and 2 are proved using techniques from Real categorical representation theory.

This work is part of a larger project whose goal is to use Real matrix factorizations to construct non-semisimple unoriented topological field theories in two dimensions. The results of this talk are described in [6].

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Simplices of higher-dimensional flops

Will Donovan

The set of crepant resolutions of a given singularity may be related by an intricate web of birational maps. In this talk, I discussed a particular sequence of singularities in dimension 4 and above, and the associated derived category structures.

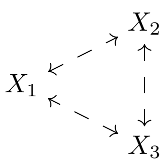
These singularities have crepant resolutions whose exceptional loci are isomorphic to cartesian powers of the projective line $\mathbb{P}^1$. In each dimension n, the resolutions naturally correspond to vertices of an $(n-2)$-simplex, and flops between them correspond to edges of the simplex. I show that each face of the simplex corresponds to a certain relation between functors of derived categories.

Singularities and resolutions. Consider the n-fold singular cone for $n \geq 3$ given by the rank 1 tensors of signature 2^{n-1} as follows.

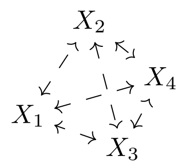
$$Z = \{v_1 \otimes \cdots \otimes v_{n-1} \in V_1 \otimes \cdots \otimes V_{n-1}\} \qquad V_i \cong \mathbb{C}^2$$

The cone Z has $n-1$ crepant resolutions, here written as X_i for $i = 1, \ldots, n-1$. These may be constructed explicitly by putting $\mathrm{Exc} = \mathbb{P}V_{i+1} \times \cdots \times \mathbb{P}V_{i+n-2}$ where subscripts for the vector spaces $V_1, \ldots, V_{n-1}$ are taken modulo $n-1$. Then the resolution X_i is given by the total space of the rank 2 bundle $V_i \otimes \mathcal{O}(-1, \ldots, -1)$ over Exc along with a natural map to Z. The exceptional locus for each X_i is the zero locus of this bundle, which may be identified with $\mathrm{Exc} \cong (\mathbb{P}^1)^{n-2}$.

For $n = 3$, we have two resolutions of the cone of singular 2×2 matrices, related by a 3-fold Atiyah flop. In general, assigning each resolution X_i to a vertex of an $(n-2)$-simplex, the edges of the simplex correspond to birational maps $X_i \dashleftarrow\dashrightarrow X_j$ as illustrated below.



(A) Four-folds for $n = 4$.



(B) Five-folds for $n = 5$.

Result. The birational maps appearing here are all family Atiyah flops, and therefore have associated flop functors, which are equivalences of derived categories of coherent sheaves. This is illustrated below (on the left) for the case $n = 4$.

(1)

For $n \geq 4$ take, without loss of generality, three resolutions X_1, X_2 and X_3. I then prove the following, by calculating the action of functors on a certain (relative) tilting bundle.

Theorem. [2] *Write flop functors as on the right in* (1)*. Then there is a natural isomorphism* $\mathsf{F}_3 \cong \mathsf{Tw}_2 \circ \mathsf{F}_2 \circ \mathsf{F}_1$ *where* Tw_2 *is for* $n = 4$ *a Seidel–Thomas spherical twist around the torsion sheaf* $\mathcal{O}_{\mathrm{Exc}}(0, -1)$ *on* X_3*, where* $\mathrm{Exc} \cong \mathbb{P}V_1 \times \mathbb{P}V_2$*, and for* $n > 4$ *a family version* [1] *of this spherical twist, over base* $\mathbb{P}V_4 \times \cdots \times \mathbb{P}V_{n-1}$.

Remark. *As an immediate corollary of the theorem, we get* $\mathsf{F}_3 \circ \mathsf{F}_1^{-1} \circ \mathsf{F}_2^{-1} \cong \mathsf{Tw}_2$*. There are many results of the form 'flop-flop = twist' (up to taking inverses) in the literature. I hope the above formula gives a hint of how they may generalize to flop cycles of length 3 and above.*

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Classifying crepant resolutions of quotient singularities and quiver varieties

Travis Schedler

(joint work with Gwyn Bellamy, Alastair Craw, Steven Rayan, Hartmut Weiß, Dan Kaplan)

The main goal of this work, based on [3], [4], and [11] is to understand crepant resolutions of symplectic singularities. These resolutions are ubiquitous, not merely in geometry, but in geometric representation theory and physics. For instance, the Springer resolution encodes the representation theory of semisimple Lie algebras. The Higgs and Coulomb branch varieties are important pieces of the moduli of vacua in supersymmetric gauge theories.

This talk was focused on two main kinds of symplectic cones: the quotient singularities V/Γ for V a symplectic complex vector space and $\Gamma \leq \mathrm{Sp}(V)$ a finite subgroup, and Nakajima quiver varieties, defined as a Hamiltonian reduction of $T^* \operatorname{Rep}_\alpha(Q)$ by GL_α, where Q is a quiver with vertex set Q_0 and arrow set Q_1, and $\alpha \in \mathbb{N}^{Q_0}$.

In [3], we found a surprising isomorphism between one of each of these examples. The talk was constructed so as to introduce each of the cases, observing similarities, and discuss the classification question in this context, before generalizing to arbitrary cones of the preceding type.

The quotient singularity V/Γ we consider takes $V = \mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^4$ and $G = Q_8 \times_{C_2} D_8$, where $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\} \subseteq \mathbb{H}$ is the quaternionic group of order eight, and D_8 is the dihedral group of order eight. Since Q_8 acts on $\mathbb{C}^2$ preserving a symplectic form, and D_8 acts on $\mathbb{C}^2$ preserving an orthogonal form, the product $Q_8 \times D_8$ acts on $V = \mathbb{C}^2 \otimes \mathbb{C}^2$ preserving the tensor product of these forms, which is symplectic. The kernel of the action is the order-two subgroup generated by $(-I, -I)$, so calling this the subgroup C_2, the quotient $Q_8 \times_{C_2} D_8 := (Q_8 \times D_8)/C_2$ acts faithfully and symplectically on V.

This quotient singularity was considered in [6], where it was shown to admit a crepant (=symplectic) resolution. Subsequently in [5], Bellamy showed that there were precisely 81 projective crepant resolutions. Then, in [7], all of these were explicitly constructed via Cox rings, with the help of computer.

Part of the idea of the classification result of [5] involves explicitly computing a hyperplane arrangement which was defined earlier in [14], together with Namikawa's Weyl group action. Briefly, for any projective crepant resolution $\rho : X \to Y$ of a Gorenstein singularity Y, one can consider the rational vector space $N^1(X/Y)$ spanned by line bundles on X modulo numerical equivalence, i.e., $L \simeq L'$ if L and L' have the same degree on every irreducible curve contracted by ρ (necessarily projective). Then one can consider the movable cone $\mathrm{Mov}(X/Y) \subseteq N^1(X/Y)$ of all line bundles such that the stable base locus (the locus where some section of a tensor power of the bundle does not vanish) has codimension at least two. By [2], the movable cone is a finite polyhedral cone, whose chambers, called Mori chambers, are the ample cones of all the other projective crepant resolutions (this says that X is a relative Mori dream space over Y). In [14], in the case that X is symplectic (equivalently, Y is a symplectic singularity), it was shown that $N^1(X/Y)$ is equipped with a finite hyperplane arrangement and a Weyl group action W, preserving the hyperplanes, so that a fundamental region for the action is identified with $\mathrm{Mov}(X/Y)$, sending the hyperplanes precisely to the boundary walls of the Mori chambers. As a result, the number of projective crepant resolutions equals the number of chambers in the complement of this hyperplane arrangement, divided by the order of W.

Finally, in [5], this combinatorial data was computed for an arbitrary quotient singularity V/Γ. The main point is that there is a natural deformation of V/Γ defined by Etingof and Ginzburg [9] (whose construction appears earlier in work of Drinfeld [8]), by the spectrum of the so-called spherical symplectic reflection algebra, $eH_{0,c}e$. The base of this family consists of conjugation-invariant functions $S \to \mathbb{C}$ for $S \subseteq \Gamma$ the subset of *symplectic reflections*, i.e., γ such that V^γ has codimension two (the minimum nonzero value, since Γ acts symplectically). In the case that this generic deformation is smooth (which is actually equivalent to the existence of a projective crepant resolution thanks to [10], [13]), the hyperplane arrangement consists of the loci of $c : S \to \mathbb{C}$ such that the deformation is singular.

In the talk, I illustrated these ideas by explicitly describing the hyperplane arrangement for the aforementioned example V/Γ. First we briefly review some facts about Γ: this group of size 32 has seventeen conjugacy classes; aside from $\{I\}$ and $\{-I\}$, they all have order two. The symplectic reflections are the order-two elements other than $-I$, and there are ten of these, giving five conjugacy classes. The outer automorphism group of Γ acts simply transitively on these five classes, hence is isomorphic to S_5. The quotient $\Gamma/\{\pm I\}$ is abelian and isomorphic to C_2^4, which means that there are sixteen one-dimensional characters $\chi : \Gamma \to \mathbb{C}^\times$, each valued in $\{\pm 1\}$.

Let $s_1, \ldots, s_5$ be representatives of the five conjugacy classes of symplectic reflections. The hyperplane arrangement in question consists of the five coordinate planes $\{c(s_i) = 0\}$, together with the sixteen hyperplanes $\{\sum_{s \in S} \chi(s)c(s) = 0\}$, where χ ranges over all one-dimensional characters.

On the other side, I considered the quiver in Fig. 1. This quiver Q has vertex set Q_0 of size six and arrow set Q_1 of size five. Let $\alpha = (2,1,1,1,1,1)$ be the dimension vector, and $\mathrm{GL}_\alpha = \mathrm{GL}_2 \times \mathrm{GL}_1^5$, with Lie algebra $\mathfrak{gl}_\alpha = \mathfrak{gl}_2 \oplus \mathfrak{gl}_1^5$. Consider the representation space $\mathrm{Rep}_\alpha(Q) = \mathrm{Hom}(\mathbb{C}, \mathbb{C}^2)^5$ and doubled space $T^* \mathrm{Rep}_\alpha(Q) \cong \mathrm{Hom}(\mathbb{C}, \mathbb{C}^2)^5 \oplus \mathrm{Hom}(\mathbb{C}^2, \mathbb{C})^5$. The moment map $\mu : T^* \mathrm{Rep}_\alpha(Q) \to \mathfrak{gl}_\alpha$ is given by

$$\mu((X_1, \ldots, X_5), (Y_1, \ldots, Y_5)) = (X_1Y_1 + \cdots + X_5Y_5, -Y_1X_1, -Y_2X_2, \ldots, -Y_5X_5).$$

The quiver variety $\mathfrak{M}_0(Q, \alpha)$ is defined by the categorical quotient $\mu^{-1}(0)/\!/ \mathrm{GL}_\alpha$.

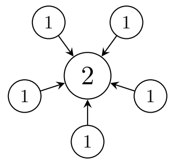

FIGURE 1. Five-branch star shaped quiver

We also consider a natural resolution of singularities, given by a stability condition $\theta \in \mathbb{Z}^6$ satisfying $\theta \cdot \alpha = 0$. We may then consider the open subsets $\mu^{-1}(0)^{\theta\text{-s}} \subseteq \mu^{-1}(0)^{\theta\text{-ss}} \subseteq \mu^{-1}(0)$ of θ-stable and semistable points, respectively. Concretely, these may be identified with the representations of the doubled quiver $\overline{Q}$ (now with ten arrows, going in both directions from the central node to the external ones) such that every dimension β of a subrepresentation satisfies $\beta \cdot \theta < 0$, or $\beta \cdot \theta \leq 0$, respectively. Then we define the quiver variety $\mathfrak{M}_\theta(Q, \alpha) := \mu^{-1}(0)^{\theta\text{-ss}} /\!/ \operatorname{GL}_\alpha$. This is a geometric invariant theory (GIT) quotient, which admits a projective morphism $\mathfrak{M}_\theta(Q, \alpha) \to \mathfrak{M}_0(Q, \alpha)$. For generic θ, every θ-semistable point is stable, which implies that $\mathfrak{M}_\theta(Q, \alpha)$ is smooth and this morphism is a crepant resolution of singularities.

The quiver varieties $\mathfrak{M}_\theta(Q, \alpha)$ are hyperkähler analogues of Kirwan and Klyachko's moduli spaces of pentagons in $\mathbb{R}^3$ with prescribed edge lengths, which are isomorphic to $\operatorname{Rep}_\alpha(Q)^{\theta\text{-ss}} /\!/ \operatorname{GL}_\alpha$, with the last five coordinates of θ giving the edge lengths (above we only allowed θ to be integral in order to have an interpretation via GIT, but this can be generalized to any real values).

Note that the condition $\theta \cdot \alpha = 0$ means that θ is uniquely determined by the five values at the external vertices. The generic values of θ are when these values are in the complement of the following explicit hyperplane arrangement in $\mathbb{R}^5$: the five coordinate planes $\theta_i = 0, 1 \leq i \leq 5$, and for every subset $I \subseteq \{1, \ldots, 5\}$, the hyperplane $\sum_{i \in I} \theta_i = \sum_{j \notin I} \theta_j$. *These hyperplanes are the same as those for the quotient singularity discussed above!* In fact, Mekareeya [12] observed that the two cones V/Γ and $\mathfrak{M}_0(Q, \alpha)$ have the same Hilbert series of rings of regular functions, suggesting they are isomorphic.

Theorem 1. [3] *There is an isomorphism* $V/\Gamma \cong \mathfrak{M}_0(Q, \alpha)$.

The idea of the proof is, while writing an explicit isomorphism is difficult, it is much easier to relate the *Cox ring* of V/Γ, which is $\mathbb{C}[V]^{[\Gamma,\Gamma]=\{\pm I\}}$ with the ring $\mathbb{C}[\mu^{-1}(0)]^{\mathrm{SL}_2}$ of "semi-invariants", which in turn are closely related to the Cox ring of resolutions $\mathfrak{M}_\theta(Q, \alpha)$. In more detail, we define a surjection $\mathbb{C}[\mu^{-1}(0)]^{\mathrm{SL}_2} \to \mathbb{C}[V]^{\pm I}$. Under this surjection, the subring $\mathbb{C}[\mu^{-1}(0)]^{\mathrm{GL}_\alpha}$ has image $\mathbb{C}[V]^\Gamma$, and since these are both integral domains of the same dimension, it is an isomorphism.

In [3], using techniques of [1], we show that all projective crepant resolutions of $\mathfrak{M}_0(Q, \alpha)$ are of the form $\mathfrak{M}_\theta(Q, \alpha)$, by identifying the movable cone $\operatorname{Mov}(\mathfrak{M}_\theta(Q, \alpha)/\mathfrak{M}_0(Q, \alpha))$ for a fixed generic θ with a region of the above hyperplane arrangement in $\mathbb{R}^5$—the region where all coordinates are nonnegative. This we did also for the analogous quiver Q^n with any number n of branches, and $\alpha^n = (2, 1, \dots, 1) \in \mathbb{N}^n$. The resolutions are then called hyperpolygon spaces, in analogy with the polygon spaces of Kirwan and Kylachko. The hyperplane arrangement is also defined in exactly the same way, replacing 5 by n.

Theorem 2. [3] *The projective crepant resolutions of* $\mathfrak{M}_0(Q^n, \alpha^n)$ *are in bijection with the chambers of the hyperplane arrangement lying in the region* $\mathbb{R}^n_{\geq 0}$.

In particular this gives a conceptual and computational-free construction of all of the projective crepant resolutions of $V/\Gamma \cong \mathfrak{M}_0(Q^5, \alpha^5)$. Moreover, as there are well known techniques for computing with hyperplane arrangements, we can count resolutions (with some computational help):

Corollary 3. *There are precisely* 81 *projective crepant resolutions of* $V/\Gamma \cong \mathfrak{M}_0(Q^5, \alpha^5)$. *There are precisely* 1684 *projective crepant resolutions of* $\mathfrak{M}_0(Q^6, \alpha^6)$.

The techniques used to prove Theorem 2 apply to quite general GIT quotients. The main point is to show that, fixing generic θ, the linearization map $L_\theta : \mathrm{Hom}(\mathrm{GL}_\alpha / \mathbb{C}^\times, \mathbb{C}^\times) \otimes_{\mathbb{Z}} \mathbb{Q} \to N^1(\mathfrak{M}_\theta(Q, \alpha)/\mathfrak{M}_0(Q, \alpha))$ identifies a region of the GIT fan with the movable cone. In work in progress with Bellamy and Craw, we carry this out generally, and as a consequence prove:

Theorem 4 (Bellamy–Craw–Schedler, in progress)**.** *Fix an arbitrary quiver* Q *with dimension vector* α. *If there is a projective crepant resolution of the form* $\mathfrak{M}_\theta(Q, \alpha) \to \mathfrak{M}_0(Q, \alpha)$, *then every projective crepant resolution is of this form. Under a mild minimality assumption on* α, *two values* θ, θ' *give isomorphic resolutions if and only if their GIT chambers are images of each other under an element of the Namikawa Weyl group.*

We note that the minimality assumption on α can always be arranged up to replacing α with some $\alpha' < \alpha$ with $\mathfrak{M}_0(Q, \alpha) \cong \mathfrak{M}_0(Q, \alpha')$; equivalently, we could leave α the same and pass to a linear subspace of the GIT parameter space. Also, we understand the conditions under which some crepant resolution is given as above. For example, if α is not just minimal but indecomposable (we cannot write $\mathfrak{M}_0(Q, \alpha)$ as a product of quiver varieties for $\alpha' < \alpha$), then the condition is that α is indivisible, i.e., $\gcd(\alpha_i) = 1$.

Finally, I briefly discussed joint work in progress with D. Kaplan. The point here is that many moduli spaces have singularities locally given by quiver varieties. By the above, we can classify projective crepant resolutions of these singularities (under mild assumptions). Our joint work explains what the obstructions are to gluing these to a global crepant resolution. The main point is that, if these local resolutions do glue, they glue uniquely. This is because, if $f : X \to Y$ and $f' : X' \to Y$ are two birational maps, then there can exist at most one isomorphism $g : X \to X'$ such that $f = f' \circ g$, as g is uniquely determined on a dense open subset. Hence the only problem with gluing resolutions is to see whether the choices of local resolutions are compatible with each other. This leads to a combinatorial description of all global locally projective crepant resolutions in terms of the local hyperplane arrangements and Namikawa Weyl group actions. See the extended abstract by Kaplan for more details on this.

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Projectivity of good moduli spaces of semistable quiver representations and vector bundles

Pieter Belmans

(joint work with C. Damiolini, H. Franzen, V. Hoskins, S. Makarova, T. Tajakka)

1. Parallels between quivers and curves

There are many parallels between moduli spaces $\mathrm{M}_Q^{\theta\text{-ss}}(\mathbf{d})$ of semistable quiver representations and moduli spaces $\mathrm{M}_C(r,d)$ of semistable vector bundles on a curve C. Here we consider a quiver $Q = (Q_0, Q_1)$ and finite-dimensional representations of Q with dimension vector $\mathbf{d}$, which are semistable with respect to a stability function $\theta\colon \mathbb{Z}^{Q_0} \to \mathbb{Z}$ such that $\theta(\mathbf{d}) = 0$ (resp. semistable bundles of rank r and degree d, where C has genus $g \geq 2$). Some of these parallels are discussed in [6], and there are others (such as similarities in the structure of their Brauer groups, or rationality questions).

The parallel I wish to focus on is their (usual) construction in algebraic geometry via geometric invariant theory (GIT). Using the notion of (semi)stability for quiver representations [7] (resp. for vector bundles [8]) one considers

- the open locus of $\mathrm{Rep}(Q, \mathbf{d}) = \prod_{i \in Q_0} \mathbb{A}_k^{\mathbf{d}(i)}$, respectively
- a suitable Quot scheme (by making all bundles globally generated)

corresponding to semistable objects, and then quotients out

- the conjugation action by $\mathrm{GL}_{\mathbf{d}} = \prod_{i \in Q_0} \mathrm{GL}_{\mathbf{d}(i)}$, respectively
- the GL_N-action with N the dimension of the global sections after twisting.

The moduli space of semistable quiver representations is *projective-over-affine*, the base affine variety being the spectrum of the invariant ring (from the zero stability function), which by Le Bruyn–Procesi is generated by traces along cycles. In particular, if Q is acyclic the resulting moduli space of semistable representations is *projective*. The moduli space of semistable vector bundles is always projective.

Nowadays there are many (more complicated) moduli spaces being studied, e.g. moduli spaces of Bridgeland-semistable objects [9]. For these *no* GIT-construction is available, so a GIT-free construction is needed. The general program is to:

(1) interpret the moduli problem as an algebraic stack $\mathcal{M}$ of finite type;
(2) prove that there exists a good moduli space M, and show it is a proper algebraic space via the valuative criterion for universal closedness for $\mathcal{M}$;
(3) descend a line bundle to M and check its ampleness.

For step (2), if $\mathcal{M}$ has finite stabilisers, then the Keel–Mori theorem provides coarse moduli space M. If $\mathcal{M}$ has infinite stabilisers, one can use the recent Alper–Halpern-Leistner–Heinloth existence criterion [2, Theorem A].

For step (3) one uses the moduli-theoretic interpretation of the space and the line bundle to produce sections thereof, in order to check ampleness.

By implementing this program for well-known moduli spaces one can start to understand more complicated constructions, and also obtain additional results in these classical cases. The program is explained for $\mathrm{M}_C(r, d)$ in [1]. For the Deligne–Mumford compactification $\overline{\mathrm{M}}_g$ of the moduli space of curves (which is also usually constructed using GIT) it is explained in [4].

In the next section I will briefly explain the structure of the program in the case of quiver representations, which is the novel joint work I'm reporting on [3]. In this abstract we work over an algebraically closed field k of characteristic 0, so that we avoid adequate moduli spaces and geometrically stable representations, but op. cit. is written in greater generality.

2. Projectivity for moduli spaces of quiver representations

Let us assume that Q is acyclic in what follows. Step (1) consists of writing the usual setup in a suitable functor-of-points language and quickly deducing the necessary properties. For step (2) one applies the Alper–Halpern-Leistner–Heinloth existence criterion for the moduli stack $\mathcal{M}_Q^{\theta\text{-ss}}(\mathbf{d})$, by explicitly checking Θ-reductivity, S-completeness, and the valuative criterion for universal closedness. The latter is done by giving a version of Langton's semistable reduction for quiver representations. The existence criterion then yields a good moduli space $\mathrm{M}_Q^{\theta\text{-ss}}(\mathbf{d})$, and shows it is a proper algebraic space.

In order to show projectivity we need a line bundle on $\mathrm{M}_Q^{\theta\text{-ss}}(\mathbf{d})$ for which we can prove ampleness. In the setting of vector bundles on a curve such line bundles are provided by descending *determinantal line bundles* from the moduli stack: considering the Fourier–Mukai functor given by the universal vector bundle, a vector bundle F on C gives a 2-term complex of vector bundles on the moduli stack. One can take the determinant of this complex to obtain a line bundle on the moduli stack, which only depends on the numerical invariants of F. If the rank and degree of F are chosen appropriately so that $\chi(C, E \otimes F) = 0$ (where E has rank r and degree d) it is possible to descend this line bundle to the good moduli space, and moreover construct a section of this line bundle (which does depend on the isomorphism class of F).

A similar construction can be done for moduli of quiver representations using the universal representation. Interpreted in concrete terms (which is how Schofield originally introduced them): if M is a $\mathbf{d}$-dimensional and N is an $\mathbf{e}$-dimensional representation, define $d_N^M : \bigoplus_{i \in Q_0} \mathrm{Hom}_k(M_i, N_i) \to \bigoplus_{\alpha \in Q_1} \mathrm{Hom}_k(M_{t(\alpha)}, N_{h(\alpha)})$ as $(\phi_i)_{i \in Q_0} \mapsto (\phi_{h(\alpha)} \circ M_\alpha - N_\alpha \circ \phi_{t(\alpha)})_{\alpha \in Q_1}$. If $\langle \mathbf{d}, \mathbf{e} \rangle = 0$, then d_N^M is in fact a square matrix, and following Schofield we define the *determinantal semi-invariant* $c(M, N) := \det d_N^M$. In what follows we will usually fix some N (or more precisely try to construct one with special properties) so that $c(\_, N)$ can be seen as a section of a determinantal line bundle on $\mathcal{M}_Q^{\theta\text{-ss}}(\mathbf{d})$. One important result on these determinantal semi-invariants is that by varying over all N of dimension vector $\mathbf{e}$ orthogonal to $\mathbf{d}$ they span the ring of semi-invariants (as a vector space), a result independently proven by Derksen–Weyman, Schofield–Van den Bergh and Domokos–Zubkov. The semi-invariant $c(\_, N)$ has weight $-\langle \_, \mathbf{e} \rangle$.

The next step is to produce enough determinantal semi-invariants to show that the determinantal line bundles (for an appropriate choice of multiple of the dimension vector $\mathbf{e}$) is basepoint-free. This can be done using the analogue of Faltings's characterization of semistable vector bundles, which says that a vector bundle is semistable if and only if there exists a vector bundle orthogonal in the sense from above for which $\mathrm{Hom}_C(F^\vee, E) = \mathrm{Ext}_C^1(F^\vee, E) = 0$. Such characterizations were known already to Schofield(–Van den Bergh) and Crawley-Boevey. But interestingly, from our proof we also obtain effective bounds on which power of this semiample determinantal line bundle becomes basepoint-free.

The final step, where truly new ingredients are needed, is to prove for an acyclic quiver Q that determinantal line bundle is *ample*, and not just semiample. This is done by performing a dimension count, which for curves is done in [5]. The semiample determinantal line bundle provides us with a morphism from a proper algebraic space to some $\mathbb{P}^N$, and by constructing suitable determinantal semi-invariants we can separate enough points to prove that this map is finite, thus the determinantal line bundle is ample. This part of the argument builds upon the (limited) compatibility between Auslander–Reiten functors and semistability.

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Soergel and Springer: Motives and Correspondences

Jens Niklas Eberhardt

(joint work with Catharina Stroppel)

In this talk, I presented joint work with Catharina Stroppel, see [9], in which we explain how many equivalences between categories of sheaves on spaces and representations of convolution algebras can be understood in terms of formality of so-called *Springer motives.*

1. Convolution

1.1. **A toy example.** As a first motivation, we start with a toy example of convolution of functions on finite sets. For a finite set X, denote by $\mathbb{Q}^X$ the vector space of $\mathbb{Q}$-valued functions on X. There is a bilinear point wise product $\cap$ as well as pullbacks f^* and pushforwards $f_!$ (summing along the fibers of f) for maps $f : X \to Y$. For three finite sets X, Y and Z one can define the *convolution product*

$$* : \mathbb{Q}^{X\times Y} \times \mathbb{Q}^{Y\times Z} \to \mathbb{Q}^{X\times Z},\ \alpha * \beta = \pi_!(\Delta^*(p^*(\alpha) \cap q^*(\beta))) \tag{1}$$

using the natural maps

$$\begin{matrix} X\times Y \xleftarrow{p} \\ Y\times Z \xleftarrow[q]{} \end{matrix} X\times Y\times Y\times Z \xleftarrow{\Delta} X\times Y\times X \xrightarrow{\pi} X\times Y.$$

In fact, this convolution product is just matrix multiplication.

1.2. **Cohomological Convolution Algebras.** The toy example has a vast generalization by replacing finite sets X by spaces (manifolds, varieties, stacks, ...) and the vector space of functions $\mathbb{Q}^X$ by some cohomology theory (singular cohomology, K-theory, Chow groups,...). Since these cohomology theories are (under appropriate assumptions) also equipped with analogues the operations $\cap, f^*, f_!$ one can define a convolution product exactly as in (1). For example, the Chow groups $(\mathrm{CH}_\bullet(X \times_S X)_\mathbb{Q}, *)$ with convolution form a graded associative algebra [4].

1.3. **Examples.** Let $\mathcal{N} \subset \mathfrak{g}$ be the nilpotent cone in a complex reductive Lie algebra and denote by $\mu : T^*X \to \mathcal{N}$ the *Springer resolution*, where X is flag variety of $\mathfrak{g}$. Then there is an isomorphism $(\mathrm{CH}_{\dim T^*X}(T^*X \times_\mathcal{N} T^*X)_\mathbb{Q}, *) = \mathbb{Q}[W]$ where W is the Weyl group. Using equivariant Chow groups and algebraic K-theory yields Lusztig's (graded) affine Hecke algebra, see [17, 10]. Similarly, quiver Hecke/Schur algebras arise from quiver flag varieties, [12, 19, 18]. Another example is *Soergel theory*: Consider the *Bott–Samelson resolutions* of *Schubert varieties* in the flag variety $\mu : BS = \biguplus_{w \in W} BS(\underline{w}) \to X$ where $\underline{w}$ is a reduced expression for $w \in W$. The category of finite-dimensional modules over $(\mathrm{CH}_\bullet(BS \times_X BS)_\mathbb{C}, *)$ is equivalent to $\mathcal{O}_0(\mathfrak{g}^L)$, the principal block of category $\mathcal{O}$ associated to the Langlands dual Lie algebra, see [14].

1.4. **Chow Motives.** Convolution also appears prominently in Grothendieck's category of Chow motives. Let S/k be variety over a field k. The category of correspondences $\mathrm{Corr}(S)_\mathbb{Q}$ consists of objects $M(X/S)$ for any $\pi : X \to S$ for X/k smooth and π projective. Its morphism are $\mathrm{Hom}(M(X/S), M(Y/S)) = \mathrm{CH}_{\dim X}(X \times_S Y)_\mathbb{Q}$ and are composed by convolution. By passing to the idempotent completion and formally inverting the reduced motive of $\mathbb{P}^1$ one obtains the category of Chow motives $\mathrm{Chow}(S)_\mathbb{Q}$.

1.5. **Mixed Motives.** Chow motives can be considered *pure motives*, since they are tied to smooth projective varieties. Correspondingly, they only form an additive category. However, they are contained in the triangulated category of *mixed motives* $\mathrm{Chow}(S)_\mathbb{Q} \subset \mathrm{DM}(S)_\mathbb{Q}$, see [20, 5]. The categories $\mathrm{DM}(S)_\mathbb{Q}$ come equipped with a six functor formalism and realisation functors to constructible sheaves on the complex points $S^{an}(\mathbb{C})$ or ℓ-adic sheaves on $S/\overline{\mathbb{F}}_q$. Moreover, under the correct assumptions, they are equipped with a weight structure w whose heart $\mathrm{DM}(S)_\mathbb{Q}^{w=0} = \mathrm{Chow}(S)_\mathbb{Q}$ is the category of Chow motives, see [2]. This implies the existence of a weight complex functor $\mathrm{DM}(S) \to K^b(\mathrm{Chow}(S)_\mathbb{Q})$ to the homotopy category of complexes of Chow motives.

2. Motivic Springer Theory

2.1. **Springer Motives.** Let $k = \mathbb{F}_q$, $\mu : X \to S$ be a projective map of varieties over k and X/k be smooth. Moreover, assume that μ is equivariant for the action of a linear algebraic group G. Then we define the category of *Springer motives*

$$\mathrm{DM}_G^{Spr}(S)_\mathbb{Q} = \langle \mu_!(\mathbb{Q}_X) \rangle_{\Delta, \in, (1)} \subset \mathrm{DM}_G(S)$$

as the full triangulated generated by the motive $\mu_!(\mathbb{Q}_X)$ its direct summands and Tate twists. Here, $\mathrm{DM}_G(S)$ is the G-equivariant version of the category DM [15].

2.2. **Formality.** Under the appropriate purity assumptions on the fibers of μ the weight complex functor becomes an equivalence when restricted to Springer motives:

Theorem (See [9]). *Assume that X has finitely many G-orbits and that the motives $M(\mu^{-1}(\{x\})/k)$ of the fibers are direct sums of $\mathbb{Q}(n)[2n]$ for $n \in \mathbb{Z}$. Then there is an equivalence $\mathrm{DM}_G^{Spr}(S)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{D}^{\mathbb{Z}}_{\mathrm{perf}}(E)$ with the perfect derived category of graded modules of the convolution algebra $E = (\mathrm{CH}^G_\bullet(X \times_S X)_{\mathbb{Q}}, *)$.*

This theorem can be seen as a prototypical formality result, as it show that $\mathrm{DM}_G^{Spr}(S)$ is governed by a *formal* dg-algebra E.

2.3. **Examples.** The assumptions of the theorem are fulfilled in many of the examples explained in Section 1.3. For example, the motives of Springer fibers are pure Tate by [6] and one obtains an equivalence $\mathrm{DM}^{Spr}_{G\times\mathbb{G}_m}(\mathcal{N})_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{D}^{\mathbb{Z}}_{\mathrm{perf}}(\overline{\mathbb{H}}(G)_{\mathbb{Q}})$ where $\overline{\mathbb{H}}(G)_{\mathbb{Q}}$ is the graded affine Hecke algebra. Similar results hold for quiver Hecke/Schur algebras in types A, D, E and for cyclic quivers using cell decomposition of quiver flag varieties, see [3, 11, 13, 9]. Moreover, $\mu : BS \to X$ recovers the Koszul equivalence $\mathrm{DM}^{Spr}(X) \xrightarrow{\sim} \mathrm{D}^b(\mathcal{O}_0(\mathfrak{g}^L))$ from [14, 1].

2.4. **Further Directions.** These results are the starting point for a further investigation in *motivic representation theory* pioneered in [16]. A logical next step is to replace (equivariant) Chow group by K-theory and to consider integral coefficients, yielding a wide range of equivalences not achievable with classical categories of sheaves, see [7, 8].

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The reconstruction theorem for AS-regular 3-dimensional cubic $\mathbb{Z}$-algebras

Shinnosuke Okawa

(joint work with Takuya Kitamura)

$\mathbb{Z}$-graded algebras, or more generally $\mathbb{Z}$-algebras, are noncommutative generalizations of polarized varieties. To each such algebra A with good enough properties we can associate the category $\operatorname{qgr} A = \operatorname{grmod} A / \operatorname{tors} A$, which should be regarded as the category of coherent sheaves on a noncommutative projective variety.

Since the data of "polarization" is lost in the process $A \mapsto \operatorname{qgr} A$, in general one can not reconstruct A from the category $\operatorname{qgr} A$. Hence it is natural to ask when two such algebras A and A' satisfy the equivalence of categories:

$$\operatorname{qgr} A \simeq \operatorname{qgr} A. \tag{1}$$

It is shown by Van den Bergh that if A and A' are AS-regular 3-dimensional quadratic $\mathbb{Z}$-algebras, then (1) holds if and only if $A \simeq A'$. Note that the categories $\operatorname{qgr} A$ for those algebras are noncommutative projective planes in that they are flat deformations of the abelian category $\operatorname{coh} \mathbb{P}^2$.

On the contrary, in [6] Van den Bergh introduced two involutions on the set of isomorphism classes of AS-regular 3-dimensional cubic $\mathbb{Z}$-algebras (cubic $\mathbb{Z}$-algebras, for short) which do not change the equivalence classes of qgr. In this case, the categories should be considered as noncommutative $\mathbb{P}^1 \times \mathbb{P}^1$.

One of the two involutions is the degree shift $A \mapsto A(1)$, where $A(n)_{ij} = A_{i+n,j+n}$, which turns out to be an involution since we always have an isomorphism $A(2) \simeq A$ for cubic $\mathbb{Z}$-algebras. The other involution, which is called ω in [6], is best explained in terms of the geometric data corresponding to the algebras (or the moduli space of points) but we will come back to this later. The two involutions are independent, and hence yield an action of the infinite dihedral group $D_\infty = C_2 * C_2 \simeq \mathbb{Z} \rtimes C_2$ on the set of isomorphism classes of cubic $\mathbb{Z}$-algebras. Our main result (of [4] in preparation) is as follows:

Theorem 1. *Two AS-regular 3-dimensional cubic $\mathbb{Z}$-algebras A and A' satisfy* $\operatorname{qgr} A \simeq \operatorname{qgr} A'$ *if and only if they belong to the same orbit of the D_∞-action.*

For the proof of Theorem 1, we introduce the notion of line bundles.

Definition 2. An object $\mathcal{L} \in \operatorname{qgr} A$ is called a *line bundle* if it is torsion free, of rank 1, and satisfies $\chi(\mathcal{L}, \mathcal{L}) = 1$.

On the other hand, for each $(a, b) \in \mathbb{Z}^2$ we define the object $\mathcal{O}(a, b) \in \operatorname{qgr} A$ inductively, starting with the case where $0 \leq b - a \leq 1$; in this case, they are defined to be the irreducible projective modules of A regarded as objects of $\operatorname{qgr} A$. Indeed, it is an acyclic helix (as we call it in [5]) of the derived category $D^b \operatorname{qgr} A$. Note that these objects are identified with the vertices of the "zig-zag" or "stairs" in the plane whose edges are of length 1 and are parallel to either of the coordinate axes, and contains the origin as a "right bottom corner". To proceed, we define the notion of even mutation and odd mutation for cubic $\mathbb{Z}$-algebras. If we identify the original algebra A with the zig-zag, then the even/odd mutation correspond to flipping the zig-zag along the line $b - a = 1$ and $b - a = 0$, respectively. It follows that the result of even/odd mutation of a cubic $\mathbb{Z}$-algebra is again a cubic $\mathbb{Z}$-algebra, which is identified with the "flipped zig-zag" and have the same qgr as the original $\mathbb{Z}$-algebra. Therefore by iterating the even/odd mutations alternatively, we can fill the plane by the repeatedly flipped zig-zags and thus obtain the objects $\mathcal{O}(a, b)$ for all $a, b \in \mathbb{Z}$ corresponding to their vertices.

The key ingredient of the proof is the following

Theorem 3. *An object $\mathcal{L} \in \operatorname{qgr} A$ is a line bundle if and only if $\mathcal{L} \simeq \mathcal{O}(a, b)$ for some $(a, b) \in \mathbb{Z}^2$.*

Moreover, we can rephrase the notion of torsion-freeness and rank in Definition 2 by those intrinsic to the category $\operatorname{qgr} A$. Combined with Theorem 3, this implies that any equivalence as in (1) preserves the collection of objects $\{\mathcal{O}(a, b)\}_{(a,b)\in\mathbb{Z}^2}$. Note that this set of objects is partially ordered by the order $\mathcal{M} \leq \mathcal{L} \iff \operatorname{Hom}(\mathcal{M}, \mathcal{L}) \neq 0$, and the induced bijection must respect this structure. Note that this poset is isomorphic to $\left(\mathbb{Z}^2, \leq\right)$, where $(a, b) \leq (a', b') \iff a \leq a'$ and $b \leq b'$.

Thus we conclude that the group $G := \operatorname{Aut}\left(\mathbb{Z}^2, \leq\right) = \mathbb{Z}^2 \rtimes C_2$ acts transitively on the set of possible $\mathbb{Z}^2$-markings of line bundles of $\operatorname{qgr} A$, which are indeed equivalent to the data of the original $\mathbb{Z}$-algebra A. However, it is known that the action of the element $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] \in G$, which is the generator of the center of G, sends the algebra A to its shift $A(2)$. Thus we obtain an (effective) action of the quotient group $G/Z(G) = \left\langle \overline{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} \right\rangle \rtimes \langle s \rangle \simeq \mathbb{Z} \rtimes C_2$ on the set of isomorphism classes of cubic $\mathbb{Z}$-algebras whose orbits consist exactly of those algebras yielding the same qgr.

From this point of view, the two involutions (1) and ω are nothing but the elements $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] \circ s$ and s, respectively. In particular, they generate the whole group $G/Z(G)$. This finishes the proof of Theorem 1.

Lastly, we point out that the D_∞-action above originates from the autoequivalence group of the bounded derived category of coherent sheaves of the stack $\mathbb{P}(1,1,2)$. To see this, we use the fact that for any cubic $\mathbb{Z}$-algebra A there is a flat family of abelian categories over a smooth curve whose central fiber is equivalent to $\operatorname{coh} \mathbb{P}(1,1,2)$ and also there is a fiber which is equivalent to $\operatorname{qgr} A$. The helix

$$\left(\mathcal{O}_{\mathbb{P}(1,1,2)}(i)\right)_{i\in\mathbb{Z}} \tag{2}$$

on the central fiber deforms to the standard helix (corresponding to the zig-zag) of $\operatorname{qgr} A$. On the other hand, the autoequivalence group of the bounded derived category $D^b \operatorname{coh} \mathbb{P}(1,1,2)$ send the helix (2) to various helices of $D^b \operatorname{coh} \mathbb{P}(1,1,2)$. There is the derived McKay correspondence for the A_1-singularity $D^b \operatorname{coh} \mathbb{P}(1,1,2) \simeq D^b \operatorname{coh} \Sigma_2$, where Σ_2 is the Hirzebruch surface of degree 2. It contains a (-2)-curve C, and line bundles on C yield 2-spherical twists of the derived category. Since the action of any spherical twist on the helix (2) is realized by a sequence of mutations (see, say, the proof of [3, Theorem 6.4]), we confirm that any of the helices obtained by applying autoequivalences of the central fiber deforms to a helix of the derived category of $\operatorname{qgr} A$, which turns out to be in the heart of the standard t-structure and corresponds to a polarization of $\operatorname{qgr} A$. Thus we obtain the action of the autoequivalence group of $D^b \operatorname{coh} \Sigma_2$ on the set of polarizations(=helices) of $\operatorname{qgr} A$, hence a surjective group homomorphism to the group D_∞ (autoequivalence groups of smooth projective toric surfaces are classified in [1], based on [2]).

For example, the even/odd mutation of the zig-zag mentioned above are obtained from the spherical twist by $\mathcal{O}_C$ and $\mathcal{O}_C(-1)$, respectively. Hence they generate a subgroup of D_∞ of index 2 (the gap is filled by $- \otimes \mathcal{O}_{\mathbb{P}(1,1,2)}(1)$), which is also isomorphic to D_∞. In fact, it is more appropriate to regard it as the affine Weyl group of type A_1. It is widely recognized that the geometry of noncommutative del Pezzo surface should be governed by the affine Weyl group whose Dynkin type is specified by the orthogonal complement of the anti-canonical class in the Picard group, and this is an explanation of why that is the case (in the case of noncommutative $\mathbb{P}^1 \times \mathbb{P}^1$) via degeneration.

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Multiplicative preprojective algebras and quiver varieties

Daniel Kaplan
(joint work with Travis Schedler)

1. The strong free product property

The exposition follows Etingof–Ginzburg [13, Section 3.3], which in turn utilizes the cotangent sequence of Cuntz–Quillen [11] and Anick's resolution [2].

Let k be a field, K a separable k-bimodule, and A a graded k-algebra with K-bimodule structure. Suppose further that A has a presentation as $A \cong B/(R)$ where $B := T_K(V)$ is the tensor algebra, R is the K-bimodule of relations, and $(R) \subset T_K(V)^{\geq 2}$ is the two-sided ideal generated by R. Consider the following comparison map between two A-bimodule complexes of A:

$$\begin{array}{ccccccccc}
P_\bullet := 0 & \longrightarrow & A \otimes_K R \otimes_K A & \xrightarrow{f_1} & A \otimes_K V \otimes_K A & \xrightarrow{f_0} & A \otimes_K A & \xrightarrow{\mu} & A \\
\downarrow{\scriptstyle \varphi_\bullet} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\
Q_\bullet := 0 & \longrightarrow & (R)/(R)^2 & \xrightarrow{g_1} & A \otimes_B \Omega^1(B) \otimes_B A & \xrightarrow{g_0} & A \otimes_K A & \xrightarrow{\mu} & A
\end{array}$$

where $\varphi_1(1 \otimes r \otimes 1) = r$ and $\varphi_0(1 \otimes v \otimes 1) = 1 \otimes [v \otimes 1 - 1 \otimes v] \otimes 1$. Note that φ_0 is an isomorphism of A-bimodules so $\varphi_\bullet$ is an isomorphism of complexes if φ_1 is an isomorphism of A-bimodules. In this case, $(R)/(R)^2$ is a projective bimodule, and Etingof–Ginzburg call A a *noncommutative complete intersection* (NCCI).

NCCIs have homological dimension at most two. If further $P_\bullet$ is self-dual, then A is 2-Calabi–Yau. In this case, $\dim_{A^e}(A \otimes_K R \otimes_K A) = \dim_{A^e}(A \otimes_K A) = 1$ so we can write $R = K\langle r \rangle$, for a single relation r.

We want to generalize the above analysis to the *ungraded* setting where B is a localization of a path algebra of a quiver and $r \in B$ need not be homogeneous. One can still ask if (I) $(R)/(R)^2$ is projective as an A-bimodule, an *ungraded* NCCI condition, and stronger if (II) φ_1 is an isomorphism of K-bimodules. We propose a mild strengthening of this condition, that relies on the auxiliary data of a K-bimodule section $\sigma : B/(r) \to B$ of the projection map $\pi : B \to B/(r)$.

Definition 1 ([15, Definition 3.1])**.** We say the triple (B, r, σ) satisfies the *strong free product property* (SFPP) if the map $\tilde{\sigma} : B/(r) *_K K\langle t \rangle \to B$, $\tilde{\sigma}\,|_{B/(r)} \equiv \sigma$, $\tilde{\sigma}(t) = r$ extended multiplicatively is a K-bimodule isomorphism. We say that (B, r) (or sometimes $B/(r)$) satisfies the SFPP if there exists σ such that (B, r, σ) satisfies the SFPP.

Remark 2. Later, we will fix a quiver Q with vertex set Q_0 and arrow set Q_1 and consider the case $K = kQ_0$ and $B = k\overline{Q}[(1 + a^*a)^{-1}]_{a \in Q_1}$. The relation $r := r_{\mathrm{mult}} \in B$ has $r_{\mathrm{mult}} + q$ invertible for some $q \in K$, (see Definition 4). In this setting, we say (B, r, σ) satisfies the SFPP if $\tilde{\sigma} : B *_{kQ_0} kQ_0[t, (t+q)^{-1}] \to B$ is an isomorphism of kQ_0-bimodules.

Notice if (B,r) satisfy the SFPP then $\tilde{\sigma}$ preserves the t-adic filtration on the LHS and the r-adic filtration on the RHS. Taking the associated graded map and considering the first filtered piece gives a $B/(r)$-bimodule isomorphism

$$\mathrm{gr}(\tilde{\sigma})_1 : B/(r) \otimes_K K\langle t\rangle \otimes_K B/(r) \to (r)/(r)^2 \qquad 1 \otimes t \otimes 1 \mapsto r + (r)^2.$$

So $(r)/(r)^2$ is projective and $B/(r)$ is an (ungraded) NCCI. Note that in the graded setting $\mathrm{gr}(\tilde{\sigma})_1 = \varphi_1$ and in the following three notions are equivalent: (I) $B/(r)$ is an NCCI, (II) $\mathrm{gr}(\tilde{\sigma})_1$ is a K-bimodule isomorphism, and (III) (B,r) satisfies the SFPP. In general, (III) implies (II) which in turn implies (I). The advantage of the SFPP is that one can often prove it (and build σ explicitly) using a system of reductions and Bergman's Diamond Lemma for ring theory [5].

Example 3. (a) The pair $(\mathbb{C}[x],\, r = x^2)$ does not satisfy the SFPP since

$$\tilde{\sigma} : \mathbb{C}[x]/(x^2) * \mathbb{C}[t] \to \mathbb{C}[x] \qquad \tilde{\sigma}(xt - tx) = \sigma(x)x^2 - x^2\sigma(x) = 0$$

for any $\sigma : \mathbb{C}[x]/(x^2) \to \mathbb{C}[x]$ and hence no such $\tilde{\sigma}$ is injective.

(b) The triple $(\mathbb{C}\langle x,y\rangle, r = y, \sigma_1)$ where $\sigma_1(x + (y)) = x$ satisfies the SFPP but the triple $(\mathbb{C}\langle x,y\rangle, r = y, \sigma_2)$ with $\sigma_2(x + (y)) = x(1-y)$ does not satisfy the SFPP since $x = \sigma_2(x)(1-y)^{-1}$ is not in the image of $\tilde{\sigma_2}$.

(c) Fix a connected quiver Q with non-empty arrow set. Let $K := kQ_0$ and $B := k\overline{Q}$ the path algebra of the double quiver. Then $(k\overline{Q}, r_{\mathrm{add}} := \sum_{a\in Q_1}[a,a^*])$ satisfies the SFPP if and only if Q is not ADE Dynkin, see [16, Propositions 5.1.9, 5.2.1]. These were known to be NCCIs from [13, Theorem 1.3.1, Proposition 6.3.1] following [7]. Moreover, one can deform the additive preprojective relation using $\lambda \in k^{Q_0}$ and the corresponding pair satisfies the SFPP, giving an alternate proof of [9, Theorem 2.7].

2. Multiplicative preprojective algebras

In this section we recall the definition of the multiplicative preprojective algebra following Crawley-Boevey and Shaw and give a summary of results.

Definition 4 ([10, Definition 1.2, Section 2])**.** Let $L := k\overline{Q}[(1+a^*a)^{-1}]_{a\in Q_1}$ be a localized path algebra and define the multiplicative preprojective algebra with parameter $q \in (k^*)^{Q_0}$ to be the quotient $\Lambda^q(Q) := L/(r_{\mathrm{mult}})$ where

$$r_{\mathrm{mult}} := \prod_{a\in Q_1} (1+aa^*)(1+a^*a)^{-1} - \sum_{i\in Q_0} q_i e_i.$$

The product is taken with respect to some choice of order $\leq$ on the set of arrows Q_1, but the isomorphism class of $\Lambda^q(Q)$ is independent of this choice.

A summary of results from [15, 14]:

(1) If Q is ADE Dynkin then $\Lambda^1(Q) \cong \Pi(Q)$ if and only if $\mathrm{char}(k)$ is good for Q meaning not 2 in type D, not 2 or 3 in type E, and not 2, 3, or 5 for E_8, (see [14] and references therein.)

(2) If $Q = \tilde{A}_n$ then (L, r_{mult}) satisfies the SFPP and hence $\Lambda^q(Q)$ is an ungraded NCCI. Furthermore it is prime, 2-Calabi–Yau, and if $q = 1$ it is an NCCR over

$$Z(\Lambda^1(\tilde{A}_n)) \cong e_0\Lambda^1(\tilde{A}_n)e_0 \cong k[x, y, z]/(z^n + xy + xyz)$$

where the second isomorphism is due to Shaw [17, Theorem 1.3.5].

(3) If $Q \supsetneq \tilde{A}_n$ then (L, r_{mult}) satisfies the SFPP so $\Lambda^q(Q)$ is an ungraded NCCI. Further $\Lambda^q(Q)$ is 2-Calabi–Yau, prime, and $Z(\Lambda^q(Q)) = k$ [15].

3. Symplectic resolutions of multiplicative quiver varieties

The goal of this section is to use the 2-Calabi–Yau property of preprojective algebras to prove results about multiplicative quiver varieties.

For any algebra A over kQ_0 let $\mathrm{Rep}_d^\theta(A)$ denote the set of θ-semistable representations of A of dimension vector d. The group $\mathrm{GL}_d := \prod_{i\in Q_0} \mathrm{GL}_{d_i}(k)$ acts on $\mathrm{Rep}_d^\theta(A)$ by conjugation and let $\mathcal{M}_d^\theta(A)$ denote the GIT quotient, $\mathrm{Rep}_d^\theta(A)//\mathrm{GL}_d$. If $A = \Pi(Q)$ (respectively $A = \Lambda^q(Q)$) then we call $\mathcal{M}_d^\theta(A)$ a *quiver variety* (respectively *multiplicative quiver variety*).

Results about the structure of $\mathcal{M}_d^\theta(A)$ for A a 2-Calabi–Yau algebra:

- Prehistory: A *graded* 2-Calabi–Yau algebra is isomorphic to $\Pi_k(Q)$ for Q with connected components not ADE Dynkin.
- Van den Bergh : A *complete* 2-Calabi–Yau algebra is isomorphic to $\hat{\Pi}(Q)$ for Q connected not ADE Dynkin [18, Corollary 9.3].
- Bocklandt–Galluzzi–Vaccarino: If A is a 2-Calabi–Yau algebra and M is a semisimple A-module, then $H_0(\mathrm{Ext}_A^\bullet(M, M)^!)$ is a completed preprojective algebra, where $\mathcal{A}^! := \mathrm{RHom}_{kQ_0}(\mathcal{A}, \mathcal{A})$ denotes the Koszul dual of $\mathcal{A}$. In words, the formal neighborhood of $M \in \mathcal{M}_d^0(A)$ is isomorphic to the formal neighborhood of $0 \in \mathcal{M}_{d'}^0(\Pi(Q))$ for some d', Q [6, Theorem 6.6].
- Kaplan–Schedler: The previous result holds for any $\theta \neq 0$ and $M \in \mathcal{M}_d^\theta(A)$ θ-polystable, or more generally if $\mathrm{End}_A(M)$ is semisimple [15].
- Davison: The moduli space of objects in a (nice) 2-Calabi–Yau category is formally locally a quiver variety [12].

In the case $A = \Lambda^q(Q)$, we have the following result for $\mathcal{M}_d^\theta(A)$:

Theorem 5 ([15, Theorem 5.4])**.** *Multiplicative quiver varieties are formally locally isomorphic to the neighborhood of the zero representation in a quiver variety.*

Proof. By [15], if Q contains a cycle, then $\Lambda^q(Q)$ is 2-Calabi–Yau and the theorem holds. Otherwise, define $\tilde{Q}$ from Q by adding an additional vertex v, adding an arrow $a : v \to w$ for some $w \in Q_0$ a vertex in the original quiver, and adding $b : v \to v$ a loop at vertex v. Now any (neighborhood of a) representation in $\mathcal{M}_d^\theta(\Lambda^q(Q))$ can be regarded as a (neighborhood of a) representation in $\mathcal{M}_{(d,0)}^{(\theta,0)}(\Lambda^{(q,1)}(\tilde{Q}))$, which is formally locally isomorphic to the zero representation in a quiver variety. □

A recent program aims to classify which symplectic singularities have projective symplectic resolutions, and classify all such resolutions when they exist. Bellamy–Schedler [4] show that the quiver variety $\mathcal{M}_d^0(\Pi(Q))$ has symplectic singularities and classify which pairs (Q,d) admit projective symplectic resolutions. Work in progress by Bellamy–Craw–Schedler [3] aims to prove that if a single projective symplectic resolution of $\mathcal{M}_d^0(A)$ is given by VGIT (i.e., via a map $\mathcal{M}_d^\theta(A) \to \mathcal{M}_d^0(A)$ for some stability parameter θ) then all projective symplectic resolutions arise this way. Therefore, as explained in Schedler's talk, one can classify all projective symplectic resolutions of X by giving an identification $X \cong \mathcal{M}_d^\theta(\Pi(Q))$ where some projective symplectic resolution is given by VGIT.

For $X = \mathcal{M}_d^\theta(\Lambda^q(Q))$, one no longer has an identification $X \cong \mathcal{M}_{d'}^0(\Pi(Q'))$, but Theorem 5 ensures the statement is true for every formal local neighborhood of X. Hence we can use Bellamy–Schedler to classify symplectic resolutions in every formal neighborhood, after which we hope to classify all global resolutions, by developing a local-to-global obstruction theory for symplectic resolutions. This should give classifications of symplectic resolutions in new, interesting examples.

Example 6. Let $X = (\mathbb{C}^*)^n \times (\mathbb{C}^*)^n$ and consider the the action of $\Gamma := (C_2)^n \rtimes S_n$ where S_n is permuting the factors and C_2 acts by $(z,w) \mapsto (z^{-1}, w^{-1})$. For simplicity take $n=1$, although, with more work, our analysis holds for general n. The singularities of X/Γ (i.e., the fixed points of the Γ-action) are the four points $(\pm 1, \pm 1)$. Formally locally each singularity is simple: $\mathbb{C}^* \times \mathbb{C}^*/C_2 \cong \mathbb{C}^2/C_2$. The simple singularity has a unique resolution, and hence the same is true for X, as these resolutions glue to a global symplectic resolution.

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Faithful ADE braid group actions on triangulated categories

Anya Nordskova
(joint work with Yury Volkov)

Let k be an infinite field and $\mathfrak{D}$ be a k-linear triangulated category and with a fixed dg-enhancement. A so-called ADE-configuration of n-spherical objects in $\mathfrak{D}$ gives rise to an action of a generalised braid group (Artin group) on $\mathfrak{D}$ via spherical twists. Together with Y. Volkov, we showed that this action is faithful for any integer $n \neq 1$. Although various results on the faithfulness of this action have been obtained over the years (e.g. for type A or n=2, and in some other cases), the approach we present allows us to establish it for any ADE diagram, any enhanced triangulated category, and any integer $n \neq 1$ almost simultaneously. Let us now introduce the problem and state the result in more detail.

Let $\mathrm{Hom}^k(A, B)$ denote $\mathrm{Hom}_{\mathfrak{D}}(A, B[k])$. By $\mathrm{Hom}^*(A, B)$ we denote the graded space $\bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}^k(A, B)$.

Definition. Let $n \in \mathbb{Z}$. An object $P \in \mathfrak{D}$ is called *n-spherical* if

(i) $\dim_{\mathrm{k}} \mathrm{Hom}^*(P, X) < \infty$ for any object $X \in \mathfrak{D}$.
(ii) $\mathrm{Hom}^*(P, P) \cong k[t]/(t^2)$ as graded k-algebras, where $deg(t) = n$.
(iii) P is an n-Calabi–Yau object, which means that for any $X \in \mathfrak{D}$ there is a perfect pairing

$$\mathrm{Hom}^*(P, X) \times \mathrm{Hom}^*(X, P) \xrightarrow{\circ} \mathrm{Hom}^*(P, P)/\langle Id_P\rangle \cong \mathrm{k}.$$

Definition. Let P be a spherical object. The *spherical twist* t_P along P is an endofunctor of $\mathfrak{D}$ defined by

$$t_P(-) = cone(P \otimes \mathrm{Hom}^*(P, -) \xrightarrow{\varepsilon(-)} (-)),$$

where ε is the counit of the adjunction $\mathrm{Hom}^*(P, -) : \mathfrak{D} \longleftrightarrow D(\mathrm{Vect}-\mathrm{k}) : P \otimes (-)$ and $D(\mathrm{Vect}-\mathrm{k})$ denotes the derived category of finite dimensional k-vector spaces.

Since we assumed that the category $\mathfrak{D}$ comes with a fixed dg-enhancement, one can indeed correctly define t_P as a functor, considering cones that descend from the dg level (details can be found in the literature). In fact, it is well-known that if P is a spherical object, then t_P is an *autoequivalence* of $\mathfrak{D}$.

Let Γ be an ADE Dynkin diagram. By Γ_0 we denote the set of vertices of Γ.

Definition. The *Artin group* (*generalised braid group*) B_Γ of type Γ is generated by $s_i, i \in \Gamma_0$ subject to braid relations $s_i s_j s_i = s_j s_i s_j$ for i, j adjacent in Γ and commutation relations $s_i s_j = s_j s_i$ for i, j not adjacent in Γ. The *braid monoid* B_Γ^+ is a monoid given by the same generators and relations.

Definition. A collection of n-spherical objects $\{P_i\}_{i\in\Gamma_0}$ labeled by the vertices of Γ is called a Γ-*configuration* if for any $i \neq j$

(1) $\dim_{\mathrm{k}} \operatorname{Hom}^*(P_i, P_j) = 1$ if i and j are adjacent in Γ;
(2) $\operatorname{Hom}^*(P_i, P_j) = 0$ otherwise.

Proposition (Seidel, Thomas [4])**.** Let $\{P_i\}_{i\in\Gamma_0}$ be a Γ-configuration. Then the functors t_{P_i} satisfy braid relations of type Γ up to a natural isomorphism. In other words, there is a group homomorphism

$$F\colon B_\Gamma \to \operatorname{Aut}(\mathfrak{D})$$

where $\operatorname{Aut}(\mathfrak{D})$ is the group of exact autoequivalences of $\mathfrak{D}$ modulo natural isomorphisms.

We have just defined an action of the braid group B_Γ on the category $\mathfrak{D}$. The main result of this talk says that this action is faithful if $n \neq 1$:

Theorem (Nordskova, Volkov [2])**.** *Let Γ be a simply-laced Dynkin diagram, $n \in \mathbb{Z}$, $n \neq 1$ and $\{P_i\}_{i\in\Gamma_0}$ a Γ-configuration of n-spherical objects in a triangulated category $\mathfrak{D}$ with a fixed dg-enhancement. Then the action of B_Γ on $\mathfrak{D}$ generated by the spherical twists t_{P_i} is faithful.*

Remark.

(1) For $n = 1$ this action can be unfaithful, see [4] for a counterexample.
(2) The faithfulness of this action had been established in some cases before, for instance, see [4] (type A, $n \geq 2$), [1] ($n = 2$), [3] ($n \geq 2$, $\mathfrak{D}$ is the derived category of a Ginzburg dg-algebra).

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Non-Fourier-Mukai functors via twisted Hodge diamonds

Alice Rizzardo

This is a report on work by my PhD student, Felix Küng. Throughout we work over a base field of characteristic zero.

Consider an exact functor F between two triangulated categories $\mathcal{A}$ and $\mathcal{B}$, and assume that these have corresponding DG- or A_∞-enhancements $\mathfrak{a}$ and $\mathfrak{b}$: in other words, $\mathfrak{a}$ and $\mathfrak{b}$ are pre-triangulated DG or A_∞-categories such that $H^0(\mathfrak{a}) = \mathcal{A}$ and $H^0(\mathfrak{b}) = \mathcal{B}$.

Definition 1. *We say that $F : \mathcal{A} \to \mathcal{B}$ is a Fourier-Mukai functor if there exists a DG or A_∞-functor $f : \mathfrak{a} \to \mathfrak{b}$ such that $F \cong H^0(f)$ as graded functors.*

Note that in the case where $\mathcal{A}$ and $\mathcal{B}$ are the bounded derived categories of two smooth projective varieties X and Y, by a result of Toën [5], this definition coincides with the more classical definition of a Fourier-Mukai functor as a functor of the shape

$$F \cong Rp_{2*}(\mathcal{K} \overset{L}{\otimes}_{X\times Y} Lp_1^*(-))$$

for some $\mathcal{K} \in D^b(\mathrm{coh}(X \times Y))$, where p_1 and p_2 are the projections from $X \times Y$ onto X and Y respectively.

The question of which functors are Fourier-Mukai has a relatively long history, both in the algebraic setting where $\mathcal{A}$ and $\mathcal{B}$ are bounded derived categories of finite dimensional algebras, and in the geometric setting where $\mathcal{A}$ and $\mathcal{B}$ are bounded derived categories of smooth projective varieties. Let us focus on the geometric setting here.

First of all, let us remind the reader that the bounded derived category of a smooth projective variety always admits a unique DG enhancement [2]. This allows us to quote the celebrated result by Orlov [1] using our definition of Fourier-Mukai functor: any fully faithful exact functor $F : D^b(\mathrm{coh}\, X) \to D^b(\mathrm{coh}\, Y)$ is a Fourier-Mukai functor. For some time, there was some hope that every exact functor would have this shape, so that there would be a perfect dictionary between the derived categorical level and the level of enhancements, but this turned out not to be the case.

Indeed, together with my coauthors Van den Bergh and Neeman, we exhibited a counterexample in [4]:

Theorem 2 (Rizzardo, Van den Bergh, Neeman)**.** *There exists an exact, non-Fourier-Mukai functor*

$$F : D^b(\mathrm{coh}\, Q) \to D^b(\mathrm{coh}\, \mathbb{P}^4)$$

where Q is a smooth quadric hypersurface in $\mathbb{P}^4$.

The proof that the functor F constructed in the paper is non-Fourier-Mukai hinges on an obstruction theory that proved harder to control in higher dimension: this is the reason why the original paper [4] only provided one example, despite the construction of candidate non-Fourier-Mukai functors being potentially very general.

Intuitively, it should be possible to obtain non-Fourier-Mukai functors as noncommutative deformations of a Fourier-Mukai functor. This suggests that there should be plenty of examples out there, and indeed there are! First of all we have the following result from [3]:

Theorem 3 (Raedschelders, Rizzardo, Van den Bergh)**.** *Let X be a smooth projective scheme of dimension $m \geq 3$ which has a tilting bundle. Then there exists an exact, non-Fourier Mukai functor*

$$D^b(\mathrm{coh}(X)) \to D^b(\mathrm{coh}(Y)) \tag{1}$$

where Y is a smooth projective scheme.

This theorem provides an infinite amount of non-Fourier-Mukai functors, but this is done at the price of allowing Y to have very high dimension.

The result of my PhD student Felix Küng builds on the original result of [4] by providing an infinite family of non-Fourier-Mukai functors with controlled source and target, one for each odd-dimensional quadric hypersurface:

Theorem 4 (Küng)**.** *Let Q be a smooth quadric hypersurface in $\mathbb{P}^{2k}$, for $k > 2$. Then there exists an exact, non-Fourier-Mukai functor*

$$F : D^b(\mathrm{coh}\, Q) \to D^b(\mathbb{P}^n).$$

Küng also provides a way to construct "candidate" non-Fourier-Mukai functors with source equal to the derived category of an arbitrary hypersurface X. In particular he exhibits a candidate non-Fourier-Mukai functor for each element in a vector space of dimension equal to a certain entry in a twisted Hodge diamond of X. These twisted Hodge numbers are easy to calculate using a computer algebra program. One can see in this way that the number of candidate functors is often extremely high, even in low dimension. Unfortunately it is still not clear how to show that these yield non-Fourier-Mukai functors, and how to see which of these choices give nonisomorphic functors.

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Singularity categories and cluster categories

Osamu Iyama
(joint work with Norihiro Hanihara)

1. Cohen-Macaulay representations and cluster categories

Let R be a commutative Gorenstein ring of dimension d. One of the main objects in Cohen-Macaulay representation theory is the *singularity category* $\mathrm{D}_{\mathrm{sg}}(R) := \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, R)/\operatorname{per} R$ of R, the Verdier quotient of the bounded derived category by the perfect derived category. It is enhanced by the Frobenius category $\mathrm{CM}\, R$ of Cohen-Macaulay R-modules [5] and thus triangle equivalent to the stable category $\underline{\mathrm{CM}}\, R$. Moreover it is $(d-1)$-Calabi-Yau when R is an isolated singularity [3].

We have another important class of Calabi-Yau triangulated categories, namely *cluster categories* [4, 1, 7, 12, 13]. Let A be a finite dimensional algebra over a field k, and let $d \in \mathbb{Z}$. Then the *d-cluster category* $\mathrm{C}_d(A)$ of A is the triangulated hull [12] of the orbit category $\operatorname{per} A/-\otimes_A^{\mathrm{L}} DA[-d]$, where $D = \operatorname{Hom}_k(-,k)$ is the k-dual. Under a mild assumption on A, it is a d-Calabi-Yau triangulated category.

Recently there have been extensive studies in relating these two classes of Calabi-Yau triangulated categories [10]. Based on [9], we will present a general strategy to construct equivalences between singularity categories and cluster categories, and give some applications.

2. Results

2.1. **Main theorem.** Our approach is to consider a grading on our Gorenstein ring R. When R is graded by an abelian group G we form the G-graded singularity $\mathrm{D}^G_{\mathrm{sg}}(R) := \mathrm{D}^{\mathrm{b}}(\mathrm{mod}^G R)/\operatorname{per}^G R$. For simplicity, we explain our results in [9] in the following typical setting: Let $R = \bigoplus_{i \geq 0} R_i$ be a positively graded commutative Gorenstein isolated singularity of dimension d such that $R_0 = k$ is a field and has Gorenstein parameter $a \neq 0$. In this setting we have the following main result.

Theorem 1. *Suppose we have a triangle equivalence $\mathrm{D}^{\mathbb{Z}}_{\mathrm{sg}}(R) \simeq \operatorname{per} A$ for a finite dimensional algebra A. Then it extends to a commutative diagram*

$$\begin{array}{ccccc} \operatorname{per} A & \longrightarrow & \mathrm{C}_d(A) & \longrightarrow & \mathrm{C}_d^{(1/a)}(A) \\ \wr\| & & \wr\| & & \wr\| \\ \mathrm{D}^{\mathbb{Z}}_{\mathrm{sg}}(R) & \longrightarrow & \mathrm{D}^{\mathbb{Z}/a\mathbb{Z}}_{\mathrm{sg}}(R) & \longrightarrow & \mathrm{D}_{\mathrm{sg}}(R). \end{array}$$

Here, $\mathrm{C}_d^{(1/a)}(A)$ is the triangulated hull of $\operatorname{per} A/F$ for an autoequivalence of F satisfying $F^a \simeq -\otimes_A^{\mathrm{L}} DA[-d]$ (cf. [14, 8]).

2.2. **Examples.** Let us give some examples to apply our result.

Example 2. Let k be an algebraically closed field of characteristic 0 and $G \subset \mathrm{SL}(d,k)$ a finite subgroup. Then G acts on the polynomial ring $S = k[x_1, \ldots, x_d]$. Giving a grading on S by $\deg x_i = 1$, the invariant subring $R = S^G$ inherits a grading so that its Gorenstein parameter is d. We assume that R is an isolated singularity. In this case it is shown in [11] that we have a triangle equivalence $\mathrm{D}^{\mathbb{Z}}_{\mathrm{sg}}(R) \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, A)$ for some finite dimensional algebra A of finite global dimension. We conclude that there is a commutative diagram

$$\begin{array}{ccccc} \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, A) & \longrightarrow & \mathrm{C}_{d-1}(A) & \longrightarrow & \mathrm{C}^{(1/d)}_{d-1}(A) \\ \wr\| & & \wr\| & & \wr\| \\ \mathrm{D}^{\mathbb{Z}}_{\mathrm{sg}}(R) & \longrightarrow & \mathrm{D}^{\mathbb{Z}/d\mathbb{Z}}_{\mathrm{sg}}(R) & \longrightarrow & \mathrm{D}_{\mathrm{sg}}(R). \end{array}$$

Let us look at the following two cases in more detail.

(1) Let $d = 3$ and $G \subset \mathrm{SL}(3,k)$ the cyclic subgroup of order 3 generated by $\mathrm{diag}(\omega,\omega,\omega)$, where ω is a primitive third root of unity. In this case, $A = kQ_3 \times kQ_3 \times kQ_3$ for the Kronecker quiver $Q_3\colon \circ \xrightarrow{3} \circ$ with 3 arrows. By the main theorem and an interpretation of $\mathrm{C}^{(1/3)}_2(A)$, we obtain equivalences

$$\mathrm{D}_{\mathrm{sg}}(R) \simeq \mathrm{C}^{(1/3)}_2(A) \simeq \mathrm{C}_2(kQ_3),$$

as was established in [15].

(2) Let $d = 4$ and $G \subset \mathrm{SL}(4,k)$ the cyclic subgroup of order 2 generated by $\mathrm{diag}(-1,-1,-1,-1)$. In this case we can take $A = kQ_6 \times kQ_6$ for the Kronecker quiver $Q_6\colon \circ \xrightarrow{6} \circ$ with 6 arrows. We similarly obtain equivalences

$$\mathrm{D}_{\mathrm{sg}}(R) \simeq \mathrm{C}^{(1/4)}_3(A) \simeq \mathrm{C}^{(1/2)}_3(kQ_6).$$

This recovers a result of [14].

Example 3. Let $d = 0$, so that $R = \bigoplus_{i\geq 0} R_i$ be an Artinian Gorenstein ring with $R_0 = k$. Then it has Gorenstein parameter $a = -\max\{i \geq 0 \mid R_i \neq 0\}$. It is shown in [16] that $T := \bigoplus_{i=1}^{-a} R(i)_{\geq 0}$ is a tilting object, and $A := \mathrm{End}_{\mathrm{D}^{\mathbb{Z}}_{\mathrm{sg}}(R)}(T)$ has finite global dimension. It follows that we have a commutative diagram

$$\begin{array}{ccccc} \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, A) & \longrightarrow & \mathrm{C}_{-1}(A) & \longrightarrow & \mathrm{C}^{(1/a)}_{-1}(A) \\ \wr\| & & \wr\| & & \wr\| \\ \mathrm{D}^{\mathbb{Z}}_{\mathrm{sg}}(R) & \longrightarrow & \mathrm{D}^{\mathbb{Z}/a\mathbb{Z}}_{\mathrm{sg}}(R) & \longrightarrow & \mathrm{D}_{\mathrm{sg}}(R). \end{array}$$

We have further examples for quotient singularities with different gradings [2], Gorenstein rings of dimension 1 [6], and so on.

2.3. **Key points.** We mention two important steps toward the main theorem. Although everything is hidden in the statement of the main result, we essentially work in differential graded (dg) enhancements of the relevant categories. The first one is the following property of the canonical enhancement of the singularity category of R.

Theorem 4. *Let $\mathcal{C}$ be the canonical dg enhancement of $\mathrm{D}^{\mathbb{Z}}_{\mathrm{sg}}(R)$. Then there exists an isomorphism in $\mathrm{D}(\mathcal{C}^e)$:*

$$D\mathcal{C} \simeq \mathcal{C}(-a)[d-1].$$

Taking the 0-th cohomology we recover the classical Calabi-Yau property of $\mathrm{D}^{\mathbb{Z}}_{\mathrm{sg}}(R)$ [3].

The second point concerns an enhancement of the cluster category. Note that an enhancement of the cluster category $\mathrm{C}_d(A)$ is given by the dg orbit category $\mathcal{A}/F$, where $\mathcal{A} = \mathcal{C}^{\mathrm{b}}(\operatorname{proj} A)$ the enhancement of $\operatorname{per} A$ and $F = - \otimes_A X$ for a bimodule projective resolution $X \to DA[-d]$ [12]. Notice that dg orbit categories have a natural additional $\mathbb{Z}$-grading, leading to study such $\mathbb{Z}$-graded dg categories. The second key step is the following characterization of dg orbit categories among dg categories with an additional $\mathbb{Z}$-grading.

Theorem 5. *Let $\mathcal{B}$ be a $\mathbb{Z}$-graded dg category. Assume that $\operatorname{per}^{\mathbb{Z}}\mathcal{B}$ is generated by $\{\mathcal{B}(-, B) \mid B \in \mathcal{B}\}$. Put $\mathcal{A} = \mathcal{B}_0$ and $V = \mathcal{B}_{-1}$. Then there exists a $\mathbb{Z}$-graded quasi-equivalence $\mathcal{B} \simeq \mathcal{A}/V$.*

The main result is obtain by a combination of Theorem 4 and Theorem 5.

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Feigin and Odesskii's Elliptic Algebras

S. Paul Smith
(joint work with Alexandru Chirvasitu, Ryo Kanda)

We reported on results from [4, 5, 6, 7, 8, 9] by Chirvasitu-Kanda-Smith (CKS) that establish some fundamental properties of the algebras $Q_{n,k}(E,\xi)$, which were defined in the late 80's by Feigin and Odesskii. [11, 13]. Almost all these properties had been stated by Feigin and Odesskii in their inaugural and subsequent papers, but proofs were often omitted and some of their claims are not true in the generality they are stated. Some of their statements to the effect that something is true "generically" cry out for more precision. Nevertheless, Feigin and Odesskii's papers are a reliable guide as to what one should expect of these algebras.

1. Definition and first properties of $Q_{n,k}(E,\xi)$

1.1. **Preliminary definitions.** Fix relatively prime integers $n > k \geq 1$ and $(\eta \,|\, \tau) \in \mathbb{C} \times \mathbb{H}$, where $\mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$. Define $\Lambda_\tau := \mathbb{Z} + \mathbb{Z}\tau$ and $E_\tau := \mathbb{C}/\Lambda_\tau$. We write $E[r]$ for the r-torsion subgroup of E and $e(z) := e^{2\pi i z}$.

Let $H_n := \langle S, T, \epsilon \mid S^n = T^n = \epsilon^n = 1,\ [S,T] = \epsilon, \epsilon \text{ is central} \rangle$ be the Heisenberg group of order n^3. By standard complex analysis, $\dim_{\mathbb{C}} \Theta_n(\tau) = n$, where $\Theta_n(\tau)$ is the space of holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ such that

$$\begin{cases} f(z+1) &= f(z) \\ f(z+\tau) &= -\,e(-nz)f(z) \end{cases} \tag{1}$$

for all $z \in \mathbb{C}$. Every $0 \neq f \in \Theta_n(\tau)$ has (counted with multiplicity) exactly n zeros in a fundamental parallelogram for Λ_τ; the sum of the zeros is $\frac{n-1}{2}$ modulo Λ_τ.

The conditions in (1) imply that the map $\mathbb{C} \to \mathbb{P}^{n-1}$, $z \mapsto (\theta_0(z), \ldots, \theta_{n-1}(z))$, descends to a well-defined holomorphic map

$$\varphi : E_\tau \longrightarrow \mathbb{P}^{n-1}, \qquad \varphi(z) = (\theta_0(z), \ldots, \theta_{n-1}(z)). \tag{2}$$

This is a closed immersion when $n \geq 3$.[1] Each point $z \in \mathbb{C}$ determines a codimension-one subspace of $\Theta_n(\tau)$, namely $H_z := \{f \in \Theta_n(\tau) \mid f(z) = 0\}$, and hence a 1-dimensional subspace of $\Theta_n(\tau)^*$; as z varies, $z \mapsto H_z$ is a map from $\mathbb{C}$ to the projective space $\mathbb{P}\big(\Theta_n(\tau)^*\big)$ of lines in $\Theta_n(\tau)^*$. But this is the map φ, so

$$\varphi : E_\tau \longrightarrow \mathbb{P}\big(\Theta_n(\tau)^*\big) \;\cong\; \mathbb{P}^{n-1}. \tag{3}$$

See also [4, §2.4]. Let $\mathcal{L} := \varphi^* \mathcal{O}_{\mathbb{P}^{n-1}}(1) = \mathcal{O}_{\mathbb{P}^{n-1}}(1)\big|_E$. Since the sum of the zeros of a non-zero function $f \in \Theta_n(\tau)$ equals $\frac{n-1}{2}$ modulo Λ_τ, $\mathcal{L} \cong \mathcal{O}_E\big(n(\frac{n-1}{2n})\big)$.

The group H_n acts on the space of holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ via

$$\begin{cases} (S \cdot f)(z) &:= f\big(z + \frac{1}{n}\big) \\ (T \cdot f)(z) &:= e\big(z + \frac{1}{2n} - \frac{n-1}{2n}\tau\big) f\big(z + \frac{1}{n}\tau\big). \end{cases}$$

[1] We will assume that $n \geq 3$ in what follows. The case $n = 2$ is not interesting because $Q_{2,1}(E,\xi)$ is a polynomial ring on 2 variables.

It is easy to check that $\Theta_n(\tau)$ is an n-dimensional irreducible representation of H_n under this action and that it has a basis $\{\theta_\alpha \mid \alpha \in \mathbb{Z}_n\}$ (which is unique up to a common non-zero scalar multiple) of functions $\theta_\alpha(z) = \theta_\alpha(z \,|\, \tau)$ such that

$$S \cdot \theta_\alpha \;=\; e\big(\tfrac{\alpha}{n}\big)\theta_\alpha, \qquad \text{and} \qquad T \cdot \theta_\alpha \;=\; \theta_{\alpha+1}.$$

The function $\theta(z \,|\, \tau)$ is a basis for $\Theta_1(\tau)$ and it has simple zeros at the points of Λ_τ and no other zeros. The zeros of θ_α therefore occur at the points $\frac{\alpha}{n}\tau + \frac{1}{n}\mathbb{Z} + \Lambda_\tau$ and each of these is a simple zero.

1.2. **Definition of $Q_{n,k}(\eta \,|\, \tau)$.** Fix again $(n \geq 3, k, \eta, \tau)$. Let V be a complex vector space with basis $\{x_\alpha \mid \alpha \in \mathbb{Z}_n\}$ and let $R(z) = R_\eta(z) = R_{n,k,\eta,\tau}(z) : V^{\otimes 2} \to V^{\otimes 2}$

$$R_\eta(z)(x_i \otimes x_j) \;:=\; \frac{\theta_0(-z)\ldots\theta_{n-1}(-z)}{\theta_1(0)\ldots\theta_{n-1}(0)} \sum_{r \in \mathbb{Z}_n} \frac{\theta_{j-i+r(k-1)}(-z+\eta)}{\theta_{j-i-r}(-z)\theta_{kr}(\eta)}\, x_{j-r} \otimes x_{i+r} \tag{4}$$

for all $(i,j) \in \mathbb{Z}_n^2$. The family of linear operators $R(z)$, $z \in \mathbb{C}$, satisfies the quantum Yang-Baxter equation (QYBE) with spectral parameter. The family $R_\eta(z)$ is Belavin's elliptic solution to the quantum Yang-Baxter equation [3].

Given the description of the zeros of the θ_α's, $R_{n,k,\eta,\tau}(z)$ is not defined when $\eta \in \frac{1}{n}\Lambda_\tau$ so assume for a moment that $\eta \notin \frac{1}{n}\Lambda_\tau$; the term $\theta_{j-i-r}(-z)$ in the denominator of (4) is sometimes zero, but the numerator before the Σ sign was chosen so as to cancel the $\theta_{j-i-r}(-z)$ term, thereby ensuring that the singularities in $R_{n,k,\eta,\tau}(z)$ are removable; thus, $R_\eta(z)$ makes sense for all $z \in \mathbb{C}$ when $\eta \notin \frac{1}{n}\Lambda_\tau$. A finer analysis shows that one can define $R_{n,k,\eta,\tau}$ for all $\eta \in \mathbb{C}$ and so make sense of the operators $R_{n,k,\eta,\tau}(z)$ for all η and z.

We now define, for all (n,k,η,τ),

$$Q_{n,k}(\eta \,|\, \tau) \;:=\; \frac{\mathbb{C}\langle x_0, \ldots, x_{n-1}\rangle}{(\text{image of } R(\eta))} \;:=\; \frac{TV}{(\text{image of } R(\eta))}.$$

Because $Q_{n,k}(\eta \,|\, \tau)$ is defined by homogeneous relations of degree two it has a quadratic dual which we denote by $Q_{n,k}(\eta \,|\, \tau)^!$. By Theorem 1.3 below, $Q_{n,k}(\eta \,|\, \tau)$ is usually (conjecturally, always) a Koszul algebra with Hilbert series $(1-t)^{-n}$ which implies that the Hilbert series of $Q_{n,k}(\eta \,|\, \tau)^!$ is $(1+t)^n$. The finite dimensional algebras $Q_{n,k}(\eta \,|\, \tau)^!$ have a rich representation theory (see [16] for example).

Feigin and Odesskii observed that H_n acts as $\mathbb{C}$-algebra automorphisms via

$$S \cdot x_\alpha \;=\; e\big(\tfrac{\alpha}{n}\big)x_\alpha \qquad \text{and} \qquad T \cdot x_\alpha \;=\; x_{\alpha+1}.$$

The H_n-modules V and $\Theta_n(\tau)$ are isomorphic via $x_\alpha \longleftrightarrow \theta_\alpha$. Thus, it makes sense to identify the degree-one component of $Q_{n,k}(\eta \,|\, \tau)$ with $\Theta_n(\tau)$. The map φ in (2) therefore gives a natural embedding $E_\tau \to \mathbb{P}(V^*)$.[2]

[2]Translation by a point $\zeta \in E_\tau[n]$ extends to an automorphism of $\mathbb{P}(V^*)$ and these are the same automorphisms as those coming from the projective action of $H_n/\langle\epsilon\rangle$ on $\mathbb{P}(V^*)$; on E_τ, S acts as translation by $\frac{1}{n}$ and T acts as translation by $\frac{1}{n}\tau$.

1.3. **Definition of $Q_{n,k}(E,\xi)$.** Fix a complex elliptic curve E and a translation automorphism $\xi : E \to E$.

The group $\mathrm{SL}(2,\mathbb{Z})$ acts on $\mathbb{C} \times \mathbb{H}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\eta \,|\, \tau) \;=\; \left(\frac{\eta}{c\tau + d} \;\middle|\; \frac{a\tau + b}{c\tau + d} \right)$$

Theorem 1.1. [9] *There are graded $\mathbb{C}$-algebra isomorphisms*

$$Q_{n,k}(\eta \,|\, \tau) \;\cong\; Q_{n,k}\left(\frac{\eta}{c\tau + d} \;\middle|\; \frac{a\tau + b}{c\tau + d} \right).$$

Choose any point $(\eta \,|\, \tau) \in \mathbb{C} \times \mathbb{H}$ such that there is an isomorphism $\psi : E_\tau \to E$ with the property that $\psi(\eta + \Lambda_\tau) = \xi$; such a point $(\eta \,|\, \tau)$ exists. Another point $(\eta' \,|\, \tau')$ "has the same property" if and only if $(\eta' \,|\, \tau') = \gamma \cdot (\eta \,|\, \tau)$ for some $\gamma \in \mathrm{SL}(2,\mathbb{Z})$. It therefore follows from Theorem 1.1 that the definition

$$Q_{n,k}(E,\xi) \;:=\; Q_{n,k}(\eta \,|\, \tau)$$

defines $Q_{n,k}(E,\xi)$ up to isomorphism of $\mathbb{N}$-graded $\mathbb{C}$-algebras.

1.4. **Properties of $Q_{n,k}(E,\xi)$.** When $k = 1$, Feigin and Odesskii gave an alternative geometric description of the relations for $Q_{n,k}(E,\xi)$ (see [4, §3.2.5] and [13, §2(a)]). Tate and Van den Bergh used this description to define algebras $Q_{n,1}(E,\xi)$ for elliptic curves E over any field [20].

It would be good to have an analogous geometric description of the relations when $k > 1$. This might allow us to use Tate and Van den Bergh's notion of a basis of I-type for $Q_{n,k}(E,\xi)$, which was the key tool in proving Theorem 1.2, when $k \geq 2$ and so prove the next theorem for all k. (See [1] and [12] for some of the terminology.)

Theorem 1.2. [20] *The algebra $A = Q_{n,1}(E,\xi)$ is a noetherian domain which is Koszul and has Hilbert series equal to $(1-t)^{-n}$. It is a finitely generated module over its center $\Leftrightarrow$ ξ has finite order. Moreover it is Artin-Schelter regular, satisfies the Auslander condition, is Cohen-Macaulay and a maximal order;*

We expect Theorem 1.2 is true for all $Q_{n,k}(E,\xi)$, but can only prove a slightly sharper version of the next result (which is, of course, over $\mathbb{C}$).

Theorem 1.3. [7] *Assume $\xi \in E$ is not a torsion point. The algebra $Q_{n,k}(E,\xi)$ is Koszul with Hilbert series $(1-t)^{-n}$. It is a Frobenius algebra for almost all $\xi \in E$.*

2. The characteristic variety of $Q_{n,k}(E,\xi)$

2.1. **Representation theory of a connected graded algebra.** Let $\Bbbk$ be an algebraically closed field and $A = \Bbbk \oplus A_1 \oplus A_2 \oplus \cdots$ a finitely presented connected graded $\Bbbk$-algebra that is generated by A_1. If $x_1, \ldots, x_n$ is a basis for A_1, then $A = \Bbbk\langle x_1, \ldots, x_n \rangle / (f_1, \ldots, f_r)$ for certain homogeneous elements f_i in the free algebra. Clearly, $d \times d$ matrix solutions to the system of equations $f_1 = \cdots = f_r = 0$ are the same things as d-dimensional A-modules and, since the f_i's are homogeneous, if $(M_1, \ldots, M_n)$ is such a solution, then so is $(\lambda M_1, \ldots, \lambda M_n)$ for all $\lambda \in \Bbbk^\times$. For that reason we consider the category $\mathrm{Gr}(A)$ of graded left A-modules. As one notes in the context of projective algebraic geometry, there is a trivial solution to the system $f_1 = \cdots = f_r = 0$, namely $(0, \dots, 0)$ and, in analogy with omitting the origin when passing from affine to projective space, one performs an algebraic operation that "excises" the trivial solution: one defines the quotient category

$$\mathrm{QGr}(A) \;:=\; \frac{\mathrm{Gr}(A)}{\mathrm{Fdim}(A)}$$

where $\mathrm{Fdim}(A)$ denotes the full subcategory of $\mathrm{Gr}(A)$ consisting of those modules that are the sum of their finite dimensional submodules or, equivalently, those modules all of whose elements are killed by a suitably high power of the ideal $A_{\geq 1}$ (informally, the modules supported at $\{0\}$.) *When we talk about the representation theory of A we mean the category $QGr(A)$.* If A is also commutative, then there is an equivalence of categories $\mathrm{QGr}(A) \;\equiv\; \mathrm{Qcoh}(X)$, where $\mathrm{Qcoh}(X)$ is the category of quasi-coherent sheaves on the projective scheme $X := \mathrm{Proj}(A) \subseteq \mathbb{P}^{n-1}$ (Serre in FAC). The simplest modules over a $\Bbbk$-algebra are those of dimension one. The simplest objects in $\mathrm{Qcoh}(X)$ are the skyscraper sheaves (they are the simple objects in $\mathrm{Qcoh}(X)$). If $X = \mathrm{Proj}(A)$ (where A is as above and commutative!), then the skyscraper sheaf $\mathcal{O}_x$ for a point $x = (\alpha_1, \dots, \alpha_n) \in X \subseteq \mathbb{P}^{n-1}$ corresponds to the graded A-module

$$M_x \;:=\; \frac{A}{(\alpha_j x_i \,-\, \alpha_i x_j \mid 1 \leq i, j \leq n)} \;=\; \frac{A}{Ax^{\perp}}\,.$$

This module has the following properties: (1) it is generated by its degree-zero component, and (2) $\dim_{\Bbbk}(M_x)_i = 1$ for all $i \geq 0$.

Now, for any A as defined at the start of this subsection, we call a graded left A-module M a point module if $M = AM_0$ and $\dim_{\Bbbk}(M_i) = 1$ for all $i \geq 0$. Such a module becomes a simple object in $\mathrm{QGr}(A)$ and if M and M' are point modules, then $M \cong M'$ in $\mathrm{Gr}(A)$ if and only if $M \cong M'$ in $\mathrm{QGr}(A)$. Thus point modules give "skyscraper sheaves" in $\mathrm{QGr}(A)$ (or "points in $\mathrm{Proj}_{nc}(A)$).[3]

For many of the non-commutative algebras A having the "good properties" in Theorems 1.2 and/or 1.3, there is a subvariety (not necessarily irreducible) $X \subseteq \mathbb{P}(A_1^*)$ with the following property: if $x \in X$, then the module $M_x := A/Ax^{\perp}$, where $x^{\perp} := \{a \in A_1 \mid a \text{ vanishes at } x\}$, is a point module and all point modules are of this form. (This is the case for all $Q_{n,1}(E,\xi)$, for example.) In that case we call X the point variety for A.

The characteristic variety that we define below is "a large part" of the point variety for $Q_{n,k}(E,\xi)$ and, conjecturally, is "often" all of it.[4]

[3]There will usually be other simple objects in $\mathrm{QGr}(A)$, e.g., the fat point modules in [15].

[4]Feigin and Odesskii thought that the characteristic variety coincided with the point variety but that is not true for $Q_{4,1}(E,\xi)$ or $Q_{8,3}(E,\xi)$, for example.

2.2. **The integers** $n_1, \dots, n_g$ **and the subgroup** $\Sigma_{n/k} \subseteq \mathrm{Aut}(E^g)$**.** If $n_1, \dots, n_g$ are integers ≥ 2, we use the notation

$$[n_1, \dots, n_g] \;=\; n_1 - \cfrac{1}{n_2 - \cfrac{1}{\ddots - \frac{1}{n_g}}}. \tag{5}$$

There are unique integers $g \geq 1$ and $n_1, \dots, n_g$, all ≥ 2, such that $\frac{n}{k} \;=\; [n_1, ..., n_g]$. We will consider the g-fold product $E^g = E \times \cdots \times E$. Let $\mathcal{L} = \mathcal{O}_E((0))$. Odesskii and Feigin defined the invertible $\mathcal{O}_{E^g}$-module

$$\mathcal{L}_{n/k} \;:=\; \Big(\mathcal{L}^{n_1} \boxtimes \cdots \boxtimes (\mathcal{L}^{n_g}\Big)\left(\bigotimes_{i=1}^{g-1} \mathrm{pr}_{i,i+1}^* \mathcal{P}\right)$$

where $\mathrm{pr}_{i,i+1} : E^g \to E \times E$ is the map $(z_1, \dots, z_g) \mapsto (z_i, z_{i+1})$ and $\mathcal{P}$ is the Poincaré bundle $\mathcal{O}_{E^2}(\Delta - \{0\} \times E - E \times \{0\})$,

The subgroup

$$E^g \;\equiv\; \{(z_1, \dots, z_{g+1}) \in E^{g+1} \mid z_1 + \cdots + z_{g+1} = 0\} \;\subseteq\; E^{g+1}$$

is stable under the permutation action of the symmetric group Σ_{g+1} of order $(g+1)!$ on E^{g+1}. Let $s_i \in \Sigma_{g+1}$ be the transposition $(i, i+1)$ and define

$$\Sigma_{n/k} \;:=\; \text{the subgroup generated by } \{s_i \mid n_i = 2,\; 1 \leq i \leq g\} \;\subseteq\; \Sigma_{g+1}.$$

Proposition 2.1. [5] We have $\dim H^0(E^g, \mathcal{L}_{n/k}) = n$. Moreover, $\mathcal{L}_{n/k}$ is ample and base-point free (i.e., ample and generated by its global sections). It is very ample if and only if all n_i are ≥ 3.

It follows from (1) and (2) that the complete linear system $|\mathcal{L}_{n/k}|$ provides a morphism $E^g \to \mathbb{P}H^0(E^g, \mathcal{L}_{n/k})^* \cong \mathbb{P}^{n-1}$. We define the characteristic variety

$$X_{n/k} \;:=\; \text{the image of } |\mathcal{L}_{n/k}| \;\subseteq\; \mathbb{P}H^0(E^g, \mathcal{L}_{n/k})^* \cong \mathbb{P}^{n-1}. \tag{6}$$

Theorem 2.2. [5] $X_{n/k} \;\cong\; E^g/\Sigma_{n/k}$.

One can give a more explicit description of $X_{n/k}$ in terms of products of symmetric powers of E [8].

Theorem 2.3 (Feigin-Odesskii)**.** *Each point* $x \in X_{n/k}$ *determines a point module for* $Q_{n,k}(E, \xi)$.[5]

2.3. **A twisted homogeneous coordinate ring.** Using the integers $n_1, \dots, n_g$, one may define an automorphism $\sigma : E^g \to E^g$ that descends to an automorphism $\sigma' : X_{n/k} \to X_{n/k}$. Let $\mathcal{L}'$ denote $\mathcal{O}_{\mathbb{P}^{n-1}}(1)\big|_{X_{n/k}}$, via (6). Following [2], we denote by $B(X_{n/k}, \sigma', \mathcal{L}')$ the twisted homogeneous coordinate ring associated to this data.

Theorem 2.4. [6]

[5]It is not clear that the point module associated to x is of the form $A/Ax^{\perp}$, though it is certainly a quotient of this.

(1) There are graded $\mathbb{C}$*-algebra homomorphisms*

$$Q_{n,k}(E,\xi) \longrightarrow B(X_{n/k},\sigma',\mathcal{L}'_{n/k}) \xrightarrow{\sim} B(E^g,\sigma,\mathcal{L}_{n/k})^{\Sigma_{n/k}} \subseteq B(E^g,\sigma,\mathcal{L}_{n/k})$$

that are surjective in degree one.

(2) $\mathrm{QGr}(B(X_{n/k},\sigma',\mathcal{L}'_{n/k}))$ *and* $\mathrm{QGr}(B(E^g,\sigma,\mathcal{L}_{n/k}))$ *are equivalent to* $\mathrm{Qcoh}(X_{n/k})$ *and* $\mathrm{Qcoh}(E^g)$*, respectively.*

We could show further results, in effect, that there is a closed immersion

$$i : X_{n/k} \longrightarrow \mathrm{Proj}_{nc}\big(Q_{n,k}(E,\xi)\big).$$

Phrases like this can be made precise: see [22] and [17].

2.4. **Connections with $U_q(\mathfrak{sl}_2)$.** The remarks in the introduction to [13] suggest a relation between a degeneration of $Q_{n^2,n-1}(E,\xi)$ and $U_q(\mathfrak{sl}_n)$. The case $n = 2$, i.e., $Q_{4,1}(E,\xi)$ and $U_q(\mathfrak{sl}_2)$, is examined in some detail in [10]. The structure constants for $Q_{4,1}(E,\xi) = Q_{4,1}(\eta\,|\,\tau)$ (in Sklyanin's presentation [14]) can be expressed in terms of ratios of products of theta functions having half-integer characteristics; if one replaces these by their limits as $\tau \to i\infty$ one obtains an algebra S, or rather a family of algebras depending on the parameter η. There are normal degree-one elements K and K' in S such that KK' is central. It turns out that $S[(KK')^{-1}]_0 \cong U_q(\mathfrak{sl}_2)$ where q depends on η. It turns out that there is an exact functor $j^* : \mathrm{QGr}(S) \to \mathrm{Mod}\big(U_q(\mathfrak{sl}_2)\big)$ having a right adjoint $j_* : \mathrm{Mod}\big(U_q(\mathfrak{sl}_2)\big) \to \mathrm{QGr}(S)$. In geometric terms, after removing the two "noncommutative hyperplanes" $\{KK' = 0\}$ in $\mathrm{Prop}_{nc}(S)$, which is a non-commutative analogue of $\mathbb{P}^3$, one obtains a "noncommutative affine variety" with "coordinate ring" $U_q(\mathfrak{sl}_2)$; there is an open immersion $j : \mathrm{Spec}_{nc}(U_q(\mathfrak{sl}_2) \to \mathrm{Prop}_{nc}(S)$. The remarks in [13] suggest that something like this will happen when one degenerates $Q_{n^2,n-1}(\eta\,|\,\tau)$ by taking $\tau \to i\infty$. We also remind the reader of Van den Bergh's remarkable paper [21] which establishes a translation principle for $Q_{4,1}(E,\xi)$ that is analogous to that for $U(\mathfrak{sl}_2)$. It would not be surprising if there were a similar result for $Q_{n^2,n-1}(E,\xi)$.

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Coadjoint orbits for the Virasoro algebra

Susan J. Sierra
(joint work with Alexey V. Petukhov)

Unless otherwise referenced, all results in this extended abstract are proved in [4].

Let $W = \mathbb{C}[t,t^{-1}]\partial_t$ be the *Witt algebra* of algebraic vector fields on $\mathbb{C}^\times$, and let *Vir*, the *Virasoro algebra* be its unique nontrivial central extension $Vir = \mathbb{C}[t,t^{-1}]\partial_t \oplus \mathbb{C}z$, with Lie bracket given by

$$[f\partial_t, g\partial_t] = (fg' - f'g)\partial_t + \mathrm{Res}_0(f'g'' - g'f'')z, \quad z \text{ is central.}$$

It has been known since 2013 that the universal enveloping algebras of these Lie algebras have badly behaved one-sided structure:

Theorem 1 ([5]). *$U(W)$ and thus $U(Vir)$ are neither left or right noetherian.*

(This theorem was first proved by relating $U(W)$ to a graded domain of cubic growth which was known, through the (partial) classification of noncommutative projective surfaces, to be non-noetherian.)

However, two-sided structures of these enveloping algebras appear more tractable. As a shadow of this, we study Poisson ideals (under the Kostant–Kirillov Poisson bracket) in their symmetric algebras. Concretely, we write

$$Vir = \mathbb{C} \cdot (z, \{e_i : i \in \mathbb{Z}\}),$$

where $e_i = t^{i+1}\partial_t$. Then we have $S(Vir) = \mathbb{C}[z, \{e_i\}]$, with Poisson bracket

$$\{z, -\} = 0, \qquad \{e_i, e_j\} = (j-i)e_{i+j} + \delta_{i,-j}\frac{i^3 - i}{12}z.$$

Standard algebraic geometry on $\operatorname{MSpec} S(Vir)$ is difficult, as maximal ideals correspond to elements of the uncountable-dimensional vector space Vir^* via the extended Nullstellensatz:

$$\chi \in Vir^* \leftrightarrow \mathfrak{m}_\chi = \{f : f(\chi) = 0\}.$$

It turns out, however, that *Poisson geometry* on $\operatorname{MSpec} S(Vir)$ has much more structure. Here recall that a two-sided ideal $I \triangleleft S(Vir)$ is a *Poisson ideal*, written $I \triangleleft_P S(Vir)$, if I is a Lie ideal for the Poisson bracket:

$$\{I, S(Vir)\} \subseteq I.$$

Poisson ideals in $S(Vir)$ are well-behaved in important ways. For example:

Proposition 2 ([3])**.** *$S(Vir)$ satisfies ACC on radical Poisson ideals.*

Theorem 3 ([2])**.** *Let $\lambda \in \mathbb{C}$ and let $I \triangleleft_P S(Vir)$ be a Poisson ideal which strictly contains $(z-\lambda)$. Then $S(Vir)/I$ has polynomial growth: that is, finite Gelfand–Kirillov dimension.*

A moral corollary of these two results is that if P is a prime Poison ideal of $S(Vir)$ which strictly contains some $(z-\lambda)$, then

$$V(P) \subseteq \operatorname{MSpec} S(Vir) = Vir^*$$

is a finite-dimensional variety, and one can hope to do algebraic geometry on $V(P)$.

We are particularly interested in $V(P)$ where P is a so-called *Poisson primitive* ideal of $S(Vir)$: that is, there is some $\chi \in Vir^*$ so that P is the maximal Poisson ideal contained in $\mathfrak{m}_\chi$. We write $P = \operatorname{PCore}(\mathfrak{m}_\chi)$ for this situation.

Recall that if G is an algebraic group with $\mathfrak{g} = \operatorname{Lie}(G)$, and $\chi \in \mathfrak{g}^*$, then $\operatorname{PCore}(\mathfrak{m}_\chi) \triangleleft_P S(\mathfrak{g})$ is the defining ideal of the coadjoint orbit $G \cdot \chi$ in $\mathfrak{g}^* = \operatorname{MSpec} S(\mathfrak{g})$. Further, $\eta \in G \cdot \chi$ if and only if $\operatorname{PCore}(\mathfrak{m}_\eta) = \operatorname{PCore}(\mathfrak{m}_\chi)$. Although Vir famously has no adjoint group, we are motivated by this classical relationship to define:

Definition. Let $\chi \in Vir^*$.The *pseudo-orbit* of χ is

$$\mathbb{O}(\chi) := \{\eta \in Vir^* : \operatorname{PCore}(\mathfrak{m}_\eta) = \operatorname{PCore}(\mathfrak{m}_\chi)\}.$$

There are then two natural questions:

Question.

(1) Classify the Poisson primitive ideals $\mathrm{PCore}(\mathfrak{m}_\chi)$.
(2) Given χ*, compute* $\mathbb{O}(\chi)$.

We are able to do both. In particular, a sub-question of Question (1) is:

Question. *(1a) For which* $\chi \in Vir^*$ *does* $\mathrm{PCore}(\mathfrak{m}_\chi) \supsetneqq (z - \chi(z))$*?*

The answer to Question (1a) involves local functions, where $\chi \in Vir^*$ is *local* if χ is a sum of functions of the form:

$$\chi_{x;\alpha_0,\ldots,\alpha_n} : z \mapsto 0, f\partial_t \mapsto \alpha_0 f(x) + \cdots + \alpha_n f^{(n)}(x).$$

One of our main results is:

Theorem 4. $\chi \in Vir^*$ *is local if and only if* $\mathrm{PCore}(\mathfrak{m}_\chi) \neq (z - \chi(z))$.
In particular, if $\lambda \in \mathbb{C}^\times$ *then* $S(Vir)/(z-\lambda)$ *is Poisson simple.*

To prove this theorem, we also consider local functions on W (under the obvious definition), and show

Proposition 5. *For* $\chi \in W^*$*, the following are equivalent:*

(a) χ *is local;*
(b) $\dim W \cdot \chi < \infty$*;*
(c) There is $0 \neq h \in \mathbb{C}[t, t^{-1}]$ *so that* $W^\chi \supseteq hW$*, where* W^χ *is the isotropy subalgebra of* χ*;*
(d) $\mathrm{PCore}(\mathfrak{m}_\chi)$ *is nontrivial.*

From this result, we can deduce:

Corollary 6. *Let* $\mathfrak{g}$ *be a Lie subalgebra of* W *of finite codimension. Then* $\mathfrak{g}$ *contains some nontrivial* hW*.*

Corollary 7. *If* $\mathfrak{g}$ *is a Lie subalgebra of* Vir *of finite codimension, then* $z \in [\mathfrak{g}, \mathfrak{g}]$.

Corollary 7 and Theorem 3 allow us to conclude that if $\chi \in Vir^*$ is such that $\mathrm{PCore}(\mathfrak{m}_\chi) \neq (z - \chi(z))$, then $\chi(z) = 0$, reducing the characterisation in Theorem 4 to that in Proposition 5.

We are further able to show that for a local function $\chi \in Vir^*$, the pseudo-orbit $\mathbb{O}(\chi)$ is isomorphic to the orbit of a finite-dimensional algebraic group on an affine variety. We use this to explicitly describe $\mathbb{O}(\chi)$. We also give a parameterisation of Poisson prime ideals in $S(W) = S(Vir)/(z)$. (Recall that by Theorem 4, $S(Vir)/(z-\lambda)$ is Poisson simple for $\lambda \neq 0$.) Open questions include:

Question.

(A) The containment relations among Poisson primitive ideals.
(B) If $\lambda \neq 0$*, is* $U(Vir)/(z-\lambda)$ *simple?*
(C) Does $U(Vir)$ *satisfy the ascending chain condition on two-sided ideals?*

As moral evidence towards Question (B), consider the following recent result:

Theorem 8 ([1]). *Let $\mathfrak{g}$ be a finite-dimensional simple complex Lie algebra (for example, $\mathfrak{g} = \mathfrak{sl}_2$), and let $\widehat{L\mathfrak{g}}$ be the unique nontrivial central extension of the loop algebra of $\mathfrak{g}$. If $\lambda \neq 0$ then $U(\widehat{L\mathfrak{g}})/(z-\lambda)$ is simple.*

As representations of $\widehat{L\mathfrak{g}}$ and of Vir are linked through the Sugawara construction, and $\widehat{L\mathfrak{g}}$ is significantly more commutative than Vir, it is difficult to believe that Theorem 8 is compatible with a negative answer to Question B.

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Spaces of quasi-invariants of compact Lie groups

Yuri Berest

(joint work with Ajay C. Ramadoss)

Quasi-invariants are natural geometric generalizations of invariant polynomials of finite reflection groups. In the case of Coxeter groups, they first appeared in mathematical physics in the early 1990s (see [5]), and since then have found applications in other areas: most notably, representation theory, noncommutative algebra and combinatorics (see, e.g., [8], [7], [2], [1]). For general pseudoreflection groups, quasi-invariants were introduced in [1]. This last paper extended the results of [2] linking quasi-invariants to representation theory of Cherednik algebras.

The goal of this work (see [3]) is to realize the algebras of quasi-invariants topologically: as (equivariant) cohomology rings of certain spaces naturally attached to compact connected Lie groups. Our main result is a generalization of a well-known theorem of A. Borel [4] that realizes the algebra of invariant polynomials of a Weyl group W as the cohomology ring of the classifying space BG of the corresponding Lie group G:

$$H^*(BG, \mathbb{Q}) \cong H^*(BT, \mathbb{Q})^W, \tag{1}$$

where BT is the classifying space of the maximal torus T in G. Replacing cohomology with equivariant K-theory gives a multiplicative (exponential) analogues of quasi-invariants, and in fact, quasi-invariants can be defined for an arbitrary (complex oriented) generalized cohomology theory.

We first recall the algebraic definition of quasi-invariants. Let W be a Coxeter group acting in its complexified reflection representation V. Denote by $\mathcal{A} := \{H_\alpha\}$ the set of reflection hyperplanes of W in V and write $s_\alpha \in W$ for the reflection in

H_α. For each $H_\alpha \in \mathcal{A}$, fix a linear form $\alpha_H \in V^*$ such that $H_\alpha = \mathrm{Ker}(\alpha_H)$ and choose a non-negative integer $m_\alpha \in \mathbb{Z}_+$ to be referred to as a *multiplicity* of H_α. The group W acts naturally on $\mathcal{A}$ and we assume that $m_\alpha = m_{\alpha'}$ whenever H_α and $H_{\alpha'}$ are in the same W-orbit in $\mathcal{A}$: thus, we define a W-invariant function $m : \mathcal{A} \to \mathbb{Z}_+$ by $H_\alpha \mapsto m_\alpha$. Now, following [5] (see also [2]), we call a polynomial $p \in \mathbb{C}[V]$ a *W-quasi-invariant of multiplicity* m if it satisfies

$$s_\alpha(p) \equiv p \mod \langle \alpha_H \rangle^{2m_\alpha}, \ \forall\, \alpha \in \mathcal{A}, \tag{2}$$

where $\langle \alpha_H \rangle$ is the principal ideal in $\mathbb{C}[V]$ generated by the form α_H. We write $Q_m(W)$ for the space of all W-quasi-invariants in $\mathbb{C}[V]$ of multiplicity m. It is easy to see that $Q_m(W)$ is a f.g. graded subalgebra of $\mathbb{C}[V]$ containing $\mathbb{C}[V]^W$ and stable under the action of W. A much deeper fact (proved in [8] for rank two groups and [7] and [2] in general) is that $Q_m(W)$ is a free module over $\mathbb{C}[V]^W$ of rank $|W|$ for all m. For $m = 0$, this is a well-known theorem due to Chevalley.

Now, let's turn to topology. It is a general consequence of Quillen's rational homotopy theory [9] that every reduced, locally finite, graded commutative algebra A defined over a field k of characteristic zero is topologically realizable, i.e. $A \cong H^*(X, k)$ for some (simply-connected) space X. Thus, if we put on $\mathbb{C}[V]$ the cohomological grading ($\deg(v) = 2$ for all $v \in V$), then the natural question: "For which values of m is the algebra $Q_m(W) \subseteq \mathbb{C}[V]$ realizable?" has an immediate answer: for all m. A more interesting question is whether we can realize the *family* of algebras $Q_m(W)$ *together* with additional structure that these algebras possess. To make this precise denote by $\mathcal{M}(W) := \mathbb{Z}_+[\mathcal{A}/W]$ the set of all W-invariant multiplicity functions $m : \mathcal{A} \to \mathbb{Z}_+$ and put on this set the natural partial order:

$$m' \geq m \quad \overset{\text{def}}{\Longleftrightarrow} \quad m'_\alpha \geq m_\alpha \ , \ \forall\, \alpha \in \mathcal{A}.$$

Then the algebras $Q_m(W)$ (with fixed W but varying 'm') form a contravariant diagram on the poset $\mathcal{M}(W)$, i.e. a functor $\mathcal{M}(W)^{\mathrm{op}} \to \mathtt{CommAlg}_{\mathbb{C}}$ with values in (graded) commutative algebras that we can visualize as a filtration

$$\mathbb{C}[V] = Q_0(W) \supseteq \ldots \supseteq Q_m(W) \supseteq Q_{m'}(W) \supseteq \ldots \supseteq \mathbb{C}[V]^W \tag{3}$$

If we apply 'Spec' to (3), we get a diagram of W-equivariant affine schemes $V_m(W) := \mathrm{Spec}\,[Q_m(W)]$ (called the *varieties of quasi-invariants*) over $\mathcal{M}(W)$:

$$V = V_0(W) \to \ldots \to V_m(W) \xrightarrow{\pi_{m,m'}} V_{m'}(W) \to \ldots \tag{4}$$

that has many nice geometric properties (see [2]). We would like to realize the diagrams (3) and (4) topologically, modeling geometric properties of schemes in (4) by homotopy-theoretic properties of classifying spaces of compact Lie groups. To make it precise we state our realization problem axiomatically.

Problem. Given a compact connected Lie group G with maximal torus $T \subseteq G$ and associated Weyl group $W = W_G(T)$, construct a diagram of spaces $X_m(G, T)$ over the poset $\mathcal{M}(W)$:

$$BT = X_0(G, T) \to \ldots \to X_m(G, T) \xrightarrow{\pi_{m,m'}} X_{m'}(G, T) \to \ldots \tag{5}$$

together with natural maps $p_m : X_m(G, T) \to BG$ (one for each m), such that

(P1) Each $X_m(G,T)$ is a W-space (i.e., a CW complex equipped with an action of W), and all maps are W-equivariant. The map $p_0 : X_0(G,T) \to BG$ coincides with the canonical map $p : BT \to BG$, and for all $m' \geq m$, we have $p_{m'} \circ \pi_{m,m'} = p_m$. Thus, (5) is a diagram of W-spaces over BG.

(P2) The diagram (5) 'converges' to BG in the sense that the maps p_m induce a weak homotopy equivalence of spaces:

$$\mathrm{hocolim}_{\mathcal{M}(W)}[X_m(G,T)] \xrightarrow{\sim} BG\,.$$

(P3) Each map $p_m : X_m(G,T) \to BG$ factors (naturally in m) through the fibre inclusion into the space $X_m(G,T)_{hW}$ of homotopy W-orbits in $X_m(G,T)$, inducing algebra isomorphisms for all $m \in \mathcal{M}(W)$:

$$H^*_W(X_m,\,\mathbb{Q}) \cong H^*(BG,\,\mathbb{Q})$$

(P4) Each map $\pi_{m,m'}$ in (5) induces an injective homomorphism on cohomology so that the Borel map (1) factors into a $\mathcal{M}(W)^{\mathrm{op}}$-diagram (filtration) of algebra maps

$$H^*(BT,\mathbb{Q}) \hookleftarrow \ldots \hookleftarrow H^*(X_m,\mathbb{Q}) \overset{\pi^*_{m,m'}}{\hookleftarrow} H^*(X_{m'},\mathbb{Q}) \hookleftarrow \ldots \hookleftarrow H^*(BG,\mathbb{Q})$$

(P5) With natural identification $H^*(BT,\mathbb{Q}) = \mathbb{Q}[V]$ (where $V = H_2(BT,\mathbb{Q})$), the maps $\pi^*_{0,m} : H^*(X_m,\mathbb{Q}) \hookrightarrow H^*(BT,\mathbb{Q})$ in (P4) induce isomorphisms

$$H^*(X_m,\mathbb{Q}) \otimes \mathbb{C} \cong Q_m(W)$$

where $Q_m(W)$ are the subalgebras of W-quasi-invariants in $\mathbb{C}[V]$.

Note that (P4) and (P5) show that the diagram of spaces (5) provides a topological realization for the diagram of algebras (3). The first three properties are natural homotopy-theoretic analogues of properties of the diagram of varieties (4).

Now, the main result of our work can be encapsulated into the following

Theorem ([3])**.** *For any compact connected Lie group G, there exists a diagram of spaces $X_m(G,T)$ satisfying all properties* (P1)–(P5) *listed above, with* (P4) *and* (P5) *holding* (*at least*) *for even-dimensional cohomology.*

We expect that axioms (P1)-(P5) characterize the spaces $X_m(G,T)$ uniquely, up to (rational) homotopy equivalence (i.e. the above realization problem has a unique solution for any Lie group G). Most interesting perhaps is the fact that $X_m(G,T)$ can be constructed functorially in a purely homotopy-theoretic way (using the so-called 'fibre-cofibre' construction due to T. Ganea). A generalization of this construction allows us to define analogues of the spaces $X_m(G,T)$ for some non-Coxeter (p-adic) pseudoreflection groups, in which case the compact Lie groups are replaced by the *p-compact groups* a.k.a. homotopy Lie groups (see [6]).

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Representations of cohomological Hall algebras via stable pairs

Francesco Sala
(joint work with Duiliu-Emanuel Diaconescu and Mauro Porta)

Let $\mathcal{A}$ be an abelian category and denote by $\mathcal{M}_{\mathcal{A}}$ the corresponding *moduli stack of objects*. Under certain conditions on $\mathcal{A}$, the stack $\mathcal{M}_{\mathcal{A}}$ is an Artin stack locally of finite type over $\mathbb{C}$ parametrizing *families* of flat objects in $\mathcal{A}$. In particular, its groupoid of $\mathbb{C}$-points $\mathcal{M}_{\mathcal{A}}(\mathbb{C})$ coincides with the groupoid of objects of $\mathcal{A}$. Similarly, we can consider the moduli stack $\mathcal{M}_{\mathcal{A}}^{\mathsf{ext}}$ parametrizing *families* of short exact sequences in $\mathcal{A}$ and form the diagram

$$\mathcal{M}_{\mathcal{A}} \times \mathcal{M}_{\mathcal{A}} \xleftarrow{\ p\ } \mathcal{M}_{\mathcal{A}}^{\mathsf{ext}} \xrightarrow{\ q\ } \mathcal{M}_{\mathcal{A}} \tag{1}$$

where p and q map $0 \to E_1 \to E \to E_2 \to 0$ to (E_1, E_2) and E, respectively.

The map q is representable by proper schemes. Assume that p is "sufficiently well behaved", then passing to (an oriented[1]) Borel–Moore homology yields a product map

$$q_* \circ p^* \colon \mathsf{H}_*^{\mathsf{BM}}(\mathcal{M}_{\mathcal{A}}) \otimes \mathsf{H}_*^{\mathsf{BM}}(\mathcal{M}_{\mathcal{A}}) \longrightarrow \mathsf{H}_*^{\mathsf{BM}}(\mathcal{M}_{\mathcal{A}}) \ , \tag{2}$$

which can after been proven to be associative. We refer to the above multiplicative structure as a "cohomological Hall algebra" attached to $\mathcal{A}$.

Examples of cohomological Hall algebras are those associated to finite-dimensional representations of a quiver (cf. [KS11]), finite-dimensional representations of the preprojective algebra of a quiver (cf. [SV20] and references therein), properly supported coherent sheaves on a smooth surface (cf. [KV19] and references therein), Higgs sheaves on a smooth projective curve (cf. [SS20] and references therein), etc.

In [PS19], in the curve and surface cases, we defined suitable natural derived enhancements $\mathbb{R}\mathcal{M}_{\mathcal{A}}$ and $\mathbb{R}\mathcal{M}_{\mathcal{A}}^{\mathsf{ext}}$ of the moduli stacks $\mathcal{M}_{\mathcal{A}}$ and $\mathcal{M}_{\mathcal{A}}^{\mathsf{ext}}$, such that

[1] Examples of oriented Borel–Moore homology theories are the G_0-theory (i.e., the Grothendieck group of coherent sheaves), Chow groups, elliptic cohomology.

p is *derived lci*. Thus, the derived enhancement of the convolution diagram (1) induces a $\mathbb{E}_1$-monoidal structure

$$\circledast_{\mathrm{Hall}} \colon \mathsf{Coh}^{\mathsf{b}}(\mathbb{R}\mathcal{M}_{\mathcal{A}}) \otimes \mathsf{Coh}^{\mathsf{b}}(\mathbb{R}\mathcal{M}_{\mathcal{A}}) \longrightarrow \mathsf{Coh}^{\mathsf{b}}(\mathbb{R}\mathcal{M}_{\mathcal{A}}) \tag{3}$$

on the dg-category of complexes of sheaves with bounded coherent cohomology on $\mathbb{R}\mathcal{M}_{\mathcal{A}}$. We call this monoidal dg-category the *categorified Hall algebra of* $\mathcal{A}$.

Two-dimensional cohomological Hall algebras represent a geometrical approach to the construction of *Yangians*, i.e., certain quantizations of the universal enveloping algebra of the current algebra of a Lie algebra. A different approach to the geometric realization of Yangians is due to Maulik–Okounkov, which developed the theory of *stable envelopes* associated to any symplectic variety. Applying this theory to *Nakajima quiver varieties* of a quiver $\mathcal{Q}$, they produced a R-matrix, and then, thanks to the RTT formalism, they defined an associative algebra, which is the Yangian of a graded Lie algebra associated to $\mathcal{Q}$ (whose degree zero part is the usual Kac–Moody algebra of $\mathcal{Q}$). In [SV17], the authors established a relation between Maulik–Okounkov approach and the cohomological Hall algebra of the preprojective algebra of $\mathcal{Q}$: one of the key points is that the latter acts on the cohomology of Nakajima quiver varieties of $\mathcal{Q}$.

To apply Maulik–Okounkov formalism of stable envelope to other cases (such as the "surface case") and establish a relation to the corresponding cohomological Hall algebra, a first step consists of providing a representation of the latter to the cohomology of certain moduli space. In addition, the searched moduli space should have a symplectic structure and a proper map to a singular (affine) space, so that it could be interpreted as a "surface analog" of a Nakajima quiver variety. Let S be a smooth projective irreducible complex surface. In this case, a candidate is the moduli space of *Pandharipande–Thomas stable pairs*.

A *stable pair* is a pair consisting of a pure one-dimensional sheaf $\mathcal{F}$ on S and a section $s \colon \mathscr{O}_S \to \mathcal{F}$ with zero-dimensional cokernel. Let $\mathbf{P}(S)$ be the moduli space of stable pairs on S. One can define a natural derived enhancement $\mathbb{R}\mathbf{P}(S)$ of $\mathbf{P}(S)$. One of the main results of [DPS] is the following:

Theorem ([DPS])**.** *Assume that*

(1) $H^0(S, \omega_S \otimes \mathscr{O}_S(-C)) = 0$ *for any non-zero effective divisor* C*; or*

(2) $H^0(S, \omega_S) = 0$.

Then, $\mathsf{Coh}^{\mathsf{b}}_{\mathsf{pro}}(\mathbb{R}\mathbf{P}(S))$ *is a right categorical module over the categorified Hall algebra of the category of torsion sheaves on* S*. In particular,* $G_0(\mathbf{P}(S))$ *and* $H^{\mathsf{BM}}_*(\mathbf{P}(S))$ *are right modules of the K-theoretical and cohomological Hall algebras of the category of torsion sheaves on* S*, respectively.*

The derived moduli space $\mathbb{R}\mathbf{P}(S)$ has a natural 0-shifted symplectic structure. Moreover, the classical stack $\mathbf{P}(S)$ has a *Hilbert–Chow map* to the Chow variety of S, sending the underlying one-dimensional sheaf of a stable pair to its fundamental one-cycle. Thus, although $\mathbf{P}(S)$ has some of the geometrical properties of a Nakajima quiver variety, one cannot apply directly Maulik–Okounkov formalism since the classical moduli space is singular: a way to overcome this could be to extend their theory to symplectic quasi-smooth derived schemes.

Let X be a smooth projective complex curve and let $S \coloneqq \mathbb{P}(\mathscr{O}_X \oplus \omega_X)$ be the compactification of the cotangent space T^*X of X. Denote by D the compactifying divisor. In this case, we can restrict the above action to an action of the categorified (resp. K-theoretical, cohomological) Hall algebra of the category of torsion sheaves on S, which are disjoint from D, on $\mathsf{Coh}^{\mathsf{b}}_{\mathsf{pro}}(\mathbb{R}\mathbf{P}(S)^\circ)$ (resp. $G_0(\mathbf{P}(S)^\circ)$ and $H_*^{\mathsf{BM}}(\mathbf{P}(S)^\circ)$). Here, $\mathbf{P}(S)^\circ$ is the open scheme of $\mathbf{P}(S)$ consisting of those stable pairs, which are disjoint from D.

The spectral correspondence yields a correspondence between torsion sheaves (resp. stable pairs) on S whose support is disjoint from D and Higgs sheaves (resp. cyclic Higgs bundles) on X. Denote by $\mathbf{Higgs}^{\mathsf{cyc}}(X)$ the moduli space of cyclic Higgs bundles on X.

For example, if $X = \mathbb{P}^1$, via the derived McKay correspondence the moduli space $\mathbf{Higgs}^{\mathsf{cyc}}(\mathbb{P}^1)$ can be realized as a closed subscheme of a certain Nakajima quiver variety of the affine type A_1 quiver associated to a non-dominant (!) stability condition. The previous result reduces to:

Theorem ([DPS])**.** *The stable ∞-pro-category $\mathsf{Coh}^{\mathsf{b}}_{\mathsf{pro}}(\mathbb{R}\mathbf{Higgs}^{\mathsf{cyc}}(\mathbb{P}^1))$ is a right categorical module over the categorified Hall algebra of Higgs sheaves on X. In particular, $G_0(\mathbf{Higgs}^{\mathsf{cyc}}(\mathbb{P}^1)))$ and $H_*^{\mathsf{BM}}(\mathbf{Higgs}^{\mathsf{cyc}}(\mathbb{P}^1))$ are right modules of the K-theoretical and cohomological Hall algebras of Higgs sheaves on X, respectively.*

This represents the first construction of a representation of a cohomological Hall algebra related to a Nakajima quiver variety which is not (!) associated to the dominant stability condition.

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Approaching symplectic/orthogonal orbit closure relations

Magdalena Boos
(joint work with Giovanni Cerulli Irelli)

The notion of a symmetric quiver was first introduced by Derksen and Weyman [7] in 2002. Symmetric quiver representations are collected in so-called symmetric representation varieties which are acted on by reductive groups via change of basis [7, 10, 11]. We motivate our interest in understanding orbit closure relations, i.e. symmetric degenerations of said actions. Our main result describes them explicitly in case the symmetric quiver is of finite representation type.

Symmetric Representation Theory

We fix a symmetric quiver $(\mathcal{Q},\sigma)$, that is, a finite quiver $\mathcal{Q}$ together with an arrow-reversing involution σ on $\mathcal{Q}_0 \cup \mathcal{Q}_1$ such that $\sigma(\mathcal{Q}_0) = \mathcal{Q}_0$ and $\sigma(\mathcal{Q}_1) = \mathcal{Q}_1$ [7]. Let $I \subset \mathbf{C}\mathcal{Q}$ be an admissible ideal with $\sigma(I) = I$ and denote by $\mathcal{A} := \mathbf{C}\mathcal{Q}/I$ the finite-dimensional associative quotient algebra. The involution σ then induces an isomorphism $\mathcal{A} \cong \mathcal{A}^{\mathrm{op}}$.

Let $V = \bigoplus_{i\in\mathcal{Q}_0} V_i$ be a $\mathcal{Q}_0$-graded $\mathbf{C}$-vector space of dimension vector $\underline{d} = (d_i)_i$. There is a natural change of basis action of the reductive group $\mathrm{GL}^\bullet(V) := \prod_{i\in\mathcal{Q}_0}\mathrm{GL}(V_i)$ on the representation variety $R(\mathcal{A},V) \subseteq \prod_{\alpha:i\to j}\mathrm{Hom}(V_i,V_j)$. Its orbits are in bijection with the isomorphism classes of representations in the representation category $\mathrm{rep}(\mathcal{A})(\underline{d})$.

Let ε be 1 or -1 and fix a non-degenerate bilinear ε-form

$$\langle\ ,\ \rangle : V \times V \to \mathbf{C}$$

which fulfills $\langle\ ,\ \rangle|_{V_i\oplus V_j} = 0$ unless $i = \sigma(j)$. For an endomorphism $f \in \mathrm{End}(V)$, denote by f^* its adjoint with respect to the form $\langle\ ,\ \rangle$.

We define $R^\varepsilon(\mathcal{A},V) := \{M \mid M = -M^*\} \subseteq R(\mathcal{A},V)$ to be the subvariety of so-called symmetric representations. The subgroup $G^\bullet(V,\varepsilon) := \{g \mid g = (g^*)^{-1}\} \subseteq \mathrm{GL}^\bullet(V)$ of graded isometries acts on $R^\varepsilon(\mathcal{A},V)$ by change of basis. The following theorem relates the orbits of both described base change actions and thus relates representations and symmetric representations [7, 8, 4].

Theorem. *Let $M,N \in R^\varepsilon(\mathcal{A},V)$. Then*

$$\mathrm{GL}^\bullet(V).M = \mathrm{GL}^\bullet(V).N \iff G^\bullet(V,\varepsilon).M = G^\bullet(V,\varepsilon).N$$

There is a self-duality ∇ on the representation category $\mathrm{rep}(\mathcal{A})$ and every symmetric representation is self-dual with respect to ∇ [7]. Thus, orbits are well understood and we focus on the description of orbit closure relations now.

Definition. *For $M,N \in R(\mathcal{A},V)$ set $M \leq_{\mathrm{deg}} N$ whenever $\mathrm{GL}^\bullet(V).N \subseteq \overline{\mathrm{GL}^\bullet(V).M}$. Similarly, for $M,N \in R^\varepsilon(\mathcal{A},V)$ set $M \leq^\varepsilon_{\mathrm{deg}} N$ whenever $G^\bullet(V,\varepsilon).N \subseteq \overline{G^\bullet(V,\varepsilon).M}$.*

Our aim is to figure out the interrelation between the partial orders $\leq_{\mathrm{deg}}$ and $\leq^\varepsilon_{\mathrm{deg}}$.

Motivation

Our motivation to study symmetric representations and their degenerations is to a certain degree self-contained since we believe that Symmetric Representation Theory will play a bigger role in understanding symplectic and orthogonal setups - in a similar way as Type A Representation Theory has impact on different fields of mathematics. However, there are two particular interests.

(1) Algebraic group actions: In Type A, for several algebraic group actions there are associated fibre bundle translations to setups in Representation Theory. In a similar manner, the restrictions to classical groups can then be translated to setups in Symmetric Representation Theory. Thus, in oder to understand orbit closure relations of said actions, we need to understand (ε)-degenerations. One example is the conjugation action of a Borel subgroup of a classical Lie group on the variety of 2-nilpotent matrices in its Lie algebra [5].

(2) Linear degenerations of flag varieties: In [6], Cerulli Irelli, Fang, Feigin, Fourier and Reineke construct a projective $\mathrm{GL}^\bullet(V)$-equivariant family $Y \to R(\mathbf{C}\mathcal{Q}, V)$, where $\mathcal{Q}$ is a linearly oriented Type A Dynkin quiver. The generic fibre of this map is the complete flag variety and every other fibre is a quiver Grassmannian. To transfer these results to classical Lie types, the first step is to understand orbit closure relations in the base, that is, ε-degenerations in $R^\varepsilon(\mathbf{C}\mathcal{Q}, V)$.

Main result

Main Theorem. *Let $(\mathcal{Q}, \sigma)$ be a quiver of finite ε-representation type. Then for $M, N \in R^\varepsilon(V, \mathcal{A})$:*

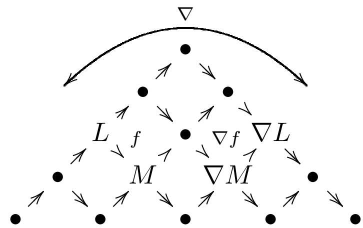
$$M \leq_{\mathrm{deg}} N \Leftrightarrow M \leq^\varepsilon_{\mathrm{deg}} N$$

The proof of our Main Theorem [4] is constructive. Given two ε-representations $M, N \in R^\varepsilon(V, \mathcal{A})$ such that $M \leq_{\mathrm{deg}} N$, it inductively provides ε-representations $M = M(0), M(1), \cdots, M(k) = N$ such that for every i there is a one parameter subgroup $\lambda_i(t) \in \mathrm{G}^\bullet(V, \varepsilon)$ which fullfills $\lim_{t\to 0} \lambda_i(t) \cdot M(i) = M(i+1)$.

This strategy is inspired by work of Bongartz [2]. In Type A, there is another partial order $\leq_{\mathrm{ext}}$ on $R(\mathcal{A}, V)$. It is defined as the transitive closure of the relation given by $M \leq N$ if there is an exact sequence $0 \to L \to M \to V \to 0$ such that $U \oplus V \cong N$. Then if $M \leq_{\mathrm{ext}} N$ holds for two representations, Bongartz shows $M \leq_{\mathrm{deg}} N$ [2] by construction of an explicit one parameter subgroup which goes to the smaller orbit in the limit. In case of a Dynkin quiver, in the same article he proves that both orders coincide.

For ε-degenerations, in a similar way we define a partial order $\leq^\varepsilon_{\mathrm{ext}}$ as the transitive closure of the relation given by $M \leq^\varepsilon N$ if there is an exact sequence $0 \to L \to M \to V \to 0$ where $L \to M$ is an isotropic embedding and $M \simeq L \oplus \nabla L \oplus L^\perp/L$. Then $M \leq^\varepsilon_{\mathrm{ext}} N$ implies $M \leq^\varepsilon_{\mathrm{deg}} N$ [4] and the proof provides the before mentioned one parameter subgroups.

The induction of the proof of our Main Theorem then makes use of deep Auslander-Reiten combinatorics. The reason why the Auslander-Reiten quiver plays such big role, is the fact that it inherits the symmetry of the quiver deeply. In more detail, the functor ∇ can be read off the Auslander-Reiten quiver as can be seen nicely in the following example of the linearly oriented Dynkin quiver of type A_5:



Next steps

Our next aim is to figure out symplectic and orthogonal linear degenerations of flag varieties. Furthermore, we want to understand ε-degenerations for representation-finite algebras with symmetries. As shown in [3], in general $\leq_{\mathrm{deg}}$ and $\leq^{\varepsilon}_{\mathrm{deg}}$ do not coincide. We conjecture that for representation-directed algebras they do.

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Attractors of torus actions on quiver moduli

Hans Franzen
(joint work with Magdalena Boos)

The talk is based on the paper [2]. We investigate torus actions on quiver moduli. We compute the weight spaces of the action on tangent spaces at fixed points and determine an explicit description of the attracting sets. This allows us in special cases to determine generic normal forms of stable representations.

Let Q be a quiver, d a dimension vector and $\theta : \mathbb{Z}^{Q_0} \to \mathbb{Z}$ a stability function. Fix complex vector spaces V_i of dimension d_i. Let $R(Q,d) = \bigoplus_{a \in Q_1} \operatorname{Hom}(V_{s(a)}, V_{t(a)})$, regarded as an affine space. It is acted upon by the algebraic group $G_d = \prod_{i \in Q_0} \operatorname{GL}(V_i)$ by conjugation. We look at the GIT quotient

$$M^{\theta-\mathrm{st}}(Q,d) = R(Q,d)^{\theta-\mathrm{st}}/G_d$$

and call it, following King [3], the θ-stable moduli space. If Q is acyclic and if d and θ satisfy a certain genericity assumption, then $M^{\theta-\mathrm{st}}(Q,d)$ is smooth and projective.

Let $T = \mathbb{C}^\times$. Choose weights $w_a \in \mathbb{Z}$ for $a \in Q_1$. The linear action of T on $R(Q,d)$ given by $t.M = (t^{w_a} M_a)$ leaves the stable locus invariant and it commutes with the G_d-action. It hence descends to an action of T on $M^{\theta-\mathrm{st}}(Q,d)$. The fixed points of this action were described by Weist [4]. He defines a covering quiver $Q(w)$, depending on the set $w = \{w_a\}_a$ of weights. His result asserts that the decomposition of the fixed point locus into connected components

$$M^{\theta-\mathrm{st}}(Q,d)^T = \bigsqcup_\beta F_\beta$$

is indexed by dimension vectors β of $Q(w)$ which cover d, up to translation. The fixed point component F_β is isomorphic to the moduli space $M^{\theta-\mathrm{st}}(Q(w),\beta)$.

Let $[M] \in M^{\theta-\mathrm{st}}(Q,d)^T$, and let $\dot{M}$ be a lift of M to $Q(w)$ according to Weist's result. The torus T acts on the tangent space $T_{[M]}(M^{\theta-\mathrm{st}}(Q,d)) \cong \operatorname{Ext}_{\mathbb{C}Q}(M,M)$. We show that for $n \in \mathbb{Z}$, the weight n weight space of the tangent space at $[M]$ is

$$(T_{[M]}M^{\theta-\mathrm{st}}(Q,d))_n \cong \operatorname{Ext}^1_{\mathbb{C}Q(w)}(\dot{M}, s_{-n}(\dot{M})).$$

Here, s_{-n} is an auto-equivalence which lifts the translation action on the quiver $Q(w)$. As a consequence, the dimension of the weight space can be computed as $\delta_{n,0} - \langle \beta, s_{-n}(\beta) \rangle_{Q(w)}$.

For a lift $\dot{M}$ of M to the covering quiver, we obtain decompositions $V_i = \bigoplus_{n \in \mathbb{Z}} V_{i,n}$. We define the ascending filtration $F_{i,*}$ of V_i by $F_{i,n} = \bigoplus_{m \leq n} V_{i,m}$. The collection $F_* = \{F_{i,*}\}_i$ is called the filtration associated with $\dot{M}$. We define

$$\begin{aligned} R_{F_*} &:= \{N \in R(Q,d) \mid N_a(F_{s(a),n}) \subseteq F_{t(a),n+w_a-1} \text{ (all } a)\} \\ \mathfrak{u}_{F_*} &:= \{x \in \mathfrak{g}_d \mid x_i(F_{i,n}) \subseteq F_{i,n-1} \text{ (all } i)\} \\ [\mathfrak{u}_{F_*}, M] &:= \operatorname{im}\Big(\mathfrak{u}_{F_*} \to R_{F_*},\ x \mapsto [x,M]\Big). \end{aligned}$$

The vector spaces R_{F_*} and $\mathfrak{u}_{F_*}$ are graded by $\mathbb{Z}_{>0}$ and $[\mathfrak{u}_{F_*}, M]$ is a graded subspace of R_{F_*}. We show that if we choose a $\mathbb{Z}_{>0}$-graded complement R' of $[\mathfrak{u}_{F_*}, M]$ inside R_{F_*}, then the map

$$R' \to R(Q,d)^{\theta-\mathrm{st}} \to M^{\theta-\mathrm{st}}(Q,d)$$

which sends $N \in R'$ to the isomorphism class $[M+N]$ is well-defined and induces an isomorphism onto the attractor of the fixed point $[M]$ in the Białynicki-Birula decomposition [1].

Assume now that there exists a dimension vector β of $Q(w)$ which covers d, such that β is a real root of $Q(w)$, the stable moduli space $M^{\theta-\mathrm{st}}(Q(w),\beta)$ is non-empty, and $\langle \beta, s_{-m}(\beta)\rangle_{Q(w)} = 0$ for all $m < 0$. Then, for R' as above, the subset $\{M\} + R'$ of $R(Q,d)$ is a generic normal form for θ-stable representations of Q. Such a covering dimension vector exists for example for any generalized Kronecker quiver and dimension vectors of the form $d = (2, 2r+1)$.

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What do abelian categories form?

Dmitry Kaledin

The talk is based on my recent paper [1]. Its title is taken from D. Tamarkin's paper [2] where an analogous question was asked for DG-categories over a fixed field k. The answer Tamarkin gave is essentially a 2-category of small DG-categories over k, with all the natural higher structures. We want to do the same for abelian categories, both in the k-linear and in the absolute context, where no base field is fixed. Namely, for any finitely presentable abelian categories $\mathcal{A}$, $\mathcal{B}$, with the full subcategories $\mathcal{A}_c \subset \mathcal{A}$, $\mathcal{B}_c \subset \mathcal{B}$ of compact objects, we define an abelian category $\mathrm{Mor}(\mathcal{A},\mathcal{B})$ of functors $E : \mathcal{A} \to \mathcal{B}$ that are continuous – that is, commute with filtered colimits – and satisfy the sheaf condition for an appropriate Grothendieck topology on $\mathcal{A}_c^o$ that we call "the single-ei topology" (coverings are epimorphisms). Note that we do not require our functors to be additive! Nevertheless, both conditions are obviously closed under compositions – indeed, the sheaf condition amounts to requiring that for any injective map $f : A \to B$ in $\mathcal{A}$, $E(f) : E(A) \to E(B)$ is injective, and so is the natural map

$$E(B) \oplus_{E(A)} E(B) \to E(B \oplus_A B),$$

where we let $B \oplus_A B$ be the cokernel of the map $a \oplus (-a) : A \to B \oplus B$, and similarly for $E(B) \oplus_{E(A)} E(B)$. Therefore we obtain a well-defined 2-category of finitely presentable abelian categories, with $\mathrm{Mor}(-,-)$ as the categories of 1-morphisms. We can then define the "absolute Hochschild cohomology" $HH^{\bullet}_{abs}(\mathcal{C})$ of a finitely presentable abelian category $\mathcal{C}$ as $HH^{\bullet}_{abs}(\mathcal{C}) = \mathrm{Ext}^{\bullet}_{\mathrm{Mor}(\mathcal{C},\mathcal{C})}(\mathrm{Id},\mathrm{Id})$, and it is expected that these groups control deformations of $\mathcal{C}$ in the appropriate sense (including in particular square-zero extensions such as $\mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$). If $\mathcal{C}$ is the category of k-vector spaces for a perfect field k, then $HH^{\bullet}(\mathcal{C})$ recovers the Mac Lane cohomology of k.

We also explore the derived category $\mathcal{D}(\mathrm{Mor}(\mathcal{A},\mathcal{B}))$ of our category of morphisms. Morally, the full subcategory $\mathcal{D}^{+}_{add}(\mathrm{Mor}(\mathcal{A},\mathcal{B})) \subset \mathcal{D}^{+}(\mathrm{Mor}(\mathcal{A},\mathcal{B}))$ of functors additive on the derived level is an explicit purely homological model for the category of stable functors $\mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ (in any homotopically-enhanced sense). While we do not prove this – that would require fixing a particular model for homotopical enhancement, and none that exist at the moment are too appealing – we prove the following two statements that at the end of the day, amount to the same thing. Let $C^{\geq 0}(\mathcal{A})$, $C^{\geq 0}(\mathcal{B})$ be the categories of chain complexes in $\mathcal{A}$, $\mathcal{B}$ concentrated in non-negative cohomological degrees, and say that a functor $C^{\geq 0}(\mathcal{A}) \to C^{\geq 0}(\mathcal{B})$ is *homotopical* if it sends quesiisomorphisms to quasiisomorphisms. Then (i) a continuous functor $E : \mathcal{A} \to \mathcal{B}$ that is a morphism in our sense extends to a continuous homotopical functor $\mathsf{D}(E) : C^{\geq 0}(\mathcal{A}) \to C^{\geq 0}(\mathcal{B})$, uniquely up to a pointwise quasiisomorphism, and every continuous homotopical functor arises in this way, and (ii) $\mathsf{D}(E)$ descends to an additive functor on the level of derived categories if and only if it commutes with homological shifts, thus extends to all complexes bounded from below.

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Noncommutative Surfaces and Stacky Surfaces

Colin Ingalls

(joint work with Eleonore Faber, Shinnosuke Okawa, Matthew Satriano)

We first recall some well known examples that relate categories of modules with categories of coherent sheaves on stacks. The first two follow immediately from definitions.

Example 1. *Let k be a field and let G be an algebraic group over k. The classifying stack $BG = [\mathrm{Spec} k/G]$ is the stacky quotient of* Spec$k$ *by G. Let kG be the group algebra. Then there is an equivalence of monoidal categories of finitely generated modules and coherent sheaves* $\mathrm{mod}\, kG \simeq \mathrm{coh}\, BG$ *since both are equivalent to the category of representations of G.*

Example 2. *Let X be a scheme with an action by a finite group G. We define the skew group algebra $\mathcal{O}_X \rtimes G = \bigoplus_{g\in G} \mathcal{O}_X g$ with multiplication $(xg)(yg) =$*

$(xg(y))(gh)$ for $x, y \in \mathcal{O}_X$ and $g, h \in G$. Let $\mathcal{X} = [X/G]$ be the quotient stack. Then the monoidal categories $\operatorname{mod} \mathcal{O}_X \rtimes G \simeq \operatorname{coh}[X/G]$ *are both equivalent to the category of G-equivariant coherent sheaves on X*

This example (partially described in [2]) is not immediate, but is well known.

Example 3. *Let $\mathcal{A}$ be an Azumaya algebra over a scheme X. Then*

$$\mathcal{A} \in \mathrm{H}^1(X, \mathrm{PGL}_n) \to \mathrm{H}^2(X, \mu_n)$$

gives rise to a μ_n-gerbe $\mathcal{X}$ over X. Let $\operatorname{coh}^{(i)} \mathcal{X}$ be the category coherent sheaves on $\mathcal{X}$ where μ_n acts by the character $\chi(z) = z^i$. There is an equivalence of categories

$$\operatorname{mod} \mathcal{A} \simeq \operatorname{coh}^{(1)} \mathcal{X}$$

and an equivalence of monoidal categories

$$\operatorname{mod} \prod_{i=0}^{n-1} \mathcal{A}^{\otimes i} \simeq \operatorname{coh} \mathcal{X}.$$

Alain Connes defined a convolution algebra of a groupoid of topological spaces by using integration on fibres. In the algebraic setting, we can consider a groupoid of schemes where the source and target maps $s, t : R \rightrightarrows U$ of the groupoid are finite and flat. Let $\mathcal{X} = [U/R]$ be the quotient stack. In this case [1], we have the convolution algebra $\mathcal{O}_R^\vee = \mathcal{H}\mathrm{om}(s_* \mathcal{O}_R, \mathcal{O}_U)$ and an equivalence of monoidal categories $\operatorname{mod} \mathcal{O}_R^\vee \simeq \operatorname{coh} \mathcal{X}$. We are interested in understanding which noncommutative algebras are convolution algebras of finite flat groupoids.

Let X be an integral scheme over a base field k. An *order* $\mathcal{A}$ over $\mathcal{O}_X$ is a sheaf of $\mathcal{O}_X$-central algebras that is coherent and torsion free as a sheaf such that $\mathcal{A} \otimes_X k(X)$ is simple algebra. We say that $\mathcal{A}$ is *tame* if it is reflexive and hereditary in codimension one. If the inclusion of sheaves $\mathcal{O}_X \to \mathcal{A}$ is split, we say that $\mathcal{A}$ is *split.* The following is our main result. This was previously known locally in [3, Section 5], and in dimension one [1]. Below we sketch the proof.

Theorem 1. *Let k be an algebraicaly closed field of characteristic ≥ 7. Let X be a projective surface over k. Let $\mathcal{A}$ be a tame split order with centre $\mathcal{O}_X$ of global dimension two. Then there is a tame algebraic stack $\mathcal{X}$ with generic stablizer μ_n and coarse moduli space X such that* $\operatorname{mod} \mathcal{A} \simeq \operatorname{coh}^{(1)} \mathcal{X}$.

Proof. We first follow [3] and we define the graded algebra $\mathcal{A}_r := \oplus_{i \in \mathbb{Z}} \omega_A^{\otimes i}$. Let $\mathcal{X}_r = [\mathcal{S}\mathrm{pec}_X Z(\mathcal{A}_r)/\mathbb{G}_m]$ whic is the root stack on the discriminant. Then $\mathcal{A}_r$ is a Gorenstein order on $\mathcal{X}$ and is Azumaya in codimension one. Furthermore $\operatorname{mod} \mathcal{A}_r \simeq \operatorname{mod} \mathcal{A}$. Next we take the canonical stack $\pi : \mathcal{X}_c \to \mathcal{X}_r$ and pullback $\mathcal{A}_c = \pi^* \mathcal{A}_r$. Now $\mathcal{A}_c$ is an Azumaya algebra and again we have $\operatorname{mod} \mathcal{A}_c \simeq \operatorname{mod} \mathcal{A}_r$. Lastly, we let $\mathcal{X}$ be a μ_n-gerbe over $\mathcal{X}_r$ to obtain $\operatorname{mod} \mathcal{A}_c \simeq \operatorname{coh}^{(i)} \mathcal{X}$. □

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Reporter: Lukas Bonfert

Participants

Dr. Jenny August
Department of Mathematics
Aarhus University
Ny Mukegade 118
8000 Aarhus
DENMARK

Prof. Dr. Jason P. Bell
Department of Pure Mathematics
University of Waterloo
N2L 3G1 Waterloo
CANADA

Prof. Dr. Gwyn Bellamy
School of Mathematics and Statistics
University of Glasgow
University Place
Glasgow G12 8QQ
UNITED KINGDOM

Prof. Dr. Pieter Belmans
Université du Luxembourg
Campus Belval, Maison du Nombre
6, Avenue de la Fonte
4364 Esch-sur-Alzette
LUXEMBOURG

Dr. Raphael Bennett-Tennenhaus
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld
GERMANY

Prof. Dr. Yuri Berest
Department of Mathematics
Cornell University
Malott Hall
Ithaca, NY 14853-4201
UNITED STATES

Prof. Dr. Roman Bezrukavnikov
Department of Mathematics
Massachusetts Institute of
Technology
77 Massachusetts Avenue
Cambridge, MA 02139-4307
UNITED STATES

Prof. Dr. Rafael Bocklandt
Korteweg-de Vries Institute for
Mathematics
Science Park 904
1098 XH Amsterdam
NETHERLANDS

Lukas Bonfert
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Dr. Magdalena Boos
Fakultät für Mathematik
Ruhr-Universität Bochum
Universitätsstr. 150
44801 Bochum
GERMANY

Prof. Dr. Ken A. Brown
Department of Mathematics and
Statistics
University of Glasgow
University Place
Glasgow G12 8QQ
UNITED KINGDOM

Dr. Daniel Chan
School of Mathematics and Statistics
UNSW, Sydney
Sydney NSW 2052
AUSTRALIA

Prof. Dr. William Crawley-Boevey
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld
GERMANY

Dr. Ben D. J. Davison
School of Mathematics
University of Edinburgh
James Clerk Maxwell Building
King's Buildings
Peter Guthrie Tait Road
Edinburgh EH9 3FD
UNITED KINGDOM

Prof. Dr. Will Donovan
Department of Mathematical Sciences
Tsinghua University
Beijing 100 084
CHINA

Dr. Jens Eberhardt
FB C: Mathematik u.
Naturwissenschaften
Bergische Universität Wuppertal
Gaußstr. 20
42119 Wuppertal
GERMANY

Dr. Eleonore Faber
School of Mathematics
University of Leeds
Leeds LS2 9JT
UNITED KINGDOM

Dr. Hans Franzen
Fakultät für Mathematik
Ruhr-Universität Bochum
Universitätsstraße 150
44780 Bochum
GERMANY

Prof. Dr. Victor Ginzburg
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago, IL 60637-1514
UNITED STATES

Prof. Dr. Kenneth R. Goodearl
Department of Mathematics
University of California at Santa Barbara
South Hall
Santa Barbara, CA 93106
UNITED STATES

Prof. Dr. Iain Gordon
School of Mathematics
University of Edinburgh
James Clerk Maxwell Building
King's Buildings
Mayfield Road
Edinburgh EH9 3FD
UNITED KINGDOM

Prof. Dr. Birge Huisgen-Zimmermann
Department of Mathematics
University of California at Santa Barbara
South Hall
Santa Barbara, CA 93106
UNITED STATES

Prof. Dr. Colin Ingalls
School of Mathematics and Statistics
Carleton University
4229 Herzberg Laboratories
Ottawa ON K1S 5B6
CANADA

Prof. Dr. Osamu Iyama
Department of Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914
JAPAN

Prof. Dr. Dmitry Kaledin
Department of Algebraic Geometry
Steklov Mathematical Institute
Gubkina 8
Moscow 119 991
RUSSIAN FEDERATION

Dr. Daniel Kaplan
Department WNI
Hasselt University
3590 Diepenbeek
BELGIUM

Prof. Dr. Mikhail Kapranov
Kavli Institute for the Physics and Mathematics of the Universe (Kavli IPMU)
5-1-5 Kashiwanoha
Kashiwa 277-8583
JAPAN

Prof. Dr. Bernhard Keller
U.F.R. de Mathématiques
Université Paris Cité
Bâtiment Sophie Germain, Case 7012
8, place Aurélie Nemours
75013 Paris Cedex 13
FRANCE

Prof. Dr. Ellen E. Kirkman
Department of Mathematics
Wake Forest University
Box 7388 Reynolda Station
Winston-Salem, NC 27109
UNITED STATES

Dr. Yanki Lekili
Imperial College London
Department of Mathematics
Huxley Building
180 Queen's Gate
London SW7 2AZ
UNITED KINGDOM

Prof. Dr. Wendy Lowen
Department of Mathematics
University of Antwerpen
Middelheimlaan 1
2020 Antwerpen
BELGIUM

Prof. Dr. Valery A. Lunts
Department of Mathematics
Indiana University at Bloomington
3-d street
Bloomington, IN 47405
UNITED STATES

Karen Lizeth Martinez Acosta
Fakultät für Mathematik
Ruhr-Universität Bochum
Universitätsstr. 150
44801 Bochum
GERMANY

Jonas Nehme
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Anya Nordskova
Department WNI
Hasselt University
3500 Hasselt
BELGIUM

Dr. Shinnosuke Okawa
Department of Mathematics
Graduate School of Science
Osaka University
Machikaneyama 1-1, Toyonaka
Osaka 560-0043
JAPAN

Prof. Dr. Dmitri Orlov
Steklov Mathematical Institute
Russian Academy of Sciences
Gubkina 8
119 991 Moscow GSP-1
RUSSIAN FEDERATION

Dr. Tomasz Przezdziecki
University of Edinburgh
Peter Guthrie Tate Rd
EH9 3FD Edinburgh EH9 3FD
UNITED KINGDOM

Prof. Dr. Markus Reineke
Fakultät für Mathematik
Ruhr-Universität Bochum
Universitätsstraße 150
44801 Bochum
GERMANY

Dr. Alice Rizzardo
Department of Mathematical Sciences
University of Liverpool
Peach Street
Liverpool L69 7ZL
UNITED KINGDOM

Dr. Francesco Sala
Università di Pisa
Dipartimento di Matematica
Largo Bruno Pontecorvo, 5
56127 Pisa
ITALY

Dr. Julia Sauter
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld
GERMANY

Prof. Dr. Travis Schedler
Department of Mathematics
Imperial College London
Huxley Building
180 Queen's Gate
London SW7 2AZ
UNITED KINGDOM

Prof. Dr. Sarah Scherotzke
Université du Luxembourg
Unite de Recherche en Mathématiques
1359 Belvaux
LUXEMBOURG

Prof. Dr. Olivier Schiffmann
Département de Mathématiques
Université de Paris-Saclay
Bâtiment 307
91405 Orsay Cedex
FRANCE

David Sembowski
Fakultät für Mathematik
Ruhr-Universität Bochum
Universitätsstr. 150
44801 Bochum
GERMANY

Dr. Susan J. Sierra
School of Mathematics
University of Edinburgh
James Clerk Maxwell Bldg.
Edinburgh EH9 3FD
UNITED KINGDOM

Prof. Dr. S. Paul Smith
Department of Mathematics
University of Washington
Padelford Hall
Box 354350
Seattle, WA 98195-4350
UNITED STATES

Prof. Dr. Agata Smoktunowicz
School of Mathematics
University of Edinburgh
James Clerk Maxwell Bldg.
King's Buildings, Mayfield Road
Edinburgh EH9 3JZ
UNITED KINGDOM

Prof. Dr. Yan Soibelman
Department of Mathematics
Kansas State University
Manhattan, KS 66506-2602
UNITED STATES

Dr. Spela Spenko
Department of Mathematics
Université Libre de Bruxelles
1050 Bruxelles
BELGIUM

Prof. Dr. J. Toby Stafford
Department of Mathematics
The University of Manchester
Oxford Road
Manchester M13 9PL
UNITED KINGDOM

Prof. Dr. Catharina Stroppel
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Michel Van den Bergh
Universiteit Hasselt
Martelarenlaan 42
Hasselt 3500
BELGIUM

Prof. Dr. Michael Wemyss
School of Mathematics and Statistics
University of Glasgow
University Place
Glasgow G12 8QQ
UNITED KINGDOM

Prof. Dr. Quan-Shui Wu
Institute of Mathematics
Fudan University
Shanghai Shi 200 433
CHINA

Prof. Dr. Amnon Yekutieli
cellphone +972525254633
Department of Mathematics
Ben-Gurion University of the Negev
Beer-Sheva 84 105
ISRAEL

Dr. Matthew Young
Dept. of Mathematics and Statistics
Utah State University
Logan, UT 84322-3900
UNITED STATES

Prof. Dr. James Zhang
Department of Mathematics
University of Washington
Padelford Hall
P.O. Box 354350
Seattle WA 98195-4350
UNITED STATES