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Domain-scaled regular variation: Mathematical foundations for a new tail process

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Abstract

Threshold exceedances of stochastic processes in space and time often appear to be more localized the more extreme they are. While classical regularly varying stochastic processes cannot model this effect, we introduce an adapted version of regular variation, where a suitable domain-scaling can be incorporated to accommodate this behaviour. Our theory is inspired by the triangular array convergence of domain-scaled maxima of Gaussian processes to a Brown–Resnick process and turns out to be natural in this context. We study key properties of the resulting tail approximating process and demonstrate its ability to approximate conditional exceedance probabilities of Gaussian processes. Mathematical convenience arises from the recently rediscovered concept of vague convergence based on boundedness.

Contents

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1 Introduction

The concept of regular variation, introduced by Karamata (1930) and widely popularized through Feller (1971), has a long history in probability and statistics, including textbook treatments spanning, e.g., Seneta (1976), Resnick (2008), Bingham et al. (1989), de Haan and Ferreira (2006), or, more recently, Kulik and Soulier (2020) and Mikosch and Wintenberger (2024) for time series. The theory of regularly varying spatial stochastic processes has been scattered across the literature for some time (de Haan and Lin, 2001; Hult and Lindskog, 2005; Davis and Mikosch, 2008; Ferreira and de Haan, 2014) culminating in the unified presentation of Dombry and Ribatet (2015). Part of the mathematical appeal stems from the equivalence of multiple characterizations of regular variation, which in turn can be exploited for statistical inference and tail extrapolation. Generally, if X is a stochastic process on a compact domain $S \subset \mathbb{R}^d$, there is equivalence of (i) regular variation of X, (ii) weak convergence of the collection of renormalized independent copies of X to a Poisson point process, (iii) weak convergence of renormalized pointwise maxima of independent copies of X to a max-stable process, and (iv) weak convergence of renormalized threshold exceedances of X to a generalized Pareto process.

On the other hand, the asymptotic stability properties of a regularly varying process are rather rigid and are appropriate only for the modelling of the tail behaviour of so-called asymptotically dependent stochastic processes, i.e., loosely speaking, those X, where it is not unlikely that a high threshold exceedance of X at some location s_0 necessitates threshold exceedances of X of a similar order of magnitude across the entire domain of X. Instead, empirical evidence suggests that spatial extremes often exhibit weakening of dependence that is not compatible with pure asymptotic dependence, that is, they become more localized as the conditioning threshold increases (Wadsworth and Tawn, 2022; Huser and Wadsworth, 2022; Huser et al., 2025). The present manuscript introduces a refined notion of regular variation that overcomes this limitation via domain-scaling.

This idea has been inspired, amongst others, by the works of Kabluchko et al. (2009) and Dombry and Ribatet (2015), as the former establishes the convergence of maxima of Gaussian processes to Brown–Resnick processes, while the latter provides the road map for the (yet) unexplored counterparts that complement the maxima convergence. At a first glance, the maxima convergence from Kabluchko et al. (2009) may seem contradictory, since Gaussian processes exhibit asymptotic independence, while the max-stable Brown–Resnick processes are asymptotically dependent. However, the key feature of this convergence is the consideration of triangular arrays instead of sequences through the introduction of a spatial scaling in the underlying Gaussian processes, cf. also Brown and Resnick (1977), Hüsler and Reiss (1989) or Hsing et al. (1996). We shall see below that this modification can conveniently be incorporated into the theory of regular variation for spatial stochastic processes to make it more flexible and better reflect empirical evidence.

Definitions of regular variation of random vectors or stochastic processes are based on concepts of vague convergence as treated in classical textbooks such as Daley and Vere-Jones (1988) and Kallenberg (2017). This was, for instance, advocated early by Resnick (2008) for random vectors to facilitate establishing connections to related point processes. However, the characterization of regular variation from Resnick (2008) via vague convergence is based on relative compactness and, thus, comes at the price of technical completion arguments that permeate the extremes literature to date. In spatial settings, these technicalities appear for instance in de Haan and Ferreira (2006) and Dombry and Ribatet (2015), and also Hult and Lindskog (2005), which includes to the best of our knowledge the first definition of regular variation for spatial stochastic processes. Subsequently, the same authors discovered that this completion is in fact unnecessary (Hult and Lindskog, 2006), and that their earlier definition via $\hat{w}$ convergence for continuous stochastic processes on compact domains is equivalent to a formulation based on sets bounded away from the origin, cf. also Segers et al. (2017) and Dombry et al. (2018) and references therein. More recently, new light has been shed on the matter by Basrak and Planinić (2019) who show that all such definitions can be framed as vague convergence based on boundedness (Hu, 1949, 1966), which brings mathematical convenience to the theory of regular variation. The textbook by Kulik and Soulier (2020) makes this new viewpoint available for a range of broad settings, although predominantly with time series analysis in view. Here, we make the theory for continuous stochastic processes on convex domains in $\mathbb{R}^d$ explicit in this framework, cf. also Soulier (2022) and Bladt et al. (2022) for extensions to càdlàg processes.

Thereby, this manuscript has two aims: In Section 2, we lay the foundations for the needed triangular array convergences for continuous stochastic processes in light of the new convergence framework. Thereafter, we introduce the new concept of domain-scaled regular variation in Section 3 and, in Section 4, study its associated tail approximating process. The Gaussian case arising from the maxima convergence to a Brown–Resnick process is detailed in Section 5. Statistical inference based on these new tail approximating processes is left for future research.

Notation

The following standard notation and conventions will be used throughout this manuscript. We write $\mathbb{1}\{\text{condition}\}$ for the indicator function, which takes the value 1 if the condition is fulfilled and 0 else. The cumulative distribution function F of a standard Pareto distribution is $F(t) = (1-t^{-1})\mathbb{1}\{t \geq 1\}$; for a standard Fréchet distribution it is $F(t) = \exp(-t^{-1})\mathbb{1}\{t > 0\}$, and for a standard Gumbel distribution we have $F(t) = \exp(-\exp(-t))$ for $t \in \mathbb{R}$. If F is standard Fréchet and $\sigma > 0$, we call $F(\cdot/\sigma)$ a unit Fréchet distribution. The standard GEV distribution with extreme value index $\xi \in \mathbb{R} \setminus \{0\}$ is given by $F(t) = \exp(-[(1+\xi t)\mathbb{1}\{1+\xi t > 0\}]^{-1/\xi})$; for $\xi = 0$ it coincides with the standard Gumbel distribution. The standard exponential distribution is $F(t) = (1-\exp(-t))\mathbb{1}\{t \geq 0\}$. We denote the cumulative distribution function of a standard normal distribution by Φ; its density is $\varphi(t) = (2\pi)^{-1/2}\exp(-t^2/2)$ for $t \in \mathbb{R}$. For $\alpha > 0$, let ν_α be the Borel measure on $(0, \infty)$ given by

$$\nu_\alpha((t,\infty)) = t^{-\alpha}, \quad t > 0.$$

The law of a random object X will be denoted by $\mathcal{L}(X)$. The density of a random vector X will be denoted by f_X. If F is a cumulative distribution function, we write $\overline{F} = 1 - F$ for its survival function. We will use throughout the text that the law of a standard Pareto variable P is given by ν_1 restricted to the interval $[1, \infty)$. Moreover, we recall its threshold stability property that $\mathcal{L}(P \mid P > t) = \mathcal{L}(tP)$ for $t \geq 1$, as follows from the (-1)-homogeneity of ν_1, i.e. for a Borel subset $A \subset \mathbb{R}$

$$\mathbb{P}\{P \in A \mid P > t\} = \frac{\nu_1(A \cap [t,\infty) \cap [1,\infty))}{\nu_1([t,\infty) \cap [1,\infty))} = \frac{\nu_1(A \cap [t,\infty))}{\nu_1([t,\infty))} = \nu_1(t^{-1}A \cap [1,\infty)) = \mathbb{P}\{tP \in A\}.$$

2 Triangular array convergence

The theory of regular variation for continuous stochastic processes (de Haan and Lin, 2001; Hult and Lindskog, 2005; de Haan and Ferreira, 2006; Davis and Mikosch, 2008; Dombry and Ribatet, 2015) is very technical. In this section, we simplify its exposition making use of the rediscovered vague convergence (Basrak and Planinić, 2019), which seems naturally suited to describe regular variation in this context. We need to do so while considering triangular arrays instead of sequences in order to introduce the domain-scaled regular variation presented in Section 3. As we simplify and generalize previously known regular variation arguments, we try to keep this section as self-contained as possible by including many proofs explicitly. We do not claim complete originality in such arguments. But to the best of our knowledge the reinterpretation of the regular variation of continuous stochastic processes as in Dombry and Ribatet (2015) in view of the vague convergence from Basrak and Planinić (2019) cannot be found explicitly in the literature otherwise, and the arguments are typically derived for sequences instead of triangular arrays. The formulation presented in this section has several advantages, for example, it also provides a shorter proof for the continuity of Brown–Resnick processes than in Kabluchko et al. (2009).

2.1 Technical background

We proceed similarly to Chapter 7 and Appendix B in Kulik and Soulier (2020), which are largely based on Basrak and Planinić (2019), to simplify and unify the convergence concepts relevant for typical limiting arguments in the theory of regular variation. This setup is based on the concept of boundedness (Hu, 1949, 1966).

First, we introduce the relevant spaces and their properties. Let S be a compact subset of $\mathbb{R}^d$ for some dimension $d \geq 1$. We consider the following spaces of functions; some of them being complete separable metric spaces (c.s.m.s.):

- $C(S) = C(S, \mathbb{R})$, the space of real-valued continuous functions on S, equipped with with supremum norm $\|f\|_\infty = \sup_{s \in S}|f(s)|$, a Banach space, hence c.s.m.s.,
- $C_\geq(S) = C(S, [0,\infty))$, a closed subset of $C(S)$, hence c.s.m.s.,
- $C_+(S) = C(S, (0,\infty))$, an open subset of $C_\geq(S) \subset C(S)$,
- $C_0(S) = C_\geq(S) \setminus \{0\}$.

A subset $B \subset C_0(S)$ is called bounded away from 0 if $\inf\{\|f\|_\infty : f \in B\} > 0$. As explained in Kulik and Soulier (2020, Example B.1.4), the metric

$$d_0(f,g) = (\|f-g\|_\infty \wedge 1) \vee \left| \frac{1}{\|f\|_\infty} - \frac{1}{\|g\|_\infty} \right|$$

induces the trace topology of $C(S)$ on $C_0(S)$, whilst $(C_0(S), d_0)$ is complete and $B \subset C_0(S)$ is bounded away from 0 if and only if it is bounded for d_0.

The sequence of subsets $U_k = \{f \in C_0(S) : \|f\|_\infty > k^{-1}\}$, $k \in \mathbb{N}$, is properly localizing $C_0(S)$, i.e., each U_k is open and bounded, while U_{k+1} contains the closure of U_k, and the union of all U_k, $k \in \mathbb{N}$, recovers $C_0(S)$ (Kulik and Soulier, 2020, Definition B.1.1). Hence, $C_0(S)$ is a properly localized Polish space. Let

$$\mathbb{S} = \{f \in C_{\geq}(S) : \|f\|_\infty = 1\}$$

be the unit sphere in $C_{\geq}(S) \subset C(S)$, which is again a closed subset of $C_{\geq}(S)$ and hence a c.s.m.s. The multiplication map

$$m : (0, \infty) \times \mathbb{S} \to C_0(S), \qquad m(t, f) = tf \tag{1}$$

is a homeomorphism between $(0, \infty) \times \mathbb{S}$ and $C_0(S)$, with inverse

$$m^{-1} : C_0(S) \to (0, \infty) \times \mathbb{S}, \qquad m^{-1}(f) = (\|f\|_\infty, f/\|f\|_\infty), \tag{2}$$

usually known as polar decomposition. By $M(C_0(S))$ we denote the space of all d_0-boundedly finite Borel measures μ on $C_0(S)$, i.e., measures satisfying $\mu(B) < 0$ for d_0-bounded Borel sets $B \subset C_0(S)$, endowed with the $v^\#$-topology, i.e.,

$$\mu_n \xrightarrow{v^\#} \mu \quad \text{if and only if} \quad \mu_n(B) \to \mu(B) \text{ for all } d_0\text{-bounded Borel sets } B \text{ with } \mu(\partial B) = 0,$$

which turns $M(C_0(S))$ into a Polish space (Kulik and Soulier, 2020, Theorem B.1.20).

Now we are in a position to introduce all limiting objects for the triangular array convergences in Section 2.2. To begin with, since $M(C_0(S))$ is a Polish space, we may consider random measures and (Poisson) point processes on $C_0(S)$ in a natural way (Kulik and Soulier, 2020, Theorem 7.1.11). The following example of a Poisson point process with (-1)-homogeneous mean measure will accompany us throughout this text as the basis for an associated max-stable process, which we define subsequently. Here, a measure $\mu \in M(C_0(S))$ is called (-1)-homogeneous if $\mu(cB) = c^{-1}\mu(B)$ for all $c > 0$ and all d_0-bounded Borel sets $B \subset C_0(S)$.

Lemma 2.1. *(a) A non-zero Borel measure Λ on $C_0(S)$ is d_0-boundedly finite and (-1)-homogeneous if and only if it can be represented as*

$$\Lambda = (\nu_1 \otimes \theta\sigma) \circ m^{-1}, \tag{3}$$

for a probability measure σ on $\mathbb{S}$ and a constant $\theta > 0$, i.e.

$$\Lambda(B) = (\nu_1 \otimes \theta\sigma)(m^{-1}(B)) = \theta \int_0^\infty \sigma(\{f \in \mathbb{S} : tf \in B\})\, \nu_1(\mathrm{d}t) \tag{4}$$

for Borel sets $B \subset C_0(S)$. The pair (θ, σ) is uniquely identified from Λ via

$$\theta = \Lambda(\{f \in C_0(S) : \|f\|_\infty > 1\}), \quad \sigma(B) = \frac{1}{\theta}\Lambda\big(\{f \in C_0(S) : \|f\|_\infty > 1\,,\ f/\|f\|_\infty \in B\}\big) \tag{5}$$

for Borel sets $B \subset \mathbb{S}$.

(b) Suppose Λ is of the form (3). The associated Poisson point process $\mathrm{PPP}(\Lambda)$ on $C_0(S)$ with mean measure Λ can be represented as the sum of random point masses

$$\sum_{i=1}^\infty \epsilon_{\theta\Gamma_i^{-1}V_i}, \tag{6}$$

where $(\Gamma_i)_{i\in\mathbb{N}}$ are the arrival times of a renewal process on $(0, \infty)$ with standard exponential interarrival times, and independently, $(V_i)_{i\in\mathbb{N}}$ are independent copies of V distributed according to the measure σ on $\mathbb{S}$, i.e. $\mathbb{P}\{V \in B\} = \sigma(B)$ for all Borel sets $B \subset \mathbb{S}$.

Proof. (a) Suppose a measure Λ is of the form (3). Then it is necessarily d_0-boundedly finite and (-1)-homogeneous. In addition, since m and m^{-1} are inverse to each other, $\Lambda \circ m = \nu_1 \otimes \theta\sigma$ (where we mean by m the preimage function of the inverse of m, which coincides here with the image of m). This gives $\Lambda\big(\{f \in C_0(S) : \|f\|_\infty > t\,,\ f/\|f\|_\infty \in B\}\big) = \theta t^{-1}\sigma(B)$ for Borel sets $B \subset \mathbb{S}$ and $t > 0$, hence (5).

Conversely, suppose Λ is d_0-boundedly finite and (-1)-homogeneous. Define θ and σ as in (5), which is possible with $\theta \in (0, \infty)$, since Λ is d_0-boundedly finite (so that $\theta < \infty$) and Λ is non-zero and (-1)-homogeneous (so that $\theta > 0$). Then it suffices to note that $(\nu_1 \otimes \theta\sigma) \circ m^{-1}$ and Λ coincide on a Dynkin system that is closed under intersections and generates the Borel σ-algebra on $C_0(S)$. Such a system is given by sets of the form $m((t, \infty) \times B)$ for $t > 0$ and Borel sets $B \subset \mathbb{S}$. Since $m((t, \infty) \times B) = tm((1, \infty) \times B)$, we have with the (-1)-homogeneity of Λ that

$$\Lambda\big(m((t, \infty) \times B)\big) = t^{-1}\Lambda(m((1, \infty) \times B)) = t^{-1}\theta\sigma(B) = \nu_1((t, \infty))\theta\sigma(B),$$

as desired.

(b) The result is a direct application of the solution to Problem 7.2 in Kulik and Soulier (2020) from the transformation of points for Poisson processes, cf. Kulik and Soulier (2020, Proposition 7.1.12). □

Lemma 2.2. *Let $\Lambda = (\nu_1 \otimes \theta\sigma) \circ m^{-1}$ and $V \sim \sigma$ be given as in Lemma 2.1. Let $r : C_\geq(S) \to [0, \infty)$ be continuous and 1-homogeneous, i.e. $r(cf) = cr(f)$ for all $f \in C_\geq(S)$ and $c > 0$. Then the set $\{f \in C_0(S) : r(f) > r_0\}$ is a continuity set of Λ for any $r_0 > 0$ and*

$$\Lambda(\{f \in C_0(S) : r(f) > r_0\}) = \frac{\theta}{r_0}\,\mathbb{E}[r(V)]$$

is finite.

Proof. It suffices to consider $r_0 = 1$, as r/r_0 has the same properties as r. By continuity of r in $0 \in C_\geq(S)$ and its homogeneity, there exists $a > 0$ such that $r(f) \leq a$ for all $f \in \mathbb{S}$, and thereby $\mathbb{P}\{r(V) > a\} = 0$. Thus,

$$\begin{aligned}\Lambda(\{f \in C_0(S) : r(f) > 1\}) &= \Lambda\left(\left\{f \in C_0(S) : r\left(\frac{f}{\|f\|_\infty}\right) > \frac{1}{\|f\|_\infty}\right\}\right)\\ &= \theta \int_{1/a}^{\infty} \mathbb{P}\left\{r(V) > \frac{1}{t}\right\}\nu_1(\mathrm{d}t) = \theta \int_0^a \mathbb{P}\{r(V) > x\}\mathrm{d}x = \theta\,\mathbb{E}[r(V)] < \infty.\end{aligned}$$

Similarly,

$$\Lambda(\{f \in C_0(S) : r(f) = 1\}) = \theta \int_0^a \mathbb{P}\big\{r(V) = x\big\}\mathrm{d}x = 0.$$

The last equality follows, since a random variable $r(V)$ may have at most countably many atoms, i.e. the integrand can be non-zero at most on a set of Lebesgue-measure zero. By continuity of r, the sets $r^{-1}((1, \infty))$ and $r^{-1}((0, 1))$ are open in $C_0(S)$. Hence, the boundary of $r^{-1}((1, \infty)) = \{f \in C_0(S) : r(f) > 1\}$ must be included in the Λ-null set $r^{-1}(\{1\}) = \{f \in C_0(S) : r(f) = 1\}$. □

Lemma 2.3. *Let $\Lambda = (\nu_1 \otimes \theta\sigma) \circ m^{-1}$ and $V \sim \sigma$ be given as in Lemma 2.1. Let P be a standard Pareto variable independent of V and $a > 0$. Then the measure*

$$\frac{\Lambda(\cdot \cap \{f \in C_0(S) : \|f\|_\infty > a\})}{\Lambda(\{f \in C_0(S) : \|f\|_\infty > a\})}$$

coindices with the law of the product aPV on $C_\geq(S)$.

Proof. Let B be a Borel set in $C_\geq(S)$. Then

$$\begin{aligned}\mathbb{P}\{aPV \in B\} &= \int_1^\infty \mathbb{P}\{atV \in B\}\mathbb{P}\{P \in \mathrm{d}t\} = \int_1^\infty \sigma\left(\left\{\frac{f}{t} : f \in a^{-1}B,\ \|f\|_\infty = t\right\}\right)\nu_1(\mathrm{d}t)\\ &= \frac{\Lambda(a^{-1}B \cap \{f \in C_0(S) : \|f\|_\infty > 1\})}{\theta} = \frac{a\Lambda(B \cap \{f \in C_0(S) : \|f\|_\infty > a\})}{\Lambda(\{f \in C_0(S) : \|f\|_\infty > 1\})}\\ &= \frac{\Lambda(B \cap \{f \in C_0(S) : \|f\|_\infty > a\})}{\Lambda(\{f \in C_0(S) : \|f\|_\infty > a\})},\end{aligned}$$

where we have used the homogeneity property of Λ in the last steps. □

The process PV is often called a Pareto process associated with the (-1)-homogeneous measure Λ. We may also associate to Λ a semi-simple max-stable process; a stochastic process Z with realizations in $C_{\geq}(S)$ is called *semi-simple max-stable* if for each $n \in \mathbb{N}$ the law of the normalized pointwise maximum process

$$\frac{1}{n}\bigvee_{i=1}^{n} Z_i$$

coincides with the law of Z, when $Z_1, \ldots, Z_n$ are independent copies of Z. In particular, each marginal distribution $\mathcal{L}(Z(s))$, $s \in S$, is either degenerate at 0 or a unit Fréchet distribution. If, in addition, Z has standard Fréchet marginal distributions, we call Z *simple max-stable.* The term *semi-simple max-stable* has been used in this way, for instance, in Molchanov (2008) for random vectors.

Lemma 2.4. *Let $\Lambda = (\nu_1 \otimes \theta\sigma) \circ m^{-1}$ and the point process $\sum_{i=1}^{\infty} \epsilon_{\theta\Gamma_i^{-1}V_i}$ be given as in Lemma 2.1. Then the pointwise maximum process*

$$Z = \theta \bigvee_{i=1}^{\infty} \Gamma_i^{-1} V_i, \tag{7}$$

is a semi-simple max-stable process in $C_{\geq}(S)$ with

$$\mathbb{P}\{r(Z) \leq t\} = \exp\Big(-\frac{\theta\,\mathbb{E}[r(V)]}{t}\Big), \qquad t > 0, \tag{8}$$

for any continuous and 1-homogeneous $r : C_{\geq}(S) \to [0,\infty)$, where $V \sim \sigma$. Moreover, the positive constant θ and the probability measure σ on $\mathbb{S}$, i.e., the law of V, are both identified from the law of Z.

Proof. Since Λ is d_0-boundedly finite, we have that for each $n \in \mathbb{N}$ almost surely only finitely many $\theta\Gamma_i^{-1}V_i$ lie in $m([1/n,\infty) \times \mathbb{S}) = \{f \in C_0(S) : \|f\|_\infty \geq 1/n\}$, so that $Z \vee n^{-1}$ is continuous almost surely, since it arises as the maximum of finitely many continuous functions. It follows that $Z \vee n^{-1}$ is continuous for all $n \in \mathbb{N}$ almost surely, and hence $Z \in C_{\geq}(S)$ almost surely.

The max-stability of Z is then a direct consequence of the homogeneity of $\Lambda = (\nu_1 \otimes \theta\sigma) \circ m^{-1}$, which implies that the superposition of n independent copies of the Poisson point processes (6), each with mean measure Λ, is a Poisson point process with mean measure $n\Lambda = (\nu_1 \otimes (n\theta)\sigma) \circ m^{-1}$. So, if $Z_1, \ldots, Z_n$ are independent copies of Z,

$$\mathcal{L}\Big(\bigvee_{i=1}^{n} Z_i\Big) = \mathcal{L}\Big((n\theta)\bigvee_{i=1}^{\infty} \Gamma_i^{-1} V_i\Big) = \mathcal{L}\big(nZ\big).$$

Finally, we may express $\mathbb{P}\{r(Z) \leq t\}$ as the probability that the Poisson point process (6) has no points in $\{f \in C_0(S) : r(f) > t\}$ and it remains to note that $\Lambda(\{f \in C_0(S) : r(f) > t\}) = \theta\mathbb{E}[r(V)]/t$, cf. Lemma 2.2.

To recover θ from the law of Z, let r be the supremum-norm $\|\cdot\|_\infty$ and observe that

$$\theta = -\log \mathbb{P}\{\|Z\|_\infty \leq 1\}. \tag{9}$$

Moreover, any non-negative continuous function $\mathsf{b} : \mathbb{S} \to [0,\infty)$ can be obtained as the restriction of a continuous and 1-homogeneous $r : C_{\geq}(S) \to [0,\infty)$ to $\mathbb{S}$ by setting $r(0) = 0$ and $r(f) = \|f\|_\infty \mathsf{b}(f/\|f\|_\infty)$ for $f \in C_0(S)$, hence $\mathbb{E}[\mathsf{b}(V)] = -\log \mathbb{P}\{r(Z) \leq 1\}/\theta$ by (8), which identifies the law of V by allowing b to vary. $\square$

Remark 2.5 (Stable tail dependence functional)**.** Setting the functional $r(f) = \sup_{s\in S} w(s)f(s)$ in Lemma 2.4 for an upper semicontinuous weight function $w : S \to [0,\infty)$ retrieves the stable tail dependence functional $\ell(w) = \theta\,\mathbb{E}\sup_{s\in S}[w(s)V(s)]$ as the scaling coefficient in the Fréchet law (8), cf. e.g. Molchanov and Strokorb (2016). In particular, choosing $w = \bigvee_{i=1}^{m} w_i \mathbb{1}\{s = s_i\}$ for $w_i \in (0,\infty)$ and mutually distinct $s_i \in S$, $i = 1, \ldots, m$, retrieves the finite dimensional distributions of Z.

In particular, the process Z in Lemma 2.4 is simple max-stable if and only if $\mathbb{E}[V(s)] = 1/\theta$ for all $s \in S$. Conversely, any simple max-stable process can be represented in this form — a result that dates back to the very fundamental contributions of de Haan (1984), Vatan (1985), de Haan and Pickands (1986), Norberg (1986) and Giné et al. (1990).

Theorem 2.6 (de Haan and Ferreira (2006) Theorem 9.4.1; Giné et al. (1990) Proposition 3.2 and Corollary 3.4). *Let Z be a simple max-stable process with realizations in $C_{\geq}(S)$. Then the following holds:*

(a) $\mathbb{P}\{Z \in C_{+}(S)\} = 1$.

(b) There exists a probability measure σ on $\mathbb{S}$ and $\theta > 0$, such that the law of Z equals the law of

$$\theta \bigvee_{i=1}^{\infty} \Gamma_i^{-1} V_i, \tag{10}$$

where $(\Gamma_i)_{i\in\mathbb{N}}$ and $(V_i)_{i\in\mathbb{N}}$ are as in Lemma 2.1, and $\mathbb{E}[V(s)] = 1/\theta$ for all $s \in S$.

The measure Λ of the form (3), its associated Poisson point process (6), its associated max-stable process (7) and its associated Pareto processes from Lemma 2.3 are now well-defined and we may proceed to consider triangular arrays that give rise to convergence to such limiting objects.

2.2 Triangular array setup

In what follows, we consider the following triangular array:

- $(X_n^{\text{orig}})_{n\in\mathbb{N}}$ – a sequence of pathwise continuous stochastic processes on S with identical continuous marginal distributions F^{orig},
- $X_{n,i}^{\text{orig}}$ – independent copies of X_n^{orig} for $1 \leq i \leq n$,
- $M_n^{\text{orig}}(s) = \max\left\{X_{n,1}^{\text{orig}}(s), \dots, X_{n,n}^{\text{orig}}(s)\right\}$, $s \in S$,

Its marginally transformed version is given as follows:

- $X_n(s) = 1/(1 - F^{\text{orig}}(X_n^{\text{orig}}(s)))$ for $s \in S$, so that each $X_n(s)$ has standard Pareto margins,
- $X_{n,i}$ – independent copies of X_n for $1 \leq i \leq n$,
- $M_n(s) = \max\left\{X_{n,1}(s), \dots, X_{n,n}(s)\right\}$, $s \in S$.

The following lemma links the convergence of the original and transformed maxima processes. Although we consider a triangular array here, the proof is very similar to the well-documented sequence case.

Lemma 2.7. *Let $a_n > 0$ and $b_n \in \mathbb{R}$ for $n \in \mathbb{N}$. Let Z^{orig} be a sample-continuous stochastic process on S with standard GEV marginal distribution G^{orig} for some extreme value index $\xi \in \mathbb{R}$. Consider*

$$Z(s) = \begin{cases} (1 + \xi Z^{\text{orig}}(s))^{1/\xi}, & s \in S, \quad \xi \neq 0, \\ \exp(Z^{\text{orig}}), & s \in S, \quad \xi = 0, \end{cases}$$

so that each $Z(s)$ has standard Fréchet margins. The following two statements are equivalent:

(i) $a_n^{-1}(M_n^{\text{orig}} - b_n) \to Z^{\text{orig}}$ *weakly in $C(S)$.*

(ii) The distribution $(F^{\text{orig}})^n(a_n \cdot + b_n)$ converges weakly to the distribution G^{orig} and $n^{-1}M_n \to Z$ weakly in $C_{\geq}(S) \subset C(S)$.

Proof. The result can be deduced along the same lines as de Haan and Ferreira (2006, Theorem 9.2.1). First note that $Z = f(Z^{\text{orig}})$ for $f(t) = (1 + \xi t)^{1/\xi}\mathbb{1}\{1 + \xi t > 0\}$ and $n^{-1}M_n = f_n(a_n^{-1}(M_n^{\text{orig}} - b_n))$ for $f_n(t) = n^{-1}(1 - F^{\text{orig}}(a_n t + b_n))^{-1}$. Now, (i) implies the marginal convergence of $(F^{\text{orig}})^n(a_n \cdot + b_n)$ to G^{orig} as well as the convergence of $f_n(t)$ to $f(t)$ locally uniformly for all t with $1 + \xi t > 0$, hence we obtain (ii). Conversely, if (ii) holds, the marginal convergence of $(F^{\text{orig}})^n(a_n \cdot + b_n)$ to G^{orig} implies also the locally uniform convergence of the generalized inverse of f_n to the inverse of f on $(0, \infty)$. Hence, we retrieve (i) from the weak convergence of $f_n^{-1}(n^{-1}M_n)$ to $f^{-1}(Z)$. □

Without additional assumption on the limiting process Z, the latter is necessarily max-infinitely divisible (Norberg, 1986, Theorem 5.2.). For our purposes we will even impose max-stability from here. As in the sequence formulation, the convergence of the maxima process $n^{-1}M_n$ to Z is closely linked to the theory of regular variation. To make this explicit, we extend the notion of regular variation to triangular arrays as follows.

Definition 2.8. We call a sequence $(X_n)_{n\in\mathbb{N}}$ of pathwise continuous stochastic processes on S *regularly varying with index* $\alpha > 0$ *and spectral process* V *in the triangular array sense* if $V \in \mathbb{S}$ almost surely, and for all $t > 0$ and Borel sets $B \subset \mathbb{S}$ with $\mathbb{P}\{V \in \partial B\} = 0$

$$\frac{1}{\mathbb{P}\{\|X_n\|_\infty > n\}}\,\mathbb{P}\Big\{\frac{X_n}{\|X_n\|_\infty} \in B\,,\ \frac{\|X_n\|_\infty}{n} > t\Big\} \to \mathbb{P}\{V \in B\}\,t^{-\alpha} \qquad \text{as} \quad n \to \infty.$$

Remark 2.9. Consider the setting of Definition 2.8 with $\alpha = 1$. Since $\mathbb{P}\{V \in \mathbb{S}\} = 1$, we have $\|V\|_\infty = 1$ almost surely, and so $\mathbb{E}[\|V\|_\infty] = 1$ and $\mathbb{E}[V(s)] \in [0,1]$ for all $s \in S$. By pathwise continuity of V, there is at least one $s_0 \in S$ such that $\mathbb{E}[V(s_0)] > 0$. For any such s_0 we obtain from the continuous mapping theorem

$$\lim_{n\to\infty} \frac{\mathbb{P}\{X_n(s_0) > n\}}{\mathbb{P}\{\|X_n\|_\infty > n\}} = \int_0^\infty \mathbb{P}\Big\{V(s_0) > \frac{1}{t}\Big\}\nu_1(\mathrm{d}t) = \int_0^\infty \mathbb{P}\big\{V(s_0) > x\big\}\,\mathrm{d}x = \mathbb{E}[V(s_0)] \in (0,1].$$

If each process X_n has identical marginal distributions, then this implies also $\mathbb{E}[V(s)] = \mathbb{E}[V(s_0)]$ for all $s \in S$. In the case of standard Pareto margins, we obtain

$$\lim_{n\to\infty} n\mathbb{P}\{\|X_n\|_\infty > n\} = \frac{1}{\mathbb{E}[V(s)]} \in [1,\infty), \qquad s \in S.$$

We now provide a similar collection of equivalent characterizations as in Dombry and Ribatet (2015, Theorem 1). The main differences are that we consider triangular arrays instead of independent copies of processes and the slightly more convenient topological setup inspired by Basrak and Planinić (2019) and Kulik and Soulier (2020). Standardization to unit Pareto margins simplifies the scaling constants compared to Dombry and Ribatet (2015). A subtlety that may be easily overlooked is that, with triangular arrays, establishing the equivalence with characterization (v) requires considering the statement for all scaling constants $a > 0$, while, in the original formulation, the result follows from the case $a = 1$, which implies equivalence for all $a > 0$.

Theorem 2.10. *Let $\{X_{n,i},\ 1 \le i \le n,\ n \in \mathbb{N}\}$ be the triangular array of pathwise continuous stochastic processes with standard Pareto margins as specified above, where each row consists of independent copies of X_n, and M_n is the corresponding sequence of pointwise row-maxima. Let Z be a simple max-stable process with spectral representation* (10) *as in Theorem 2.6, which is specified via a law σ on $\mathbb{S}$ and $\theta > 0$. Let $\Lambda = (\nu_1 \otimes \theta\sigma) \circ m^{-1}$, where m is the multiplication map* (1). *Let V be distributed according to σ on $\mathbb{S}$ and P a standard Pareto variable independent of V.*

The following statements are equivalent:

- *(i)* $n^{-1}M_n \to Z$ *weakly in* $C_+(S)$;
- *(ii)* $n\mathbb{P}\{n^{-1}X_n \in \cdot\} \xrightarrow{v^\#} \Lambda$ *in* $M(C_0(S))$;
- *(iii)* $\sum_{i=1}^n \epsilon_{n^{-1}X_{n,i}} \to \sum_{i=1}^\infty \epsilon_{\theta\Gamma_i^{-1}V_i} \sim \mathrm{PPP}(\Lambda)$ *weakly in* $M(C_0(S))$;
- *(iv)* $(X_n)_{n\in\mathbb{N}}$ *is regularly varying with index* 1 *and spectral process V in the triangular array sense;*
- *(v)* $\mathcal{L}\big(n^{-1}X_n \,|\, \|X_n\|_\infty > an\big) \to \mathcal{L}(aPV)$ *weakly in* $C_{\ge}(S)$ *for all* $a > 0$.

Proof. The equivalence of (i) and (ii) follows exactly in the same manner as de Haan and Ferreira (2006, Theorem 9.3.1) except that we are dealing here with triangular arrays instead of i.i.d. sequences, cf. also the proof of the equivalence of (2a) and (2b) of Theorem 9.5.1 therein on page 312. The convergence of measures considered therein is indeed equivalent to the $v^\#$-convergence considered here, cf. also de Haan and Ferreira (2006, Remark 9.3.3). The equivalence of (ii) and (iii) is an immediate consequence of Kulik and Soulier (2020, Theorem 7.1.21), since the null-array condition therein amounts to $\lim_{n\to\infty} \mathbb{P}\{n^{-1}X_n \in B\} = 0$ for all d_0-bounded Borel sets $B \subset C_0(S)$. This shows that (i), (ii) and (iii) are equivalent. The remaining equivalences can be deduced as follows: Recall that m is a homeomorphism between $(0,\infty) \times \mathbb{S}$ and $C_0(S)$ with inverse the polar decomposition m^{-1} from (2).

(ii) implies (iv) For $t > 0$ and $B \subset \mathbb{S}$ Borel with $\mathbb{P}\{V \in \partial B\} = 0$

$$\frac{1}{\mathbb{P}\{\|X_n\|_\infty > n\}}\,\mathbb{P}\bigg\{\frac{X_n}{\|X_n\|_\infty} \in B\,,\ \frac{\|X_n\|_\infty}{n} > t\bigg\} = \frac{n\mathbb{P}\{n^{-1}X_n \in m((t,\infty)\times B)\}}{n\mathbb{P}\{n^{-1}X_n \in m((1,\infty)\times \mathbb{S})\}},$$

and by (ii) the latter converges to $\theta t^{-1}\sigma(B)/\theta = \mathbb{P}\{V \in B\}t^{-1}$.

(iv) implies (ii) First, we observe from (iv) as in Remark 2.9 that

$$n\,\mathbb{P}\{\|X_n\|_\infty > n\} \to 1/\mathbb{E}[V(s)] = \theta, \quad s \in S,$$

where the last equality follows from the specification of V and θ in this theorem, cf. Theorem 2.6. Since the limiting measure in (ii) has the product representation $\Lambda = (\nu_1 \otimes \theta\sigma)\circ m^{-1}$, it is sufficient to consider test sets of the form $m((t,\infty)\times B)$, where $t > 0$ and $B \subset \mathbb{S}$ is a Borel set with $\sigma(\partial B) = 0$, in order to establish the convergence of (ii). Indeed, we obtain

$$\begin{aligned} &n\,\mathbb{P}\{n^{-1}X_n \in m((t,\infty)\times B)\}\\ &= n\,\mathbb{P}\{\|X_n\|_\infty > n\}\frac{1}{\mathbb{P}\{\|X_n\|_\infty > n\}}\mathbb{P}\bigg\{\frac{X_n}{\|X_n\|_\infty} \in B\,,\ \frac{\|X_n\|_\infty}{n} > t\bigg\}\end{aligned}$$

converges to $\theta\,\mathbb{P}\{V \in B\}t^{-1} = \theta\sigma(B)t^{-1} = \Lambda(m((t,\infty))\times B)$ by (iv), hence we have (ii).

(ii) implies (v) Let $A = \{f \in C_0(S) : \|f\|_\infty > a\}$. For the limiting distribution in (v) we have $\mathbb{P}\{aPV \in A\} = 1$ and as for Λ there is no mass on the boundary of A, cf. Lemma 2.3 and Lemma 2.2. So let $B \subset A$ be a continuity set of $\mathbb{P}\{aPV \in \cdot\}$, then it is also a continuity set for Λ by Lemma 2.3. In addition, A and B are d_0-bounded. Hence, by (ii)

$$\mathbb{P}\{n^{-1}X_n \in B \,|\, \|X_n\|_\infty > an\} = \frac{n\mathbb{P}\{n^{-1}X_n \in B\cap A\}}{n\mathbb{P}\{n^{-1}X_n \in A\}} \to \frac{\Lambda(B\cap A)}{\Lambda(A)} = \mathbb{P}\{aPV \in B\},$$

where the last equality follows directly from Lemma 2.3.

(v) implies (iv) Let $t > 0$. First we show that (v) implies

$$\frac{\mathbb{P}\{\|X_n\|_\infty > nt\}}{\mathbb{P}\{\|X_n\|_\infty > n\}} = t^{-1}.$$

Indeed, if $t \geq 1$, the left-hand side may be written as conditional probability

$$\mathbb{P}\{n^{-1}X_n \in \{f \in C_0(S) : \|f\|_\infty > t\} \,|\, \|X_n\|_\infty > n\},$$

which converges by (v), Lemma 2.3 and the homogeneity property of Λ to

$$\mathbb{P}\{PV \in \{f \in C_0(S) : \|f\|_\infty > t\}\} = \frac{\Lambda(\{f \in C_0(S) : \|f\|_\infty > t\})}{\Lambda(\{f \in C_0(S) : \|f\|_\infty > 1\})} = \frac{\theta t^{-1}}{\theta} = t^{-1}.$$

If $t < 1$, the left-hand side may be written as

$$1/\mathbb{P}\{n^{-1}X_n \in \{f \in C_0(S) : \|f\|_\infty > 1\} \,|\, \|X_n\|_\infty > nt\},$$

which converges instead to

$$\frac{1}{\mathbb{P}\{tPV \in \{f \in C_0(S) : \|f\|_\infty > 1\}\}} = \frac{\Lambda(\{f \in C_0(S) : \|f\|_\infty > 1\})}{\Lambda(\{f \in C_0(S) : \|f\|_\infty > t^{-1}\})} = \frac{\theta}{\theta t} = t^{-1}.$$

In order to establish (iv), let in addition $B \subset \mathbb{S}$ be a continuity set for the law of V. Then

$$\begin{aligned}&\frac{1}{\mathbb{P}\{\|X_n\|_\infty > n\}}\,\mathbb{P}\bigg\{\frac{X_n}{\|X_n\|_\infty} \in B\,,\ \frac{\|X_n\|_\infty}{n} > t\bigg\}\\ =&\frac{\mathbb{P}\{\|X_n\|_\infty > nt\}}{\mathbb{P}\{\|X_n\|_\infty > n\}}\,\mathbb{P}\bigg\{n^{-1}X_n \in \bigg\{f \in C_0(S) : \frac{f}{\|f\|_\infty} \in B\bigg\}\,\bigg|\, \|X_n\|_\infty > nt\bigg\}.\end{aligned}$$

We have seen that the first factor converges to t^{-1}. The second factor converges by (v) to

$$\begin{aligned}&\mathbb{P}\{tPV \in \{f \in C_0(S) : f/\|f\|_\infty \in B\}\}\\ =&\frac{\Lambda(\{f \in C_0(S) : f/\|f\|_\infty \in B,\, \|f\|_\infty > t\})}{\Lambda(\{f \in C_0(S) : \|f\|_\infty > t\})} = \frac{\theta t^{-1}\sigma(B)}{\theta t^{-1}} = \sigma(B).\end{aligned}$$

Taken together, this shows (iv).

This finishes the proof. ☐

Remark 2.11. Under the setup of Theorem 2.10 the constant θ satisfies

$$\theta = \lim_{n\to\infty} n\,\mathbb{P}\{\|X_n\|_\infty > n\},$$

and $\theta \geq 1$, cf. Remark 2.9 and Theorem 2.6. It is typically called the *extremal coefficient* of Z, since $\|Z\|_\infty/\theta$ follows a standard Fréchet distribution, cf. (9). If the processes X_n in the triangular array were distributed according to an identical process X for all $n \in \mathbb{N}$, i.e., if we are in the framework of classical regular variation, the constant θ is also understood as the extremal coefficient of X.

A trivial example for the equivalent conditions of Theorem 2.10 to hold arises from the max-stable limit itself, since a max-stable Z lies in its own domain of attraction.

Example 2.12. [Regular variation of max-stable processes] Let $X_{n,i}^{\mathrm{orig}} = Z_{n,i}$, where $Z_{n,i}$ are independent copies of a simple max-stable process $Z \in C_{\geq}(S)$, so that $X_{n,i} = T(Z_{n,i})$, where $T(x) = 1/(1 - \exp(-x^{-1}))$ is the transformation from standard Fréchet to standard Pareto margins. By the max-stability of Z, we have the distributional identity $\mathcal{L}(n^{-1}M_n^{\mathrm{orig}}) = \mathcal{L}(Z)$ for all $n \in \mathbb{N}$. Hence, in view of Lemma 2.7, $n^{-1}M_n$ converges weakly to Z, so that statement (i) from Theorem 2.10 holds and the others follow by equivalence.

2.3 Risk functionals

Rather than conditioning on the norm being large, we consider a more general notion of exceedance in this section. To this end, let

$$r : C_{\geq}(S) \to [0, \infty)$$

be a continuous and 1-homogeneous functional, i.e. $r(cf) = cr(f)$ for all $f \in C_{\geq}(S)$ and $c > 0$. In particular, $r(0) = 0$ by continuity. We call such a functional a *risk functional*. Common examples are an $\mathcal{L}^p(S, \mu)$-norm for a finite measure μ on S or a weighted infimum or supremum over S for a bounded non-negative weight function. The point evaluation risk functional , $f \mapsto f(s_0)$ for an $s_0 \in S$, will also be key to the rest of the paper. Similarly to Section 2.1, we first review relevant spaces and distances. Let

$$N_r(S) = \{f \in C_{\geq}(S) \,:\, r(f) = 0\} \qquad \text{and} \qquad C_r(S) = \{f \in C_{\geq}(S) \,:\, r(f) \in (0, \infty)\},$$

so that $C_{\geq}(S)$ is the disjoint union of $N_r(S)$ and $C_r(S)$. In particular, $0 \in N_r(S)$ and $C_r(S)$ is a subset of $C_0(S)$. Suppose that r is not identically zero, so that $C_r(S) \neq \emptyset$. Denote the distance of a function $f \in C_r(S)$ to $N_r(S)$ by

$$d(f, N_r(S)) = \inf\{\|f - g\|_\infty \,:\, g \in N_r(S)\}.$$

A subset $B \subset C_r(S)$ is called bounded away from $N_r(S)$ if $\inf\{d(f, N_r(S)) \,:\, f \in B\} > 0$. The metric

$$d_r(f, g) = \big(\|f - g\|_\infty \wedge 1\big) \vee \left| \frac{1}{d(f, N_r(S))} - \frac{1}{d(g, N_r(S))} \right|$$

induces the trace topology of $C(S)$ on $C_r(S)$, while $(C_r(S), d_r)$ is complete and $B \subset C_r(S)$ is bounded away from $N_r(S)$ if and only if B is bounded for d_r, cf. Kulik and Soulier (2020, Example B.1.6). The sequence of subsets $\{f \in C_r(S) \,:\, d(f, N_r(S)) > k^{-1}\}$, $k \in \mathbb{N}$, is properly localizing $C_r(S)$. Hence, $C_r(S)$ is a properly localized Polish space.

The following lemma implies that all sets of the form $\{f \in C_r(S) \,:\, r(f) > t\}$, where $t > 0$, are bounded away from $N_r(S)$ if r is uniformly continuous or if $N_r(S)$ is compact.

Remark 2.13. The condition that r is uniformly continuous is not very restrictive in practice. For instance, supremum or infimum over S or point evaluation at some $s_0 \in S$ are even Lipschitz continuous with constant 1, while the constant will need to be adapted for weighted versions. An integral over S has Lipschitz constant equal to the total mass of the integrating measure.

Lemma 2.14. *Suppose that the risk functional r is uniformly continuous or that $N_r(S)$ is compact. Then there exists a constant $\kappa \in (0, \infty)$, such that*

$$\kappa\, r(f) \le d(f, N_r(S)) \tag{11}$$

for all $f \in C_r(S)$.

Proof. By the definition of $C_r(S)$ we have that $f \in C_r(S)$ satisfies $r(f) \in (0, \infty)$. Hence,

$$\begin{aligned} d(f, N_r(S)) &= \inf\{\|f - g\|_\infty \,:\, g \in N_r(S)\} \\ &= r(f) \inf\{\|f/r(f) - g/r(f)\|_\infty \,:\, r(g) = 0\} \\ &= r(f) \inf\{\|f/r(f) - g'\|_\infty \,:\, r(g') = 0\} \\ &\ge r(f) \inf\{\|f' - g'\|_\infty \,:\, r(f') = 1 \text{ and } r(g') = 0\}. \end{aligned}$$

If r is uniformly continuous or $N_r(S)$ is compact (and r possibly just continuous, not necessarily uniformly continuous), then we have that $r^{-1}(\{0\})$ and $r^{-1}(\{1\})$ are disjoint closed subsets with a positive distance. So, setting $\kappa = \inf\{\|f' - g'\|_\infty \,:\, r(f') = 1 \text{ and } r(g') = 0\}$ gives the assertion. $\square$

Remark 2.15. A situation, in which a bound κ as in (11) does not exist, arises, for instance, when we consider the risk functional $r(f) = (f(s_1)f(s_2))^{1/2}$, where S contains at least two distinct points $s_1 \neq s_2$. Consider then a sequence of continuous functions with $f_n(s_1) = n$ and $f_n(s_2) = 1/n$, so that $r(f_n) = 1$. But then we also have that $g_n = \max(f_n - 1/n, 0)$ satisfies $g_n \in N_r(S)$ with $\|f_n - g_n\|_\infty \le 1/n$. Hence, $d(f_n, N_r(S)) \to 0$.

Even, if we are in a situation, where (11) does hold, conversely, $d(\cdot, N_r(S))$ need not be bounded by a multiple of r. Consider, for instance, $S = [0, 1]$ and r the integral over S. Then $N_r(S) = \{0\}$, and so $d(f, N_r(S)) = \|f\|_\infty$. One may easily construct a sequence of functions f_n, $n \ge 1$, on S, whose integrals $r(f_n)$ are bounded, whilst their supremum norms $\|f_n\|_\infty$ are unbounded.

On the other hand, if r is either the supremum over S, the infimum over S or the point evaluation at some $s_0 \in S$, it is easily seen that we even have $d(f, N_r(S)) = r(f)$, cf. Lemma A.1. Weighted versions of supremum, infimum and point evaluation with a non-trivial non-negative bounded weight function satisfy that $d(f, N_r(S)) \le K r(f)$ for a constant $K \in (0, \infty)$.

Let

$$\mathbb{S}_r = \{f \in C_{\ge}(S) \,:\, r(f) = 1\},$$

which is again a closed subset of $C_{\ge}(S)$ (hence c.s.m.s.). Then the multiplication map

$$m_r : (0, \infty) \times \mathbb{S}_r \to C_r(S), \qquad m_r(t, f) = tf \tag{12}$$

is a homeomorphism between $(0, \infty) \times \mathbb{S}_r$ and $C_r(S)$ with inverse

$$m_r^{-1} : C_r(S) \to (0, \infty) \times \mathbb{S}_r, \qquad m_r^{-1}(f) = (r(f), f/r(f)),$$

also known as pseudo-polar decomposition. By $M(C_r(S))$ we denote the space of all d_r-boundedly finite Borel measures μ on $C_r(S)$, i.e. measures satisfying $\mu(B) < 0$ for d_r-bounded Borel sets $B \subset C_0(S)$. Endowed with the $v_r^{\#}$-topology, i.e.

$$\mu_n \xrightarrow{v_r^{\#}} \mu \quad \text{if and only if} \quad \mu_n(B) \to \mu(B) \text{ for all } d_r\text{-bounded Borel sets } B \text{ with } \mu(\partial B) = 0,$$

$M(C_r(S))$ is a Polish space (Kulik and Soulier, 2020, Theorem B.1.20). Lemma 2.16 is an immediate consequence of the circumstance that a set $B \subset C_r(S) \subset C_0(S)$ bounded away from $N_r(S)$ is also bounded away from 0.

Lemma 2.16. *Let $\mu, \mu_n \in M(C_0(S))$, $n \in \mathbb{N}$. Then their restrictions to $C_r(S) \subset C_0(S)$ are d_r-boundedly finite, and we have the implication*

$$\mu_n \xrightarrow{v^{\#}} \mu \quad \Longrightarrow \quad \mu_n\big|_{C_r(S)} \xrightarrow{v_r^{\#}} \mu\big|_{C_r(S)}.$$

Remark 2.17 (Completeness and preservation of boundedness)**.** The product topology on $(0,\infty)\times\mathbb{S}_r$ is also induced by the metric

$$\widetilde{d}_r\big((t_1,f_1),(t_2,f_2)\big)=\big(|t_1-t_2|\wedge 1\big)\vee\left|\frac{1}{t_1}-\frac{1}{t_2}\right|\vee\big(\|f_1-f_2\|_\infty\wedge 1\big).$$

In particular, $(0,\infty)\times\mathbb{S}_r$ is complete with metric $\widetilde{d}_r$. However, the homeomorphism m_r^{-1} need not necessarily preserve boundedness. For instance, $m_r^{-1}(\{f\in C_r(S)\,:\,d(f,N_r(S))>k^{-1}\})$ is unbounded for $\widetilde{d}_r$ if $S=[0,1]$ and r is the integral over S, cf. Remark 2.15. On the other hand, m_r^{-1} does preserve boundedness if there exists a $K\in(0,\infty)$ such that $d(f,N_r(S))\le Kr(f)$ as is the case for (weighted) supremum, infimum or point evaluation. For such risk functionals, the sequence of subsets $\{f\in C_r(S)\,:\,r(f)>k^{-1}\}$, $k\in\mathbb{N}$, is also properly localizing $C_r(S)$.

2.4 Homogeneous measures, risk functionals and triangular arrays

Let us consider again a measure

$$\Lambda=(\nu_1\otimes\theta\sigma)\circ m^{-1}$$

as given in Lemma 2.1, where σ is a probability measure on $\mathbb{S}$ and $\theta>0$. By construction, Λ is (-1)-homogeneous, d_0-boundedly finite, and Λ does not vanish on $C_0(S)$. The restriction of Λ to $C_r(S)$,

$$\Lambda_r=\Lambda\big|_{C_r(S)},\tag{13}$$

is also (-1)-homogeneous and necessarily d_r-boundedly finite, cf. Lemma 2.16. If Λ has no mass on $N_r(S)$, we can even retrieve Λ from Λ_r. In many cases, e.g., when the risk functional r is a point evaluation, it is however possible that Λ might have mass on $N_r(S)$. Lemma 2.18 gives the necessary and sufficient conditions on $V\sim\sigma$, which ensure that Λ_r is not a zero measure or, even more, that no mass ist lost by restricting Λ to $C_r(S)$.

Lemma 2.18. *Let $V\sim\sigma$. Then*

$$\begin{aligned}\Lambda(C_r(S))\neq 0 &\iff \mathbb{P}\{r(V)>0\}>0,\\ \Lambda(N_r(S))=0 &\iff \mathbb{P}\{r(V)>0\}=1.\end{aligned}$$

Proof. Recall that we can express $\Lambda(B)$ by the definition of Λ as (4). For $B=C_r(S)$, the integrand on the right-hand-side in (4) equals

$$\sigma(\{f\in\mathbb{S}\,:\,tf\in C_r(S)\})=\mathbb{P}\{r(tV)>0\}=\mathbb{P}\{tr(V)>0\}=\mathbb{P}\{r(V)>0\}.$$

for all $t\in(0,\infty)$, while for $B=N_r(S)$ the integrand equals

$$\sigma(\{f\in\mathbb{S}\,:\,tf\in N_r(S)\})=\mathbb{P}\{r(tV)=0\}=\mathbb{P}\{tr(V)=0\}=\mathbb{P}\{r(V)=0\}$$

for all $t\in(0,\infty)$. Hence, if $\mathbb{P}\{r(V)>0\}>0$ or $r(V)>0$ almost surely, we see that $\Lambda(C_r(S))\neq 0$ or $\Lambda(N_r(S))=0$, respectively. Conversely, if $\Lambda(C_r(S))\neq 0$, the integrand $\sigma(\{f\in\mathbb{S}\,:\,tf\in C_r(S)\})$ needs to be positive on a set of positive ν_1-measure, hence $\mathbb{P}\{r(V)>0\}>0$. And if $\Lambda(N_r(S))=0$, the integrand $\sigma(\{f\in\mathbb{S}\,:\,tf\in N_r(S)\})$ can only be positive on a ν_1-null set, hence $\mathbb{P}\{r(V)=0\}=0$. ☐

The condition $\mathbb{P}\{r(V)>0\}>0$ means that r does not vanish σ-almost everywhere. We assume henceforth that it is satisfied, so that Λ_r is not a zero measure.

Lemma 2.19. *Suppose that $\mathbb{P}\{r(V)>0\}>0$. Then we may represent Λ_r as*

$$\Lambda_r=(\nu_1\otimes\theta_r\sigma_r)\circ m_r^{-1},\tag{14}$$

that is,

$$\Lambda_r(B)=(\nu_1\otimes\theta_r\sigma_r)(m_r^{-1}(B))=\theta_r\int_0^\infty\sigma_r(\{f\in\mathbb{S}_r\,:\,tf\in B\})\,\nu_1(\mathrm{d}t)$$

for Borel sets $B \subset C_r(S)$, where

$$\theta_r = \Lambda_r(\{f \in C_r(S) \,:\, r(f) > 1\} = \Lambda(\{f \in C_0(S) \,:\, r(f) > 1\}) = \theta\, \mathbb{E}[r(V)] \in (0, \infty), \tag{15}$$

and σ_r is the probability measure on $\mathbb{S}_r$ given by

$$\sigma_r(B) = \frac{1}{\theta_r} \Lambda_r(\{f \in C_r(S) \,:\, f/r(f) \in B,\, r(f) > 1\}) = \frac{1}{\theta_r} \Lambda(\{f \in C_0(S) \,:\, f/r(f) \in B,\, r(f) > 1\}) \tag{16}$$

for Borel sets $B \subset \mathbb{S}_r$.

Proof. The condition $\mathbb{P}\{r(V) > 0\} > 0$ ensures that Λ_r is not a zero measure, cf. Lemma 2.18, and so, by Lemma 2.2,

$$\theta_r = \Lambda_r(\{f \in C_r(S) \,:\, r(f) > 1\} = \Lambda(\{f \in C_0(S) \,:\, r(f) > 1\}) = \theta\, \mathbb{E}[r(V)] \in (0, \infty).$$

Therefore, we may define a probability measure σ_r on $\mathbb{S}_r$ by (16). By the homogeneity of Λ and Λ_r, we have

$$\theta_r\, \sigma_r(B)\, t^{-1} = \Lambda_r\big(\{f \in C_r(S) \,:\, r(f) > t\,,\, f/r(f) \in B\}\big) = \Lambda\big(\{f \in C_0(S) \,:\, r(f) > t\,,\, f/r(f) \in B\}\big)$$

for Borel sets $B \subset \mathbb{S}_r$ and $t > 0$. Thus (by a similar Dynkin system argument as in Lemma 2.1), we may represent Λ_r as $\Lambda_r = (\nu_1 \otimes \theta_r \sigma_r) \circ m_r^{-1}$. □

The measure σ_r in (16) can be obtained directly from σ as follows, cf. also Dombry and Ribatet (2015, Theorem 3).

Lemma 2.20. *The measure σ_r from* (16) *can be obtained from σ via*

$$\sigma_r(B) = \frac{1}{\mathbb{E}[r(V)]} \int_{\mathbb{S}} r(f)\, \mathbb{1}\{f \in r(f) B\}\, \sigma(\mathrm{d}f) \tag{17}$$

for Borel sets $B \subset \mathbb{S}_r$. If $r' : C_{\geq}(S) \to [0, \infty)$ is measurable and homogeneous and $V^r \sim \sigma_r$, then

$$\mathbb{E}\big[r'(V^r)\big] = \frac{\mathbb{E}[r'(V) \mathbb{1}\{r(V) > 0\}]}{\mathbb{E}[r(V)]}.$$

In particular, $\mathbb{E}[r'(V^r)]$ is finite if r' is another risk functional, and

$$\mathbb{E}\big[\|V^r\|_\infty\big] = \frac{\mathbb{P}\{r(V) > 0\}}{\mathbb{E}[r(V)]} \in (0, \infty).$$

Proof. By definition of σ_r, we have

$$\sigma_r(B) = \frac{1}{\theta\, \mathbb{E}[r(V)]} \Lambda(\{f \in C_0(S) \,:\, f/r(f) \in B,\, r(f) > 1\}).$$

Recall from the proof of Lemma 2.2 that there exists $a > 0$ such that $\mathbb{P}\{r(V) > a\} = 0$. Hence, an application of Fubini's theorem yields

$$\begin{aligned}
\sigma_r(B) &= \frac{1}{\theta\, \mathbb{E}[r(V)]} \int_{C_0(S)} \mathbb{1}\{f/r(f) \in B,\, r(f) > 1\}\, \Lambda(\mathrm{d}f) \\
&= \frac{\theta}{\theta\, \mathbb{E}[r(V)]} \int_{1/a}^{\infty} \int_{\mathbb{S}} \mathbb{1}\{tf/r(tf) \in B,\, r(tf) > 1\}\, \sigma(\mathrm{d}f)\, \nu_1(\mathrm{d}t) \\
&= \frac{1}{\mathbb{E}[r(V)]} \int_{\mathbb{S}} \int_{1/a}^{\infty} \mathbb{1}\{r(f) > t^{-1}\}\, \nu_1(\mathrm{d}t)\, \mathbb{1}\{f/r(f) \in B\}\, \sigma(\mathrm{d}f) \\
&= \frac{1}{\mathbb{E}[r(V)]} \int_{\mathbb{S}} r(f)\, \mathbb{1}\{f \in r(f) B\}\, \sigma(\mathrm{d}f),
\end{aligned}$$

which shows the claimed relation (17) between σ and σ_r. By this relation, we obtain further

$$\begin{aligned}\mathbb{E}[r'(V^r)] &= \int_0^\infty \mathbb{P}\{r'(V^r) > t\}\,\mathrm{d}t = \int_0^\infty \sigma_r(\{f \in \mathbb{S}_r : r'(f) > t\})\,\mathrm{d}t \\ &= \int_0^\infty \frac{1}{\mathbb{E}[r(V)]} \int_{\mathbb{S}} r(f)\,\mathbb{1}\{r'(f) > r(f)t\}\,\sigma(\mathrm{d}f)\,\mathrm{d}t \\ &= \frac{1}{\mathbb{E}[r(V)]} \int_{\mathbb{S}} \mathbb{1}\{r(f) > 0\} \int_0^\infty \mathbb{1}\{r'(f)/r(f) > t\}\,\mathrm{d}t\, r(f)\,\sigma(\mathrm{d}f) = \frac{\mathbb{E}[r'(V)\mathbb{1}\{r(V) > 0\}]}{\mathbb{E}[r(V)]}.\end{aligned}$$

We know already that the denominator lies in $(0, \infty)$. If r' is another risk functional, the enumerator is bounded by $\mathbb{E}[r'(V)] < \infty$, cf. Lemma 2.2. Choosing $r' = \|\cdot\|_\infty$ yields the last assertion. □

Remark 2.21. Similarly to (17), if r' and r are two risk functionals and σ_r and $\sigma_{r'}$ are both derived from Λ as in (16), while $\mathbb{P}\{r(V) > 0\} > 0$ and $\mathbb{P}\{r'(V) > 0\} = 1$, i.e., Λ has no mass on $N_{r'}(S)$, then

$$\sigma_r(B) = \frac{1}{\mathbb{E}[r(V^{r'})]} \int_{\mathbb{S}_{r'}} r(f)\,\mathbb{1}\{f \in r(f)B\}\,\sigma_{r'}(\mathrm{d}f), \tag{18}$$

cf. also Dombry and Ribatet (2015, Proposition 2).

The following lemma is analogous to Lemma 2.3 with almost identical proof except for replacing the norm $\|\cdot\|_\infty$ by the general risk functional $r(\cdot)$ and the constant θ by the constant θ_r.

Lemma 2.22. *Let V^r be distributed according to the probability measure σ_r from (16). Let P be a standard Pareto variable independent of V^r and let $a > 0$. Then the measure*

$$\frac{\Lambda(\,\cdot\, \cap \{f \in C_0(S) : r(f) > a\})}{\Lambda(\{f \in C_0(S) : r(f) > a\})} = \frac{\Lambda_r(\,\cdot\, \cap \{f \in C_r(S) : r(f) > a\})}{\Lambda_r(\{f \in C_r(S) : r(f) > a\})}$$

coindices with the law of the product aPV^r on $C_{\geq}(S)$.

Next, we consider consequences from the triangular array convergence from Theorem 2.10 when conditioning on an exceedance of a risk functional. The result is analogous to Dombry and Ribatet (2015, Theorem 3).

Proposition 2.23. *Consider the setup of Theorem 2.10, and assume that one of the statements (i)-(v) in Theorem 2.10 (hence any) holds true. In addition, let $r : C_{\geq}(S) \to [0, \infty)$ be a risk functional, such that $\mathbb{P}\{r(V) > 0\} > 0$, and let V^r and P be as in Lemma 2.22. Then we have the following implications:*

(a) For all $t > 0$ and Borel sets $B \subset \mathbb{S}_r$ with $\mathbb{P}\{V^r \in \partial B\} = 0$

$$\frac{1}{\mathbb{P}\{r(X_n) > n\}} \mathbb{P}\left\{\frac{X_n}{r(X_n)} \in B\,,\ \frac{r(X_n)}{n} > t\right\} \to \mathbb{P}\{V^r \in B\}\,t^{-1} \qquad \text{as} \quad n \to \infty.$$

(b) $\mathcal{L}\big(n^{-1}X_n \,|\, r(X_n) > an\big) \to \mathcal{L}\big(aPV^r\big)$ weakly in $C_{\geq}(S)$ for all $a > 0$.

Proof. Let $t > 0$ and $A \subset \mathbb{S}_r$ be Borel. Then, by the continuity of r, any f in the closure of $m_r((t, \infty) \times A)$ in $C_0(S)$ satisfies that $r(f) \geq t$ and that $f/r(f)$ is in the closure of A. Therefore,

$$\partial m_r((t, \infty) \times A) \subset m_r((\{t\} \times A) \cup ([t, \infty) \times \partial A)) = m_r(\{t\} \times A) \cup m_r([t, \infty) \times \partial A) \subset C_r(S). \tag{19}$$

(a) The left-hand side in (a) equals

$$\frac{n\mathbb{P}\{n^{-1}X_n \in m_r((t, \infty) \times B)\}}{n\mathbb{P}\{n^{-1}X_n \in m_r((1, \infty) \times \mathbb{S}_r)\}}.$$

By Equation (19), the boundaries of the sets $m_r((t, \infty) \times B)) \subset C_r(S)$ and $m_r((1, \infty) \times \mathbb{S}_r)) \subset C_r(S)$ in $C_0(S)$ are contained in $m_r(\{t\} \times B) \cup m_r([t, \infty) \times \partial B) \subset C_r(S)$ and $m_r(\{1\} \times \mathbb{S}_r) \subset C_r(S)$, respectively, all of which are null-sets for $\Lambda\big|_{C_r(S)} = \Lambda_r$, cf. (13) and (14). Moreover, continuity and homogeneity of r ensure that both sets are d_0-bounded and, consequently, by (ii) of Theorem 2.10, the left-hand side in (a) convergences to

$$\frac{\Lambda_r(\{f \in C_r(S) \,:\, f/r(f) \in B,\, r(f) > t\})}{\Lambda_r(\{f \in C_r(S) \,:\, r(f) > 1\})} = \frac{\theta_r \sigma_r(B) t^{-1}}{\theta_r} = \mathbb{P}\{V^r \in B\}\, t^{-1}.$$

(b) Let $A_r = \{f \in C_0(S) \,:\, r(f) > a\} \subset C_r(S)$. For the limiting distribution we have the property $\mathbb{P}\{aPV^r \in A_r\} = 1$ and as for Λ there is no mass on the boundary of A_r, cf. Lemma 2.22 and Lemma 2.2. So let $B \subset A_r$ be a continuity set of $\mathbb{P}\{aPV^r \in \cdot\}$, then it is also a continuity set for Λ by Lemma 2.22. In addition, A_r and B are bounded away from $\{0\}$. Hence, by (ii) of Theorem 2.10,

$$\mathbb{P}\{n^{-1}X_n \in B \,|\, r(X_n) > an\} = \frac{n\mathbb{P}\{n^{-1}X_n \in B \cap A_r\}}{n\mathbb{P}\{n^{-1}X_n \in A_r\}} \to \frac{\Lambda(B \cap A_r)}{\Lambda(A_r)} = \mathbb{P}\{aPV^r \in B\},$$

where the last equality follows directly from Lemma 2.22.

□

Remark 2.24. If we evaluate Theorem 2.10 (ii) on the continuity set $\{f \in C_0(S) \,:\, r(f) > 1\}$, we obtain with (15) that

$$\theta_r = \theta\, \mathbb{E}[r(V)] = \Lambda\big(\{f \in C_0(S) \,:\, r(f) > 1\}\big) = \lim_{n\to\infty} n\, \mathbb{P}\{r(X_n) > n\},$$

cf. also Remark 2.9. This shows how the constant θ_r can be directly obtained from the law $\mathcal{L}(X_n)$ of each row in the triangular array from Theorem 2.10.

Instead of considering a d_0-boundedly finite (-1)-homogeneous measure Λ on $C_0(S)$ and investigating properties of its restriction to $C_r(S)$, suppose for a moment that we are only given a non-vanishing (-1)-homogeneous measure Λ_r of the form

$$\Lambda_r = (\nu_1 \otimes \theta_r \sigma_r) \circ m_r^{-1}$$

on $C_r(S)$ for a constant $\theta_r \in (0, \infty)$ and a probability measure σ_r on $\mathbb{S}_r$. The motivation for doing so is as follows. We have seen already in Lemma 2.4 that the LePage representation (7), i.e., the pointwise maximum process

$$Z = \theta \bigvee_{i=1}^{\infty} \Gamma_i^{-1} V_i,$$

derived from the measure Λ is a semi-simple max-stable process in $C_{\geq}(S)$. This respresentation of Z is supremum-normalized, since $\|V_i\|_\infty = 1$ almost surely, as each V_i is a copy of $V \sim \sigma$ with support on $\mathbb{S}$. Naturally, we may ask when a process of the form

$$Z^r = \theta_r \bigvee_{i=1}^{\infty} \Gamma_i^{-1} V_i^r, \tag{20}$$

where $\theta_r \in (0, \infty)$, $(\Gamma_i)_{i\in\mathbb{N}}$ are the arrival times of a renewal process on $(0, \infty)$ with standard exponential interarrival times, and independently, $(V_i^r)_{i\in\mathbb{N}}$ are independent copies of a stochastic process V^r distributed according to a probability measure σ_r on $\mathbb{S}_r$, is semi-simple max-stable in $C_{\geq}(S)$. This would imply that the max-series on the right-hand side of (20) is an r-normalized representation of an almost surely continuous semi-simple max-stable process Z^r. It turns out that the key to this question lies in the d_0-boundedly finiteness of the extension of Λ_r to $C_0(S)$ via zero mass on $C_0(S) \backslash C_r(S) = N_r(S) \backslash \{0\}$. Lemma 2.25 gives the characterization of this property in terms of $V^r \sim \sigma_r$. Note that in general a measure Λ_r of the form $\Lambda_r = (\nu_1 \otimes \theta_r \sigma_r) \circ m_r^{-1}$ need not even be d_r-boundedly finite, cf. Remark 2.15 and Remark 2.17, let alone d_0-boundedly finite.

Lemma 2.25. *Let $\theta_r \in (0,\infty)$ and let σ_r be a probability measure on $\mathbb{S}_r$ and $V^r \sim \sigma_r$. Let Λ be the extension of $\Lambda_r = (\nu_1 \otimes \theta_r\sigma_r) \circ m_r^{-1}$ to $C_0(S)$ by zero mass on $C_0(S) \setminus C_r(S) = N_r(S) \setminus \{0\}$.*

(a) Then Λ is d_0-boundedly finite if and only if

$$\mathbb{E}\big[\|V^r\|_\infty\big] = \mathbb{E}\Big[\sup_{s\in S} V^r(s)\Big] < \infty. \tag{21}$$

(b) If (21) is met, then $\Lambda = (\nu_1 \otimes \theta\sigma) \circ m^{-1}$, where

$$\theta = \Lambda_r(\{f \in C_r(S) \,:\, \|f\|_\infty > 1\}) = \theta_r\, \mathbb{E}[\|V^r\|_\infty] \in (0,\infty), \tag{22}$$

and

$$\sigma(B) = \frac{1}{\theta}\,\Lambda_r\big(\{f \in C_r(S) \,:\, \|f\|_\infty > 1\,,\; f/\|f\|_\infty \in B\}\big) \tag{23}$$

for Borel sets $B \subset \mathbb{S}$.

(c) If (21) is met, then the associated Poisson point process $\mathrm{PPP}(\Lambda)$ on $C_0(S)$ with mean measure Λ can be represented as the sum of random point masses

$$\sum_{i=1}^{\infty} \epsilon_{\theta_r \Gamma_i^{-1} V_i^r}, \tag{24}$$

where $(\Gamma_i)_{i\in\mathbb{N}}$ are the arrival times of a renewal process on $(0,\infty)$ with standard exponential interarrival times, and independently, $(V_i^r)_{i\in\mathbb{N}}$ are independent copies of V^r.

Proof. (a) By definition of Λ, we have that $\Lambda(B) = \Lambda_r(B \cap C_r(S))$ for Borel sets $B \subset C_0(S)$. So, the extension Λ is d_0-boundedly finite if and only if $\Lambda_r(\{f \in C_0(S) \,:\, \|f\|_\infty > a\} \cap C_r(S))$ is finite for all $a > 0$. By definition of Λ_r, we have for $a > 0$

$$\begin{aligned}\Lambda_r(\{f \in C_r(S) \,:\, \|f\|_\infty > a\}) &= \theta_r \int_0^\infty \sigma_r(\{f \in \mathbb{S}_r \,:\, \|tf\|_\infty > a\})\,\nu_1(\mathrm{d}t)\\ &= \theta_r \int_0^\infty \int_{\mathbb{S}_r} \mathbb{1}\{\|f\|_\infty > a/t\}\,\sigma_r(\mathrm{d}f)\nu_1(\mathrm{d}t),\end{aligned}$$

which may be finite or infinite. By Tonelli's theorem we may rewrite this expression as

$$\theta_r \int_{\mathbb{S}_r} \int_0^\infty \mathbb{1}\{\|f\|_\infty > a/t\}\,\nu_1(\mathrm{d}t)\sigma_r(\mathrm{d}f) = \theta_r\, \mathbb{E} \int_0^\infty \mathbb{1}\{\|V^r\|_\infty/a > x\}\,\mathrm{d}x = \frac{\theta_r}{a}\mathbb{E}\big[\|V^r\|_\infty\big],$$

hence the first assertion.

(b) If (21) is met, then Λ_r is the restriction of Λ to $C_r(S)$, while Λ is non-vanishing, d_0-boundedly finite and (-1)-homogeneous. Hence, we may represent Λ as in Lemma 2.1 as $(\nu_1 \otimes \theta\sigma) \circ m^{-1}$, where in this situation Λ has no mass on $N_r(S)$, so that (22) and (23) follow from Lemma 2.1 and the previous calculation from the proof of part (a).

(c) Since Λ has no mass on $N_r(S)$ and the restriction of Λ to the set $C_r(S) = C_0(S) \setminus N_r(S)$ is $\Lambda_r = (\nu_1 \otimes \theta_r\sigma_r) \circ m_r^{-1}$, the result follows similarly as the solution to Problem 7.2 in Kulik and Soulier (2020) from the transformation of points for Poisson processes, cf. Kulik and Soulier (2020, Proposition 7.1.12).

□

The decisive requirement in Lemma 2.25 is condition (21), which can be found in the literature in the context of simple max-stable processes in de Haan and Ferreira (2006, Corollary 9.4.5). In other words, condition (21) ensures that almost surely no points of the Poisson point process in (24) accumulate in proximity of $N_r(S)$ unless the accumulation point is $0 \in C_{\geq}(S)$, where we mean proximity with respect to the original supremum norm. In particular, a continuous max-stable processes can be constructed from the spectral processes V_i^r that are almost surely in $\mathbb{S}_r$, i.e. $r(V_i^r) = 1$ almost surely, rather than being confined to supremum-normalized processes V_i with $V_i \in \mathbb{S}$ almost surely as in Lemma 2.4.

Corollary 2.26. *Let $\Lambda_r = (\nu_1 \otimes \theta_r \sigma_r) \circ m_r^{-1}$ be as in Lemma 2.25, such that $V^r \sim \sigma_r$ satisfies condition (21). Furthermore, let $\sum_{i=1}^{\infty} \epsilon_{\theta_r \Gamma_i^{-1} V_i^r}$ be the point process as in Lemma 2.25 part (c). Then the pointwise maximum process*

$$Z^r = \theta_r \bigvee_{i=1}^{\infty} \Gamma_i^{-1} V_i^r,$$

is a semi-simple max-stable process in $C_{\geq}(S)$ with

$$\mathbb{P}\{r'(Z^r) \leq t\} = \exp\left(- \frac{\theta_r \, \mathbb{E}[r'(V^r)]}{t} \right), \qquad t > 0, \tag{25}$$

for any continuous and 1-homogeneous $r' : C_{\geq}(S) \to [0, \infty)$. Moreover, the positive constant θ_r and the probability measure σ_r on $\mathbb{S}_r$, i.e. the law of V^r, are both uniquely identified from the distribution of Z^r.

Proof. Let Λ be the zero mass extension of Λ_r as in Lemma 2.25. Then, by Lemma 2.25, Λ is d_0-boundedly finite and can be represented as $(\nu_1 \otimes \theta\sigma) \circ m^{-1}$, where θ is as in (22) and σ as in (23). Moreover, the point process $\sum_{i=1}^{\infty} \epsilon_{\theta_r \Gamma_i^{-1} V_i^r}$ is in fact a Poisson point process with mean measure Λ. Therefore, the law of Z^r coincides in fact with the law of the semi-simple max-stable process Z from Lemma 2.4 if the measure Λ therein coincides with Λ here. Sample continuity and max-stability of Z^r follow. Moreover,

$$\mathbb{P}\{r'(Z^r) \leq t\} = \mathbb{P}\{r'(Z) \leq t\} = \exp\left(- \frac{\theta \, \mathbb{E}[r'(V)]}{t} \right), \qquad t > 0,$$

where $V \sim \sigma$. Since Λ is the zero mass extension of Λ_r to $C_0(S)$, we have by Lemma 2.18 that $r(V) > 0$ almost surely. Hence, by applying Lemma 2.20 twice and finally (22), we arrive at

$$\theta \, \mathbb{E}[r'(V)] = \theta \, \mathbb{E}[r(V)] \mathbb{E}[r'(V^r)] = \frac{\theta}{\mathbb{E}[\|V^r\|_\infty]} \mathbb{E}[r'(V^r)] = \theta_r \mathbb{E}[r'(V^r)],$$

as desired.

To identify θ_r from the law of Z^r, we use Equation (25) with $r' = r$. Recall that $\mathbb{E}[r(V^r)] = 1$ since $V^r \in \mathbb{S}_r$ almost surely, hence $\theta_r = -\log \mathbb{P}\{r(Z^r) \leq 1\}$. Moreover, any non-negative bounded continuous function $\mathsf{b} : \mathbb{S}_r \to [0, \infty)$ arises as the restriction to $\mathbb{S}_r$ for a continuous and 1-homogeneous $r' : C_{\geq}(S) \to [0, \infty)$ by setting $r'(f) = 0$ for $f \in N_r(S)$ and $r'(f) = r(f)\mathsf{b}(f/r(f))$ for $f \in C_r(S)$, hence $\mathbb{E}[\mathsf{b}(V^r)] = -\log \mathbb{P}\{r'(Z^r) \leq 1\}/\theta_r$, which identifies the law of V^r by allowing b to vary. $\square$

Remark 2.27. The proof of Corollary 2.26 also reveals that the law of Z^r coincides with the law of Z from Lemma 2.4 if Λ therein is the zero mass extension of Λ_r to $C_0(S)$. Moreover, we have for $V^r \sim \sigma_r$ and $V \sim \sigma$ with σ as in (23) and θ as in (22) that

$$\theta_r \, \mathbb{E}\big[V^r(s)\big] = \theta \, \mathbb{E}\big[V(s)\big], \quad s \in S.$$

So the process Z^r is simple max-stable if and only if $\mathbb{E}\big[V^r(s)\big] = 1/\theta_r$ for all $s \in S$. If r is the point evaluation at $s_0 \in S$ and we denote $\theta_r = \theta_{s_0}$ and $V^r = V^{(s_0)}$, then $V \sim \sigma$ satisfies $V(s_0) > 0$ almost surely, so that with Lemma 2.20 $\mathbb{E}[V^{(s_0)}(s)] = \mathbb{E}[V(s)]/\mathbb{E}[V(s_0)] = (1/\theta)/(1/\theta) = 1$ for $s \in S$, and the condition $\mathbb{E}[V^{(s_0)}(s)] = 1/\theta_{s_0}$ means that we have necessarily $\theta_{s_0} = 1$.

3 Domain-scaled regular variation

If in all rows of the triangular array setup of Section 2, the processes $X_{n,i}$ on the compact domain $S \subset \mathbb{R}^d$ are independent copies a stochastic process X with unit Pareto margins, irrespective of the row index n, then Theorem 2.10 retrieves classical regular variation results and the process X is called regularly varying, cf. Dombry and Ribatet (2015). However, Theorem 2.10 also extends this framework and covers, for instance, the maxima convergence of suitably rescaled Gaussian processes to a Brown–Resnick process, cf. Section 5. In this case, a triangular array arises from scaling of the underlying domain, which motivates the following definition.

3.1 Definition for processes on compact domains

We start by defining domain-scaled regular variation for processes on compact convex domains $S \subset \mathbb{R}^d$.

Definition 3.1. Let $s_0 \in \mathbb{R}^d$ and $S \subset \mathbb{R}^d$ compact and convex such that $s_0 \in S$. Let X be a pathwise continuous stochastic process on S with standard Pareto margins. We call X *domain-scaled regularly varying on* S *with respect to* s_0 if there exists a *scaling sequence* $\delta_n \in [1, \infty)$, such that there exists a simple max-stable process Z with spectral representation (10) as in Theorem 2.6, and accordingly $\Lambda = (\nu_1 \otimes \theta\sigma) \circ m^{-1}$, $V \sim \sigma$ and P as in Theorem 2.10, such that one (hence any) of the equivalent conditions of Theorem 2.10 is satisfied when we set $X_n(s)$ therein equal to

$$X_n^{(s_0)}(s) = X\left(s_0 + \frac{s - s_0}{\delta_n}\right), \qquad s \in S, \tag{26}$$

and accordingly $X_{n,i}(s)$ equal to

$$X_{n,i}^{(s_0)}(s) = X_i\left(s_0 + \frac{s - s_0}{\delta_n}\right), \qquad s \in S,$$

where X_i, $1 \le i \le n$, are independent copies of X.

Remark 3.2 (Convexity of the domain)**.** In principle, one might define domain-scaled regular variation on star-shaped domains that are star-shaped with respect to the reference point s_0 rather than convex ones. This condition is necessary to ensure that $s_0 + (S - s_0)/\delta_n$ remains within S for all $n \ge 1$. Convexity of $S \subset \mathbb{R}^d$ ensures that S is star-shaped with respect to any reference point $s_0 \in S$. Starting from Section 3.2, we consider different reference points $s_0 \in S$, so that the convexity of the domain S becomes relevant.

If X is regularly varying on S in the classical sense, it is domain-scaled regularly varying with the constant scaling sequence $\delta_n \equiv 1$. Trivial examples arise, for instance, from max-stable processes, cf. Example 2.12. The point evaluation at s_0,

$$r_{s_0}(f) = f(s_0), \tag{27}$$

applied to the triangular array (26) always returns the same random variable

$$r_{s_0}\big(X_n^{(s_0)}\big) = X_n^{(s_0)}(s_0) = X(s_0) = r_{s_0}(X),$$

i.e., the scaling operation in (26) does not change the outcome of r_{s_0}, which is simply $X(s_0)$. Consequently, the term $r_{s_0}\big(X_n^{(s_0)}\big)$, which appears in Proposition 2.23 in the conditioning part as $r(X_n)$, depends only on X and s_0, and not on $n \in \mathbb{N}$, so that Corollary 3.3 is a direct consequence of Proposition 2.23 if we plug in as risk functional r the point evaluation r_{s_0}. In what follows, we use the following suggestive notation when $r = r_{s_0}$:

$$\theta_{s_0} = \theta_r,\ \sigma_{s_0} = \sigma_r,\ \mathbb{S}_{s_0} = \mathbb{S}_r \text{ and } V^{(s_0)} = V^r,$$

and we call $V^{(s_0)}$ the *point-spectral process* of X *with respect to* $s_0 \in S$.

Corollary 3.3. *Let X be a stochastic process with standard Pareto margins that is domain-scaled regularly varying on a compact convex $S \subset \mathbb{R}^d$ with respect to $s_0 \in S$ and let $X_n^{(s_0)}$ be as in (26). Let P be a standard Pareto variable independent of $V^{(s_0)}$. Then we have the following implications:*

(a) For all $t > 0$ and Borel sets $B \subset \mathbb{S}^{(s_0)}$ with $\mathbb{P}\{V^{(s_0)} \in \partial B\} = 0$

$$n\,\mathbb{P}\left\{\frac{X_n^{(s_0)}}{X(s_0)} \in B\,,\ X(s_0) > nt\right\} \to \mathbb{P}\big\{V^{(s_0)} \in B\big\}\,t^{-1} \qquad \text{as} \quad n \to \infty.$$

(b) $\mathcal{L}\big(n^{-1}X_n^{(s_0)} \,\big|\, X(s_0) > na\big) \to \mathcal{L}\big(aPV^{(s_0)}\big)$ weakly in $C_{\ge}(S)$ for all $a > 0$.

Part (a) and (b) of Corollary 3.3 are equivalent to the weak convergences

$$\mathcal{L}\bigg(\frac{X_n^{(s_0)}}{X(s_0)}\,\bigg|\,X(s_0)>nt\bigg)\to\mathcal{L}\big(V^{(s_0)}\big)\qquad\text{and}\qquad\mathcal{L}\bigg(\frac{X_n^{(s_0)}}{nt}\,\bigg|\,X(s_0)>nt\bigg)\to\mathcal{L}\big(PV^{(s_0)}\big),\tag{28}$$

respectively, as $n\to\infty$. By definition, the point-spectral process $V^{(s_0)}$ satisfies $V^{(s_0)}(s_0)=1$ almost surely, and

$$\theta_{s_0}=\Lambda(\{f\in C_0(S)\,:\,f(s_0)>1\})=\theta\,\mathbb{E}[V(s_0)]=1,$$

where the last equality is due to the marginal standardization in Theorems 2.6 or 2.10, respectively. Further, Lemma 2.20 gives that

$$\mathbb{E}\big[V^{(s_0)}(s)\big]=\frac{\mathbb{E}[V(s)\mathbb{1}\{V(s_0)>0\}]}{\mathbb{E}[V(s_0)]},\qquad s\in S,$$

which simplifies to $\mathbb{E}[V^{(s_0)}(s)]=1$ if $V(s_0)>0$ almost surely.

3.2 Consequences under stationarity

The following lemma summarises a few consequences when the domain-scaling process X is stationary on S, i.e. when the distribution of

$$\big\{X(s_1+h)\big\}_{h\in(S-s_1)\cap(S-s_2)}\qquad\text{and}\qquad\big\{X(s_2+h)\big\}_{h\in(S-s_1)\cap(S-s_2)}$$

coincide for any $s_1,s_2\in S$.

Lemma 3.4. *Let X be as in Corollary 3.3 and assume that X is stationary on S. Let $s_0,s_1,s_2\in S$ with $s_1\neq s_2$. Let P be a standard Pareto variable independent of $V^{(s_1)}$ and $V^{(s_2)}$. Then the following holds:*

(a) We have the distributional equality

$$\mathcal{L}\Big(\big\{V^{(s_1)}(s_1+h)\big\}_{h\in(S-s_1)\cap(S-s_2)}\Big)=\mathcal{L}\Big(\big\{V^{(s_2)}(s_2+h)\big\}_{h\in(S-s_1)\cap(S-s_2)}\Big).\tag{29}$$

For $s_1,s_2\in S$ and any $t>0$

$$\mathbb{P}\big\{PV^{(s_1)}(s_2)>t\big\}=t^{-1}\,\mathbb{P}\big\{PV^{(s_2)}(s_1)>t^{-1}\big\},\tag{30}$$

and

$$\mathbb{E}\big[V^{(s_1)}(s_2)\big]=\mathbb{P}\big\{V^{(s_2)}(s_1)>0\big\}.\tag{31}$$

For any $t_1,t_2\geq 1$

$$\mathbb{P}\big\{PV^{(s_1)}(s_1)>t_1,PV^{(s_1)}(s_2)>t_2\big\}=\mathbb{P}\big\{PV^{(s_2)}(s_1)>t_1,PV^{(s_2)}(s_2)>t_2\big\}.\tag{32}$$

(b) Suppose that, additionally, all the bivariate distributions of X are symmetric, i.e. $\mathcal{L}(X(s),X(s'))=\mathcal{L}(X(s'),X(s))$ for all $s,s'\in S$. Then for any $h\in(S-s_0)\cap(-S+s_0)$

$$\mathcal{L}\big(V^{(s_0)}(s_0+h)\big)=\mathcal{L}\big(V^{(s_0)}(s_0-h)\big).\tag{33}$$

For any $t\geq 0$

$$\mathbb{P}\big\{PV^{(s_1)}(s_2)>t\big\}=\mathbb{P}\big\{PV^{(s_2)}(s_1)>t\big\},\tag{34}$$

and

$$\mathbb{E}\big[V^{(s_1)}(s_2)\big]=\mathbb{P}\big\{V^{(s_1)}(s_2)>0\big\}.\tag{35}$$

For any $t_1,t_2\geq 1$

$$\mathbb{P}\big\{PV^{(s_1)}(s_1)>t_1,PV^{(s_1)}(s_2)>t_2\big\}=\mathbb{P}\big\{PV^{(s_2)}(s_1)>t_2,PV^{(s_2)}(s_2)>t_1\big\}.\tag{36}$$

Proof. (a) The distributional equality (29) follows directly from (28) and the stationarity of X. By Lemma 2.22 and Lemma 2.2, the set $\{f \in C_0(S) \,:\, f(s_i) > u\}$ is a continuity set of $PV^{(s_j)}$ for any $u > 0$ and $(i,j) \in \{(1,2),(2,1)\}$. So Corollary 3.3 (b) gives

$$\begin{aligned}\mathbb{P}\big\{PV^{(s_1)}(s_2) > t\big\} &= \lim_{n\to\infty} \mathbb{P}\big\{X_n^{(s_1)}(s_2) > nt \,\big|\, X(s_1) > n\big\} \\ &= \lim_{n\to\infty} n\,\mathbb{P}\bigg\{X\bigg(s_1 + \frac{s_2 - s_1}{\delta_n}\bigg) > nt\,,\, X(s_1) > n\bigg\} \\ &= t^{-1} \lim_{n\to\infty} nt\,\mathbb{P}\bigg\{X(s_1) > n\,,\, X\bigg(s_1 + \frac{s_2 - s_1}{\delta_n}\bigg) > nt\bigg\}\end{aligned}$$

Since X is stationary,

$$\mathbb{P}\big\{PV^{(s_1)}(s_2) > t\big\} = t^{-1} \lim_{n\to\infty} nt\,\mathbb{P}\bigg\{X(s_1 + h) > n\,,\, X\bigg(s_1 + \frac{s_2 - s_1}{\delta_n} + h\bigg) > nt\bigg\}$$

for any h such that $s_1 + h$ and $s_1 + (s_2 - s_1)/\delta_n + h$ lie in S, e.g. for $h = (1 - 1/\delta_n)(s_2 - s_1)$, which gives

$$\begin{aligned}\mathbb{P}\big\{PV^{(s_1)}(s_2) > t\big\} &= t^{-1} \lim_{n\to\infty} nt\,\mathbb{P}\bigg\{X\bigg(s_2 + \frac{s_1 - s_2}{\delta_n}\bigg) > n\,,\, X(s_2) > nt\bigg\} \\ &= t^{-1} \lim_{n\to\infty} \mathbb{P}\big\{X_n^{(s_2)}(s_1) > n \,\big|\, X(s_2) > nt\big\} = t^{-1}\mathbb{P}\big\{tPV^{(s_2)}(s_1) > 1\big\},\end{aligned}$$

where the last identity is again implied by Corollary 3.3 (b). This shows (30).

Using (30) (with exchanged roles for s_1 and s_2) and Lemma A.2 we obtain

$$\mathbb{E}\big[V^{(s_1)}(s_2)\big] = \lim_{t\to 0} \mathbb{P}\big\{PV^{(s_2)}(s_1) > t\big\} = \mathbb{P}\big\{PV^{(s_2)}(s_1) > 0\big\},$$

hence (31) with $P \geq 1$ almost surely.

The identity (32) means in fact

$$\mathbb{P}\big\{P > t_1, PV^{(s_1)}(s_2) > t_2\big\} = \mathbb{P}\big\{PV^{(s_2)}(s_1) > t_1, P > t_2\big\}, \tag{37}$$

since $V^{(s)}(s) = 1$ for all $s \in S$ almost surely by definition. Using the threshold-stability of P, the left-hand-side of (37) equals

$$\mathbb{P}\big\{P > t_1, PV^{(s_1)}(s_2) > t_2\big\} = t_1^{-1}\mathbb{P}\big\{PV^{(s_1)}(s_2) > t_2 \,|\, P > t_1\big\} = t_1^{-1}\mathbb{P}\big\{t_1 PV^{(s_1)}(s_2) > t_2\big\},$$

and the same calculation means the right-hand side of (37) equals $t_2^{-1}\mathbb{P}\big\{t_2 PV^{(s_2)}(s_1) > t_1\big\}$, so that their equality follows directly from (30).

(b) The distributional equality (33) follows directly from (28) together with the stationarity and symmetry of X; more precisely we employ in (28) that

$$\mathcal{L}\big(X(s_0 + h/\delta_n), X(s_0)\big) = \mathcal{L}\big(X(s_0), X(s_0 - h/\delta_n)\big) = \mathcal{L}\big(X(s_0 - h/\delta_n), X(s_0)\big).$$

In order to show (34), we use again

$$\mathbb{P}\big\{PV^{(s_1)}(s_2) > t\big\} = \lim_{n\to\infty} n\,\mathbb{P}\bigg\{X\bigg(s_1 + \frac{s_2 - s_1}{\delta_n}\bigg) > nt\,,\, X(s_1) > n\bigg\},$$

where due to X's bivariate symmetry and stationarity

$$\begin{aligned}\mathbb{P}\big\{PV^{(s_1)}(s_2) > t\big\} &= \lim_{n\to\infty} n\,\mathbb{P}\bigg\{X(s_1) > nt\,,\, X\bigg(s_1 + \frac{s_2 - s_1}{\delta_n}\bigg) > n\bigg\} \\ &= \lim_{n\to\infty} n\,\mathbb{P}\bigg\{X\bigg(s_2 + \frac{s_1 - s_2}{\delta_n}\bigg) > nt\,,\, X(s_2) > n\bigg\} = \mathbb{P}\big\{PV^{(s_2)}(s_1) > t\big\}.\end{aligned}$$

In this situation, the equality extends to $t = 0$, so that (35) follows from (31).

As for (36), recall that

$$\mathbb{P}\big\{P > t_1, PV^{(s_1)}(s_2) > t_2\big\} = t_1^{-1}\mathbb{P}\big\{PV^{(s_1)}(s_2) > t_2 \,|\, P > t_1\big\} = t_1^{-1}\mathbb{P}\big\{t_1 PV^{(s_1)}(s_2) > t_2\big\}.$$

Likewise, $\mathbb{P}\big\{P > t_1, PV^{(s_2)}(s_1) > t_2\big\} = t_1^{-1}\mathbb{P}\big\{t_1 PV^{(s_2)}(s_1) > t_2\big\}$. So (34) gives

$$\mathbb{P}\big\{P > t_1, PV^{(s_1)}(s_2) > t_2\big\} = \mathbb{P}\big\{P > t_1, PV^{(s_2)}(s_1) > t_2\big\},$$

hence (36). □

3.3 Definition for processes on $\mathbb{R}^d$

So far, we have only considered stochastic processes defined on a fixed compact subset $S \subset \mathbb{R}^d$. In what follows, we will need to revert the domain-scaling and work with stochastic processes on $\mathbb{R}^d$ in order to define an appropriate tail approximating process for a domain-scaled regularly varying process. We start by explicitly stating how the set-specific domain-scaled regular variation is compatible with restrictions to a smaller subset.

Lemma 3.5. *Let X be a pathwise continuous stochastic process on a compact convex $S \subset \mathbb{R}^d$ with standard Pareto margins that is domain-scaled regularly varying with respect to $s_0 \in S$ for the scaling sequence $\delta_n \in [1, \infty)$. Let $V^{(s_0)}$ be the corresponding point-spectral process on S. Let $S' \subset S$ be a compact convex subset with $s_0 \in S'$. Then the restriction of X to S' is domain-scaled regularly varying on S' with respect to s_0 for the same scaling sequence $\delta_n \in [1, \infty)$, and the corresponding point-spectral process for the restriction of X to S' is the restriction of $V^{(s_0)}$ to S'.*

Proof. That the restriction of X to S' is domain-scaled regularly varying as stated follows from Definition 3.1 and the characterization of this property via part (i) of Theorem 2.10. The second implication follows then from Corollary 3.3, cf. also (28). □

The following definition extends the concept of domain-scaled regular variation to processes on $\mathbb{R}^d$.

Definition 3.6. Let X be a pathwise continuous stochastic process on $\mathbb{R}^d$ with standard Pareto margins. We call X *domain-scaled regularly varying with respect to $s_0 \in \mathbb{R}^d$ for the scaling sequence $\delta_n \in [1, \infty)$* if it is domain-scaled regularly varying on S with respect to s_0 for the scaling sequence δ_n for *any* compact convex $S \subset \mathbb{R}^d$ such that $s_0 \in S$.

Remark 3.7. In particular, if a stationary stochastic process X on $\mathbb{R}^d$ is domain-scaled regularly varying with respect to *some* $s_0 \in \mathbb{R}^d$, then it is domain-scaled regularly varying with respect to *any* $s_0 \in \mathbb{R}^d$.

Lemma 3.8. *Let X be domain-scaled regularly varying on $\mathbb{R}^d$ with respect to $s_0 \in \mathbb{R}^d$ for the scaling sequence $\delta_n \in [1, \infty)$ as in Definition 3.6. Then there exists a pathwise continuous stochastic process $V^{(s_0)}$ on $\mathbb{R}^d$, such that the restriction of $V^{(s_0)}$ to S is the point-spectral process of the restriction of X to S.*

Proof. Let $S_1 \subset S_2 \subset \ldots$ be a series of compact convex subsets of $\mathbb{R}^d$ with $\bigcup_{k \in \mathbb{N}} S_k = \mathbb{R}^d$ and $s_0 \in S_1$. For each $k \in \mathbb{N}$, let $V_k^{(s_0)}$ be the point-spectral process of the restriction of X to S_k, which exists due to the domain-scaled regular variation of X on $\mathbb{R}^d$. Due to Lemma 3.5 we have that for $k \leq n$ the law of the process $V_k^{(s_0)}$ is the law of the restriction of $V_n^{(s_0)}$ to S_k. The existence of the law of $V^{(s_0)}$ is then ensured as the projective limit of the respective probability measures of $V_k^{(s_0)}$, $k \in \mathbb{N}$, cf. Corollary 6.15 of Kallenberg (2002). It necessarily satisfies all the stated properties, since for any compact convex $S \subset \mathbb{R}^d$ with $s_0 \in S$, we can find an index $k \in \mathbb{N}$, such that $S \subset S_k$, so that we may apply Lemma 3.5 again. □

Hence, if X is domain-scaled regularly varying on $\mathbb{R}^d$ with respect to a reference point $s_0 \in \mathbb{R}^d$, then there exists a *global point-spectral process* $V^{(s_0)}$ on $\mathbb{R}^d$ for X that is compatible with respect to restrictions to compact convex subsets containing s_0. When no ambiguity arises, we may omit the term global. In particular, $V^{(s_0)}(s_0) = 1$ almost surely. If, in addition, X is stationary, then Lemma 3.4 implies that for any $s_0 \in \mathbb{R}^d$

$$\mathcal{L}\Big(\big\{V^{(s_0)}(s_0 + h)\big\}_{h \in \mathbb{R}^d}\Big) = \mathcal{L}\Big(\big\{V^{(0)}(h)\big\}_{h \in \mathbb{R}^d}\Big), \tag{38}$$

and so the distribution of any point-spectral processes $V^{(s_0)}$ can be recovered from the point spectral process $V^{(0)}$ with reference point the origin $0 \in \mathbb{R}^d$. If, furthermore, the bivariate distributions of X are symmetric, Lemma 3.4 implies that

$$\mathcal{L}\big(V^{(0)}(h)\big) = \mathcal{L}\big(V^{(0)}(-h)\big), \qquad h \in \mathbb{R}^d. \tag{39}$$

Remark 3.9 (Connection to Basrak et al. (2025)). Let $C_{\geq}(\mathbb{R}^d)$ be the space of continuous non-negative function on $\mathbb{R}^d$ equipped with the topology of uniform convergence on compact sets, and let $\mathbb{S}^{(s_0)}(\mathbb{R}^d) = \{f \in C_{\geq}(\mathbb{R}^d) \,:\, f(s_0) = 1\}$. Define the multiplication map

$$m_{s_0} : (0, \infty) \times \mathbb{S}^{(s_0)}(\mathbb{R}^d) \to C_{\geq}(\mathbb{R}^d), \qquad m_{s_0}(t, f) = tf.$$

Then, a pathwise continuous stochastic process X on $\mathbb{R}^d$ with standard Pareto margins is domain-scaled regularly varying with respect to $s_0 \in \mathbb{R}^d$ for the scaling sequence δ_n (as in Definition 3.6) if and only if there exists a continuous stochastic process $V^{(s_0)}$ on $\mathbb{R}^d$ with $V^{(s_0)}(s_0) = 1$ almost surely, such that the sequence of measures

$$n\mathbb{P}\left\{n^{-1}X\left(s_0 + \frac{\cdot - s_0}{\delta_n}\right) \in \cdot\right\} \qquad \text{converges vaguely to} \qquad \left(\nu_1 \otimes \mathcal{L}\big(V^{(s_0)}\big)\right) \circ m_{s_0}^{-1}$$

in the sense of Section 4.4.4. of Basrak et al. (2025). More precisely, this means that the convergence holds for continuity sets of the limit from the metric exclusion ideal of the zero function with respect to the Fréchet metric induced by the supremum norms of a sequence of compact subsets $S_1 \subset S_2 \subset \dots$ with $\bigcup_{k\in\mathbb{N}} S_k = \mathbb{R}^d$ and $s_0 \in S_1$. If this is the case, the law of $V^{(s_0)}$ coincides necessarily with the law of the global point-spectral process from Lemma 3.8.

However, note that domain-scaled regular variation is usually not a form of regular variation in the sense of Definition 3.14 of Basrak et al. (2025), as the operation

$$T_t(f) = tf(s_0 + \delta(t)(\cdot - s_0))$$

usually does not satisfy the semigroup property $T_s \circ T_t = T_{st}$, unless $\delta_s \delta_t = \delta_{st}$. In what follows, this requirement is only met for the exceptional case $\delta \equiv 1$, see Assumption 4.3.

4 A domain-scaled tail approximating process

Classical regular variation motivates a widely known tail process, the Pareto process $PV^{(s_0)}$, that is used in order to approximate the behaviour of a regularly varying process X around s_0 when $X(s_0)$ is large. Corollary 3.3 enables to extend this class by inversion of the inner domain-scaling, such that the rescaled Pareto process mimics again locally the original process X around s_0. To do so, we replace scaling sequences by scaling functions: a function $\delta : [1, \infty) \to [1, \infty)$ *interpolates* a sequence δ_n, $n \in \mathbb{N}$, if $\delta(u) \in [\delta_{\lfloor u \rfloor}, \delta_{\lceil u \rceil}]$ for $u \in [1, \infty)$. In particular, $\delta(n) = \delta_n$ for $n \in \mathbb{N}$.

Definition 4.1. Let X be domain-scaled regularly varying on $\mathbb{R}^d$ with respect to $s_0 \in \mathbb{R}^d$ for the scaling sequence δ_n and with global point-spectral process $V^{(s_0)}$ on $\mathbb{R}^d$. Let $\delta : [1, \infty) \to [1, \infty)$ be a function that interpolates the sequence δ_n. And let P be a standard Pareto random variable independent from $V^{(s_0)}$. Based on these quantities, we define the *(domain-scaled) tail approximating process $Y^{(s_0)}$ for X at s_0* as follows:

$$Y^{(s_0)}(s) = P\, V^{(s_0)}\big(s_0 + \delta(P)(s - s_0)\big), \quad s \in \mathbb{R}^d.$$

At the reference point s_0, the tail approximating process $Y^{(s_0)}$ is standard Pareto, i.e. $Y^{(s_0)}(s_0) = P$. Otherwise, for (P-almost all) $u \geq 1$, the law of the tail approximating process $Y^{(s_0)}$ conditional on this value is given by

$$\mathcal{L}\big(Y^{(s_0)} \,\big|\, Y^{(s_0)}(s_0) = u\big) = \mathcal{L}\Big(\big\{uV^{(s_0)}\big(s_0 + \delta(u)(s - s_0)\big)\big\}_{s\in\mathbb{R}^d}\Big). \tag{40}$$

Conditioning on being larger than a threshold $u \geq 1$ leads instead to

$$\begin{aligned}
\mathcal{L}\big(Y^{(s_0)} \,\big|\, Y^{(s_0)}(s_0) > u\big) &= \mathcal{L}\left(\left\{P\, V^{(s_0)}\Big(s_0 + \delta(P)(s - s_0)\Big)\right\}_{s\in\mathbb{R}^d} \,\middle|\, P > u\right) \\
&= \mathcal{L}\left(\left\{uP\, V^{(s_0)}\Big(s_0 + \delta(uP)(s - s_0)\Big)\right\}_{s\in\mathbb{R}^d}\right).
\end{aligned} \tag{41}$$

Since $\delta(1) = 1$ and $Y^{(s_0)}(s_0)$ is standard Pareto,

$$\mathcal{L}\big(Y^{(s_0)} \,\big|\, Y^{(s_0)}(s_0) = 1\big) = \mathcal{L}\big(V^{(s_0)}\big) \qquad \text{and} \qquad \mathcal{L}\big(Y^{(s_0)} \,\big|\, Y^{(s_0)}(s_0) > 1\big) = \mathcal{L}\big(Y^{(s_0)}\big).$$

Figures 1 and 2 illustrate the localizing effect that the domain-scaling can have on a tail approximating process $Y^{(s_0)}$ when the conditioning threshold increases. Figure 1 displays sample paths of (40) when the influence of the domain-scaling becomes significant, while Figure 2 serves as reference and displays the exact same samples with the classical tail process $PV^{(s_0)}$.

4.1 Stability properties

By definition, the tail approximating process $Y^{(s_0)}$ satisfies a range of stability properties. The following stability relations are an immediate consequence of (40) and (41), respectively.

Lemma 4.2. *Let $u_0 \geq 1$ and $u \geq 1$. Then the conditional laws of $Y^{(s_0)}$ conditioned on either $Y^{(s_0)}(s_0) = u_0$ or $Y^{(s_0)}(s_0) = u$ are related as follows*

$$\mathcal{L}\big(Y^{(s_0)} \,\big|\, Y^{(s_0)}(s_0) = u_0\big) = \mathcal{L}\bigg(\bigg\{\frac{u_0}{u}\, Y^{(s_0)}\bigg(s_0 + \frac{\delta(u_0)}{\delta(u)}(s - s_0)\bigg)\bigg\}_{s\in\mathbb{R}^d} \,\bigg|\, Y^{(s_0)}(s_0) = u\bigg),$$

while the relation between conditioning on either $Y^{(s_0)}(s_0) > u_0$ or $Y^{(s_0)}(s_0) > u$ reads as

$$\mathcal{L}\big(Y^{(s_0)} \,\big|\, Y^{(s_0)}(s_0) > u_0\big) = \mathcal{L}\bigg(\bigg\{\frac{u_0}{u}\, Y^{(s_0)}\bigg(s_0 + \frac{\delta\big(\frac{u_0}{u} Y^{(s_0)}(s_0)\big)}{\delta\big(Y^{(s_0)}(s_0)\big)}(s - s_0)\bigg)\bigg\}_{s\in\mathbb{R}^d} \,\bigg|\, Y^{(s_0)}(s_0) > u\bigg).$$

In particular, for $u = 1$,

$$\begin{aligned}
\mathcal{L}\big(Y^{(s_0)} \,\big|\, Y^{(s_0)}(s_0) = u_0\big) &= \mathcal{L}\Big(\big\{u_0\, Y^{(s_0)}\big(s_0 + \delta(u_0)(s - s_0)\big)\big\}_{s\in\mathbb{R}^d} \,\big|\, Y^{(s_0)}(s_0) = 1\Big),\\
\mathcal{L}\big(Y^{(s_0)} \,\big|\, Y^{(s_0)}(s_0) > u_0\big) &= \mathcal{L}\bigg(\bigg\{u_0\, Y^{(s_0)}\bigg(s_0 + \frac{\delta\big(u_0 Y^{(s_0)}(s_0)\big)}{\delta\big(Y^{(s_0)}(s_0)\big)}(s - s_0)\bigg)\bigg\}_{s\in\mathbb{R}^d}\bigg).
\end{aligned}$$

For $\delta \equiv 1$, when we have no scaling of the domain, the stability relations from Lemma 4.2 simplify to classical identities for standard tail processes

$$\begin{aligned}
\mathcal{L}\big(Y^{(s_0)} \,\big|\, Y^{(s_0)}(s_0) = u_0\big) &= \mathcal{L}\bigg(\frac{u_0}{u}\, Y^{(s_0)} \,\bigg|\, Y^{(s_0)}(s_0) = u\bigg) = \mathcal{L}\big(u_0\, Y^{(s_0)} \,\big|\, Y^{(s_0)}(s_0) = 1\big),\\
\mathcal{L}\big(Y^{(s_0)} \,\big|\, Y^{(s_0)}(s_0) > u_0\big) &= \mathcal{L}\bigg(\frac{u_0}{u}\, Y^{(s_0)} \,\bigg|\, Y^{(s_0)}(s_0) > u\bigg) = \mathcal{L}\big(u_0\, Y^{(s_0)}\big).
\end{aligned}$$

4.2 Domain-scaled regular variation of the tail approximating process

Henceforth, we will often assume the following regular behaviour of the scaling function δ:

Assumption 4.3. *The scaling function $\delta : [1, \infty) \to [1, \infty)$ is normalized with $\delta(1) = 1$, and it is non-decreasing and slowly varying, i.e.*

$$\lim_{n\to\infty} \frac{\delta(nu)}{\delta(n)} = 1, \quad u > 0.$$

In particular, this gives

$$\lim_{u\to\infty} \delta(u) = \sup_{u\in[1,\infty)} \delta(u) \in [1, \infty].$$

Assumption 4.3 ensures that, apart from marginal standardization, the tail approximating process $Y^{(s_0)}$ is domain-scaled regularly varying. Let $S \subset \mathbb{R}^d$ be compact convex with $s_0 \in S$ and recall that $\mathbb{S}^{(s_0)} = \{f \in C(S, [0, \infty)) \,:\, f(s_0) = 1\}$. Denote the restriction of $V^{(s_0)}$ to S by $V^{(s_0)}\big|_S$. Similarly to (26), let

$$Y_n^{(s_0)}(s) = Y^{(s_0)}\bigg(s_0 + \frac{s - s_0}{\delta(n)}\bigg), \quad s \in \mathbb{R}^d.$$

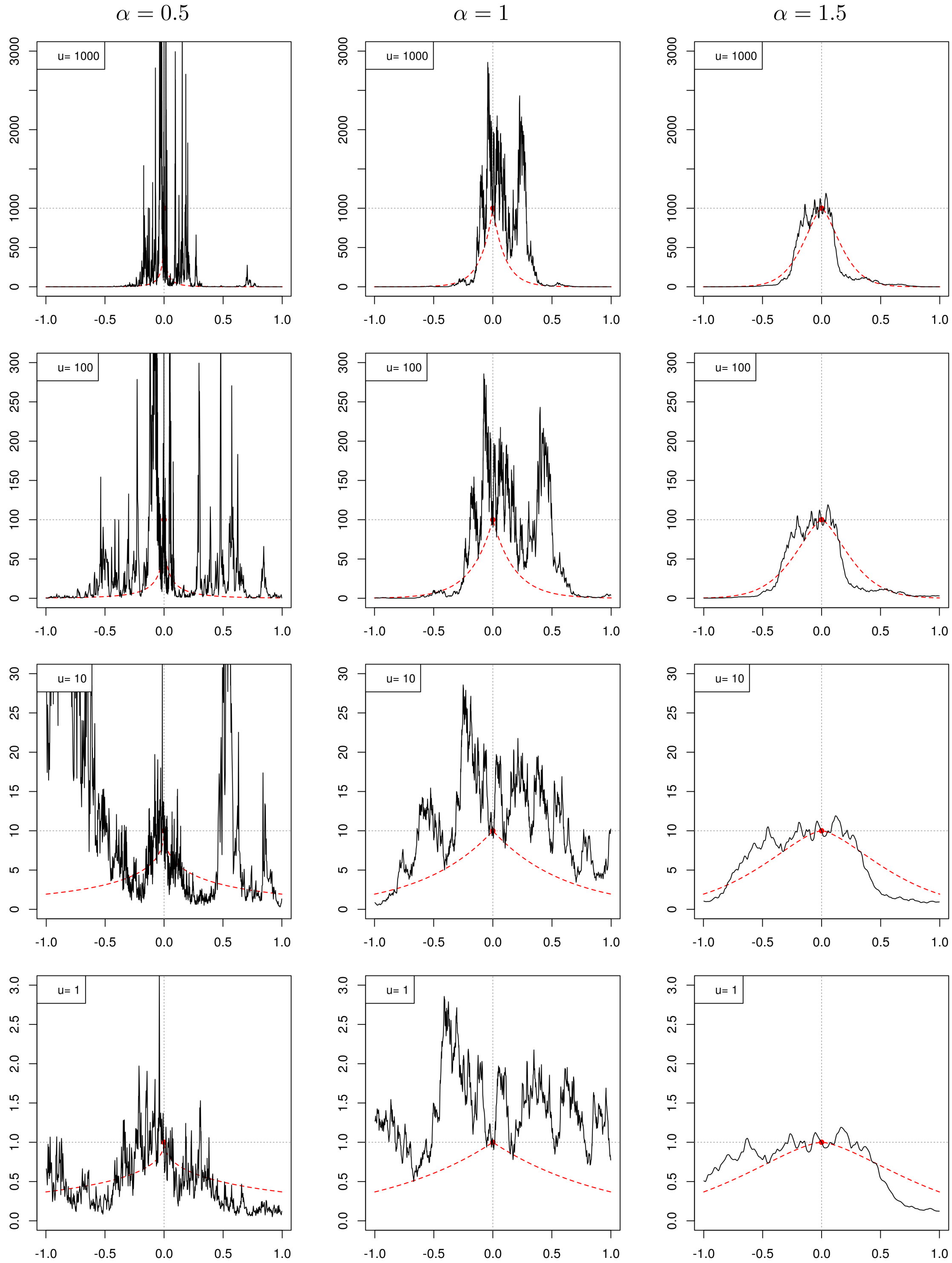

Figure 1: Each plot contains one sample path from a conditioned tail approximating process $Y^{(0)} \mid Y^{(0)}(0) = u$ from (40), where the underlying point-spectral process $V^{(0)}$ is of log-Gaussian type as in (57) based on the semi-variogram $\gamma(h) = |h|^\alpha$ for $\alpha \in \{0.5, 1, 1.5\}$ (from left to right), and the scaling function δ is chosen as in (61). The conditioning threshold u ranges from 1 to 1000 (99.99%-quantile of the Pareto variable $Y^{(0)}(0)$). Each sample path passes through the level u at 0, as indicated by a red dot in the background. Red dashed lines illustrate the multiplicative trends given by $h \mapsto u \exp(-\gamma(\delta(u)h))$. The extreme events around the conditioning location 0 become more localized as u increases — contrasting classical tail process behaviour without domain-scaling, cf. Figure 2.

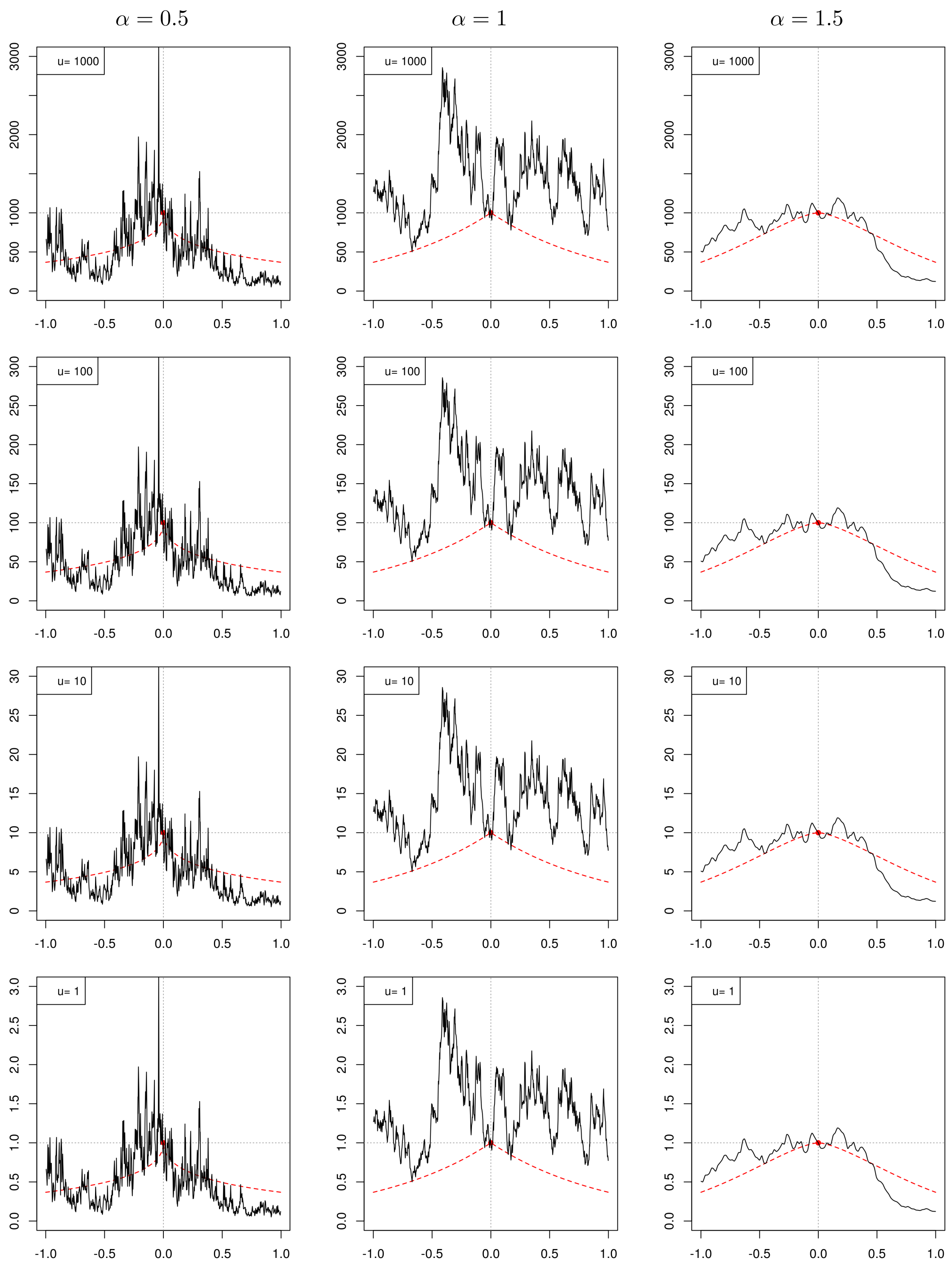

Figure 2: Each plot contains one sample path from a conditioned classical tail process $Y^{(0)} \mid Y^{(0)}(0) = u$ from (40) without domain-scaling, where the underlying point-spectral process $V^{(0)}$ is of log-Gaussian type as in (57) based on the semi-variogram $\gamma(h) = |h|^\alpha$ for $\alpha \in \{0.5, 1, 1.5\}$ (from left to right). The conditioning threshold u ranges from 1 to 1000 (99.99%-quantile of the Pareto variable $Y^{(0)}(0)$). Each sample path passes through the level u at 0, as indicated by a red dot in the background. Red dashed lines illustrate the multiplicative trends given by $h \mapsto u \exp(-\gamma(h))$. The probabilistic behaviour of extreme events around the conditioning location 0 does not change as u increases — contrasting the localization when we consider domain-scaling, cf. Figure 1.

In particular, the value at the reference point does not change through this inner scaling operation, i.e.

$$Y_n^{(s_0)}(s_0) = Y^{(s_0)}(s_0).$$

The following result is analogous to Corollary 3.3.

Proposition 4.4. *Under Assumption 4.3 the tail approximating process $Y^{(s_0)}$ satisfies:*

(a) For all $t > 0$ and Borel sets $B \subset \mathbb{S}^{(s_0)}$ with $\mathbb{P}\{V^{(s_0)}|_S \in \partial B\} = 0$

$$n\,\mathbb{P}\bigg\{\frac{Y_n^{(s_0)}|_S}{Y^{(s_0)}(s_0)} \in B\,,\, Y^{(s_0)}(s_0) > nt\bigg\} \to \mathbb{P}\{V^{(s_0)}|_S \in B\}\, t^{-1} \qquad \text{as} \quad n \to \infty.$$

(b) $\mathcal{L}\big(n^{-1}Y_n^{(s_0)}|_S \,\big|\, Y^{(s_0)}(s_0) > na\big) \to \mathcal{L}\big(aPV^{(s_0)}|_S\big)$ weakly for all $a > 0$.

Proof. (a) Consider n large enough such that $nt \geq 1$. Then

$$\begin{aligned}
n\,\mathbb{P}\bigg\{\frac{Y_n^{(s_0)}|_S}{Y^{(s_0)}(s_0)} \in B\,,\, Y^{(s_0)}(s_0) > nt\bigg\} &= n\int_{nt}^{\infty} \mathbb{P}\bigg\{\frac{Y_n^{(s_0)}|_S}{Y^{(s_0)}(s_0)} \in B\,\bigg|\, Y^{(s_0)}(s_0) = u\bigg\}\nu_1(\mathrm{d}u)\\
&= \int_t^{\infty} \mathbb{P}\bigg\{\frac{Y_n^{(s_0)}|_S}{Y^{(s_0)}(s_0)} \in B\,\bigg|\, Y^{(s_0)}(s_0) = nu\bigg\}\nu_1(\mathrm{d}u)\\
&= \int_t^{\infty} \mathbb{P}\bigg\{\bigg\{V^{(s_0)}\bigg(s_0 + \frac{\delta(nu)}{\delta(n)}(s - s_0)\bigg)\bigg\}_{s\in S} \in B\bigg\}\nu_1(\mathrm{d}u).
\end{aligned}$$

Now, for any $s \in S$ we have that

$$s_0 + \frac{\delta(nu)}{\delta(n)}(s - s_0) = s + \bigg(\frac{\delta(nu)}{\delta(n)} - 1\bigg)(s - s_0),$$

where the norm of $s - s_0$ is bounded by the diameter of S and $\delta(nu)/\delta(n) - 1$ converges to 0 since δ is slowly varying. Hence,

$$V^{(s_0)}\bigg(s_0 + \frac{\delta(nu)}{\delta(n)}(\cdot - s_0)\bigg)$$

converges almost surely to $V^{(s_0)}$ on S. In particular,

$$\lim_{n\to\infty} \mathbb{P}\bigg\{\bigg\{V^{(s_0)}\bigg(s_0 + \frac{\delta(nu)}{\delta(n)}(s - s_0)\bigg)\bigg\}_{s\in S} \in B\bigg\} = \mathbb{P}\Big\{\big\{V^{(s_0)}(s)\big\}_{s\in S} \in B\Big\} = \mathbb{P}\{V^{(s_0)}|_S \in B\}.$$

By dominated convergence, we obtain

$$\lim_{n\to\infty} n\,\mathbb{P}\bigg\{\frac{Y_n^{(s_0)}|_S}{Y^{(s_0)}(s_0)} \in B\,,\, Y^{(s_0)}(s_0) > nt\bigg\} = \mathbb{P}\{V^{(s_0)}|_S \in B\}\nu_1\big([t,\infty)\big) = \mathbb{P}\{V^{(s_0)}|_S \in B\}\, t^{-1},$$

as desired.

(b) Let B be a continuity set of the law of $aPV^{(s_0)}|_S$, then it is necessarily also a continuity set for $auV^{(s_0)}|_S$ for ν_1-almost all $u > 1$. Similarly, to part (a) we may deduce

$$\begin{aligned}
\mathbb{P}\{n^{-1}Y_n^{(s_0)}|_S \in B \,|\, Y^{(s_0)}(s_0) > na\} &= na\,\mathbb{P}\{n^{-1}Y_n^{(s_0)}|_S \in B\,,\, Y^{(s_0)}(s_0) > na\}\\
&= na\int_{na}^{\infty} \mathbb{P}\{n^{-1}Y_n^{(s_0)}|_S \in B \,|\, Y^{(s_0)}(s_0) = u\big)\nu_1(\mathrm{d}u)\\
&= \int_1^{\infty} \mathbb{P}\{n^{-1}Y_n^{(s_0)}|_S \in B \,|\, Y^{(s_0)}(s_0) = nau\}\nu_1(\mathrm{d}u)\\
&= \int_1^{\infty} \mathbb{P}\bigg\{\bigg\{au\,V^{(s_0)}\bigg(s_0 + \frac{\delta(nau)}{\delta(n)}(s - s_0)\bigg)\bigg\}_{s\in S} \in B\bigg\}\nu_1(\mathrm{d}u).
\end{aligned}$$

By the same argument as in part (a), the integrand convergences for ν_1-almost all $u > 1$ to

$$\mathbb{P}\Big\{\big\{au\,V^{(s_0)}(s)\big\}_{s\in S} \in B\Big\} = \mathbb{P}\{au\,V^{(s_0)}|_S \in B\}.$$

By dominated convergence, this gives

$$\lim_{n\to\infty} \mathbb{P}\{n^{-1}Y_n^{(s_0)}\big|_S \in B \,\big|\, Y^{(s_0)}(s_0) > na\} = \int_1^\infty \mathbb{P}\{au\,V^{(s_0)}\big|_S \in B\}\nu_1(\mathrm{d}u) = \mathbb{P}\{aP\,V^{(s_0)}\big|_S \in B\},$$

as desired.

□

Remark 4.5. Let m_{s_0} be the multiplication map m_r from (12) for r being the point evaluation (27) and recall $\theta_{s_0} = 1$ in our setup. Part (a) of Proposition 4.4 implies the vague convergence

$$n\,\mathbb{P}\{n^{-1}Y_n^{(s_0)}\big|_S \in \cdot\} \xrightarrow{v^\#_{s_0}} \Big(\nu_1 \otimes \mathcal{L}\big(V^{(s_0)}\big|_S\big)\Big) \circ m_{s_0}^{-1},$$

which resembles

$$n\,\mathbb{P}\{n^{-1}X_n^{(s_0)}\big|_S \in \cdot\} \xrightarrow{v^\#_{s_0}} \Big(\nu_1 \otimes \mathcal{L}\big(V^{(s_0)}\big|_S\big)\Big) \circ m_{s_0}^{-1}$$

(the latter follows from Theorem 2.10, Lemma 2.16 and (14)). However, note that $Y_n^{(s_0)}$ need not have standard Pareto margins everywhere, but only at the reference point s_0. A similar argument as in the proof of Lemma 2.2 ensures that

$$n\,\mathbb{P}\{Y_n^{(s_0)}(s) > n\} \to \theta_{s_0}\mathbb{E}\big[V^{(s_0)}(s)\big] = \mathbb{E}\big[V^{(s_0)}(s)\big],$$

which recovers the classical Pareto-like tail asymptotics of the marginal distributions of the tail process without domain-scaling, i.e., when $Y_n^{(s_0)}(s) = Y^{(s_0)}(s)$ for all $n \in \mathbb{N}$.

Remark 4.6. Proposition 4.4 and Corollary 3.3 yield identical distributional limits

$$\lim_{n\to\infty} \mathcal{L}\left(\frac{X_n^{(s_0)}}{X(s_0)} \,\middle|\, X(s_0) > nt\right) \;=\; \mathcal{L}\big(V^{(s_0)}\big) \;=\; \lim_{n\to\infty} \mathcal{L}\left(\frac{Y_n^{(s_0)}}{Y(s_0)} \,\middle|\, Y(s_0) > nt\right)$$

and

$$\lim_{n\to\infty} \mathcal{L}\left(\frac{X_n^{(s_0)}}{nt} \,\middle|\, X(s_0) > nt\right) \;=\; \mathcal{L}\big(PV^{(s_0)}\big) \;=\; \lim_{n\to\infty} \mathcal{L}\left(\frac{Y_n^{(s_0)}}{nt} \,\middle|\, Y(s_0) > nt\right)$$

in the sense of uniform convergence on compact sets for all $t > 0$. This translates into the modelling paradigm of approximating $X|X(s_0) > u_0$ by $Y^{(s_0)}|Y^{(s_0)}(s_0) > u_0$ for some sufficiently high threshold $u_0 \geq 1$ around $s_0 \in \mathbb{R}^d$.

4.3 Joint and conditional exceedance probabilities

A priori, we know for the (bivariate) joint exceedance probability of the tail approximating process $Y^{(s_0)}$ at $s \in \mathbb{R}^d$ that for $u > 0$ and $u_0 \geq 1$

$$\begin{aligned}\mathbb{P}\{Y^{(s_0)}(s) > u,\, Y^{(s_0)}(s_0) > u_0\} &= u_0^{-1}\,\mathbb{P}\{Y^{(s_0)}(s) > u \,\big|\, Y^{(s_0)}(s_0) > u_0\}\\ &= u_0^{-1}\mathbb{P}\{u_0P\,V^{(s_0)}\big(s_0 + \delta(u_0P)(s-s_0)\big) > u\}. \end{aligned} \tag{42}$$

We denote the conditional exceedance probability of $Y^{(s_0)}$ and its asymptotic version by

$$\chi_Y^{(s_0)}(s;u) = \mathbb{P}\{Y^{(s_0)}(s) > u \,\big|\, Y^{(s_0)}(s_0) > u\} \qquad \text{and} \qquad \chi_Y^{(s_0)}(s) = \lim_{u\to\infty}\chi_Y^{(s_0)}(s;u), \qquad s \in \mathbb{R}^d. \tag{43}$$

Setting $u = u_0$, we obtain from (42) that for $u \geq 1$

$$\begin{aligned}\chi_Y^{(s_0)}(s;u) &= \mathbb{P}\{P\,V^{(s_0)}\big(s_0 + \delta(uP)(s-s_0)\big) > 1\}\\ &= \int_1^\infty \mathbb{P}\{\,V^{(s_0)}\big(s_0 + \delta(ut)(s-s_0)\big) > t^{-1}\}\nu_1(\mathrm{d}t), \end{aligned} \tag{44}$$

which depends on u via its appearance in the domain scaling $\delta(ut)$ in the previous term.

Under Assumption 4.3, we find ourselves always in one of the following two cases: Either (i) $\lim_{u\to\infty}\delta(u)=c$ for some constant $c\in[1,\infty)$, where the case $c=1$ implies that the function δ is identically 1 ($\delta\equiv 1$), or (ii) $\lim_{u\to\infty}\delta(u)=\infty$; the latter case corresponds to δ being unbounded. Accordingly, different behaviours of the conditional exceedance probabilities are possible.

Proposition 4.7. *Let $Y^{(s_0)}$ be the tail approximating process for a domain-scaled regularly varying process X on $\mathbb{R}^d$ with point-spectral process $V^{(s_0)}$. Let $s\in\mathbb{R}^d$ and*

$$\chi_X^{(s_0)}(s)=\lim_{u\to\infty}\mathbb{P}\big\{X(s)>u\,\big|\,X(s_0)>u\big\}.$$

Then:

(a) If $\lim_{u\to\infty}\delta(u)=c\in[1,\infty)$, then

$$\chi_Y^{(s_0)}(s)=\mathbb{P}\big\{P\,V^{(s_0)}\big(s_0+c(s-s_0)\big)>1\big\}=\chi_X^{(s_0)}(s).$$

If additionally $c=1$, then $\chi_Y^{(s_0)}(s;u)$ is constant in $u\geq 1$.

(b) If $\lim_{u\to\infty}\delta(u)=\infty$ and $\lim_{r\to\infty}V^{(s_0)}\big(s_0+r(s-s_0)\big)=0$ in probability, then

$$\chi_Y^{(s_0)}(s)=\begin{cases}1 & \text{if } s=s_0,\\ 0 & \text{if } s\in\mathbb{R}^d\setminus\{s_0\}.\end{cases}$$

If additionally there exists $r_0>0$ and $u_0>1$, such that

$$\mathbb{P}\big\{X(s_0+r(s-s_0))>u\,\big|\,X(s_0)>u\big\}=u\,\mathbb{P}\big\{X(s_0+r(s-s_0))>u\,,\,X(s_0)>u\big\}\tag{45}$$

is non-increasing in $r>r_0$ for all $u\geq u_0$, then also $\chi_Y^{(s_0)}(s)=\chi_X^{(s_0)}(s)$.

Proof. We will use in each case that we can rewrite

$$\chi_X^{(s_0)}(s)=\lim_{n\to\infty}\mathbb{P}\big\{X(s)>n\,\big|\,X(s_0)>n\big\}=\lim_{n\to\infty}\mathbb{P}\left\{\frac{X_n^{(s_0)}(s_n)}{n}>1\,\middle|\,X(s_0)>n\right\}$$

for $s_n=s_0+\delta(n)(s-s_0)$.

(a) Since $\lim_{u\to\infty}\delta(u)=c\in(0,\infty)$, we also have $\lim_{u\to\infty}\delta(ut)=c\in(0,\infty)$ for any $t\geq 1$, and there exists a compact convex set $S\subset\mathbb{R}^d$, such that $s_0+\delta(ut)(s-s_0)\in S$ for all $t\geq 1$ and all $u\geq 1$. In particular, also $s_c=s_0+c(s-s_0)\in S$. Now, $\{f\in C(S,[0,\infty)):f(s_c)>1\}$ is a continuity set of the law of $PV^{(s_0)}\big|_S$ (by an almost identical argument as presented in the proof of Lemma 2.2). Therefore, the almost sure convergence of $V^{(s_0)}\big(s_0+\delta(ut)(s-s_0)\big)$ to $V^{(s_0)}\big(s_0+c(s-s_0)\big)$ on S implies that the integrand in (44) converges pointwise to $\mathbb{P}\big\{V^{(s_0)}\big(s_0+c(s-s_0)\big)>t^{-1}\big)$, so that the assertion $\chi_Y^{(s_0)}(s)=\mathbb{P}\big\{P\,V^{(s_0)}(s_c)>1\big\}$ follows by dominated convergence.

When $\delta\equiv 1$, the assertion $\chi_Y^{(s_0)}(s;u)\equiv\chi_Y^{(s_0)}(s)=\mathbb{P}\big\{P\,V^{(s_0)}(s)>1\big\}$ follows directly from (44).

Further, recall that the restrictions of $n^{-1}X_n^{(s_0)}|X(s_0)>n$ and $PV^{(s_0)}$ to S obey the distributional convergence (28). Let η_n and η be random elements in $C(S,[0,\infty))$ whose laws are given by those restrictions, i.e. η_n has the same law as $n^{-1}X_n^{(s_0)}|X(s_0)>n$ restricted to S, and η the law of $PV^{(s_0)}$ restricted to S, so that $\eta_n\to\eta$ in distribution. In addition, $PV^{(s_0)}(s_c)\neq 1$ almost surely, i.e. $\eta(s_c)\neq 1$ almost surely. Moreover, consider the maps $M_n,M:C(S,[0,\infty))\to\{0,1\}$ that are given by

$$M_n(f)=\mathbb{1}\{f(s_n)>1\}\qquad\text{and}\qquad M(f)=\mathbb{1}\{f(s_c)>1\}.$$

According to Lemma A.3, we know that $M_n(f_n)\to M(f)$ if $f_n\to f$ in $C(S,[0,\infty))$ and $f(s_c)\neq 1$. So, we may conclude by an extended continuous mapping theorem (Theorem 4.27 in Kallenberg (2002)) that $M_n(\eta_n)$ converges in distribution to $M(\eta)$ as random variables. Since these variables have binary outcomes in $\{0,1\}$ only, this implies that $\mathbb{P}\{M_n(\eta_n)=1\}$ converges to $\mathbb{P}\{M(\eta)=1\}$. Finally, note that by construction

$$\mathcal{L}\big(M_n(\eta_n)\big)=\mathcal{L}\big(\mathbb{1}\{n^{-1}X_n^{(s_0)}(s_n)>1\}\big|X(s_0)>n\big)\quad\text{and}\quad\mathcal{L}\big(M(\eta)\big)=\mathcal{L}\big(\mathbb{1}\{PV^{(s_0)}(s_c)>1\}\big).$$

Hence,

$$\chi_X^{(s_0)}(s) = \lim_{n\to\infty} \mathbb{P}\{n^{-1}X_n^{(s_0)}(s_n) > 1 \,|\, X(s_0) > n\} = \mathbb{P}\{PV^{(s_0)}(s_c) > 1\},$$

as desired.

(b) It suffices to consider $s \neq s_0$. The integrand in (44) converges pointwise to 0 as $u \to \infty$, and so the assertion that $\chi_Y^{(s_0)}(s) = 0$ follows by dominated convergence. Further, the monotonicity condition (45) gives that, for suffcently large $c > 0$ and sufficiently large $n \in \mathbb{N}$,

$$\mathbb{P}\{n^{-1}X_n^{(s_0)}(s_n) > 1 \,|\, X(s_0) > n\} \leq \mathbb{P}\{n^{-1}X_n^{(s_0)}(s_0 + \min(\delta(n), c)(s - s_0)) > 1 \,|\, X(s_0) > n\}.$$

Similarly to (b), the right-hand side converges to

$$\mathbb{P}\{PV^{(s_0)}(s_c) > 1\} = \mathbb{P}\{PV^{(s_0)}(s_0 + c(s - s_0)) > 1\}$$

as $n \to \infty$. Hence,

$$\limsup_{n\to\infty} \mathbb{P}\{n^{-1}X_n^{(s_0)}(s_n) > 1 \,|\, X(s_0) > n\} \leq \liminf_{c\to\infty} \mathbb{P}\{PV^{(s_0)}(s_0 + c(s - s_0)) > 1\} = 0,$$

where the last identity follows from $\lim_{r\to\infty} V^{(s_0)}\big(s_0 + r(s - s_0)\big) = 0$ in probability. □

Remark 4.8. The monotonicity condition (45) on conditional exceedance probabilities is natural in practice: It means that the joint exceedance of a large level $u \geq u_0$ at s_0 and $s_0 + r(s - s_0)$ is at least not more likely, the further these two points are apart, i.e. the larger r is. For example, considering a process with Gaussian dependence structure, condition (45) holds under a mild monotonicity condition on its correlation function, cf. Lemma 5.9. In spatial statistics, this condition could be akin to the First Law of Geography, "everything is related to everything else, but near things are more related than distant things" (Tobler, 1970).

Suppose now that the process $V^{(s_0)}$ arises as the global point-spectral process of a stationary(!) domain-scaled regularly varying process X on $\mathbb{R}^d$. Then (44) simplifies to

$$\chi_Y^{(s_0)}(s; u) = \mathbb{P}\{P\, V^{(0)}\big(\delta(uP)(s - s_0)\big) > 1\} = \int_1^\infty \mathbb{P}\{\, V^{(0)}\big(\delta(ut)(s - s_0)\big) > t^{-1}\}\nu_1(\mathrm{d}t), \tag{46}$$

such that $\chi_Y^{(s_0)}(s; u)$ depends only on the difference $s - s_0$. In case $\delta \equiv 1$, $\chi_Y^{(s_1)}(s_2; u)$ is always symmetric in the pair (s_1, s_2), i.e.

$$\chi_Y^{(s_1)}(s_2; u) = \chi_Y^{(s_1)}(s_2) = \chi_Y^{(s_2)}(s_1) = \chi_Y^{(s_2)}(s_1; u)$$

for any $s_1, s_2 \in \mathbb{R}^d$, as we can see from (30) with $t = 1$ therein. Otherwise, the symmetry

$$\chi_Y^{(s_1)}(s_2; u) = \chi_Y^{(s_2)}(s_1; u)$$

would be guaranteed, for instance, if we had $\mathcal{L}(V^{(0)}(h)) = \mathcal{L}(V^{(0)}(-h))$ for $h \in \mathbb{R}^d$, as is the case when the bivariate distributions of the underlying process X are symmetric, cf. Lemma 3.4. Similarly, the joint exceedance probabilities exhibit the following symmetries, when derived from a point-spectral process.

Lemma 4.9. *Let $Y^{(s_0)}$ be the tail approximating process of a stationary domain-scaled regularly varying process X on $\mathbb{R}^d$ with global point-spectral process $V^{(s_0)}$. Let $t_1, t_2 \geq 1$. Then*

$$\mathbb{P}\{Y^{(s_1)}(s_1) > t_1,\, Y^{(s_1)}(s_2) > t_2\} = \mathbb{P}\{Y^{(0)}(0) > t_1,\, Y^{(0)}(s_2 - s_1) > t_2\}.$$

(a) If, in addition, $\delta \equiv 1$, then

$$\mathbb{P}\{Y^{(s_1)}(s_1) > t_1,\, Y^{(s_1)}(s_2) > t_2\} = \mathbb{P}\{Y^{(s_2)}(s_1) > t_1,\, Y^{(s_2)}(s_2) > t_2\}.$$

(b) If, in addition, the bivariate distributions of X are symmetric (but not necessarily $\delta \equiv 1$), then

$$\mathbb{P}\{Y^{(s_1)}(s_1) > t_1,\, Y^{(s_1)}(s_2) > t_2\} = \mathbb{P}\{Y^{(s_2)}(s_1) > t_2,\, Y^{(s_2)}(s_2) > t_1\}.$$

Proof. Stationarity of X implies via (29) that

$$\begin{aligned}\mathbb{P}\big\{Y^{(s_1)}(s_1) > t_1,\, Y^{(s_1)}(s_2) > t_2\big\} &= t_1^{-1}\mathbb{P}\big\{t_1 P\, V^{(s_1)}\big(s_1 + \delta(t_1 P)(s_2 - s_1)\big) > t_2\big\} \\ &= t_1^{-1}\mathbb{P}\big\{t_1 P\, V^{(0)}\big(\delta(t_1 P)(s_2 - s_1)\big) > t_2\big\} \\ &= \mathbb{P}\big\{Y^{(0)}(0) > t_1,\, Y^{(0)}(s_2 - s_1) > t_2\big\}.\end{aligned}$$

(a) When $\delta \equiv 1$ the assertion is a direct consequence of (32).

(b) Similarly, as above, $\mathbb{P}\big\{Y^{(s_2)}(s_1) > t_2,\, Y^{(s_2)}(s_2) > t_1\big\} = t_1^{-1}\mathbb{P}\big\{t_1 P\, V^{(0)}\big(\delta(t_1 P)(s_1 - s_2)\big) > t_2\big\}$, and hence the assertion follows from the symmetry of X via (33). □

Remark 4.10. In either of the situations (a) and (b) in Lemma 4.9, we have, for $t \geq 1$,

$$\mathbb{P}\big\{Y^{(s_1)}(s_1) > t,\, Y^{(s_1)}(s_2) > t\big\} = \mathbb{P}\big\{Y^{(s_2)}(s_1) > t,\, Y^{(s_2)}(s_2) > t\big\}. \tag{47}$$

In other words, we may choose the reference point to be either of the evaluation points, the joint exceedance probability depends only on $s_2 - s_1 \in \mathbb{R}^d$ rather than the pair $(s_1, s_2) \in \mathbb{R}^d \times \mathbb{R}^d$, and the expression is symmetric in s_1 and s_2.

5 Application to Gaussian processes

A prominent example of a triangular array as in Theorem 2.10 arises from the convergence of maxima of suitably rescaled and domain-scaled Gaussian processes to a Brown–Resnick process (Kabluchko et al., 2009), cf. Theorem 5.4 below. First, we recall the definition of the max-stable limiting process.

Definition 5.1 (Brown–Resnick process on standard Fréchet scale)**.** Let W be a pathwise continuous zero-mean Gaussian process on $\mathbb{R}^d$ with semi-variogram

$$\gamma(s_1, s_2) = \frac{1}{2}\,\mathbb{E}\big[(W(s_1) - W(s_2))^2\big], \quad s_1, s_2 \in \mathbb{R}^d.$$

Then the stochastic process

$$Z(s) = \bigvee_{i=1}^{\infty} \Gamma_i^{-1} \exp\left(W_i(s) - \frac{1}{2}\mathrm{Var}(W_i(s))\right), \quad s \in \mathbb{R}^d,$$

where $(\Gamma_i)_{i\in\mathbb{N}}$ are the arrival times of a renewal process on $(0, \infty)$ with standard exponential interarrival times, and independently, $(W_i)_{i\in\mathbb{N}}$ are independent copies of W, is called *Brown–Resnick process associated to the semi-variogram* γ.

The Brown–Resnick process associated to the semi-variogram γ is well-defined, as it has been established in Kabluchko et al. (2009) and Kabluchko (2011) that the law of Z in Definition 5.1 depends indeed only on its associated semi-variogram γ. Conversely the law of Z is identified from γ, i.e. two different semi-variograms γ give rise to two different laws of Brown–Resnick processes. Note that the sample continuity of W implies the continuity of γ, cf. e.g. Adler and Taylor (2007, Section 1.3). While the law of Z is uniquely identified from γ, the same cannot be said for the law of the underlying Gaussian process W in its construction. For instance, we may modify a given W by subtracting $W(s_0)$ everywhere for some $s_0 \in \mathbb{R}^d$ to obtain the random field $W^{(s_0)} = W - W(s_0)$. Then $W^{(s_0)}$ has the same semi-variogram as W, but satisfies in addition the constraint $W^{(s_0)}(s_0) = 0$ almost surely. This results in the representation of the law of Z as the law of the LePage series

$$Z^{(s_0)} = \bigvee_{i=1}^{\infty} \Gamma_i^{-1} V_i^{(s_0)},$$

where $V_i^{(s_0)}$ are independent copies of

$$V^{(s_0)}(s) = \exp\big(W^{(s_0)}(s) - \gamma(s, s_0)\big), \quad s \in \mathbb{R}^d. \tag{48}$$

Since γ is continuous, the variance of $W^{(s_0)}$ is bounded on any compact $S \subset \mathbb{R}^d$. Taking into account general tail asymptotics for the supremum of a Gaussian process on a compact domain, cf. e.g. Debicki et al. (2010, (1.2)) and references therein, we may conclude that

$$\mathbb{E}\Big[\sup_{s\in S} V^{(s_0)}(s)\Big] < \infty \tag{49}$$

for any compact $S \subset \mathbb{R}^d$. So, the condition (21) with r being the point evaluation at s_0 is met for any compact $S \subset \mathbb{R}^d$, while at the same time $\mathbb{E}\big[V^{(s_0)}(s)\big] = 1$ for $s \in \mathbb{R}^d$. This confirms that $Z^{(s_0)}$ is a simple max-stable process with continuous sample paths, cf. Corollary 2.26 and Remark 2.27. This argument also abbreviates the deduction of sample-continuity of $Z^{(s_0)}$ provided in Kabluchko et al. (2009, Proposition 13), which underlines the importance of condition (21) in Lemma 2.25.

Before we get to the prominent triangular array convergence, it is worth remembering that the Brown–Resnick process itself yields an example of a domain-scaled regularly varying stochastic process in the sense of Definition 3.6, albeit a trivial one, as in this instance no inner domain scaling is needed, i.e. $\delta_n = 1$, $n \in \mathbb{N}$. For ease of reference, we state this example here explicitly.

Lemma 5.2. *Let Z be a Brown–Resnick process with semi-variogram γ. Then the corresponding Brown–Resnick copula process with standard Pareto margins,*

$$X(s) = \frac{1}{1-\exp(-1/Z(s))}, \qquad s \in \mathbb{R}^d,$$

is domain-scaled regularly varying with respect to any $s_0 \in \mathbb{R}^d$ for the identical scaling sequence $\delta_n \equiv 1$ (in the sense of Definition 3.6). The associated point-spectral process $V^{(s_0)}$ is given by (48).

Proof. Each restriction of X to a compact S is regularly varying as explained in Example 2.12, which confirms the stated regular variation. Now fix a compact convex $S \subset \mathbb{R}^d$ with $s_0 \in S$. Since the process $V^{(s_0)}$ from (48) satisfies (49) and $V^{(s_0)}(s_0) = 1$ almost surely, its restriction to S coincides with the point-spectral process for X when restricted to S, cf. Corollary 2.26 for r being the point evaluation at s_0. Since this is true for any such $S \subset \mathbb{R}^d$, this identifies the point spectral process for X as $V^{(s_0)}$ from (48), cf. Section 3.3. □

Let us review some elementary properties of the point-spectral process (48). While $V^{(s_0)}(s_0) = 1$ almost surely, the multivariate log-normal marginal distributions of $V^{(s_0)}$ are given by

$$\begin{pmatrix} \log(V^{(s_0)}(s_1)) \\ \log(V^{(s_0)}(s_2)) \\ \vdots \\ \log(V^{(s_0)}(s_m)) \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu^{(s_0)}(s_1) \\ \mu^{(s_0)}(s_2) \\ \vdots \\ \mu^{(s_0)}(s_m) \end{pmatrix}, \begin{pmatrix} \Sigma^{(s_0)}(s_1,s_1) & \Sigma^{(s_0)}(s_1,s_2) & \dots & \Sigma^{(s_0)}(s_1,s_m) \\ \Sigma^{(s_0)}(s_2,s_1) & \Sigma^{(s_0)}(s_2,s_2) & \dots & \Sigma^{(s_0)}(s_2,s_m) \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma^{(s_0)}(s_m,s_1) & \Sigma^{(s_0)}(s_m,s_2) & \dots & \Sigma^{(s_0)}(s_m,s_m) \end{pmatrix} \right),$$

where

$$\mu^{(s_0)}(s_i) = -\gamma(s_i, s_0),$$
$$\Sigma^{(s_0)}(s_i, s_j) = \gamma(s_i, s_0) + \gamma(s_j, s_0) - \gamma(s_i, s_j)$$

for $s_i, s_j \in \mathbb{R}^d$, $1 \le i, j \le m$, $m \in \mathbb{N}$.

Lemma 5.3. *Let $V^{(s_0)}$ be the point-spectral process from* (48). *Then $\lim_{n\to\infty} V^{(s_0)}(s_n) = 0$ in probability if and only if $\lim_{n\to\infty} \gamma(s_n, s_0) = \infty$.*

Proof. Abbreviate $\gamma_n = \gamma(s_n, s_0)$. If $\gamma_n = 0$ for some $n \in \mathbb{N}$, then $V^{(s_0)}(s_n) = 1$ almost surely, and so we may assume without loss of generality that eventually $\gamma_n > 0$. Let $\varepsilon > 0$ and assume, again without loss of generality, that $\varepsilon < 1$, so that $A = -\log(\varepsilon) > 0$. Then

$$\mathbb{P}\{V^{(s_0)}(s_n) > \varepsilon\} = \mathbb{P}\{-\log(V^{(s_0)}(s_n)) < A\} = \Phi\left(\frac{A-\gamma_n}{\sqrt{2\gamma_n}}\right) = \Phi\left(\frac{A}{\sqrt{2\gamma_n}} - \sqrt{\frac{\gamma_n}{2}}\right).$$

If $\gamma_n \to \infty$, then it follows readily that $\mathbb{P}\{V^{(s_0)}(s_n) > \varepsilon\} \to 0$. Otherwise, there exists a subsequence of γ_n, which is bounded and therefore has an accumulation point in $[0, \infty)$. In particular, $\mathbb{P}\{V^{(s_0)}(s_n) > \varepsilon\}$ does not converge to 0. □

By slight abuse of notation, we write $\gamma(s_1 - s_2) = \gamma(s_1, s_2)$ if $\gamma(s_1, s_2)$ depends on s_1 and s_2 only via the difference $s_1 - s_2 \in \mathbb{R}^d$. This is precisely the case, when the corresponding Brown–Resnick process Z is stationary, or, equivalently, the underlying W in its definition is intrinsically stationary, i.e., the law of $\{W(s + s_0) - W(s_0) : s \in \mathbb{R}^d\}$ does not depend on $s_0 \in \mathbb{R}^d$. The family of point spectral processes $\{V^{(s_0)} : s_0 \in \mathbb{R}^d\}$ satisfies then both (38) and (39). In particular,

$$\mathbb{E}\big[V^{(s_0)}(s)\big] = \mathbb{P}\big\{V^{(s_0)}(s) > 0\big\} = 1, \qquad s, s_0 \in \mathbb{R}^d, \tag{50}$$

cf. (35).

5.1 Domain-scaled regular variation of the Gaussian copula process

In Kabluchko et al. (2009) the Brown–Resnick process is considered on a Gumbel scale, which is the natural scale for convergence of maxima of Gaussian random variables, i.e., one considers $\log(Z)$ instead of Z as the limiting process. As for the marginal convergence, we recall that $\Phi^n(a_n \cdot + b_n)$ converges weakly to the standard Gumbel distribution if $a_n \sim 1/b_n$ and $b_n = b(n)$, where b is, for instance, one of the following functions $b \in \{b^{(1)}, b^{(2)}, b^{(3)}\}$, each of which is well-defined for sufficiently large u:

$$b^{(1)}(u) = \Phi^{-1}\left(1 - \frac{1}{u}\right) \quad \text{or} \quad b^{(2)}(u) = \sqrt{2\log u} - \frac{1}{\sqrt{2\log u}}\left(\frac{1}{2}\log\log u + \log\left(2\sqrt{\pi}\right)\right), \tag{51}$$

or $b^{(3)}$ is a function satisfying $b^{(3)}(u) = u\varphi(b^{(3)}(u))$, cf. e.g. Resnick (2008).

Theorem 5.4 (Convergence of maxima of triangular array of domain-scaled Gaussian processes; consequence of Theorems 17 and 20 of Kabluchko et al. (2009))**.** *Let G be a pathwise continuous zero-mean variance-one Gaussian process on $\mathbb{R}^d$, whose covariance function $\rho(s_1, s_2) = \mathbb{E}(G(\varepsilon s_1)G(\varepsilon s_2))$ satisfies for all compact $S \subset \mathbb{R}^d$ that*

$$\lim_{\varepsilon \to 0} \sup_{s_1, s_2 \in S} \left| \frac{1 - \rho(\varepsilon s_1, \varepsilon s_2)}{\varepsilon^\alpha} - \gamma^{(\rho)}(s_1 - s_2) \right| = 0 \tag{52}$$

for a continuous and α-homogeneous function $\gamma^{(\rho)} : \mathbb{R}^d \to [0, \infty)$, where $\alpha \in (0, 2]$. Then $\gamma^{(\rho)}$ is a semi-variogram. Further, let $(G_i)_{i \in \mathbb{N}}$ be independent copies of G, and let $a_n > 0$ and $b_n \in \mathbb{R}$ be such that the marginal convergence of $\Phi^n(a_n \cdot + b_n)$ to a standard Gumbel law holds. Let $\delta_n > 0$ be a sequence, such that

$$\lim_{n \to \infty} \frac{\delta_n}{b_n^{2/\alpha}} = c \in (0, \infty). \tag{53}$$

Then

$$\left\{ a_n^{-1} \left(\max_{i=1}^{n} G_i(s/\delta_n) - b_n \right) \right\}_{s \in S} \to \{\log(Z(s))\}_{s \in S} \tag{54}$$

weakly in $C(S, \mathbb{R})$ as $n \to \infty$ for any compact set $S \subset \mathbb{R}^d$, where Z is the Brown–Resnick process associated to the semi-variogram $\gamma = c^{-\alpha}\gamma^{(\rho)}$.

Remark 5.5. The sequence δ_n from (53) is necessarily slowly varying. In fact, a function δ that satisfies

$$\lim_{u \to \infty} \frac{\delta(u)}{b(u)^{2/\alpha}} \in (0, \infty),$$

for some $\alpha \in (0, 2]$ is necessarily slowly varying, as slow variation is closed under asymptotic equivalence and the respective b from (51) is slowly varying. Assumption 4.3 is satisfied if we set $\delta(u) = \max(1, b(u)^{2/\alpha})$. Setting $\delta_n = \max(1, b(n)^{2/\alpha})$ gives $c = 1$ in (53).

Example 5.6. Let the Gaussian process G in Theorem 5.4 be stationary, so that we may write by a slight abuse of notation $\rho(s_1 - s_2) = \rho(s_1, s_2)$. Suppose, G is additionally isotropic and that

$$\rho(h) = 1 - c'\|h\|^\alpha + o\big(\|h\|^\alpha\big) \quad \text{as} \quad \|h\| \to 0,$$

for some $\alpha \in (0, 2]$ and $c' > 0$, for instance, $\rho(h) = \exp(-c'\|h\|^\alpha)$. Assume further that the inner domain scaling sequence is chosen as $\delta_n \sim c b_n^{2/\alpha}$, then the convergence (54) holds true with Z being the Brown–Resnick process associated to the power semi-variogram

$$\gamma(h) = c^{-\alpha}\gamma^{(\rho)}(h) = c' c^{-\alpha}\|h\|^\alpha, \quad h \in \mathbb{R}^d.$$

Theorem 5.4 provides sufficient conditions for convergence of the pointwise maxima of a triangular array of domain-scaled Gaussian processes to a Brown–Resnick process on the Gumbel scale. This result translates into domain-scaled regular variation of the associated Gaussian copula process as follows.

Theorem 5.7. *Let G be a Gaussian process on $\mathbb{R}^d$ and δ_n a scaling sequence, both as in Theorem 5.4. Then the corresponding Gaussian copula process with standard Pareto margins,*

$$X(s) = \frac{1}{1 - \Phi(G(s))}, \quad s \in \mathbb{R}^d, \tag{55}$$

is domain-scaled regularly varying with respect to the origin $0 \in \mathbb{R}^d$ for the scaling sequence δ_n (in the sense of Definition 3.6). The associated point-spectral process $V^{(0)}$ is given by

$$V^{(0)}(s) = \exp\Big(W^{(0)}(s) - \gamma(s)\Big), \quad s \in \mathbb{R}^d, \tag{56}$$

where $W^{(0)}$ is a zero-mean Gaussian process with semi-variogram $\gamma = c^{-\alpha}\gamma^{(\rho)}$ as in Theorem 5.4 and $W^{(0)}(0) = 0$ almost surely.

Proof. Let $S \subset \mathbb{R}^d$ be compact and convex, such that $0 \in \mathbb{R}^d$. The convergence (54) implies that condition (i) in Lemma 2.7 holds true with a_n and b_n therein as a_n and b_n in (54), Z^{orig} therein as $\log(Z)$ from (54) and $X^{\text{orig}}_{n,i}(\cdot) = G_i(\cdot/\delta_n)$, $1 \le i \le n$, $n \in \mathbb{N}$. Thus, by Lemma 2.7, the independent marginally transformed Gaussian copula processes on standard Pareto scale

$$X_i(s) = \big(1 - \Phi(G_i(s))\big)^{-1}, \quad s \in S, \quad i \in \mathbb{N},$$

satisfy

$$\big\{n^{-1} \max_{i=1}^{n} X_i(s/\delta_n)\big\}_{s\in S} \to \{Z(s)\}_{s\in S}$$

weakly in $C = C(S, \mathbb{R})$ as $n \to \infty$. Hence, Theorem 2.10 (i) holds for the triangular array

$$\{X_{n,i}(s)\}_{s\in S} = \big\{\big(1 - \Phi(G_i(s/\delta_n))\big)^{-1}\big\}_{s\in S}, \qquad 1 \le i \le n, \quad n \in \mathbb{N},$$

and Z being the Brown–Resnick process in (54) on the standard Fréchet scale. Therefore, also all the other convergence statements of Theorem 2.10 and Proposition 2.23 apply to this situation. In other words, the Gaussian copula process (55) is domain-scaled regularly varying in the stated sense. Further, the law of the point spectral process $V^{(0)}$ when restricted to a compact convex S with $0 \in S$ is uniquely identified from the law of the limiting max-stable process Z that appears in condition (i) of Theorem 2.10, cf. Corollary 2.26. Hence, that $V^{(0)}$ is given by (56) follows almost verbatim as in the proof of Lemma 5.2 with $s_0 = 0$ therein. □

In what follows, we will assume that the underlying Gaussian process G — and hence its copula process X from (55) — is stationary. Then X is domain-scaled regularly varying with respect to any $s_0 \in \mathbb{R}^d$, cf. Remark 3.7. Similarly as shown above for the origin $0 \in \mathbb{R}^d$, the associated point-spectral process for general $s_0 \in \mathbb{R}^d$ is then given by

$$V^{(s_0)}(s) = \exp\Big(W^{(s_0)}(s) - \gamma(s - s_0)\Big), \quad s \in \mathbb{R}^d, \tag{57}$$

where $W^{(s_0)}$ is an intrinsically stationary zero-mean Gaussian process with semi-variogram γ and $W^{(s_0)}(s_0) = 0$ almost surely, i.e., $V^{(s_0)}$ coincides with the point-spectral process from (48), where no domain-scaling was applied.

5.2 Joint and conditional exceedance probabilities

Recall that the tail approximating process $Y^{(s_0)}$ associated with point-spectral process $V^{(s_0)}$ and scaling function δ is given by

$$Y^{(s_0)}(s) = P\, V^{(s_0)}\big(s_0 + \delta(P)(s - s_0)\big), \quad s \in \mathbb{R}^d, \tag{58}$$

where P is standard Pareto and independent from $V^{(s_0)}$, cf. Definition 4.1. In this section, we assume that the point-spectral process $V^{(s_0)}$ in (58) is log-Gaussian as in (48) and (57). For the scaling function $\delta : [1,\infty) \to [1,\infty)$, we distinguish again the following two cases, cf. Section 3: (i) $\lim_{u\to\infty} \delta(u) = c \in [1,\infty)$ and (ii) $\lim_{u\to\infty} \delta(u) = \infty$. Case (i) includes the no scaling case $\delta \equiv 1$ arising, for instance, if $Y^{(s_0)}$ is the tail approximating process of the Brown–Resnick copula process from Lemma 5.2, or more generally the tail approximating process of a marginally standardized stochastic process in the domain of attraction of the Brown–Resnick process. Case (ii) appears when $Y^{(s_0)}$ is the tail approximating process of a Gaussian copula process X in (55), cf. Theorem 5.7.

Since the family of point spectral processes $\{V^{(s_0)} : s_0 \in \mathbb{R}^d\}$ from (57) satisfies both (38) and (39), it satisfies the symmetry (47) in the bivariate joint exceedance probabilities. Similarly, the conditional exceedance probabilities from (43), depend only on $s - s_0$ and are symmetric in s and s_0. Therefore, we write by slight abuse of notation

$$\chi_Y(s - s_0; u) = \chi_Y^{(s_0)}(s; u) \qquad \text{and} \qquad \chi_Y(s - s_0) = \chi_Y^{(s_0)}(s)$$

for the tail dependence functions (43) of the tail approximating process $Y^{(s_0)}$. By (46) and (57) we may express $\chi_Y(h; u)$ as

$$\chi_Y(h; u) = \int_1^\infty \overline{\Phi}\left(\frac{-\log(t) + \gamma(\delta(ut)h)}{\sqrt{2\gamma(\delta(ut)h)}}\right)\nu_1(\mathrm{d}t).$$

If the semi-variogram $\gamma(rh)$ is monotonously increasing in $r > 0$ for $h \in \mathbb{R}^d \setminus \{0\}$, as is the case when γ is α-homogeneous and δ satisfies Assumption 4.3, it is clear from Lemma A.6 and Lemma A.5 that $\chi_Y(h; u)$ has the following upper bound

$$\chi_Y(h; u) \le \int_1^\infty \overline{\Phi}\left(\frac{-\log(t) + \gamma(\delta(u)h)}{\sqrt{2\gamma(\delta(u)h)}}\right)\nu_1(\mathrm{d}t) = \mathbb{P}\big\{P\,V^{(0)}\big(\delta(u)h\big) > 1\big\} = 2\overline{\Phi}\left(\sqrt{\frac{\gamma(\delta(u)h)}{2}}\right).$$

If $\delta \equiv 1$, the behaviour of $\chi_Y(h; u) = \chi_Y(h)$ is well-studied. The general case is given in Corollary 5.8.

Corollary 5.8. *Let $Y^{(s_0)}$ be the tail approximating process* (58) *with $V^{(s_0)}$ being the log-Gaussian point-spectral process from* (57).

(a) Let $\lim_{u\to\infty} \delta(u) = c \in [1,\infty)$. Then

$$\chi_Y(h) = 2\overline{\Phi}\left(\sqrt{\frac{\gamma(ch)}{2}}\right), \qquad h \in \mathbb{R}^d.$$

(b) Let $\lim_{u\to\infty} \delta(u) = \infty$ and $\lim_{h\to\infty} \gamma(h) = \infty$. Then

$$\chi_Y(h) = \begin{cases} 1 & \text{if } h = 0, \\ 0 & \text{if } h \in \mathbb{R}^d \setminus \{0\}. \end{cases}$$

Proof. Part (a) follows from Proposition 4.7 (a) and (70) with $u = 1$, while part (b) follows from Proposition 4.7 (b) and Lemma 5.3. □

For $c = 1$ (i.e. $\delta \equiv 1$), we recover the familiar tail-correlation function of a stochastic process in the domain of attraction of a Brown–Resnick process associated to the semi-variogram γ. Likewise, one might have expected part (b) at least in the situation when δ is chosen as in Remark 5.5, since $Y^{(s_0)}$ is then the tail approximating process of a Gaussian copula process X as in (55), which is known to exhibit *asymptotic independence*, i.e., $\chi_X(h) = \chi_Y(h)$, where $\chi_Y(h)$ is as stated in part (b) of Corollary 5.8. A priori, it is however not clear if these two quantities — χ_X and χ_Y — are necessarily equal or not in this case. In Proposition 4.7, we formulated the sufficient condition (45), which guarantees that this is not merely a coincidence. Lemma 5.9 confirms that the Gaussian copula process (55) satisfies this condition under a mild monotonicity condition on the correlation function. To see this, note that for X as in (55) the condition (45) means that $\mathbb{P}\big\{G(s_0 + r(s - s_0)) > u \,\big|\, G(s_0) > u\big\}$ is eventually non-increasing in $r > 0$ for $u \ge u_0$ for some $u_0 \in \mathbb{R}$.

Lemma 5.9. *Let G be a zero-mean variance-one Gaussian process on $\mathbb{R}^d$, whose covariance function $\rho(s_1, s_2) = \mathbb{E}(G(s_1)G(s_2))$ satisfies that there exists $r_0 > 0$, such that $\rho(s_0+r(s-s_0), s_0) \in (-1,1)$ is non-increasing in $r > r_0$ for $s \neq s_0$. Then*

$$\mathbb{P}\{G(s_0 + r(s - s_0)) > u \,|\, G(s_0) > u\} = \overline{\Phi}(u)^{-1}\mathbb{P}\{G(s_0 + r(s - s_0)) > u\,,\, G(s_0) > u\}$$

is also non-increasing in $r > r_0$ for $u > 0$ and $s \neq s_0$.

Proof. We abbreviate $s_r = s_0 + r(s - s_0)$ and $\rho(r) = \rho(s_0 + r(s - s_0), s_0)$, so that

$$\mathbb{P}\{G(s_r) > u \,|\, G(s_0) > u\} = \int_u^\infty \mathbb{P}\{G(s_r) > u \,|\, G(s_0) = t\}\varphi(t)\,\mathrm{d}t = \int_u^\infty \overline{\Phi}\left(\frac{u - \rho(r)t}{\sqrt{1-\rho(r)^2}}\right)\varphi(t)\,\mathrm{d}t.$$

Hence, it suffices to show that $(u-\rho t)/\sqrt{1-\rho^2}$ is non-increasing in $\rho \in (-1,1)$ for $t \geq u > 1$, or, equivalently, that $(a-\rho)/\sqrt{1-\rho^2}$ is decreasing in $\rho \in (-1,1)$ for $a \in (0,1]$. The latter can be seen from

$$\frac{\partial}{\partial\rho}\left(\frac{a-\rho}{\sqrt{1-\rho^2}}\right) = \frac{-1+a\rho}{(1-\rho^2)^{3/2}} < 0,$$

hence the desired monotonicity. □

5.3 Refined conditional exceedance probabilities

A consequence from Corollary 5.8 is that when $\lim_{u\to\infty}\delta(u) = \infty$, the conditional exceedance probability χ becomes uninformative. Instead, we consider in this section the refined conditional exceedance probabilities

$$\mathbb{P}\{Y^{(s_0)}(s) > au \,|\, Y^{(s_0)}(s_0) = u\} \qquad a > 0, \quad u \geq 1, \quad s \in \mathbb{R}^d \setminus \{s_0\}, \tag{59}$$

and investigate to what extent they are a reasonable approximation of their original non-limiting counterparts

$$\mathbb{P}\{X(s) > au \,|\, X(s_0) = u\} \qquad a > 0, \quad u \geq 1, \quad s \in \mathbb{R}^d \setminus \{s_0\}, \tag{60}$$

when X is the Gaussian copula process (55) and $Y^{(s_0)}$ its tail approximating process.

To make this precise, we need to choose a scaling function δ. As observed in Remark 5.5, we may use

$$\delta(u) = \max(1, b(u))^{2/\alpha} \quad \text{with} \quad b(u) = \Phi^{-1}\left(1 - \frac{1}{u}\right), \quad u \geq 1, \tag{61}$$

where $\alpha \in (0,2]$ is the degree of homegeneity of the variogram γ involved in the definition of the point-spectral process $V^{(s_0)}$ and its tail approximating process $Y^{(s_0)}$, respectively. This choice also ensures that $\gamma = \gamma^{(\rho)}$ in Theorem 5.4 and we do not need to distinguish between the two semi-variograms for this purpose, cf. also Remark 5.5; otherwise, the linking scaling factor $c^{-\alpha}$ would cancel out in the calculations below (as it should). To avoid degenerate cases, we assume henceforth that

$$\gamma(h) > 0 \quad \text{and} \quad \rho(h) \in (-1,1) \quad \text{for} \quad h = s - s_0 \in \mathbb{R}^d \setminus \{0\}.$$

Due to (40) and (57) we can express the conditional exceedance probabilities of interest as

$$\mathbb{P}\{Y^{(s_0)}(s_0 + h) > au \,|\, Y^{(s_0)}(s_0) = u\} = \overline{\Phi}\left(\frac{\log(a) + \gamma(\delta(u)h)}{\sqrt{2\gamma(\delta(u)h)}}\right) = \overline{\Phi}\left(\frac{\log(a) + \delta(u)^\alpha\gamma(h)}{\delta(u)^{\alpha/2}\sqrt{2\gamma(h)}}\right) \tag{62}$$

and

$$\mathbb{P}\{X(s_0 + h) > au \,|\, X(s_0) = u\} = \overline{\Phi}\left(\frac{b(au) - b(u)\rho(h)}{\sqrt{1-\rho(h)^2}}\right). \tag{63}$$

Proposition 5.10 examines the relative error made when approximating (63) by (62). Its proof is straightforward and uses that $\delta(u)^\alpha = b(u)^2$ for $u \geq 1/\overline{\Phi}(1)$, while employing $\overline{\Phi}(x) \sim \varphi(x)/x$ as $x \to \infty$. The explicit calculations are postponed to Lemma A.8.

Proposition 5.10. *For $a > 0$, and as $u \to \infty$, the ratio of conditional exceedance probabilities for the Gaussian copula process X and its tail approximating process $Y^{(s_0)}$ behave as*

$$\frac{\mathbb{P}\big\{X(s_0+h) > au \,\big|\, X(s_0) = u\big\}}{\mathbb{P}\big\{Y^{(s_0)}(s_0+h) > au \,\big|\, Y^{(s_0)}(s_0) = u\big\}}$$
$$\sim a^{-\left[1/(1+\rho(h)) - 1/2\right]} \sqrt{\frac{\gamma(h)}{2} \Big/ \frac{1-\rho(h)}{1+\rho(h)}} \exp\left(\frac{b(u)^2}{2}\left[\frac{\gamma(h)}{2} - \frac{1-\rho(h)}{1+\rho(h)}\right]\right).$$

This relation calls for taking a closer look at the special case when

$$\frac{\gamma(h)}{2} = \frac{1-\rho(h)}{1+\rho(h)} \tag{64}$$

holds for h in a compact convex neighbourhood, say S, of the origin $0 \in \mathbb{R}^d$. Note that (52) is directly implied by (64), if γ is α-homogeneous. In this situation, as $u \to \infty$, and for $h \in S$

$$\frac{\mathbb{P}\big\{X(s_0+h) > au \,\big|\, X(s_0) = u\big\}}{\mathbb{P}\big\{Y^{(s_0)}(s_0+h) > au \,\big|\, Y^{(s_0)}(s_0) = u\big\}} \sim a^{-\left[1/(1+\rho(h)) - 1/2\right]}. \tag{65}$$

For $a = 1$, we even have equality

$$\mathbb{P}\big\{X(s_0+h) > u \,\big|\, X(s_0) = u\big\} = \mathbb{P}\big\{Y^{(s_0)}(s_0+h) > u \,\big|\, Y^{(s_0)}(s_0) = u\big\}$$

for $u \geq 1/\overline{\Phi}(1)$, i.e., from the $\approx 84\%$-quantile of the standard Pareto law; see Lemma A.8. In Figure 3 this precision for $a = 1$ is illustrated in the top row of plots of a range of such probabilities as functions of $\rho \in (0, 1)$. In addition, Figure 3 includes plots for $a = 2/3 < 1$ and $a = 3/2 > 1$ for comparison. A priori, we know that, as $\rho \to 1$ (i.e. $\gamma \to 0$), both (63) and (62) converge to 1, $1/2$ and 0 in the cases $a < 1$, $a = 1$ and $a > 1$, respectively, and we find this confirmed here. But even for $a \neq 1$ and across a range of several quantiles u (we consider 90% to 99.99%), the approximations seem surprisingly close. From (65), we see that for $a \neq 1$ we make a relative error when approximating (63) by (62), which disappears as $\rho \to 1$. This is visible in Figure 4, where we display such probabilities on a log-scale. A situation, in which (64) holds, is for instance given when

$$\rho(h) = \begin{cases} \dfrac{1 - c'\|h\|^\alpha/2}{1 + c'\|h\|^\alpha/2} & \text{for } \|h\| \leq (2/c')^{1/\alpha}, \\ 0 & \text{else.} \end{cases} \tag{66}$$

General statements, in which dimensions d and for which $\alpha \in (0, 2]$ the function (66) constitutes a valid covariance function in $\mathbb{R}^d$, are difficult to obtain. However, to give an example, it is valid for $d = 1$ and $\alpha = 1$, since $(1 + c'\|h\|^\alpha/2)^{-1}$ is a member of both, the Dagum and the Cauchy family for any $\alpha \in (0, 2]$, cf. Berg et al. (2008), while the validity of $(1 - c'\|h\|^\alpha/2)_+$ for $d = 1$ and $\alpha = 1$ goes back to Golubov (1981), cf. also Gneiting (2002). Another standard relation between correlation function and variogram that obeys (52) is

$$\rho(h) = \exp(-\gamma(h)), \qquad h \in \mathbb{R}^d. \tag{67}$$

Figure 3 displays (63) and (62) when they are linked in this way in the bottom row, while Figure 4 includes the resulting values for (62) on a log-scale with shorter dashes. The larger the correlation ρ, the closer we are to the previously described patterns, when ρ and γ are linked via (64) instead. Since (52) is a local property of the correlation function, this is precisely what we can expect from the tail approximating process $Y^{(s_0)}$ for the Gaussian copula process X from (55), that is, to approximate X well for large values of u in a neighbourhood of conditioning sites s_0.

For comparison, let us also state the refined conditional exceedance probabilities (59) and (60) for a different process X returning to a situation, when no domain-scaling is required and X is in the max-domain of attraction of the Brown–Resnick process, cf. Definition 5.1. The easiest candidate is the Brown–Resnick process itself, albeit on the slightly modified standard Pareto scale, i.e.

$$X(s) = T(Z(s)), \qquad s \in \mathbb{R}^d, \qquad \text{where} \qquad T(z) = 1/\big(1 - \exp(-1/z)\big),$$

cf. Lemma 5.2. Let $\gamma = \gamma(h) > 0$ as above. From the well-known bivariate marginal distribution of Z, that is,

$$\mathbb{P}\{Z(s_0+h) \leq z\,,\, Z(s_0) \leq z_0\} = \exp\left(-\left\{\frac{1}{z}\Phi\left(\frac{\log(z_0/z) + \gamma}{\sqrt{2\gamma}}\right) + \frac{1}{z_0}\Phi\left(\frac{\log(z/z_0) + \gamma}{\sqrt{2\gamma}}\right)\right\}\right),$$

we obtain readily the conditional distribution

$$\begin{aligned}
&\mathbb{P}\big\{Z(s_0+h) > z \,|\, Z(s_0) = z_0\big\} \\
&= 1 - \Phi\bigg(\frac{\log(z/z_0)+\gamma}{\sqrt{2\gamma}}\bigg) \exp\bigg(-\bigg\{\frac{1}{z}\Phi\bigg(\frac{\log(z_0/z)+\gamma}{\sqrt{2\gamma}}\bigg) + \frac{1}{z_0}\Phi\bigg(\frac{\log(z/z_0)+\gamma}{\sqrt{2\gamma}}\bigg) - \frac{1}{z_0}\bigg\}\bigg),
\end{aligned}$$

and hence

$$\mathbb{P}\big\{X(s_0+h) > au \,|\, X(s_0) = u\big\} = \mathbb{P}\big\{Z(s_0+h) > T^{-1}(au) \,|\, Z(s_0) = T^{-1}(u)\big\}. \tag{68}$$

On the other hand, when $\delta \equiv 1$, the conditional exceedance probability of the tail approximating process $Y^{(s_0)}$ of the Brown–Resnick process is simply

$$\mathbb{P}\big\{Y^{(s_0)}(s_0+h) > au \,\big|\, Y^{(s_0)}(s_0) = u\big\} = \overline{\Phi}\bigg(\frac{\log(a)+\gamma(h)}{\sqrt{2\gamma(h)}}\bigg) \tag{69}$$

and does not depend on $u \geq 1$. Figures 5 and 6 are analogous to Figures 3 and 4 and display these conditional exceedance probabilities as functions of $\exp(-\gamma) \in (0,1)$. In this situation we have

$$\lim_{u\to\infty} \frac{\mathbb{P}\big\{X(s_0+h) > au \,\big|\, X(s_0) = u\big\}}{\mathbb{P}\big\{Y^{(s_0)}(s_0+h) > au \,\big|\, Y^{(s_0)}(s_0) = u\big\}} = 1$$

and any relative error is quickly absorbed for high quantiles, cf. Figure 6.

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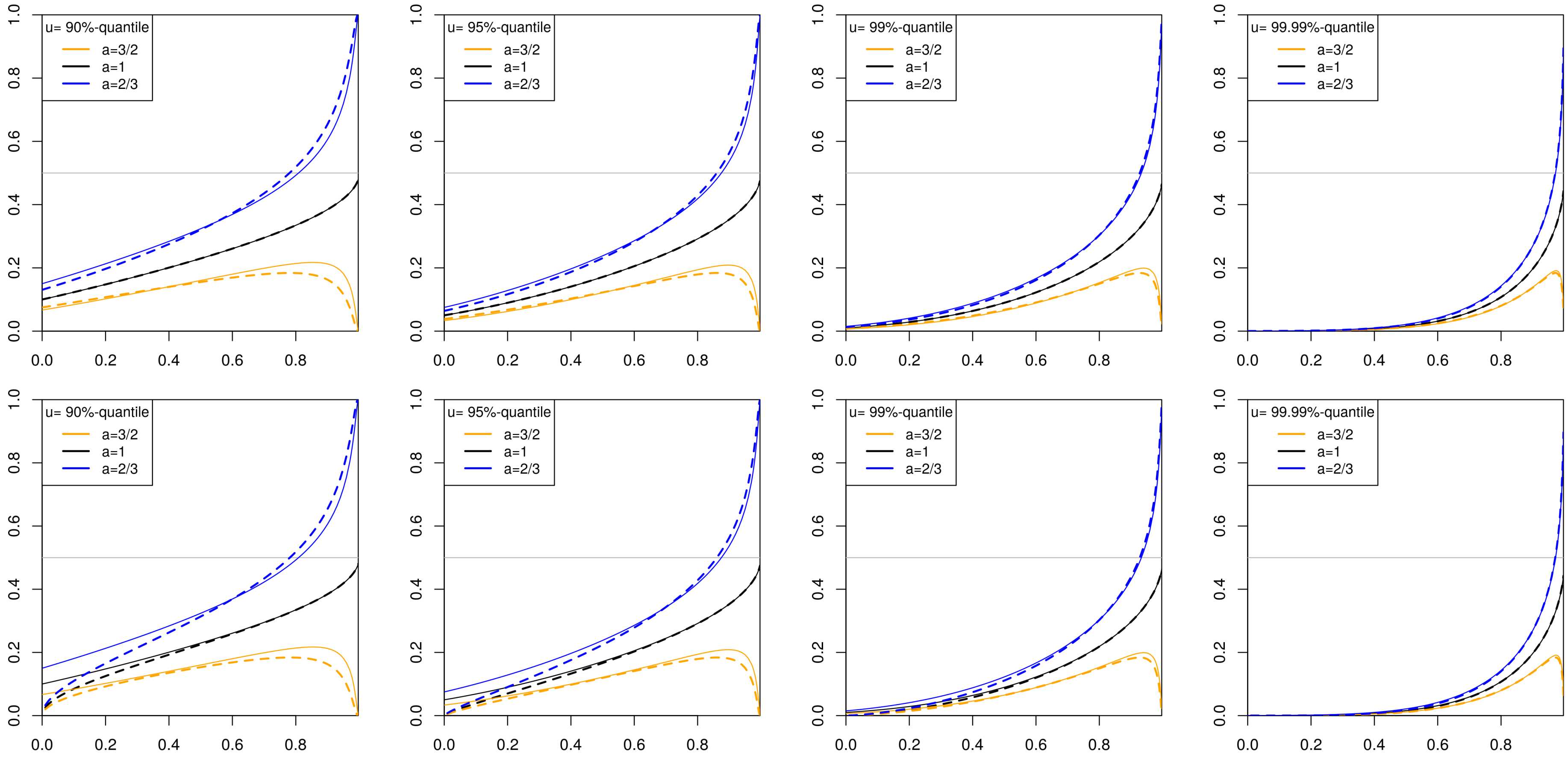

Figure 3: Conditional exceedance probabilities $\mathbb{P}\{X(s) > au \,|\, X(s_0) = u\}$ of the Gaussian copula process from (63) (solid) versus their tail approximations $\mathbb{P}\{Y^{(s_0)}(s) > au \,|\, Y^{(s_0)}(s_0) = u\}$ from (62) (dashed) as functions of the correlation $\rho \in (0, 1)$ (x-axis) for different values of u and a; top row: the correlation ρ of X and the semi-variogram γ of $Y^{(s_0)}$ are related via $\gamma/2 = (1-\rho)/(1+\rho)$ as locally in (64); bottom row: they are related via $\rho = \exp(-\gamma)$ as in (67). A refined plot with probabilities on a log-scale for the two upper-most u is given in Figure 4

.

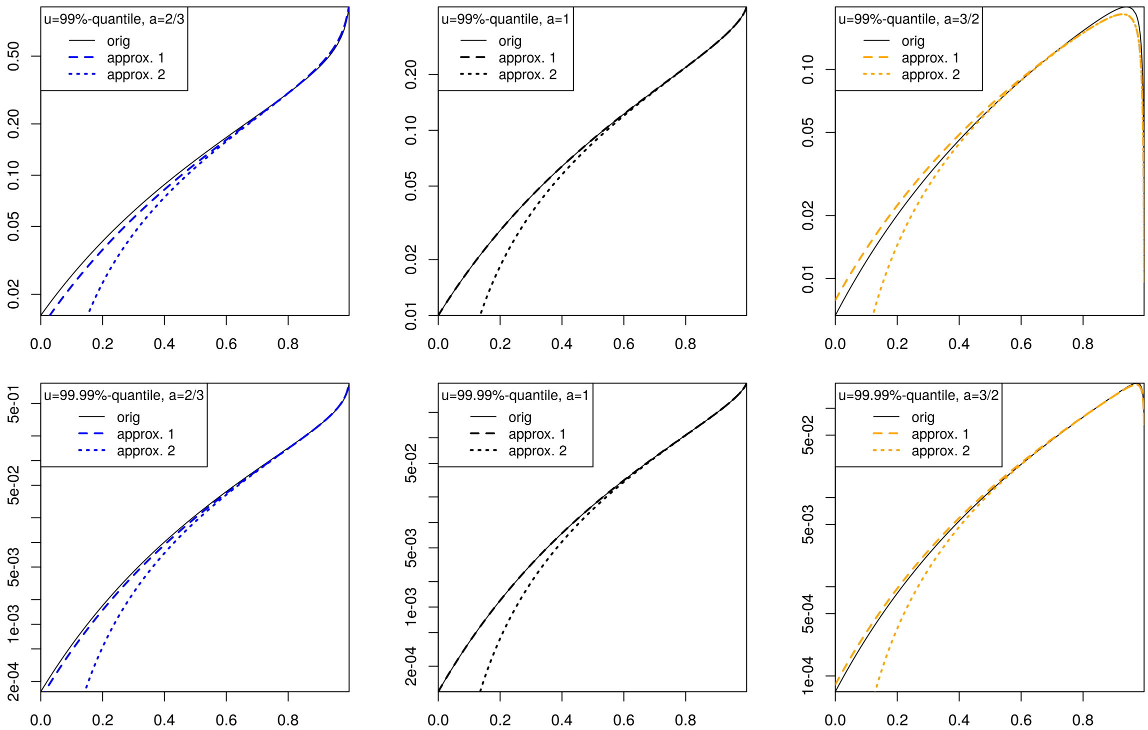

Figure 4: Conditional exceedance probabilities $\mathbb{P}\{X(s) > au \mid X(s_0) = u\}$ of the Gaussian copula process from (63) (solid, black) versus their tail approximations $\mathbb{P}\{Y^{(s_0)}(s) > au \mid Y^{(s_0)}(s_0) = u\}$ from (62) (coloured, dashed) as functions of the correlation $\rho \in (0, 1)$ (x-axis) for different values of u and a displayed on a log-scale (y-axis); top row: u is the 99%-quantile of the conditioning variable, bottom row: u is the 99.99%-quantile. In approximation 1 the semi-variogram γ of $Y^{(s_0)}$ and the correlation ρ of X are related via $\gamma/2 = (1-\rho)/(1+\rho)$ as locally in (64); in approximation 2 they are related via $\rho = \exp(-\gamma)$ as in (67). In Figure 3 these probabilities are displayed on their original scale.

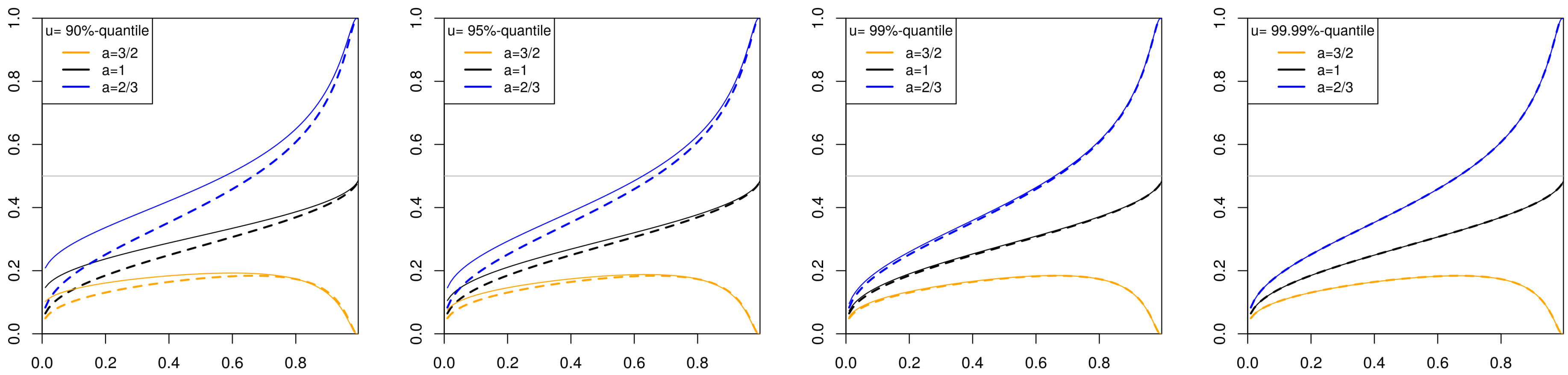

Figure 5: Conditional exceedance probabilities $\mathbb{P}\{X(s) > au \,|\, X(s_0) = u\}$ of the Brown–Resnick copula process from (68) (solid) versus their tail approximations $\mathbb{P}\{Y^{(s_0)}(s) > au \,|\, Y^{(s_0)}(s_0) = u\}$ from (69) (dashed) as functions of $\exp(-\gamma) \in (0,1)$ (x-axis) for different values of u and a. A refined plot with probabilities on a log-scale for the two upper-most u is given in Figure 6

.

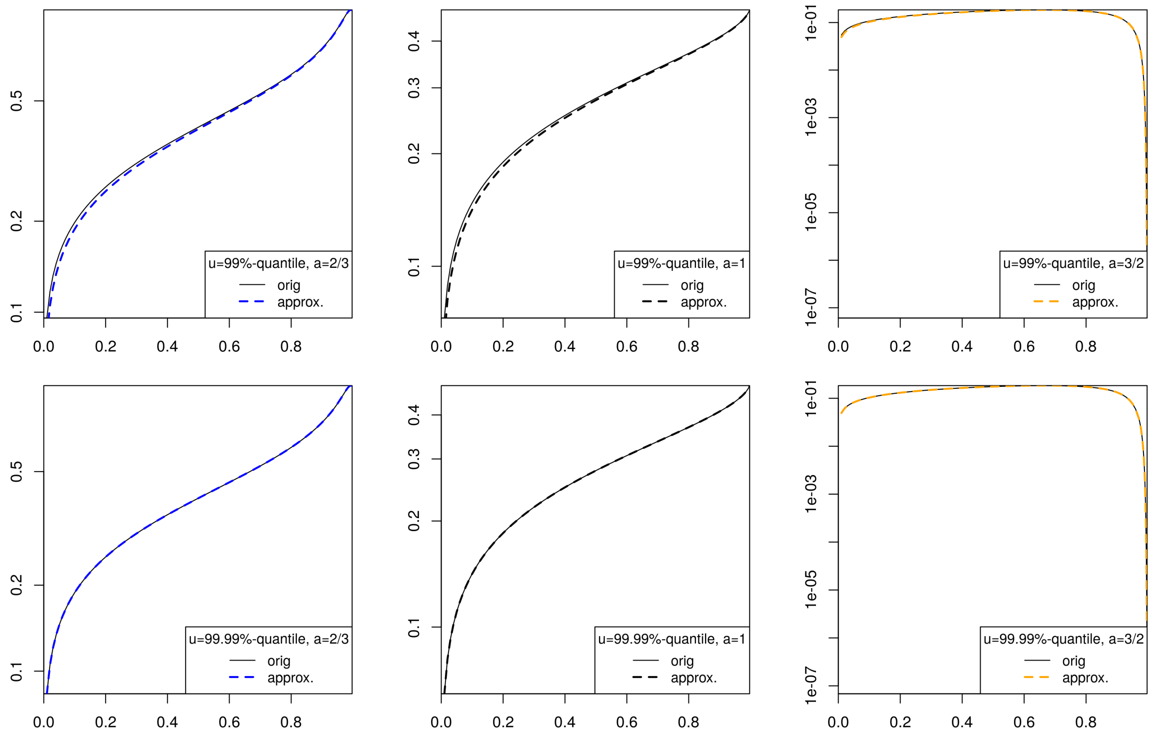

Figure 6: Conditional exceedance probabilities $\mathbb{P}\{X(s) > au \,|\, X(s_0) = u\}$ of the Brown–Resnick copula process from (68) (solid, black) versus their tail approximations $\mathbb{P}\{Y^{(s_0)}(s) > au \,|\, Y^{(s_0)}(s_0) = u\}$ from (69) (coloured, dashed) as functions of $\exp(-\gamma) \in (0,1)$ (x-axis) for different values of u and a displayed on a log-scale (y-axis); top row: u is the 99%-quantile of the conditioning variable, bottom row: u is the 99.99%-quantile. In Figure 5 these probabilities are displayed on their original scale.

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A Auxiliary results

Lemma A.1. *Let $S \subset \mathbb{R}^d$ be compact and $r : C_{\geq}(S) \to [0, \infty)$ be either $r(f) = \|f\|_\infty = \sup_{s\in S} f(s)$, $r(f) = f(s_0)$ for some $s_0 \in S$ or $r(f) = \inf_{s\in S} f(s)$. Then*

$$\inf\left\{\|f - g\|_\infty \,:\, g \in C_{\geq}(S),\, r(g) = 0\right\} = r(f)$$

for $f \in C_{\geq}(S)$.

Proof. When $r(f) = \|f\|_\infty$, the only $g \in C_{\geq}(S)$ that satisfies $r(g) = 0$ is $g = 0$, and so the claimed identity is immediate. Let $r(f) = f(s_0)$ for some $s_0 \in S$, then

$$\begin{aligned}\inf\left\{\|f - g\|_\infty \,:\, g \in C_{\geq}(S),\, r(g) = 0\right\} &\geq \inf\left\{|f(s_0) - g(s_0)| \,:\, g \in C_{\geq}(S),\, g(s_0) = 0\right\}\\ &= f(s_0) = r(f).\end{aligned}$$

On the other hand, the function g given by $g(s) = (f(s)-f(s_0))_+ = \max(f(s)-f(s_0), 0)$ satisfies $g \in C_{\geq}(S)$ and $r(g) = 0$, so that

$$\begin{aligned}\inf\left\{\|f - g\|_\infty \,:\, g \in C_{\geq}(S),\, r(g) = 0\right\} &\leq \|f - (f - f(s_0))_+\|_\infty\\ &= \|\min(f(s_0), f)\|_\infty = f(s_0) = r(f).\end{aligned}$$

Finally, let $r(f) = \inf_{s\in S} f(s)$. Then $g = f - r(f)$ satisfies $g \in C_{\geq}(S)$ and $r(g) = 0$, so that

$$\inf\left\{\|f - g\|_\infty \,:\, g \in C_{\geq}(S),\, r(g) = 0\right\} \leq \|f - (f - r(f))\|_\infty = r(f).$$

On the other hand, let $g \in C_{\geq}(S)$ with $r(g) = \inf_{s\in S} g(s) = 0$. Then, by continuity of g, there exists $s_0 \in S$ such that $g(s_0) = 0$. Hence,

$$\|f - g\|_\infty \geq |f(s_0) - g(s_0)|_\infty = f(s_0) \geq r(f),$$

and the assertion follows. □

Lemma A.2. *Consider a random variable X such that $X \geq 0$ almost surely, and P a standard Pareto variable independent from X. Then*

$$\mathbb{E}[X] = \lim_{u\to\infty} u\,\mathbb{P}\{PX > u\} \in [0, \infty].$$

Proof. Since $X \geq 0$ almost surely,

$$\begin{aligned}\mathbb{E}[X] &= \lim_{u\to\infty} \int_0^u \mathbb{P}\{X > t\}\,\mathrm{d}t = \lim_{u\to\infty} u \int_0^1 \mathbb{P}\{t^{-1}X > u\}\,\mathrm{d}t\\ &= \lim_{u\to\infty} u \int_1^\infty \mathbb{P}\{tX > u\}\,\nu_1(\mathrm{d}t) = \lim_{u\to\infty} u\,\mathbb{P}\{PX > u\}.\end{aligned}$$ □

Lemma A.3. *Let $S \subset \mathbb{R}^d$ be compact. Let $f_n, f \in C(S, [0, \infty))$ with $f_n \to f$. Let $s_n, s^* \in S$ with $s_n \to s^*$ and suppose $f(s^*) \neq 1$. Then $\mathbb{1}\{f_n(s_n) > 1\} \to \mathbb{1}\{f(s^*) > 1\}$.*

Proof. We have either $f(s^*) > 1$ (case 1) or $f(s^*) < 1$ (case 2).

Case 1: Let $f(s^*) > 1$. Then $\varepsilon = f(s^*) - 1 > 0$. In particular, there exists $N_1 \in \mathbb{N}$, such that for all $n \geq N_1$ we have $f(s_n) > 1 + \varepsilon/2$. In addition, there exists N_2 such that for all $n \geq N_2$ we have $|f_n(s) - f(s)| < \varepsilon/2$ for all $s \in S$. So for $n \geq \max(N_1, N_2)$

$$f_n(s_n) = (f_n(s_n) - f(s_n)) + f(s_n) > -\varepsilon/2 + (1 + \varepsilon/2) > 1.$$

Case 2: Similarly, $f(s^*) < 1$ implies that eventually $f_n(s_n) < 1$ for $n \geq N$ for some $N \in \mathbb{N}$.

In either case, the assertion follows. □

The following lemma is a special form of exponential tilting for Gaussian densities.

Lemma A.4. *Let $\boldsymbol{N} = (N_1, N_2, \dots, N_m)^\top$ be a non-degenerate zero-mean variance-one Gaussian random vector with $\rho_{ij} = \mathbb{E}(N_i N_j)$ for $1 \le i,j \le m$. In addition, let $\sigma_i > 0$, $i = 1, \dots, m$. Denote*

$$\begin{aligned}\boldsymbol{\sigma N} &= (\sigma_1 N_1, \sigma_2 N_2, \dots, \sigma_m N_m)^\top,\\ \boldsymbol{\sigma N} \pm \boldsymbol{\sigma}\sigma_i \boldsymbol{\rho}_i &= (\sigma_1 N_1 \pm \sigma_1\sigma_i\rho_{1i}, \sigma_2 N_2 + \sigma_2\sigma_i\rho_{2i}, \dots, \sigma_m N_m \pm \sigma_m\sigma_i\rho_{mi})^\top.\end{aligned}$$

Then

$$e^{\pm t_i} f_{\boldsymbol{\sigma N}}(t_1, \dots, t_m) = e^{\sigma_i^2/2} f_{\boldsymbol{\sigma N} \pm \boldsymbol{\sigma}\sigma_i\boldsymbol{\rho}_i}(t_1, \dots, t_m), \qquad (t_1, \dots, t_m)^\top \in \mathbb{R}^m.$$

Proof. Without loss of generality let $i = 1$. Observe that $\boldsymbol{\rho}_1 = (\rho_{11}, \rho_{21}, \dots, \rho_{m1})^\top$ is the first column of the correlation matrix Σ of $\boldsymbol{N}$, i.e. $\boldsymbol{\rho}_1 = \Sigma \boldsymbol{e}_1$, where $\boldsymbol{e}_1 = (1, 0, \dots, 0)^\top \in \mathbb{R}^m$. Since $\boldsymbol{N}$ is non-degenerate, we may conclude that $\boldsymbol{e}_1 = \Sigma^{-1}\boldsymbol{\rho}_1$. Setting

$$\boldsymbol{y} = \left(\frac{t_1}{\sigma_1}, \dots, \frac{t_m}{\sigma_m}\right)^\top,$$

we may rewrite

$$\begin{aligned}e^{\pm t_1} f_{\boldsymbol{\sigma N}}(t_1, \dots, t_m) &= e^{\pm t_1} f_{\boldsymbol{N}}\left(\frac{t_1}{\sigma_1}, \dots, \frac{t_m}{\sigma_m}\right) = (2\pi)^{-d/2} \exp\left(\pm t_1 - \frac{1}{2}\boldsymbol{y}^\top \Sigma^{-1} \boldsymbol{y}\right)\\ e^{\sigma_1^2/2} f_{\boldsymbol{\sigma N} \pm \boldsymbol{\sigma}\sigma_1\boldsymbol{\rho}_1}(t_1, \dots, t_m) &= e^{\sigma_1^2/2} f_{\boldsymbol{N}}\left(\frac{t_1}{\sigma_1} \mp \sigma_1\rho_{11}, \dots, \frac{t_m}{\sigma_m} \mp \sigma_1\rho_{m1}\right)\\ &= (2\pi)^{-d/2} \exp\left(\frac{1}{2}\sigma_1^2 - \frac{1}{2}(\boldsymbol{y} \mp \sigma_1\boldsymbol{\rho}_1)^\top \Sigma^{-1} (\boldsymbol{y} \mp \sigma_1\boldsymbol{\rho}_1)\right)\\ &= (2\pi)^{-d/2} \exp\left(\frac{1}{2}\sigma_1^2 - \frac{1}{2}\boldsymbol{y}^\top \Sigma^{-1}\boldsymbol{y} \pm \sigma_1 \boldsymbol{y}^\top\Sigma^{-1}\boldsymbol{\rho}_1 - \frac{1}{2}\sigma_1^2 \boldsymbol{\rho}_1^\top \Sigma^{-1}\boldsymbol{\rho}_1\right),\\ &= (2\pi)^{-d/2} \exp\left(\frac{1}{2}\sigma_1^2 - \frac{1}{2}\boldsymbol{y}^\top \Sigma^{-1}\boldsymbol{y} \pm \sigma_1 \boldsymbol{y}^\top\boldsymbol{e}_1 - \frac{1}{2}\sigma_1^2 \boldsymbol{\rho}_1^\top \boldsymbol{e}_1\right),\\ &= (2\pi)^{-d/2} \exp\left(\frac{1}{2}\sigma_1^2 - \frac{1}{2}\boldsymbol{y}^\top \Sigma^{-1}\boldsymbol{y} \pm \sigma_1 \frac{t_1}{\sigma_1} - \frac{1}{2}\sigma_1^2 \rho_{11}\right),\end{aligned}$$

and so the assertion follows with $\rho_{11} = 1$. □

Lemma A.5. *Let $V^{(s_0)}$ be the point-spectral process from* (57) *and P an independent Pareto variable. Let us abbreviate $\gamma = \gamma(s - s_0)$ for $s \in \mathbb{R}^d$ and assume $\gamma > 0$. Then for $u > 0$*

$$\mathbb{P}\{PV^{(s_0)}(s) > u\} = \Phi\left(\frac{\log(1/u) - \gamma}{\sqrt{2\gamma}}\right) + \frac{1}{u}\Phi\left(\frac{\log(u) - \gamma}{\sqrt{2\gamma}}\right), \tag{70}$$

and for $u > 0$, $u_0 \ge 1$

$$\mathbb{P}\{PV^{(s_0)}(s) > u,\, P > u_0\} = \frac{1}{u_0}\Phi\left(\frac{\log(u_0/u) - \gamma}{\sqrt{2\gamma}}\right) + \frac{1}{u}\Phi\left(\frac{\log(u/u_0) - \gamma}{\sqrt{2\gamma}}\right). \tag{71}$$

Proof. Let $t = \log(u)$, $\sigma = \sqrt{2\gamma}$ and $N \sim \mathcal{N}(0,1)$ independent of a standard exponential random variable E. Then

$$\begin{aligned}\mathbb{P}\{PV^{(s_0)}(s) > u\} &= \mathbb{P}\{E + \sigma N > t + \gamma\}\\ &= \mathbb{P}\{\sigma N > t + \gamma\} + \mathbb{P}\{E + \sigma N > t + \gamma,\, \sigma N \le t + \gamma\}\\ &= 1 - \Phi\left(\frac{t+\gamma}{\sigma}\right) + \int_{-\infty}^{t+\gamma}\int_{t+\gamma-x}^{\infty} e^{-y} f_{\sigma N}(x)\,\mathrm{d}y\,\mathrm{d}x,\end{aligned}$$

where $f_{\sigma N}$ is the density of σN. Using Lemma A.4 for $m = 1$ and $\sigma^2 = 2\gamma$, we obtain

$$\begin{aligned}\int_{-\infty}^{t+\gamma}\int_{t+\gamma-x}^{\infty} e^{-y} f_{\sigma N}(x)\,\mathrm{d}y\,\mathrm{d}x &= \int_{-\infty}^{t+\gamma}\int_{t+\gamma}^{\infty} e^{-(y-x)} f_{\sigma N}(x)\,\mathrm{d}y\,\mathrm{d}x\\ &= \int_{t+\gamma}^{\infty} e^{-y}\,\mathrm{d}y \int_{-\infty}^{t+\gamma} e^{x} f_{\sigma N}(x)\,\mathrm{d}x = e^{-(t+\gamma)}\int_{-\infty}^{t+\gamma} e^{\gamma} f_{\sigma N + 2\gamma}(x)\,\mathrm{d}x,\end{aligned}$$

so that

$$\mathbb{P}\big\{PV^{(s_0)}(s) > u\big\} = 1 - \Phi\left(\frac{t+\gamma}{\sigma}\right) + e^{-t}\,\mathbb{P}\{\sigma N + 2\gamma \le t + \gamma\} = \Phi\left(\frac{-t-\gamma}{\sigma}\right) + e^{-t}\,\Phi\left(\frac{t-\gamma}{\sigma}\right),$$

from which we obtain (70) by substituting $\log(u)$ for t and $\sqrt{2\gamma}$ for σ again.

The bivariate joint exceedance probability (71) is then obtained from (42) with $\delta \equiv 1$ as

$$\mathbb{P}\big\{PV^{(s_0)}(s) > u,\ PV^{(s_0)}(s_0) > u_0\big\} = u_0^{-1}\mathbb{P}\big\{PV^{(s_0)}(s) > u/u_0\big\}.$$

This finishes the proof. □

Lemma A.6. *For fixed $t \in \mathbb{R}$, the probability*

$$\overline{\Phi}\left(\frac{t+\gamma}{\sqrt{2\gamma}}\right)$$

is strictly increasing in γ for $\gamma \in (0,t)$ and strictly decreasing in γ for $\gamma \in (t,\infty)$.

Proof. Denote $\gamma = \sigma^2/2$, such that

$$\overline{\Phi}\left(\frac{t+\gamma}{\sqrt{2\gamma}}\right) = \overline{\Phi}\left(\frac{t}{\sigma} + \frac{\sigma}{2}\right),$$

whose derivative in σ equals

$$\frac{1}{\sigma^2}\varphi\left(\frac{t}{\sigma} + \frac{\sigma}{2}\right)(t-\gamma),$$

hence the assertion. □

Lemma A.7. *Let $b(u) = \Phi^{-1}(1-1/u)$. Then*

$$\lim_{u\to\infty} b(u)\big(b(au) - b(u)\big) = \log(a), \qquad a > 0.$$

Proof. Recall that b is slowly varying. Hence $\lim_{u\to\infty} b(au)/b(u) = 1$. So,

$$b(u)\big(b(au) - b(u)\big) \sim b(u)\big(b(au) - b(u)\big)\cdot\frac{1}{2}\left(\frac{b(au)}{b(u)} + 1\right) = \frac{1}{2}\big(b(au)^2 - b(u)^2\big),$$

and so the assertion follows from

$$\exp\left(\frac{1}{2}\big(b(au)^2 - b(u)^2\big)\right) = \frac{\varphi(b(u))}{\varphi(b(au))} \sim \frac{b(u)}{b(au)}\cdot\frac{\overline{\Phi}(b(u))}{\overline{\Phi}(b(au))} \sim 1\cdot\frac{1/u}{1/(au)} = a.$$

□

Lemma A.8. *Let $b(u) = \Phi^{-1}(1-1/u)$, $\gamma > 0$ and $\rho \in (-1,1)$ and $a > 0$. Then, as $u \to \infty$,*

$$\left(\frac{b(au) - b(u)\rho}{\sqrt{1-\rho^2}}\right)^{-1}\left(\frac{\log(a) + b(u)^2\gamma}{b(u)\sqrt{2\gamma}}\right) \sim \sqrt{\frac{\gamma}{2}\Big/\frac{1-\rho}{1+\rho}},$$

and

$$\varphi\left(\frac{b(au) - b(u)\rho}{\sqrt{1-\rho^2}}\right)\left\{\varphi\left(\frac{\log(a) + b(u)^2\gamma}{b(u)\sqrt{2\gamma}}\right)\right\}^{-1} \sim a^{-\left[1/(1+\rho) - 1/2\right]}\exp\left(\frac{b(u)^2}{2}\left(\frac{\gamma}{2} - \frac{1-\rho}{1+\rho}\right)\right).$$

In both situations, we have equality for all $u > 1$ when $a = 1$.

Proof. We may rewrite

$$\left(\frac{b(au)-b(u)\rho}{\sqrt{1-\rho^2}}\right)^{-1}\left(\frac{\log(a)+b(u)^2\gamma}{b(u)\sqrt{2\gamma}}\right) = \sqrt{\frac{(1-\rho^2)}{2\gamma}}\frac{\log(a)+b(u)^2\gamma}{b(u)(b(au)-b(u))+b(u)^2(1-\rho)}$$
$$= \sqrt{\frac{\gamma(1+\rho)}{2(1-\rho)}}\frac{\log(a)/\gamma+b(u)^2}{b(u)(b(au)-b(u))/(1-\rho)+b(u)^2},$$

so that the first assertion follows with Lemma A.7 and $\lim_{u\to\infty} b(u)=\infty$. In case $a=1$ the second fraction equals indeed 1. Further,

$$\varphi\left(\frac{b(au)-b(u)\rho}{\sqrt{1-\rho^2}}\right)\left\{\varphi\left(\frac{\log(a)+b(u)^2\gamma}{b(u)\sqrt{2\gamma}}\right)\right\}^{-1}$$
$$= \exp\left(\frac{1}{2}\left\{\left(\frac{\log(a)+b(u)^2\gamma}{b(u)\sqrt{2\gamma}}\right)^2-\left(\frac{b(u)(b(au)-b(u))+b(u)^2(1-\rho)}{b(u)\sqrt{1-\rho^2}}\right)^2\right\}\right)$$
$$= \exp\left(\frac{1}{2}\left\{\frac{1}{b(u)^2}\left(\frac{\log(a)^2}{2\gamma}-\frac{[b(u)(b(au)-b(u))]^2}{1-\rho^2}\right)\right.\right.$$
$$\left.\left.+\left(\log(a)-\frac{2}{1+\rho}b(u)(b(au)-b(u))\right)+b(u)^2\left(\frac{\gamma}{2}-\frac{1-\rho}{1+\rho}\right)\right\}\right).$$

For $a=1$, we can read off the claimed equality. Else, we obtain with $\lim_{u\to\infty} b(u)=\infty$ and Lemma A.7 that

$$\varphi\left(\frac{b(au)-b(u)\rho}{\sqrt{1-\rho^2}}\right)\left\{\varphi\left(\frac{\log(a)+b(u)^2\gamma}{b(u)\sqrt{2\gamma}}\right)\right\}^{-1} \sim \exp\left(\frac{1}{2}\left\{\log(a)\left(1-\frac{2}{1+\rho}\right)+b(u)^2\left(\frac{\gamma}{2}-\frac{1-\rho}{1+\rho}\right)\right\}\right),$$

as desired. □

B Marginal distributions of the tail approximating process

In Section 4.2 we have seen that the domain-scaled tail approximating process behaves like a domain-scaled regularly varying process (Proposition 4.4), but without marginal standardisation. Here, we investigate the tail behaviour of the univariate marginal distributions of the domain-scaled tail approximating process $Y^{(s_0)}$ from Definition 4.1. By definition, $Y^{(s_0)}(s_0)$ is standard Pareto, that is,

$$\mathbb{P}\{Y^{(s_0)}(s_0)>u\}=u^{-1}, \quad u\geq 1.$$

For $s\in\mathbb{R}^d\setminus\{s_0\}$ the tail behaviour of $Y^{(s_0)}(s)$ depends on the domain-scaling. A priori we know the following. By definition of $Y^{(s_0)}$, we have for a standard Pareto variable P independent from the point-spectral process $V^{(s_0)}$ that

$$\begin{aligned} u\,\mathbb{P}\{Y^{(s_0)}(s)>u\} &= u\,\mathbb{P}\{PV^{(s_0)}(s_0+\delta(P)(s-s_0))>u\} \\ &= u\int_1^\infty \mathbb{P}\{tV^{(s_0)}(s_0+\delta(t)(s-s_0))>u\}\nu_1(\mathrm{d}t) \\ &= \int_{1/u}^\infty \mathbb{P}\{tV^{(s_0)}(s_0+\delta(ut)(s-s_0))>1\}\nu_1(\mathrm{d}t) \\ &= \int_0^u \mathbb{P}\{V^{(s_0)}(s_0+\delta(u/x)(s-s_0))>x\}\mathrm{d}x. \end{aligned} \tag{72}$$

If there is no domain-scaling ($\delta\equiv 1$), it is immediate and well-known that $Y^{(s_0)}(s)$ will be either Pareto-like or $Y^{(s_0)}(s)=0$ almost surely, in other words,

$$\lim_{u\to\infty} u\,\mathbb{P}\{Y^{(s_0)}(s)>u\}=\mathbb{E}[V^{(s_0)}(s)], \qquad s\in\mathbb{R}^d, \tag{73}$$

cf. also Remark 4.5. If, instead, the domain-scaling becomes relevant, we distinguish again the two cases: (i) $\lim_{u\to\infty}\delta(u)=c\in[1,\infty)$ and (ii) $\lim_{u\to\infty}\delta(u)=\infty$, cf. Section 3.

Lemma B.1. *Let $Y^{(s_0)}$ be a domain-scaled tail approximating process with point-spectral process $V^{(s_0)}$ and scaling function δ, such that δ is bounded and satisfies $\lim_{u\to\infty}\delta(u) = c \in [1,\infty)$. Then*

$$\liminf_{u\to\infty} u\,\mathbb{P}\{Y^{(s_0)}(s) > u\} \geq \mathbb{E}\big[V^{(s_0)}(s_0 + c(s-s_0))\big], \qquad s \in \mathbb{R}^d.$$

Proof. From (72) we see that

$$u\,\mathbb{P}\{Y^{(s_0)}(s) > u\} \geq \int_0^\infty \mathbb{P}\{tV^{(s_0)}(s_0 + \delta(ut)(s-s_0)) > 1\}\mathbb{1}\{t > 1/u\}\nu_1(\mathrm{d}t).$$

Hence, by Fatou's lemma, we have that

$$\liminf_{u\to\infty} u\,\mathbb{P}\{Y^{(s_0)}(s) > u\} \geq \int_0^\infty \liminf_{u\to\infty}\mathbb{P}\{tV^{(s_0)}(s_0 + \delta(ut)(s-s_0)) > 1\}\mathbb{1}\{t > 1/u\}\nu_1(\mathrm{d}t).$$

From here, we can continue similarly to the proof of Proposition 4.7. Let $t > 0$. Since $\lim_{u\to\infty}\delta(u) = c \in [1,\infty)$, we also have $\lim_{u\to\infty}\delta(ut) = c \in [1,\infty)$, and, since δ is bounded, there exists a compact convex set $S \subset \mathbb{R}^d$, such that $s_0 + \delta(ut)(s-s_0) \in S$ for all $u > 1/t$. In particular, also $s_c = s_0 + c(s-s_0) \in S$. Recall that $\{f \in C(S,[0,\infty)) : f(s_c) > 1\}$ is a continuity set of the law of $PV^{(s_0)}\big|_S$ (by an almost identical argument as presented in the proof of Lemma 2.2). Therefore, the almost sure convergence of $V^{(s_0)}\big(s_0 + \delta(ut)(s-s_0)\big)$ to $V^{(s_0)}\big(s_0 + c(s-s_0)\big)$ on S implies that

$$\begin{aligned}\liminf_{u\to\infty} u\,\mathbb{P}\{Y^{(s_0)}(s) > u\} &\geq \int_0^\infty \mathbb{P}\{tV^{(s_0)}(s_0 + c(s-s_0)) > 1\}\nu_1(\mathrm{d}t)\\ &= \int_0^\infty \mathbb{P}\{V^{(s_0)}(s_0 + c(s-s_0)) > x\}\mathrm{d}x = \mathbb{E}\big[V^{(s_0)}(s_0 + c(s-s_0))\big],\end{aligned}$$

which gives the assertion. □

In order to gain more specific insights, we need to specify the point-spectral process $V^{(s_0)}$. If we plug in the Brown–Resnick point-spectral process (57), this means that we obtain again Pareto-like tails when $\lim_{u\to\infty}\delta(u) = c < \infty$.

Proposition B.2. *Let $Y^{(s_0)}$ be a domain-scaled tail approximating process with point-spectral process $V^{(s_0)}$ and scaling function δ. Suppose that $V^{(s_0)}$ is given by the Brown–Resnick point-spectral process* (57)*, such that the underlying semi-variogram γ satisfies $\gamma(r(s-s_0)) > 0$ for $r > 0$, and that the scaling function δ is bounded and satisfies $\lim_{u\to\infty}\delta(u) = c \in [1,\infty)$. Then*

$$\lim_{u\to\infty} u\,\mathbb{P}\{Y^{(s_0)}(s) > u\} = 1, \qquad s \in \mathbb{R}^d.$$

Proof. It is already clear from Lemma B.1 and (50) that $\liminf_{u\to\infty} u\,\mathbb{P}\{Y^{(s_0)}(s) > u\} = 1$. Moreover, we obtain from (72) that

$$u\,\mathbb{P}\{Y^{(s_0)}(s) > u\} \leq \int_0^\infty \mathbb{P}\Big\{V^{(s_0)}\big(s_0 + \delta(\max(u/x,1))(s-s_0)\big) > x\Big\}\mathrm{d}x. \tag{74}$$

If we can find an integrable majorant for the integrand on the right-hand side, the assertion will follow via dominated convergence, since then

$$\begin{aligned}\limsup_{u\to\infty} u\,\mathbb{P}\{Y^{(s_0)}(s) > u\} &\leq \int_0^\infty \lim_{u\to\infty}\mathbb{P}\Big\{V^{(s_0)}\big(s_0 + \delta(\max(u/x,1))(s-s_0)\big) > x\Big\}\mathrm{d}x\\ &= \int_0^\infty \mathbb{P}\Big\{V^{(s_0)}\big(s_0 + c(s-s_0)\big) > x\Big\}\mathrm{d}x = \mathbb{E}\Big[V^{(s_0)}\big(s_0 + c(s-s_0)\big)\Big] = 1,\end{aligned}$$

where the first equality follows by a similar argument as in the proof of Proposition 4.7.

Due to our choice of point-spectral process $V^{(s_0)}$ as (57), the integrand on the right-hand side of (74) equals

$$\overline{\Phi}\bigg(\frac{\log(x) + \gamma(\delta(\max(u/x,1))(s-s_0))}{\sqrt{2\gamma(\delta(\max(u/x,1))(s-s_0))}}\bigg).$$

Setting $\gamma^* = \sup_{r\in[\min\delta,\max\delta]} \gamma(r(s-s_0))$, which is possible by the boundedness of δ and the continuity of γ, we claim that it has the following integrable majorant

$$f(x) = \begin{cases} 1 & \text{if } x \le e^1, \\ \overline{\Phi}\big(\log(x)/\sqrt{2\gamma^*}\big) & \text{if } x > e^1. \end{cases}$$

Clearly, f is a majorant. To confirm integrability, note that with $\sigma^* = \sqrt{2\gamma^*}$ and N standard normal

$$\begin{aligned}\int_{e^1}^{\infty} f(x)\mathrm{d}x = \int_{e^1}^{\infty} \overline{\Phi}\Big(\frac{\log(x)}{\sigma^*}\Big)\mathrm{d}x &\le \int_0^{\infty} \overline{\Phi}\Big(\frac{\log(x)}{\sigma^*}\Big)\mathrm{d}x \\ &= \int_0^{\infty} \mathbb{P}\big\{\exp(\sigma^* N) > x\big\}\mathrm{d}x = \mathbb{E}\big(\exp(\sigma^* N)\big) = \exp((\sigma^*)^2/2).\end{aligned}$$

This finishes the proof. □

Lastly, let us consider the case when $\lim_{u\to\infty}\delta(u) = \infty$. The following lemma reduces the problem, so that we will only have to investigate the integral (72) from 1 to u instead of 0 to u.

Lemma B.3. *Let $Y^{(s_0)}$ be the domain-scaled tail approximating process with point-spectral processes $V^{(s_0)}$ and scaling function δ. Suppose $\lim_{u\to\infty}\delta(u) = \infty$ and that $\lim_{r\to\infty} V^{(s_0)}\big(s_0 + r(s-s_0)\big) = 0$ in probability. Then*

$$\lim_{u\to\infty}\Big[u\,\mathbb{P}\big\{Y^{(s_0)}(s) > u\big\} - \int_1^u \mathbb{P}\big\{V^{(s_0)}(s_0+\delta(u/x)(s-s_0)) > x\big\}\,\mathrm{d}x\Big] = 0.$$

Proof. By (72), it suffices to show that

$$\lim_{u\to\infty}\int_0^1 \mathbb{P}\big\{V^{(s_0)}(s_0+\delta(u/x)(s-s_0)) > x\big\}\,\mathrm{d}x = 0.$$

Hence the assertion follows by dominated convergence from $\lim_{r\to\infty}\mathbb{P}\big\{V^{(s_0)}\big(s_0+r(s-s_0)\big) > x\big\} = 0$ and $\lim_{u\to\infty}\delta(u/x) = \infty$ for any $x \in (0,1]$. □

In general, the interplay between point-spectral process $V^{(s_0)}$ and scaling function δ in the domain-scaled tail approximating process $Y^{(s_0)}$ can be quite subtle. Here, we focus again on the Brown–Resnick point spectral process from (57).

Proposition B.4. *Let $Y^{(s_0)}$ be a domain-scaled tail approximating process with point-spectral process $V^{(s_0)}$ and scaling function δ. Suppose that $V^{(s_0)}$ is given by the Brown–Resnick point-spectral process* (57)*, such that the underlying semi-variogram γ satisfies that $\gamma(r(s-s_0)) > 0$ is non-decreasing in $r > 0$ and $\lim_{r\to\infty}\gamma(r(s-s_0)) = \infty$ for $s \in \mathbb{R}^d\setminus\{s_0\}$, and that the scaling function δ satisfies Assumption 4.3 with $\lim_{u\to\infty}\delta(u) = \infty$. Then*

$$\limsup_{u\to\infty} u\,\mathbb{P}\big\{Y^{(s_0)}(s) > u\big\} \le 1, \qquad s \in \mathbb{R}^d\setminus\{s_0\}.$$

Proof. By Lemma 5.3 we have $\lim_{r\to\infty} V^{(s_0)}\big(s_0+r(s-s_0)\big) = 0$ in probability, so that by Lemma B.3 it suffices to show that

$$\limsup_{u\to\infty}\int_1^u \mathbb{P}\big\{V^{(s_0)}(s_0+\delta(u/x)(s-s_0)) > x\big\}\,\mathrm{d}x \le 1.$$

We abbreviate $h = s - s_0$ henceforth, so that

$$\int_1^u \mathbb{P}\big\{V^{(s_0)}(s_0+\delta(u/x)(s-s_0)) > x\big\}\,\mathrm{d}x = \int_1^u \overline{\Phi}\Big(\frac{\log(x)+\gamma(\delta(u/x)h)}{\sqrt{2\gamma(\delta(u/x)h)}}\Big)\mathrm{d}x$$

For suffiently large u we have that $\log(u)$ is larger than $\gamma(h)$. For such u we note that on the interval $[1,u]$ the function $\log(x)$ is strictly increasing from 0 to $\log(u)$, while $\gamma(\delta(u/x)h)$ is continuously decreasing from $\gamma(\delta(u)h)$ to $\gamma(h)$. Hence there exits $x_u \in (1,u)$, such that

$$\begin{aligned}\gamma(\delta(u/x)h) &\ge \log(x) \quad \text{for} \quad x \le x_u,\\ \gamma(\delta(u/x)h) &\le \log(x) \quad \text{for} \quad x \ge x_u.\end{aligned}$$

In fact, this means we even have

$$\begin{aligned}\gamma(\delta(u/x)h) &\geq \gamma(\delta(u/x_u)h) = \log(x_u) > \log(x) \quad \text{for} \quad x < x_u,\\ \gamma(\delta(u/x)h) &\leq \gamma(\delta(u/x_u)h) = \log(x_u) < \log(x) \quad \text{for} \quad x > x_u.\end{aligned}$$

By Lemma A.6, we obtain

$$\overline{\Phi}\left(\frac{\log(x)+\gamma(\delta(u/x)h)}{\sqrt{2\gamma(\delta(u/x)h)}}\right) \leq \overline{\Phi}\left(\frac{\log(x)+\gamma(\delta(u/x_u)h)}{\sqrt{2\gamma(\delta(u/x_u)h)}}\right),$$

which entails

$$\begin{aligned}\int_1^u \mathbb{P}\big\{V^{(s_0)}(s_0+\delta(u/x)(s-s_0)) > x\big\}\,\mathrm{d}x &\leq \int_1^u \mathbb{P}\big\{V^{(s_0)}(s_0+\delta(u/x_u)(s-s_0)) > x\big\}\,\mathrm{d}x\\ &< \int_0^\infty \mathbb{P}\big\{V^{(s_0)}(s_0+\delta(u/x_u)(s-s_0)) > x\big\}\,\mathrm{d}x = 1,\end{aligned}$$

cf. (50), and hence the assertion. □

In fact, the argument in the proof of Proposition B.4 shows that at finite levels $u > e^{\gamma(h)}$

$$u\,\mathbb{P}\big\{Y^{(s_0)}(s) > u\big\} \leq \int_0^u \overline{\Phi}\left(\frac{\log(x)+\gamma(\delta(u/x_u)h)}{\sqrt{2\gamma(\delta(u/x_u)h)}}\right)\mathrm{d}x, \tag{75}$$

where x_u is a solution to $\gamma(\delta(u/x_u)h) = \log(x_u)$. The following lemma can help to get more tractable upper bounds, see Example B.6.

Lemma B.5. *Let $a, b \in \mathbb{R}$ and $\sigma > 0$. Then*

$$\int_a^b \overline{\Phi}\left(\frac{\log(x)}{\sigma}+\frac{\sigma}{2}\right)\mathrm{d}x = \int_0^\infty \mathbb{P}\left\{N \in \left(\frac{t+\log(a)}{\sigma}-\frac{\sigma}{2}, \frac{t+\log(b)}{\sigma}-\frac{\sigma}{2}\right)\right\}e^{-t}\,\mathrm{d}t,$$

where N is a standard normal variable. For $a = -\infty$ and $b = \infty$ the interval boundaries on the right-hand side are instead $-\infty$ and ∞, respectively.

Proof. Substituting x by e^y gives

$$\begin{aligned}\int_a^b \overline{\Phi}\left(\frac{\log(x)}{\sigma}+\frac{\sigma}{2}\right)\mathrm{d}x &= \int_{\log(a)}^{\log(b)} \overline{\Phi}\left(\frac{y}{\sigma}+\frac{\sigma}{2}\right)e^y\,\mathrm{d}y\\ &= \int_{\log(a)}^{\log(b)} \mathbb{P}\{\sigma N - \sigma^2 > y - \sigma^2/2\}e^y\,\mathrm{d}y = \int_{\log(a)}^{\log(b)}\int_{y-\sigma^2/2}^\infty f_{\sigma N-\sigma^2}(t)\mathrm{d}t\,e^y\,\mathrm{d}y.\end{aligned}$$

Using Lemma A.4 we see that the latter equals

$$\begin{aligned}\int_{\log(a)}^{\log(b)}\int_{y-\sigma^2/2}^\infty f_{\sigma N}(t)e^{-t}\mathrm{d}t\,e^{y-\sigma^2/2}\,\mathrm{d}y &= \int_{\log(a)-\sigma^2/2}^{\log(b)-\sigma^2/2}\int_y^\infty f_{\sigma N}(t)e^{-(t-y)}\mathrm{d}t\,\mathrm{d}y\\ &= \int_{\log(a)-\sigma^2/2}^{\log(b)-\sigma^2/2}\int_0^\infty f_{\sigma N}(t+y)e^{-t}\mathrm{d}t\,\mathrm{d}y.\end{aligned}$$

Finally, applying Fubini's theorem gives

$$\begin{aligned}\int_a^b \overline{\Phi}\left(\frac{\log(x)}{\sigma}+\frac{\sigma}{2}\right)\mathrm{d}x &= \int_0^\infty\int_{\log(a)-\sigma^2/2}^{\log(b)-\sigma^2/2} f_{\sigma N}(t+y)\mathrm{d}y\,e^{-t}\,\mathrm{d}t\\ &= \int_0^\infty\int_{t+\log(a)-\sigma^2/2}^{t+\log(b)-\sigma^2/2} f_{\sigma N}(y)\mathrm{d}y\,e^{-t}\,\mathrm{d}t\end{aligned}$$

and hence the assertion. □

By Lemma B.5 the bound (75) reads as

$$u\,\mathbb{P}\big\{Y^{(s_0)}(s) > u\big\} \leq \int_0^\infty \Phi\left(\frac{t+\log(u)-\gamma(\delta(u/x_u)h)}{\sqrt{2\gamma(\delta(u/x_u)h)}}\right)e^{-t}\,\mathrm{d}t.$$

For further (non-asymptotic) refinements, we will need to choose the scaling function δ explicitly.

Example B.6. Suppose γ is α-homogeneous and $\delta(u) = \max(1, 2\log(u))^{1/\alpha}$, cf. Remark 5.5. Let $u > e^{\gamma(h)+1/2}$ and set

$$\beta(h) = \frac{2\gamma(h)}{2\gamma(h)+1}.$$

Then

$$\gamma\big(\delta(u/u^{\beta(h)})h\big) = \big[\delta(u^{1-\beta(h)})\big]^{\alpha}\gamma(h) = 2\gamma(h)(1-\beta(h))\log(u) = \beta(h)\log(u) = \log\big(u^{\beta(h)}\big),$$

so that x_u in (75) is given by $x_u = u^{\beta(h)}$ and

$$\begin{aligned}\frac{\log(u) - \gamma(\delta(u/x_u)h)}{\sqrt{2\gamma(\delta(u/x_u)h)}} &= \frac{\log(u) - \beta(h)\log(u)}{\sqrt{2\beta(h)\log(u)}} \\ &= \frac{1}{\sqrt{2}}\left(\frac{1-\beta(h)}{\sqrt{\beta(h)}}\right)\sqrt{\log(u)} = \frac{\sqrt{\log(u)}}{\sqrt{2}\sqrt{2\gamma(h)}\sqrt{2\gamma(h)+1}},\end{aligned}$$

which gives

$$u\,\mathbb{P}\big\{Y^{(s_0)}(s) > u\big\} \le \int_0^\infty \Phi\left(\frac{t}{\sqrt{2\beta(h)\log(u)}} + \frac{\sqrt{\log(u)}}{\sqrt{2}\sqrt{2\gamma(h)}\sqrt{2\gamma(h)+1}}\right)e^{-t}\,\mathrm{d}t. \tag{76}$$

Note that, by dominated convergence, the right-hand side of (76) converges to 1 as $u \to \infty$. This reveals that the strategy of proof applied for Proposition B.4 cannot lead to a sharper upper bound for $\limsup_{u\to\infty} u\mathbb{P}\{Y^{(s_0)}(s) > u\}$, even when it might exist. Moreover, the key monotonicity argument in Proposition B.4 can also be applied to obtain an a priori lower bound from Lemma A.6, namely

$$u\,\mathbb{P}\big\{Y^{(s_0)}(s) > u\big\} \ge \int_1^{x_u} \overline{\Phi}\left(\frac{\log(x) + \gamma(\delta(u)h)}{\sqrt{2\gamma(\delta(u)h)}}\right)\mathrm{d}x + \int_{x_u}^{u} \overline{\Phi}\left(\frac{\log(x) + \gamma(h)}{\sqrt{2\gamma(h)}}\right)\mathrm{d}x.$$

By Lemma B.5 we obtain

$$\begin{aligned}u\,\mathbb{P}\big\{Y^{(s_0)}(s) > u\big\} \ge \int_0^\infty & \mathbb{P}\left\{N \in \left(\frac{t - \gamma(\delta(u)h)}{\sqrt{2\gamma(\delta(u)h)}}, \frac{t + \log(x_u) - \gamma(\delta(u)h)}{\sqrt{2\gamma(\delta(u)h)}}\right)\right\} \\ &+ \mathbb{P}\left\{N \in \left(\frac{t + \log(x_u) - \gamma(h)}{\sqrt{2\gamma(h)}}, \frac{t + \log(u) - \gamma(h)}{\sqrt{2\gamma(h)}}\right)\right\}e^{-t}\,\mathrm{d}t.\end{aligned}$$

However, this bound does not give sufficient insight to tighten the possible limits for the left-hand side between 0 and 1, as the right-hand side converges to 0 by dominated convergence. The observation that $\liminf_{u\to\infty} u\mathbb{P}\{Y^{(s_0)}(s) > u\} \ge 0$ is a trivial one that needs no such analysis. Overall, this demonstrates the intricacy of the tail behaviour of the marginal distribution of the tail approximating process $Y^{(s_0)}$, once domain-scaling is at play. Proposition B.4 shows that the tail of $Y^{(s_0)}(s)$ is at least not heavier than the Pareto tail of $Y^{(s_0)}(s_0)$ when the underlying point-spectral process is of Brown–Resnick type.