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SOME NOTES ON PONTRYAGIN DUALITY OF ABELIAN TOPOLOGICAL GROUPS

KARL HEINRICH HOFMANN AND LINUS KRAMER

Some observations on topological abelian groups and their duality appear to be useful to recall, notably those with an emphasis on commutative pro-Lie groups. Some of our comments appear to be new and some of the items we note here have been known for some time, yet appear to be worth while to be recalled. A good deal of the necessary background is available in sources like [9] and [10].

The circle group $\mathbb{R}/\mathbb{Z} = \mathbb{T}$ (written additively), for each abelian topological group G , gives rise to its *character group*

$$\widehat{G} = \text{Hom}(G, \mathbb{R}/\mathbb{Z}),$$

which is again a topological abelian group when we endow it with the topology of uniform convergence on compact subsets of G . The repetition of this first step leads to the creation of the bidual $\widehat{\widehat{G}} = \text{Hom}(\text{Hom}(G, \mathbb{R}/\mathbb{Z}), \mathbb{R}/\mathbb{Z})$, and if $g \in G$ and $\chi \in \widehat{G}$ yield the element $\chi(g) \in \mathbb{R}/\mathbb{Z}$, then the following *evaluation homomorphism* is immediately present:

$$\eta_G: G \rightarrow \widehat{\widehat{G}} \text{ defined by } \eta_G(g)(\chi) = \chi(g).$$

A central portion of the classical theory of PONTRYAGIN DUALITY is the statement that

(A) η_G is an isomorphism (algebraically and topologically) whenever G is locally compact.

Yet many items in the literature lead us beyond these limitations. Another significant aspect of that classical duality is that

(B) the category of locally compact abelian groups $\mathbf{lca}$ is closed under the passage from G to $\widehat{G}$.

While significant progress about duality of abelian topological groups beyond local compactness was indeed achieved (as documented in the recent monograph [2] by AUSSENHOFER, DIKRANJAN and GIORDANO BRUNO), the overall picture is neither clear nor complete. As a test of this claim we address here what is known for the category of *abelian pro-Lie groups* $\mathbf{apl}$ in the regard of duality. This category, which contains all $\mathbf{lca}$ groups, is in contrast to the category of $\mathbf{lca}$ groups complete, that is, closed under the

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formation of all limits. It contains the category of all weakly complete real vector spaces, being isomorphic to $\mathbb{R}^I$ (for an arbitrary set I) with the product topology. If $G = \mathbb{R}^I$, then $\widehat{G} \cong \mathbb{R}^{(I)}$ (the direct sum of cardinality I -many copies of $\mathbb{R}$ with the finest locally convex vector space topology) does not belong to the category of pro-Lie groups. This example illustrates that, even on an elementary level, *the category of abelian pro-Lie groups **apl** differs significantly from its dual category*. Even though some information is provided in Chapter 4 of [10], the status of information on duality of **apl** groups is far from satisfactory. This encourages us to offer in this note some complementary pieces of information on that duality.

It will be useful to begin the discussion by giving the details of an example of a comparatively minute nondiscrete but prodiscrete (hence pro-Lie) group G for which η_G fails to be continuous, further $\widehat{G}$ fails to be complete, while its bidual $\widehat{\widehat{G}}$ is discrete. This example, due to LEPTIN ([14] 1955), later mentioned by NOBLE ([16] Chapter 1, 1967, and [17], Example 1.6, 1970), BANASZCZYK ([3] 1991) also illustrates certain aspects of the general progress that was contributed to duality by AUSSENHOFER ([1], 1999).

Subsequently, we point out that the environment of pro-Lie groups permits a different access to AUSSENHOFER's insight that for large classes of abelian topological groups G including **apl** groups the evaluation morphism η_G is *bijective and open*, in other words, that η_G^{-1} exists and is continuous. We explain in which sense duality is necessary and sufficient for the continuity of η_G for pro-Lie groups.

While η_G may be discontinuous even for **apl** groups, as LEPTIN's example had illustrated for 70 years, it had been essentially pointed out in the proof of [9], Theorem 7.7 (iii) that

(C) *for each compact subspace C of G , the restriction $\eta_G|_C: C \rightarrow \widehat{G}$ is continuous,*

which points into the direction of what became known as k -groups. According to NOBLE [16, 17], a group homomorphism defined on a topological group G is called *k -continuous* if each restriction to a compact subset of G is continuous. Accordingly, η_G is *k -continuous*. NOBLE calls a topological group G a *k -group* if each k -continuous homomorphism from G to a topological group is continuous. In this terminology, for every k -group G , the evaluation morphism η_G is continuous. This justifies our review of the validity of duality in the context of topological abelian k -groups (see [16, 17, 18]). Indeed we present here a new category theoretical aspect, namely, that *the subcategory of k -groups is coreflexive in the category of topological groups* and what this means in concrete circumstances. All locally compact groups are k -groups. The example of a pro-Lie group we shall discuss in our first section fails to be a k -group.

Theorem. (A) *For each topological group G there is a topological abelian k -group kG arising functorially from G by refining the topology in such a fashion that the natural bijection $\kappa_G: kG \rightarrow G$ preserves all compact subspaces of G and that each morphism $f: H \rightarrow G$ from a k -group H to G factors via $f': H \rightarrow kG$ through kG such that $f = \kappa_G \circ f'$.*

(B) For each abelian pro-Lie k -group G the evaluation morphism $\eta_G: G \rightarrow \widehat{\widehat{G}}$ is an isomorphism.

One might say that an abelian pro-Lie group G satisfies PONTRYAGIN DUALITY if it is a k -group, and that no abelian pro-Lie group is ever far away from a k -group. Indeed we conclude our discourse with the open question whether for any abelian pro-Lie group G the double dual $\widehat{\widehat{G}}$ is automatically a k -group.

In what follows, all topological spaces are assumed to be Hausdorff unless stated otherwise.

1. THE LEPTIN-NOBLE-BANASZCZYK EXAMPLE OF AN EXPONENT 2 GROUP

As indicated in the introduction the instance of an interesting example of an evaluation function $\eta_E: E \rightarrow \widehat{\widehat{E}}$ deserves to be explained explicitly right in the beginning.

Definition 1.1. Let I denote the set of all ordinals called α, β, γ etc. less than the first uncountable ordinal ω_1 , and let $\mathbb{Z}(2)^{(I)} = \bigoplus_{\alpha} \mathbb{Z}_{\alpha}$, $\mathbb{Z}_{\alpha} = \mathbb{Z}(2)$. For the elements write $g = (g_{\alpha})_{\alpha \in I}$.

Let H_{α} denote the subgroup of $\mathbb{Z}(2)^{(I)}$ of all elements $g = (g_{\beta})_{\beta \in I}$ such that $g_{\beta} = 0$ for $\beta < \alpha$. Next, let K_{α} denote the subgroup of $\mathbb{Z}(2)^{(I)}$ of all elements $g = (g_{\beta})_{\beta \in I}$ such that $g_{\beta} = 0$ for $\beta \geq \alpha$.

Then $\{H_{\alpha} : \alpha \in I\}$ is a basis of identity neighborhoods of a group topology making $\mathbb{Z}(2)^{(I)}$ into a topological abelian group E of exponent 2. The cardinality of I is that of the first uncountable cardinal¹ $\aleph_1$. Note that

$$(\forall \alpha) \quad E = K_{\alpha} \oplus H_{\alpha}$$

and that the subgroups K_{α} are discrete in that topology.

Also observe that the projective limit of the projective system of discrete groups $\{E/H_{\beta} \rightarrow E/H_{\alpha}, \alpha \leq \beta\}$ up to natural isomorphism agrees with E . This fact shows that E is indeed a prodiscrete and thus, in particular, a pro-Lie group. In detail, this follows from Lemma 1.6 below. Recall that an F_{σ} -set in a topological space is a countable union of closed sets.

Lemma 1.2. *In E , every F_{σ} -set is closed.*

Proof. Let C_n , for $n \in \mathbb{N}$, be a sequence of closed subsets of E and set $C = \bigcup_{n \in \mathbb{N}} C_n$. Let $g \in E - C$. Then for each $n \in \mathbb{N}$ we find an $\alpha_n \in I$ such that $(g + H_{\alpha_n}) \cap C_n = \emptyset$. Then we find a $\beta \in I$ such that $\alpha_n < \beta$ for all $n \in \mathbb{N}$. But then $(g + H_{\beta}) \cap C = \emptyset$, and therefore C is closed. $\square$

Spaces with this property are sometimes called *pseudo-discrete spaces* or *P-spaces*.

¹If we accept the Continuum Hypothesis $\aleph_1 = 2^{\aleph_0}$, this is the cardinality of $\mathbb{R}$. Then E and $\mathbb{Z}(2)^{(\mathbb{R})}$ are *algebraically* isomorphic.

Corollary 1.3. *Every compact subset of E is finite.*

Proof. By Lemma 1.2 every countable subset of E is closed and discrete, because it cannot have any accumulation point. $\square$

In particular, for any topological space X the compact-open topology on $C(E, X)$ is the topology of pointwise convergence induced by X^E .

Definition 1.4. We introduce the group $A = \prod_{\alpha \in I} K_\alpha$ with the product topology, as an uncountable product of discrete groups and define

$$\varphi: E \rightarrow A, \quad \varphi((z_\alpha)_{\alpha \in I}) = (g_\alpha)_{\alpha \in I} \text{ where } g_\alpha = \sum_{\beta < \alpha} z_\beta.$$

Recall that each factor K_α carries the discrete topology. Clearly, φ is injective.

Lemma 1.5. *The subset $\varphi(E)$ of $A = \prod_{\alpha \in I} K_\alpha$ consists of all $(y_\alpha)_{\alpha \in I}$, where $y_\alpha = \sum_{\beta < \alpha} x_{\alpha\beta}$ such that*

$$\forall_{\alpha, \alpha' > \beta} \quad x_{\alpha\beta} = x_{\alpha'\beta}.$$

Accordingly, $\varphi(E)$ is closed in A .

Proof. For a proof we shall show that $x_{\alpha\beta} = 0$ for almost all β . Indeed otherwise we would have elements $x_{a_n, \beta_n} \neq 0$ for $n \in \mathbb{N}$ and we would choose a $\gamma > \beta_n$ for all n and conclude that $\sum_{\beta < \gamma} x_{\gamma\beta} \notin K_\gamma$, which is a contradiction. $\square$

Lemma 1.6. *The morphism $\varphi: E \rightarrow A$ is an isomorphism onto its image.*

Proof. For the continuity of φ we argue that the composition $p \circ \varphi$ for each composition with any projection p onto any factor of the product is continuous, which is clear.

For the openness of the corestriction $E \rightarrow \varphi(E)$ we note that each $\varphi(H_\gamma)$ is open in the image. Let $p_\gamma: \prod_{\beta \in I} K_\beta \rightarrow K_\gamma$ denote the projection. We note that $p_\gamma^{-1}(0) \cap \varphi(E) = \varphi(H_\gamma)$ is open in $\varphi(E)$, since $\{0\}$ is open in K_γ by the discreteness of K_γ . $\square$

We identify the algebraic dual of $\mathbb{Z}(2)^{(I)}$ with $\mathbb{Z}(2)^I$ in the familiar way. Since K_α is discrete for $\alpha \in I$, we may identify $\widehat{K_\alpha}$ with the group $\mathbb{Z}(2)^{\{\gamma \in I: \gamma < \alpha\}}$ of all $y = (y_\gamma)_{\gamma \in I}$ such that $y_\gamma = 0$ for $\gamma \geq \alpha$. Now for $x = (x_\beta)_{\beta \in I} \in E$ we have

$$x \cdot y = \sum_{\gamma < \alpha} x_\gamma y_\gamma,$$

which is well defined, since only finitely many x_γ are nonzero. We note also that every character $\chi: E \rightarrow \mathbb{R}/\mathbb{Z}$ takes its values in $\mathbb{Z}(2) \subseteq \mathbb{R}/\mathbb{Z}$, because E has exponent 2. A morphism $\chi: E \rightarrow \mathbb{Z}(2)$ is continuous if and only if there is an α such that $H_\alpha \subseteq \ker \chi$. Thus we may summarize:

Proposition 1.7. *Let E be $\mathbb{Z}(2)^{(I)}$ with the topology introduced in Definition 1.1. Then*

$$\widehat{E} = \bigcup_{\beta \in I} \widehat{K_\beta} \cong \{(g_\alpha)_{\alpha \in I} : (\exists \beta \in I)(\forall \alpha \geq \beta) g_\alpha = 0\} \subseteq \mathbb{Z}(2)^I$$

is dense in $\mathbb{Z}(2)^I$ with the product topology, which induces on $\widehat{E}$ the topology of pointwise convergence. Each $\widehat{K}_\alpha$ carries the topology of pointwise convergence, i.e. the topology induced by the inclusion $\widehat{K}_\alpha \subseteq \mathbb{Z}(2)^I$. Note that $\widehat{E} \cong \mathbb{Z}(2)^{(I)}$.

We shall recall a generally well accepted fact.

Lemma 1.8. *Let T and F be topological groups, with F complete, D a dense subgroup of T , and $\psi: D \rightarrow F$ a morphism. Then ψ has a unique continuous extension to a morphism $\bar{\psi}: T \rightarrow F$,*

$$\begin{array}{ccc} D & \xrightarrow{\psi} & F \\ \downarrow & \nearrow \exists! \bar{\psi} & \\ T & & \end{array}$$

Proof. See [4], Chap. III, Corollaire de la Proposition 8, p. TGIII.25, or [20], Cor.8.48. $\square$

Accordingly, each character of $\widehat{E}$, i.e., each element of $\widehat{E}$ uniquely extends to a character of $\mathbb{Z}(2)^I$, i.e. to an element of $\mathbb{Z}(2)^{(I)}$. Thus from $\widehat{E} \rightarrow \mathbb{Z}(2)^I$ we obtain a bijective morphism of abelian topological groups $\mathbb{Z}(2)^{(I)} \rightarrow \widehat{E}$. At this point we note the following.

Lemma 1.9. *$\widehat{E}$ is discrete and hence*

$$\eta_E^{-1}: \widehat{E} \rightarrow E$$

is a bijective morphism of abelian topological groups.

Proof. For $\alpha \in I$ let $\delta_\alpha: E \rightarrow \mathbb{Z}(2)$ be defined by $\delta_\alpha(g) = g_\alpha$. Let $\Delta = \{0\} \cup \{\delta_\alpha : \alpha \in I\} \subseteq \widehat{E}$. Then Δ is compact and $\{\psi \in \widehat{E} : \psi(\Delta) = \{0\}\}$ is open as a basis element of the compact open topology. But $\psi(\Delta) = \{0\}$ implies $\psi = 0$ because Δ generates a dense subgroup of $\widehat{E}$. $\square$

Let us summarize the features of the group E we have discussed now! Recall that I denotes the set of all countable ordinals. In the introduction we introduced the concept of a *k-group* which we shall discuss in detail in Section 3 below.

Theorem 1.10. *There is a nondiscrete abelian group topology on the exponent 2 group $\mathbb{Z}(2)^{(I)}$ making it into an abelian topological group E with the following properties.*

- (1) *E is isomorphic as a topological group to a closed subgroup F of the group A of Definition 1.4, which is an uncountable product of discrete groups. In particular, E is pro-discrete, pro-Lie and complete.*
- (2) *E is not discrete, but every compact subset of E is finite.*
- (3) *The character group*

$$\widehat{E} = \{(x_\alpha)_{\alpha \in I} : (\exists \beta \in I)(\forall \beta \leq \alpha)x_\alpha = 0\} \subseteq \mathbb{Z}(2)^I$$

is a dense proper subgroup of the compact group $\mathbb{Z}(2)^I$. In particular, it is incomplete.

(4) Its bidual

$$\widehat{\widehat{E}} \cong \mathbb{Z}(2)^{(I)}$$

is discrete.

(5) The evaluation morphism $\eta_E: E \rightarrow \widehat{\widehat{E}}$ is bijective, open, discontinuous, and is (trivially) continuous on every compact subset of E .

(6) The group A is a k -group, but the closed subgroup $F \subseteq A$ is not a k -group.

In particular, the example E and its dual $\widehat{E}$ show that *the dual of an abelian pro-Lie group may be incomplete*. We recall that a group which satisfies Pontryagin duality is called *reflexive*. Kaplan [11] proves that a product of reflexive groups is again reflexive. This applies in particular to the group A , which is a product of discrete abelian groups. Thus $F \subseteq A$ is also an example of *a closed subgroup of a reflexive group which is not reflexive*. NOBLE shows in [17] Corollary 3.5 that a closed subgroup of a *countable* product of locally compact groups satisfies Pontryagin duality. In our case, the ambient group A is an uncountable product of (countable) discrete groups.

As we mentioned above, the discovery of the group E goes back to LEPTIN in [14]. It is mentioned in the literature repeatedly, e.g. by NOBLE in [16, 17], and by BANASZCZYK in [3].

2. THE EVALUATION η REVISITED FOR ABELIAN PRO-LIE GROUPS

According to [10], Proposition 4.40, the group homomorphism $\eta_G: G \rightarrow \widehat{\widehat{G}}$ is injective for abelian pro-Lie groups. From AUSSENHOFER's fundamental source [1], however, we know much more accurately its role as the core of the PONTYAGIN DUALITY. Indeed for a wide category of abelian topological groups her results show that η_G is bijective and that its inverse is continuous. In the following we observe by a short direct approach, in which the surjectivity of η_G suffices in the case of pro-Lie groups that its inverse is, in fact, a continuous bijective morphism. Theorem 1.10 shows that this result cannot be improved.

Proposition 2.1. *For all abelian pro-Lie groups G the morphism of abelian groups $\eta_G: G \rightarrow \widehat{\widehat{G}}$ is bijective. Its inverse $\eta_G^{-1}: \widehat{\widehat{G}} \rightarrow G$ is continuous, i.e., is a morphism in the category **tab** of topological abelian groups.*

Proof. Assume that in the category **tab** of topological abelian groups we have $G = \lim_{j \in J} G_j$ for a projective system $\{G_j : j \in J\}$ of Lie groups as in [10], p. 81. Such a presentation is possible since G is a pro-Lie group. For each $j \in I$ let $p_j: G \rightarrow G_j$ denote the morphism in the system. Since G_j is an abelian Lie group we record that

$$\eta_{G_j}: G_j \rightarrow \widehat{\widehat{G_j}}$$

is an *isomorphism* for each $j \in I$. (See e.g. [9], Theorem 7.63.) Hence for each $j \in J$ we have a morphism

$$\widehat{\widehat{G}} \xrightarrow{\widehat{p_j}} \widehat{\widehat{G_j}} \xrightarrow{\eta_{G_j}^{-1}} G_j,$$

and so, by the limit property of $G = \lim_{j \in I} G_j$, we have a unique morphism of topological abelian groups $\eta_G^!: \widehat{\widehat{G}} \rightarrow G$ such that, for each $j \in I$, we have the commutative diagram

$$\begin{array}{ccc} \widehat{\widehat{G}} & \xrightarrow{\eta_G^!} & G \\ \widehat{p_j} \downarrow & & \downarrow p_j \\ \widehat{\widehat{G_j}} & \xrightarrow{\eta_{G_j}^{-1}} & G_j, \end{array} \quad j \in J.$$

We denote the forgetful functor from the category **tab** of topological abelian groups to the category **ab** of abelian groups by $G \mapsto UG$. In the category of abelian groups **ab**, for each $j \in I$ we have the commutative diagram

$$\begin{array}{ccc} UG & \xrightarrow{U\eta_G} & U\widehat{\widehat{G}} \\ U p_j \downarrow & & \downarrow U\widehat{p_j} \\ UG_j & \xrightarrow{U\eta_{G_j}} & U\widehat{\widehat{G_j}}, \end{array} \quad j \in J,$$

by the naturality of η . Staying in the category of abelian groups we combine these two diagrams and obtain

$$\begin{array}{ccccc} UG & \xrightarrow{U\eta_G} & U\widehat{\widehat{G}} & \xrightarrow{U\eta_G^!} & UG \\ U p_j \downarrow & & \downarrow U\widehat{p_j} & & \downarrow U p_j \\ UG_j & \xrightarrow{U\eta_{G_j}} & U\widehat{\widehat{G_j}} & \xrightarrow{U\eta_{G_j}^{-1}} & UG_j, \end{array} \quad j \in J.$$

Accordingly, since we have $\eta_{G_j}^{-1} \circ \eta_{G_j} = \text{id}_{G_j}$ for all $j \in J$, we have a commutative diagram of abelian groups

$$\begin{array}{ccc} UG & \xrightarrow{U\eta_G^! \circ U\eta_G} & UG \\ U p_j \downarrow & & \downarrow U p_j \\ UG_j & \xlongequal{\quad} & UG_j, \end{array} \quad j \in J.$$

The grounding functor U of the category of topological abelian groups to the category of abelian groups is right adjoint to the functor which attaches to an abelian group the discrete abelian topological group it supports. Right adjoint functors preserve limits (see e.g. [9], Theorem A3.52). Thus $G = \lim_{j \in J} G_j$ implies that $UG = \lim_{j \in J} UG_j$ holds in

the category of abelian groups. Since the fill-in morphism of limits is unique, we conclude that

$$\eta_G^! \circ \eta_G = \text{id}_G. \quad (*)$$

However, at this point we need to recall a consequence of AUSSENHOFER's result [1] Corollary 21.5., namely, that for abelian pro-Lie groups, η_G is *surjective*. Then $(*)$ implies that $U\eta_G^! = (U\eta_G)^{-1}$, i.e., that $U\eta_G$ is invertible in the category of abelian groups and so that it is bijective, and that its inverse $\eta_G^{-1} = \eta_G^!$ is a morphism in the category of topological abelian groups, as asserted. $\square$

Again we emphasize that it is illustrated by the example of the abelian pro-Lie group E of the preceding section that η_E itself may fail to be continuous.

The existence of the **tab**-morphism $\eta_G^{-1}: \widehat{\widehat{G}} \rightarrow G$ of (2.1) allows us to pass to the duals to obtain the morphism

$$\widehat{\eta_G^{-1}}: \widehat{G} \rightarrow \widehat{\widehat{G}}, \quad (\forall \chi \in \widehat{G}, \omega \in \widehat{\widehat{G}}) \widehat{\eta_G^{-1}}(\chi) \cdot \omega = \omega(\chi). \quad (1)$$

The topological abelian group $H = \widehat{G}$ has its own evaluation morphism η_H which by its very definition is given by

$$\eta_H: H \rightarrow \widehat{\widehat{H}}, \quad (\forall \chi \in H, \omega \in \widehat{\widehat{H}}) \eta_H(\chi)(\omega) = \omega(\chi). \quad (2)$$

Comparing statements (1) and (2) we conclude the following

Corollary 2.2. *For any topological abelian group G for which η_G has an inverse η_G^{-1} which is a morphism of topological abelian groups,*

$$\widehat{\eta_G^{-1}}: \widehat{G} \rightarrow \widehat{\widehat{G}} \quad \text{equals} \quad \eta_{\widehat{G}}: \widehat{G} \rightarrow \widehat{\widehat{G}}.$$

This corollary allows us to formulate and prove the following result.

Theorem 2.3. *For an abelian pro-Lie group G the following statements are equivalent.*

- (1) *G is the character group of an abelian topological group H for which η_H is bijective and open.*
- (2) *$\eta_G: G \rightarrow \widehat{\widehat{G}}$ is an isomorphism of abelian pro-Lie groups.*

Proof. (1) implies (2): Since $\eta_H^{-1}: \widehat{\widehat{H}} \rightarrow H$ is a morphism by hypothesis, so is its dual $\widehat{\eta_H^{-1}}: \widehat{H} \rightarrow \widehat{\widehat{\widehat{H}}}$, agreeing with the morphism $\eta_g: G \rightarrow \widehat{\widehat{G}}$ by Corollary 2.2.

(2) implies (1): Condition (2) allows us to apply Corollary 2.2, to set $H = \widehat{G}$, and to conclude that $\widehat{H} \cong \widehat{\widehat{G}} \cong G$. $\square$

In Theorem 1.10 above we saw the example of a nondiscrete prodiscrete topological abelian group E on the underlying group $\mathbb{Z}(2)^{(I)}$, which showed that η_E can fail to be continuous in general, and so, accordingly, that η_E^{-1} can fail to be open.

That example also illustrates $\widehat{\widehat{E}} \cong \mathbb{Z}(2)^{(I)}$ with the discrete topology and $\widehat{\widehat{\widehat{E}}} \cong \mathbb{Z}(2)^I$.

The subcategory of all real topological vector spaces G which are pro-Lie groups is the category of *weakly complete real vector spaces*, whose dual category is the category of all real topological vector spaces $\widehat{G}$ endowed with the finest possible vector space topology, as is shown in [9], Appendix 7, pp. 932ff., or [10], Appendix 3, pp. 737ff., and such topological vector spaces are pro-Lie groups only as long as they are finite dimensional. Accordingly, by way of example, if $G = \mathbb{R}^{(J)}$ for an infinite set J (e.g. $J = \mathbb{N}$), then G , equipped with the finest locally convex topology, is the dual of the abelian pro-Lie group $\mathbb{R}^J$ (see [9], p.932ff., [10], p.737ff.) illustrating Theorem 2.3 and showing that the category of abelian pro-Lie groups fails to be closed under passage to the duals.

A core result on any abelian pro-Lie group G says that G is the direct product of a weakly complete real vector group and a closed subgroup whose identity component is compact. (For more details see [10], Theorem 4.22, pp. 144, 145.)

3. K-GROUPS

We recall that a *k-space* (sometimes called a *Kelley space*) is a Hausdorff space X with the following property: a map $f: X \rightarrow Y$ to any other topological space Y is continuous if and only if the restriction of f to every compact subspace $C \subseteq X$ is continuous. These spaces were introduced by HUREWICZ; they play a major role in algebraic topology because they have many favorable properties. Every locally compact space and every first countable space is a k-space. We refer to [5] XI.9 or [6] 3.3 for more results about these spaces. If X is any Hausdorff space, with topology $\mathcal{T}$, then there is a finer topology $\mathcal{T}_{\max} \supseteq \mathcal{T}$ such that $(X, \mathcal{T}_{\max})$ is a k-space. The topology $\mathcal{T}_{\max}$ has an explicit description as follows. A subset $A \subseteq X$ is $\mathcal{T}_{\max}$ -closed if and only if $A \cap C$ is closed, for every compact subset $C \subseteq X$, that is,

$$\mathcal{T}_{\max} = \{U \subseteq X : (X - U) \cap C \text{ is closed for every compact } C \subseteq X\}.$$

It follows that $\mathcal{T}_{\max}$ is the unique largest topology containing $\mathcal{T}$ that has the same compact sets as $\mathcal{T}$. The assignment $(X, \mathcal{T}) \mapsto (X, \mathcal{T}_{\max})$ is functor which is right adjoint to the inclusion functor of k-spaces into Hausdorff spaces. These categorical aspects are discussed in [19] and in [15], VII.8. The category of k-spaces also has some drawbacks. Notably, a product of k-spaces need not be a k-space in the product topology, see Remark 3.12 below. Also, $\mathcal{T}_{\max}$ need not be a group topology if $\mathcal{T}$ is a group topology. Nevertheless, one may study the group objects in the category of k-spaces. This is carried out in LAMARTIN's work [13].

NOBLE [16, 17] introduced a variation of this construction in the context of topological groups in his 1967 PhD thesis and considered k-groups. However, he did not discuss the categorical aspects of his construction. In what follows, we give a self-contained introduction to k-groups from a categorical viewpoint.

The following general lemma is certainly well-known.

Lemma 3.1. *Let G be a group and let τ be a set of group topologies on G (which need not be Hausdorff). Then τ has a unique supremum $\mathcal{T} = \sup \tau$ in the partially ordered set of all topologies on G , and $\mathcal{T}$ is a group topology.*

Proof. For each $\mathcal{S} \in \tau$ let $G_{\mathcal{S}}$ denote the topological group G with topology $\mathcal{S}$. We put $H = \prod_{\mathcal{S} \in \tau} G_{\mathcal{S}}$ and we consider the diagonal map $d: G \rightarrow H$, and $\mathcal{T} = \{d^{-1}(U) : U \text{ open in } H\}$. Then $\mathcal{T}$ is a group topology and an upper bound for τ , because each map $p_{\mathcal{S}} \circ d: G \rightarrow G_{\mathcal{S}}$ is continuous and bijective. If some topology $\mathcal{T}'$ on G is an upper bound for τ , then each map $p_{\mathcal{S}} \circ d$ is $\mathcal{T}'$ -continuous and hence d is $\mathcal{T}'$ -continuous. Thus $\mathcal{T}' \supseteq \mathcal{T}$. This shows that the group topology $\mathcal{T}$ is a least upper bound for τ in the set of all topologies on G . $\square$

The following notion is due to NOBLE [16, 17].

Definition 3.2. We call a homomorphism $f: G \rightarrow H$ between topological groups G, H *k-continuous* if the restriction of f to any compact subset $C \subseteq G$ is a continuous map. A k-continuous homomorphism is thus sequentially continuous. We call G a *k-group* if every k-continuous homomorphism $f: G \rightarrow H$ is continuous, for every topological group H .

For example, every locally compact group and every first countable topological group is a k-group. More generally, a topological group whose underlying topology is a k-space is a k-group.

Lemma 3.3. *The group E in Theorem 1.10 is not a k-group.*

Proof. In this group, every compact set is finite. The largest group topology for which the compact sets are finite is the discrete topology, and thus kE is a discrete group, whereas E is not discrete. $\square$

Note that every morphism $f: E \rightarrow H$ of topological groups is k-continuous while E is not a k-group.

We denote the full subcategory of k-groups in the category of topological groups **tg** by **ktg**.

Construction 3.4. Let G be a topological group, with group topology $\mathcal{T}$. By Lemma 3.1, the set τ of all group topologies on G which have the same compact sets as $\mathcal{T}$ has a unique supremum $k\mathcal{T} = \sup \tau$, and $k\mathcal{T}$ is a group topology. Since $\mathcal{T}_{\max}$ is an upper bound for τ , the group topology $k\mathcal{T}$ has the same compact sets as $\mathcal{T}$, that is, $k\mathcal{T} = \max \tau$. We will see below in 3.12 that $k\mathcal{T}$ may be strictly smaller than $\mathcal{T}_{\max}$.

We denote the resulting topological group by kG for short.

Note that the group topology $k\mathcal{T}$ does not have an explicit description, in contrast to the topology $\mathcal{T}_{\max}$ introduced above.

Lemma 3.5. *The group kG is a k-group, and kG has the same compact subsets as G .*

Proof. Let $\mathcal{T}$ denote the group topology of G . By Construction 3.4, the topological group kG has the same compact subsets as G . Let $f: kG \rightarrow H$ be a k -continuous group homomorphism. Since kG and G have the same compact subsets, f is also k -continuous as a map from G to H . If $B \subseteq H$ is a closed subset, then $f^{-1}(B) \cap C$ is closed for every compact subset $C \subseteq G$, because f is k -continuous. The (possibly non-Hausdorff) group topology $\mathcal{S} = \{f^{-1}(U) : U \subseteq H \text{ open}\}$ on G is therefore contained in the topology $\mathcal{T}_{\max}$. Hence $\mathcal{T}' = \sup\{\mathcal{T}, \mathcal{S}\} \subseteq \mathcal{T}_{\max}$ is a Hausdorff group topology having the same compact sets as $\mathcal{T}$, and f is continuous with respect to $\mathcal{T}'$. But then f is also continuous with respect to $k\mathcal{T} \supseteq \mathcal{T}'$. $\square$

The identity map on the underlying group G is a morphism $\kappa_G: kG \rightarrow G$, which plays a special role.

Proposition 3.6. *Let G be a topological group. Then $\kappa_G: kG \rightarrow G$ has the following universal property. If H is a k -group, and if $f: H \rightarrow G$ is a morphism, then f factors uniquely as $f = \kappa_G \circ f'$,*

$$\begin{array}{ccc} H & & \\ f' \downarrow & \searrow f & \\ kG & \xrightarrow{\kappa_G} & G. \end{array}$$

In particular, G is a k -group if and only if $G = kG$.

Proof. Since kG and G have the same compact subspaces, f is k -continuous as a map from H to kG . Hence $f = f': H \rightarrow kG$ is a morphism. Since κ_G is the identity map on the underlying group G , the map f' is uniquely determined by f . $\square$

Let us now consider the category $\mathbf{tg}$ of topological groups (and continuous group morphisms), and the full subcategory $\mathbf{ktg}$ of k -groups and continuous group morphisms, with the inclusion functor $\iota: \mathbf{ktg} \rightarrow \mathbf{tg}$. We shall now use standard category theoretical notation as is presented e.g. in [9] (Theorem A3.28 ff., in [9] p. 814ff.). The machinery of adjoint functors (see e.g. [9], Definition A3.29ff., see also [10], A1.40 and A1.41) shows the following from Proposition 3.6.

Theorem 3.7. *The inclusion functor $\iota: \mathbf{ktg} \rightarrow \mathbf{tg}$ has a right adjoint $k: \mathbf{tg} \rightarrow \mathbf{ktg}$ that maps a group G to the k -group kG .*

It may be useful to repeat explicitly what this adjunction of functors means. For each topological group G there is a k -group kG and a natural morphism $\kappa_G: \iota kG \rightarrow G$ of topological groups such that for every k -group H and each morphism $f: \iota H \rightarrow G$ of topological groups there is a unique k -morphism $f': H \rightarrow kG$ such that $f = \kappa_G \circ \iota f'$ and that

$$f \mapsto f': \mathbf{tg}(\iota H, G) \rightarrow \mathbf{ktg}(H, kG)$$

is a natural bijection.

Repeated again in other words: for any topological group G we obtain functorially a k -group kG , and a morphism $\kappa_G: \iota kG \rightarrow G$ of topological groups (in fact turning out to be *bijective*). Then the universal property explained above is summarized in the following diagram:

$$\begin{array}{ccc}
 & \text{tg} & \text{ktg} \\
 \hline
 G & \xleftarrow{\kappa_G} \iota kG & kG \\
 & \nwarrow f & \uparrow \exists! f' \\
 & \iota H & H
 \end{array}$$

The diagram might be expressed briefly by saying (as in Proposition 3.6) that *any morphism of topological groups from any k -group H into the topological group G factors through κ_G* .

Remark 3.8. The subcategory $\iota: \text{ktg} \rightarrow \text{tg}$ is what is called a *coreflective* subcategory, with coreflector $k: \text{tg} \rightarrow \text{ktg}$. Such a subcategory has several important properties which we now recall. Suppose that $D: J \rightarrow \text{ktg}$ is any small diagram.

- (1) If $\iota D: J \rightarrow \text{tg}$ has a colimit K , then $K \cong \iota K'$ for some K' in ktg —every colimit that exists in tg already exists in ktg . (For the proof one puts $K' = kK$. From the colimit property of K , there is a universal morphism $K \rightarrow \iota K'$ and from the adjunction there is a universal morphism $\iota K' = \iota kK \rightarrow K$.)
- (2) In particular, if $N \trianglelefteq G$ is a closed normal subgroup in a k -group G , then G/N is a k -group with respect to the quotient topology.
- (3) If $\iota D: J \rightarrow \text{tg}$ has a limit L , then kL is the limit of D in ktg . (This holds because the right adjoint k preserves limits.)
- (4) In particular, if $N \trianglelefteq G$ is a closed normal subgroup of a k -group G , then $kN \rightarrow G$ is the ktg -equalizer of the diagram $G \begin{smallmatrix} \xrightarrow{p} \\ \xrightarrow{\text{const}} \end{smallmatrix} G/N$, with $p(g) = gN$, in ktg . Also, if G_i for $i \in I$ is a family of k -groups, then $k(\prod_{i \in I} G_i)$ is the categorical product of the G_i in ktg .
- (5) By (1) and (3), the category ktg is complete and cocomplete, since tg is complete and cocomplete.

Analogous remarks apply to the category tab of abelian topological groups and the full subcategory of abelian k -groups kab . The inclusion $\text{kab} \rightarrow \text{tab}$ also has the right adjoint k . It follows in this case that a finite product of abelian k -groups (which is also coproduct in tab) is a k -group.

In view of Remark 3.8(4), the following result by NOBLE [16, 18] is rather surprising. We present a simplified version of the proof² given in [16].

Theorem 3.9. *Products of k -groups are again k -groups, with respect to the product topology.*

²The author L.K. did not understand the line of proof presented in [18].

Proof. Let $(G_i)_{i \in I}$ be a family of k -groups, with product $G = \prod_{i \in I} G_i$, with projections $p_i: G \rightarrow G_i$ and with product topology $\mathcal{T}$. We need to show that $\mathcal{T} = k\mathcal{T}$. For $g \in G$ we put $\text{supp}(g) = \{i \in I : p_i(g) \neq e_i\}$, and $G' = \{g \in G : \text{supp}(g) \text{ is countable}\}$. For $J \subseteq I$ we put $G'_J = \{g \in G' : \text{supp}(g) \cap J = \emptyset\}$.

Claim 1. For every $k\mathcal{T}$ -identity neighborhood $V \subseteq G$, there is a finite set $J \subseteq I$ such that $G'_J \subseteq V$.

Assume that the claim is false. We pick an element $g_0 \in G' - V$, and enumerate the countable set $\text{supp}(g_0)$ by a surjective map $\mathbb{N} \rightarrow \text{supp}(g_0)$. Inductively, we choose $g_{n+1} \in G' - V$ in such a way that $\text{supp}(g_{n+1})$ contains none of first n elements in each of the sets in $\text{supp}(g_0), \text{supp}(g_1), \dots, \text{supp}(g_n)$, and we fix a surjective map $\mathbb{N} \rightarrow \text{supp}(g_{n+1})$. Each g_n has its support contained in the countable set $\bigcup_{m \in \mathbb{N}} \text{supp}(g_m) = K \subseteq I$. Moreover, for each $k \in K$ there exists some $m \in \mathbb{N}$ such that $p_k(g_n) = e_k$ holds for all $n \geq m$. Therefore the sequence $(g_n)_{n \in \mathbb{N}}$ converges in the product topology $\mathcal{T}$ to the identity element e . The set $C = \{g_n \mid n \in \mathbb{N}\} \cup \{e\} \subseteq G$ is thus compact (in $\mathcal{T}$, and hence also in $k\mathcal{T}$) with $V \cap C = \{e\}$. This is a contradiction to the fact that V is a $\mathcal{T}_{\max}$ -neighborhood of e , because e is not isolated in C . Hence there is some finite set $J \subseteq I$ of cardinality m , with $G'_J \subseteq V$, and Claim 1 is true.

Claim 2. Let $J \subseteq I$ be a subset. The $k\mathcal{T}$ -closure of G'_J contains $\prod_{j \in J} \{e_j\} \times \prod_{i \in I-J} G_i$.

Let $g = (g_j)_{j \in J} \in \prod_{j \in J} \{e_j\} \times \prod_{i \in I-J} G_i$ and consider the $\mathcal{T}$ -compact set

$$D = \prod_{j \in J} \{e_j\} \times \prod_{i \in I-J} \{e_i, g_i\},$$

which contains g . Then $G'_J \cap D$ is dense in D . Since D is also $k\mathcal{T}$ -compact, g is in the $k\mathcal{T}$ -closure of G'_J , and Claim 2 is true.

Now we prove the theorem. Let $U \subseteq G$ be any $k\mathcal{T}$ -identity neighborhood, and let $V \subseteq U$ be a $k\mathcal{T}$ -identity neighborhood with $VVV \subseteq U$. By Claim 1, there is a finite subset $J \subseteq I$ with $G'_J \subseteq V$, of cardinality m . Let $W \subseteq G$ be a $k\mathcal{T}$ -identity neighborhood with $W^m = W \cdots W$ (m times) $\subseteq V$. For each $j \in J$ we have the inclusion morphism $\iota_j: G_j \rightarrow G$. We choose identity neighborhoods $W_j \subseteq G_j$ such that $\iota_j(W_j) \subseteq W$. Then $Z = \prod_{j \in J} W_j \times \prod_{i \in I-J} G_i$ is a $\mathcal{T}$ -identity neighborhood which is contained in the $k\mathcal{T}$ -closure of $W^m G'_J$, and $W^m G'_J \subseteq VV$. The $k\mathcal{T}$ -closure of VV is contained in VVV , whence

$$Z \subseteq VVV \subseteq U,$$

which shows that U is a $\mathcal{T}$ -identity neighborhood. Thus $\mathcal{T} \supseteq k\mathcal{T} \supseteq \mathcal{T}$. $\square$

Caveat 3.10. We noticed in Remark 3.8 that the category of abelian k -groups forms a full and coreflexive subcategory $\iota: \mathbf{kab} \rightarrow \mathbf{tab}$ of the category of abelian topological groups. The coreflector is the ‘ k -fication’ functor $G \mapsto kG$. The category $\mathbf{kab}$ contains all metrizable abelian groups and all locally compact abelian groups (in fact, it contains

all Čech-complete abelian groups by [6] 3.9.5). Being coreflexive, the category **kab** is complete and cocomplete.

The inclusion functor ι preserves colimits, since it is left adjoint to k . It does *not* preserve limits, as we recall now. Therefore, it is rather surprising that ι preserves products by Theorem 3.9, which are after all special limits.

Example 3.11 (Failures of limit preservation). We list some examples of limits that are not preserved by $\iota: \mathbf{kab} \rightarrow \mathbf{tab}$. All our examples are based on the group E in Theorem 1.10 and the closed injective morphism $\varphi: E \rightarrow A$.

- (1) *Preservation of projective limits fails.* The group E is (in **tab**) a projective limit of the discrete groups $E/H_\alpha \cong K_\alpha$. These groups K_α are k -groups, but E is not.
- (2) *Preservation of equalizers fails.* The morphism $\varphi: E \rightarrow A$ is the **tab**-equalizer of the diagram

$$A \begin{array}{c} \xrightarrow{p} \\ \xrightarrow{\text{const}} \end{array} A/\varphi(E), \quad p(g) = g + \varphi(E),$$

and both A and $A/\varphi(E)$ are k -groups, but E is not.

- (3) *Preservation of intersections fails.* Put

$$D = \{(\varphi(g), -\varphi(g)) : g \in E\} \subseteq A \times A \quad \text{and} \quad B = (A \times A)/D.$$

In B we have the closed subgroups $P = (A \times \varphi(E))/D \cong A$ and $Q = (\varphi(E) \times A)/D \cong A$. Then B, P, Q are k -groups, but $P \cap Q = (\varphi(E) \times \varphi(E))/D \cong E$ is not.

We note that a functor that preserves products and equalizers (or products and intersections) preserves all limits.

Remark 3.12. The underlying topological space of the group A introduced in Definition 1.4 is not a k -space, since otherwise E would also be a k -space, as shown in [6] Theorem 3.3.25. On the other hand, A is a k -group by Theorem 3.9. It follows for the topology $\mathcal{T} = k\mathcal{T}$ of A that $\mathcal{T} \subsetneq \mathcal{T}_{\max}$, and that $\mathcal{T}_{\max}$ is not a group topology for A . Thus, k -groups are not necessarily k -spaces, and closed subgroups of k -groups are not necessarily k -groups. We note also that the weakly complete vector spaces $\mathbb{R}^I$ are k -groups by Theorem 3.9, but that $\mathbb{R}^I$ is not a k -space if I is uncountable, as is noted in Kelley's book [12] p. 240, Exercise J(b).

However, the following is true ³.

Theorem 3.13. *Let $H \subseteq G$ be an open subgroup of the Hausdorff group G . Then H is a k -group if G is a k -group. Conversely, if H is a central k -group, then G is a k -group.*

Proof. Let $\mathcal{T}$ denote the topology on G . If the open subgroup H is a k -group, then $H \rightarrow kG$ is continuous. Since H is open in G then $G \rightarrow kG$ is continuous, whence $\mathcal{T} = k\mathcal{T}$.

³The proof presented in [16] appears to be incomplete.

Conversely, let G be a k -group and let $H \subseteq G$ be a central open subgroup, with subspace topology $\mathcal{S}$. The topology of kH , which refines the subspace topology of H , extends in a unique way to group topology $\mathcal{T}'$ refining the topology $\mathcal{T}$ such that kH is an open subgroup of G in $\mathcal{T}'$. Here we use that H is central in G .

If $C \subseteq G$ is $\mathcal{T}$ -compact, then $C \cap gH$ is $\mathcal{T}'$ -compact for every coset gH . Moreover, C intersects only finitely many such cosets nontrivially. Thus C is also $\mathcal{T}'$ -compact. This shows that $\mathcal{T}' \supseteq \mathcal{T}$ is a group topology having the same compact sets as $\mathcal{T}$, whence $\mathcal{T}' = \mathcal{T}$. $\square$

Remark 3.14. One might also consider the category $k\text{-}\mathbf{tg}$ whose objects are topological groups, and whose morphisms are k -continuous homomorphisms. In this category $k\text{-}\mathbf{tg}$, whose hom-sets are much larger than those of $\mathbf{tg}$, the natural transformation $\kappa_G: kG \rightarrow G$ becomes invertible and gives an equivalence of categories $k\text{-}\mathbf{tg} \simeq \mathbf{ktg}$.

4. ABELIAN K -GROUPS

Now we consider the full subcategory $\mathbf{kab}$ of $\mathbf{tab}$ of abelian k -groups in the category of topological abelian groups.

Remark 4.1. For every topological abelian group G , the group morphism

$$\eta_G: G \rightarrow \widehat{\widehat{G}}$$

is k -continuous.

Proof. See the proof in [9], Theorem 7.7(iii) or [2] Proposition 13.4.1. $\square$

From this remark and Proposition 2.1 we get immediately a result stated in the introduction.

Proposition 4.2. *If G is a pro-Lie group which is also a k -group, then there is an isomorphism of topological groups*

$$G \xrightarrow[\cong]{\eta_G} \widehat{\widehat{G}}.$$

In particular, $\widehat{\widehat{G}}$ is then a k -group.

Let G be a topological abelian group. If $\eta_G: G \rightarrow \widehat{\widehat{G}}$ has a continuous inverse, as is the case for abelian pro-Lie groups by Proposition 2.1, we have also the $\mathbf{kab}$ -morphism $k(\eta_G^{-1}): k\widehat{\widehat{G}} \rightarrow kG$. Accordingly, we get

Proposition 4.3. *For an abelian pro-Lie group G , there is an isomorphism*

$$kG \xrightarrow[\cong]{k\eta_G} k\widehat{\widehat{G}}$$

inside **kab** and there is a commutative diagram

$$\begin{array}{ccc} kG & \xleftarrow[k\eta_G^{-1}]{\cong} & \widehat{\widehat{kG}} \\ \kappa_G \downarrow & & \downarrow \kappa_{\widehat{\widehat{G}}} \\ G & \xleftarrow[\eta_G^{-1}]{} & \widehat{\widehat{G}} \end{array}$$

of bijective **tab**-morphisms.

Quite generally, we have the morphism $\widehat{\kappa_G}: \widehat{G} \rightarrow \widehat{kG}$ and its dual $\widehat{\widehat{\kappa_G}}: \widehat{\widehat{kG}} \rightarrow \widehat{\widehat{G}}$.

We note that there are 3 commutative square diagrams of **kab**-morphisms as follows:

Firstly,

$$\begin{array}{ccc} kG & \xrightarrow{\eta_{kG}} & \widehat{\widehat{kG}} \\ \kappa_G \downarrow & & \downarrow \widehat{\widehat{\kappa_G}} \\ G & \xrightarrow{(\eta_G)} & \widehat{\widehat{G}}. \end{array}$$

Secondly,

$$\begin{array}{ccc} kG & \xrightarrow{k\eta_G} & \widehat{\widehat{kG}} \\ \kappa_G \downarrow & & \downarrow \kappa_{\widehat{\widehat{G}}} \\ G & \xrightarrow{(\eta_G)} & \widehat{\widehat{G}}. \end{array}$$

Thirdly,

$$\begin{array}{ccc} kG & \xrightarrow{\eta_{kG}} & \widehat{\widehat{kG}} \\ k\eta_G \downarrow & & \downarrow \widehat{\widehat{\kappa_G}} \\ \widehat{\widehat{kG}} & \xrightarrow{\kappa_{\widehat{\widehat{G}}}} & \widehat{\widehat{G}}. \end{array}$$

In the commutative diagrams, the dotted arrow may not be continuous, but all arrows are morphisms in **ab**. All of them have one and the same diagonal morphism

$$d_G = \widehat{\widehat{\kappa_G}} \circ \eta_{kG} = \kappa_{\widehat{\widehat{G}}} \circ k\eta_G = \eta_G \circ \kappa_G.$$

By Proposition 3.6 the morphism η_{kG} factors through $\kappa_{\widehat{\widehat{kG}}}: \widehat{\widehat{kG}} \rightarrow \widehat{\widehat{kG}}$, so that $\eta_{kG} = \kappa_{\widehat{\widehat{kG}}} \circ \eta_{kG}$ and we obtain $d_G = \widehat{\widehat{\kappa_G}} \circ \kappa_{\widehat{\widehat{kG}}} \circ \eta_{kG}'$.

The morphism η_G in the category of abelian groups **ab** is k -continuous by Remark 4.1 and is bijective and open if G is an abelian pro-Lie group by Proposition 2.1. Since both κ_G and η_G are bijective in this case, we obtain

Remark 4.4. For every abelian pro-Lie group G , the morphism $d_G: kG \rightarrow \widehat{\widehat{G}}$ is bijective.

The example of Section 1 with $G = E$ from Definition 1.1 illustrates the present circumstances. We had $\widehat{E} = \bigcup_{\beta \in I} \widehat{K_\beta}$ in Proposition 1.6, which is dense in $\mathbb{Z}(2)^I$. Here $kE \cong \mathbb{Z}(2)^{(I)}$ with the discrete topology and, accordingly, $\widehat{kE} \cong \mathbb{Z}(2)^I$, as was already indicated in 1.6, and so we have $\widehat{\widehat{kE}} \cong \mathbb{Z}(2)^{(I)} \cong kE$. In a diagram, we have the following situation:

$$\begin{array}{ccc}
 \mathbb{Z}(2)^{(I)} \cong kE & \xrightarrow[\cong]{\eta_{kE}} & \widehat{\widehat{kE}} \cong \mathbb{Z}(2)^{(I)} \\
 \downarrow \kappa_E & \searrow d_E & \downarrow \widehat{\kappa_E} \\
 E & \xrightarrow{(\eta_E)} & \widehat{\widehat{E}} \cong \mathbb{Z}(2)^{(I)}, \\
 & & \uparrow \widehat{\kappa_E} \\
 & & \widehat{E} \subset \mathbb{Z}(2)^I.
 \end{array}$$

This discussion has led to various representations of the significant morphism $d_G = kG \rightarrow \widehat{\widehat{G}}$. Chances are that it is an isomorphism for abelian pro-Lie groups. Since we know from Proposition 4.3 that $k\eta_G$ is an isomorphism for abelian pro-Lie groups, we record the following observations:

Corollary 4.5. *For an abelian pro-Lie group G , the following statements are equivalent:*

- (i) $\widehat{\widehat{G}}$ is a k -group.
- (ii) $\kappa_{\widehat{\widehat{G}}}$ is an isomorphism.
- (iii) d_G is an open morphism.
- (iv) d_G is an isomorphism of topological groups.

The Example E of Theorem 1.10 shows that these statements will not imply that $\eta_G: G \rightarrow \widehat{\widehat{G}}$ is an isomorphism.

These aspects encourage us to ask the following question, which may be an indication that the duality of abelian topological groups is still a source of challenges:

Question. *Is the double dual $\widehat{\widehat{G}} = \text{Hom}(\text{Hom}(G, \mathbb{R}/\mathbb{Z}), \mathbb{R}/\mathbb{Z})$ of an abelian pro-Lie group G always a k -group?*

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